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CONFORMALLY FLAT RIEMANNIAN MANIFOLDS ADMITTING A ONE-PARAMETER GROUP OF CONFORMAL TRANSFORMATIONS

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Introduction

It is a classical theorem of H. A. Schwarz [4] that a compact Riemann surface of genus greater than 1 has only a finite number of conformal transformations. It is furthermore known that in the case of genus 1, i.e. on a torus, there is no one-parameter group of non-isometric conformal transformations.

For Riemannian manifolds the following is a well-known conjecture:

Conjecture. *If a compact Riemannian n-manifold M, $n > 2$, admits an essential one-parameter group of conformal transformations, then M is conformorphic to a Euclidean sphere S^n.*

Here a one-parameter group and the vector field defined by it are said to be *essential* if there is no Riemannian metric, conformal to the original one, with respect to which the group is a group of isometries, and by "conformorphic" we mean "conformally diffeomorphic". Furthermore, throughout this paper manifolds under consideration are assumed to be connected and of C^∞.

In a previous paper [3], we have proved the following Theorems A and B:

Theorem A. *Let M be a Riemannian n-manifold, $n > 2$, admitting an essential one-parameter graup f_t of conformal transformations. Then*

(i) *f_t has a fixed point,*

(ii) *M is conformorphic to either a Euclidean n-sphere S^n or a once-punctured n-sphere $S^n - \{p_\infty\}$ provided that the vector field defined by f_t has nonvanishing divergence at each of the fixed points of f_t.*

The vector field induced by a one-parameter group of conformal transformations is called a *conformal* vector field, and a fixed point of a one-parameter group is called a *zero* or a *singular point* of the corresponding vector field.

Theorem B. *Let u be an essential conformal vector field on a Euclidean n-sphere S^n. Then u satisfies one of the following two properties:*

(i) *u has exactly one singular point p_0, at which the divergence of u vanishes, and the orbit $f_t(p)$ of u through a point p satisfies*

$$\lim_{t \to \pm\infty} f_t(p) = p_0 ,$$

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(ii) *u has exactly two singular points p_0 and p_∞, at each of which the divergence is not zero, and the orbit $f_t(p)$ through a point $p \notin \{p_0, p_\infty\}$ connects p_0 and p_∞.*

The vector field in Theorem A corresponds to a vector field of the second type in Theorem B, and the conformal-flatness of a Riemannian n-manifold, $n > 2$, is an implication of the existence of such a vector field.

The purpose of the present paper is to establish the

Theorem. *Let M be a conformally flat Riemannian n-manifold, $n > 2$, with finite fundamental group. If M admits an essential one-parameter group f_t of conformal transformations, then M is conformorphic to either a Euclidean n-sphere S^n or a once-punctured Euclidean n-sphere $S^n - \{p_\infty\}$.*

This is a parital solution to the conjecture and a generalization of a result of Nagano [2] for the transitive group of conformal transformations.

1. Lemmas

Lemma 1. *Let G and Γ be respectively groups of isometries and conformal transformations of a Riemannian manifold (M, g). If G and Γ are commutative element-wise, and Γ is compact, then there is a Riemannian metric g^* conformal to g such that both G and Γ are groups of isometries of (M, g^*).*

Proof. We put

$$g^* = \int_\Gamma \gamma^* g d\gamma\ , \qquad \gamma \in \Gamma,$$

where $d\gamma$ denotes the invariant measure on Γ with $\int_\Gamma d\gamma = 1$. Then g^* is a Riemannian metric conformal to g since each $\gamma \in \Gamma$ is a conformal transformation. Obviously Γ is a group of isometries of (M, g^*). For any $\sigma \in G$, we have

$$\sigma^* g^* = \int_\Gamma \sigma^* \gamma^* g d\gamma = \int_\Gamma \gamma^* \sigma^* g d\gamma = \int_\Gamma \gamma^* g d\gamma = g^*\ .$$

Thus G is a group of isometries of (M, g^*)

Lemma 2. *Let G be an essential one-parameter group of conformal transformations of a Riemannian manifold (M, g). If the fundamental group Γ of M is finite, then G is essential as a group of conformal transformations of the universal covering space $(\tilde{M}, \tilde{g})$ of (M, g), where $\tilde{g}$ is locally isometric to g by means of the natural projection.*

Proof. Since $(\tilde{M}, \tilde{g})$ is the Riemannian covering of (M, g), Γ is a finite group of isometries. G acts on $(\tilde{M}, \tilde{g})$ as a group of conformal transformations and G and Γ are commutative element-wise.

Assume that G is inessential, so that on $\tilde{M}$ there is a Riemannian metric $\tilde{h}$ conformal to $\tilde{g}$ such that G is a group of isometries with respect to $\tilde{h}$. Obviously Γ is a group of conformal transformations of $(\tilde{M}, \tilde{h})$. Since Γ is finite and commutative with G element-wise, by Lemma 1, on $\tilde{M}$ there is a Riemannian metric $\tilde{h}^*$ conformal to $\tilde{h}$ and hence to $\tilde{g}$, such that both Γ and G are groups of isometries of $(M, \tilde{h}^*)$. Since Γ is a group of isometries, $\tilde{h}^*$ can be projected to a Riemannian metric h on M so that $\pi^*h = \tilde{h}^*$, where π denotes the natural projection $\tilde{M} \to M$. Therefore G is a group of isometries of (M, h), contrary to our assumption that G is essential on (M, g).

2. Proof of the theorem

Let $(\tilde{M}, \tilde{g})$ be the Riemannian universal covering space of (M, g). Then f_t acts on $(\tilde{M}, \tilde{g})$ as a group of conformal transformations and is essential by Lemma 2. Let $\tilde{u}$ be the corresponding conformal vector field on $(\tilde{M}, \tilde{g})$. Clearly $\pi\tilde{u} = u$. Since $(\tilde{M}, \tilde{g})$ is conformally flat and simply connected, it is conformorphic to an open set W of S^n[1], and by this conformorphism, $\tilde{u}$ corresponds to a vector field U on W. Let $\tilde{p}_0$ be a singular point of $\tilde{u}$, and $\tilde{P}_0$ the corresponding singular point of U. Since U is uniquely determined by the values of U, the covariant derivatives ∇U of U, and the divergence Φ of U, at P_0, [2], U can be extended to a conformal vector field $\tilde{U}$ on S^n. It is clear that $\tilde{U}$ is essential on S^n. Let $\tilde{f}_t$ be the one-parameter group generated by $\tilde{U}$. Then W must be invariant by $\tilde{f}_t$ so that $\tilde{f}_t(W) \subset W$.

i) If U has vanishing divergence at P_0, then P_0 is the only singular point of $\tilde{U}$, and by Theorem B there is no invariant open subset of S^n except S^n itself. Therefore $W = S^n$, and $(\tilde{M}, \tilde{g})$ is conformorphic to S^n.

Let $p_0 = \pi(\tilde{p}_0)$. Then any points $\tilde{p} \in \pi^{-1}(p_0)$ are singular points of $\tilde{u}$ on $\tilde{M}$. However $\tilde{u}$ has only one singular point and therefore the fundamental group is trivial. Hence (M, g) itself is conformorphic to S^n.

ii) If U has nonvanishing divergence at p_0, then on account of i) the divergence never vanishes at any other singular points of $\tilde{U}$, if any. Therefore u has the same property as U, and by Theorem A, (M, g) is conformorphic to either S^n or $S^n - \{p_\infty\}$.

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