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THE SPHERICAL TWO-PIECE PROPERTY AND TIGHT SURFACES IN SPHERES

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A set A in E^3 is said to have the *spherical two-piece property* (STPP) if no plane or sphere in E^3 separates A into more than two pieces. Examples of such sets include a sphere, a plane, a circle, an annulus, and a torus of revolution. In the first part of this paper we give a characterization of all closed sets in the plane with the STPP.

A surface M^2 in E^n is said to be *tight* if no hyperplane separates M^2 into more than two pieces. (This is equivalent to the hypothesis that M has minimal total absolute curvature in the sense of Chern and Lashof.) We proceed to identify all tight smooth surfaces which are subsets of spheres in E^n. By a result of Kuiper, the only tight surface in a sphere S^{n-1} which is not contained in a 3-sphere S^3 is a Veronese surface, which is an algebraic surface in S^4. We show as the main result of this paper that the only tight surfaces in S^3 other than S^2 are the images under inverse stereographic projection of cyclides of Dupin, a class of algebraic tori in E^3.

Several of the results in the first part of this paper have been generalized by S. Glass [8].

1. The circle-two-piece property

A set A in the plane E^2 has the *circle-two-piece-property* (CTPP) if every circle or straight line S in E^2 separates A into at most two pieces. In an earlier study [2] we examined objects with the ordinary two-piece property, i.e., those which are separated into at most two pieces by every line in E^2, and we related this property to the study of total absolute curvature. The CTPP is also related to total absolute curvature, but for objects in a sphere rather than in Euclidean space, and we make this relationship more precise in the latter sections of this paper. In this section however, we give a treatment which is independent of the total absolute curvature theory, and also independent of the ordinary TPP.

The whole space E^2 has the CTPP, since any circle or straight line S determines precisely two open complementary components D_1 and D_2, with closures $\bar{D}_i = D_i \cup S, i = 1, 2$. The empty set, a one-point set, and a set consisting of just two points all have the CTPP trivially. The last-mentioned

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set is the only CTPP set which is not path-connected, where a set A is called *path-connected* if for every two points p and q of A, there is a path γ, the continuous image of a closed interval, from p to q in A. Using this notion, we make the definition of the CTPP more precise:

Definition. A set A in E^2 has the CTPP if $A \cap \bar{D}_i$ is path-connected, for either complementary component D_i of any circle or straight line S in E^2.

Proposition 1.1. *A circle A has the* CTPP.

Proof. A circle A and a distinct circle or straight line S meet in at most two points, so $A \cap \bar{D}_i$ must be either a closed circular arc, the whole circle A, or the empty set. In any case $A \cap \bar{D}_i$ is path-connected so A has the CTPP.

Note that a straight line does *not* have the CTPP, and in fact the only non-empty subsets of a straight line which can have the CTPP are the sets consisting of one or two points.

Proposition 1.2. *A closed disc A (i.e., a disc together with its circular boundary ∂A) and a closed half-plane both have the* CTPP.

Proof. If A is a closed disc, and p and q lie in $A \cap \bar{D}_1$, then the segment $\overline{pq}$ meets D_2 in an open subinterval (ab) with endpoints on S (which may be empty, and which must be empty if S is a straight line). If S is a circle, then one of the circular arcs, say $\widehat{ab}$, from a to b along S must lie in A, since S has the CTPP. Then we may connect p to q by $\overline{pa} \cup \widehat{ab} \cup \overline{bq}$. Similarly the closed half-plane has the CTPP.

Other examples of CTPP objects are an annulus, a pair of internally tangent circles together with the region between them, and generally, a closed disc with a collection of open discs removed. The fact that these have the CTPP follows from the following proposition.

Proposition 1.3. *If A is a closed disc, a closed half-plane, or the whole space, and if $\{B^j\}$ is a collection of disjoint open discs in A, then $A - \bigcup B^j$ has the* CTPP.

Proof. If p and q lie in $(A - \bigcup B^j) \cap D$, then, as in Proposition 1.2, there is a path $\gamma = \overline{pa} \cup \widehat{ab} \cup \overline{bq}$ from p to q in $A \cap D_1$. The path γ meets the disc B^j in at most two disjoint subpaths of γ. Let a_j be the first point at which γ meets B^j and b_j the last. As in the previous proposition, we may find an arc $\widehat{a_j b_j}$ from a_j to b_j on B_j and lying within D_1, and we may form a new "path" γ' from p to q by replacing the subpath from a_j to b_j by the arc $\widehat{a_j b_j}$. In the case where the number of discs is finite, the set γ' is automatically a path, but in the case of an infinite (necessarily countable) collection $\{B^j\}$ we must still show that γ' is the continuous image of a closed interval. (The analogous theorem in the case of the ordinary two-piece property requires additional hypotheses to insure the continuity, cf. [2].) Continuity follows however since for any m, there are only finitely many disjoint discs of radius greater than $1/m$ which will meet γ. Thus any sequence of points $\{p'_n\}$ of γ' which converge to a point p' of A either has all but a finite number of points in a finite number of arcs, in which case p' is in γ', or $\{p_n\}$ has points in a sequence of discs with radii converging to zero, so again p lies in γ', and in fact in $\gamma \cap \gamma'$.

An interesting use of a CTPP set obtained by removing an infinite number of disjoint open discs from the disc so that no interior points remain is the so-called "Swiss Cheese" used as a counterexample in the theory of function algebras. In this context it is usually required that the sum of the radii of the B_j be finite to insure that γ' has finite length, but we do not require any finite length condition for our paths; cf. [7].

Note that all the examples considered in Proposition 1.3 are closed sets. Recall the following definitions: The frontier of $A = \partial A = \{p$ in A such that any disc about p in E^2 meets A and $E^2 - A\}$. The *closure* of A is $\bar{A} = A \cup \partial A$, and A is said to be *closed* if $A = \bar{A}$. The *interior* of $A = \mathring{A} = A - (\partial A \cap A)$, and A is *open* if $A = \mathring{A}$.

We shall prove as our main theorem in this section, a sort of converse of Proposition 1.3:

Theorem 1.4. *The only connected closed sets in E^2 with the* CTPP *are a closed disc, a closed half-plane, or the whole plane, each with a collection of disjoint open discs removed.*

This class includes a single point, as a disc with zero radius, and the empty set—a disc with negative radius. Also included is a circle, as a closed disc with its interior (an open disc) removed.

The first step in the proof is to show that any simple closed curve with the CTPP must be a circle. We recall some facts about simple closed curves from the topology of plane sets. A *simple closed curve A* in E^2 is a $1-1$ continuous image of a circle. Each such curve determines precisely two complementary domains, one bounded B_1 and the other unbounded B_2. Four distinct points a, b, c, d on A are said to be in *cyclic order* on A if b lies on one of the subpaths of A from a to c and d lies on the other. If $a, b, c,$ and d are in cyclic order on A, then any path from a to c in $\bar{B}_i$ meets any path from b to d in $\bar{B}_i$, for $i = 1, 2$. Also, if A is a simple infinite curve, then A separates E^2 to two complementary components B_1 and B_2, and four points a, b, c, d on A are *in order* if any path in $\bar{B}_i$ from a to c meets any path in $\bar{B}_i$ from b to d.

(Basic) Proposition 1.6. *If A is a simple closed curve in E^2 with the* CTPP, *then A is a circle.*

Proof. If A is a simple closed curve which is not a circle, then we may find four points a, b, c, d in cyclic order on A such that d lies in one of the complementary components D_1 of the circle (or straight line) S determined by $a, b,$ and c. We may then find a circle S' through a and c sufficiently close to S such that b and d both lie in D_1'. By changing the radius by a sufficiently small amount, we obtain a circle S'' with b and d in D_1'' and a and c in D_2''. Thus any path from a to c on A meets D_1'', so there is no path from a to c in $A \cap \bar{D}_2''$, so A does not have the CTPP.

Remark. The same reasoning shows that if A is a simple infinite curve which is not a straight line, then it is possible to find a circle S and four points a, b, c, and d in order on A such that a and c lie in D_1, and b and d in D_2.

Proposition 1.7. *If A is a region bounded by a simple (closed or infinite) curve ∂A, and has the* CTPP, *then ∂A is a circle or a straight line.*

Proof. If ∂A is a simple closed curve which is not a circle, then as in Proposition 1.6, we may find a, b, c, d in (cyclic) order on ∂A, and a circle S with a and c in D_1 and b and d in D_2. If there is no path from b to d in $A \cap \bar{D}_2$, then A does not have the CTPP and we are done. If there is a path γ from b to d in $A \cap \bar{D}_2$, then we may find a circle S' of slightly different radius so that a and c lie in D_1' and γ lies in D_2'. Then any path from a to c in the region A meets γ, so there is no path from a to c in $A \cap \bar{D}_1'$ and A does not have the CTPP. For a simple infinite curve ∂A, the analogous reasoning shows that ∂A must be a straight line if A has the CTPP.

We now recall a proposition which properly belongs with the study of the ordinary two-piece property (TPP), but which we prove here for completeness. (A set A has the TPP if $A \cap \bar{H}_1$ is connected for every half-space in E^2.)

Lemma 1.8. *If A is a set with the* TPP *with unbounded complementary components $\{C^j\}$, then $E^2 - \bigcup C^j$ is convex.*

Proof. First of all, $E^2 - \bigcup C^j$ is contained in the convex hull $H(A)$ of A ($=$ the intersection of all closed half-planes containing the bounded set A), since a point in a bounded complementary component necessarily lies in an interval with endpoints in A. If $E^2 - \bigcup C^j \neq H(A)$, then there must be a point p on $\partial H(A)$ which is in some C^j. But any such point must lie on a segment $\overline{ab}$ for a and b in A, where the line determined by a and b bounds a half-plane $\bar{H}_1$ which contains A. But then $A \cap \bar{H}_2$ is not connected, so A does not have the TPP. Note that for an unbounded convex set $H(A)$, we must have $\partial H(A) = \emptyset$, a pair of parallel straight lines, or a simple infinite curve.

Proposition 1.9. *If A is a bounded closed connected* CTPP *set with unbounded complementary component C, then $E^2 - C$ is a closed disc.*

Proof. By the previous proposition, $E^2 - C$ is convex. If $E^2 - C$ is contained in a line, then $E^2 - C$ must be a bounded closed interval $\overline{pq}$. If $p \neq q$, then for the disc D_1 with diameter $\overline{pq}$, we have $A \cap \bar{D}_2 = \{p, q\}$ and $A \cap \bar{D}_2$ is not connected. Thus either $E^2 - C$ is a point (a disc with 0 radius), or a convex set with interior, bounded by a simple closed curve, which must be a circle, by the previous proposition.

To show that all bounded components must also have circles as boundaries, we use a transformation which reduces the problem to the previous case.

We recall the definition and properties of inversion with respect to a circle with center x and radius ρ. The map $I_{x,\rho}\colon E^2 - \{x\} \to E^2 - \{x\}$ sends p to a point $I_{x,\rho}(p)$ on the ray from x through p such that $|p - x| \cdot |I_{x,\rho}(p) - x| = \rho^2$. This map sends the set of circles not through x into itself, maps the set of circles through x and the set of lines not through x into one another, and preserves lines through x. It follows that if A is a CTPP set and $x \notin A$, then for any ρ, the set $I_{x,\rho}(A)$ also has the CTPP.

We now proceed to the proof of the main theorem:

Proof of Theorem 1.4. If A is a bounded connected CTPP set, then by Proposition 1.9, either A is a point or the boundary of the unbounded complementary component is a circle. If x is a point in a bounded complementary component B of an arbitrary CTPP set A, then $I_{x,\rho}(A)$ is a bounded CTPP set which has $I_{x,\rho}(B)$ as its unbounded complementary component, so $I_{x,\rho}(B)$, and B as well, will have circles as boundaries.

If A is unbounded, then $\partial H(A)$ is either empty, a pair of parallel straight lines, or a simple infinite curve. If $\partial H(A)$ is empty, then A is $E^2 - \bigcup B^j$. If $\partial H(A)$ is an infinite strip, then A does not have the CTPP. If $\partial H(A)$ is a simple infinite curve, then by Proposition 1.7, ∂A must be a straight line and A is a half-plane with a collection of disjoint open discs removed. If A is bounded, then A is a closed disc with discs removed.

2. Additional results on the CTPP

In this section, we present a collection of results and techniques which will be used in the latter section of the paper.

Definition. A connected set A on the sphere S^2 has the CTPP iff $A \cap \bar{D}_i$ is pathwise connected for any complementary component D_i of a circles S on S^2.

Proposition 2.1. *A connected closed set A on S^2 has the* CTPP *if and only if A is S^2 with a collection of disjoint open discs B^j removed, where each disc B^j is the intersection of S^2 with an open half-space in E^3.*

Proof. We reduce this to a previous problem by using stereographic projection $\pi_p\colon S^2 - \{p\} \to E^2$, where E^2 is the plane through the origin perpendicular to the vector to p, and $\pi_p(x)$ is the point in E^2 collinear with p and x. This projection sends circles not through p to circles in E^2, and circles through p to straight lines in E^2. Thus a closed set A in $S^2 - \{p\}$ has the CTPP if and only if $\pi_p(A)$ is a bounded CTPP set in E^2.

Now if A is not S^2 itself, we may choose p in $E^2 - A$ to get a bounded connected closed set with the CTPP in E^2. By Theorem 1.4, such a set is a closed disc with a collection of open circular discs removed, and therefore A itself is either a point or the complement of a number of discs with circular boundaries.

Using stereographic projection, we may present an additional proof of the (Basic) Proposition 1.6, and tie the subject in with total absolute curvature.

The fundamental observation is that a set B on S^2 has the CTPP on S^2 if and only if B has the TPP as a subset of E^3, since a plane E^2 in E^3 separates B into at most two pieces if and only if the circle $E^2 \cap S^2$ separates B into at most two pieces in S^2.

Proof 2 *of Proposition* 1.6. If A is a simple closed curve in E^2 with the CTPP, then $\pi_p^{-1}(A)$ is a simple closed curve with the CTPP in $S^2 - \{p\}$, and therefore is a simple closed curve in E^3 with the TPP. But a simple closed curve with the TPP has at most one local maximum for every linear (height) function on E^3, and this, for smooth curves, is equivalent to the condition of minimal total absolute curvature, i.e., $\int_{s \in C} |k(s)|\, ds = 2\pi$. In [2] we showed that a TPP simple closed curve must lie in a plane and be convex, generalizing the classical result of Fenchel [6] for twice-differentiable curves. Therefore $\pi_p^{-1}(A)$, as a TPP curve, must lie in a plane as well as on the sphere, so $\pi_p^{-1}(A)$, and A as well, must be circles.

Using the concept of inversion with respect to a circle we may prove some propositions:

Proposition 2.2. *If A is a simple closed curve in the plane such that $I_{x,\rho}(A)$ is convex for any x not in A and any ρ, then A is a circle.*

Proof. If A is not a circle, then we can find a circle S which separates A into more than two pieces. But then for a point x on S not in A, the image of S under $I_{x,\rho}$ is a straight line which separates $I_{x,\rho}(A)$ into more than two pieces, so $I_{x,\rho}(A)$ is not convex.

Proposition 2.3. *If A is a simple closed curve in the sphere S, which is not a circle, then in each complementary component of A, there is a circle which meets A in a non-connected set.*

Proof. First we project A by stereographic projection π_p from a point p of S not on A to obtain a simple closed curve in the plane, which is not a circle. It follows from the proof of Proposition 2.2 that in each complementary component there is an x such that the image under $I_{x,\rho}$ is non-convex. But for any non-convex simple closed curve in the plane, there is a support line which meets the curve in a non-connected set. In this way we obtain a pair of circles (or line segments), one in each complementary component of $\pi_p(A)$, each meeting $\pi_p(A)$ is a non-connected set, and this gives a pair of circles on A, which bound discs containing A and meet A in non-connected sets.

Corollary 2.4. *If A is a closed connected set on S^2 without the* STPP, *then there is a circle C on S^2 which bounds a disc D containing A and such that $C \cap A$ is not connected.*

Proof. Choose a complementary component B of A, which is bounded by a simple closed curve which is not a circle. By the previous proposition, we may find a circle C in B meeting A in a non-connected set, as required.

In [1] we examined the critical points of the angular coordinate function θ_x of a polar coordinate system centered at x, and showed that θ_x has exactly two critical points on a curve A for every point x not within A if and only if A is convex. Here we may consider instead the radial coordinate function $\rho_x\colon E^2 \to E^2$, $\rho_x(p) = |x - p|$.

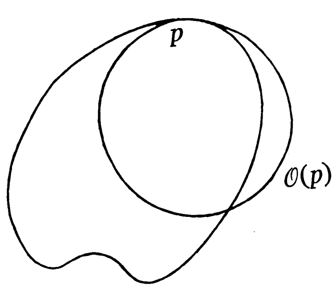
Proposition 2.3. *If A is a simple closed curve and ρ_x has at most two critical points for almost every x not on A, then A is a circle.*

Proof. If A is not a circle, we may find a circle S with center x which separates A into at least four pieces, so ρ_x must have an extremum on each piece. This is also true for all circles sufficiently close to S, so for an open set of x in E^2, the function ρ_x has more than two extrema on A.

We now consider a twice-differentiable curve A in E^2 with a well-defined curvature $k(p)$ at every point p of A. At every point p, we have a well-defined osculating circle $\mathcal{O}(p)$, tangent to A at p with radius (of curvature) $r(p) = 1/k(p)$, which agrees with A up to the second order (i.e., the Taylor expansions of A and $\mathcal{O}(p)$ about p have identical second order terms). It follows that for any circles S different from $\mathcal{O}_p(k)$ and tangent to A at p, there is a neighborhood ω of p on A such that $\omega - \{p\} \subset D_i$ for one of the complementary components of D_i of S. We describe the circles tangent to A at the vector $\vec{p}$ by choosing a unit normal $\vec{n}(p)$ to A at p, and letting $S(p, k)$ be the circle with center $\vec{p} + \vec{n}(p)/k$ and radius $|1/k|$. (Here k can be any real number, so $S(p, 0)$ denotes the tangent line of A at p, and $S(p, k(p)) = \mathcal{O}(p)$.) Since the curvatures at the points of a closed curve are bounded, the circles $S(p, k)$ for $|k|$ sufficiently large meet the curve A at just the one point p. Let $\underline{k}(p)$ and $\bar{k}(p)$ be the two numbers such that $S(p, k) \cap A \neq \{p\}$ if $\underline{k}(p) < k < \bar{k}(p)$ and $S(p, k) \cap A = \{p\}$ if $k < \underline{k}(p)$ or $\underline{k}(p) < k$. Thus A is contained between the circles $S(p, \underline{k}(p))$ and $S(p, \bar{k}(p))$ tangent to A at p.

Proof of Proposition 1.6 *for twice-differentiable curves.* If A has the CTPP, then $\underline{k}(p) = k(p)$. Otherwise $\bar{D}_i(p, \underline{k}(p)) \cap \omega = \{p\}$ for some neighborhood ω of p on A, and $\bar{D}_i(p, \underline{k}(p)) \cap (A - \omega)$ is non-empty or we could find a larger value for $\underline{k}(p)$. Thus $\bar{D}_i(p, k(p)) \cap A$ is not connected so A does not have the CTPP. Similarly $\bar{k}(p) = k(p)$, so A, which lies between $S(p, \underline{k}(p))$ and $S(p, \bar{k}(p))$, must equal $\mathcal{O}(p)$.

Remark. The osculating circle $\mathcal{O}(p)$ has the property that it separates the plane into two regions such that $I_{x,\rho}(A)$ is convex at $I_{x,\rho}(p)$ if x is in one complementary component of $\mathcal{O}(p)$, and concave if x is in the other component (where a closed curve C is said to be convex at a point p if there is a segment through p not containing any points interior to C). This follows since the circle



through x tangent to A at p will support the image of A locally, and the support segment will lie inside or outside the region bounded by $I_{x,p}$ depending on the position of x relative to $\mathcal{O}(p)$. It is *only* for points x on $\mathcal{O}(p)$ that the image curve can have a tangent which has order of contact greater than two at the image of p.

Remark. If we restrict consideration to closed bounded 2-manifolds-with-boundary in E^2 or S^2, then we may express the main theorem of § 1 in a more symmetrical form. A 2-manifold-with-boundary A in E^2 is a non-empty set such that every point p of ∂A has a neighborhood B in E^2 such that $B \cap A$ is in $1 - 1$ continuous correspondence with the intersection of an open disc about the origin with the closed upper half-plane. A bounded 2-manifold-with-boundary A is characterized by the fact that A is closed, A has no components which are simple closed curves, and every component of ∂A is a simple closed curve.

Theorem 1.4 for Manifolds-with-Boundary. *A 2-manifold-with-boundary A has the* CTPP *if and only if A is pathwise connected and every component of ∂A has the* CTPP.

3. TPP surfaces in spheres and the STPP

Definition. A set A in E^{n+1}(or S^{n+1}) has the spherical-two-piece-property (STPP) if $A \cap \bar{D}_i$ is pathwise connected for any complementary component $D_i (i = 1, 2)$ of a hyperplane or hypersphere S in E^{n+1}(or S^{n+1}).

Proposition 3.1. *An n-sphere S^n in E^{n+1} has the* STPP *if $n > 0$. A hyperplane H^n in E^{n+1} has the* STPP *if $n > 1$.*

Proof. For any $S, \bar{D}_i \cap S^n$ is either S^n, the empty set, a single point, or an n-disc on S^n bounded by an $(n - 1)$-sphere, and for $n > 0$, these sets are all connected. Similarly, for any $S \neq H^n$, $S \cap H^n$ is either the empty set, an $(n - 1)$-sphere, or an $(n - 1)$-plane, and each of these will separate H^n into at most two pieces if $n > 1$.

We recall from [2] that a set A is said to have the two-piece-property (TPP) if $A \cap \bar{D}_i$ is pathwise connected for any complementary component $D_i (i = 1, 2)$ of a hyperplane H^n of E^{n+1}.

Proposition 3.2. *A set A contained on an S^n in E^{n+1} has the* TPP *as a subset of E^{n+1} if and only if A has the* STPP *as a subset of S^n.*

Proof. This follows since each hyperplane H^n in E^{n+1} meets S^n in a point, the empty set, or an $(n - 1)$-sphere, and $A \cap \bar{D}_i$ for a complementary component D_i of H^n in E^{n+1} is identical with $A \cap \bar{D}'_i$ for a complementary component $D'_i = D_i \cap S^n$ of $H^n \cap S^n$ in S^n.

Theorem 3. *If A is a smoothly embedded 2-sphere in E^n which has the* STPP, *then A must be (Euclidean) 2-sphere.*

Proof. If A has the STPP in E^n, then $\pi_p^{-1}(A)$ is an STPP set in $S^n - \{p\}$ in E^{n+1}, so in particular, $\pi_p^{-1}(A)$ is a smooth 2-sphere in E^{n+1}. By a result of

Chern and Lashof [4], the set $\pi_p^{-1}(A)$ must be the boundary of a convex set in an E^3 in E^{n+1}. But then $\pi_p^{-1}(A)$ lies in $E^3 \cap (S^{n+1} - \{p\})$, so $\pi_p^{-1}(A)$ is a Euclidean 2-sphere and therefore so is A.

Proposition 3. *If A is a smooth 2-dimensional surface embedded in S^n with the* STPP, *then either $A = S^2$, $A =$ the real projective plane embedded as a Veronese surface in S^4, or A is an orientable surface in S^3.*

Proof. This follows from a result of Kuiper—any TPP smooth surface must be contained in an E^5, and the only such surface not contained in an E^4 is the Veronese surface [11]. If A is a smooth surface embedded in S^3, then for $p \notin A$, the image $\pi_p(A)$ is a smooth STPP surface imbedded in E^3, so $\pi_p(A)$ and A will be orientable.

The problem of finding all TPP surfaces contained in spheres is thus reduced to finding all orientable surfaces in E^3 with the STPP.

4. STPP surfaces embedded in E^3

For smooth surfaces M^2 in E^3, an alternative characterization of the TPP is that every local support plane at a point p is a global support plane. (A plane H through a point p is said to be a *local support plane* of M^2 at p if p has a neighborhood U in M^2 which lies in a closed component, say $\bar{D}_1$, of H, and H is a *global support plane* if $H \cap M^2 = \emptyset$ and $M^2 \subset \bar{D}_1$). In the study of smooth surface with the STPP, the corresponding concept is that of a support sphere.

Definition. A sphere (or plane) S is said to be a *local support sphere* of M^2 at p if $p \in S \cap M^2$ and $U \subset \bar{D}_1$ for some neighborhood U of p in M^2. If U can be chosen so that $S \cap U = \{p\}$, then S is said to be a *strict* local support sphere at p. A *global support sphere* S is one such that $S \cap M^2 \neq \emptyset$ and $M^2 \subset \bar{D}_1$.

Proposition 4.1. *A surface M^2 has the* STPP *if and only if every local support sphere at a point p is a global support sphere.*

Proof. If S is a local support sphere at p, which is not global, then there is a point q of M^2 in D_2. For a sphere S' in $\bar{D}_2$ tangent to S at p but with a slightly different radius, we still have q in D_2', but now $S' \cap U = \{p\}$, and we have no path from p to q in $\bar{D}_2$, so M^2 does not have the STPP. Conversely, if M^2 does not have the STPP, then for some $S, M^2 \cap \bar{D}_1$ is not connected so we have at least two components C_1 and C_2, both bounded since M^2 is a closed surface. Change S radially into $\bar{D}_1$ until the last sphere S' which meets both C_1 and C_2. Then $\bar{D}_1' \cap M_2$ is still disconnected, but S' is a local support sphere for a point p either on C_1 or C_2.

As in the case of curves in the plane, we may introduce for each point p the spheres $S(p, k)$ tangent to M^2 at p, with center $\vec{p} + \vec{n}(p)/k$ and radius $|1/k|$, where k may be any real number, with $S(p, 0)$ denoting the tangent plane at p. (The unit normal vector $\vec{n}(p)$ may be selected continuously over the entire embedded surface M^2, but at this point we work only locally.)

Let $\underline{k}(p)$ and $\bar{k}(p)$ be the two numbers such that $S(p, k)$ is a strict global support plane for $k < \underline{k}(p)$ or $\bar{k}(p) < k$, and such that for any k in the interval $(\underline{k}(p), \bar{k}(p)), S(p, k) \cap M^2 \neq \{p\}$. Thus at each point p, the surface M^2 is contained in the region between $S(p, \underline{k}(p))$ and $S(p, \bar{k}(p))$. Since we assume that M^2 has continuously defined curvature, both $\underline{k}(p)$ and $\bar{k}(p)$ are finite.

Recall that for a smooth surface with continuous curvature, the normal sections at p obtained by intersecting M^2 with a plane through $\vec{n}(p)$ are plane curves with well-defined curvatures, so each has an osculating circle. The maximum and minimum values $k_1(p)$ and $k_2(p)$ of these curvatures are the principal curvatures; and if they are distinct, then they correspond to the normal sections from a pair of orthogonal planes. The osculating circle for the normal section with radius $k_i(p)$ is the equator of the sphere $S(p, k_i(p))$, and in fact all of the osculating circles of the normal sections must lie in the closed region between the spheres $S(p, k_1(p))$ and $S(p, k_2(p))$. From the basic property of osculating circles, it follows that if k is not in $[k_1(p), k_2(p)]$, then $S(p, k)$ is a local support sphere at p, since every normal section at p will contain an arc about p which except for p, lies entirely in a complementary component D_1 of $S(p, k)$, and the union of these arcs will contain a disc neighborhood U of p lying, except for p, in D_1.

Proposition 4.2. *If M^2 is a smooth surface in E^3 with the* STPP, *then at every point, $\underline{k}(p) = k_2(p)$ and $\bar{k}(p) = k_1(p)$.*

Proof. Since every local support sphere must be global, it follows that $k_2(p) \leq \underline{k}(p) \leq \bar{k}(p) \leq k_1(p)$, and we need only show that for any k in $(k_1(p), k_2(p))$, we have $M^2 \cap S(p, k) \neq \{p\}$. This follows since $S(p, k)$ meets the osculating circles with curvatures $k_1(p)$ and $k_2(p)$ just at p, so we have a neighborhood ω_1 of p in the first normal section such that $\omega_1 - \{p\} \subset D_1$ and ω_2 of p in the other normal section with $\omega_2 - \{p\} \subset D_2$. Thus any embedded disc neighborhood U of p in M^2 must meet $S(p, k)$ in points other than p, since its boundary curve contains parts of both complementary components of $S(p, k)$.

Theorem 4.3. *If M^2 is a smooth surface embedded in E^3 with the* STPP, *then M^2 is either a (Euclidean) 2-sphere or a smooth torus.*

Proof. By Proposition 4.2, if M^2 has *any* umbilic p (where $k_1(p) = k_2(p)$), then M^2 lies "between" the identical spheres $S(p, k_1(p))$ and $S(p, k_2(p))$, so M^2 is a sphere. It follows that if M^2 is not the sphere, then M^2 has *no* umbilics, so at every point p there is a larger principal curvature $k_1(p)$ determining a certain principal direction in the tangent space of M^2 at p. Since we have no umbilics, this yields a differentiable field of directions, without singularities, on the orientable surface M^2. But the only orientable surface admitting such a field is a torus (the sum of the indices of singularities of any direction field is the Euler-Poincaré characteristic, and since we have no singularities $\chi(M^2) = 0$, and M^2 is orientable so M^2 is a torus. For a clear development of this result compare H. Hopf [9]).

Remark. The support spheres $S(p, k_1(p)) = S_1$ and $S(p, k_2(p)) = S_2$ separate E^3 into three regions, such that for any point x between S_1 and S_2, the image $I_{x,\rho}(M^2)$ has positive curvature at $I_{x,\rho}(p)$, while the curvature is negative for any point x outside the closed region between S_1 and S_2. This is independent of the radius ρ, and it follows simply from the fact that the maximum and minimum curvatures of the normal sections will be non-zero, and with different signs if the point x lies between S_1 and S_2 and with the same signs if x lies in one of the other open complementary components of $S_1 \cup S_2$.

This demonstrates in particular that any sphere S between $S_1(p)$ and $S_2(p)$ cuts a sufficiently small disc neighborhood U of p into exactly four parts, two inside S and two outside and each curve in a principal direction contains points in two of the components. In fact, if we invert with respect to a sphere centered at a point x on S and not on M^2, then the image of $S \cap U$ will be the intersection of the tangent plane to $I_{x,\rho}(M^2)$ at $I_{x,\rho}(p)$, and at such a negatively curved region, the intersection lines are tangent to the asymptotes of the Dupin indicatrix.

5. Tori in E^3 with the STPP

We first show that there exist tori in E^3 with the STPP.

Proposition 5.1. *A torus of revolution T^2, formed by revolving a circle around an axis which it does not meet, has the* STPP.

Proof. Any point p of T^2 lies on a circle of latitude $C_1(p)$ and a circle of longitude $C_2(p)$, and for any other point q, we obtain $r = C_1(p) \cap C_2(q)$ and $s = C_2(p) \cap C_1(q)$. The points p, r, q, s taken in order form an isosceles trapezoid in E^3 (which may degenerate to a doubly covered segment), and this can be inscribed in a circle C. If S is a sphere or plane in E^3, and p and q lie in $T^2 \cap \bar{D}_1$, then since C has the STPP, at least one of the points r or s, say r, lies in $\bar{D}_1$. Since $C_1(p)$ has the STPP, there is at least one arc γ_1 from p to r on $C_1(p)$ in $\bar{D}_1$ and similarly an arc γ_2 from p to q in $C_2(q)$ in $\bar{D}_1$. Then the path $\gamma_1 \cup \gamma_2$ connects p to q in $\bar{D}_1$, so $T^2 \cap \bar{D}_1$ is pathwise connected.

Remark. An alternate proof of this proposition may be obtained by expressing T^2 as the image $\pi_p(S^1(r_1) \times S^1(r_2))$ under stereographic projection from a 3-sphere of a flat torus $S^1(r_1) \times S^1(r_2) \subset E^2 \times E^2 = E^4$, where $S^1(r_i) = \{(x, y) \in E^2 \mid x^2 + y^2 = r_i^2\}$ so $S^1(r_1) \times S^1(r_2) \subset S^3(\sqrt{r_1^2 + r_2^2}) = \{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 = r_1^2 + r_2^2\}$. By a result of Kuiper [10], such a flat torus has the TPP in $S^3(\sqrt{r_1^2 + r_2^2})$ so it must have the STPP, and then so does T^2. Conversely, since T^2 has the STPP by Proposition 5.1, any flat torus of the form $S^1(r_1) \times S^1(r_2)$ in a 3-sphere has the TPP.

We may obtain a further collection of examples by inversion. The image of a torus of revolution T^2 under an inversion $I_{x,\rho}\colon E^3 - \{x\} \to E^3 - \{x\}$ in a sphere of radius ρ and center x not on T^2 must also have the STPP. Such surfaces are known as *Cyclides of Dupin*, and they are characterized by the fact that all their lines of curvature in both systems are circles. (Cf. Eisenhart [5, pp. 312–313].) We proceed to make use of this characterization to conclude that the only STPP tori are Dupin cyclides.

We must first make a closer examination of the intersections of support spheres with M^2.

Proposition 5.2. *For any support sphere S of an* STPP *surface M^2, the intersection $S \cap M^2$ has the* CTPP *on the sphere S.*

Proof. If $S \cap M^2$ does not have the CTPP, then by the remark after Proposition 2.2, we can find a circle C on S bounding a disc $\bar{D}_1$ containing $S \cap M^2$ and such that $C \cap M^2$ is not connected. Choose a and c on different components of $C \cap M^2$ and points b and d in $C - (C \cap M^2)$ such that a, b, c, and d are in cyclic order on C. Let γ be the geodesic arc from b to d in D_2, and let B be a neighborhood of γ in E^3 not meeting M^2. Then for a sphere S' sufficiently close to S meeting S in C, the points a and c will be in different components of $D' \cap M^2$—there can be no path from a to c in $M^2 \cap D' \cap D$ by the choice of C, and any path from a to c in $D' - (D \cap D')$ must pass through B.

Proposition 5.3. *If M^2 is a torus in E^3 with the* STPP, *then for any support sphere S, $S \cap M^2$ is a point or a circle.*

Proof. By the previous proposition, $S \cap M^2$ is a (connected) closed set with the CTPP. But we know all the CTPP connected closed sets of a 2-sphere S by Theorem 1.4—i.e., S, the empty set, a single point, or S with a union of disjoint open circular discs removed. But $S \cap M^2 \neq S$ since M^2 is a torus, and $S \cap M^2 \neq \emptyset$ since S is a support sphere. If $M^2 \cap S$ contained any interior points, then these would be interior points of a sphere S, so umbilics on M^2 and then M^2 would be a sphere by Theorem 4.3, a contradiction. Thus $M^2 \cap S$ is either a point or a set without interior obtained by removing disjoint open circular discs from S. One such set is a circle, but for any other such set we must remove infinitely many open discs, as in the "Swiss cheese" example. But then we may find at least three boundary circles of the removed discs which do not intersect, and any three disjoint curves on a torus must disconnect the torus into more than two pieces, so $M \cap D_1$ would be disconnected and so would $M \cap \bar{D}_1'$ for a sphere S' obtained by changing the radius of S by a sufficiently small amount.

Proposition 5.4. *If M^2 is a torus with the* STPP *in E^3, then for every support sphere S, $S \cap M^2$ is a circle.*

Proof. If $S(p, k_1(p)) \cap M^2 = \{p\}$, then for any value of $\tilde{k}$ in $(k_2(p), k_1(p))$ there is a neighborhood U of p such that $S(p, \tilde{k}) \cap U$ consists of four paths from p separating the pair of regions in $D_1 \cap U$ (containing a deleted neighborhood $\omega_1 - \{p\}$ of the normal section with curvature $k_1(p)$) from the pair of regions $D_2 \cap U$. Since $S(0, k_1(p)) \cap M^2 = \{p\}$, for k' sufficiently close to $k_1(p)$, the sets $S(p, k') \cap M$ and $D_1' \cap M^2$ will lie in the neighborhood U of

p, and we may find points r and s of $(\omega_1 - \{p\}) \cap D_1'$ such that any path from r to s in $M^2 \cap \bar{D}_1'$ passes through p. For a sphere S'' with radius slightly different from that of $S(p, k')$, the points r and s will still be in D_1'', but p will be in D_2''; so there is no path from r to s in $\bar{D}_1'' \cap M^2$ and M^2 does not have the STPP.

Theorem 5.5. *If M^2 is a smooth torus in E^3 with the* STPP, *then M^2 is a cyclide of Dupin.*

Proof. By Proposition 5.4, the curve through p with principal curvature $k_1(p)$ is a circle, and the same is true for the other direction of curvature. Since this is true at *all* points of the surface, all the lines of curvature are circles, and the theorem follows.

Remark. The family of cyclides obtained by stereographic projection of the standard flat torus from various positions on S^3 is the subject of a computer graphics film, described in [3].

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