J. DIFFERENTIAL GEOMETRY
3 (1969) 493-510

PERIODIC GEODESICS ON COMPACT RIEMANNIAN MANIFOLDS

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The interest in periodic geodesics arose at a very early stage of differential geometry, and has grown rapidly since then. It is a basic general problem to estimate the number of distinct periodic geodesics $c\colon \boldsymbol{R} \to M$, $c(t + 1) = c(t)$, on a complete riemannian manifold M in terms of topological invariants. Here periodic geodesics are always understood to be non-constant, and two such curves c_1, c_2 will be said to be distinct if they are geometrically different, $c_1(\boldsymbol{R}) \neq c_2(\boldsymbol{R})$.

If M is non-compact, then it is even difficult to find reasonable conditions for the existence of at least one periodic geodesic, see § 4. However, in the compact case many results are available. It is classical and rather elementary that any non-trivial conjugacy class of the fundamental group $\pi_1(M)$ gives rise to periodic geodesics, and very often the existence of a larger number of distinct periodic geodesics can be deduced from further properties of $\pi_1(M)$; compare [4, p. 240], [7], and also § 4. When M is simply connected, the problem is getting much more delicate. Here the first result was obtained in 1905 by Poincaré, who proved that there exists a periodic geodesic on every surface analytically equivalent to the euclidean sphere S^2. Yet rather late, in 1952, Fet and Lusternik established the theorem that on any compact riemannian manifold M at least one geodesic is periodic. Several authors have proved the existence of certain finite numbers of distinct periodic geodesics for special topological types of manifolds, partly under restrictive metric conditions. We mention the work of Lusternik, Schnirelmann, Morse, Fet, Alber, and Klingenberg; for references see [11].

In § 4 of this paper we shall prove: There always exist infinitely many distinct periodic geodesics on an arbitrary compact manifold, provided some weak topological condition holds. In fact, it seems that our condition will be satisfied except in comparatively few cases. Until now it was not even possible to decide whether there is some compact simply connected differentiable manifold M with infinitely many distinct periodic geodesics for all riemannian structures on M.

As yet the most natural and successful way of dealing with periodic geodesics

Communicated by F. Hirzebruch, February 24, 1969 and, in revised form, June 19, 1969. The first author was a Miller Fellow.

is to apply calculus of variations in the large, which was already developed by Morse in [17] to a very large extent. Let us agree with the standard terminology and call a geodesic c: $[0, 1] \to M$ closed if it is differentiably closed, $c(0) = c(1)$, $\dot{c}(0) = \dot{c}(1)$. Of course, a nonconstant closed geodesic determines a periodic geodesic and vice versa; we will often use both terms interchangeably. Now closed geodesics are precisely the critical points of the energy functional E on the free loop space Ω of M, and Morse theory provides information about the existence and the number of critical points of E in terms of the topology of Ω. For the simpler corresponding variational problem on the path space Ω_{pq} of curves $[0, 1] \to M$ with fixed end points $p, q \in M$, Serre proved in [19] that Ω_{pq} contains infinitely many geodesics if M is compact. Unfortunately, one cannot use the same procedure for our poblem since, in a trivial way, any nonconstant closed geodesic c gives rise to infinitely many distinct critical points of E in Ω by taking the iterates of c, which however, do not lead to distinct periodic geodesics in the above geometric sense. Besides this, the structure of the homology $H_*\Omega$ is not so well-known as $H_*\Omega_{pq}$. Yet basically, our approach is more related to Serre's arguments and rather opposite to all the previous work on periodic geodesics. From the assumption that there exist only finitely many distinct periodic geodesics on M, we will derive a strong condition for the sequence of Betti numbers of Ω. We construct a uniform bound for the homology arising from the tower of iterates generated by every closed geodesic. This has been made possible by results of Bott [2] on index and nullity of iterated closed geodesics and by some quantitative extension of non-degenerate Morse theory to the degenerate situation, given by the authors in [5]. In the following, differentiable will always mean sufficiently smooth, and one may think of C^∞ for convenience.

We appreciate some conversations with Ralph Abraham, Peter Klein, and Wilhelm Klingenberg, and are also indebted to James Eells for pointing out a technical mistake in our manuscript.

1. Index and nullity of periodic geodesics

Let M be a riemannian manifold of dimension $n + 1 \geq 2$. Consider a closed non-constant geodesic c: $[0, 1] \to M$, and denote by $\mathscr{V}_c^0$ the vector space of closed absolutely continuous vector fields with square summable first derivatives along c which are orthogonal to the tangent field $\dot{c}$. The index form I: $\mathscr{V}_c^0 \times \mathscr{V}_c^0 \to \boldsymbol{R}$ of c is defined by

$$(1) \qquad I(X, Y) = \int_0^1 (\langle X', Y' \rangle - \langle R(X, \dot{c})\dot{c}, Y \rangle) dt \,.$$

Here $X' = \nabla_D X = \nabla X$ denotes the covariant derivative of X, and R the curvature tensor of M. I is clearly symmetric and bilinear. If the vector field

X is differentiable and differentiably closed, then (1) can be written as

$$I(X, Y) = -\int_0^1 \langle X'' + R(X, \dot{c})\dot{c}, Y\rangle dt .$$

The index and nullity of I are called the index and nullity of c, and will be denoted by $\lambda(c)$ and $\nu(c)$. Now let ω be the positive definite scalar product on $\mathscr{V}_c^0$ defined by

$$\omega(X, Y) = \int_0^1 \langle X, Y\rangle dt .$$

There is a unique selfadjoint operator A given by

$$\omega(AX, Y) = I(X, Y) .$$

A is an elliptic differential operator, and the following classical result is well-known.

Theorem 1. *For $\mu \in \boldsymbol{R}$ and $X \in \mathscr{V}_c^0$ the following two conditions are equivalent:*

(a) $AX = -\mu X$,

(b) *X is differentiable and differentiably closed, and $X'' + R(X, \dot{c})\dot{c} = \mu X$.*

Moreover:

(c) *The spectrum of A is a discrete subset of $\boldsymbol{R}$ which is bounded from below.*

(d) *The dimension $\Theta(\mu)$ of the kernel of $A - \mu$ id is bounded by $2n$.*

Clearly $\Theta(\mu)$ is the number of linearly independent solutions of the differential equation $X'' + R(X, \dot{c})\dot{c} = \mu X$ subject to the boundary condition $X(t + 1) = X(t)$. As a consequence of (c) and (d) we obtain that $\lambda(c)$ and $\nu(c)$ are finite numbers, more precisely,

$$\lambda(c) = \sum_{\mu<0} \Theta(\mu) , \qquad \nu(c) = \Theta(0) .$$

Due to the first statement of Theorem 1, we may restrict ourselves to differentiable vector fields along c in order to study the index and nullity of c. $\mathscr{V}_c$ will denote the vector space of (not necessarily closed) differentiable vector fields along c which are orthogonal to $\dot{c}$.

If $c\colon \boldsymbol{R} \to M$ is a periodic geodesic, $c(t + 1) = c(t)$, the m-th iterate c_m of c is defined by

$$c_m(t) = c(mt) , \qquad t \in \boldsymbol{R}, m > 0 ,$$

and we set

$$\lambda(c_m) = \lambda(c_m \,|\, [0, 1]) , \qquad \nu(c_m) = \nu(c_m \,|\, [0, 1]) .$$

$\mathscr{L}: \mathscr{V}_c \to \mathscr{V}_c$ will be the second order differential operator with $-\mathscr{L}X = X'' + R(X, \dot{c})\dot{c}$. For any positive integer m the restriction of $\mathscr{L}$ to the vector fields of period m is selfadjoint with respect to the scalar product ω_m defined by $\omega_m(X, Y) = \int_0^1 \langle X(mt), Y(mt)\rangle dt$. Now if $\Theta_m(\mu)$ denotes the number of the linearly independent solutions of the differential equation

$$\mathscr{L}X = \mu X$$

subject to the boundary condition $X(t + m) = X(t)$ for $t \in \boldsymbol{R}$, we have the formulas for the index and nullity:

$$\lambda(c_m) = \sum_{\mu<0} \Theta_m(\mu) , \qquad \nu(c_m) = \Theta_m(0) .$$

In [2], Bott has studied the sequences $\lambda(c_m)$ and $\nu(c_m)$, and some of his results will be important for our investigation. One of the basic ideas for the study of the above sequences is to consider equivalently a hermitian operator L instead of $\mathscr{L}$. Let $V_c = \mathscr{V}_c \otimes \boldsymbol{C}$ be the complexification of $\mathscr{V}_c$. Then $L: V_c \to V_c$ is the $\boldsymbol{C}$-linear extension of $\mathscr{L}$, i.e., $L(X + iY) = \mathscr{L}(X) + i\mathscr{L}(Y)$. For a complex number $z \in S^1 \subset \boldsymbol{C}$, a real number μ and a positive integer m we consider the differential equation

$$LY = \mu Y$$

subject to the boundary condition

$$Y(t + m) = zY(t) , \qquad t \in \boldsymbol{R} . \tag{2}$$

Set

$\Theta_m^z(\mu)$ = complex dimension of the subspace of vector fields Y in V_c satisfying $LY = \mu Y$ and (2),

$$\Lambda(z) = \sum_{\mu<0} \Theta_1^z(\mu) , \qquad N(z) = \Theta_1^z(0) .$$

One can check that $\Theta_m^z(\mu) = \sum_{w^m=z} \Theta_1^w(\mu)$; see [2, § 1.7]. Therefore we have the formulas for the index and nullity of c_m:

$$\lambda(c_m) = \sum_{\mu<0} \Theta_m^1(\mu) = \sum_{z^m=1} \Lambda(z) , \qquad \nu(c_m) = \sum_{z^m=1} N(z) . \tag{3}$$

It follows that the sequences $\lambda(c_m)$ and $\nu(c_m)$ are completely determined by the nonnegative integer valued functions Λ and N on the unit circle.

We list some important properties of Λ and N:

$$\Lambda(z) = \Lambda(\bar{z}) , \qquad N(z) = N(\bar{z}) , \tag{4}$$

(5) $\qquad N(z) = 0$, except for at most $2n$ points, the so-called Poincaré points,

(6) $\qquad \Lambda$ is locally constant except possibly at Poincaré points,

(7) $\qquad \lim_{z \to z_0^{\pm}} \Lambda(z) \geq \Lambda(z_0)$.

In fact, Bott proves some more properties in his paper, but we will only use these quite elementary facts, which are contained in Proposition 1.3 of [2, p. 178]. (4) is clear since L is a C-linear extension of a real operator. To check (5), Bott introduces an endomorphism P of a $2n$-dimensional vector space, the so-called Poincaré map. Then the Poincaré points are the eigenvalues of P with absolute value 1. Let E denote the complexification of the orthogonal complement of $\dot{c}(0)$ in the tangent space $M_{c(0)}$. Then $P: E \oplus E \to E \oplus E$ is obtained as follows: For $u \oplus v \in E \oplus E$, let Y be the unique complex Jacobi field (i.e., a solution of $LY = 0$) satisfying $Y(0) = u$, $Y'(0) = v$; then $P(u \oplus v) = Y(1) \oplus Y'(1)$. Now we have $N(z) = \dim_C \ker (P - z\, id)$ so that (5) is settled. (6) is a consequence of the following "continuity theorem", which is contained in Theorem 3.2 of Morse [7, p. 91].

Theorem 2. *Let J be a bounded interval such that the end points are not in the spectrum of L subject to the boundary condition* (2). *Then there is a neighborhood U of z in S^1 such that the end points of J are not in the spectrum of L subject to the boundary condition $Y(t + m) = wY(t)$ for $w \in U$ and*

$$\sum_{\mu \in J} \Theta_1^z(\mu) = \sum_{\mu \in J} \Theta_1^w(\mu) \ .$$

The relation (7) will also follow from this theorem, since for z close to z_0 and $\varepsilon > 0$ small we have

$$\sum_{\mu<0} \Theta_1^z(\mu) = \sum_{\mu<-\varepsilon} \Theta_1^z(\mu) + \sum_{-\varepsilon \leq \mu < 0} \Theta_1^z(\mu) = \sum_{\mu<-\varepsilon} \Theta_1^{z_0}(\mu) + \sum_{-\varepsilon \leq \mu<0} \Theta_1^z(\mu) \ .$$

We draw some consequences for $\lambda(c_m)$ and $\nu(c_m)$ using the properties of Λ and N. Let $z_1, \cdots, z_r$ be the Poincaré points, $z_j = e^{2\rho_j \pi i}$. The numbers $0 < \rho_j \leq 1$ are called Poincaré exponents, and we assume $0 < \rho_1 < \cdots < \rho_r \leq 1$.

Lemma 1. *Either $\lambda(c_m) = 0$ for all m or there are numbers $\varepsilon > 0$ and $a > 0$ such that*

$$\lambda(c_{m+s}) - \lambda(c_m) \geq s\varepsilon - a \qquad \textit{for all } m, s \ .$$

Proof. Set $\rho_0 = 0$, $\rho_{r+1} = 1$, and let $a_j = \Lambda(z)$ for $z \in (\rho_{j-1}, \rho_j)$ and $1 \leq j \leq r + 1$, and $a_{r+1} = 0$ if $\rho_r = 1$; compare (6). By (3) we obtain

$$\lambda(c_{m+s}) - \lambda(c_m) = \sum_{z^{m+s}=1} \Lambda(z) - \sum_{z^m=1} \Lambda(z) \geq \sum_{j=1}^{r+1} [(m+s)(\rho_j - \rho_{j-1})]a_j$$
$$- \sum_{j=1}^{r+1} [m(\rho_j - \rho_{j-1})]a_j - a' \geq \sum_{j=1}^{r+1} [s(\rho_j - \rho_{j-1})]a_j - a' \, .$$

Here $[\alpha]$ is the greatest integer $\leq \alpha$, and, for example, the constant a' may be chosen $\geq 3(r+1) \max \Lambda$. Now if $\lambda(c_m) \neq 0$ for some m, then we find j_0 such that $\rho_{j_0-1} < \rho_{j_0}$ and $a_{j_0} > 0$ by using (3), (6), (7). Hence

$$\lambda(c_{m+s}) - \lambda(c_m) \geq [s(\rho_{j_0} - \rho_{j_0-1})]a_{j_0} - a'$$
$$\geq s(\rho_{j_0} - \rho_{j_0-1})a_{j_0} - a_{j_0} - a' \, .$$

Lemma 2.

(a) $\nu(c_m) = 0$ *for all* m *iff all Poincaré exponents are irrational.*

(b) *There are positive integers* $k_1, \cdots, k_s$ *and sequences* $m_j^i \in Z^+$, $i > 0$, $j = 1, \cdots, s$, *such that the numbers* $m_j^i k_j$ *are mutually distinct,* $m_j^1 = 1$, $\{m_j^i k_j\} = Z^+$, *and*

$$\nu(c_{m_j^i k_j}) = \nu(c_{k_j}) \, .$$

The Poincaré points contained in $\{z \mid z^{m_j^i k_j} = 1\}$ are exactly the Poincaré points contained in $\{z \mid z^{k_j} = 1\}$.

Proof. (a) follows from (3) and (5). If all Poincaré exponents are irrational, (b) follows from (a). If there are rational Poincaré exponents, let Q denote the set of denominators of the rational exponents (we assume that all exponents are of the form p/q with p, q relatively prime). For $A \subset Q$ let $k(A)$ denote the least common multiple of all elements in A. Choose distinct numbers $k_1, \cdots, k_s$ such that $\{k_1, \cdots, k_s\} = \{k(A) \mid A \subset Q\} \cup \{1\}$. Keeping $j \in \{1, \cdots, s\}$ fixed, we select from the sequence mk_j, $m > 0$, the greatest subsequence $m_j^i k_j$, $i > 0$, satisfying $q \nmid m_j^i k_j$, whenever $q \in Q$ and $q \nmid k_j$. Then the claim is obvious.

2. Equivariant Morse theory on the free loop space ΩM

From now on, M is a compact riemannian manifold without boundary, and $\dim M = n + 1 \geq 2$. Morse theory on ordinary path spaces Ω_{pq} of M can be treated quite easily by means of suitable finite dimensional subspaces of broken geodesics in Ω_{pq}. Since Morse a similar, though more complicated, procedure has also been used successfully for the free loop space Ω of M. However, in our investigation the introduction of a Hilbert manifold structure on Ω seems to be the natural approach. For a discussion of the basic facts about Hilbert manifolds which we will need here, compare [18] and [11].

We consider $\Omega M = \Omega$ as the complete riemannian Hilbert manifold of absolutely continuous maps $S^1 \to M$ with square summable first derivatives.

Here S^1 will also be viewed as the identification space $[0, 1]/\{0, 1\}$. Note that the tangent functor T commutes with Ω, i.e., $T(\Omega M) = \Omega(TM)$. Furthermore, the tangent space $T_c\Omega$ of ΩM at a curve c consists of the absolutely continuous vectors fields along c with square summable first derivatives. A curve $X \in \Omega(TM)$ belongs to $T_c(\Omega M)$ iff $\pi \circ X = c$, where $\pi\colon TM \to M$ is the bundle projection. The riemannian structure is given by

$$\langle\!\langle X, Y \rangle\!\rangle = \int_0^1 (\langle X, Y\rangle + \langle X', Y'\rangle) dt \ ,$$

where $X, Y \in T_c\Omega$, and $c \in \Omega$. On Ω we have the energy, a differ entiable function $E\colon \Omega \to \boldsymbol{R}$ defined by

$$E(c) = \frac{1}{2} \int_0^1 \| \dot{c}(t) \|^2 \, dt \ .$$

One can check that E satisfies condition (C) of Palais and Smale [18]. It is well known that c is a critical point of E iff c is a closed geodesic. There is an equivariant and also isometric operation of the orthogonal group $0(2)$ on Ω via the operation of $0(2)$ on the parameter circle S^1. This action is continuous, but not differentiable. However, if $c \in \Omega$ is a C^r-curve, then the orbit $0(2)c$ is a C^{r-1}-submanifold, $r \geq 1$. A critical point c of E in Ω lies always on a critical submanifold of Ω, the orbit $0(2)c$, which in turn consists of two disjoint copies of S^1 when c is not constant. Note that in this case the isotropy group Γ at c is finite cyclic. In the following we consider only critical points which are non-constant geodesics.

The hessian H_c of E at a critical point c is given by

$$H_c(X, Y) = \int_0^1 (\langle X', Y'\rangle - \langle R(X, \dot{c})\dot{c}, Y\rangle) dt \ .$$

Now the restriction of H_c to a fiber of the normal bundle $\mathscr{N}$ of the critical orbit $0(2)c$ has same index and nullity as the index form of c. For this observe that the fiber $\mathscr{N}_c$ of $\mathscr{N}$ over c is $\mathscr{V}_c^0 \oplus \mathscr{T}_c$, where $\mathscr{V}_c^0$ is the vector space introduced in § 1, and $\mathscr{T}_c$ is the subspace of vector fields $a\dot{c}$ in $T_c\Omega$ with $\int_0^1 a(t) dt = 0$. Furthermore, H_c is positive definite on $\mathscr{T}_c$, the sum $\mathscr{V}_c^0 \oplus \mathscr{T}_c$ is orthogonal with respect to H_c, and $H_c \,|\, \mathscr{V}_c^0 \oplus \mathscr{V}_c^0$ is just the index form. So if the critical orbit $0(2)c$ is isolated, then the index and nullity of $0(2)c$ as a critical submanifold of ΩM are well defined and equal to the index and nullity of the geodesic c. We should mention that the normal bundle $\mathscr{N}$ of $0(2)c$ is differentiably trivial, since the orthogonal group of Hilbert space is connected. However, in the equivariant Morse theory we have to consider $\mathscr{N}$

also as a continuous bundle with the natural structure group Γ with respect to which $\mathscr{N}$ is not trivial in general. The 0(2)-operation extends canonically to $\Omega(TM)$, and the bundle map $0(2) \times \mathscr{N}_c \to \mathscr{N}$ with $(\alpha, X) \to \alpha X$ induces locally a continuous trivialization. In fact, this map is a covering, and Γ acts on the trivial bundle $0(2) \times \mathscr{N}_c$ by covering transformations.

In [5], we have discussed Morse theory of differentiable functions on Hilbert manifolds which have only isolated but possibly degenerate critical points. We will use these results here to get information about homological invariants of isolated critical orbits of the energy E on Ω. In fact, our treatment of this special example contains the main ideas how to handle the general case of isolated critical orbits of a function with respect to an equivariant continuous action of a compact Lie group G, at least when critical orbits are smooth and G acts by isometries.

We have to study the energy E in a neighborhood of an isolated critical orbit $0(2)c$. Consider a tubular neighborhood $\mathscr{D}$ of $0(2)c$ which is a normal disc bundle such that the action of 0(2) on Ω transforms fibers equivariantly into each other. So the normal space $\mathscr{N}_c$ of $0(2)c$ is the tangent space of the fiber $\mathscr{D}_c$ at c, and $\alpha\mathscr{D}_c = \mathscr{D}_{\alpha c}$ for $\alpha \in 0(2)$. Clearly, the bundle $\mathscr{D}$ has the structure group Γ viewed as a continuous bundle. The existence of such tubes $\mathscr{D}$ can be proved in various ways. Since 0(2) acts by isometries, one may take for $\mathscr{D}$ the diffeomorphic image of a sufficiently small tubular neighborhood of the zero section in the normal bundle $\mathscr{N}$ of $0(2)c$ under the exponential map Exp of Ω; compare also [13]. The exponential map exp of M may be used as well. Observe that the map $\mathscr{N} \to \Omega$ with $Y \to \exp \circ Y$ is a local diffeomorphism along the zero section of $\mathscr{N}$.

From the viewpoint of Morse theory it is now sufficient to study the energy function in just one fiber of a tube $\mathscr{D}$ as above, since we will see that the local invariants of the critical orbit $0(2)c$ are completely determined by the orbit, the restriction E_c of E to the fiber $\mathscr{D}_c$, and the action of the isotropy group Γ on $\mathscr{D}_c$. All constructions involving the energy, which are carried out in some fiber, can be extended equivariantly to the whole bundle. For the hessian $\tilde{H}_c$ of E_c at c we obtain immediately $\tilde{H}_c = H_c | \mathscr{N}_c \oplus \mathscr{N}_c$. Our next lemma makes sure that the results of [5] may be applied to the function E_c.

Lemma 3. *Let $c \in \Omega$ be a closed geodesic. Then the operator $A\colon T_c\Omega \to T_c\Omega$ defined by*

$$\langle\langle AX, Y \rangle\rangle = H_c(X, Y)$$

admits a decomposition $A = id + k$ with a compact operator k.

Clearly, the corresponding operator $\tilde{A}$ for $\tilde{H}_c$ is also of the form $\tilde{A} = id + \tilde{k}$, where $\tilde{k}$ compact.

Proof. Imbed M isometrically in $\boldsymbol{R}^p$. Then $T_c\Omega$ may be considered as a closed linear subspace of $\Omega\boldsymbol{R}^p$. One can show that the inner product on $T_c\Omega$

induced from the product of $\Omega \boldsymbol{R}^p$ is equivalent to our product $\langle\!\langle , \rangle\!\rangle$. Now let $H^0(S^1, \boldsymbol{R}^p)$ denote the Hilbert space of square summable measurable functions $S^1 \to \boldsymbol{R}^p$ with the usual inner product $\langle , \rangle_0$ given by $\langle \varphi, \psi \rangle_0 = \int_0^1 \langle \varphi, \psi \rangle \, dt$, where $\langle , \rangle$ is the ordinary scalar product in $\boldsymbol{R}^p$. We use the fact that the inclusion $\Omega \boldsymbol{R}^p \subset H^0(S^1, \boldsymbol{R}^p)$ is completely continuous, i.e., the unit sphere $\boldsymbol{S}$ in $\Omega \boldsymbol{R}^p$ is relatively compact in the topology of $H^0(S^1, \boldsymbol{R}^p)$, a consequence of a theorem of Rellich. Note the identity $\langle\!\langle AX, Y \rangle\!\rangle = \langle\!\langle X, Y \rangle\!\rangle - \langle X + R(X, \dot{c})\dot{c}, Y \rangle_0$, and define an operator $k \colon T_c\Omega \to T_c\Omega$ by $\langle\!\langle kX, Y \rangle\!\rangle = -\langle X + R(X, \dot{c})\dot{c}, Y \rangle_0$. For k we have an estimate $\langle\!\langle kX, Y \rangle\!\rangle^2 \leq K \| X \|_0^2 \| Y \|_0^2$, since the curvature along c is bounded. Hence $\langle\!\langle kX, kX \rangle\!\rangle \leq K \| X \|_0^2$. Observe $\| Y \|_0^2 \leq \langle\!\langle Y, Y \rangle\!\rangle$. Then the relative compactness of $\boldsymbol{S}$ in $H^0(S^1, \boldsymbol{R}^p)$ and the last inequality imply that $k(\boldsymbol{S})$ is relatively compact, so k is compact.

As a consequence, condition (C) holds for E_c in some neighborhood of c in $\mathscr{D}_c$; this can be seen, for example, by means of the splitting Lemma 1 in [5]. Now by restricting attention to sufficiently small bundles $\mathscr{D}$ we may also assume that c is the only critical point of E_c in $\mathscr{D}_c$. Our argument depends on an orbit version of the splitting lemma. First observe that the normal bundle splits into an orthogonal sum $\mathscr{N} = F \oplus N$ with smooth subbundles F, N which are invariant under $0(2)$. Let A denote the smooth section in the bundle of selfadjoint operators over $0(2)c$ associated with $\mathscr{N}$ such that $\langle A_c x, y \rangle = \bar{H}_c(x, y)$, and consider the characteristic function χ of an intervall $(-\varepsilon, \varepsilon)$ on $\boldsymbol{R}$ such that the non-zero part of the spectrum of A lies outside $(-\varepsilon, \varepsilon)$. Then $Q = \chi(A)$ is a smooth section of projection operators; compare [15]. Set $N = Q\mathscr{N}$, and $F = (I - Q)\mathscr{N}$, and let $\exp_{\alpha c}$ be the exponential map of the fiber $\mathscr{D}_{\alpha c}$ of $\mathscr{D}$ at c. If $\mathscr{D}$ is chosen sufficiently small, there is a tubular neighborhood $\tilde{\mathscr{N}}$ of the zero section in $\mathscr{N}$ such that the map $\Psi \colon \tilde{\mathscr{N}} \to \mathscr{D}$ with $\Psi \,|\, \tilde{\mathscr{N}}_{\alpha c} = \exp_{\alpha c} |\, \tilde{\mathscr{N}}_{\alpha c}$ is a diffeomorphism onto $\mathscr{D}$. Clearly $\Psi(\alpha_* x) = \alpha\Psi(x)$, since $0(2)$ acts by isometries. The induced $0(2)$-operation on $\tilde{\mathscr{N}}$ is equivariant with respect to the function $\tilde{E} = E \circ \Psi$, and the null space of the hessian of $\tilde{E} \,|\, \tilde{\mathscr{N}}_c$ at 0 is the fiber N_c of N. The construction leading to the splitting lemma applied fiberwise now yields the result: There are a fiber preserving bundle diffeomorphism $\Phi \colon \tilde{\mathscr{N}} \to \tilde{\mathscr{N}}$, a fiber preserving differentiable map $h \colon N \cap \tilde{\mathscr{N}} \to F$, and an orthogonal bundle projection $P \colon F \to F$ such that

$$\tilde{E} \circ \Phi(x \oplus y) = \|Px\|^2 - \|(I - P)x\|^2 + \tilde{E}(h(y) \oplus y) ,$$

where $\mathscr{N} = F \oplus N$, and $\mathscr{D}$ is small. Moreover, Φ commutes with the $0(2)$-action, i.e., $\alpha_* \circ \Phi = \Phi \circ \alpha_*$, which depends on the fact that α_* is an isometry and $\tilde{E}(x) = \tilde{E}(\alpha_* x)$. All the critical points of $\tilde{E} \circ \Phi \,|\, \tilde{\mathscr{N}}_c$ lie on N_c. Since the points of N_c are precisely the periodic Jacobi fields along c, their $0(2)$-orbits in $\mathscr{N}$ are differentiable submanifolds. Hence, the orbits of critical points of

E_c in $\mathscr{D}$ are always smooth. The last statement also follows from the classical theorem of Bochner and Montgomery implying that the 0(2)-action on the finite dimensional smooth submanifold N of $\mathscr{N}$ is differentiable. If $\mathscr{D}$ is small enough, then ∇E will be non-zero and perpendicular to $\mathscr{D}_c$ at any critical point c_0 of E_c if $c_0 \neq c$. So the level surface $E^{-1}(a)$ is tangent to $\mathscr{D}_c$ at c_0, where $a = E(c_0)$. On the other hand, the orbit $0(2)c_0$ intersects $\mathscr{D}_c$ and hence $E^{-1}(a)$ transversally, which is impossible since $0(2)c_0 \subset E^{-1}(a)$. This completes the argument.

Now we will define a local homological invariant $\mathscr{H}(E, 0(2)c)$ of the energy E at the isolated critical orbit $0(2)c$ by using the constructions and the notations of [5]. Choose a disc bundle $\mathscr{D}$ as above and an admissible region W_c for the function E_c on the fiber $\mathscr{D}_c$ at the isolated critical point c, and recall that the local invariant $\mathscr{H}(E_c, c)$ is well defined, i.e., $\mathscr{H}(E_c, c) = H_*(W_c, W_c^-)$, here we may use singular homology with arbitrary coefficients. Set $W = 0(2)W_c$, $W^- = 0(2)W_c^-$, and

$$\mathscr{H}(E, 0(2)c) = H_*(W, W^-) \ . \tag{8}$$

It remains to check that the definition (8) does not depend upon the choice of the bundle $\mathscr{D}$ and the admissible region W_c. Let $\hat{W}_c \subset \mathscr{D}_c$ be another admissible region for E_c. Then we have $H_*(\hat{W}, \hat{W}^-) = H_*(W, W^-)$ by a canonical equivariant version of the proof of the proposition in [5], where all constructions have to be done fiberwise. To prove now that the definition (8) is independent of $\mathscr{D}$, consider a tubular neighborhood U of the zero section in $\mathscr{N}$, which is mapped diffeomorphically onto some neighborhood V of $0(2)c$ in Ω under the exponential map Exp of Ω. Define an equivariant smooth function g on V by $g(p) = \|(\mathrm{Exp} \,|\, U)^{-1}(p)\|^2$. If $\tilde{\mathscr{D}}$ is a bundle as above, then $g \,|\, \mathscr{D}_c$ and $g \,|\, \tilde{\mathscr{D}}_c$ are cut functions at c, which may be used to define admissible regions $W_c, \tilde{W}_c$ such that $(W, W^-) = (\tilde{W}, \tilde{W}^-)$. A modification of Lemma 4 below will provide another argument.

Lemm 4. *Let M be compact, and b the only critical value of the energy E in $[b - \varepsilon, b + \varepsilon]$ for some $\varepsilon > 0$. Assume that the critical set in $f^{-1}(b)$ consists of finitely many critical orbits $0(2)c^1, \cdots, 0(2)c^r$. Then*

$$H_*(\Omega^{b+\varepsilon}, \Omega^{b-\varepsilon}) = \sum_{i=1}^{r} \mathscr{H}(E, 0(2)c^i) \ . \tag{9}$$

Proof. Ω is complete and condition (C) holds for the energy E by the compactness of M. It is sufficient to consider the case where $r = 1$, and $c^1 = c$. We define W with the smooth function g as above and apply exactly the same technique used for the proof of Lemma 3 in [5] to obtain $H_*(\Omega^{b+\varepsilon}, \Omega^{b-\varepsilon}) = H_*(W, W^-)$.

From now on, for convenience all homological invariants are taken with respect to coefficients in a field of characteristic zero. We may write (W, W^-)

$= (0(2) \times W_c, 0(2) \times W_c^-)/\Gamma$, where the isotropy group Γ acts on the trivial bundle $0(2) \times W_c$ by covering transformations as described before. Hence, $H_*(W, W^-)$ is isomorphic to the subspace $H_*(0(2) \times W_c, 0(2) \times W_c^-)^\Gamma$ of all elements in $H_*(0(2) \times W_c, 0(2) \times W_c^-)$, which are kept fixed under the induced operation of Γ on the homology. Observing that Γ acts trivially on $H_*(0(2))$ we obtain

$$(10) \quad \mathscr{H}(E, 0(2)c) = H_*(0(2)) \otimes H_*(W_c, W_c^-)^\Gamma \subset H_*(0(2)) \otimes \mathscr{H}(E_c, c) .$$

The invariant $\mathscr{H}(E, 0(2)c)$ is of finite type as $\mathscr{H}(E_c, c)$, i.e., $\mathscr{H}_k$ is finite dimensional and $\mathscr{H}_k = 0$ for almost all k. In [5], there was also introduced the characteristic invariant $\mathscr{H}^0$, which together with the index λ of c determines $\mathscr{H}$ completely by the shifting theorem, so that $\mathscr{H}_{k+\lambda}(E_c, c) = \mathscr{H}_k^0(E_c, c)$. The last equality and (10) yield

$$(11) \quad \mathscr{H}_k(E, 0(2)c) \subset V_k \oplus V_k , \qquad V_k = \mathscr{H}_{k-\lambda}^0(E_c, c) \oplus \mathscr{H}_{k-\lambda-1}(E_c, c) .$$

Define the type numbers $B_k(c)$ of an isolated critical orbit $0(2)c$ of the energy E to be the dimension of the vector space $\mathscr{H}_k(E, 0(2)c)$, and further $B_k^0(c) = \dim \mathscr{H}_k^0(E_c, c)$. In terms of these numerical invariants, (11) reads as

$$(12) \qquad B_k(c) \le 2[B_{k-\lambda}^0(c) + B_{k-\lambda-1}^0(c)] .$$

Let $a<b$ be regular values of the energy E such that the critical set in $E^{-1}[a, b]$ consists of finitely many critical orbits $0(2)c^1, \cdots, 0(2)c^r$. Then we have the Morse inequalities

$$(13) \qquad \mathrm{b}_k(\Omega^b, \Omega^a) \le \sum_{i=1}^{r} B_k(c^i) ,$$

where $b_k(\Omega^b, \Omega^a) = \dim H_k(\Omega^b, \Omega^a)$. If $E(c^i) = E(c^1)$ for $1 \le i \le r$, equality holds in (13) by the definition of $B_k(c^i)$ and (9). To prove (13) in general, let $a_1 < a_2 < \cdots < a_s$ be the critical values of E in $[a, b]$. Set $a_0 = a$, and $a_{s+1} = b$, and choose $\varepsilon > 0$ such that $\varepsilon < a_i - a_{i-1}$, $i = 1, \cdots, s + 1$. We have an increasing sequence $\Omega^{a_0} \subset \Omega^{a_1+\varepsilon} \subset \cdots \subset \Omega^{a_s+\varepsilon} \subset \Omega^{a_{s+1}}$ of subspaces in Ω. Clearly $H_k(\Omega^{a_1+\varepsilon}, \Omega^{a_0}) = H_k(\Omega^{a_1+\varepsilon}, \Omega^{a_1-\varepsilon})$, $H_k(\Omega^{a_i+\varepsilon}, \Omega^{a_{i-1}+\varepsilon}) = H_k(\Omega^{a_i+\varepsilon}, \Omega^{a_i-\varepsilon})$ for $i = 1, \cdots, s$, and $H_k(\Omega^{a_{s+1}}, \Omega^{a_s+\varepsilon}) = 0$. All these vector spaces are of finite dimension by (9). Since b_k is a subadditive function, (13) follows; compare [16, §5].

3. The type numbers of a periodic geodesic

For any positive integer m we will introduce the iteration map $m: \Omega \to \Omega$, which plays an important role in the Morse theory on Ω. A point c in Ω may be considered as the restriction of the periodic curve $\tilde{c}: \boldsymbol{R} \to M$ with $\tilde{c}(k + t) = c(t)$ for $k \in \boldsymbol{Z}$ and $t \in [0, 1]$. The iterates $\tilde{c}_m$ of $\tilde{c}$, $\tilde{c}_m(t) = \tilde{c}(mt)$, determine points $\tilde{c}_m = c_m \mid [0, 1]$ in Ω. We also refer to c_m as the m-th iterate of c. Now the iteration map m is defined by $m(c) = c_m$. Obviously, m is equivariant up to the constant factor m^2,

$$E(c_m) = m^2 E(c) .$$

Furthermore, m is an imbedding of Hilbert manifolds. As a main step we study the sequences $B_k(c_m)$ of type numbers for the iterates c_m of c, provided that all $O(2)c_m$ are isolated critical orbits. The numbers $B_k(c_m)$ may also be called type numbers of the corresponding periodic geodesic. Since the sequence $\lambda(c_m)$ has been treated in § 1, it remains to handle the sequences $B_k^0(c_m)$ in order to estimate $B_k(c_m)$.

For technical reasons we are interested in the following slightly different riemannian structure on ΩM:

$$\langle\!\langle X, Y \rangle\!\rangle_\alpha = \int_0^1 (\alpha \langle X, Y \rangle + \langle X', Y' \rangle) dt , \tag{14}$$

with a real number $\alpha > 0$. Clearly, all these metrics are equivalent, so that $\langle\!\langle\ ,\ \rangle\!\rangle_1 = \langle\!\langle\ ,\ \rangle\!\rangle$.

Lemma 5. *If $\alpha = m^2$, then the gradient $\overset{\alpha}{\nabla} E$ with respect to* (14) *is tangent along the image $m\Omega \subset \Omega$ under the iteration map m,*

$$\overset{\alpha}{\nabla} E|_{c_m} = m_* \nabla E|_c \qquad \textit{for all } c \in \Omega .$$

Note that $m\colon (\Omega, \langle\!\langle\ ,\ \rangle\!\rangle) \to (\Omega, \langle\!\langle\ ,\ \rangle\!\rangle_\alpha)$ is an isometric imbedding up to the constant factor m^2.

Proof. It suffices to consider the gradient $\overset{\alpha}{\nabla} E$ only at the iterates c_m of C^2-curves c in Ω. Then using calculus of variations, $X = \overset{\alpha}{\nabla} E|_{c_m}$ is given as the unique periodic solution of the equation

$$X'' - m^2 X = \nabla \dot{c}_m . \tag{15}$$

Now let $Y = \nabla E|_c$, so $Y'' - Y = \nabla \dot{c}$, and the iterate Y_m of Y satisfies (15). Hence, $m_* Y = Y_m = X$ is tangent to $m\Omega$.

Theorem 3. *Let c be a closed geodesic in Ω such that $O(2)c_m$ is an isolated critical orbit and $\nu(c) = \nu(c_m)$ for some m. Then $B_k^0(c) = B_k^0(c_m)$ for all k.*

Proof. Choose a sufficiently small normal disc bundle $\mathscr{D} = O(2)\mathscr{D}_c$ as in the preceding section, and let $\mathscr{N}$ be the normal bundle of $m\mathscr{D}$ in Ω with respect to the metric $\langle\!\langle\ ,\ \rangle\!\rangle_\alpha$, $\alpha = m^2$. The exponential map Exp of Ω with respect to $\langle\!\langle\ ,\ \rangle\!\rangle_\alpha$ maps a neighborhood of the zero section in $\mathscr{N}$ diffeomorphically onto a neighborhood of $m\mathscr{D}$ in Ω. The image under Exp of a suitable neighborhood of the zero section in $\mathscr{N} \mid m\mathscr{D}_c$ will be a disc $\mathscr{D}_{c_m}$ containing

$m\mathscr{D}_c$ as a submanifold. For this observe also that $m:(\Omega, \langle\!\langle\ ,\ \rangle\!\rangle) \to (\Omega, \langle\!\langle\ ,\ \rangle\!\rangle_a)$ is a conformal map. If $\mathscr{D}_{c_m}$ has been chosen small, $\mathscr{D}_m = 0(2)\mathscr{D}_{c_m}$ is a smooth normal disc bundle over $0(2)c_m$ with respect to the metric $\langle\!\langle\ ,\ \rangle\!\rangle_a$ and with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$ as well, and moreover $m\mathscr{D} \subset \mathscr{D}_m$. According to Lemma 4, $\overset{a}{\nabla} E$ is tangent to $m\mathscr{D}$; thus $\overset{a}{\nabla} E_{c_m}$ is tangent to $m\mathscr{D}_c$ by the choice of $\mathscr{D}_{c_m}$ where $E_{c_m} = E \mid \mathscr{D}_{c_m}$.

It is clear that m_* maps the null space of the hessian $\bar{H}_c$ of E_c injectively and hence, by our assumption on the nullities, isomorphically onto the null space of the hessian $\bar{H}_{c_m}$ of E_{c_m}. Moreover, c_m is an isolated critical point of $E_{c_m} \mid m\mathscr{D}_c$, for $0(2)c$ is with $0(2)c_m$ an isolated critical orbit. Now the energy E_{c_m} on $\mathscr{D}_{c_m}$ and the submanifold $m\mathscr{D}_c$ of $\mathscr{D}_{c_m}$ satisfy the hypotheses of Lemma 7 in [5], so we obtain $\mathscr{H}^0(E_{c_m}, c_m) = \mathscr{H}^0(E_{c_m} \mid m\mathscr{D}_c, c_m)$. On the other hand, $\mathscr{H}^0(E \mid m\mathscr{D}_c, c_m) = \mathscr{H}^0(E \mid \mathscr{D}_c, c)$, since $m\colon \mathscr{D}_c \to m\mathscr{D}_c$ is a diffeomorphism and $E(\bar{c}_m) = m^2 E(\bar{c})$ for all $\bar{c}$. This completes the proof.

Combining Lemma 2 and Theorem 3 we obtain

Corollary 1. *Let c be a closed geodesic in Ω, and assume that all the critical orbits $0(2)c_m$ are isolated. Then $B_k^0(c_m)$ is uniformly bounded; more precisely,*

$$B_k^0(c_m) \le B \qquad \text{for all } k, m$$

with some constant B. Furthermore, there is a number k_0 such that

$$B_k^0(c_m) = 0 \quad \textit{for} \quad k > k_0 \quad \textit{and all } m.$$

From (3), (5), (6) in §1 it is clear that the sequence $\lambda(c_m)$ is determined by finitely many constants, the different values of the function Λ on S^1. The corollary shows that $B_k^0(c_m)$ takes on only finitely many values. We should mention that all type numbers $B_k(c_m)$ are determined by finitely many constants via (10), though explicit computations are difficult.

Corollary 2. *Under the hypotheses of Corollary 1, for the resulting constants B and k_0 the type numbers $B_k(c_m)$ are uniformly bounded by $4B$. Moreover, given $k > k_0 + 1$, the number of orbits $0(2)c_m$ such that $B_k(c_m) \neq 0$ is bounded by a constant C which does not depend on k.*

Proof. We use (12), Lemma 1, and Corollary 1. Observe that $B_k(c_m) = 0$ for $k > k_0 + 1$ if $\lambda(c_m) = 0$ for all m. If the index of some iterate of c is not zero, then we have to estimate the number of orbits $0(2)c_m$ with $B^0_{k-\lambda(c_m)}(c_m) + B^0_{k-\lambda(c_m)-1}(c_m) \neq 0$. Now $B_k^0(c_m) = 0$ whenever $k > k_0$ or $k < 0$. Therefore we need an estimate for the number of the orbits satisfying $k - 1 - k_0 \le \lambda(c_m) \le k$. Let ε and a be the constants in Lemma 1, and $s > a/\varepsilon$ an integer. Then $C = s\left(\dfrac{1 + k_0}{s\varepsilon - a} + 1\right)$ will serve as a uniform upper bound.

4. Existence of closed geodesics

Let M be a compact simply connected riemannian manifold without boundary, and $\dim M \geq 2$. Then the free loop space $\Omega = \Omega M$ is connected. It is known that the homotopy type of ΩM depends only on the homotopy type of M. In fact, the inclusion of Ω into the space of all continuous maps $S^1 \to M$ with the compact open topology is a homotopy equivalence. The Betti numbers $b_k(\Omega) = \dim H_k(\Omega)$ are finite; this follows, for example, from [19, p. 465] by using the fibration $\Omega_* \to \Omega \to M$, where Ω_* is the ordinary loop space with fixed base point.

Theorem 4. *If the sequence $b_k(\Omega)$ is not bounded, then there exist infinitely many geometrically distinct periodic geodesics in M.*

Proof. Suppose that there are only finitely many such geodesics. Then we find simply closed geodesics $c^1, \cdots, c^r$ such that any non-constant closed geodesic in Ω lies on some orbit $0(2)c^i_m$. So all critical orbits $0(2)c^i_m$ are isolated, and Ω^b contains only a finite number of them for any given b. By Corollary 2 we find the constant $\hat{B} = \max\limits_{i,k,m} B_k(c^i_m)$. Choose k^i_0 and C^i for the geodesic c^i according to Corollaries 1 and 2, and set $\hat{k}_0 = \max k^i_0$, and $\hat{C} = \sum\limits_{i=1}^{r} C^i$. Now given any $k > \hat{k}_0 + 1$, the constant $\hat{C}$ is an upper bound for the number of orbits $0(2)c^i_m$ with $B_k(c^i_m) \neq 0$. So we obtain, from the Morse inequalities (13),

$$b_k(\Omega^b, \Omega^a) \leq \hat{C}\hat{B} \qquad \text{for } k > \hat{k}_0 + 1$$

and for all regular values $0 < a < b$. Let $0 < a < \min\limits_i E(c^i)$. Then $\Omega^0 = M$ is a strong deformation retract of Ω^a, and therefore $b_k(\Omega^b, \Omega^a) = b_k(\Omega^b, M)$. Since M is of finite dimension, an exact sequence argument shows that $b_k(\Omega^b, M) = b_k(\Omega^b)$ for almost all k. Choose b large such that all geodesics c^i_m with $B_k(c^i_m) \neq 0$ and $B_{k+1}(c^i_m) \neq 0$ are contained in Ω^b. Then $b_k(\Omega^d, \Omega^b) = 0$ and $b_{k+1}(\Omega^d, \Omega^b) = 0$ for all regular values $d \geq b$, and therefore $b_k(\Omega, \Omega^b) = 0$ and $b_{k+1}(\Omega, \Omega^b) = 0$. Thus $b_k(\Omega) = b_k(\Omega^b)$. Combining the last conclusions we get

$$b_k(\Omega) \leq \hat{C}\hat{B} \qquad \text{for almost all } k\,,$$

which contradicts the hypothesis of the theorem.

In the following we discuss the topological conditions in Theorem 4 and add some further remarks. Recall that we are working with a coefficient field of characteristic zero. Clearly, the property of the sequence $b_k(\Omega)$, to be bounded or not, is a homotopy invariant of M. Let us start with examples.

(i) If G is a compact simply connected Lie group, then $\Omega = G \times \Omega_*$, hence $H^*(\Omega) = H^*(G) \otimes H^*(\Omega_*)$. Since $H^*(\Omega_*)$ is a polynomial algebra with rank(G) generators, $b_k(\Omega)$ is unbounded iff rank$(G) \geq 2$, i.e., iff $G \neq Sp(1) = S^3$. So when M has the homotopy type of such a group G, the theorem applies.

(ii) For the sphere S^n the cohomology of Ω is given by

$$H^*(\Omega) = H^*(S^n) \otimes H^*(\Omega_*)$$

if n is odd. The ring $H^*(\Omega_*)$ is well known: $\dim H^k(\Omega_*) = 1$ for $k \equiv 0 \bmod n-1$, and $H^k(\Omega_*) = 0$ otherwise. In the case of even dimensional spheres one has $b_0(\Omega) = b_{(2i-1)(n-1)}(\Omega) = b_{(2i-1)(n-1)+1}(\Omega) = 1$ for $i \geq 1$, and $b_k(\Omega) = 0$ otherwise. This implies that $b_k(\Omega(S^m \times S^n)) = b_k(\Omega S^m \times \Omega S^n)$ is always unbounded whenever $m, n \geq 2$. Of course, $b_k(\Omega S^n) \leq 1$. Compare [21] and [8] for the cohomology structure of ΩS^n.

(iii) More generally, our condition on the Betti numbers $b_k(\Omega)$ is satisfied for all manifolds M having one of the following properties; see [8]:

1) The smallest positive integer k_0 with $b_{k_0}(M) \neq 0$ is odd, and $b_{k_0}(M) \geq 2$.

2) $H^*(M)$ is a tensor product of at least two truncated polynomial algebras and possibly exterior algebras.

Special cases of 1) and 2) are manifolds with the cohomology of a product of spheres or projective spaces.

Recently Klein has obtained new results on the cohomology of ΩM with characteristic zero coefficients; see [9]. His results imply that $b_k(\Omega)$ is not bounded under weak conditions on the cohomology ring $H^*(M)$, when M is simply connected. Without any further assumptions on the compact simply connected manifold M, Klein shows for example: There is an arithmetic progression k_i with $b_{k_i}(\Omega) \neq 0$ for all i. So in particular, $b_k(\Omega)$ is unbounded if M is homotopically a product of two compact simply connected manifolds. In fact, the only examples of compact simply connected manifolds M known to us, where the sequence $b_k(\Omega)$ is bounded and hence Theorem 4 does not apply, have the homotopy type of a symmetric space of rank 1. This is the more surprising, since the latter spaces always carry a metric for which even all geodesics are periodic. As yet it is not known whether for arbitrary metrics on such spaces there are infinitely many distinct periodic geodesics, even when restricting attention to perturbations of the standard symmetric structure.

Now let us drop the assumption that M is simply connected. In case the fundamental group $\pi_1(M)$ is finite, we may pass to the compact universal riemannian covering $\tilde{M}$. If $b_k(\Omega\tilde{M})$ is unbounded, then Theorem 4 applies to $\tilde{M}$, and infinitely many distinct periodic geodesics of $\tilde{M}$ project down to infinitely many distinct periodic geodesics of M. The problem is more delicate when $\pi_1(M)$ has infinite order; compare [7] for a further discussion.

Without the compactness of M our methods break down in general, since the energy E on ΩM will not satisfy condition (C) any more. The funnel surface $(x^2 + y^2)z^2 = 1, z < 0$, in euclidean space $\boldsymbol{R}^3$ is complete and not simply connected, yet there is no periodic geodesic at all. For complete open riemannian manifolds M we know only of some existence and non-existence theorems under the additional hypothesis that the sectional curvature K does not change sign on M. If $K \leq 0$ and M is simply connected, then by the theorem of Hadamard-Cartan, any regular geodesic $\boldsymbol{R} \to M$ is an imbedding and hence not periodic. In the case $K < 0$ and $\pi_1(M) \neq 1$, it follows from a theorem of Preissmann that the connected component of ΩM determined by any conjugacy class of $\pi_1(M)$ contains at most one closed geodesic up to parametrization; see also [4, pp. 210–212]. So for example, if $\pi_1(M) = \boldsymbol{Z}$, there exists at most one periodic geodesic on M. The authors have shown that all closed geodesics are constant when $K > 0$; compare [6], Lemma 3 and Theorem 4, but there may be lots of closed geodesics with a corner. Now let $K \geq 0$. From the structure theory for complete open manifolds of nonnegative curvature (see [3]), one can deduce that there exist infinitely many distinct periodic geodesics in M when $\pi_1(M)$ is infinite, otherwise the same conclusion holds, provided $b_k(\Omega\tilde{M})$ is not bounded for the finite universal covering $\tilde{M}$ of M. To see this, one may use Theorem 4 and the fact that there is a strong deformation retract of M onto a compact totally geodesic submanifold S of M. One can also derive a result quite similar to Theorem 4 for compact riemannian manifolds with locally convex boundary; see [7].

Let us look again at the proof of Theorem 4. Assume that all the critical orbits of the energy E on ΩM are non-degenerate; then the metric of M is said to be bumpy. In that case our arguments can be simplified by combining more directly Lemma 1 and the Morse inequalities for non-degenerate critical manifolds; see also [15], [22]. Now Abraham has shown that the set of bumpy metrics is dense in the space of all metrics with the C^∞-topology; compare [1]. However, this fact does not seem to be of any help in the proof of Theorem 4 for arbitrary metrics. Periodic geodesics of bumpy metrics may fade away when taking limits. We should point out that a bumpy manifold always carries arbitrarily long simply closed geodesics as soon as there are infinitely many distinct periodic geodesics. Of course, this need no longer be true in general, say when M is the euclidean sphere. We did not emphasize the best possible order of differentiability in all our procedures, as Theorem 4 is already interesting for C^r-metrics, $r \geq 2$. It cannot be deduced from the C^∞-case by an approximation method in an obvious way, since in most cases periodic geodesics are unstable under deformations of the metric. However, all techniques involved work without any change at least for $r \geq 6$.

Another question in this context is whether there are closed geodesics without self-intersections on M, which are called simple closed geodesics. Results dealing with this problem were obtained by Lusternik-Schnirelmann [14] and Klingenberg [12], [10]. In dimensions ≥ 3 there is a procedure of resolving intersections of geodesics by arbitrarily small approximations of the underlying metric. This yields the following: There exists a dense subset of bumpy metrics in the space of all metrics on M for which all simply closed geodesics are simple and mutually disjoint. So, if $0 < k \leq \infty$ is a lower bound for the number of distinct periodic geodesics on M with respect to all metrics, almost any metric will carry at least k mutually disjoint simple closed geodesics; see [7].

Looking for distinct periodic geodesics in M is equivalent to asking for closed orbits of the geodesic flow on the tangent sphere bundle over M. This flow is known to be a hamiltonian dynamical system. In the general theory of such systems there are some strong, but very special existence theorems for closed orbits. The strongest result is essentially due to Anosov, implying that periodic orbits of the geodesic flow are dense if the sectional curvature of M is negative; compare [20]. For arbitrary metrics there is no structural stability, and the theory of dynamical systems does not provide any existence theorem as yet. After the preparation of this paper, A. Weinstein gave a simple construction of metrics on any manifold for which periodic orbits of the geodesic flow are not dense; see [23].

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