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HOLOMORPHIC VECTOR FIELDS AND THE FIRST CHERN CLASS OF A HODGE MANIFOLD

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In a recent paper [2] Bott has proved that if a connected compact complex manifold admits a nonvanishing holomorphic vector field, then all the Chern numbers of M vanish.

In this paper we first prove the following theorem.

Theorem 1. *Let M be a connected Hodge manifold, and suppose that there exists a nonvanishing holomorphic vector field X in M. Then there exists a nonvanishing holomorphic 1-form ω in M such that $\omega(X) \neq 0$. In particular, the first Betti number $b_1(M)$ of M is different from zero.*

We shall then study the structure of a Hodge manifold with zero first Chern class. We denote by $c_1(M)$ and $q(M)$ the first Chern class and the irregularity (i.e., one half of the first Betti number) of M respectively, and by G the identity component of the group of all holomorphic transformations of M. The group G is a connected complex Lie group.

We shall prove the following two theorems which sharpen some of the recent results of Lichnerowicz [5].

Theorem 2. *Let M be a connected Hodge manifold such that $c_1(M) = 0$. Then the group G is an abelian variety of dimension $q(M)$ and the isotropy subgroup of G at any point in M is a finite group.*

Theorem 3. *Let M be a connected Hodge manifold and assume that $c_1(M) = 0$ and $q(M) > 0$. Then there exist an abelian variety A and a connected Hodge manifold F with the following properties.*

a) *$c_1(F) = 0$ and $q(F) = 0$;*

b) *$A \times F$ is a finite regular covering space of M and the group of covering transformations is solvable.*

After having finished this work, the author learned that Calabi stated these two theorems in his paper [4] as his well-known conjecture, and proved them under the assumption that M is a connected compact Kähler manifold with vanishing Ricci curvature tensor.

1. Let M be a connected compact Kähler manifold, and $\mathfrak{h}$ and $\mathfrak{g}$ denote, respectively, the complex vector space of all holomorphic 1-forms and the complex Lie algebra of all holomorphic vector fields in M. Then $\dim \mathfrak{h} = q(M)$ and we can identify $\mathfrak{g}$ with the Lie algebra of the group G. If $\omega \in \mathfrak{h}$ and $X \in \mathfrak{g}$, then

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$\omega(X)$ is a holomorphic function on M and hence a constant. Therefore $(\omega, X) \to \omega(X)$ defines a bilinear form B on $\mathfrak{h} \times \mathfrak{g}$.

Now let α be the canonical holomorphic mapping of M into the Albanese variety $A(M)$ of M [1], [6]. There exists also a complex Lie group homomorphism $\hat{\alpha}$ of G into the complex torus $A(M)$ such that $\alpha(\varphi x) = \hat{\alpha}(\varphi)\alpha(x)$ for any $\varphi \in G$ and $x \in M$. Let I be the kernel of the homomorphism $\hat{\alpha}: G \to A(M)$, and I^0 the identity component of I. The subalgebra $\mathfrak{i}$ of $\mathfrak{g}$ corresponding to I consists of all holomorphic vector fields X in M such that $\omega(X) = 0$ for all $\omega \in \mathfrak{h}$. In particular, if zero(X), $X \in \mathfrak{g}$, is non-empty, then $X \in \mathfrak{i}$, where zero(X) denotes the set of zero points of X. Assume now that M is a Hodge manifold, and let $\varphi: M \to P^N$ be a projective imbedding of M into a complex projective space P^N. Let G_φ be the group of all holomorphic transformations of M induced by the projective transformations of the ambient space P^N which leave stable the submanifold M. The subalgebra $\mathfrak{g}_\varphi$ of $\mathfrak{g}$ corresponding to G_φ consists of the restriction in M of all holomorphoic vector fields in P^N tangent to M. By a fixed point theorem of Borel [3], every $X \in \mathfrak{g}_\varphi$ has a zero point and hence $\mathfrak{g}_\varphi \subset \mathfrak{i}$ for any projective imbedding φ. On the other hand, a theorem of Blanchard [1, Theoreme principal I] asserts that there exists a projective imbedding φ_0 such that $I \subset G_{\varphi_0}$. It follows from these that 1) $\mathfrak{i} = \mathfrak{g}_{\varphi_0}$ and hence $\mathfrak{i}$ consists of all $X \in \mathfrak{g}$ such that zero(X) is non-empty; 2) $[I: I^0] < \infty$, because I^0 coincides with the identity component $G^0_{\varphi_0}$ of G_{φ_0} and, since G_{φ_0} is an algebraic group, we have $[G_{\varphi_0}: G^0_{\varphi_0}] < \infty$. We have thus proved

Proposition 1. *Let M be a connected Hodge manifold, and I the kernel of the homomorphism $\hat{\alpha}: G \to A(M)$. Then the number of connected components of I is finite, and the Lie algebra $\mathfrak{i}$ of I consists of all holomorphic vector fields X in M such that zero(X) is non-empty.*

Now let X be a nonvanishing holomorphic vector field in M. Then X does not belong to $\mathfrak{i}$ by Proposition 1, and there exists a holomorphic 1-form ω such that $\omega(X) \neq 0$, because $\mathfrak{i}$ consists of all $Y \in \mathfrak{g}$ such that $\omega(Y) = 0$ for all $\omega \in \mathfrak{h}$. Since $\omega(X) \neq 0$, ω is nonvanishing, which proves Theorem 1.

Remark. Let M be an even-dimensional connected compact semi-simple Lie group. Then there exists a left invariant complex structure on M, a right invariant vector field in M is a nonvanishing holomorphic vector field, and the first Betti number of M is zero. This example shows that the existence of a nonvanishing holomorphic vector field does not necessarily imply the nonvanishing of the first Betti number of a connected compact complex manifold. However, in this example, for any right invariant vector field X there exists a right invariant 1-form ω such that $\omega(X) \neq 0$, and ω is holomorphic although ω is not a closed form.

2. Let M be a connected compact Kähler manifold such that $c_1(M) = 0$. Then by a theorem of Lichnerowicz [5, *a*] the bilinear form $B: \mathfrak{h} \times \mathfrak{g} \to C$ is nondegenerate. In particular, we have $\dim \mathfrak{g} = \dim \mathfrak{h} = q(M)$ and every non-zero holomorphic vector field in M has no zero point. Hence from Proposition 1 we obtain the following

Proposition 2. *Let M be a connected Hodge manifold such that $c_1(M) = 0$. Then the homomorphism $\hat{\alpha}\colon G \to A(M)$ is an isogeny, that is, a surjective homomorphism with a finite kernel I. In particular, G is an abelian variety of dimension $q(M)$. If $\varphi\colon M \to P^N$ is a projective imbedding, then the group G_φ of holomorphic transformations of M induced by the projective transformations of P^N is finite.*

The assertion of Theorem 2 is included in Propositions 1 and 2.

3. Let M be a connected Hodge manifold such that $c_1(M) = 0$ and let $M_1 = \alpha^{-1}(e)$, where $\alpha\colon M \to A(M)$ is the canonical holomorphic mapping and e denotes the identity element of the torus $A(M)$. From the universality of the mapping α, we can easily conclude that M_1 is connected [5, b] and see that $c_1(M_1) = 0$. Since the finite group I acts on M_1, let E be the holomorphic fibre bundle over $A(M)$ with fibre M_1 associated with the holomorphic principal bundle $0 \to I \to G \to A(M) \to 0$. Then E is the quotient of $G \times M_1$ by the action of I defined by $\psi(\varphi, u) = (\varphi \cdot \psi^{-1}, \psi(u))(\psi \in I, \varphi \in G, u \in M_1)$. Let $\tilde{\beta}$ be the holomorphic mapping of $G \times M_1$ into M defined by $\tilde{\beta}(\varphi, u) = \varphi(u)$. Then it is easy to see that $\tilde{\beta}$ is surjective and that $\tilde{\beta}$ induces a bijective holomorphic mapping of E onto M such that the diagram

$$\begin{tikzcd} E \arrow[rr, "\beta"] \arrow[dr] & & M \arrow[dl, "\alpha"] \\ & A(M) & \end{tikzcd}$$

is commutative. Thus, M is a fibre bundle over $A(M)$ with projection α and fibre M_1. It follows also from the above that $G \times M_1$ is a finite covering space of M with I as the group of covering transformations. If $q(M_1) = 0$, then we have completed the proof of Theorem 3, because I is abelian. Assume $q(M_1) > 0$. Then we can find a connected Hodge manifold M_2 such that $c_1(M_2) = 0$ and that $G_1 \times M_2$ is a covering space of M_1 with a finite abelian covering transformation group, where G_1 denotes the identity component of the group of holomorphic transformations of M_1. Continuing in this way we get a sequence $\{M_i\}$ such that $c_1(M_i) = 0$, $\dim M_{i+1} = \dim M_i - q(M_i)$ for $i = 0, 1, \cdots$, where $M_0 = M$. Therefore, there must exist an integer k such that $q(M_k) = 0$ (the dimension of M_k might be zero). Let $A = G \times G_1 \times \cdots \times G_{k-1}$ and $F = M_k$. Then A is an abelian variety and $A \times F$ is a covering manifold of M with a finite solvable covering transformation group. Hence Theorem 3 is proved.

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