J. DIFFERENTIAL GEOMETRY
3 (1969) 461-465

SOME PROPERTY OF CLOSED HYPERSURFACES IN RIEMANNIAN MANIFOLDS

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1. The main result of the present paper is the

Theorem. *Let M^3 be a three-dimensional symmetric Riemannian manifold whose sectional curvature $K(P,\sigma)$ satisfies $(1-\delta)T \leq K(P,\sigma) \leq T$, where T is a positive constant and $0 \leq \delta < 1/2$. Let M be a closed surface in M^3 with the mean curvature H satisfying $H = C$, C being a positive constant, and assume that M is strictly convex, and the second fundamental form of M is positive. Let the total volume or the total area of M be denoted by V_M, and the volume of the sebset M_L of M, where the difference of the principal curvatures exceeds $2L$, be denoted by $V(L)$. If $(1-\delta)L^2 > \delta C^2$, then $V(L)$ satisfies*

$$\frac{V(L)}{V_M} \leq \frac{\delta^2 C^2}{\delta(25\delta - 16)C^2 + 8(1-\delta)(2-3\delta)L^2} . \tag{0}$$

Corollary. *If M^3 is a space of constant curvature with positive scalar curvature, then the surface M of the above theorem is totally umbilical.*

2. Let M^{n+1} be a Riemannian manifold of dimension $n+1$, K_{kjih} the curvature tensor of M^{n+1}, and M^n a hypersurface of M^{n+1}, whose equation is given by $x^h = x^h(u^a)$ locally. Throughout this paper all the indices run as follows: $h, i, j, k = 1, \cdots, n+1$; $a, b, c, d = \dot{1}, \cdots, \dot{n}$.

We define $B_a{}^h$ as usual by $B_a{}^h = \partial_a x^h$ where $\partial_a = \partial/\partial u^a$. From $B_a{}^h$ and the unit normal vector N^h we can construct a matrix $(B_a{}^h, N^h)$ and denote its reciprocal matrix by $(B^a{}_h, N_h)$. $g_{ba} = B_b{}^i B_a{}^h g_{ih}$ is the first fundamental tensor of M^n. Using the Van der Waerden-Bortolotti operator ∇ we get $\nabla_b B_a{}^h = h_{ba}N^h$, $\nabla_b N^h = -h_b{}^a B_a{}^h$, where h_{ba} is the second fundamental tensor of M^n. The equation of Codazzi is

$$\nabla_c h_{ba} - \nabla_b h_{ca} = K_{kjih} B^{kji}_{cba} N^h , \tag{1}$$

and the equation of Gauss is

$$'K_{dcba} = K_{kjih} B^{kjih}_{dcba} + h_{cb}h_{da} - h_{db}h_{ca} , \tag{2}$$

where $'K_{dcba}$ is the curvature tensor of the Riemannian manifold M^n.

If M^n is a closed hypersurface, then

Communicated by K. Yano, August 5, 1968.

(3) $$\int (\nabla_c h_{ba})\nabla^c h^{ba} dV = \int (-\nabla^c\nabla_c h_{ba})h^{ba} dV\,,$$

where dV is the volume element of M^n, $dV = (\det(g_{ba}))^{\frac{1}{2}}du^1 \cdots du^n$, and the integration is performed over M^n.

By virtue of (1) and the Ricci identity in M^n the second member of (3) becomes

$$\begin{aligned}
&\int [-\nabla^c(\nabla_b h_{ca} + K_{kjih}B^{kji}_{cba}N^h)]h^{ba}dV \\
&= \int [-(\nabla_b\nabla^c h_{ca})h^{ba} + {}'K^c{}_{bce}h^e{}_a h^{ba} + {}'K^c{}_{bae}h_c{}^e h^{ba} \\
&\qquad - (\nabla^l K_{kjih})B^{ckji}_{lcba}N^h h^{ba} - K_{kjih}(h^c{}_c N^k B^{ji}_{ba}N^h \\
&\qquad + h^c{}_b N^j B^{ki}_{ca}N^h - B^{kjih}_{cbae}h^{ce})h^{ba}]dV\,.
\end{aligned}$$

Using (1) again we reduce the last member to

$$\begin{aligned}
&\int [-(\nabla_b(\nabla_a h^c{}_c + K^k{}_{jih}B^{cji}_{kac}N^h))h^{ba} \\
&\qquad + {}'K^c{}_{bce}h^e{}_a h^{ba} + {}'K^c{}_{bae}h_c{}^e h^{ba} - (\nabla^k K_{kjih})N^h B^{ji}_{ba}h^{ba} \\
&\qquad + N^l(\nabla_l K_{kjih})N^k N^h B^{ji}_{ba}h^{ba} - h_c{}^c K_{kjih}N^k N^h B^{ji}_{ba}h^{ba} \\
&\qquad + K_{kjih}N^k N^h B^{ji}_{ba}h^{bc}h_c{}^a + K_{kjih}B^{kjih}_{dcba}h^{da}h^{cb}]dV\,.
\end{aligned}$$

By (2) and

$$\begin{aligned}
-\nabla_b(K^k{}_{jih}B^{cji}_{kac}N^h) &= \nabla_b(K_{jh}B_a{}^j N^h) \\
&= (\nabla_k K_{jh})B^{kj}_{ba}N^h + K_{jh}h_{ba}N^j N^h - K_{jh}B^{jh}_{ac}h_b{}^c\,, \\
\nabla^k K_{kjih} = \nabla_h K_{ji} - \nabla_i K_{jh}\,,
\end{aligned}$$

a straightforward calculation gives

(4) $$\begin{aligned}
&\int (\nabla_c h_{ba})\nabla^c h^{ba} dV \\
&= \int [-(\nabla_b\nabla_a h_c{}^c)h^{ba} \\
&\qquad + (2N^h\nabla_j K_{ih} - N^l\nabla_l K_{ji} + N^l N^k N^h \nabla_l K_{kjih})B^{ji}_{ba}h^{ba} \\
&\qquad - h_c{}^b h_b{}^a h_a{}^c h_e{}^e + (h_{ba}h^{ba})^2 \\
&\qquad + K_{kjih}N^k N^h B^{ji}_{ba}(g^{ba}h_{dc}h^{dc} - h^{ba}h_e{}^e) \\
&\qquad + 2K_{kjih}B^{kjih}_{dcba}(h^{da}h^{cb} - h^{ce}h_e{}^b g^{da})]dV\,.
\end{aligned}$$

If M^{n+1} is a symmetric Riemannian manifold, then

$$\int (\nabla_c h_{ba})\nabla^c h^{ba} dV$$

(5)
$$= \int [-(\nabla_b \nabla_a h_c{}^c)h^{ba} - h_c{}^b h_b{}^a h_a{}^c h_e{}^e + (h_{ba}h^{ba})^2$$
$$+ K_{kjih}N^k N^h B^{ji}_{ba}(g^{ba}h_{dc}h^{dc} - h^{ba}h_e{}^e)$$
$$+ 2K_{kjih}B^{kjih}_{dcba}(h^{da}h^{cb} - h^{ce}h_e{}^b g^{da})]dV .$$

3. Let us consider the case where M^{n+1} is a symmetric space and the mean curvature of the hypersurface M^n is constant, that is, $\nabla_a h_c{}^c = 0$, and at each point P of M^n take an orthonormal frame so that

$$g_{ba} = \delta_{ba} , \qquad h_{ba} = k_a \delta_{ba} .$$

If we use the notation

(6)
$$T_{Na} = K_{kjih}N^k N^h B^{ji}_{aa} ,$$
$$T_{ba}(= T_{ab}) = K_{kjih}B^{kjih}_{baab} ,$$

we can write the integrand in the second member of (5) in the form

$$f = -\sum_a (k_a)^3 \sum_b k_b + [\sum_a (k_a)^2]^2$$
$$+ \sum_a T_{Na} \sum_b (k_b)^2 - \sum_a T_{Na} k_a \sum_b k_b$$
$$+ 2\sum_{a,b} T_{ba} k_b k_a - 2\sum_{a,b} T_{ba}(k_b)^2 .$$

Hence

(7)
$$f = -\frac{1}{2}\sum_{a,b} k_b k_a (k_b - k_a)^2$$
$$+ \frac{1}{2}\sum_{a,b} (T_{Na}k_b - T_{Nb}k_a)(k_b - k_a)$$
$$- \sum_{a,b} T_{ba}(k_b - k_a)^2 .$$

If the sectional curvature $K(P, \sigma)$ satisfies

$$(1 - \delta)T \leq K(P, \sigma) \leq T ,$$

where $T > 0$, $0 \leq \delta \leq 1$, and if $k_a \geq 0$, then

(8)
$$f \leq -\frac{1}{2}\sum_{a,b} k_b k_a (k_b - k_a)^2$$
$$+ T\sum_{b>a} (k_b - (1 - \delta)k_a)(k_b - k_a)$$
$$- (1 - \delta)T \sum_{a,b} (k_b - k_a)^2$$

$$= -\frac{1}{2}\sum_{a,b} k_b k_a (k_b - k_a)^2 + \delta T \sum_{b>a} k_b (k_b - k_a)$$
$$- \frac{1}{2}(1-\delta) T \sum_{a,b} (k_b - k_a)^2 ,$$

where the principal curvatures are arranged in the order

$$0 \le k_1 \le k_2 \le \cdots \le k_n .$$

4. Now let us consider the case $n = 2$, $0 < k_1 \le k_2$. Then f satisfies

$$f \le -k_1 k_2 (k_2 - k_1)^2 - (1-\delta) T (k_2 - k_1)^2 + \delta T k_2 (k_2 - k_1) .$$

Let us put $k_2 - k_1 = 2x$. Then, since we have $k_1 + k_2 = 2C$, we get

$$(9) \qquad k_1 = C - x , \qquad k_2 = C + x , \qquad 0 \le x < C .$$

Now define $g(x)$ by

$$(10) \qquad g(x) = -4(C^2 - x^2)x^2 - 4(1-\delta)Tx^2 + 2\delta T x (C + x) .$$

If P is a point of M^2 such that at P the principal curvatures satisfy (9) for a given number x, then we have

$$(11) \qquad f(P) \le g(x) .$$

Hence, if $g(x)$ satisfies $g(x) \le G$ for $0 \le x < C$, we get $f \le G$ on M^2.

Let the total volume of M^2 be V_M, and A any positive number, and denote by V_A the volume of the subset of M^2 on which $x = \frac{1}{2}(k_2 - k_1)$ satisfies $g(x) \le -A$. Then we have

$$0 \le \int (\nabla_c h_{ba}) \nabla^c h^{ba} dV = \int f dV \le G(V_M - V_A) - A V_A ,$$

and can conclude

$$(12) \qquad \frac{V_A}{V_M} \le \frac{G}{G+A} .$$

Let us now estimate G. If we put

$$\varphi(x) = -4(1-\delta)Tx^2 + 2\delta T x (C + x) ,$$

we have $f(P) \le \varphi(x)$, and the maximum M_φ of $\varphi(x)$ for $0 \le x \le C$ is given by

$$M_\varphi = \frac{\delta^2 C^2 T}{2(2 - 3\delta)} , \qquad \text{if} \quad \delta \le \frac{4}{7} ,$$
$$M_\varphi = 4(2\delta - 1) C^2 T , \qquad \text{if} \quad \delta \ge 4/7 .$$

We take M_φ for G, although a better estimate will be possible when C^2 is large compared with T.

Now we suppose $\delta < \frac{1}{2}$, and let L be a positive number such that

$$C > L > \left(\frac{\delta}{1-\delta}\right)^{\frac{1}{2}} C .$$

If x satisfies $C \geq x \geq L$, then $g(x)$ satisfies

$$g(x) \leq \varphi(x) \leq -4T((1-\delta)L^2 - \delta C^2) .$$

Hence we can put

$$A = 4T((1-\delta)L^2 - \delta C^2), \qquad G = \frac{\delta^2 C^2 T}{2(2-3\delta)} ,$$

so that (0) holds.

Example. If $\delta = 0.2$ and $L = 0.75C$, we have

$$\frac{V(L)}{V_M} \leq \frac{1}{71} .$$

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