J. DIFFERENTIAL GEOMETRY
3 (1969) 213-229

# MORSE THEORY OF CERTAIN SYMMETRIC SPACES

S. RAMANUJAM

## Introduction

T. T. Frankel applied Morse theory to the classical groups, which we shall denote by $G$, and Stiefel manifolds [1] by taking the matrix representation of the classical groups and using the "trace function" as Morse function. The present author [4] obtained a Morse decomposition for certain symmetric spaces $G/K$ by using methods similar to those of Frankel with the same trace function since those symmetric spaces are imbedded in the group $G$. M. Takeuchi [6] has considered the same problem and others from a more general point of view.

The purpose of this paper is to eliminate the heavy manipulation of [4] and to point out alternate methods with the symmetric spaces described in terms of matrices, so that this paper differs from [4] in the following three ways:

First, the critical submanifolds of $G/K$ are shown to be the intersection of the space $G/K$ and the critical submanifolds of $G$.

Secondly, the indices of the critical submanifolds of $G/K$ are immediately obtained from that of the critical submanifolds of $G$.

Thirdly, "Floyd Theorem B" is used not to a great extent but only for the case $U(2n)/Sp(n)$.

## 1. Preliminaries

We briefly describe how the coset space $G/K$ arises as a symmetric space. Let $G$ be a compact connected Lie group with a left and right invariant Riemannian metric. (In all our discussions $G$ will denote the classical groups, i.e., $SO(n)$, $U(n)$ or $Sp(n)$.) Let $\theta$ be an involution on $G$, $K$ the full fixed set, and $K'$ the identity component of $K$. Then $G/K$ is a symmetric space.

Let $\mathfrak{g}$ be the Lie algebra of $G$. Then $\mathfrak{g}$ decomposes into a natural direct sum, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k} = \{x \in \mathfrak{g} \mid s(x) = x\}$ and $\mathfrak{p} = \{x \in \mathfrak{g} \mid s(x) = -x\}$ i.e., into eigenspaces of eigenvalue $+1$ and $-1$ for $s$, the differential action induced by $\theta$ on $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{p}$ be a maximal subalgebra of $\mathfrak{p}$. Then $\mathfrak{h}$ is abelian and is called *Cartan subalgebra*.

Define $\eta\colon G \to G$ by $\eta(g) = g \cdot \theta(g^{-1})$. Then $\eta(gK) = \eta(g)$, which means $\eta$ is constant along the left cosets of $K$. Hence, $\eta$ induces a map $\eta_*\colon G/K \to G$.

Received April 11, 1968. This research was supported by NSF Grant GP-5874.

Let $M$ be the image of $\mathfrak{p}$ under the exponential mapping. Then $\eta_*$ is a homeomorphism of $G/K$ onto $M$, and the natural action of $K$ on $G/K$ (under the imbedding $gK \to g\cdot\theta(g^{-1})$) becomes the adjoint action of $K$ on $G$ restricted to $M$ for

$$k(gK) \to k\cdot g\cdot\theta(g^{-1}k^{-1}) = kg\theta(g^{-1})\theta(k^{-1}) = kg\theta(g^{-1})k^{-1} .$$

So we think of $G/K$ as a submanifold $M$ of $G$, and $M$ is a totally geodesic submanifold. In particular the geodesics of $M$ through $e$ (the identity element) coincide with the 1-parameter subgroups of $G$, which lie in $M$.

Let $T$ be the image of $\mathfrak{h}$ under the exponential map. Any torus of this form will be called a maximal torus of $G/K = M$, and these tori have the following two properties:

1) Every point of $M$ lie on a maximal tori of $M$.
2) If $T$ and $T'$ are two maximal tori, then $T' = kTk^{-1}$ for some $k \in K'$.

The dimension of the maximal torus is the rank of $G/K$. If the rank of $G/K =$ rank of $G$, then $G/K$ is said to be of maximal rank.

## 2. Critical submanifolds of the classical groups

Frankel considered the function $f(g) = \operatorname{Re} \operatorname{tr} g$, $g \in G$ on the classical groups $G$. Here, of course, $G$ is represented by matrices. Frankel showed [1] that *the critical set for this "trace function" consists of all matrices in $G$ satisfying the condition $g^2 = e$.* (We call such a matrix a Grassmann matrix in $G$). Thus for

$SO(n)$, the critical sets are real Grassmannians $\dfrac{SO(n)}{SO(n)\cap\{O(2k)\times O(n-2k)\}}$,

$U(n)$, the critical sets are complex Grassmannians $\dfrac{U(n)}{U(k)\times U(n-k)}$,

$Sp(n)$, the critical sets are quaternionic Grassmannians $\dfrac{Sp(n)}{Sp(k)\times Sp(n-k)}$,

$$k = 0, 1, \cdots, n .$$

Before we describe these critical sets in terms of matrices, we give the following well-known definitions.

**Definition.** An $n \times n$ matrix $X$ (with real or complex entries) is said to be *symmetric* if $X = X^t$ ($X^t =$ transpose of $X$).

A matrix $X$ (real or complex) will be said to be *skew-symmetric* if $X = -X^t$. (Remark: A skew-symmetric matrix of odd order is singular. We will deal only with even order matrices in this case.)

A complex matrix $X$ is *Hermitian symmetric* if $\bar{X} = X^t$ ($\bar{X}$ means complex conjugate of $X$).

A complex matrix $X$ is *skew-Hermitian* if $\bar{X} = -X^t$.

Now, following Steenrod [5, p. 205] we show that the *real Grassmannians* $G_{n,2k}$, $k = 0, 1, \cdots, [n/2]$, *which arise as critical submanifolds of* $SO(n)$, *are the set of symmetric matrices in* $SO(n)$. Suppose $h$ is any symmetric matrix in $SO(n)$. Then for any $g \in SO(n)$, $ghg^{-1}$ is again symmetric and conversely. Thus any symmetric matrix in $SO(n)$ is (special) orthogonally equivalent to a symmetric matrix on the maximal torus. But symmetric matrix on the maximal torus are precisely those matrices which have $+1$ or $-1$ on the diagonal. Since we are in $SO(n)$, the determinant must be $+1$. Hence the $-1$'s must occur in pairs. Further the diagonal matrices with the same number of $+1$'s and $-1$'s on the diagonal are conjugate to each other. Thus, if

$$\sigma = I(n-2k) \times -I(2k) , \quad \text{where } I(2k) \text{ is } 2k \times 2k \text{ identity matrix} ,$$

then the set of all symmetric matrices in $SO(n)$ in obtained by the adjoint action of the group $SO(n)$ on $\sigma$ for all $\sigma$. This is precisely how the critical manifolds of the function $f$ are obtained. One obtains the critical points on the standard maximal torus of $SO(n)$ (these turn out to be $\sigma$'s) and then the critical manifolds for $SO(n)$ are obtained by letting $SO(n)$ act on $\sigma$ by conjugation.

The standard maximal torus of $SO(n)$ is

$$\begin{aligned} &\{R_2(\theta_1) \times \cdots \times R_2(\theta_m) \times 1\} , && \text{if } n = 2m+1 , \\ &\{R_2(\theta_1) \times \cdots \times R_2(\theta_m)\} , && \text{if } n = 2m, \text{ where} \\ &R_2(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} . \end{aligned}$$

A similar argument shows that the complex Grassmannians $W_{n,k}$, $k = 0, 1, \cdots, n$, which are critical submanifolds of $U(n)$ (for the trace function), are the set of all Hermitians symmetric matrices in $U(n)$. The standard maximal torus of $U(n)$ is

$$\{e^{i\theta_1} \times \cdots \times e^{i\theta_n}\} .$$

We imbed $Sp(n)$ in $U(2n)$ (more precisely in $SU(2n)$) under the correspondence

$$A + Bj \leftrightarrow \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} .$$

Now, we can think of the quaternionic Grassmannians $Q_{n,k}$, $k = 0, 1, \cdots, n$, as the set of "hermitian symmetric" matrices in $Sp(n) \subset U(2n)$. If one defines the quaternionic conjugation in the usual way (i.e., $\overline{a + bi + cj + dk} = a - bi - cj - dk$), then quaternionic Grassmannian is the set of all matrices

$X \in Sp(n)$ such that $X = \overline{\overline{X^t}}$. Confusion, if any, can be avoided by defining "complex hermitian symmetric matrices" (for the case $U(n)$ and "quaternionic hermitian symmetric matrices" (for the case $Sp(n)$). The standard maximal torus for $Sp(n) \subset SU(2n)$ is

$$\{e^{i\theta_1} \times \cdots \times e^{i\theta_n} \times e^{-i\theta_1} \times \cdots \times e^{-i\theta_n}\} .$$

## 3. Description of the symmetric spaces

$U(n)/O(n)$. On $U(n)$ consider the involution $\theta(X) = \bar{X}$, $\bar{X} =$ complex conjugate of $X$. The fixed set is $O(n)$, the identity component is $SO(n)$, and the standard maximal torus for $U(n)/O(n)$ is $\{e^{i\theta_1} \times \cdots \times e^{i\theta_n}\}$. *$U(n)/O(n) \subset U(n)$ can be identified with the manifold of all symmetric matrices in $U(n)$.* Since it is imbedded in $U(n)$ by $g \cdot O(n) \to g \cdot \theta(g^{-1}) = g \cdot \bar{g}^{-1} = gg^t$, the coset space $U(n)/O(n)$ is sent into symmetric matrices in $U(n)$.

Let $A$ be a symmetric matrix in $U(n)$, $\varepsilon_i$ be an eigenvalue, and $x_i$ the corresponding eigenvector. Then $Ax_i = \varepsilon_i x_i$, $\bar{A}\bar{x}_i = \bar{\varepsilon}_i \bar{x}_i$, $\bar{x}_i = \bar{A}^{-1}\bar{\varepsilon}_i\bar{x}_i = A\dfrac{1}{\varepsilon_i}\bar{x}_i$, because $\bar{A}^{-1} = A$ and $\varepsilon_i\bar{\varepsilon}_i = 1$. Thus $A\bar{x}_i = \varepsilon_i\bar{x}_i$, and we conclude that the eigenspace of $A$ is real, so that we can choose vectors $x_1, \cdots, x_n$ with $x_i x_j = \delta_{ij}$ such that $A((x_{ij})) = ((x_{ij}))\begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{pmatrix}$. Hence

$$A = ((x_{ij}))\begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix}((x_{ij}))^{-1} \in U(n)/O(n) .$$

This proof is due to H. Iwamoto [3].

$Sp(n)/U(n)$. $Sp(n)$ can be imbedded in $SU(2n)$ under the correspondence $A + Bj \leftrightarrow \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$. The involution on $Sp(n)$ is $\theta(X) = \bar{X} = J_n X J_n^{-1}$, $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, and the fixed set is $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ ($A$, $B$ real). This is precisely the imbedding of $U(n)$ in $SO(2n) \subset SU(2n)$. The imbedding of $Sp(n)/U(n)$ in $Sp(n)$ is given by $g \cdot U(n) \to g \cdot g^t$.

As in the case of $U(n)/O(n)$ it can be shown that if $A$ is a symmetric matrix in $Sp(n)$, i.e., $\bar{A} = A^{-1}$, then the eigenspace of $A$ is real. Moreover, $A \in Sp(n)$ if and only if $\bar{A} = J_n A J_n^{-1}$. It follows that if $x$ is a (real) eigenvector corresponding to the eigenvalue $\varepsilon$ for $A$, then $J_n^{-1}x$ is an eigenvector corresponding to the eigenvalue $\bar{\varepsilon}$. In other words $A = gtg^{-1}$, where $g \in U(n)$ and $t$ is a diagonal matrix of the form $e^{i\theta_1} \times \cdots \times e^{i\theta_n} \times e^{-i\theta_1} \times \cdots \times e^{-i\theta_n}$, so $A \in Sp(n)/U(n)$, and hence $Sp(n)/U(n)$ can be thought of as the manifold of symmetric matrices in $Sp(n)$.

$U(2n)/Sp(n)$. The involution on $U(2n)$ is $\theta(X) = J_n \bar{X} J_n^{-1}$, and the fixed set is $\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$ which is $Sp(n)$. As a symmetric space $U(2n)/Sp(n)$ is imbedded in $U(2n)$ under the mapping $gSp(n) \to gJ_n g^t J_n^{-1}$.

We define $X \in U(2n)$ to be a transvection if and only if $X^{-1} = J_n \bar{X} J_n^{-1}$. A simple computation shows that between the set of transvections (denoted by $X$) and the set of all skew-symmetric matrices (denoted by $A$) in $U(2n)$ there is a one-to-one correspondence given by $A = XJ_n$, $X = -AJ_n$. Under the imbedding $g \cdot Sp(n) \to gJ_n g^t J_n^{-1}$ the coset space $U(2n)/Sp(n)$ is sent into transvections.

Now, if $x$ is an eigenvector for the eigenvalue $\varepsilon$ of a tranvection, then $J_n^{-1}\bar{x}$ is also in the eigenspace corresponding to the eigenvalue $\varepsilon$. This means that $X = gtg^{-1}$, where $g \in Sp(n)$, and $t$ is a diagonal matrix of the form $\{e^{i\theta_1} \times \cdots \times e^{i\theta_n} \times e^{i\theta_1} \times \cdots \times e^{i\theta_n}\}$. Since all the elements in $U(2n)/Sp(n)$ are transvections, there is a one-to-one correspondence between $U(2n)/Sp(n)$ and the set of all skew-symmetric matrices in $U(2n)$.

$SO(2n)/U(n)$. The involution on $SO(2n)$ is $\theta(X) = J_n X J_n^{-1}$. Since here we think of $U(n)$ as imbedded in $SO(2n)$ under the correspondence $A + iB \leftrightarrow \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$, the symmetric space $SO(2n)/U(n)$ is imbedded in $SO(2n)$ under the correspondence $gU(n) \to gJ_n g^{-1} J_n^{-1}$.

We follow Steenrod [5, p. 213] to establish a correspondence between $W_n$, the set of skew-symmetric matrices in $SO(2n)$, and $SO(2n)/U(n)$. Let $\phi(X) = XJ_n X^t$ be defined on $SO(2n)$. Then $\phi$ is onto $W_n$. Also $\phi(X) = J_n$ if and only if $X = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$, i.e., $X \in U(n)$. Hence $\phi$ induces an identification between the coset space $SO(2n)/U(n)$ and $W_n$. In this identification $SO(2n)/U(n)$ is imbedded in $SO(2n)$ under $g \cdot U(n) \to gJ_n g^{-1}$. Thus the matrices $X \in SO(2n)/U(n)$ (considered as a symmetric space imbedded in $SO(2n)$) are in a one-to-one correspondence given by $X = -AJ_n$, $A = XJ_n$ with the matrices $A \in W_n$.

**Remarks.** (1) The set of skew-hermitian matrices in $U(n)$ can be identified with the complex Grassmannians $W_{n,k}$, $k = 0, 1, \cdots, n$. The argument is very much similar to the case of hermitian matrices in $U(n)$. In the case of hermitian matrices in $U(n)$, $W_{n,k}$ are obtained by conjugating a *real* diagonal matrix with $\pm 1$ along the diagonal, whereas in the case of skew-hermitian matrices the $W_{n,k}$ are obtained by conjugating *purely imaginary* diagonal matrices with $\pm i$ along the diagonal.

(2) The set of skew-hermitian matrices in $Sp(n)$ can also be identified with the quaternionic Grassmannians $Q_{n,k}$, $k = 0, 1, \cdots, n$. As in the previous

case such $Q_{n,k}$ arise by conjugating *purely imaginary* diagonal matrices by the group $Sp(n)$.

(3) Lastly, we find the skew-symmetric matrices in $Sp(n)$, and know that in $U(2n)$ between transvections $X$ (i.e., matrices such that $J_n\bar{X}J_n^{-1} = X^{-1}$) and skew-symmetric matrices $A$ there is a one-to-one correspondence given by $X = -AJ_n$, $A = XJ_n$. We also know that $U(2n)/Sp(n)$ is the space of all transvections in $U(2n)$. Now, if $A \in Sp(n)$, then $J_n\bar{A}J_n^{-1} = A \Longleftrightarrow J_n\bar{X}J_nJ_n^{-1} = XJ_n \Longleftrightarrow J_n\bar{X}J_n^{-1} = X$, so that the transvections $X$ must be in $Sp(n)$ and $X = X^{-1}$. Thus these $X$ are the $Q_{n,k}$ in $Sp(n)$. Hence the skew-symmetric matrices $A$ in $Sp(n)$ are in a one-to-one correspondence with all the $Q_{n,k}$, and this correspondence is given by $A = XJ_n$, $X = -AJ_n$, $X \in Q_{n,k}$, $k = 0, 1, \cdots, n$.

*Real Grassmannians.* Consider the involution $\theta$ on $SO(p + q)$ defined by $\theta(X) = I_{p,q}XI_{p,q}$, where $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, $p + q = n$. The fixed set $K$ is $SO(p + q) \cap \{O(p) \times O(q)\}$, and the identity component is $SO(p) \times SO(q)$. The real Grassmannian $G_{p+q,p}$ arises as a symmetric space, and the imbedding of $G_{p+q,p}$ in $SO(p + q)$ is given by $gK \to gI_{p,q}g^{-1}I_{p,q}$.

The Grassmann manifolds $G_{n,p}$ and $\dfrac{O(n)}{O(p) \times O(q)}$ are essentially the same, but $\dfrac{O(n)}{O(p) \times O(q)}$, $p + q = n$, are all the symmetric matrices in $O(n)$ of trace $p - q$. Hence $G_{p+q,p}$ can be identified with all symmetric matrices in $O(n)$ of trace $p - q$. If $S_{n,p}$ is the set of all symmetric matrices in $O(n)$ orthogonally equivalent to $I_{p,q}$, then there is a one-to-one correspondence between $S_{n,p}$ and $G_{n,p}$ given by

$$Y = SI_{pq}\ , \quad YI_{pq} = S\ , \quad Y \in G_{p+q,p}, \quad S \in S_{n,p}\ .$$

*The complex Grassmannian* $W_{p+q,p}$. Consider the involution $\theta(X) = I_{p,q}XI_{p,q}$ on $U(p + q)$. The fixed set $K$ is $U(p) \times U(q)$, and the coset space $W_{p+q,p} = \dfrac{U(p + q)}{U(p) \times U(q)}$ is imbedded in $U(p + q)$ under $XK \to XI_{p,q}X^{-1}I_{p,q}$.

Let $H_{p+q,p}$ be the set of hermitian matrices in $U(p + q)$ unitarily equivalent to $I_{p,q}$. Then there is a one-to-one correspondence between $W_{p+q,p}$ and $H_{p+q,p}$ given by

$$Y = HI_{p,q}\ , \quad YI_{p,q} = H\ , \quad Y \in W_{p+q,p}\ , \quad H \in H_{p+q,p}\ .$$

*The quaternionic Grasmannian* $Q_{p+q,p}$. Since we imbed $Sp(n)$ in $U(2n)$, this case is much similar to the previous one.

## 4. Critical submanifolds of symmetric spaces

Frankel showed [1] that for the function $f(g) = Re\ tr\ g$, $g \in G$, the critical set consists of all matrices $g$ such that $g^2 = e$, i.e., of what we have called

Grassmann matrices. The symmetric space $M = G/K$ is imbedded in $G$, and we can consider the restriction of $f$ to $M$.

**Lemma** [4]. grad $f$ *is tangent to* $M$ *at each point* $m \in M$.

*Proof.* Consider the anti-involution $\tau: G \to G$ defined by $\tau(g) = \theta(g^{-1})$. Then $\tau$ leaves $M$ fixed, because $M$ consists of all elements of the form $g \cdot \theta(g^{-1})$, $g \in G$ and

$$\tau(g \cdot \theta(g^{-1})) = \theta(\theta(g^{-1})^{-1} g^{-1}) = \theta(\theta(g))\theta(g^{-1}) = g\theta(g^{-1}) .$$

For $\dot{\tau}$, the differential action of $\tau$,

$$\dot{\tau}(\mathfrak{g}) = s(-\mathfrak{g}) = s(-\mathfrak{k} - \mathfrak{p}) = -\mathfrak{k} + \mathfrak{p} ,$$

where $s$ is the differential action induced by $\theta$. Hence $\dot{\tau}$ leaves $\mathfrak{p}$ fixed. But $\tau$ is an isometry. Hence $F_\tau$, the fixed set of $\tau$, is a totally geodesic submanifold of $G$, and $M$, the identity component of $F_\tau$, is totally geodesic in $G$.

It can be verified that $Re\, tr\, g = Re\, tr\, \tau(g)$. In a bi-invariant Riemannian metric, this means that $\dot{\tau}$ leaves grad $f$ fixed. Thus grad $f$ is tangent to $M$.

This lemma means that the critical points of $f$ on $M \subset G$ are the same as those of $f$ restricted to $M$. (Here $f$ is still a function defined on $G$.) Since the critical points of $f$ on $M \subset G$ are the matrices $m \in M$ such that $m^2 = e$ and these matrices are precisely the critical points of $f$ restricted to $M$, we have the following

**Proposition.** *Consider the function* $f(m) = Re\, tr\, m$ *on* $M$. *Then* $m \in M$ *is critical for* $f$ *if and only if* $m^2 = e$. *Let* $\Gamma_f(G) =$ *critical points of* $f$ *on* $G$. *Then* $\Gamma_f(M) = \Gamma_f(G) \cap M$.

Next we observe that to locate all such matrices $m \in M$, it is enough to find such matrices on the standard maximal torus of $M$ and conjugate them by $K'$, the identity component of the full fixed set of the involution $\theta$.

The standard maximal tori are (see [2]):

$U(n)/O(n)$. $e^{i\theta_1} \times \cdots \times e^{i\theta_n}$ .

$Sp(n)/U(n)$. $e^{i\theta_1} \times \cdots \times e^{i\theta_n} \times e^{-i\theta_1} \times \cdots \times e^{-i\theta_n}$ .

$U(2n)/Sp(n)$. $e^{i\theta_1} \times \cdots \times e^{i\theta_n} \times e^{i\theta_1} \times \cdots \times e^{i\theta_n}$ .

$SO(2n)/U(n)$. $R_2(\theta_1) \times \cdots \times R_2(\theta_m) \times R_2(-\theta_1) \times \cdots \times R_2(-\theta_m)$ , if $n = 2m$ ;

$R_2(\theta_1) \times \cdots \times R_2(\theta_m) \times 1 \times R_2(-\theta_1) \times \cdots \times R_2(-\theta_m) \times 1$ , if $n = 2m + 1$ ,

where

$$R_2(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} .$$

$$\left.\begin{array}{l} G_{p+q,p} \\ W_{p+q,p} \end{array}\right\}. \quad T = \begin{pmatrix} \cos\theta_1 & & & & \sin\theta_1 & & \\ & \ddots & & & & \ddots & 0 \\ & & & & 0 & & \\ & & \cos\theta_q & & & & \sin\theta_q \\ & & & \ddots & & & \\ & & & & 1 & & \\ -\sin\theta_1 & & & & \cos\theta_1 & & \\ & \ddots & 0 & & & \ddots & 0 \\ 0 & & & & 0 & & \\ & & -\sin\theta_q & & & & \cos\theta_q \end{pmatrix}.$$

$$Q_{p+q,p}. \qquad \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}.$$

Thus we immediately see that the critical manifolds of the respective spaces are as follows:

$\dfrac{U(n)}{O(n)}$: real Grassmannian $= \dfrac{SO(n)}{SO(n) \cap \{O(r) \times O(n-r)\}}$,

$r = 0, 1, \cdots, n$.

$\dfrac{Sp(n)}{U(n)}$: complex Grassmannian $= \dfrac{U(n)}{U(r) \times U(n-r)}$,

$r = 0, 1, \cdots, n$.

$\dfrac{U(2n)}{Sp(n)}$: quaternionic Grassmannian $= \dfrac{Sp(n)}{Sp(r) \times Sp(n-r)}$,

$r = 0, 1, \cdots, n$.

$$\frac{SO(2n)}{U(n)}: \text{ complex Grassmannian } = \begin{cases} \dfrac{U(2m)}{U(2r) \times U(2m-2r)}, \\ \quad \text{if } n = 2m,\ r = 0, 1, \cdots, m, \\ \dfrac{U(2m+1)}{U(2k+1) \times U(2m-2r)}, \\ \quad \text{if } n = 2m+1, r = 0, 1, \cdots, m. \end{cases}$$

$G_{p+q,p}$: product of real Grassmannians $= G_{p,r} \times G_{q,r}$,

$r = 0, 1, \cdots, q$.

$W_{p+q,p}$: product of complex Grassmannians $= W_{p,r} \times W_{q,r}$,

$r = 0, 1, \cdots, q$.

$Q_{p+q,p}$: product of quaternionic Grassmannians $= Q_{p,r} \times Q_{q,r}$,

$r = 0, 1, \cdots, q$.

The critical submanifolds of $G/K$ are Grassmann matrices in $K$; this can be readily seen as follows:

Any matrix in $M$ is $m = g\cdot\theta(g^{-1})$, $g \in G$. Suppose $m^2 = e$, i.e., $m = m^{-1}$. Then

$$g\cdot\theta(g^{-1}) = \theta(g^{-1})^{-1}g^{-1} = \theta(g)\cdot g^{-1} = \theta(g\cdot\theta(g^{-1})) .$$

Thus $m$ is left fixed by $\theta$, and a Grassmann matrix in $G/K$ must be a Grassmann matrix in $K$. The case $SO(2n)/U(n)$ illustrates that the converse is not true.

Our results can be stated in terms of intersections. For example, the first of the results says

$$\frac{U(n)}{O(n)} \cap \frac{U(n)}{U(p)\times U(q)} = \frac{SO(n)}{SO(n)\cap\{O(p)\times O(q)\}}\,, \qquad p+q=n .$$

It should be emphasised that the coset space $U(n)/O(n)$ is always considered to be imbedded in the group $U(n)$.

One can discuss a case by case analysis, and find $\Gamma_f(M) = \Gamma_f(G)\cap M$. For example, let us find $\Gamma_f(W_{p+q,p})$ the critical set of $W_{p+q,p}$ (the complex Grassmannian as imbedded in $U(p+q)$, $p+q=n$). Since $\Gamma_f(W_{p+q,p}) = \Gamma_f(U(p+q))\cap W_{p+q,p}$, and $W_{p+q,p}$ is imbedded in $U(p+q)$ as $XI_{pq}X^{-1}I_{pq}$, $X \in U(p+q)$, to obtain the required intersection we use the fact that $\Gamma_f(U(p+q))$ is the set of all hermitian symmetric matrices in $U(p+q)$, so that

$$XI_{pq}X^{-1}I_{pq} = (XI_{pq}X^{-1}I_{pq})^{-1} = I_{pq}XI_{pq}X^{-1}\,,$$

and $XI_{pq}X^{-1}$ commutes with $I_{pq}$, i.e., $XI_{pq}X^{-1} \in U(p)\times U(q)$. Thus we have to find all hermitian symmetric matrices in $U(p)\times U(q)$ with trace $p-q$. These matrices form the manifolds $\dfrac{U(p)}{U(k)\times U(p-k)} \times \dfrac{U(q)}{U(k')\times U(q-k')}$, and must have trace $p-q$ so that $k' = q-k$. Hence the required intersection is $\dfrac{U(p)}{U(k)\times U(p-k)} \times \dfrac{U(q)}{U(k)\times U(q-k)}$, $k = 0, 1, \cdots, q$.

## 4. Non-degeneracy and index of the critical submanifolds

Frankel considers $\sigma = I(p) \times -I(q)$, $p+q=n$, a critical point of $f$ on $G$. Such a $\sigma$ is in fact in the standard maximal torus of $G$, and the critical submanifold obtained by conjugating $\sigma$ by the group $G$ is $G/C(\sigma) = M_\sigma$ where $C(\sigma)$ is the centralizer of $\sigma$ in $G$. In order to show that the critical submanifolds are non-degenerate and to find their indices, Frankel shows that $f$ has a non-degenerate absolute maximum at $\sigma$ on the manifold $G_p \times -I(q)$ ($\sigma C^-(\sigma)$ in Frankel's notations), and has a non-degenerate absolute minimun at $\sigma$ on the manifold $I(p) \times G_q$, where $G_p$ denotes the classical group, which is of the same type as $G$, and has $p \times p$ matrix representation, while $G$ has $n \times n$ matrix representation. In the language of differential equations, this means that in $C(\sigma)$, $G_p \times -I(q)$ is the stable manifold for $\sigma$ (i.e., the submanifold of $C(\sigma)$ formed by all trajections of grad $f$, which end at $\sigma$) and $I(p) \times G_q$ is the unstable manifold for $\sigma$. By dimensional reasons, it follows that $M_\sigma$ is non-degenerate. Also, at $\sigma$ the tangent space of $G_p \times -I(q)$ is the maximal subspace on which the Hessian is negative definite, whence the index of $M_\sigma$ is the dimension of $G_p$. If $\tau = g\sigma g^{-1}$ is another point in $M_\sigma$, then the stable manifold for $\tau$ is $g\sigma C^-(\sigma)g^{-1}$.

Let $C_G(\sigma)$ be the centralizer of $\sigma$ in $G$, and $C_M(\sigma)$ the centralizer of $\sigma$ in $M$. Then Frankel has shown that grad $f$ is tangent to $C_G(\sigma)$ and we have shown that grad $f$ is tangent to $M$. Hence grad $f$ is tangent to $C_G(\sigma) \cap M$, which is precisely $C_M(\sigma)$.

Hence, due to dimensional reasons, the stable manifold for $\sigma$ in $C_M(\sigma)$ is the (stable manifold for $\sigma$ in $C_G(\sigma)) \cap M$, i.e., $\sigma C^-(\sigma) \cap M = \sigma C_M^-(\sigma)$ say. A similar statement holds for the unstable manifold for $\sigma$ in $C_M(\sigma)$.

For example, if $M = U(n)/O(n)$ and $\sigma = I(p) \times -I(q)$, $p + q = n$, then $C_M(\sigma) = \dfrac{U(p)}{O(p)} \times \dfrac{U(q)}{O(q)}$, because $U(n)/O(n)$ is the set of all $n \times n$ symmetric matrices in $U(n)$. The stable manifold for $\sigma$ in $C_M(\sigma)$ is $\dfrac{U(p)}{O(p)} \times I(q)$, and the unstable manifold for $\sigma$ in $C_M(\sigma)$ is $I(p) \times \dfrac{U(q)}{O(q)}$.

If $M = U(2n)/Sp(n)$, and $\sigma = I_{pq} \times I_{pq}$, $I_{pq} = I(p) \times -I(q)$, then $C_M(\sigma)$ is the set of all matrices $XJ_n\bar{X}^{-1}J_n^{-1}$ (for all $X \in U(2n)$) which commute with $\sigma$. Since $\sigma$ and $J_n$ commute, $C_M(\sigma)$ is the set of all $XJ_n\bar{X}^{-1}$ which commute with $\sigma$. These are all skew-hermitian matrices of the form

$$\begin{array}{c} \begin{array}{cccc} p & q & p & q \end{array} \\ \begin{pmatrix} * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & * & 0 \\ 0 & * & 0 & * \end{pmatrix} \end{array}.$$

Hence multiplying on the right by $J_n^{-1}$ we get $C_M(\sigma) = \dfrac{U(2p)}{Sp(p)} \times \dfrac{U(2q)}{Sp(q)}$, because of the one-to-one correspondence between skew-hermitian matrices in $U(2k)$ and $U(2k)/Sp(k)$. By a similar reasoning the stable manifold for $\sigma$ in $C_M(\sigma)$ is the set of all matrices of the form

$$\begin{pmatrix} A & 0 & B & 0 \\ 0 & -I_q & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & -I_q \end{pmatrix},$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is $\dfrac{U(2p)}{Sp(p)}$. In the same way we also get the unstable manifold for $\sigma$ in $C_M(\sigma)$.

If $M = W_{p+q,p}$ and $\sigma = I_{p-k,k} \times I_{q-k,k}$, then $C_M(\sigma)$ is the set of all $XI_{pq}X^{-1}I_{pq}$, which commute with $\sigma$ for all $X \in U(p+q)$. Since $I_{pq}$ and $\sigma$ commute, $C_M(\sigma)$ is the set of all $XI_{pq}X^{-1}$ which commute with $\sigma$ for all $X \in U(p+q)$. Thus $C_M(\sigma)$ is the set of all hermitian symmetric matrices obtained by conjugating $I_{pq}$ by all $X \in U(p+q)$, which commute with $\sigma$. All elements in $C_M(\sigma)$ are of the form

$$\begin{array}{c} \begin{array}{cccc} k & p-k & k & q-k \end{array} \\ \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix} \end{array},$$

where $\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ is $2k \times 2k$ hermitian symmetric matrices of trace 0 (hence, all such matrices form $W_{2k,k}$) and $\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$ is $p+q-2k \times p+q-2k$ hermitian symmetric matrices of trace $p-q$ (hence all such matrices form $W_{p+q-2k,p-k}$). Hence $C_M(\sigma) = W_{2k,k} \times W_{p+q-2k,p-k}$, and the stable manifold for $\sigma$ in $C_M(\sigma)$ is

$$\begin{array}{c} \begin{array}{cccc} \phantom{-I_k} & p-k & \phantom{-I_k} & q-k \end{array} \\ \begin{pmatrix} -I_k & 0 & 0 & 0 \\ 0 & A_2 & 0 & B_2 \\ 0 & 0 & -I_k & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix} \end{array},$$

where the set of all matrices of the form $\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$ is $W_{p+q-2k,p-k}$.

As in the case of the classical groups, if we consider another critical point $\tau = k\sigma k^{-1}$, $k \in K'$, then the stable manifold for $\tau$ would be $k\sigma C_M^-(\sigma)k^{-1}$. The representation for $\tau$ would not be unique, and if $\tau = \bar{k}\sigma\bar{k}^{-1}$ is another representation, then $\bar{k} = kc$, where $c \in C_{K'}(\sigma)$, the centralizer of $\sigma$ in $K'$. Each of the two homeomorphisms $x \to kxk^{-1}$ and $x \to \bar{k}x\bar{k}^{-1}$ sending $\sigma C_M^-(\sigma)$ onto the stable manifold of $k\sigma C_M^-(\sigma)k^{-1}$ induces an orientation on this stable manifold by means of a fixed orientation in $\sigma C_M^-(\sigma)$. These two orientations would agree if $C_{K'}(\sigma)$ is connected. But in the case of $U(n)/O(n)$ and the real Grassmannian, $C_{K'}(\sigma)$ consists of two disjoint pieces, and in these two cases the negative normal bundles of the critical submanifolds need not be orientable and we have to use $Z_2$ for coefficients in discussing homology.

We summarize the results obtained so far:

| Space | Critical submanifolds (all non-degenerate) | Index |
|---|---|---|
| $\dfrac{U(n)}{O(n)}$ | $\dfrac{SO(n)}{SO(n)\cap\{O(p)\times O(q)\}}$ | $\dfrac{p(p+1)}{2}$ |
| $\dfrac{Sp(n)}{U(n)}$ | $\dfrac{U(n)}{U(p)\times U(q)}$ | $p(p+1)$ |
| $\dfrac{U(2n)}{Sp(n)}$ | $\dfrac{Sp(n)}{Sp(p)\times Sp(q)}$ | $p(2p-1)$ |

In the above three cases $p = 0, 1, \cdots, n$.

| | | |
|---|---|---|
| $\dfrac{SO(2n)}{U(n)}$ | $\dfrac{U(2m)}{U(2p)\times U(2q)}\begin{pmatrix} n=2m \\ p+q=m\end{pmatrix}$ | $2p(2p-1)$ |
| | $\dfrac{U(2m+1)}{U(2p+1)\times U(2q)}\begin{pmatrix} n=2m+1 \\ p+q=m\end{pmatrix}$ | $2p(2p+1)$ |

In the above case $p = 0, 1, \cdots, m$.

| | | |
|---|---|---|
| $G_{p+q,p} = \dfrac{SO(p+q)}{SO(p+q)\cap\{(O(p)\times O(p)\}}$ | $G_{p,r}\times G_{q,r}$ | $(p-r)(q-r)$ |
| $W_{p+q,p} = \dfrac{U(p+q)}{U(p)\times U(q)}$ | $W_{p,r}\times W_{q,r}$ | $2(p-r)(q-r)$ |
| $Q_{p+q,p} = \dfrac{Sp(p+q)}{Sp(p)\times Sp(q)}$ | $Q_{p,r}\times Q_{q,r}$ | $4(p-r)(q-r)$ |

In the above three cases $p \geq q$ and $r = 0, 1, \cdots, q$.

## 5. Morse-Bott inequalities and applications of fixed points theory

The results obtained so far can be expressed in terms of the following inequalities:

$$b_i\left(\frac{U(n)}{O(n)}; Z_2\right) \leq \sum_{p=0}^{n} b_{i-p(p+1)/2}(G_{n,p}; Z_2) ,$$

$$b_i\left(\frac{Sp(n)}{U(n)}; K\right) \leq \sum_{p=0}^{n} b_{i-p(p+1)}(W_{n,p}; K) ,$$

$$b_i\left(\frac{U(2n)}{Sp(n)}; K\right) \leq \sum_{p=0}^{n} b_{i-p(2p-1)}(Q_{n,p}; K) ,$$

$$b_i\left(\frac{SO(2n)}{U(n)}; K\right) \leq \sum_{p=0}^{m} b_{i-2p(2p-1)}(W_{2m,2p}; K), \qquad \text{if } n = 2m$$

$$\leq \sum_{p=0}^{m} b_{i-2p(2p+1)}(W_{2m+1,2k+1}; K), \qquad \text{if } n = 2m+1 ,$$

$$b_i(G_{p+q,p}; Z_2) \leq \sum_{r=0}^{q} b_{i-(p-r)(q-r)}(G_{p,r} \times G_{q,r}; Z_2) ,$$

$$b_i(W_{p+q,p}; K) \leq \sum_{r=0}^{q} b_{i-2(p-r)(q-r)}(W_{p,r} \times W_{q,r}; K) ,$$

$$b_i(Q_{p+q,p}; K) \leq \sum_{r=0}^{q} b_{i-4(p-r)(q-r)}(Q_{p,r} \times Q_{q,r}; K) ,$$

where $b_i$ is the $i$-th Betti number, $Z_2$ the field of the integers mod. 2, and $K$ any field of coefficients.

Now we show that the Morse-Bott inequalities obtained for the classical structures of symmetric spaces are in fact equalities; this is done by means of

**Theorem A** (*Floyd*). *If a transformation of period* 2 *acts on a compact manifold $M$, and $F$ is the fixed set, then*

$$\sum_i b_i(F; Z_2) \leq \sum_i b_i(M; Z_2) .$$

On $G/K$ consider the transformation of period 2: $m \to m^{-1}$. $G/K$ is imbedded in $G$ by $gK \to g\cdot\theta(g^{-1})$, $g \in G$, and $\theta$ is the involution on $G$. Under the mapping $m \to m^{-1}$,

$$\begin{aligned} g\cdot\theta(g^{-1}) \to \theta(g^{-1})^{-1}g^{-1} = \theta(g)\cdot g^{-1} &= \theta(g)\cdot\theta(\theta(g^{-1})) \\ &= \theta(g)\cdot\theta(\theta(g)^{-1}) . \end{aligned}$$

So $m^{-1}$ corresponds to the coset $\theta(g)\cdot K$ and the transformation of period 2 is really $gK \to \theta(g)K$.

We know that the critical submanifolds of the trace function on $M$ are all $m \in M$ such that $m^2 = e$, i.e., $m = m^{-1}$, and these are precisely the fixed set of the transformation $m \to m^{-1}$. Thus Morse-Bott inequalities and the inequalities in Theorem A are opposing inequalities, and putting them together we get equalities, which can be expressed in terms of Poincaré polynomials as follows:

(1) $$P\left(\frac{U(n)}{O(n)}; t\right) = \sum_{p=0}^{n} t^{p(p+1)/2} P(G_{n,p}; t) .$$

(2) $$P\left(\frac{Sp(n)}{U(n)}; t\right) = \sum_{p=0}^{n} t^{p(p+1)} P(W_{n,p}; t) .$$

(3) $$P\left(\frac{U(2n)}{Sp(n)}; t\right) = \sum_{p=0}^{n} t^{p(2p-1)} P(Q_{n,p}; t) .$$

$$(4)\quad \begin{aligned} P\left(\frac{SO(2n)}{U(n)}; t\right) &= \sum_{p=0}^{m} t^{2p(2p-1)} P(W_{2m,2p}; t), \qquad \text{if } n = 2m, \\ &= \sum_{p=0}^{m} t^{2p(2p+1)} P(W_{2m+1,2p+1}; t), \qquad \text{if } n = 2m+1. \end{aligned}$$

$$(5)\quad P(G_{p+q,p}; t) = \sum_{r=0}^{q} t^{(p-r)(q-r)} P(G_{p,r}; t)\cdot P(G_{q,r}; t).$$

$$(6)\quad P(W_{p+q,p}; t) = \sum_{r=0}^{q} t^{2(p-r)(q-r)} P(W_{p,r}; t)\cdot P(W_{q,r}; t).$$

$$(7)\quad P(Q_{p+q,p}; t) = \sum_{r=0}^{q} t^{4(p-r)(q-r)} P(Q_{p,r}; t)\cdot P(Q_{q,r}; t).$$

It should be observed that the field of coefficients used is $Z_2$. Next, we show that (except for the cases (1) and (5)) any field of coefficients can be used. This is done by elementary considerations for the cases (2), (4), (6) and (7).

Consider $\dfrac{U(n)}{U(1) \times U(n-1)}$. By induction it is easy to see that a cellular decomposition of this space is $e^0 \cup e^2 \cup \cdots \cup e^{2n-2}$, where $e^i$ is an $i$-dimensional cell. It should be observed that this space has only even dimensional cells. Again, by induction, it can be shown that $\dfrac{U(p+q)}{U(p) \times U(q)}$ has only even dimensional cells, because its critical submanifolds have even indices and even dimensional cells by the induction hypothesis. Since the complex Grassmannian has only cells of even dimension, $\dfrac{U(p+q)}{U(p) \times U(q)}$ has no torsion and its odd betti numbers are zero. Hence the Morse-Bott equalities hold for any field $K$ of coefficients.

Also, by induction, the total betti numbers of the complex Grassmannian $W_{n,q}$ is

$$\sum_{r=0}^{q} \binom{p}{r}\binom{q}{r} = \sum_{r=0}^{q} \binom{p}{r}\binom{q}{q-r} = \binom{p+q}{q},$$

where $\binom{p}{q}$ denotes the binomial coefficients. Hence

$$(8)\quad \sum_{q=0}^{n} \sum_{i} b_i(W_{n,q}; K) = \sum_{q=0}^{n} \binom{n}{q} = 2^n.$$

Same sort of argument holds in the case of quaternionic Grassmannians. This space consists of cells of dimension $\equiv 0 (\mathrm{mod}\ 4)$, and therefore has no torsion and $b_i = 0$ if $i \not\equiv 0\ (\mathrm{mod}\ 4)$ and $b_i \neq 0$ if $i \equiv 0\ (\mathrm{mod}\ 4)$. Also

$$(9)\quad \sum_{q=0}^{n} \sum_{i} b_i(Q_{n,q}; K) = 2^n.$$

Thus, by Morse theory we obtain the two equalities (8) and (9) which were used by Frankel [1], whose proofs are, however, non-Morse theoretic.

For the cases $SO(2n)/U(n)$ and $Sp(n)/U(n)$ we observe that the critical submanifolds have cells of even dimensions and the index of the critical sets is also even. Thus the cellular decompositions which can be obtained for the two spaces have only *even* dimensional cells, and therefore they have no torsion and their odd betti numbers are zero. Hence the Morse-Bott equalities hold for any field $K$ of coefficients.

The case $U(2n)/Sp(n)$ has to be handled separately. For this case we apply Floyd's

**Theorem B.** *If a toral group operates on a compact manifold $M$, and $F$ is the fixed set (i.e., the set of points fixed under each transformation of the group), then*

$$\sum_i b_i(F; K) \leq \sum_i b_i(M; K) ,$$

*where $K$ is either $R$ or $Z_p$ with prime $p$.*

Since the group $Sp(n)$ acts on $M = U(2n)/Sp(n)$ by the adjoint action, the torus $T_{Sp(n)}$ acts on $M$ also by the adjoint action. The fixed set is $F = T_{U(2n)} \cap M$, and $T_M = (T_{U(2n)} \cap M)_e$ is the identity component. Hence

$$2^n = \sum_i b_i(T_M; K) \leq \sum_i b_i(F; K) \leq \sum_i b_i\left(\frac{U(2n)}{Sp(n)}; K\right).$$

Since $\sum_k \sum_i b_i(Q_{n,k}; K) = 2^n$, we have $\sum_k \sum_i b_i(Q_{n,k}; K) \leq \sum_i b_i\left(\frac{U(2n)}{Sp(n)}; K\right)$, which, together with the Morse-Bott inequality, gives the result for the case $U(2n)/Sp(n)$.

**Remark.** Frankel has obtained Morse decomposition for Kähler manifolds [*Fixed points and torsion on Kähler manifolds*, Ann. of Math. (2) **70** (1959) 1–8]. The complex Grassmannians, $SO(2n)/U(n)$, and $Sp(n)/U(n)$ are hermitian symmetric and hence Kählerian (see [2, p. 301]). It should be pointed out that the results of the present paper are not explicitly contained in the paper of Frankel on Kähler maniolds. For the results of Frankel on Kähler manifolds when applied to the three symmetric spaces mentioned above give only the critical points but not the critical submanifolds. In order to obtain the critical sets of Kähler manifolds, Frankel essentially used Floyd's Theorem B, and these critical sets turn out to be the fixed set of the adjoint action of certain toral group on these manifolds. For the space $\frac{Sp(n)}{U(n)}$, $\frac{SO(2n)}{U(n)}$ and $\frac{U(p+q)}{U(p) \times U(q)}$, $p + q = n$, the adjoint action of the toral group $T_{U(n)}$ (standard maximal torus of $U(n)$) has been considered in [4], and the fixed set has been obtained as follows:

| Space | Fixed set |
|---|---|
| $Sp(n)/U(n)$ | $2^n$ distinct points |
| $SO(2n)/U(n)$ | $2^{n-1}$ distinct points |
| $\dfrac{U(p+q)}{U(p)\times U(q)}$ | $\binom{p+q}{p}$ distinct points |

Hence in the two setups, we get two different results. Finally, it should be mentioned that by induction we can find cell-decompositions and the Poincaré polynomials of the symmetric spaces considered in this paper.

## Appendix

Here we mention briefly the non-degenerate critical manifold theory. Let $H_f$ be the Hessian quadratic form of $f$ at a critical point $\sigma$ of $M$; if $X$ is a tangent vector at $\sigma$, then $H_f(X) = D_X D_X(f)$, where $D_X$ is the directional derivative for $X$. In local coordinates, $D_X = \sum_i X^i \dfrac{\partial}{\partial x^i}$ and $H_f(X) = \sum_{i,j} \left(\dfrac{\partial^2 f}{\partial x^i \partial x^j}\right) X^i X^j$. *Index* $\lambda$ of $\sigma$ is defined to be the dimension of the largest subspace of the tangent space at $\sigma$ on which $H_f$ is negative definite. Critical manifold $M_\sigma$ is *non-degenerate* if the null-space of $H_f$ is exactly the tangent space of $M_\sigma$ at each of its points. (The null space of $H_f$ always contains this tangent space.) If the critical manifolds are connected, then $\lambda$ is independent of the point on $M_\sigma$; in this case $\lambda$ is called the *index* of the critical submanifold.

If $M$ is a compact manifold, and $f$ a real-valued function having only non-degenerate critical submanifolds $\{M_\alpha\}$ each with index $\lambda_\alpha$, then $M = \xi_{\lambda_1}(M_1) \cup \cdots \cup \xi_{\lambda_k}(M_k)$ (homotopy equivalence), where $\xi_{\lambda_i}(M_i)$ means a $\lambda_i$-dimensional plane bundle over $M_i$. If these negative normal bundles are orientable, then

$$b_i(M; K) \leq \sum_\alpha b_{i-\lambda_\alpha}(M_\alpha; K) ,$$

where $b_i$ is $i$th Betti number using coefficient field $K$; if they are not known to be orientable, then only $Z_2$ (integers mod 2) may be used for coefficients. These inequalities are called Morse-Bott inequalities.

## References

[1] T. T. Frankel, *Critical submanifolds of the classical groups and Stiefel manifolds*, Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), Princeton University Press, Princeton, 1965, 37–53.

[2] S. Halgaton, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.
[3] H. Iwamoto, *On integral invariants and Betti numbers of symmetric Riemannian manifolds.* I, II. J. Math. Soc. Japan **1** (1949) 91–110, 235–243.
[4] S. Ramanujam, *An application of Morse theory to certain symmetric spaces*, J. Indian Math. Soc., to appear.
[5] N. Steenrod, *The topology of fibre bundles*, Princeton University Press, Princeton, 1951.
[6] M. Takeuchi, *Cell decompositions and Morse equalities on certain symmetric spaces*, J. Fac. Sci. Univ. Tokyo Sect. I, **12** (1965) 81–192.

UNIVERSITY OF WASHINGTON