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# GROWTH OF FINITELY GENERATED SOLVABLE GROUPS AND CURVATURE OF RIEMANNIAN MANIFOLDS

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## 1. Introduction and summary

If a group $\Gamma$ is generated by a finite subset $S$, then one has the "growth function" $g_S$, where $g_S(m)$ is the number of distinct elements of $\Gamma$ expressible as words of length $\leq m$ on $S$. Roughly speaking, J. Milnor [9] shows that the asymptotic behaviour of $g_S$ does not depend on choice of finite generating set $S \subset \Gamma$, and that lower (resp. upper) bounds on the curvature of a riemannian manifold $M$ result in upper (resp. lower) bounds on the growth function of $\pi_1(M)$. The types of bounds on the growth function are

$$\textit{polynomial growth of degree} \leq E\colon \quad g_S(m) \leq c \cdot m^E \; ,$$
$$\textit{exponential growth:} \quad u \cdot v^m \leq g_S(m) \; ,$$

where $c$, $u$ and $v$ are positive constants depending only on $S$, $v > 1$, and $m$ ranges over the positive integers.

In § 3 we show that, *if a group $\Gamma$ has a finitely generated nilpotent subgroup $\Delta$ of finite index, then it is of polynomial growth, and in fact $c_1 m^{E_1(\Delta)} \leq g_S(m) \leq c_2 m^{E_2(\Delta)}$, where $0 < c_1 \leq c_2$ are constants depending on the finite generating set $S \subset \Gamma$, and $E_1(\Delta) \leq E_2(\Delta)$ are positive integers specified in* (3.3) *by the lower central series of $\Delta$*. In § 4 we consider a class of solvable groups which we call "polycyclic"; Proposition 4.1 gives eleven characterizations, all useful in various contexts; finitely generated nilpotent groups are polycyclic. We prove that *a polycyclic group, either has a finitely generated nilpotent subgroup of finite index and thus is of polynomial growth, or has no such subgroup and is of exponential growth.* We also give a workable criterion for deciding between the two cases. Applying a result of Milnor [10] which says that a finitely generated nonpolycyclic solvable group is of exponential growth, we conclude that *a finitely generated solvable group, either is polycyclic and has a nilpotent subgroup of finite index and is thus of polynomial growth, or has no nilpotent subgroup of finite index and is of exponential growth.*

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In § 5 the results of §§ 3 and 4 are applied to quotient groups of subgroups of fundamental groups of riemannian manifolds whose mean curvature[1] $\geq 0$ everywhere, resulting in fairly stringent conditions on the fundamental group of a complete riemannian manifold of non-negative mean curvature. Those conditions, however, are not strong enough to prove the conjecture that the fundamental group must be finitely generated.

In § 2, before considering growth rate, we make a few estimates and prove that *if a compact riemannian manifold M has every sectional curvature* $\leq 0$, *and* $\pi_1(M)$ *is nilpotent, then M is a flat riemannian torus.* In § 6 we combine the results of §§ 2 and 5 with a result of Cheeger and Gromoll [5], proving a result on quotient manifolds of nilmanifolds, which contains L. Auslander's conjecture that *if a compact nilmanifold M admits a riemannian metric* $ds^2$ *with every sectional curvature* $\leq 0$ *or with every sectional curvature* $\geq 0$, *then M is diffeomorphic to a torus.* In fact we prove $(M, ds^2)$ isometric to a flat riemannian torus, and we manage the diffeomorphism only assuming non-negative mean curvature. Here we do not require any invariance property on the metric, so that considerations are more delicate than the known results ([17], [19] and G. Jensen's thesis) for invariant metrics on nilmanifolds.

The results of § 2 raise the question of whether a compact riemannian manifold $M$ with every sectional curvature $\leq 0$, such that $\pi_1(M)$ has a solvable (or polycyclic) subgroup of finite index, is necessarily flat. If the answer is "yes" then one can strengthen Corollary 2.2, replacing "nilpotent" by "solvable" (or "polycyclic) in the formulation: *if M is a compact riemannian manifold with every sectional curvature* $\leq 0$, *then* $\pi_1(M)$ *has a nilpotent subgroup of finite index if and only if M is flat.*

The results of §§ 3 and 4 raise the question of whether every finitely generated group $\Gamma$, which is not of exponential growth, necessarily has a nilpotent subgroup of finite index. The answer "yes" is suggested, first by Theorem 4.8 which proves it in case $\Gamma$ is solvable, second by the work of Shub and Frank on expanding maps, and third by a personal prejudice that a finitely generated group must be of very rapid growth if it has no solvable subgroup of finite index.

## 2. Manifolds of nonpositive curvature and nilpotent fundamental group

In this section we adapt some of our earlier results [16] on bounded isometries to prove

[1] If $X$ is a tangent line at a point $x \in M$ then the mean curvature at $(x, X)$ is defined to be the average of the sectional curvatures $K(x, E)$ where $E \subset M_x$ ranges over the plane sections containing $X$. Analytically, if $n = \dim M$, $\xi = (\xi^i)$ is a unit tangent vector at $x$, and $(R_{ij})$ is the Ricci tensor, then the mean curvature $k(x, \xi) = \frac{1}{n-1} \sum_{i,j} R_{ij}\xi^i\xi^j$ in the sign convention $R_{ij} = \sum_m R^m{}_{ijm}$.

**2.1. Theorem.** *Let $M$ be a compact connected riemannian manifolds uch that*

(i) *every sectional curvature of $M$ is nonpositive, and*

(ii) *the fundamental group $\pi_1(M)$ is nilpotent.*

*Then $M$ is a flat riemannian torus. In particular, $\pi_1(M)$ is the free abelian group on* (dim $M$) *generators.*

**2.2. Corollary.** *Let $M$ be a compact connected riemannian manifold with every sectional curvature $\leq 0$. Suppose that $\pi_1(M)$ has a nilpotent subgroup $\Delta$ of finite index. Then $M$ is a compact euclidean space form and $\Delta$ is free abelian on* (dim $M$) *generators.*

*Proof of corollary from theorem.* There is a riemannian covering $\pi: M' \to M$ where $\Delta = \pi_1(M')$. The multiplicity of the covering is the index of $\Delta$ in $\pi_1(M)$, and hence finite; so $M'$ is compact. Theorem 2.1 says that $M'$ is a flat riemannian torus, and so $M$ is flat, and that $\Delta$ is free abelian on (dim $M'$) = (dim $M$) generators.

*Proof of theorem.* Let $\pi: N \to M$ denote the universal riemannian covering, and $\Gamma$ the group of deck transformations of the covering. Then $\Gamma \cong \pi_1(M)$, so $\Gamma$ is nilpotent, and $\Gamma$ is a properly discontinuous group of fixed point free isometries of $N$ such that $M = \Gamma \backslash N$. Compactness of $M$ provides us with a compact set $K \subset N$ such that $N = \Gamma \cdot K$.

If $f: N \to N$ is any map, we define the *displacement function* $\delta_f: N \to \boldsymbol{R}$ by

$$\delta_f(x) \text{ is the distance from } x \text{ to } f(x) .$$

Then $\delta_f$ is continuous if $f$ is continuous. In particular the $\delta_\gamma$, $\gamma \in \Gamma$, are continuous. Thus we have well defined bounds

$$b(\gamma, K) = \max \{\delta_\gamma(x) : x \in K\} < \infty \tag{2.3}$$

for displacement on $K$ of elements of $\Gamma$.

Let $Z_\Gamma$ denote the center of the group $\Gamma$. It is a nontrivial free abelian group because $\Gamma$ is a torsion free finitely generated nilpotent group. Let $x \in N$ and $\gamma \in Z_\Gamma$. Then we have $\xi \in \Gamma$ such that $\xi(x) \in K$, and we compute

$$\begin{aligned} \delta_\gamma(x) &= \text{distance } (x, \gamma x) = \text{distance } (\xi x, \xi\gamma x) \\ &= \text{distance } (\xi x, \gamma\xi x) = \delta_\gamma(\xi x) . \end{aligned}$$

In other words,

$$\delta_\gamma(x) \leq b(\gamma, K) < \infty \qquad \text{for all } x \in N \text{ and } \gamma \in Z_\Gamma . \tag{2.4}$$

Thus every element of $Z_\Gamma$ is a bounded isometry [16] of $N$.

$N$ is complete because $M$ is compact. Thus we have the de Rham decomposition $N = N_0 \times N_1 \times \cdots \times N_t = N_0 \times N'$ where $N_0$ is a euclidean space and $N' = N_1 \times \cdots \times N_t$ is a product of irreducible riemannian manifolds. We know [16, Theorem 1] that the bounded isometries of $N$ are just the transformations $(n_0, n') \to (\tau n_0, n')$, $n_0 \in N_0$ and $n' \in N'$, where $\tau$ is an ordinary translation of the euclidean space $N_0$. Now

(2.5) *if $\gamma \in Z_\Gamma$, then there is an ordinary translation $\tau_\gamma$ of $N_0$ such that $\gamma$ acts on $N = N_0 \times N'$ by $\gamma(n_0, n') = (\tau_\gamma n_0, n')$ .*

This decomposes $N_0$ into a product of euclidean spaces $A$ and $B$ where, from the vector space viewpoint,

(2.6) $$A \text{ is the span of } \{\tau_\gamma : \gamma \in Z_\Gamma\} \text{ and } B = A^\perp .$$

That decomposition $N_0 = A \times B$ has both factors stable under $\Gamma$, because $Z_\Gamma$ is normal in $\Gamma$. If $I$ denotes the full group of isometries, then

(2.7) $$N = A \times (B \times N') , \qquad \Gamma \subset I(A) \times I(B \times N') .$$

If $\gamma \in \Gamma$, this decomposes

(2.8) $$\gamma = \gamma_A \times \gamma', \text{ where } \gamma_A \in I(A) \text{ and } \gamma' \in I(B \times N') .$$

Note that $\gamma_A = \tau_\gamma$ and $\gamma' = 1$ in case $\gamma \in Z_\Gamma$. Thus every element of $\{\gamma_A : \gamma \in \Gamma\}$ commutes with every element of $\{\tau_\gamma : \gamma \in Z_\Gamma\}$. It follows that

(2.9) $$\text{if } \gamma \in \Gamma, \text{ then } \gamma_A \text{ is an ordinary translation of } A .$$

In particular (2.5) and (2.6) give

(2.10) $$Z_\Gamma = \{\gamma \in \Gamma : \gamma' = 1\} = \Gamma \cap I(A) .$$

Let $\Gamma'$ denote the projection of $\Gamma$ to $I(B \times N')$. Then (2.8) and (2.10) say

(2.11) $$\Gamma' = \{\gamma' : \gamma \in \Gamma\} \cong \Gamma / Z_\Gamma .$$

The quotient of a torsion free nilpotent group by its center is torsion free (see [7, p. 247]). Thus (2.11) says that

(2.12) $$\Gamma' \text{ is a torsion free nilpotent group.}$$

By (2.6), (2.8) and (2.10) we have a compact set $K_A \subset A$ such that $Z_\Gamma \cdot K_A = A$. Suppose $n' \in B \times N'$ such that $\{\gamma'(n') : \gamma' \in \Gamma'\}$ has an accumulation point. Choose $n_A \in A$ and let $n = (n_A, n') \in N$; if $\gamma' \in \Gamma'$ has $\gamma \in \Gamma$ as preimage, we replace $\gamma$ by the appropriate element of $\gamma Z_\Gamma$, and then we may assume $\gamma_A(n_A) \in K_A$. But then $\{\gamma(n) : \gamma \in \Gamma\}$ has an accumulation point, which is absurd. We conclude that

(2.13) $$\Gamma' \text{ acts discontinuously on } B \times N' .$$

Now (2.10) and compactness of $M$ say that

(2.14) $$\Gamma' \backslash (B \times N') \text{ is compact} .$$

Define $M' = \Gamma'\backslash(B \times N')$. Construction of $B \times N'$ and hypothesis on $M$ say that $B \times N'$ is a simply connected riemannian manifold[2] with every sectional curvature nonpositive. $\Gamma'$ is a properly discontinuous (by (2.13)) group of fixed point free (by (2.12)) isometries, so $M'$ is a riemannian manifold[2] such that

(i) every sectional curvature of $M'$ is nonpositive, and

(ii) $\pi_1(M') \cong \Gamma'$, nilpotent group.

Finally $M'$ is connected by construction and is compact by (2.14). Thus

(2.15) $M'$ satisfies the hypotheses of Theorem 2.1 .

We are ready to prove Theorem 2.1 by induction on dimension. Nontriviality of $Z_\Gamma$ implies $\dim A > 0$, so

$$\dim M' = \dim (B \times N') < \dim N = \dim M \ .$$

By (2.15) and induction hypothesis, now $M'$ is a flat riemannian torus. Thus $B \times N' = B$, a euclidean space, and $\Gamma'$ is a group of ordinary translations of $B$. Now $N = N_0 = A \times B$, product of euclidean spaces, and every element $\gamma \in \Gamma$ has form $\gamma_A \times \gamma'$ where $\gamma'$ is an ordinary translation of $B$. But (2.9) says that $\gamma_A$ is an ordinary translation of $A$. It follows that every $\gamma \in \Gamma$ is an ordinary translation of the euclidean space $N$. This proves $M = \Gamma\backslash N$ to be a flat riemannian torus. q.e.d.

If $M$ is a compact $n$-dimensional euclidean space form, then [18, Chapter 3] the fundamental group $\pi_1(M)$ has a normal subgroup $\Sigma$ of finite index such that $\Sigma$ is free abelian on $(\dim M)$ generators and $\pi_1(M)/\Sigma$ is isomorphic to the linear holonomy group of $M$. In particular, if $M$ has solvable linear holonomy group, then $\pi_1(M)$ is solvable. If $n = 2$ [18, p. 77] or $n = 3$ [18, Theorems 3.5.5 and 3.5.9], then the linear holonomy group is automatically solvable; so $\pi_1(M)$ is automatically solvable, although $M$ need not be a torus. Thus the version of Theorem 2.1 and Corollary 2.2, which might possibly generalize to a larger class of fundamental groups, is

**2.16. Corollary.** *Let $M$ be a compact riemannian manifold with every sectional curvature $\leq 0$. Then $\pi_1(M)$ has a nilpotent subgroup of finite index, if and only if $M$ is flat.*

Generalization of Corollary 2.16 would consist of weakening the condition "nilpotent", for example replacing it by "solvable" without strengthening the curvature condition on $M$.

## 3. The growth function for nilpotent groups —groups of polynomial growth

Let $S$ be a finite subset of a group $\Gamma$. As usual an expression

$$s_1^{a_1} \cdots s_r^{a_r}, \quad s_i \in S, \quad a_i \in Z$$

[2] We allow the dimension to be zero. In fact that is the objective of the proof.

is called a *word of length* $|a_1| + \cdots + |a_r|$ *based on* $S$. Following Milnor [9] we define the *growth function* $g_S$ to be the function on positive integers given by

(3.1) $g_S(m)$ *is the number of distinct elements of* $\Gamma$ *expressible as words of length* $\leq m$ *based on* $S$.

Milnor [9, Lemma 4] proved that, if $S$ is a finite generating set of the torsion free nilpotent group

$$\{x, y: [x, [x, y]] = [y, [x, y]] = 1\}$$

of rank 3, then $g_S$ is quartic in the sense that there are constants $0 < c_1 \leq c_2$ such that $c_1 m^4 \leq g_S(m) \leq c_2 m^4$ for all integers $m \geq 1$. Our purpose here is to prove the following extension of that result of Milnor.

**3.2. Theorem.** *Let* $\Gamma$ *be a finitely generated nilpotent group with lower central series*

$$\Gamma = \Gamma_0 \supsetneqq \Gamma_1 \supsetneqq \cdots \supsetneqq \Gamma_s \supsetneqq \Gamma_{s+1} = \{1\}, \qquad \Gamma_{k+1} = [\Gamma, \Gamma_k] .$$

*Then each* $\Gamma_k/\Gamma_{k+1}$ *is a finitely generated abelian group, say* $\Gamma_k/\Gamma_{k+1} = A_k \times B_k$ *with* $A_k$ *finite abelian and* $B_k$ *free abelian of finite rank* $n_k$, *and we define "growth exponents" by*

$$E_1(\Gamma) = \sum_{k=0}^{s} (k+1)n_k , \qquad E_2(\Gamma) = \sum_{k=0}^{s} 2^k n_k . \tag{3.3}$$

*If* $S$ *is any finite generating set for* $\Gamma$, *then there are constants* $0 < c_1 \leq c_2$ *such that*

$$c_1 m^{E_1(\Gamma)} \leq g_S(m) \leq c_2 m^{E_2(\Gamma)} \qquad \textit{for all integers } m \geq 1 .$$

*Remark.* Notice $E_1(\Gamma) \leq E_2(\Gamma)$ with equality if and only if $0 \leq s \leq 1$, and in particular $E_1(\Gamma) = E_2(\Gamma) = n_0$ if $\Gamma$ is commutative. In the general case perhaps $g_S(m) \leq c_2 m^{E_1(\Gamma)}$.

We start the proof of Theorem 3.2 with two lemmas which simplify estimates and prove $g_S$ asymptotically independent of choice of finite generating set $S$. Those lemmas are just formalizations of remarks of Milnor [9].

**3.4. Lemma.** *Let* $S$ *be a finite subset of a group* $\Gamma$, *and* $g_S$ *denote the growth function. If* $r \geq 0$ *and* $q \geq 0$ *are fixed integers such that* $g_S(m) \geq c(m-r)^q$ *for some constant* $c > 0$, *then* $g_S(m) \geq c' m^q$ *for another constant* $c' > 0$. *If* $l > 0$ *and* $q \geq 0$ *are fixed integers such that* $g_S(lm) \geq c(lm)^q$ *for some constant* $c > 0$, *then* $g_S(m) \geq c'' m^q$ *for another constant* $c'' > 0$.

*Proof.* As $1 > \dfrac{m-r}{m} \geq \dfrac{1}{r+1}$ for $m > r$ we have $(m-r)^q$

$\geq (r+1)^{-q}m^q$ for $m > r$. If $g_S(m) \geq c(m-r)^q$, then $g_S(m) \geq c'm^q$ where $c' = \min\{c(r+1)^{-q}; g_S(r)/r^q, g_S(r-1)/(r-1)^q, \cdots, g_S(1/)1^q\}$, and the first statement is proved. For the second, divide $l$ into $m$, say $m = al + b$ with $0 \leq b < l$. Then $g_S(m) \geq g_S(al) \geq c(al)^q = c(m-b)^q \geq c(m-r)^q$ where $r = l - 1$, and the assertion follows from the first statement with $c'' = \min\{c/l^q; g_S(l-1)/(l-1)^q, \cdots, g_S(1)/1^q\}$.

**3.5. Lemma.** *Let $S$ and $T$ be finite generating sets for the same group $\Gamma$. Suppose that we have constants $0 < b_1 \leq b_2$ and integers $0 \leq p \leq q$ such that $b_1 m^p \leq g_T(m) \leq b_2 m^q$. Then there are constants $0 < c_1 \leq c_2$ such that $c_1 m^p \leq g_S(m) \leq c_2 m^q$.*

*Proof.* We have integers $k$ and $l$ such that every element of $S$ is a word of length $\leq k$ based on $T$, and every element of $T$ is a word of length $\leq l$ based on $S$. Now $g_S(m) \leq g_T(km) \leq b_2 k^q m^q$ and we define $c_2 = b_2 k^q$. Also $g_S(lm) \geq g_T(m) \geq b_1 m^p = (b_1 l^{-p})(lm)^p$. So Lemma 3.4 provides $c'' > 0$ such that $g_S(m) \geq c'' m^p$, and we define $c_1 = \min\{c'', c_2\}$. q.e.d.

Milnor [9] observes that, if $\Gamma$ is the free abelian group on $T = \{\tau_1, \tau_2\}$, then $g_T(m) = 2m^2 + 2m + 1$. We give his observation a slight push.

**3.6. Proposition.** *Let $T = \{\tau_1, \cdots, \tau_n\}$ be a minimal generating set for a free abelian group $\Gamma$ of rank $n$. Then the growth function*

$$g_T(m) = \sum_{l=0}^{n} 2^l \binom{n}{l}\binom{m}{l} .$$

*If $S$ is any finite generating set for $\Gamma$ then there are constants $0 < c_1 \leq c_2$ with $c_1 m^n \leq g_S(m) \leq c_2 m^n$.*

*Proof.* For every integer $l \geq 0$ let $P_l$ denote the function on non-negative integers given by: $P_0(m) = 1$ for all integers $m \geq 0$, and $P_l(m)$, $l > 0$, is the number of distinct sequences $(a_1, \cdots, a_l)$ of positive integers with $a_1 + \cdots + a_l \leq m$. Notice $P_0(m) = \binom{m}{0}$. And if $l > 0$, then each of the $P_l(m)$ sequences $(a_1, \cdots, a_l)$ gives rise to a subset $\{a_1, a_1 + a_2, \cdots, a_1 + \cdots + a_l\}$ of cardinality $l$ in $\{1, 2, \cdots, m\}$. Conversely if a subset of cardinality $l$ in $\{1, 2, \cdots, m\}$ is put in ascending order it is seen to be of the form $\{a_1, a_1 + a_2, \cdots, a_1 + \cdots + a_l\}$. Thus $P_l(m) = \binom{m}{l}$ in general.[3] In particular $P_l$ is a polynomial of degree $l$ with positive leading coefficient on the non-negative integers.

Observe that $g_T(m) = \sum_{l=0}^{n} N_l(m)$ where $N_l(m)$ is the number of distinct expressions $\tau_1^{a_1} \cdots \tau_n^{a_n}$, $\sum |a_i| \leq m$, such that exactly $l$ of the $a_i$ are nonzero. Let $U = \{u_1, \cdots, u_l\}$ be any of the $\binom{n}{l}$ subsets of $T$ with exactly $l$ elements. By definition of $P_l$ there are precisely $P_l(m)$ distinct $u_1^{a_1} \cdots u_l^{a_l}$ with $a_i > 0$ and $\sum a_i \leq m$. Changing signs of the $a_i$ at will, we see that there are precisely

[3] This fact was pointed out to me by J. Milnor, who observed that it simplified the original version of Proposition 3.6.

$2^l P_l(m)$ distinct $u_1^{a_1} \cdots u_l^{a_l}$ with $a_i \neq 0$ and $\sum |a_i| \leq m$. Thus $N_l(m) = 2^l \binom{n}{l} P_l(m) = 2^l \binom{n}{l} \binom{m}{l}$ and the equation for $g_T$ is proved.

$g_T(m)$ now is a polynomial of degree $n$ in $m$ with leading coefficient $b > 0$. For $m$ large, say $m > m_0$, we thus have $\frac{1}{2} bm^n \leq g_T(m) \leq 2bm^n$. Now define

$$b_1 = \min\left\{\frac{b}{2}\,;\ g_T(m_0)/m_0^n, g_T(m_0 - 1)/(m_0 - 1)^n, \cdots, g_T(1)/1^n\right\},$$

$$b_2 = \max\{2b;\ g_T(m_0)/m_0^n, g_T(m_0 - 1)/(m_0 - 1)^n, \cdots, g_T(1)/1^n\}\ .$$

Then $0 < b_1 \leq b_2$, and $b_1 m^n \leq g_T(m) \leq b_2 m^n$ for every integer $m \geq 1$. Now Lemma 3.5 provides constants $0 < c_1 \leq c_2$ such that $c_1 m^n \leq g_S(m) \leq c_2 m^n$.

q.e.d.

Proposition 3.6 is the starting step for our inductive proof of Theorem 3.2. The next two lemmas provide the specific information that we need to carry out the induction step of the proof of Theorem 3.2.

**3.7. Lemma.** *Let $\Gamma$ be an $(s + 1)$-step nilpotent group with lower central series*

$$\Gamma = \Gamma_0 \supsetneqq \Gamma_1 \supsetneqq \cdots \supsetneqq \Gamma_s \supsetneqq \Gamma_{s+1} = \{1\}\ , \qquad \Gamma_{k+1} = [\Gamma, \Gamma_k]\ .$$

*Suppose that $\Gamma/\Gamma_1$ is finitely generated. Then there are finite sets $T_k = \{\tau_{k,1}, \cdots, \tau_{k,r_k}\} \subset \Gamma_k$ such that*

(i) *if $\varphi_k\colon \Gamma_k \to \Gamma_k/\Gamma_{k+1}$ denotes the projection, then $\Gamma_k/\Gamma_{k+1}$ is a finitely generated abelian group and $\{\varphi_k(\tau_{k,1}), \cdots, \varphi_k(\tau_{k,r_k})\}$ is an independent set of generators;*

(ii) *if $k > 0$, then every $\tau_{k,l} \in T_k$ is of the form $[\tau_{0,i}, \tau_{k-1,j}]$ with $\tau_{0,i} \in T_0$ and $\tau_{k-1,j} \in T_{k-1}$; and*

(iii) *$T_0$ generates $\Gamma$.*

*Proof.* To construct $T_0$ we take any independent generating set $\{a_1, \cdots, a_{r_0}\}$ of $\Gamma/\Gamma_1$ and make arbitrary choices $\tau_{0,i} \in \varphi_0^{-1}(a_i)$. Then (i) is satisfied by construction and (ii) is satisfied because it is vacuous. Now let $k > 0$ and suppose that we have constructed $T_0, T_1, \cdots, T_{k-1}$ satisfying (i) and (ii); we go on to construct $T_k$.

Let $\Delta = \Gamma/\Gamma_{k+1}$, let $\delta\colon \Gamma \to \Delta$ be the projection, observe $\Delta_l = \delta(\Gamma_l)$ and in particular that $\Delta$ is $(k + 1)$-step nilpotent, and define $S_l = \{\sigma_{l,1}, \cdots, \sigma_{l,r_l}\}$ with $\sigma_{l,m} = \delta(\tau_{l,m})$. If $\nu \in \Delta_k$ then $\nu$ is a product of commutators $[\alpha, \beta]$ with $\alpha \in \Delta$ and $\beta \in \Delta_{k-1}$. As $\Delta_k$ is central in $\Delta$ we have $[\alpha, \beta] = [\alpha', \beta']$ whenever $\alpha' \in \alpha\Delta_k$ and $\beta' \in \beta\Delta_k$; so we may assume $\alpha = \sigma_1^{a_1} \cdots \sigma_n{}^{a_n}$, $\sigma_i \in S_0$ and $a_i \in \boldsymbol{Z}$, and $\beta = \sigma_{k-1,1}{}^{b_1} \cdots \sigma_{k-1,r_{k-1}}{}^{b_{r_{k-1}}}$. Again because $\Delta_k$ is central in $\Delta$,

$$[\alpha, \beta] = \left[\prod_{i=1}^{n} \sigma_i{}^{a_i}, \prod_{j=1}^{r_{k-1}} \sigma_{k-1,j}{}^{b_j}\right] = \prod_{i,j} [\sigma_i, \sigma_{k-1,j}]^{a_i b_j}\ .$$

Thus $\Delta_k$ is generated by the $[\sigma_{0,i}, \sigma_{k-1,j}]$. In other words, $\Gamma_k/\Gamma_{k+1}$ is generated by the $\varphi_k[\tau_{0,i}, \tau_{k-1,j}]$. Choose any independent generating subset, ; that defines a finite set $T_k = \{\tau_{k,1}, \cdots, \tau_{k,r_k}\}$ in $\Gamma_k$ satisfying (i) and (ii).

We have recursively defined the sets $T_k$, $0 \leq k \leq s$, satisfying (i) and (ii); and (iii) is immediate from (ii).

*Proof of Theorem 3.2.* Retain the notation of Lemma 3.7. We first prove the lower bound for $g_{T_0}$.

Suppose $\tau_{k,l} \in T_k$ given with $k > 0$. Then we have $\tau_{0,i} \in T_0$ and $\tau_{k-1,j} \in T_{k-1}$ such that $\tau_{k,l} = [\tau_{0,i}, \tau_{k-1,j}]$. Denote

$$x = \tau_{0,i}, y = \tau_{k-1,j} \quad \text{and} \quad z = \tau_{k,l}^{-1}; \text{ so } z = [y, x] .$$

Then $z$ commutes with $x$ and $y$ modulo $\Gamma_{k+1}$. If $a$, $m$ and $b$ integers, then

$$z^{am+b} = z^{am}z^b \equiv [y^a, x^m][y, x^b] \qquad \text{modulo } \Gamma_{k+1} .$$

If $|a| \leq m^k$, then by induction on $k$ the elements $y$, $y^m$ and $y^a$ are expressible modulo $\Gamma_k$ as words of length $\leq 8^{k-1}m$ based on $T_0$; if $|am + b| \leq m^{k+1}$ it follows that $z$, $z^m$ and $z^{am+b}$ are expressible modulo $\Gamma_{k+1}$ as words of length $\leq 8^k m$ based on $T_0$. This proves:

(3.8) *every* $\tau_{k,l}{}^{p_{k,l}}$, $|p_{k,l}| \leq m^{k+1}$, *is expressible modulo* $\Gamma_{k+1}$ *as a word of length* $\leq 8^k m$ *based on* $T_0$ .

Suppose $T_k$ to be ordered so that $\{\varphi_k(\tau_{k,1}), \cdots, \varphi_k(\tau_{k,n_k})\}$ generates a free abelian subgroup of rank $n_k$ in $\Gamma_k/\Gamma_{k+1}$. Then (3.8) says that $\Gamma_k/\Gamma_{k+1}$ has at least $\{2m^{k+1} + 1\}^{n_k} \geq m^{(k+1)n_k}$ elements $\{\tau_{k,1}{}^{p_{k,1}} \cdots \tau_{k,n_k}{}^{p_{k,n_k}}\}\Gamma_{k+1}$, $|p_{k,l}| \leq m^{k+1}$, expressible as words of length $\leq (8^k n_k)m$ based on $T_0$. Denote $B = \sum_{k=0}^{s} 8^k n_k$; then $\Gamma$ has at least $m^E$ distinct elements expressible as words of length $\leq Bm$ based on $T_0$, where $E = \sum_{k=0}^{s} (k+1)n_k = E_1(\Gamma)$. Thus $g_{T_0}(Bm) \geq m^{E_1(\Gamma)} = B^{-E_1(\Gamma)} \cdot (Bm)^{E_1(\Gamma)}$. The second part of Lemma 3.4 provides a constant $b_1 > 0$ such that $g_{T_0}(m) \geq b_1 m^{E_1(\Gamma)}$ for all integers $m \geq 1$.

For the upper bound on $g_{T_0}$ we first note that every element $\gamma \in \Gamma$ has expression

$$\text{(3.9a)} \qquad \gamma = \xi_0\xi_1 \cdots \xi_s, \qquad \xi_k = \tau_{k,1}{}^{p_{k,1}}\tau_{k,2}{}^{p_{k,2}} \cdots \tau_{k,r_k}{}^{p_{k,r_k}} ,$$

which, though not unique because of torsion in the $\Gamma_k/\Gamma_{k+1}$, can be made "almost" unique by assuming $T_k$ to be ordered so that $\{\varphi_k(\tau_{k,1}), \cdots, \varphi_k(\tau_{k,n_k})\}$ generates a free abelian group of rank $n_k$ in $\Gamma_k/\Gamma_{k+1}$. Then that free abelian group has some finite index $I_k$ in $\Gamma_k/\Gamma_{k+1}$, and (3.9a) can always be chosen such that

$$\text{(3.9b)} \qquad 0 \leq p_{k,l} < I_k \qquad \text{for} \quad n_k < l \leq r_k .$$

Consider the (finite) set of all commutators $\gamma = [\tau_{t,i}{}^{\pm 1}, \tau_{u,j}]$ with $\tau_{t,i} \in T_t$, $\tau_{u,j} \in T_u$ and $0 \leq t \leq u$. Each such commutator has expression (3.9a, b) with $\xi_0 = \xi_1 = \cdots = \xi_u = 1$, which is turn provides a number $\sum_{k=u+1}^{s} \sum_l |p_{k,l}|$. Let $N$ denote the maximum of that finite collection of numbers.

It will be convenient to have the notation $T^k = T_k \cup T_{k+1} \cup \cdots \cup T_s$. Note that (3.9) expresses an element $\gamma \in \Gamma_k$ as a word of length $\sum_{t=k}^{s} \sum_l |p_{t,l}|$ based on $T^k$.

Let $\gamma \in \Gamma$ be expressed as a word $w_0$ of length $\leq m$ based on $T_0$. We operate on $w_0$ by pulling each occurrence of $\tau_{0,1}{}^{\pm 1}$ to the left, then each occurrence of $\tau_{0,2}{}^{\pm 1}$ to the left, and so on through $T_0$, representing $\gamma$ by $\tau_{0,1}{}^{a_{0,1}} \cdots \tau_{0,r_0}{}^{a_{0,r_0}} w_1$ where $\sum |a_{0,i}| \leq m$ $|a_{0,i}| < I_0$ for $n_0 < i \leq r_0$, and $w_1$ is a word on $T^1$. Each of these $\leq m$ pulls of $\tau_{0,i}{}^{\pm 1}$ to the left involves $< m^2$ crossings of elements or inverses of elements of $\{\tau_{0,i+1}, \cdots, \tau_{0,r_0}\}$, each such crossing introducing a word of length $\leq N$ based on $T^1$, and involves crossings of previously introduced words on $T^1$. If the previously introduced word is of the form $\eta = \eta_1 \cdots \eta_s$ where $\eta_k$ is a word of length $\leq \binom{v}{k} N^k$ based on $T^k$, then $\eta \cdot \tau_{0,i}{}^{\pm 1} = \tau_{0,i}{}^{\pm 1} \cdot \eta'$ where $\eta' = \eta'_1 \cdots \eta'_s$, $\eta'_1 = \eta_1$, and $\eta'_k = [\eta_{k-1}{}^{-1}, \tau_{0,i}{}^{\mp 1}]\eta_k$ for $k > 1$. Thus $\eta'_k$ is a word of length $\leq \binom{v}{k-1} N^{k-1} N + \binom{v}{k} N^k = \binom{v+1}{k} N^k$ on $T^k$. Let $|w_1|_k$ denote the sum of the absolute values of the exponents of the elements of $T_k$ in $w_1$. Then $|w_1|_k < \sum_{t=1}^{k} \binom{m^2}{t} N^t \leq k \binom{m^2}{k} N^k \leq N_1 m^{2k}$, where $N_1 > 0$ depends only on $s$ and is independent of $m$. This starts the induction.

Now suppose that $t > 0$ and $\gamma$ is represented by a word $\xi_0 \xi_1 \cdots \xi_{t-1} w_t$ where, for certain positive integers $\{1 = M_0, M_1, \cdots, M_{t-1}, N_t\}$ depending only on $\Gamma$ and independent of $m$,

(i) $\xi_i = \tau_{i,1}{}^{a_{i,1}} \tau_{i,2}{}^{a_{i,2}} \cdots \tau_{i,r_i}{}^{a_{i,r_i}}, \qquad \sum_{j=1}^{r_i} |a_{i,j}| \leq M_i m^{2i}$,

(ii) $w_t$ is a word on $T^t$ with $|w_t|_k \leq N_t m^{2k}$.

Then as above we pull the $\tau_{t,l}{}^{\pm 1}$ to the left in $w_t$; so the element of $\Gamma$ represented by $w_t$ is also represented by $\xi_t w_{t+1}$ where $\xi_t$ satisfies (i) with $M_t = N_t$ and with $w_{t+1}$ a word on $T^{t+1}$ such that

$$|w_{t+1}|_k \leq \sum_{p=t+1}^{k} \binom{\{N_t m^{2t}\}^2}{p-t} N^{p-t} \leq (k-t) \binom{N_t^2 M^{2t+1}}{k-t} N^{k-t} \leq N_{t+1} m^{2k} ,$$

where $N_{t+1} > 0$ depends only on $s$ and $t$ and is independent of $m$. That is the induction. We have proved the existence of a sequence $\{1 = M_0, M_1, \cdots, M_s\}$ of positive integers depending only on $\Gamma$, such that:

(3.10) *if $\gamma \in \Gamma$ is expressible as a word of length $\leq m$ based on $T_0$, then $\gamma$ has expression* (3.9a, b) *with each* $\sum_{l=1}^{r_k} |p_{k,l}| \leq M_k m^{2k}$.

Consider the set $G_m$ of all elements of $\Gamma$ expressible as words of length $\leq m$ on $T_0$. If $\gamma \in G_m$, then we put it in the form $\xi_0\xi_1 \cdots \xi_s$ of (3.9) with each $|p_{k,l}| \leq M_k m^{2^k}$ as given by (3.10). Now the number of possibilities for $\xi_k$ is $\leq I_k^{r_k - n_k}(2M_k m^{2^k} + 1)^{n_k}$; so

$$g_{T_0}(m) \leq \left\{\prod_{k=0}^{s} I_k^{r_k-n_k}\right\} \left\{\prod_{k=0}^{s} (2M_k m^{2^k} + 1)^{n_k}\right\}.$$

The leading term of the bounding polynomial is a positive multiple of $m^E$ where $E = \sum_{k=0}^{s} 2^k n_k = E_2(\Gamma)$. Thus there is a constant $b_2 > 0$ such that $g_{T_0}(m) \leq b_2 m^{E_2(\Gamma)}$ for every integer $m \geq 1$. $m = 1$ gives $b_2 \geq b_1$.

Recall the arbitrary finite generating set $S$ for $\Gamma$. We have constants $0 < b_1 \leq b_2$ such that $b_1 m^{E_1(\Gamma)} \leq g_{T_0}(m) \leq b_2 m^{E_2(\Gamma)}$. Now Lemma 3.5 provides us with constants $0 < c_1 \leq c_2$ such that $c_1 m^{E_1(\Gamma)} \leq g_S(m) \leq c_2 m^{E_2(\Gamma)}$ for every integer $m \geq 1$. q.e.d.

**Remark.** By using $[\Gamma_u, \Gamma_v] \subset \Gamma_{u+v+1}$ one can sharpen (3.10) and thus the upper bound on $g_S$ from $c_2 m^{E_2(\Gamma)}$ to $c_2 m^E$ where

$$E = n_0 + 2n_1 + 4n_2 + 6n_3 + 10n_4 + 14n_5 + \cdots ;$$

but for our purposes this is not necessary.

Let $\Gamma$ be a finitely generated group, $S$ a finite generating set, and $E \geq 0$ an integer. If there is a constant $c > 0$ such that

$$g_S(m) \leq cm^E \qquad \text{for every integer } m \geq 1 ,$$

then we say that $\Gamma$ has *polynomial growth of degree* $\leq E$. Lemma 3.5 says that this condition is independent of choice of $S$.

Theorem 3.2 says, among other things, that a finitely generated nilpotent group $\Gamma$ has polynomial growth of degree $\leq E_2(\Gamma)$.

**3.11. Theorem.** *Let $\Sigma$ be a finitely generated group, and $\Gamma$ a subgroup of finite index.*

1. *$\Gamma$ is finitely generated.*
2. *If $\Gamma$ has polynomial growth of degree $\leq E$, then $\Sigma$ has polynomial growth of degree $\leq E$.*
3. *If $\Gamma$ is nilpotent and has polynomial growth of degree $\leq E$, then $\Sigma$ has polynomial growth of degree $\leq \min\{E, E_2(\Gamma)\}$.*

*Proof.* Let $\Psi$ be the intersection of the conjugates $\sigma\Gamma\sigma^{-1}$ of $\Gamma$ in $\Sigma$. Then $\Psi$ is a finitely generated normal subgroup of finite index in $\Sigma$. In particular $\Gamma$ is finitely generated. Suppose that Theorem 3.11 is known for normal subgroups. Let $\Gamma$ have polynomial growth of degree $\leq E$. Let $U \subset V$ be respective finite generating sets of $\Psi \subset \Gamma$, and let $c > 0$ such that $g_V(m) \leq cm^E$. Then $g_U(m) \leq g_V(m)$ shows that $\Psi$ has polynomial growth of degree $\leq E$, so $\Sigma$ has polynomial growth of degree $\leq E$. If $\Gamma$ is nilpotent, then further

$\Psi$ is nilpotent, so $\Sigma$ has polynomial growth of degree $\leq \min\{E, E_2(\Psi)\}$. But $E_2(\Psi) = E_2(\Gamma)$. Thus the assertions follow for $\Gamma$. Now we need only consider the case where $\Gamma$ is normal in $\Sigma$.

Now $\Gamma$ is a normal subgroup of finite index in $\Sigma$. Let $U = \{\mu_1, \cdots, \mu_p\}$ be a system of representatives of $\Sigma/\Gamma$. Let $T_0$ be a finite generating set for $\Gamma$ and define

$$V = U \cup T, \text{ where } T = \{\mu_i^{-1}\tau\mu_i : \mu_i \in U, \tau \in T_0\} .$$

Let $N > 0$ be an integer large enough to satisfy the finite collection of conditions

$$\mu_i^\varepsilon \mu_j^\eta = \mu_{f(\varepsilon i, \eta j)} w_{\varepsilon i, \eta j} ,$$

where $\varepsilon = \pm 1, \eta = \pm 1$, and $w_{\varepsilon i, \eta j}$ is a word of length $\leq N$ on $T$.

If $t \in T$, say $t = \mu_i^{-1}\tau\mu_i$ with $\tau \in T_0$ and $\mu_i \in U$, then $t\mu_j = \mu_i^{-1}\tau\mu_i\mu_j = \mu_i^{-1}\tau\mu_{f(i,j)}w_{ij} = \mu_i^{-1}\mu_{f(i,j)}t'w_{ij} = \mu_{f(-i, f(i,j))}w_{-i, f(i,j)}t'w_{ij}$, where $t' \in T$. Thus $t\mu_j$ has form $\mu_k w$ where $w$ is a word of length $\leq 2N + 1$ on $T$.

Len $\gamma \in \Sigma$ be represented by a word $w$ of length $\leq m$ based on $V$. Take the first occurence of an $\mu_j^{\pm 1}$ *from the right* and pull it left until it meets another occurence of an $\mu_i^{\pm 1}$. That involves crossings of elements of $T \cup T^{-1}$, each crossing inserting a word of length $\leq 2N + 1$ on $T$; amalgamation $\mu_i^{\pm 1}\mu_j^{\pm 1} = \mu_k w_{\pm i, \pm j}$ inserts a further word of length $\leq N$ on $T$. Now push $\mu_k$ left until it hits an $\mu_l^{\pm 1}$. Continue until we have $\gamma$ represented by a word $\mu w_0$, $\mu \in U$ and $w_0$ a word of length $< mN + m(2N+1)$. That proves $g_V(m) \leq p \cdot g_T((3N+1)m)$. Thus if $\Gamma$ has polynomial growth of degree $\leq E$, then $\Sigma$ has polynomial growth of degree $\leq E$.

**Remark.** We proved, more generally, that if $\Gamma \subset \Sigma$ has finite index, then there are respective finite generating sets $T \subset V$ and a number $M > 0$ such that

(3.12) $$g_T(m) \leq g_V(m) \leq g_T(Mm) \quad \text{for all integers } m \geq 1 .$$

## 4. The growth function for solvable groups —groups of exponential growth

In this section we extend our growth function estimates from the class of finitely generated nilpotent groups to a larger class of solvable groups.

A solvable group is called *polycyclic* if it satisfies the (equivalent) conditions of the following proposition.

**4.1. Proposition.** *Let $\Gamma$ be a solvable group with derived series*

$$\Gamma = \Gamma^0 \supsetneqq \Gamma^1 \supsetneqq \cdots \supsetneqq \Gamma^d \supsetneqq \Gamma^{d+1} = \{1\} , \qquad \Gamma^{k+1} = [\Gamma^k, \Gamma^k] .$$

*Then the following conditions are equivalent.*

(1) *There is a normal series* $\Gamma = A_0 \supset A_1 \supset \cdots \supset A_t = \{1\}$ *with every quotient* $A_i/A_{i+1}$ *finite or infinite cyclic.*

(2) *There is a normal series* $\Gamma = B_0 \supset B_1 \supset \cdots \supset B_u = \{1\}$ *with every quotient* $B_i/B_{i+1}$ *finitely generated abelian.*

(3) *In any solvable normal series* $\Gamma = C_0 \supset C_1 \supset \cdots \supset C_v = \{1\}$ *each* $C_i/C_{i+1}$ *is a finitely generated abelian group.*

(4) *Each quotient* $\Gamma^k/\Gamma^{k+1}$ *in the derived series is a finitely generated abelian group.*

(5) *All derived groups* $\Gamma^k$ *of* $\Gamma$ *are finitely generated.*

(6) *Every subgroup of* $\Gamma$ *is finitely generated.*

(7) $\Gamma$ *satisfies the maximal condition for increasing sequences of subgroups.*

(8) *There is an exact sequence* $\{1\} \to \Delta \to \Gamma^* \to \Phi \to \{1\}$, *where* $\Delta$ *is a finitely generated nilpotent group,* $\Phi$ *is a finitely generated free abelian group, and* $\Gamma^*$ *is a subgroup of finite index in* $\Gamma$.

(9) $\Gamma$ *is isomorphic to a discrete subgroup of a Lie group which has only a finite number of topological components.*

(10) $\Gamma$ *is isomorphic to a discrete subgroup of a connected solvable Lie group.*

(11) $\Gamma$ *has a faithful representation by integer matrices.*

**Remark.** The proof of Proposition 4.1 consists more or less of noticing some known hard theorems at the same time.

*Proof.* K. A. Hirsch [6] proved equivalence of (2), (3) and (7). The equivalences

$$(1) \iff (2),\ (3) \Rightarrow (4) \Rightarrow (2),\ (6) \Rightarrow (5) \Rightarrow (4)$$

are obvious. For (4) $\Rightarrow$ (5) by induction on $d - k$, we first note finite generation of $\Gamma^d$ by hypothesis, then have finite generation for $\Gamma^{k+1}$ by induction and for $\Gamma^k/\Gamma^{k+1}$ by hypothesis, and finally conclude finite generation for $\Gamma^k$. For (5) $\Rightarrow$ (6), since (5) and (7) are equivalent, every subgroup $\Sigma \subset \Gamma$ satisfies the maximal condition and thus has every $\Sigma^k$ finitely generated; so every subgroup $\Sigma \subset \Gamma$ has $\Sigma = \Sigma^0$ finitely generated. Now the first 7 conditions are equivalent.

Mal'cev [8] proved (4) $\Rightarrow$ (8). Given (8), every subgroup of $\Delta$ or $\Phi$ is finitely generated; so every subgroup of $\Gamma^*$ is finitely generated, and (6) follows. Now the first 8 conditions are equivalent.

L. Auslander [3] (or see Swan [13]) proved that (4) $\Rightarrow$ (11). Given (11) we have $\Gamma \subset GL(n, Z) \subset GL(n, C)$, and conjugate $\Gamma$ into a Borel subgroup of $GL(n, C)$; thus (11) $\Rightarrow$ (10). Trivially (10) $\Rightarrow$ (9). Given (9), say with $\Gamma \subset G$ and $G/G_0$ finite, G. D. Mostow [11, Theorem 1′] proves that every subgroup of $\Gamma \cap G_0$ is finitely generated on $\leq \dim G_0$ elements, so every subgroup of $\Gamma$ is finitely generated. Thus (9) $\Rightarrow$ (6). Now all eleven conditions are equivalent.

q.e.d.

Let $\Gamma$ be a group with a finite generating set $S$. We say that $\Gamma$ is of *exponential growth* if there are constants $u > 0$ and $v > 1$ such that

$$(4.2) \qquad u \cdot v^m \leq g_S(m) \qquad \text{for every integer } m \geq 1 ,$$

which does not depend on choice of $S$. For if $T$ is another finite generating set for $\Gamma$, then every element of $S$ is expressible as a word of some length $\leq k$ on $T$; so $g_S(m) \leq g_T(km) \leq g_T(m)^k$, and (4.2) implies $u^{1/k} \cdot (v^{1/k})^m \leq g_T(m)$, for all $m \geq 1$.

J. Milnor observed that we can always take $u = 1$ in (4.2). For, given (4.2), $u \cdot v^{tm} \leq g_S(tm) \leq g_S(m)^t$ for all positive integers $m$ and $t$. Taking $t$-th root, $u^{1/t} \cdot v^m \leq g_S(m)$. Then taking the limit as $t \to \infty$ we obtain $v^m \leq g_S(m)$ for all $m \geq 1$. Milnor further observed that if $\Gamma$ is of exponential growth with finite generating set $S$, then $\lim_{m \to \infty} g_S(m)^{1/m}$ is the largest number $v$ such that $v^m \leq g_S(m)$.

**4.3. Theorem.** *Let $\Gamma$ be a polycyclic group, and $S$ a finite set of generators.*

1. *If $\Gamma$ has a nilpotent subgroup $\Delta$ of finite index, then there are constants $0 < c_1 \leq c_2$ such that, in the notation* (3.3),

$$c_1 m^{E_1(\Delta)} \leq g_S(m) \leq c_2 m^{E_2(\Delta)} \qquad \textit{for every integer } m \geq 1 ;$$

*in particular $\Gamma$ is of polynomial growth of degree $\leq E_2(\Delta)$.*

2. *If $\Gamma$ does not have a nilpotent subgroup of finite index, then there is a constant $v > 1$ such that*

$$v^m \leq g_S(m) \leq g_S(1)^m \qquad \textit{for every integer } m \geq 1 ;$$

*in particular $\Gamma$ is of exponential growth.*

In order to apply Theorem 4.3, and in fact in order to prove part 2 of it, we need a criterion for deciding whether $\Gamma$ has a nilpotent subgroup of finite index.

**4.4. Proposition.** *Let $\Gamma$ be a polycyclic group.*

1. *$\Gamma$ has a torsionfree subgroup $\Gamma^*$ of finite index and an exact sequence $\{1\} \to \Delta \to \Gamma^* \to \Phi \to \{1\}$, where $\Delta$ and $\Phi$ are finitely generated, $\Delta$ nilpotent and $\Phi$ free abelian.*

*Fix $\Gamma^*$ and the sequence. Let $D$ be the unique connected simply connected nilpotent Lie group containing $\Delta$ as a discrete subgroup with coset space $D/\Delta$ compact. If $\gamma \in \Gamma^*$, let $\zeta(\gamma)$ denote the unique Lie group automorphism of $D$ such that $\zeta(\gamma)\colon \delta \to \gamma\delta\gamma^{-1}$ for every $\delta \in \Delta$, and $\zeta_*(\gamma)$ be the induced automorphism of the Lie algebra $\mathfrak{D}$ of $D$.*

2. *A subgroup $N \subset \Gamma^*$ is nilpotent if and only if every $\zeta_*(\gamma)$, $\gamma \in N$, has every eigenvalue equal to* 1.

3. *$\Gamma$ has a nilpotent subgroup of finite index, if and only if every $\zeta_*(\gamma)$, $\gamma \in \Gamma^*$, has every eigenvalue of absolute value* 1.

*Proof.* Selberg [12, Lemma 8] proved that a finitely generated group of matrices over a field of characteristic zero has a torsionfree normal subgroup of finite index. Conditions (6) and (10) of Proposition 4.1 now say

(4.5) *in a polycyclic group, every subgroup has a torsionfree normal subgroup of finite index.*

Let $\Gamma'$ be any subgroup of finite index in $\Gamma$ such that there is an exact sequence $\{1\} \to \Delta' \xrightarrow{i} \Gamma' \xrightarrow{j} \Phi' \to \{1\}$ with $\Delta'$ nilpotent and $\Phi'$ free abelian; this is condition (8) of Proposition 4.1. Let $\Gamma^*$ be a torsionfree subgroup of finite index in $\Gamma'$. Then $\Gamma^*$ is a torsionfree subgroup of finite index in $\Gamma$, and we have an exact sequence $\{1\} \to \Delta \xrightarrow{p} \Gamma^* \xrightarrow{q} \Phi \to \{1\}$, where $\Delta = i^{-1}(\Gamma^*)$ is still nilpotent and $\Phi = j(\Gamma^*)$ remains free abelian. $\Delta$ is torsionfree because $\Gamma^*$ is so, and $\Delta$ and $\Phi$ are finitely generated because $\Gamma^*$ is polycyclic. Thus the first assertion of Proposition 4.4 is proved.

Suppose that $\Gamma$ has a nilpotent subgroup $N$ of finite index. We can intersect $N$ with its conjugates and assume $N$ to be normal in $\Gamma$. Now $\Gamma/N$ is finite, and therefore $\Gamma^*$ has finite image $\Gamma^*/(\Gamma^* \cap N)$ under $\Gamma \to \Gamma/N$; thus $\Gamma^* \cap N$ is a nilpotent normal subgroup of finite index in $\Gamma^*$. Let $\Gamma^* \cap N = L_0 \supsetneqq L_1 \supsetneqq \cdots \supsetneqq L_s \supsetneqq L_{s+1} = \{1\}$ be defined by: $L_i/L_{i+1}$ is the center of $L_{i-1}/L_{i+1}$. If $\gamma \in \Gamma^* \cap N$ and $\lambda \in L_i$, then it follows that $[\gamma, \lambda] \in L_{i+1}$, i.e. that $\gamma\lambda\gamma^{-1} = \lambda\nu$ with $\nu \in L_{i+1}$. Define $\Delta_i = \Delta \cap L_i$, and let $D_i$ be the analytic subgroup of $D$ containing $\Delta_i$ such that $D_i/\Delta_i$ is compact. Then $\gamma \in \Gamma^* \cap N$ and $d \in D_i$ implies $\zeta(\gamma)\cdot d = dn$ with $n \in D_{i+1}$. Now $\zeta_*(\gamma)$ preserves each $\mathfrak{D}_i$ and is the identity transformation on each $\mathfrak{D}_i/\mathfrak{D}_{i+1}$. Thus every $\zeta_*(\gamma)$, $\gamma \in \Gamma^* \cap N$, has every eigenvalue equal to 1. If $m$ is the index of $\Gamma^* \cap N$ in $\Gamma^*$, then every eigenvalue of every $\zeta_*(\gamma)$, $\gamma \in \Gamma^*$, is an $m$-th root of 1, and thus has absolue value 1.

The part of Proposition 4.4 just proved suggests that in general the group

(4.6) $U = \{\gamma \in \Gamma^* : \zeta_*(\gamma)$ has every eigenvalue $1\}$ *is nilpotent* .

Let $m = \dim D$. If $m = 0$, then $\Gamma^* = \Phi$ abelian; so $U$ is abelian and thus nilpotent. If $m = 1$, then $\Delta$ is central in $U$; so $U$ is nilpotent because $\Delta/U \subset \Phi$ is nilpotent. Suppose $m > 1$. Refine the lower central series of $D$ to a central series $L_0 \supsetneqq L_1 \supsetneqq \cdots \supsetneqq L_m \supsetneqq L_{m+1} = \{1\}$ with $\zeta(U)$-stable 1-dimentional quotients such that each $L_i/(\Delta \cap L_i)$ is compact. Then $\Delta \cap L_m$ is central in $U$, and $U/(\Delta \cap L_m)$ is nilpotent by induction on $m$. Thus $U$ is nilpotent, and (4.6) is proved.

If $\gamma \in \Gamma^*$, then we use the Jordan canonical form to obtain polynomials $A_\gamma$ and $U_\gamma$ in $\zeta_*(\gamma)$ such that

(i) $A_\gamma$ is diagonable over the complex numbers,

(ii) $U_\gamma$ has every eigenvalue 1, and

(iii) $A_\gamma U_\gamma = \zeta_*(\gamma) = U_\gamma A_\gamma$.

If we put the solvable group $\zeta_*(\Gamma^*)$ in simultaneous triangular form on $\mathfrak{D}^C$, then $A_\gamma$ is the diagonal of $\zeta_*(\gamma)$ and $U_\gamma - I$ is the superdiagonal. Thus $A = \{A_\gamma : \gamma \in \Gamma^*\}$ is a commutative group of linear transformations of $\mathfrak{D}$ and the map $\Gamma^* \to A$ given by $\gamma \to A_\gamma$ is a group homomorphism. By definition the kernel is the group $U$ of (4.6). Thus

$$(4.7) \qquad \Gamma^*/U \cong A .$$

Every element of $A$ is an automorphism of $\mathfrak{D}$ because $D$ is a linear algebraic group. Let $\bar{A}$ be the closure of $A$ in the automorphism group of $\mathfrak{D}$. Every $\alpha \in A$ induces an automorphism of $D$, which preserves $\Delta$; the same follows for every $\alpha \in \bar{A}$. If $B$ is a connected subgroup of $\bar{A}$, then it must centralize the discrete group $\Delta$, act trivially on $D$, and thus be the trivial group. So $\bar{A}$ is discrete. Thus $A$ is a discrete subgroup of the automorphism group of $\mathfrak{D}$.

Now suppose that every $\zeta_*(\gamma)$, $\gamma \in \Gamma^*$, has every eigenvalue of absolute value 1. In other words every $\alpha \in A$ has every eigenvalue of absolute value 1. $A$ is contained in the compact group af all automorphisms of $\mathfrak{D}$ having matrix diag $\{b_1, \cdots, b_m\}$, $|b_i| = 1$, in the basis of $\mathfrak{D}^C$, that diagonalizes $A$. As $A$ is discrete now $A$ is finite. Thus (4.7) says that $U$ has finite index in $\Gamma^*$, and hence also in $\Gamma$. But (4.6) says that $U$ is nilpotent. Now $\Gamma$ has a nilpotent subgroup of finite index, and hence the proof of Proposition 4.4 is complete.

*Proof of Theorem 4.3.* Let $\Gamma$ have a nilpotent subgroup $\Delta$ of finite index. Then Theorems 3.2 and 3.11 provide $c_2 > 0$ such that $g_S(m) \leq c_2 m^{E_2(\Delta)}$. Let $T$ be a finite generating set for $\Delta$. Then Theorem 3.2 provides $b > 0$ such that $bm^{E_1(\Delta)} \leq g_T(m)$. But $g_T(m) \leq g_{T \cup S}(m)$, and now Lemma 3.5 provides $c_1 > 0$ such that $c_1 m^{E_1(\Delta)} \leq g_S(m)$. Part 1 of Theorem 4.3 is proved.

From now, $\Gamma$ has no nilpotent subgroup of finite index. We retain the notation of Proposition 4.4 fixing

$$\{1\} \to \Delta \to \Gamma^* \to \Phi \to \{1\} , \quad D \text{ and } \mathfrak{D}$$

there. Thus we have $\gamma \in \Gamma^*$ and an eigenvalue $\lambda$ of $\zeta_*(\gamma)$ such that $|\lambda| \neq 1$. Let $D = D_0 \supset D_1 \supset \cdots \supset D_s \supset D_{s+1} = \{1\}$ be the lower central series of $D$. Then the eigenvalue $\lambda$ of $\zeta_*(\gamma)$ occurs on one of the quotients $(\mathfrak{D}_k/\mathfrak{D}_{k+1})^C$. Let $S$ denote the vector space group $D_k/D_{k+1}$, $\Sigma$ the lattice $(\Delta \cap D_k)/(\Delta \cap D_{k+1})$ in $S$, and $\beta$ the action of $\zeta_*(\gamma)$ on the Lie algebra $\mathfrak{S}$.

Let $\{\lambda_1, \cdots, \lambda_v\}$ be the eigenvalues of $\beta$ on $\mathfrak{S}$; $\lambda$ is one of them. If each $|\lambda_j| < 1$, then replace $\gamma$ by $\gamma^{-1}$; now $|\lambda| > 1$. If some $|\lambda_j| > \lambda$, then replace $\lambda$ by the $\lambda_j$ of greatest modulus; now $|\lambda| \geq |\lambda_j|$ for each $j$. After having done this, choose a nonzero $L \in \mathfrak{S}^C$ such that $\beta(L) = \lambda L$. If $\lambda$ is real we can choose $L \in \mathfrak{S}$. If $\lambda$ is not real, and $\lambda = re^{i\theta}$ with $r = |\lambda|$, then $L + \bar{L}$ and $i(L - \bar{L})$ span a real 2-plane $P \subset \mathfrak{S}$ on which $\beta$ has matrix $\begin{pmatrix} r\cos\theta & r\sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$.

Choose a minimal generating set $V = \{\nu_1, \cdots, \nu_v\}$ for the free abelian group

$\Sigma$. Let $\{Z_1, \cdots, Z_v\}$ be the basis of $\mathfrak{S}$ such that $\exp(Z_j) = \nu_j$. If $\lambda$ is real, define $X_0 = L$ and let $<, >$ be any inner product on $\mathfrak{S}$ such that each $\|Z_j\| \geq 1$. If $\lambda$ is not real, define $X_0 = L + \bar{L}$ and let $<, >$ be any inner product on $\mathfrak{S}$ such that each $\|Z_j\| \geq 1$ and $\langle L + \bar{L}, i(L - \bar{L})\rangle = 0$. Define $b = \max\{\|Z_j\|: 1 \leq j \leq v\}$. If $\sigma \in \Sigma$ is expressible as a word of length $\leq m$ on $V$, say $\sigma = \nu_1^{a_1} \cdots \nu_v^{a_v}$ with $|a_1| + \cdots + |\alpha_v| \leq m$, then $\exp(a_1 Z_1 + \cdots + a_v Z_v) = \sigma$ and $\|a_1 Z_1 + \cdots + a_v Z_v\| \leq mb$.

We have $\lambda = re^{i\theta}$, $r = |\lambda|$ and $\theta$ real. Replace $\gamma$ by a high positive power $\gamma^t$, so that first $r > 12b$ and then (modulo $2\pi$) $|\theta| < 10^{-3}$. Then $\beta(X_0) = \rho_0 X_0 + Y_0$ where $\rho_0 > 11b$, $\langle X_0, Y_0\rangle = 0$ and $\|Y_0\| < 10^{-2}\rho_0\|X_0\|$. Approximate $X_0$ by a rational linear combination $X$ of the $Z_j$. Then

(i) there is an integer $n > 0$ such that $1 \neq \sigma = \exp(nX) \in \Sigma$ and $\|nX\| \geq b$; and

(ii) $\beta(X) = \rho X + Y$, where $\rho > 11b$, $\langle X, Y\rangle = 0$, $\|Y\| < 10^{-1}\rho\|X\|$.

As $\lambda$ has maximum modulus among the eigenvalues of $\beta$ on $\mathfrak{S}$, the above condition (ii) implies

(iii) $\|\beta^k X\| > 10b\|\beta^{k-1}X\|$ for every integer $k \geq 1$.

Let $\Psi = \Sigma\cdot\{\gamma\}$ semidirect product, where commutation $\gamma\tau\gamma^{-1} = \tau'$, with $\tau, \tau' \in \Sigma$, is induced from the conjugation action of $\gamma$ on $\Delta \cap D_k$. Then $U = \{\gamma, \nu_1, \cdots, \nu_v\} = \{\gamma\} \cup V$ is a finite generating set for $\Psi$, $\gamma$ is a word of length 1 on $U$, and $\sigma = \exp(nX)$ is a word of some length $l$ on $U$. Given an integer $p > 0$, we have $10^{p+1}$ integers with decimal expansion $q = k_0 + k_1 10 + k_2 10^2 + \cdots + k_p 10^p$, $0 \leq k_j < 10$. With each such integer $q$ we associate the group element

$$\sigma_q = \sigma^{k_0}\cdot\gamma\sigma^{k_1}\gamma^{-1}\cdot\gamma^2\sigma^{k_2}\gamma^{-2}\cdot \cdots \cdot\gamma^p\sigma^{k_p}\gamma^{-p} \in \Sigma .$$

For example $1 = \sigma_0$ and $\sigma = \sigma_1$. First observe that $\sigma_q = \exp(nX_q)$ where

$$X_q = k_0 X + \beta(k_1 X) + \beta^2(k_2 X) + \cdots + \beta^p(k_p X) .$$

The above conditions (ii) and (iii) show that $\{X_q\}_{q\geq 0}$ are distinct; so $\{\sigma_q\}_{q\geq 0}$ are distinct. Thus, with $p$ we have associated $10^{p+1}$ distinct elements $\sigma_q$ of $\Sigma$. Second, observe

$$\sigma_q = \sigma^{k_0}\cdot\gamma\sigma^{k_1}\cdot\gamma\sigma^{k_2}\cdot \cdots \cdot\gamma\sigma^{k_{p-1}}\cdot\gamma\sigma^{k_p}\cdot\gamma^{-p} ,$$

which is a word of length $\leq l(k_0 + \cdots + k_p) + 2p$ on $U$. As $0 \leq k_j < 10$ now $g_U(9l(p + 1) + 2p) \geq 10^{p+1}$, which proves that $\Psi$ is of exponential growth. But $\Psi$ is a homomorphic image of the semidirect product $(\Delta \cap D_k)\cdot\{\gamma\}$, which must thus be of exponential growth. Finally $(\Delta \cap D_k)\cdot\{\gamma\}$ is a subgroup of $\Gamma$, and so $\Gamma$ is of exponential growth. q.e.d.

J. Milnor extended the scope of Theorem 4.3 by proving [10] that a finitely generated nonpolycyclic solvable group must be of exponential growth.

Combining that result with Theorem 4.3 we have an estimate on the growth of finitely generated solvable groups:

**4.8. Theorem.** *Let $\Gamma$ be a finitely generated solvable group. If $\Gamma$ has a nilpotent subgroup $\Delta$ of finite index, then $\Gamma$ is polycyclic and of polynomial growth of degree $\leq E_2(\Delta)$. If $\Gamma$ does not have a nilpotent subgroup of finite index, then $\Gamma$ is of exponential growth.*

## 5. Manifolds of non-negative curvature and solvable fundamental group

We apply Theorems 3.2 and 4.3 to complete riemanian manifolds of non-negative mean curvature. $E_1(\Delta)$ is the notation (3.3).

**5.1. Theorem.** *Let $M$ be a complete $n$-dimensional riemannian manifold whose mean curvature $\geq 0$ everywhere,* i.e. *whose Ricci tensor is positive semidefinite everywhere. Let $\Gamma$ be a quotient group of a subgroup of the fundamental group $\pi_1(M)$.*

1. *If $\Gamma$ is a finitely generated solvable group, then $\Gamma$ has a finitely generated nilpotent subgroup of finite index.*
2. *If $\Delta$ is a finitely generated nilpotent subgroup of $\Gamma$, then $E_1(\Delta) \leq n$.*

*Proof.* In the guise of the fact that any exponential map $\exp_x\colon \tilde{M}_x \to \tilde{M}$ of the universal covering must be volume decreasing, J. Milnor uses the completeness and non-negative mean curvature on $M$ to prove [9, Theorem 1]

(5.2) *every finitely generated subgroup of $\pi_1(M)$ has polynomial growth of degree $\leq n$.*

Let $\Gamma$ be a quotient group of a subgroup $\Psi \subset \pi_1(M)$. If a subgroup $\Sigma \subset \Gamma$ has a finite set $S$ of generators,, then we take an arbitrary finite set $H \subset \Psi$ which projects to $S$, and observe $g_S(m) \leq g_H(m)$. Now (5.2) says

(5.3) *every finitely generated subgroup of $\Gamma$ has polynomial growth of degree $\leq n$.*

If $\Gamma$ is finitely generated and solvable then (5.3) and Theorem 4.8 force $\Gamma$ to have a (necessarily finitely generated) nilpotent subgroup of finite index. If $\Delta \subset \Gamma$ is a finitely generated nilpotent group, then (5.3) and Theorem 3.2 say $E_1(\Delta) \leq n$.

**5.4. Corollary.** *Let $M$ be a complete riemannian manifold of non-negative mean curvature with $\pi_1(M)$ finitely generated and solvable. Then $\pi_1(M)$ has a nilpotent subgroup of finite index, and every nilpotent subgroup $\Delta \subset \pi_1(M)$ satisfies $E_1(\Delta) \leq \dim M$.*

In order to take advantage of Mostows's result, written as (9) $\Rightarrow$ (6) in Proposition 4.1, we need

**5.5. Proposition.** *Let $\tilde{M}$ be a connected simply connected homogeneous riemannian manifold. Then the Lie group of all isometries of $\tilde{M}$ has only a finite number of topological components.*

*Proof.* Let $G$ be the isometry group of $\tilde{M}$, $x \in \tilde{M}$, and $K$ the isotropy subgroup of $G$ at $x$. Then $\tilde{M}$ is diffeomorphic to $G/K$; so $K$ meets every component of $G$. Thus it suffices to show that $K$ has only a finite number of components.

Let $\mathscr{R}$ be the curvature tensor on $\tilde{M}$, and $\chi$ the linear isotropy representation of $K$ on the tangent space $\tilde{M}_x$. Then $\chi(K)$ is the set of all linear isometries of $\tilde{M}_x$ which preserve every covariant differential $(\nabla^m\mathscr{R})_x$, $m \geq 0$; so $\chi(K)$ is a closed subgroup of the orthogonal group of $\tilde{M}_x$. Thus $\chi(K)$ has only finitely many components. As $\chi$ is a faithful representation it follows that $K$ has only finitely many components.

**5.6. Corollary.** *Let $M$ be a complete connected locally homogeneous riemannian manifold. Then every solvable subgroup of $\pi_1(M)$ is polycyclic. In particular every solvable subgroup of $\pi_1(M)$ is finitely generated.*

*Proof.* Let $p: \tilde{M} \to M$ be the universal riemannian covering. Then $\tilde{M}$ is homogeneous. Let $G$ be its isometry group. Then $M = \Gamma\backslash\tilde{M}$ where $\Gamma$ is a discrete subgroup of $G$, and $\Gamma \cong \pi_1(M)$. Let $\Sigma$ be a solvable subgroup of $\Gamma$. As $G$ has only finitely many components by Proposition 5.5, condition (9) of Proposition 4.1 says that $\Sigma$ is polycyclic. q.e.d.

We combine Theorem 5.1 and Corollary 5.9:

**5.7. Theorem.** *Let $M$ be a complete connected locally homogeneous riemannian manifold with mean curvature $\geq 0$ everywhere. Then*

(1) *every solvable subgroup of $\pi_1(M)$ has a nilpotent subgroup of finite index,*

(2) *every nilpotent subgroup $\Delta \subset \pi_1(M)$ is finitely generated and satisfies $E_1(\Delta) \leq \dim M$.*

We end this section by writing down a consequence of Theorem 5.7, which we will need in [19].

**5.8. Corollary.** *Let $M$ be a compact connected locally homogeneous riemannian manifold with mean curvature $\geq 0$ everywhere. Suppose that $\pi_1(M)$ has a solvable subgroup of finite index. Let $\tilde{M} \to M$ be the universal riemannian covering, $G$ the largest connected group of isometries of $\tilde{M}$, $R$ the solvable radical of $G$, and $S$ a semisimple complement so that $G = RS$. Then $S$ is compact. Let $Z$ be the center of $R$, and $N$ the nilpotent radical of $G$. Then $R/ZN$ is compact.*

*Proof.* Let $G^*$ be the full group of isometries of $\tilde{M}$, and $\Gamma \subset G^*$ the discrete subgroup such that $M = \Gamma\backslash\tilde{M}$. As $\Gamma = \pi_1(M)$, Theorem 5.7(1) says that $\Gamma$ has a nilpotent subgroup $\Delta$ of finite index. $G^*$ having only a finite number of components by Proposition 5.5, we may assume $\Delta$ contained in the identity component $G$ of $G^*$. As $M = \Gamma\backslash\tilde{M}$ is compact, $G^*/\Gamma$ is compact; so $G/\Delta$ is compact.

Let $R$ be the solvable radical of $G$. Choose a semisimple (Levi-Whitehead) complement $S$ to $R$ in $G$. Let $S_1$ be the subgroup of $S$ generated by the compact simple normal analytic subgroups, and $S_2$ the subgroup of $S$ generated by the noncompact simple normal analytic subgroups of $S$. Then $G = R \cdot S$, almost a semidirect product except that $R \cap S$ is discrete but possibly nontrivial, and $S = S_1 \cdot S_2$ local direct product.

We use compactness of $G/\Delta$ to prove $S$ compact. Define $\Sigma = \Delta \cap (R \cdot S_1)$ and $\Psi = \Delta/\Sigma$. A result of L. Auslander [2, Theorem 1] says[4] that $\Sigma \subset R \cdot S_1$ and $\Psi \subset G/R \cdot S_1$ are discrete subgroups with $(R \cdot S_1)/\Sigma$ and $(G/R \cdot S_1)/\Psi$ compact. Let $L = S_2/(RS_1 \cap S_2) = G/R \cdot S_1$, semisimple Lie group of noncompact type. Now $\Psi$ is a nilpotent discrete subgroup such that $L/\Psi$ is compact. The latter implies [14, Theorem 3.2] that every $\psi \in \Psi$ is a semisimple element of $L$. It follows ([4, Theorem 7.6], or see [14, Remark 3.8]) that $\Psi$ normalizes a Cartan subgroup $C \subset L$. As $L/$(normalizer of $C$ in $L$) is noncompact whenever $L \neq \{1\}$ we conclude that $L = \{1\}$. Thus $S_2 = \{1\}$ and $S$ is compact.

Now we must establish some notation. $N$ is the nilpotent radical of $R$ and also of $G$. $Z$ is the center of $R$, and $T$ is a maximal compact subgroup of $R$, necessarily a torus. Let $T_N$ be a maximal compact subgroup of $N$. Then $T_N$ is central in $G$ and hence contained in $Z$, and $T = T_N \times T_{R/N}$ where $T_{R/N}$ acts effectively on $\mathfrak{R}$ under the adjoint representation of $R$. Let $Z_0$ be the identity component of $Z$. Then $T_N \subset Z_0 \subset N$; it follows that $Z_0 = Z \cap N$ by a glance at the action of $R$ on $N/T_N$.

Let $F$ be the identity component of the closure of $R\Delta$ in $G$. We check that there is a torus subgroup $U \subset S$ such that $F = R \cdot U$. For let $\pi \colon G' \to G$ be the universal covering group, $R'$ the radical of $G'$, and $S'$ a semisimple complement such that $\pi(S') = S$. Then $\Delta' = \pi^{-1}(\Delta)$ is discrete in $G'$ with compact quotient. Let $F'$ be the identity component of the closure of $R'\Delta'$ in $G'$. As $R'$ is simply connected, $F'$ is solvable [2, Proposition 2]; so $F'/R'$ is a torus. But $F'/R' = (S' \cap F')/(S' \cap R') = S' \cap F'$. Thus $F' = R' \cdot U'$ where $U' = S' \cap F'$ is a torus. Now $F = \pi(F') = \pi(R') \cdot \pi(U') = R \cdot U$ where $U = \pi(U')$ is a torus subgroup of $S$.

Let $Z_F$ be the center of $F$ and $N_F$ the nilpotent radical of $F$. Then $Z_F N_F$ is a closed normal subgroup of $F$. We check that $F/Z_F N_F$ is a torus group. Let $\varphi \colon F'' \to F$ be the universal covering group, $\Delta'' = \varphi^{-1}(\Delta \cap F)$, $Z_F''$ the center of $F''$ and $N_F''$ the nilpotent radical of $F$. $\Delta \cap F$ is a discrete nilpotent subgroup of $F$ with compact quotient, and the kernel of $\varphi$ is a discrete central subgroup of $F''$; thus $\Delta''$ is a discrete nilpotent subgroup of $F''$ with compact quotient. $F''$ is torsion free; so $\Delta'' \cap N_F''$ and $\Delta''$ are torsion free. $F''/N_F''$ is a real vector

[4] Auslander states his result only for simply connected $G$ but it follows in general by a glance at the universal covering group.

group and $\Delta''/(\Delta'' \cap N''_F)$ is a discrete subgroup with compact quotient. Now application of Proposition 4.4(2) to the exact sequence

$$\{1\} \to \Delta'' \cap N''_F \to \Delta'' \to \Delta''/(\Delta'' \cap N''_F) \to \{1\}$$

proves for every $\delta \in \Delta''$ that $ad(\delta)|_{\mathfrak{N}''_F}$ is a unipotent linear transformation. Let $\beta\colon F'' \to F''/Z''_F N''_F$ be the projection. Tnen $\beta(\Delta'')$ is discrete with compact quotient in the abelian ($[F'', F''] \subset N''_F$) group $\beta(F'')$. Thus $\beta(F'') = V^* \times T^*$, product of a vector group and a torus, where $V^*/(V^* \cap \beta(\Delta''))$ is compact. If $f \in \beta^{-1}(V^*)$, then $ad(f)|_{\mathfrak{N}''_F}$ is unipotent. Thus $\beta^{-1}(V^*)_0$ is a connected normal nilpotent subgroup of $F''$ containing $N''_F$; this shows $\beta^{-1}(V^*)_0 = N''_F$; so $V^*$ is a point and $F''/Z''_F N''_F$ is a torus. As $\varphi(F'') = F$, $\varphi(Z''_F) = Z_F$ and $\varphi(N''_F) = N_F$, now $F/Z_F N_F$ is a torus.

Recall the center $Z$ and the nilpotent radical $N$ of $R$. $Z_F \subset Z \cdot U$ and $N_F \subset N \cdot U$; so $Z_F N_F \subset ZNU$. Thus $R \cap Z_F N_F \subset ZU$. As $R$ has closed image in the torus $F/Z_F N_F$, it follows that $R/ZN$ is a torus group.

**Remark.** In the above notation, $N \cdot T$ is the semidirect product $N \cdot T_{R/N}$, and it is a normal subgroup of $R$ because $[R, R] \subset N$. It follows that $V = R/NT$ is a real vector group. Notice

$$Z/Z_0 \cong ZN/N \cong (ZNT)/(NT) \qquad \text{free abelian} .$$

Now $V$ contains $(ZNT)/(NT) \cong Z/Z_0$ as a lattice, for

$$R/ZN \to R/(ZNT) \cong V/\{(ZNT)/(NT)\}$$

is a surjective Lie group homomorphism of torus groups, with kernel $(ZNT)/(ZN) \cong T_{R/N}$. In [19] we study a condition on the metric of $M$ which, if it holds, forces $V$ to be trivial, so that $R/N$ is compact.

## 6. Application to generalized nilmanifolds

Let $G$ be a connected Lie group, and Aut $(G)$ denotes its automorphism group. We can view Aut $(G)$ as the group of all automorphisms of the Lie algebra $\mathfrak{G}$ of $G$ which preserve the kernel of the universal Lie group covering $\tilde{G} \to G$. So Aut $(G)$ is a real linear algebraic group. In particular it is a Lie group with only finitely many components. Thus every compact subgroup of Aut $(G)$ is contained in a maximal compact subgroup, and any two maximal compact subgroups are conjugate.

The *affine group* $\mathbf{A}(G)$ is defined to be the semidirect product $G \cdot \text{Aut}\,(G)$. It has manifold $G \times \text{Aut}\,(G)$ and group law $(g, \alpha)(h, \beta) = (g \cdot \alpha(h), \alpha\beta)$ for $g, h \in G$ and $\alpha, \beta \in \text{Aut}\,(G)$. $\mathbf{A}(G)$ acts effectively and differentiably on $G$ by $(g, \alpha)\colon x \to g \cdot \alpha(x)$.

Let $K$ be a maximal compact subgroup of Aut$(G)$. Then the *euclidean group* $\mathbf{E}(G) = G \cdot K$ is a closed Lie subgroup of the affine group $\mathbf{A}(G)$. As any two choices of $K$ are conjugate in Aut $(G)$, now any two choices of $\mathbf{E}(G)$ are conjugate in $\mathbf{A}(G)$ by an element of the form $(1, \alpha)$.

**6.1. Lemma.** *Let $\Gamma$ be a subgroup of $\mathbf{E}(G)$.*

(1) *$\Gamma$ is a discrete subgroup of $\mathbf{E}(G)$ if and only if the action of $\Gamma$ on $G$ is properly discontinuous.*

(2) *Suppose that $\Gamma$ is closed in $\mathbf{E}(G)$. Then the coset space $\mathbf{E}(G)/\Gamma$ is compact if and only if the identification space $\Gamma\backslash G$ is compact.*

(3) *Suppose that $G$ is acyclic*[5] *and $\Gamma$ is discrete in $\mathbf{E}(G)$. Then $\Gamma$ acts freely on $G$ if and only if $\Gamma$ is torsionfree.*

*Proof.* $\Gamma$ is discrete in $\mathbf{E}(G)$ if and only if its action on $\mathbf{E}(G)$ by left translation is properly discontinuous. Now (1) follows from the fact that the map $\mathbf{E}(G) \to G$, by $(g, \alpha) \to g$, is proper and $\Gamma$-equivariant; this also proves (2) by showing $\Gamma\backslash\mathbf{E}(G) \to \Gamma\backslash G$ to be an induced proper map. And for (3) we use the fact [15] that every compact subgroup of $\mathbf{E}(G)$ is conjugate to a subgroup of $K$. q.e.d.

Finally we define a *crystallographic group on $G$* to be a discrete subgroup $\Gamma$ of $\mathbf{E}(G)$ such that $\Gamma\backslash G$ is compact.

If it happens that $G$ is the $n$-dimensional real vector group $\mathbf{R}^n$, then Aut $(G)$ $= \mathbf{GL}(n, \mathbf{R})$ general linear group, and its maximal compact subgroup is just (a choice of) the orthogonal group $\mathbf{O}(n)$. Then $\mathbf{A}(G)$ is the usual affine group $\mathbf{A}(n) = \mathbf{R}^n \cdot \mathbf{GL}(n, \mathbf{R})$, $\mathbf{E}(G)$ is the usual euclidean group $\mathbf{E}(n) = \mathbf{R}^n \cdot \mathbf{O}(n)$, Lemma 6.1 is classical, and "crystallographic group" has its usual meaning.

Let $M$ be a differentiable manifold. Then $M$ is called a *nilmanifold* if it is diffeomorphic to a coset space $\Gamma\backslash N$ of a connected simply connected nilpotent Lie group $N$ by a discrete subgroup $\Gamma$. $M$ is called a *generalized nilmanifold* if it is diffeomorphic to an identification space $\Gamma\backslash N$, where $N$ is a connected simply connected nilpotent Lie group and $\Gamma$ is a subgroup of $\mathbf{E}(N)$ that acts freely and properly discontinuously on $N$; then

(i) $\Gamma \cong \pi_1(M)$ because $N$ is simply connected,

(ii) $M$ is a nilmanifold if and only if $(N, \Gamma)$ can be chosen so that $\Gamma \subset N \subset \mathbf{E}(N)$.

In the above definitions, simply connectivity of $N$ is just a convenience. For suppose $N' = N/\Delta$ is a nilpotent Lie group with universal covering $\pi: N \to N'$, and $\Gamma' \subset \mathbf{E}(N')$ is a closed subgroup. Then $\pi$ induces a diffeomorphism of $\pi^{-1}(\Gamma')\backslash N$ onto $\Gamma'\backslash N'$, $\Gamma'$ acts freely and properly discontinuously on $N'$ if and only if $\pi^{-1}(\Gamma')$ has the same property on $N$, and $\Gamma' \subset N'$ if and only if $\pi^{-1}(\Gamma') \subset N$.

The euclidean space forms (see [18, Chapter 3]) are the generalized nilmanifolds $M \cong \Gamma\backslash\mathbf{R}^n$ where $M$ is equipped with a flat riemannian metric induced from $\mathbf{R}^n$. L. Auslander [1] has extended the Bieberbach Theorems (see [18,

[5] $G$ is acyclic if and only if $G = S\cdot(L \times L \times \cdots \times L)$ where $S$ is a simply connected solvable Lie group and each of the $\geq 0$ copies of $L$ is the universal covering group of $\mathbf{SL}(2, \mathbf{R})$. See [15].

Chapter 3]) on compact euclidean space forms to a structure theory for compact nilmanifolds. His result can be stated as:

**6.2. Proposition.** *Let $\Gamma$ be a group. Then $\Gamma$ is isomorphic to the fundamental group of a compact generalized nilmanifold, if and only if* (i) *$\Gamma$ is finitely generated and torsionfree, and* (ii) *there is an exact sequence*

$$\{1\} \to \Delta \to \Gamma \to \Psi \to \{1\}$$

*where $\Psi$ is finite and $\Delta$ is a maximal nilpotent subgroup of $\Gamma$. In that case $\Delta$ is unique, there is an unique realization (up to isomorphism of $N$) of $\Gamma$ as fundamental group of a compact generalized nilmanifold $\Gamma\backslash N$, and $\Delta = \Gamma \cap N$.*

In the case of a compact euclidean space form, which is the case where $\Delta$ is commutative, $\Psi = \Gamma/\Delta$ is the linear holonomy group.

L. Auslander has suggested that a commpact nilmanifold $M = \Gamma\backslash N$, which admits a riemannian metric[6] $ds^2$ with all sectional curvatures of one sign, must be diffeomorphic to a torus. That conjecture is contained in the following application of our results of §§ 2 and 5.

**6.3. Theorem.** *Let $M$ be a compact generalized nilmanifold, say $M = \Gamma\backslash N$. Let $ds^2$ be any*[6] *riemannian metric on $M$.*

(1) *The following conditions are equivalent, and each implies that $N$ is a real vector group.*

(1a) *$(M, ds^2)$ has every sectional curvature $\leq 0$.*

(1b) *$(M, ds^2)$ has every sectional curvature $= 0$.*

(1c) *$(M, ds^2)$ has every sectional curvature $\geq 0$.*

(1d) *$(M, ds^2)$ is isometric to a compact euclidean space form.*

(2) *Under the conditions of* (1), *the following conditions are equivalent.*

(2a) *$M$ is a nilmanifold.*

(2b) *The fundamental group $\pi_1(M)$ is nilpotent.*

(2c) *$(M, ds^2)$ is isometric to a flat riemannian torus.*

**6.4. Theorem.** *Let $M$ be a compact generalized nilmanifold, say $M = \Gamma\backslash N$. Let $ds^2$ be any*[6] *riemannian metric on $M$. Suppose that $(M, ds^2)$ has mean curvature $\geq 0$ everywhere. Then $N$ is a real vector group, $M$ is diffeomorphic to a compact euclidean space form, and the following conditions are equivalent:* (a) *$M$ is a nilmanifold,* (b) *$\Gamma = \pi_1(M)$ is nilpotent, and* (c) *$M$ is diffeomorphic to a torus.*

*Proof.* As described in Propositon 6.2 we have an exact sequence $\{1\} \to \Delta \to \Gamma \to \Psi \to \{1\}$, where $\Delta = \Gamma \cap N$ is a normal subgroup of finite index in $\Gamma$, and $\Delta = \Gamma$ if and only if $\Gamma$ is nilpotent. We thus have a finite normal riemannian covering $\pi\colon (M', ds'^2) \to (M, ds^2)$, where $M' = \Delta\backslash N$, $\pi(\Delta n) = \Gamma n$, and $ds'^2 = \pi^* ds^2$. Hence $M'$ is a compact nilmanifold.

[6] We do not assume $ds^2$ to come from a left invariant riemannian metric on $N$. Its only presumed invariance property is the trivial one, namely, its lift to $N$ is $\Gamma$-invariant, i.e. is the lift of a riemannian metric on $M$.

First suppose that $(M, ds^2)$ has every sectional curvature $\leq 0$. Then the same is true for $(M', ds'^2)$, which has nilpotent fundamental group $\Delta$; so Theorem 2.1 tells us that $(M', ds'^2)$ is a flat riemannian torus and that $\Delta$ is free abelian on $(\dim M)$ generators. In particular $(M, ds^2)$ is flat. Let $Z$ be the center of $N$. Then $Z \cap \Delta$ is the center of $\Delta$; so $\Delta \subset Z$; as $\Delta \backslash N$ is compact and $Z \backslash N$ is acyclic, now $Z = N$; thus $N$ is a real vector group.

Second suppose that $(M, ds^2)$ has mean curvature $\geq 0$ everywhere. Then Theorem 5.1 says $E_1(\Delta) \leq \dim M = \dim N$. We compare the lower central series

$$\Delta = \Delta_0 \supsetneqq \Delta_1 \supsetneqq \cdots \supsetneqq \Delta_s \supsetneqq \Delta_{s+1} = \{1\} , \qquad \Delta_{k+1} = [\Delta, \Delta_k] ,$$

$$N = N_0 \supsetneqq N_1 \supsetneqq \cdots \supsetneqq N_t \supsetneqq N_{t+1} = \{1\} , \quad N_{k+1} = [N, N_k] ,$$

and use the fact that $Z \cap \Delta$ is the center of $\Delta$, where $Z$ is the center of $N$. It follows by induction on $t$ that (i) $s = t$ and (ii) each $\Delta_k \backslash N_k$ is compact. Thus $\Delta_k / \Delta_{k+1}$ has free abelian rank equal to the dimension of $N_k / N_{k+1}$; so

$$E_1(\Delta) = \sum_{k=0}^{s} (k + 1) \dim (N_k / N_{k+1}) .$$

Now $E_1(\Delta) \leq \dim N$ implies $k + 1 = 1$ for $0 \leq k \leq s$, proving $s = 0$; so $N$ is a real vector group. Let $d\tau^2$ be any $\Gamma$-invariant $N$-invariant riemannian metric on $N$, and $d\sigma^2$ the metric induced on $M$. Then the identity map is a diffeomorphism of $M$ onto the compact euclidean space form $(M, d\sigma^2)$.

Third suppose that $(M, ds^2)$ has every sectional curvature $\geq 0$. Then a theorem of Cheeger and Gromoll [5] says that the universal riemannian covering manifold of $(M, ds^2)$ is the product of an euclidean space and a compact simply connected manifold. It follows that $(M, ds^2)$ is flat. As $(M, ds^2)$ has every mean curvature $\geq 0$, it also follows, as just seen, that $N$ is a real vector group.

To complete the poof of Theorem 6.3, we notice that (1a) and (1c) each implies that $(M, ds^2)$ is flat and that $N$ is a real vector group, thus implying (1b). As (1b) classically implies (1d), which in turn implies both (1a) and (1c), now part 1 of Theorem 6.3 is proved. Proposition 6.2 gives equivalence of (2a), (2b) and the condition that $\Delta = \Gamma$. Under the conditions of (1), $(M', ds'^2)$ is a flat riemannian torus. Thus $\Delta = \Gamma$ if and only if $(M, ds^2)$ is a flat riemannian torus, and hence part 2 of Theorem 6.3 is proved.

To complete the proof of Theorem 6.4, we recall that $N$ is a real vector group and that $M$ is diffeomorphic to a compact euclidean space form $(M, d\sigma^2)$. Thus Proposition 6.2 gives equivalence of (a), (b) and the condition that $\Delta = \Gamma$. But $\Delta = \Gamma$ if and only if $M$ is diffeomorphic (in fact equal) to the torus $M'$. Hence Theorem 6.4 is proved.

## Added in Proof

1. B. Hartley recently sent me a manuscript consisting of an alternate proof that a polycyclic group without nilpotent subgroups of finite index is of exponential growth, as well as an interesting bound $g_S(m) \leq cm^{F(\Gamma)}$ in case $\Gamma$ is a finitely generated nilpotent group. In the latter one considers the derived series $\Gamma = \Gamma^0 \supset \Gamma^1 \cdots \supset \Gamma^k \supsetneqq \Gamma^{k+1} = \{1\}$ and defines $F(\Gamma) = f_0 + c_0 f_1 + c_1 f_2 + \cdots + c_{k-1} f_k$ where (i) $\Gamma^i$ is nilpotent of class $c_i$ and (ii) $\Gamma^i/\Gamma^{i+1}$ has free abelian part of rank $f_i$. In this connection see the Remark just after the proof of Theorem 3.2.

2. Jeff Cheeger recently sharpened the bound $E_1(\Delta) \leq n$ of Theorem 5.1 to $E_1(\Delta) < n$ in the case of strictly positive mean curvature. That sharpens corresponding statements of Corollary 5.4 and Theorem 5.7 in case the mean curvature is strictly positive.

3. M. Shub has given a short exposition [*Expanding maps*, Proceedings of the 1968 Summer Institute on Global Analysis, Amer. Math. Soc., to appear] on the relations between growth estimates, expanding maps, Anasov diffeomorphisms and generalized nilmanifolds. The main problem there is the one mentioned in the last paragraph of our Introduction.

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