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COMPACT REAL FORMS OF A COMPLEX SEMI-SIMPLE LIE ALGEBRA

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Introduction

This paper gives a new proof of an old theorem, the existence of a compact real form of a complex semi-simple Lie algebra. The theorem is a consequence of the classification of real simple Lie algebras by E. Cartan in 1914 [1]. Later H. Weyl [8] gave an intrinsic proof based on the detailed structure theory of semi-simple Lie algebras. Our proof, which is based on a suggestion of Cartan [2, p. 23], is geometric in nature. The *only* results from the theory of Lie algebras which we have used are the facts that if $\mathfrak{g}$ is a semi-simple Lie algebra, then the center of $\mathfrak{g}$ is $\{0\}$ and every derivation of $\mathfrak{g}$ is inner. On the debit side, however, our proof uses an elementary lemma from algebraic geometry and does involve one long and unedifying computation.

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1. Preliminaries

$\boldsymbol{R}$ (resp. $\boldsymbol{C}$) denotes the field of real (resp. complex) numbers. If S is a set, then S^m denotes the m-fold Cartesian product $S \times \cdots \times S$. N_n denotes the set $\{1, \cdots, n\}$. If W is a vector space over $\boldsymbol{C}$, then $W^{\boldsymbol{R}}$ is the real vector space obtained from W by restriction of scalars. If V is a vector space over $\boldsymbol{C}$, then $A^m(V, V)$ denotes the vector space of all alternating m-linear maps of V^m into V.

Let $B = \{e_1, \cdots, e_n\}$ be a basis of V. If $\varphi \varepsilon A^m(V, V)$, we write $\varphi(e_{a_1}, \cdots, e_{a_m}) = \sum_{j=1}^{m} (\varphi_{a_1 \cdots a_m j}) e_j$. The $\varphi_{a_1 \cdots a_{m+1}}$ are the "coordinates" of φ with respect to the basis B, and we often write $\varphi = \varphi(_{a_1 \cdots a_{m+1}})$. The basis B determines a positive definite Hermitian inner product on $A^m(V, V)$ as follows: If φ, $\psi \varepsilon A^m(V, V)$, then $\langle \varphi, \psi \rangle = \sum_{\boldsymbol{a}} \varphi_{\boldsymbol{a}} \bar{\psi}_{\boldsymbol{a}}$, where the sum is taken over all $\boldsymbol{a} = (a_1, \cdots, a_{m+1}) \,\varepsilon\, (N_n)^{m+1}$ and the bar denotes complex conjugation. Let $\langle \varphi, \psi \rangle_r$ denote the real part of the complex number $\langle \varphi, \psi \rangle$. Then $(\varphi, \psi) \to \langle \varphi, \psi \rangle_r$ is a positive definite inner product on the real vector space $A^m(V, V)^{\boldsymbol{R}}$. For $\varphi \varepsilon A^m(V, V)$ we write $\|\varphi\|^2 = \langle \varphi, \varphi \rangle = \langle \varphi, \varphi \rangle_r$.

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Unless stated otherwise, all summation indices are understood to range from 1 to n. Thus, for example, in the expression $\sum_{(p,q)} \varphi_{pqr} \psi_{pqr}$, the sum is taken over all $(p, q) \in (N_n)^2$.

2. Outline of the proof

A Lie algebra $\mathfrak{s} = (V, \sigma)$ consists of a vector space V and an alternating bilinear map $\sigma: V \times V \to V$ which satisfies the Jacobi identity. We denote by B_σ the Cartan-Killing form of (V, σ). The Lie algebra $\mathfrak{s}$ is *semi-simple* if the form B_σ is non-degenerate. Now let $\mathfrak{g} = (V, \mu)$ be a finite-dimensional semi-simple Lie algebra over C. Then the Cartan-Killing form B_μ is non-degenerate, and consequently we may choose a basis $B = \{e_1, \cdots, e_n\}$ of V such that $B_\mu(e_p, e_q) = -\delta_{pq}$ (δ_{pq} is the Kronecker delta). Henceforth we shall consider $A^m(V, V)$ and $A^m(V, V)^R$ to be given the inner products determined by the basis B. We denote by V_0 the vector subspace of V^R spanned by $\{e_1, \cdots, e_n\}$, and let $A^m(V_0, V_0)$ be the vector subspace of $A^m(V, V)^R$ consisting of all φ such that $\varphi(V_0{}^m) \subset V_0$. Suppose that $\varphi = (\varphi_{a_1, \cdots, a_{m+1}})$. Then $\varphi \varepsilon A^m(V_0, V_0)$ if and only if each $\varphi_{a_1 \cdots a_{m+1}}$ is real. Let $\varphi = \varphi_1 + i\varphi_2$ and $\psi = \psi_1 + i\psi_2$ be elements of $A^m(V, V)$ with $\varphi_1, \varphi_2, \psi_1, \psi_2 \varepsilon A^m(V_0, V_0)$. Then one checks easily that

$$\langle \varphi, \psi \rangle_r = \langle \varphi_1, \psi_1 \rangle_r + \langle \varphi_2, \psi_2 \rangle_r .$$

Let $\mathscr{M}$ be the algebraic set in $A^2(V, V)$ consisting of all Lie algebra multiplications, and $\mathscr{N}$ the set of all $\psi \in \mathscr{M}$ which satisfy the following conditions: (1) (V, ψ) is isomorphic to $\mathfrak{g}$ and (2) $B_\mu = B_\psi$. In order to show that $\mathfrak{g}$ admits a compact real form it suffices to show that $\mathscr{N} \cap A^2(V_0, V_0)$ is non-empty. For suppose that there exists $\eta \in \mathscr{N} \cap A^2(V_0, V_0)$, and let $\eta_0: V_0 \times V_0 \to V_0$ denote the restriction of η. Then (V_0, η_0) is a real form of (V, η), and the Cartan-Killing form of (V_0, η_0) is negative definite. By a well-known theorem (see [4, p. 122]) this implies that (V_0, η_0) is a compact Lie algebra and hence is a compact real form of (V, η). Since $\mathfrak{g}$ is isomorphic to (V, η), $\mathfrak{g}$ has a compact real form.

Now let $\eta = (\eta_{pqr}) \in \mathscr{N}$. We have

$$\begin{aligned} \eta_{pqr} &= -B_\eta(\eta(e_p, e_q), e_r) \\ &= -B_\eta(e_p, \eta(e_q, e_r)) \\ &= \eta_{qrp} . \end{aligned}$$

Since we also have $\eta_{pqr} = -\eta_{qpr}$, it follows that η_{pqr} is skew-symmetric in (p, q, r). By definition we have

$$(2.1) \qquad \delta_{pq} = -B_\eta(e_p, e_q) = -\sum_{(r,s)} \eta_{qsr} \eta_{prs} = \sum_{(r,s)} \eta_{qrs} \eta_{prs} .$$

Hence we have $\sum_{(p,q,r)} (\eta_{pqr})^2 = n$. By considering real and imaginary parts of η_{pqr} we see that

$$(2.2) \qquad \| \eta \|^2 = \sum_{(p,q,r)} | \eta_{pqr} |^2 \geq n \,.$$

Furthermore we have equality in (2.1) if and only if $\eta \varepsilon A^2(V_0, V_0)$.

This suggests that we try to minimize the function $\psi \mapsto \| \psi \|^2$ on $\mathscr{N}$. In § 3 we shall show that $\mathscr{N}$ is a closed set. Hence the function $\psi \mapsto \| \psi \|^2$ attains a minimum on $\mathscr{N}$. In § 5 we shall show that if this function has a minimum at $\eta \varepsilon \mathscr{N}$, then $\eta \varepsilon A^2(V_0, V_0)$. This will prove the existence of a compact real form of $\mathfrak{g}$.

3. First proof that $\mathscr{N}$ is closed

There is a natural representation ρ of the general linear group $G = GL(V)$ on $A^2(V, V)$ defined as follows: If $\varphi \varepsilon A^2(V, V)$ and $g \varepsilon G$, then

$$(3.1) \qquad (\rho(g) \cdot \varphi)\,(x, y) = g(\varphi(g^{-1}x, g^{-1}y))$$

for $x, y \varepsilon V$. The set $\mathscr{M}$ of Lie algebra multiplications on V is stable under the corresponding action of G and the orbits of G on $\mathscr{M}$ are just the isomorphism classes of Lie algebra structures on V. Let $H \subset GL(V)$ be the group of all automorphisms of the form B_μ, and $\mathfrak{h}$ the Lie algebra of H. In terms of matrices with respect to the basis $\{e_1, \cdots, e_n\}$, H is the group of all complex orthogonal matrices and $\mathfrak{h}$ is the Lie algebra of all complex skew-symmetric matrices. Let $g \varepsilon G$ and let $\varphi = \rho(g) \cdot \mu$. Then $B_\varphi(x, y) = B_\mu(g^{-1}x, g^{-1}y)$. Thus $B_\varphi = B_\mu$ if and only if $g \varepsilon H$. Hence we see that $\mathscr{N}$ is just the orbit $H(\mu)$.

Now let $\mathscr{N}'$ be the set of all $\psi \varepsilon M$ such that $B_\psi = B_\mu$. It is easy to see that $\mathscr{N}'$ is an algebraic set in $A^2(V, V)$. Let $\sigma \varepsilon N'$ and let $\mathfrak{s} = (V, \sigma)$. Since $B_\sigma = B_\mu$, $\mathfrak{s}$ is a semi-simple Lie algebra. The isotropy group at σ for the action of G determined by ρ is just the group $\mathrm{Aut}(\mathfrak{s})$ of all automorphisms of $\mathfrak{s}$. It follows from standard properties of the Cartan-Killing form that $\mathrm{Aut}(\mathfrak{s}) \subset H$. The Lie algebra of $\mathrm{Aut}(\mathfrak{s})$ is the Lie algebra $\mathrm{Der}(\mathfrak{s})$ of all derivations of $\mathfrak{s}$. Since $\mathfrak{s}$ is semi-simple, the center of $\mathfrak{s}$ is $\{0\}$ and every derivation of $\mathfrak{s}$ is inner. Thus $\dim \mathrm{Aut}(\mathfrak{s}) = \dim \mathrm{Der}(\mathfrak{s}) = \dim V$ (here dim denotes the complex dimension). Consequently $\dim H(\sigma) = \dim H - \dim \mathrm{Aut}(\mathfrak{s}) = \dim H - \dim V$. Thus we see that all orbits of H on $\mathscr{N}'$ have the same dimension.

We shall need the following lemma, which is a special case of an elementary result from the theory of algebraic groups.

Let $\tau : S \to GL(W)$ be a rational representation of a linear algebraic group S(over C), X be an algebraic set in W which is stable under the corresponding action of S, and $x \varepsilon X$ be such that $\dim S(x) \leq \dim S(y)$ *for every $y \varepsilon X$. Then the orbit $S(x)$ is closed.*

For a proof, see [1, Prop. 15.4].

Since the representation of H defined by ρ is rational, it follows from this lemma that each orbit of H on $\mathscr{N}'$ is closed. In particular, $\mathscr{N} = H(\mu)$ is closed.

4. Second proof that $\mathscr{N}$ is closed

For the benefit of the reader who wishes to avoid the use of algebraic geometry, we give an alternate proof that $\mathscr{N}$ is closed.

A Lie algebra $s = (V, \sigma)$ is *rigid* if the oribt $G(\sigma)$ is open in $\mathscr{M}$. Nijenhuis and Richardson have proved that a semi-simple Lie algebra is rigid [6, § 7]. We shall give the proof here for the sake of completeness.

First we introduce some notation. If $\varphi, \psi \in A^2(V, V)$, we define $\varphi \barwedge \psi \in A^3(V, V)$ by

$$\varphi \barwedge \psi(x, y, z) = \varphi(\psi(x, y), z) + \varphi(\psi(y, z), x) + \varphi(\psi(z, x), y) .$$

One sees easily that $\varphi \barwedge \varphi = 0$ if and only if φ satisfies the Jacobi identity. For later use it is convenient to express $\varphi \barwedge \psi$ in terms of coordinates. If $\varphi = (\varphi_{pqr})$ and $\psi = (\psi_{pqr})$, then we have

$$(\varphi \barwedge \psi)_{pqrs} = \textstyle\sum_t (\psi_{pqt}\varphi_{trs} + \psi_{rpt}\varphi_{tqs} + \psi_{qrt}\varphi_{tps}) . \tag{4.1}$$

Similarly, if $\tau \in A^1(V, V)$ and $\varphi \in A^2(V, V)$, we define $\varphi \barwedge \tau \in A^2(V, V)$ and $\tau \barwedge \varphi \in A^2(V, V)$ by

$$\varphi \barwedge \tau(x, y) = \varphi(\tau(x), y) - \varphi(\tau(y), x) , \qquad \tau \barwedge \varphi(x, y) = \tau(\varphi(x, y)) .$$

Now let $\mathfrak{s} = (V, \sigma)$ be a Lie algebra. An element $\varphi \in A^2(V, V)$ is a 2–cocycle of $\mathfrak{s}$ if $\sigma \barwedge \varphi + \varphi \barwedge \sigma = 0$. Similarly φ is a 2–coboundary of $\mathfrak{s}$ if there exists $\tau \in A^1(V, V)$ such that $\varphi = \sigma \barwedge \tau - \tau \barwedge \sigma$. $Z^2(\mathfrak{s}, V)$ (resp. $B^2(\mathfrak{s}, V)$) denotes the space of 2–cocycles (resp. 2–coboundaries) of $\mathfrak{s}$. One checks easily that $B^2(\mathfrak{s}, V) \subset Z^2(\mathfrak{s}, V)$. We define $H^2(\mathfrak{s}, V)$, the second cohomology group of $\mathfrak{s}$ with coefficients in V, to be the quotient space $Z^2(\mathfrak{s}, V)/B^2(\mathfrak{s}, V)$.

Proposition 4.1. *If $H^2(\mathfrak{s}, V) = 0$, then $\mathfrak{s}$ is rigid.*

Proof. Define $P: A^2(V, V) \to A^3(V, V)$ by $P(\varphi) = \varphi \barwedge \varphi$. Then $\mathscr{M} = P^{-1}(0)$, We have

$$P(\sigma + \varphi) = \sigma \barwedge \varphi + \varphi \barwedge \sigma + \varphi \barwedge \varphi .$$

Hence dP_σ, the differential of P at σ, is just the map $\varphi \mapsto \sigma \barwedge \varphi + \varphi \barwedge \sigma$. In particular, $Z^2(\mathfrak{s}, V)$ is the kernel of dP_σ. We define $Q: G \to A^2(V, V)$ by $Q(g) = \rho(g) \cdot \sigma$. It follows easily from (3.1) that the differential $dQ_e: A^1(V, V) \to A^2(V, V)$ is the map $\tau \mapsto \sigma \barwedge \tau - \tau \barwedge \sigma$. Since $G(\sigma) \subset \mathscr{M}$ we have $P \circ Q(G) = \{0\}$. Since $H^2(\mathfrak{s}, V) = 0$, the kernel of dP_σ is equal to the image of dQ_e. An elementary argument using the inverse function theorem (see [7, Lemma 1]) shows that $G(\sigma)$ is an open subset of M. It follows immediately that $G(\sigma)$ is open in M.

Proposition 4.2 (*Whitehead*). *If* $\mathfrak{g} = (V, \mu)$ *is semi-simple, then* $H^2(\mathfrak{g}, V) = 0$.

Remark. This is a special case of a theorem due to Whitehead (known in the literature as "Whitehead's second lemma"). For the proof, which is purely computational, see, e.g., [5, p. 89, Lemma 6]. For the case at hand, the proof in [5] can be shortened quite a bit as follows: Choose a basis $\{e_1, \cdots, e_n\}$ of V as in § 1. Then one can show by an easy computation using (2.1) that $-\sum_j ad\, e_j \circ ad\, e_j$ (denoted by Γ in [5]) is equal to 1_V, the identity operator on V.

Corollary 4.3. *A semi-simple Lie algebra over* C *is rigid.*

The proof that $\mathcal{N}$ is closed is now immediate. If $\sigma \in \mathcal{N}'$, then $G(\sigma) \cap \mathcal{N}'$ is relatively open in $\mathcal{N}'$. Hence the complement of $\mathcal{N} = G(\mu) \cap \mathcal{N}'$ is relatively open in $\mathcal{N}'$. Therefore $\mathcal{N}$ is relatively closed in $\mathcal{N}'$ and hence is closed in $A^2(V, V)$.

5. Conclusion of the proof

Let $F: \mathcal{N} \to R$ be defined by $F(\phi) = \langle \phi, \phi \rangle_r$ $(= \|\phi\|^2)$ and assume that F achieves a minimum at η. We may write $\eta = \alpha + i\beta$ with $\alpha, \beta \in A^2(V_0, V_0)$, and wish to show that $\beta = 0$.

Since F has a minimum at η, we must have

$$
\begin{aligned}
0 &= \frac{d}{dt} \langle \rho(\exp tX) \cdot \eta, \rho(\exp tX) \cdot \eta \rangle_r |_{t=0} \\
&= 2 \langle d\rho(X) \cdot \eta, \eta \rangle_r
\end{aligned} \tag{5.1}
$$

for every $X \in \mathfrak{h}^R$, where $\exp: \mathfrak{h}^R \to H$ denotes the expotential map of the (real) Lie group H.

Let $\mathfrak{h}_0 = \mathfrak{h} \cap A^1(V_0, V_0)$. (With respect to the basis B, $\mathfrak{h}_0$ is the Lie algebra of all real skew-symmetric matrices.) The real Lie algebra $\mathfrak{h}^R$ is a vector space direct space direct sum $\mathfrak{h}_0 + i\mathfrak{h}_0$. If $X \in \mathfrak{h}_0$, then it is easy to see that $d\rho(X)$ is skew-symmetric with respect to $\langle\ ,\ \rangle_r$, and hence (5.1) holds. If $X = iY$ with $Y \in \mathfrak{h}_0$, then (5.1) becomes

$$
\begin{aligned}
0 &= \langle d\rho(iY) \cdot \eta, \eta \rangle_r \\
&= \langle i d\rho(Y) \cdot (\alpha + i\beta), \alpha + i\beta \rangle_r \\
&= \langle d\rho(Y) \cdot (i\alpha - \beta), \alpha + i\beta \rangle_r \\
&= -\langle d\rho(Y) \cdot \beta, \alpha \rangle_r + \langle d\rho(Y) \cdot \alpha, \beta \rangle_r .
\end{aligned} \tag{5.2}
$$

Since $d\rho(Y)$ is skew-symmetric with respect to $\langle\ ,\ \rangle_r$, (5.2) gives

$$
\langle d\rho(Y) \cdot \alpha, \beta \rangle_r = 0 \qquad \text{for } Y \in \mathfrak{h}_0 . \tag{5.3}
$$

At this stage it is easier to work in terms of coordinates. Let $Y \in \mathfrak{h}_0$, and (Y_{pq})

be the corresponding skew-symmetric matrix. Since (η_{pqr}) is skew-symmetric in (p, q, r), so are (α_{pqr}) and (β_{pqr}). Equation (5.3) is equivalent to

$$
\begin{aligned}
0 &= \textstyle\sum_{(p,q,r,s)} (Y_{pq}\alpha_{prs}\beta_{qrs} + Y_{pr}\alpha_{qps}\beta_{qrs} - Y_{sp}\alpha_{qrp}\beta_{qrs}) \\
&= 3\textstyle\sum_{(p,q,r,s)} Y_{pq}\alpha_{prs}\beta_{qrs} .
\end{aligned} \tag{5.4}
$$

(The second equality follows from the skew-symmetry of (Y_{pq}), (α_{pqr}) and (β_{pqr}).) Set $S_{pq} = \sum_{(r,s)} \alpha_{prs}\beta_{qrs}$, and let $S = (S_{pq})$. Then (5.4) becomes

$$0 = \textstyle\sum_{(p,q)} Y_{pq}S_{pq} . \tag{5.5}$$

We have

$$\delta_{pq} = -\beta_\eta(e_p, e_q) = \textstyle\sum_{(r,s)} \eta_{qrs}\eta_{prs} .$$

If we take real and imaginary parts of this equation, we obtain

$$\textstyle\sum_{(r,s)} (\alpha_{qrs}\alpha_{prs} - \beta_{qrs}\beta_{prs}) = \delta_{pq}, \tag{5.6}$$

$$\textstyle\sum_{(r,s)} (\alpha_{prs}\beta_{qrs} + \alpha_{qrs}\beta_{prs}) = 0 . \tag{5.7}$$

But (5.7) gives $S_{pq} = - S_{qp}$, and hence S is a skew-symmetric matrix. Thus (5.5) is satisfied for every skew-symmetric Y if and only if S is the zero matrix or, equivalently, if and only if $s = \sum_{(p,q)} (S_{pq})^2$ is equal to 0. We shall show that this implies that $\beta = 0$.

Let $a_{pq} = \sum_{(r,s)} \alpha_{prs}\alpha_{qrs}$, $b_{pq} = \sum_{(r,s)}\beta_{prs}\beta_{qrs}$ and A (resp. B) denote the matrix (a_{pq}) (resp. (b_{pq})). Then A and B are symmetric matrices, $Tr(B) = \| \beta \|^2$ (here Tr denotes the trace), and equation (5.6) becomes

$$A - B = I , \tag{5.8}$$

where I is the $n \times n$ identity matrix. Let $c = Tr(AB)$. Then (5.8) gives $AB = B^2 + B$ and hence, taking traces, we find $c = Tr(B^2) + \| \beta \|^2$. Since B is a real symmetric matrix, $\mathrm{Tr}(B^2) \geq 0$. Consequently, we see that

$$c \geq \| \beta \|^2 . \tag{5.9}$$

Now let $\gamma_{pqrs} = \sum_t \alpha_{pqt}\beta_{trs}$. It follows from the skew-symmetry of (α_{pqr}) and (β_{pqr}) that

$$\gamma_{pqrs} = - \gamma_{qprs} = - \gamma_{pqsr} . \tag{5.10}$$

We also have

$$c = \mathrm{Tr}(AB) = \textstyle\sum_{(p,q,r,s)} (\gamma_{pqrs})^2 . \tag{5.11}$$

We define $e = \sum_{(p,q,r,s)} \gamma_{pqrs}\gamma_{qrps}$. Using (4.1), one checks easily that

$$(\alpha \barwedge \beta)_{pqrs} = \gamma_{rspq} + \gamma_{psqr} + \gamma_{qsrp} \, .$$

Hence we have

$$\begin{aligned} &\| \alpha \barwedge \beta \|^2 \\ &\quad = \textstyle\sum_{(p,q,r,s)} (\gamma_{rspq} + \gamma_{psqr} + \gamma_{qsrp})(\gamma_{rspq} + \gamma_{psqr} + \gamma_{qsrp}) \, . \end{aligned}$$

When we expand this sum we get a sum of nine terms. Using (5.11) we see that three of these terms are equal to c. Furthermore it follows from (5.10) that the remaining six terms are equal to e. Thus we get

$$\| \alpha \barwedge \beta \|^2 = 3c + 6e \, . \tag{5.12}$$

Since η satisfies the Jacobi identity, we have

$$\begin{aligned} 0 = \eta \barwedge \eta &= (\alpha + i\beta) \barwedge (\alpha + i\beta) \\ &= (\alpha \barwedge \alpha - \beta \barwedge \beta) + i(\alpha \barwedge \beta + \beta \barwedge \alpha) \, . \end{aligned}$$

This gives

$$\alpha \barwedge \beta = - \beta \barwedge \alpha \, , \qquad \alpha \barwedge \alpha = \beta \barwedge \beta \, . \tag{5.13}$$

We have

$$(\beta \barwedge \alpha)_{pqrs} = \gamma_{pqrs} + \gamma_{qrps} + \gamma_{rpqs} \, .$$

Making use of (5.10), this leads to

$$- \langle \alpha \barwedge \beta, \beta \barwedge \alpha \rangle_r = - 3 \textstyle\sum_{(p,q,r,s)} \gamma_{rspq} \gamma_{pqrs} - 6e \, .$$

But, recalling that S is skew-symmetric, one checks immediately from the definitions that

$$\textstyle\sum_{(p,q,r,s)} \gamma_{rspq} \gamma_{pqrs} = - s \, .$$

Hence

$$- \langle \alpha \barwedge \beta, \beta \barwedge \alpha \rangle_r = 3s - 6e \, . \tag{5.14}$$

Combining the information from (5.12), (5.13) and (5.14), we obtain

$$3c + 6e = 3s - 6e \, , \tag{5.15}$$

$$3s - 6e \geq 0 \, . \tag{5.16}$$

We now wish to compute $\| \alpha \barwedge \alpha \|^2 = \langle \alpha \barwedge \alpha, \beta \barwedge \beta \rangle_r$. Let $\tau_{pqrs} = \sum_t \alpha_{pqt} \alpha_{trs}$, $\theta_{pqrs} = \sum_t \beta_{pqt} \beta_{trs}$. Then one checks from the definitions that

$$\sum_{(p,q,r,s)} \tau_{pqrs}\theta_{pqrs} = s\,. \tag{5.17}$$

From (4.1) we have

$$(\alpha \barwedge \alpha)_{pqrs} = \tau_{pqrs} + \tau_{qrps} + \tau_{rpqs}\,,$$
$$(\beta \barwedge \beta)_{pqrs} = \theta_{pqrs} + \theta_{qrps} + \theta_{rpqs}\,.$$

Thus

$$\begin{aligned} 0 &\leq \| \alpha \barwedge \alpha \|^2 = \langle \alpha \barwedge \alpha, \beta \barwedge \beta \rangle_r \\ &= \textstyle\sum_{(p,q,r,s)} (\tau_{pqrs} + \tau_{qrps}\,\tau_{rpqs})(\theta_{pqrs} + \theta_{qrps} + \theta_{rpqs})\,. \end{aligned} \tag{5.18}$$

When the last expression is expanded, one gets a sum of nine terms. Using (5.17) one sees that three of these terms are equal to s. Using the fact that (α_{pqr}) and (β_{pqr}) are skew-symmetric one checks that each of the remaining six terms is equal to e. Thus (5.18) becomes

$$0 \leq 3s + 6e\,. \tag{5.19}$$

We see from (5.16) and (5.19) that

$$s \geq |2e|\,. \tag{5.20}$$

Since (5.5) is satisfied for every skew-symmetric $Y = (Y_{pq})$, we know that $s = 0$. By (5.20) we then have $e = 0$, which, by (5.15), implies that $c = 0$. Finally, by (5.9) this implies that $\| \beta \|^2 = 0$, and hence that $\beta = 0$, which completes the proof.

6. Concluding remarks

In H. Weyl's paper [8], the existence of a compact real form of a complex semi-simple Lie algebra $\mathfrak{g}$ comes as a by-product of the general structure theory of complex semi-simple Lie algebras (in particular the existence of a Weyl basis). This proof involves, among other things, Lie's theorem, Engel's theorem, the existence of Cartan subalgebras and the root space decomposition of a semi-simple Lie algebra. It seems to us that it is considerably longer than the proof we have given here. But since Weyl's proof of the existence of a compact real form of $\mathfrak{g}$ also leads to the detailed structure theory of $\mathfrak{g}$, it seems clear that for most purposes it is more satisfactory than the proof we have given here. However we hope that our proof may be useful to someone who wants to obtain the existence of a compact real form of $\mathfrak{g}$ without worrying about the structure theory of $\mathfrak{g}$.

As pointed out by S. Helgason, our proof of the existence of a compact real form of $\mathfrak{g}$ gives a proof of the existence of Cartan subalgebras of $\mathfrak{g}$ without the use of Lie's theorem. This proof goes as follows: Let $\mathfrak{g}_0$ be a compact real form of $\mathfrak{g}$, G_0 a (necessarily compact) Lie group with Lie algebra $\mathfrak{g}_0$, T be a maximal torus of G_0, and $\mathfrak{t}_0$ the Lie algebra of T. If $\mathfrak{t}$ is the complexification of $\mathfrak{t}_0$ (considered in the obvious way as a subalgebra of $\mathfrak{g}$), then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$.

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