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SOME FROBENIUS THEOREMS IN GLOBAL ANALYSIS

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Introduction

In [6] we introduced a notion of differentiability which permitted us to prove that the group of C^∞ diffeomorphisms can be given the structure of a Lie group. This notion of differentiability as distinct from the Frechet definition does not depend on a topological or quasi-topological structure on the vector space of continuous linear transformations $L(E, F)$ between topological vector spaces E, F (see § 1 below). However, in [6], to prove the fundamental elementary theorems of analysis, we used the notion of quasi-topology introduced by A. Bastiani.

In § 1 it is shown how these theorems can be established by elementary techniques.

In § 2 a version of the Frobenius theorem is proved (see Theorem 3). Although our proof of Theorem 3 differs in several essential points from an analogous proof in Dubinsky [4] of an analogous theorem, we found his ideas quite useful. In Proposition 6 it is proved that under the hypotheses of Theorem 3 a C^n differential equation admits a C^n flow.

In § 3 a second version of the Frobenius theorem is proved in the context of Banach chains.

In § 4 a Frobenius theorem on the integrability of finite codimensional sub-bundles of the tangent bundle of manifolds modelled on Banach chains is proved.

In § 5 there is given an application of §§ 3 and 4 in the context of the group of diffeomorphisms of a compact connected smooth manifold; there, it is shown that finite dimensional and finite codimensional subalgebras of the Lie algebra of the right invariant vector fields on Diff (M) are integrable.

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1. Analysis in locally convex topological vector spaces

All topological vector spaces appearing in this paper are considered to be Hausdorff locally convex topological vector spaces over the real numbers R, and continuous functions will be called C^0 functions when convenient. Let us first recall the definition of a C^n function given in [6].

Definition 1. Let $U \subset E$, $V \subset F$ be open sets in topological vector spaces E and F, and suppose that G is a third topological vector space. A function $f: U \times V \to G$ is n times differentiable at $(\xi, \eta) \in U \times V$ in the first (resp. second) variable, if f is $n-1$ times differentiable in the first (resp. second) variable at (ξ, η) and there exists a continuous symmetric n-multilinear function

$$\frac{\partial^n f}{\partial x^n}(\xi, \eta): \underbrace{E \times \cdots \times E}_{n} \to G$$

$$\left(\text{resp. } \frac{\partial^n f}{\partial y^n}(\xi, \eta): \underbrace{F \times \cdots \times F}_{n} \to G\right)$$

such that

$$F(v) = f(\xi + v, \eta) - f(\xi, \eta) - \frac{\partial f}{\partial x}(\xi, \eta)(v) - \cdots - \frac{1}{n!}\frac{\partial^n f}{\partial x^n}(\xi, \eta)(v, \cdots, v)$$

$$\left(\text{resp. } G(v) = f(\xi, \eta + v) - f(\xi, \eta) - \frac{\partial f}{\partial y}(\xi, \eta)(v) - \cdots - \frac{1}{n!}\frac{\partial^n f}{\partial y^n}(\xi, \eta)(v, \cdots, v)\right)$$

satisfies the property that

$$\phi(t, v) = F(tv)/t^n, \quad t \neq 0; \qquad \phi(t, v) = 0, \quad t = 0$$

$$(\text{resp. } \gamma(t, v) = G(tv)/t^n, \quad t \neq 0; \qquad \gamma(t, v) = 0, \quad t = 0)$$

is continuous on $R \times E$ (resp. $R \times F$) at $(0, v)$, $v \in E$ (resp, $v \in F$).

Remark 1. Setting $F = \{0\}$ we find the definition of an n-times differentiable function $f: U \to G$. It is obvious how to generalize the above definition to any finite number of variables.

Remark 2. f is said to be a C^n function in the first (resp. second) variable if f is C^{n-1}, f is n-times differentiable at each point $(\xi, \eta) \in U \times V$, and $\partial^m f/\partial x^m$ (resp. $\partial^m f/\partial y^m$) defines a continous function

$$U \times V \times \underbrace{E \times \cdots \times E}_{m} \to G \; (\text{resp. } U \times V \times \underbrace{F \times \cdots \times F}_{m} \to G)$$

for $0 \leq m \leq n$.

Remark 3. When $F = \{0\}$ we write $\frac{\partial^n f}{\partial x^n}(\xi, 0) = (D^n f)_{x=\xi}$. In the case of Banach spaces it is essentially proved in [1] that our definition of C^n is equivalent to the Frechet definition (see [5]), and that the D^r in the above case and the Frechet case are the same up to a canonical isomorphism.

Proposition 1. *Suppose $E_1, \cdots, E_n, F$ are topological vector spaces. If $f: E_1 \times \cdots \times E_n \to F$ is a continuous n-linear function, then f is C^r, $r \geq 0$, in all variables. Further suppose $E_1 = \cdots = E_n$ and $\Theta: E \to F$ is given by $\Theta(\alpha) = f(\alpha, \cdots, \alpha)$. Then Θ is C^r, $r \geq 0$.*

Proof. The function given by

$$\frac{\partial^r f}{\partial x_s^r}(\xi_1, \cdots, \xi_n; a_1, \cdots, a_r) = 0\,, \quad r > 1\,,$$

$$\frac{\partial f}{\partial x_s}(\xi_1, \cdots, \xi_n; a) = f(\xi_1, \cdots, \xi_{s-1}, a, \xi_{s+1}, \cdots, \xi_n)$$

satisfies the properties of the above definition. For the second affirmation we may suppose that f is symmetric. If f were not symmetric, we may construct its symmetrization as follows: Let S_n be the symmetric group on n ciphers and set $f_\sigma(a_1, \cdots, a_n) = f(a_{\sigma(1)}, \cdots, a_{\sigma(n)})$. Then

$$\sigma \in S_n \cdot \bar{f}(a_1, \cdots, a_n) = \frac{1}{n!} \sum_{\sigma \in S} f_\sigma(a_1, \cdots, a_n)$$

is called the symmetrization of f. Observe that $\bar{f} = (a, \cdots, a) = \Theta(a)$.

Now set $D^r\Theta(\xi, a_1, \cdots, a_r) = 0$, $r > n$. For $0 \leq r \leq n$ set

$$D^r\Theta(\xi; \alpha_1, \cdots, \alpha_r) = \frac{n!}{(n-r)!} f(\underbrace{\xi, \cdots, \xi}_{n-r}; \alpha_1, \cdots, \alpha_r)$$

and observe that

$$\Theta(\alpha) = \Theta(\xi + (\alpha - \xi)) = \sum_{j=0}^{n} \binom{n}{j} f(\underbrace{\xi, \cdots, \xi}_{j}, \underbrace{\alpha - \xi, \alpha - \xi, \cdots, \alpha - \xi}_{n-j})$$

to conclude the verification of the above proposition.

It is trivial to verify that C^r, $r \geq 0$, functions $f: U \to G$ form a vector space.

Proposition 2. *Suppose E, F, and G are topological vector spaces. If $U \subset E$, $V \subset F$ are open sets and $f: U \to V$ and $g: V \to G$ are C^r, $r > 0$, functions, then $g \circ f: U \to G$ is a C^r function and $D(g \circ f)(x; \alpha) = Dg(f(x); Df(x; \alpha))$.*

Proof. For $1 \leq s \leq r$, by definition there exist functions $\gamma_s: F \to G$ and $\phi_s: E \to F$ such that

$$\Gamma_s(t, v) = \gamma_s(tv)/t^s, \quad \Phi_s(t, v) = \phi_s(tv)/t^s, \quad t \neq 0\,,$$

and

$$\Gamma_s(0, v) = \Phi(0, v) = 0$$

are continuous and such that

$$g(f(x+th))-g(f(x))=\sum_{s\geq l\geq 1}\frac{1}{l!}D^l g(f(x);f(x+th)-f(x),\cdots,f(x+th)-f(x))$$

$$+\gamma_s(f(x+th)-f(x))$$

$$=\sum_{s\geq l\geq 1}\frac{1}{l!}D^l g\Big(f(x);\sum_{s\geq k\geq 1}\frac{1}{k!}D^k f(x;th,\cdots,th)$$

$$+\phi_s(th);\cdots;\sum_{s\geq k\geq 1}\frac{1}{k!}D^k f(x;th,\cdots,th)+\phi_s(th)\Big)$$

$$+\gamma_s\Big(\sum_{s\geq k\geq 1}\frac{1}{k!}D^k f(x;th,\cdots,th)+\phi_s(th)\Big)$$

$$=\sum_{\{k_1,\cdots k_l\}}\frac{1}{k_1!\cdots k_l!}\sum_{1\leq l\leq s}\frac{1}{l!}z_{k_1,\ldots,k_l}D^l g(f(x);$$

$$D^{k_1}f(x;th,\cdots,th);\cdots;D^{k_l}f(x;th,\cdots,th))$$

$$+\gamma_s\Big(\sum_{1\leq k\leq s}\frac{1}{k!}D^k f(x;th,\cdots,th)+\phi_s(th)\Big)+\sum(th),$$

where $\sum_{\{k_1,\cdots,k_l\}}$ designates the sum over all ordered sets of l integers $1\leq k_1\leq\cdots\leq k_l\leq s$, the integers $z_{k_1,\ldots,k_l}$ are the multinominal coefficients in the expression

$$\Big(\sum_{i=1}^{s}\alpha_i\Big)^l=\sum_{1\leq k_1\leq\cdots\leq k_l\leq s}z_{k_1,\ldots,k_l}\alpha_{k_1}\cdots\alpha_{k_l},$$

and $\sum(th)$ is the sum of all the expressions of the form $D^s g(f(x);\phi(th),\cdots)$. Now let

$$D^k(g\circ f)(x;\alpha_1,\cdots,\alpha_k)=\delta_k(g\circ f)(x;\alpha_1,\cdots,\alpha_k)$$

$$=k!\sum_{t=1}^{k}\sum_{k_1+\cdots+k_t=k}\frac{1}{t!}\frac{1}{k_1!\cdots,k_t!}z_{k_1,\ldots,k_t}D^t g(f(x);$$

$$D^{k_1}f(x;\alpha_1,\cdots,\alpha_{k_1});\cdots;D^{k_t}f(x;\alpha_{k-k_t+1},\cdots,\alpha_k)),\quad k\leq s.$$

Then we have

$$g(f(x+th))-g(f(x))=\sum_{k=1}^{s}\frac{1}{k!}D^k(g\circ f)(x;th,\cdots,th)$$

$$+\gamma_s\Big(\sum_{l=1}^{s}\frac{1}{l!}D^l f(x;th,\cdots,th)+\phi_s(th)\Big)+\sum(th),$$

where $D^k(g\circ f)$ is continuous, write

$$K(t, h) = \begin{cases} \frac{1}{t^s}\left\{r_s\left(\sum_{l=1}^{s} \frac{1}{l!} D^l f(x; th, \cdots, th) + \phi_s(th)\right) + \sum (th)\right\}, & t \neq 0, \\ 0, t = 0. \end{cases}$$

Then $K(t, h)$ is easily seen to be continuous at $(0, h)$.

Corollary of the proof of Proposition 2. *If f(resp. g) is a continuous linear function, then* $D^k(g\circ f)(x, \alpha_1, \cdots, \alpha_k) = D^k g(f(x), f(\alpha_1), \cdots, f(\alpha_k))$ (resp. $D^k(g\circ f)(x, \alpha_1, \cdots, \alpha_k) = g(D^k(f)(x, \alpha_1, \cdots, \alpha_k))$), $k \leq r$.

Proposition 3. *Let E and F be topological vector spaces with F complete, and suppose $U \subset E$ is an open convex subset. If $f: U \to F$ is C^r, then $D^s f$: $U \times E \times \cdots \times E \to F$ is C^{r-s}, $s \leq r$, in the first variable and*

$$\partial\frac{D^s f}{\partial x}(x; \alpha_1, \cdots, \alpha_s; \beta) = D^{s+1} f(x; \alpha_1, \cdots, \alpha_s, \beta).$$

The proof of Proposition 3 makes use of

Lemma.

$$\begin{aligned} &D^s f(x + \beta; \alpha_1, \cdots, \alpha_s) - D^s f(x; \alpha_1, \cdots, \alpha_s) \\ &\quad - D^{s+1} f(x; \alpha_1, \cdots, \alpha_s, \beta) - \cdots \\ &\quad - \frac{1}{(r-s-1)!} D^{r-1} f(x, \alpha_1, \cdots, \alpha_s, \beta, \cdots, \beta) \\ &= \frac{1}{(r-s-1)!}\int_0^1 (1-\rho)^{r-s-1} D^r f(x + \rho\beta; \alpha_1, \cdots, \alpha_s, \beta, \cdots, \beta) d\rho. \end{aligned}$$

Proof. Designate the dual of F by F'. Let g be the restriction of f to the finite dimensional subspace of E generated by $x, \beta, \alpha_1, \cdots, \alpha_s$, and set $g_\lambda = \lambda\circ g$, $\lambda \in F'$. We then have

$$\begin{aligned} &\lambda D^s f(x + \beta; \alpha_1, \cdots, \alpha_s) - \lambda D^s f(x; \alpha_1, \cdots, \alpha_s) \\ &\quad - \lambda D^{s+1} f(x; \alpha_1, \cdots, \alpha_s, \beta) - \cdots \\ &\quad - \frac{1}{(r-s-1)!} \lambda D^{r-1} f(x; \alpha_1, \cdots, \alpha_s, \beta, \cdots, \beta) \\ &= D^s g_\lambda(x + \beta; a_1, \cdots, \alpha_s) - D^s g_\lambda(x; \alpha_1, \cdots, \alpha_s) \\ &\quad - D^{s+1} g_\lambda(x; \alpha_1, \cdots, \alpha_s, \beta) - \cdots \\ &\quad - \frac{1}{(r-s-1)!} D^{r-1} g_\lambda(x; \alpha_1, \cdots, \alpha_s, \beta, \cdots, \beta) \\ &= \frac{1}{(r-s-1)!}\int_0^1 (1-\rho)^{r-s-1} D^r g_\lambda(x + \rho\beta; \alpha_1, \cdots, \alpha_s, \beta, \cdots, \beta) d\rho \end{aligned}$$

$$= \frac{1}{(r-s-1)!}\lambda\int_0^1 (1-\rho)^{r-s-1}D^r f(x+\rho\beta;\alpha_1,\cdots,a_s,\beta,\cdots,\beta)d\rho\,.$$

Hence from the Hahn-Banach theorem the lemma follows.

The proposition follows from the observation that

$$\frac{1}{(r-s-1)!}\int_0^1 (1-\rho)^{r-s-1}D^r f(x+\rho t\beta;\alpha_1,\cdots,\alpha_s,\beta,\cdots,\beta)d\rho$$
$$-\frac{1}{(r-s)!}D^r f(x;\alpha_1,\cdots,\alpha_s,\beta,\cdots,\beta)$$
$$=\frac{1}{(r-s-1)!}\int_0^1 (1-\rho)^{r-s-1}[D^r f(x+\rho t\beta;\alpha_1,\cdots,\alpha_s,\beta,\cdots,\beta)$$
$$-D^r f(x;\alpha_1,\cdots,\alpha_s,\beta,\cdots,\beta)]d\rho$$

is continuous in (t,β) at (O,β) and equal to O at (O,β).

Corollary of the proof of Proposition 3.

$$\frac{\partial^t D^s f}{\partial x}(x;\alpha_1,\cdots,\alpha_s;\beta_1,\cdots,\beta_t)$$
$$=D^{s+t}(x;\alpha_1,\cdots,\alpha_s,\beta_1,\cdots,\beta_t),\ t\leq r-s\,.$$

Corollary 1. *If $f\colon U\to F$ is C^r, then $D^s f\colon U\times E\times\cdots\times E\to F$, $s\leq r$, are uniquely determined.*

Note that by a classical limit argument first derivatives are unique in view of Definition 1, and thus the above lemma implies the uniqueness of the higher derivatives.

Corollary 2. *Suppose F is complete and $U\subset E$ is convex, and set $\tilde{U}=E-U$. For a given closed convex subset V of F if $f\colon U\to F$ is C^1 and $Df(x;\alpha)\in V$ for $x\in U$, $\alpha\in\tilde{U}$, then $f(x_1)-f(x_0)\in V$ for $x_0, x_1\in U$.*

Proposition 4. *Let E, F, and G be topological vector spaces, and $U\subset E$, $V\subset F$ be open and non-empty. Then $f\colon U\times V\to G$ is C^1 if and only if f is in both variables.*

Proof. Suppose f is C^1. Then $\frac{\partial f}{\partial x}(x,y;h)=Df((x,y);(h,0))$ (resp. $\frac{\partial f}{\partial y}(x,y;k)=Df((x,y);(0,k))$) obviously satisfies the definition of C^1 in the first (resp. second) variables.

Suppose now that f is C^1 in both first and second variables. Set $Df((x,y);(h,k))=\frac{\partial f}{\partial x}(x,y;h)+\frac{\partial f}{\partial y}(x,y;k)$ and observed that

$$f((x,y)+t(h,k))-f((x,y))-Df((x,y);t(h,k))$$
$$=f(x+th,y+tk)-f(x+th,y)+f(x+th,y)-f(x,y)$$

$$- \frac{\partial f}{\partial x}(x, y; th) - \frac{\partial f}{\partial y}(x, y; tk)$$
$$= t\int_0^1 \left[\frac{\partial f}{\partial y}(x + th, y + \rho tk; k) - \frac{\partial f}{\partial y}(x, y; k)\right] d\rho + t\phi(t, h),$$

where $\phi(0, h) = 0$, ϕ is continuous at $(0, h)$, and the integrand is clearly continuous in (t, k) at $(0, k)$ and is 0 at $(0, k)$.

2. Elementary Frobenius' theorems

We now recall two classical theorems which will be of use to us.

Theorem 1 [3, p. 29]. *Let R be the set of real numbers, $E_1 = [0, a_1]$, $a_1 > 0$, and F a finite dimensional vector space over the reals. Suppose $F_0 \subset F$ is an open relatively compact convex neighborhood of the origin and $T: E_1 \times F_0 \times R \to F$ is a C^n, $n \geq 1$, function linear in R. Then there exists $E_0 = [0, a_0]$, $0 \leq a_0 \leq a_1$, and a unique C^{n+1} function $f: E_0 \to F_0$ such that $f(0) = 0$ and $Df(x; 1) = T(x, f(x), 1)$.*

Theorem 2. *Let $E = R \times R$ and suppose F is a finite dimensional vector space over the reals, and $F_0 \subset F$ is an open relatively compact balanced neighborhood of the origin. Let $E_1 = [0, a] \times [0, b]$, $a, b > 0$, and suppose $T: E_1 \times F_0 \times E \to F$ is a C^n, $n \geq 1$, function linear in E such that*

$$\frac{\partial T}{\partial E}((x, y), z, h; k) + \frac{\partial T}{\partial F}((x, y), z, h; T((x, y), z; h))$$

is symmetric in $h, k \in E$. Then there exist a non-trivial interval $[0, a_0] = I_0 \subset [0, a] \cap [0, b]$ and a unique function $f: I_0 \times I_0 \to F_0$ such that $f(0, 0) = 0$,

$$f(x, y) = \int_0^1 T((\tau x, \tau y), f(\tau x, \tau y), (x, y))d\tau$$

is C^{n+1} and $Df((x, y); a) = T((x, y), f(x, y); a)$.

Remark 4. In Theorem 1 we may take $a_0 = \max\{a \leq a_1 \mid T(E_1, F_0, [0, a]) \subset F_0\}$; in Theorem 2 we may take

$$a_0 = \max\{\min\{\tfrac{1}{2}M, a, b\} \mid T(E_1, F_0, [0, M] \times [0, M]) \subset F_0\}, \quad (\text{see [3, p. 53]}).$$

Theorem 3. *Let E be a barrelled topological vector space and F a finite dimensional vector space. Let $E_1 \subset E$ and $F_0 \subset F$ be open convex neighborhoods of $x_0 \in E$ and $y_0 \in F$ respectively, and let $T: E_1 \times F_0 \times E \to F$ be a C^n, $n > 1$, function linear in the third variable such that $T(E_1 \times F_0 \times E_1)$ is relatively compact and such that for all $x \in E_1$, $y \in F_0$, $h, k \in E$,*

$$\frac{\partial T}{\partial E}(x, y, h; k) + \frac{\partial T}{\partial F}(x, y, h; T(x, y, k))$$

is symmetric in h and k. Set $I = [0, 1]$. Then there exist an open convex neighborhood of x_0, $E_0 \subset E_1$, and a unique C^{n+1} function $f: E_0 \to F$ such that $f(x_0) = y_0$ and $Df(x; h) = T(x, f(x), h)$.

Proof. As in the classical case we may suppose $x_0 = 0$, $y_0 = 0$. Since $T(E_1 \times F_0 \times E_1)$ is relatively compact there exists a real number $r > 0$ such that $rT(E_1 \times F_0 \times E_1) \subset F_0$. Let E_2 be a barrel contained in rE_1 and set $E_0 = \frac{1}{4}E_2$. For $x \in E_2$ let $T_x: I \times F_0 \times R \to F$ be given by $T_x(\tau, \alpha; 1) = T(\tau x, \alpha; x)$ where $I = [0, 1]$. Then by Theorem 1 and Remark 4 there exists a unique solution $g_x: [0, 1] \to F_0$ of T_x such that $g_x(0) = 0$, $g_x(t) = \int_0^1 T_x(\tau, g_x(\tau); 1)d\tau$. Now

$$\begin{aligned} g_x(at) &= \int_0^{at} T_x(\tau, g_x(\tau); 1)d\tau = \int_0^1 aT_x(a\tau, g_x(a\tau); 1)d\tau \\ &= \int_0^t T_{ax}(a\tau, g_x(a\tau); 1)d\tau . \end{aligned}$$

Thus $h(t) = g_x(at)$ is a solution for T_{ax} such that $h(0) = 0$, and by uniqueness we obtain $g_x(at) = h(t) = g_{ax}(t)$. Now set $f(x) = g_x(1)$.

$$\begin{aligned} f(x) = g_x(1) &= \int_0^1 T_x(\tau, g_x(\tau); 1)d\tau = \int_0^1 T_x(\tau, g_{x\tau}(1); 1)d\tau \\ &= \int_0^1 T_x(\tau, f(\tau x); 1)d\tau . \end{aligned} \tag{1}$$

In order to show that $f(x)$ satisfies T with $f(0) = 0$ we shall use the following **Lemma.**

$$\begin{aligned} &\int_0^1 T(y_1 + \sigma(y_2 - y_1), f(y_1 + \sigma(y_2 - y_1)); y_2 - y_1)d\sigma \\ &\quad = f(y_2) - f(y_1) , \qquad y_1, y_2 \in E_0 . \end{aligned}$$

Proof. For $x_1, x_2 \in \frac{1}{2}E_2$ define $S: I \times I \times F_0 \times R \times R \to F$ by

$$S((s, t), y, (u, v)) = T(sx_1 + tx_2, y, ux_1 + vx_2) .$$

S satisfies the hypotheses of Theorem 2 and, by (1), $h(s, t) = f(sx_1 + tx_2)$ satisfies

$$h(s, t) = \int_0^1 S(\tau s, \tau t, h(\tau s, \tau t), (s, t))d\tau ,$$

and $Dh((s, t); \alpha) = S((s, t), h(s, t); \alpha)$. For $y_1, y_2 \in E_0$ set $y_1 = x_1$, $y_2 - y_1 = x_2 \in \frac{1}{2}E_2$. Now

$$\int_0^1 T(y_1 + \sigma(y_2 - y_1), f(y_1 + \sigma(y_2 - y_1)), y_2 - y_1)d\sigma$$

$$\begin{aligned}
&= \int_0^1 T(x_1 + \sigma x_2, f(x_1 + \sigma x_2), x_2)d\sigma = \int_0^1 S((1, \sigma), h(1, \sigma), (0, 1))d\sigma \\
(2) \quad &= \int_0^1 Dh((1, \sigma); (0, 1))d\sigma \\
&= \int_0^1 Dh(1 + \sigma(1 - 1), 0 + \sigma(1 - 0); (1 - 1, 1 - 0))d\sigma \\
&= h(1, 1) - h(1, 0) = f(y_2) - f(y_1)\,. \qquad \text{q.e.d.}
\end{aligned}$$

Now set $y_1 = x$ and $y_2 = x + \lambda h$ and apply the above lemma to obtain

$$\frac{1}{\lambda}[f(x + \lambda h) - f(x) - T(x, f(x), \lambda h)]$$
$$= \int_0^1 [T(x + \sigma\lambda h, f(x + \sigma\lambda h), h) - T(x, f(x), h)]d\sigma.$$

To obtain the theorem it suffices to prove f to be continuous. To see this let $T: E_0 \times F_0 \to L(E, F)$ be the mapping canonically associated with T, where $L(E, F)$ is the vector space of linear transformations from E to F (i.e. $\bar{T}(x, y)(\alpha) = T(x, y, \alpha)$). Since $\bar{T}(E_0 \times F_0)$ is simply bounded it follows from the Banach-Steinhaus Theorem that $\bar{T}(E_0 \times F_0)$ is equicontinuous. Thus

$$f(y_2) - f(y_1) = \int_0^1 T(y_1 + \sigma(y_2 - y_1), f(y_1 + \sigma(y_2 - y_1)), y_2 - y_1)d\sigma$$

shows that f is continuous since $f(E_0) \subset F_0$ by construction.

Remark 5. Designate by $C_k(p) \subset F$ the cube with center $p \in F$ and side $2k$. Now when $F_0 = C_d(0)$ we have that $C_{3d/8}(0) \subset \bigcap_{y \in C_{d/8}(0)} \{C_{d/2}(0) - y\}$, and therefore that there exists a barrel E_2, with center at the origin, sufficiently small so that $T(E_2 \times \{C_{d/2}(0) - y\} \times E_2) \subset C_{3d/8}(0) \subset \{C_{d/2}(0) - y\}$ for all $y \in C_{d/8}(0)$. From the proof it follows that there exists a flow $\alpha: \{x_0 + E_2\} \times C_{d/8}(y_0) \to F_0$ of the differential equation (i.e., $\alpha_y(x) = \alpha(x, y)$ is a solution of the differential equation such that $\alpha_0(x_0) = y$).

Proposition 5. *Let* $A(x_0 + \frac{1}{2}E_0) \times S_{d/4}(y_0) \to (x_0 + \frac{1}{2}E_0) \times F_0$ *be defined by* $A(x, y) = (x, \alpha(x, y))$. *Then* A *is one-one and contains* $(x_0 + \frac{1}{2}E_0) \times S_{d/4}(y_0)$.

Proof. A is one-one, since the set of points, where $\alpha_{y_1}(x) = \alpha_{y_2}(x)$, is open by Theorem 3 and closed by the fact that both α_{y_1} and α_{y_2} are continuous. For $x \in x_0 + \frac{1}{2}E_0$ and $y \in S_{d/4}(y)$ it follows from the proof of Theorem 3 that there exists a solution $f: x + E_0 \to F_0$ such that $f(x) = y$ provided that $f(x_0) = y_0$. Note that $x_0 + \frac{1}{2}E_0 \subseteq x + E_0$. By uniqueness, $\alpha_{y_0}(x) = f(x)$, and thus $A(x, y_0) = (x, y)$.

Proposition 6. *$\alpha: E_0 \times S_{d/4}(y_0) \to F_0$ is a C^n mapping under the hypotheses of Theorem 1.*

Proof. By Theorem 3, α is C^{n+1} in the first variable. $\beta(t, y) = \alpha(x_0 + t(x = x_0), y)$ is the flow of the differential equation $S(t, y) = T(x_0 + t(x - x_0), y, (x - x_0))$. It is classical that β is C^n in the second variable, and obvious that $\frac{\partial^k \beta}{\partial y^k}(1; \eta) = \frac{\partial^k \alpha}{\partial y^k}(x; \eta)$. To conclude the proof it suffices to prove α to be continuous since $\frac{\partial \alpha}{\partial x}(x, y; \gamma) = T(x, \alpha(x, y), \gamma)$. To see that α is continuous, consider

$$\begin{aligned} &\alpha(x + h, y + k) - \alpha(x, y) \\ &\quad = \alpha(x + h, y + k) - \alpha(x, y + k) + \alpha(x, y + k) - \alpha(x, y) \\ &\quad = \int_0^1 T(x + th, \alpha(x + th, y + k), h)dt + (\alpha(x, y + k) - \alpha(x, y)) . \end{aligned}$$

As $\alpha(x, th, y + k) \in F_0$ for $h \in E$, $k \in F$ sufficiently small, $T(E_0 \times F_0) \subset L(E, F)$ is equicontinuous, and, in addition, $\alpha^x(y) = \alpha(x, y)$ is continuous, it follows that α is continuous.

Proposition 7. *Let $U \subset E$ be an open subset of a topological vector space E, and F a second topological vector space. Suppose that $T: U \times F \to F$ is a C^n, $n \geq 0$, mapping linear in the second variable such that $\bar{T}: U \to L(F, F)$ maps into the isomorphisms of F. Designate by $T^{-1}: U \times F \to F$ the map defined by $T^{-1}(u, f) = \bar{T}(u)^{-1}(f)$. If T^{-1} is continuous, then T^{-1} is C^n.*

Proof. Set

$$\frac{\partial T^{-1}}{\partial x}(x, \alpha; h) = -T^{-1}\left(x, \frac{\partial T}{\partial x}(x, T^{-1}(x, T^{-1}(x, \alpha); h)\right)$$

and observe that

$$\begin{aligned} &\frac{1}{t}\left[T^{-1}(x + th, \alpha) - T^{-1}(x, \alpha) + T^{-1}\left(x, \frac{\partial T}{\partial x}(x, T^{-1}(x, \alpha); th)\right)\right] \\ &\quad = -T^{-1}\left(x + th, \frac{\partial T}{\partial x}(x, T^{-1}(x, \alpha); h)\right) + T^{-1}\left(x, \frac{\partial T}{\partial x}(x, T^{-1}(x, \alpha); h)\right) \\ &\quad\quad - T^{-1}\left(x + th, \frac{1}{t}\left[T(x + th, T^{-1}(x, \alpha)) - T(x, T^{-1}(x, \alpha)) \right.\right. \\ &\quad\quad\quad \left.\left. - \frac{\partial T}{\partial x}(x, T^{-1}(x, \alpha); th)\right]\right) \end{aligned}$$

is continuous in (t, h) at $(0, h)$ and equal to 0 at $(0, h)$.

3. Analysis in Banach chains

Definition 2. Let J^+ denote the set of nonnegative integers. A chain of Banach spaces is a set $\{B^k\}$ of Banach spaces indexed by J^+ such that

(a) if $k > l \geq 0$, then the underlying vector space of B^k is a linear subspace of the underlying vector space of B^l and the inclusion map $B^k \to B^l$ is continuous;

(b) $B^\infty = \bigcap_k B^k$ is dense in each B^k,

B^∞ is given the topology of the inverse limit $\varprojlim_k B^k$.

Definition 3. Let $\{B_1^k\}$ and $\{B_2^k\}$ be Banach chains.

Definition 4. Let $\{B^k\}$ be a Banach chain and $U \subset B^\infty$ open, I an open interval containing $0 \in R$, and $\|\ \|_k$ the norm in B^k. We shall say that $f: I \times U \to B^\infty$ satisfies uniformly a Lipschitz condition on U uniformly with respect to I if there exists a number $L > 0$ such that $\| f(t, x) - f(t, y)\|_k \leq L\| x - y \|_k$ for all $k \geq 0$. L as usual is called the Lipschitz constant.

For $k \geq l$, let $\pi_l^k : B^k \to B^l$ be the canonical injection, and suppose

$$\| \alpha \|_k \leq \| \alpha \|_{k+1} \quad \text{for } \alpha \in B^{k+1}.$$

Definition 5. Suppose $\{B_1^k\}$ and $\{B_2^k\}$ are Banach chains, and $U \subset B_1^\infty$ is an open set. A mapping $f : U \to B_2^\infty$ is called strongly continuous when there exist an integer N and an open set U_N such that $U = (\pi_N^\infty)^{-1}(U_N)$ and further that there exists a continuous extension $f_l : (\pi_N^l)^{-1}(U_N) \to B_2^l$ for all $l \geq N$. It is obvious that any strongly continuous mapping is continuous.

Define $\{B_1^k\} \times \{B_2^k\} = \{B_1^k \times B_2^k\}$. In a canonical way every Banach space B may be considered as the B^∞ of a Banach chain by setting $B_l = B$ for all $l \geq 0$.

Proposition 8. *Let $\{B_k\}$ be a Banach chain, $U \subset B^\infty$ open, and I an open interval containing $0 \in R$. If $f: I \times U \to B^\infty$ satisfies uniformly a Lipschitz condition on U uniformly with respect to I, then f is strongly continuous.*

The proof follows easily from the definitions.

Definition 6. Suppose $\{B_1^k\}$ and $\{B_2^k\}$ are Banach chains and $U \subset B_1^\infty$ is an open set. A mapping $f: U \to B_2^\infty$ is called strongly C^p if f is strongly continuous with respect to some integer N (see Definition 5) and there exist an integer $M \geq N$ and an open set U_M such that $U = (\pi_M^\infty)^{-1}(U_M)$ and further that the continuous extensions $f_l : (\pi_M^l)(U_M) \to B_2^l$ are C^p for $l \geq M$. We leave this to the reader to verify.

Proposition 9. *Every strongly C^p function $f: U \to B_2^\infty$ is C^p.*

Theorem 4. *Let $\{B^k\}$ be a Banach chain, I an open interval containing $0 \in R$, U an open subset of B^∞, and $f: I \times U \to B^\infty$ a C^p, $p \geq 0$, function such that f satisfies uniformly a Lipschitz condition on U uniformly with re-*

spect to I. Suppose that for some $N \geq 0$ (using Proposition 8) the maps $f_l : I \times (\pi^l_N)^{-1}(U_N) \to B^l$, $l \geq N$, determined by f are C^p, and further that $x_0 \in U$. Then there exist open subsets V, J of U, I containing x_0 and 0, respectively, and a unique flow $\alpha : J \times V \to U$ of f (i.e., $\alpha_v(j) = \alpha(j, v)$ is a solution of f so that $\alpha_v(0) = V$). ($U_N \subset B^N$ is an open subset of B^N such that $U = (\pi^\infty_N)^{-1}(U_N)$.)

To prove the above theorem it suffices to prove

Lemma. *Under the hypotheses of Theorem 4 there exist an open interval $J \subset I$ containing $0 \in R$, an open subset $V \subset U_N$ containing x_0, and flows of f_k, $\alpha_k : J \times (\pi^k_N)^{-1}(V) \to (\pi^k_N)^{-1}(U)$ such that $\alpha_{k+1} = \alpha_k \cdot \pi^{k+1}_k$, $N \leq k \leq \infty$.*

Proof. Without loss of generality we may suppose $x_0 = 0$. Given $S > 0$, f_N being continuous there exist a closed subinterval J of I containing 0 in its interior and a number $0 < \alpha < 1$ such that $f_N(J_1 \times S^N_{3\alpha}(0)) \subset S^N_S(0)$. Thus by Newton's method (see [5, pp. 55–62]) there exist an interval $J = [-b, b] \subset J_1$ and a flow $\alpha_N : J \times S^N_\alpha(0) \to U_N$, where $b < \inf(1, \alpha/S)$. For $l \geq N$ let M_l be the set of the continuous mappings $\alpha : J \to (\pi^l_N)^{-1}(S^N_{2\alpha}(0))$. With the uniform topology M_l is a complete metric space. Let $S_l : M_l \to M_l$ be the operator defined by

$$(S_l \alpha)(t) = x + \int_{-0}^{t} f_l(u, \alpha(u))du \ ,$$

$x \in (\pi^\infty_N)^{-1}(S^N_\alpha(0))$, $t \in [-b, b]$. $(\pi^l_N)^{-1}(S^N_S(0))$ being closed and convex we have

$$\int_0^t f_l(u, \alpha(u))du \in b(\pi^l_N)^{-1}(S^N_S(0)) \subset (\pi^l_N)^{-1}(S^N(0)) \ .$$

Further since f_l has Lipschitz constant L it follows that S satisfies the shrinking lemma and thus there exists a unique fixed point $\alpha \in M_l$. Suppose $l' \geq l > N$, and $\alpha_{l'}(t, \pi)$ is the fixed point of $S_{l'}$. Note that

$$\begin{aligned}(\pi^{l'}_l \circ \alpha_{l'})(t, x) &= + \pi^{l'}_l \int_0^t f_{l'}(u, \alpha_{l'}(u, x))du \\ &= x + \int_0^t \pi^{l'}_l f_{l'}(u, \alpha_{l'}(u, x))du \\ &= x + \int_0^t \pi^{l'}_l f_{l'}(u, (\pi^{l'}_l \circ \alpha_{l'}))(u, x)du \\ &= x + \int_0^t f_l(u, \alpha_l(u, x))du \ .\end{aligned}$$

Thus $\pi^{l'}_l \circ \alpha_{l'}(t, x)$ is the fixed point of $S_{l'}$.

It is classical that $\alpha_l : I_0 \times (\pi_N^l)^{-1}(S_\alpha^N(0)) \rightarrow (\pi_N^l)^{-1}(S_{2\alpha}^N(0))$ is a C^p mapping.

Remark. We have proved more than we stated; indeed we have proved that there exists a strongly C^p flow $\alpha : J \times V \rightarrow U$. In the above theorem, V is taken to be $(\pi_N^\infty)^{-1}(S_\alpha^N(0))$.

4. Frobenius theorem for differentiable manifolds

For the elementary definitions of this section substitute our definition of differentiability here for that used in [5]. The objective of this section is to prove

Theorem 5. *Let $\{E^l\}$ be a chain of Banach spaces and M a connected C^p, $p \geq 2$, differentiable manifold modelled on E^∞. If B is a sub-bundle of $T(M)$ of finite codimension with fiber F such that the C^{p-1} sections of B are closed under the bracket operation of $T(M)$, then B is integrable.*

Definition 7. Suppose $\{E_1^l\}$ and $\{E_2^l\}$ are chains of Banach spaces. A linear function $f : E_1^\infty \rightarrow E_2^\infty$ will be called a *morphism* if there exist an integer N and continuous linear extensions of f, $f_l : E_1^l \rightarrow E_2^l$ for $l \geq N$. Chains of Banach spaces clearly form an additive category with this definition of morphisms; designate this category by CB. Given a Banach space B we shall designate by $\{B\}$ the trivial chain $\{B^l\}$, where $B^l = B$ for all l and $\pi_m^l : B^l \rightarrow B^m$ is the identity for all $l \geq m$.

Proposition 10. *Let $\{E^l\}$ be a Banach chain and G a subspace of E^∞ of finite codimension having H as a complementary subspace. Then there exists a Banach chain $\{G^l\}$ characterized by the property that G^l is the closure of G in E^l such that $\{E^l\} \approx \{G^l\} + \{H\}$.*

To prove Proposition 10 it suffices to prove

Lemma. *Under the hypotheses of Proposition* 10 *there exists an integer N_0 such that $G^l + H \approx E^l$ for $l \geq N_0$.*

Proof. Let $\pi : E^\infty \rightarrow H$ be the canonical projection onto H. We shall show that π is a morphism $\{E^l\} \rightarrow \{H\}$. Let U be a compact neighborhood of the origin in H. Then by continuity there exists a neighborhood of the origin $V \subset E^\infty$ such that $\pi(V) \subset U$. By definition of the topology in E^∞ there exist an integer N_0 and a bounded neighborhood of the origin $V_{N_0} \subset E^{N_0}$ such that $(\pi_{N_0}^\infty)^{-1}(V_{N_0}) \subset V$. Thus $\pi : E^\infty \rightarrow H$ is continuous for the topology on E^∞ induced by the Banachable topology on E^{N_0}, and $\pi : E^\infty \rightarrow H$ is extendable to $\pi^{N_0} : E^{N_0} \rightarrow H$. Hence π is a morphism in CB. It is easy to see that $\mathrm{Ker}(\pi^l) = Im(I - \pi^l) = G^l$, $l \geq N_0$.

Proof of Theorem 5. As in [4, p. 92] one may express the subbundle B locally in the form of an exact sequence

$$0 \rightarrow U \times V \times F \xrightarrow{\bar{f}} U \times V \times F \times G \approx U \times V \times E^\infty,$$

where $U \subset F$, $U \subset G$ are open neighborhoods of x_0 and y_0 respectively.

Furthermore we may suppose that $\tilde{f}$ is of the form

$$\tilde{f}((x, y), \alpha) = ((x, y), (\alpha, f(x, y, \alpha))),$$

where f is a C^{p-1} function linear in the third variable such that $f((x_0, y_0), \alpha) = 0$ for all $\alpha \in F$.

Let us recall that the bracket of the two given sections ζ and η of $T(M)$ is given locally by

$$[\xi, \eta](x) = D\xi(x; \eta(x)) - D\eta(x, \xi(x)).$$

For given C^{p-1} maps $\xi_1, \eta_1 : U \times V \to F$ note that the C^{p-1} maps given by $\xi(x, y) = (\xi_1(x, y), f((x, y), \xi_1(x, y)))$ and $\eta(x, y) = (\eta_1(x, y), f((x, y), \eta(x, y)))$ determine the sections of B. The closure of the sections of B under the bracket operation implies that

$$\frac{\partial f}{\partial x}((x, y), \eta_1(x, y); \xi_1(x, y)) + \frac{\partial f}{\partial y}(x, y, \eta_1(x, y); f((x, y), \xi_1(x, y)))$$

is symmetric in $\xi_1(x, y), \eta_1(x, y)$.

It follows from Theorem 3 and Proposition 6 that there exist open sets $U_0 \subset U$, $V_0 \subset V$, and a C^{p-1} flow of $f, \alpha : U_0 \times V_0 \to V$. That there exist open neighborhoods of the origin $U \subset H$, $V \subset E^\infty$ so that $f(U \times V \times U)$ is relatively compact follows from the continuity of f and the local compactness of G.

There exist open sets $0 \subset U_0$ *and* $W \subset V$ *such that* $\phi(x, y) = (x, y) = (x, \alpha(x, y))$ *is a* C^{p-1} *diffeomorphism for* $(x, y) \in 0 \times W$.

In fact, from Proposition 5 we have that $\phi : U_0 \times V_0 \to F \times G$ is an injective mapping containing in its image a neighborhood $U_1 \times V_1$ of (x_0, y_0).

Since $\frac{\partial \alpha}{\partial y}(x_0, y_0; \beta) = \beta$ for all $\beta \in G$ there exists an open set $U_2 \times V_2 \subset \phi^{-1}(U_1 \times V_1)$ containing (x_0, y_0) such that $\frac{\partial \alpha}{\partial y}(x, y; \beta)$ is an isomorphism for $(x, y) \in U_2 \times V_2$. Thus

$$D\phi((x, y); (\alpha, \beta)) = \left(\alpha, D\alpha(x, y; (\alpha, \beta) = (\alpha, f(x, \alpha(x, y); \alpha)), (0, \frac{\partial \alpha}{\partial y}(x, y; \beta)\right)$$

is a continuous isomorphism of $F \times G$ onto $F \times G$ for $(x, y) \in U_2 \times V_2$. Since $f : U_2 \times V_2 \times F \to G$ is continuous there exist neighborhoods $U_3 \subset U_2$, $V_3 \subset V_2$ of x_0 and y_0, respectively, and a neighborhood U of the origin in F such that $f(U_3 \times V_3, U)$ is relatively compact in G. It now follows from the Banach-Steinhaus theorem that the linear functions $f_{(x,y)}(\alpha) = f(x, y, \alpha)$ are equicontinuous for $(x, y) \in U_3 \times V_3$.

Let S be the unit ball in G, and suppose that B is an open set in F such that $f_{(x,y)}(\alpha) \in S$ for $(x, y) \in U_3 \times V_3$ and $\alpha \in B$. Further let $B^k \subset F^k$ (see Proposition 10) be an open set such that $(\pi_k^\infty)^{-1}(B^k) = B$; designate the continuous extension of f by $f_l : U_3 \times V_3 \times F^l \to G$, $l \geq k$. Thus $D\phi$ determines continuous maps $\phi_l : U_3 \times V_3 \times F^l \times G \to F^l \times G$ in such a way that

$$\phi_{l,(x,y)}(\alpha, \beta) = \phi_l((x, y), (\alpha, \beta)) = \left(\alpha, f_l((x, y), \alpha) + \frac{\partial \phi}{\partial y}(x, y; \beta)\right)$$

is a continuous automorphism of $F^l \times G$ for $(x, y) \in U_3 \times V_3$. Note that $\phi_{l-1,(x,y)} \circ \pi_{l-1}^l = \phi_{l,(x,y)}$. It is classical that ϕ_l determines a continuous map

$$\tilde{\phi}_l : U_3 \times V_3 \to \operatorname{Aut}(F^l \times G) .$$

Let $\rho : \operatorname{Aut}(F^l \times G) \to \operatorname{Aut}(F^l \times G)$ be the continuous map which associates its inverse with every automorphism. Designate the map $f \circ \tilde{\phi}_l$ by $\tilde{\phi}_l^{-1} : U_3 \times V_3 \to \operatorname{Aut}(F^l \times G)$. Since $\tilde{\phi}_l^{-1}$ is continuous it follows that $D\phi^{-1}(x, y, \alpha, \beta)$ is continuous and therefore C^{p-1} for $(x, y) \in U_3 \times V_3$ by Proposition 7.

Set

$$T(x, y, \alpha) = \pi \circ D\phi^{-1}(x, y, \alpha, 0) ,$$

where $\pi : F \times G \to G$ is the canonical projection. T is obviously a C^{p-1} function linear in α. We shall now show that there exist open neighborhoods U_5, V_5 of x_0 and y_0, respectively, such that for all $(x, y) \in U_5 \times V_5$

$$\frac{\partial T}{\partial x}(x, y, h; k) + \frac{\partial T}{\partial y}(x, y, h; T(x, y; k)) \tag{3}$$

is symmetric in h and k.

Let Y be the subspace of F generated by x, h, and k, and

$$t : (U_3 \cap Y) \times V_3 \times Y \to G ,$$
$$g : (U_3 \cap Y) \times V_3 \to (U_3 \cap Y) \times G$$

the restrictions of T and ϕ respectively. It follows from the inverse function theorem that g is a diffeomorphism such that

$$\begin{aligned} D(g^{-1})(x, y, \alpha, \beta) &= (Dg)^{-1}(f^{-1}(x, y), \alpha, \beta) \\ &= (D\phi)^{-1}(\phi^{-1}(x, u), \alpha, \beta) , \end{aligned} \tag{4}$$

where $(\alpha, \beta) \in Y \times G$, $(x, y) \in (U_4 \cap Y) \times V_4$, and $U_4 \subset F$, $V_4 \subset G$ are open sets such that $U_4 \times V_4 \subset \phi(U_3 \times V_3)$. Thus $(\pi \circ \phi^{-1}) | (U_4 \cap Y) \times V_4$ is a flow for t, and

$$(5)\quad \begin{aligned}\frac{\partial^2(\pi\circ\phi^{-1})}{\partial x^2}(x, y; \alpha; \beta) = {} & \frac{\partial T}{\partial x}(x, \pi\circ\phi^{-1}(x, y), \alpha; \beta) \\ & + \frac{\partial T}{\partial G}(x, \pi\circ\phi^{-1}(x, y), \alpha; T(x, \pi\circ\phi^{-1}(x, y), \beta))\end{aligned}$$

is symmetric in α, β for $(x, y) \in (U_3 \cap Y) \times V_3$.

Since $x \in U_4$, h, $k \in F$ were arbitrarily chosen, (3) is time for all $x \in U_4$, h, $k \in F$. ϕ being continuous $\phi^{-1}(U_4 \times V_4)$ contains an open set $U_5 \times V_5$. (5) now implies (3).

By Theorem 3, T has a $C^{p=1}$ flow $\psi: U_5 \times V_5 \to G$ since from $i_F \times \psi \mid (U_5 \cap Y) \times V_5 = \phi^{-1} \mid (U_5 \cap Y) \times V_5$ it follows that ϕ^{-1} is C^{p-1} on $U_5 \times V_5$. $0 \subset U_0$, $U| \subset V_0$ be open sets such that $0 \times W \subset \phi^{-1}(U_5 \times V_5)$. To prove Theorem 5 it now suffices to show that $\phi: 0 \times W \to \phi(0 \times W)$ is such that $(i_{0\times W} \times \delta\phi/\partial x): (0 \times W) \times F \to (0 \times W) \times (F \times G)$ is a C^{p-1} isomorphism onto $\bar{f}(0 \times W \times F)$ which follows immediately from $\dfrac{\partial\phi}{\partial x}(x, y; \alpha) = (\alpha, f(x, y), \alpha))$.

Corollary 1. *Let M satisfy the hypotheses of Theorem 5. Suppose that N is a C^p finite dimensional connected manifold, and let $f: M \to N$ be a C^p onto mapping. If $f^*: TM \to TN$ is onto, then* Ker(f^*) *is an integrable sub-bundle of TM, and $f^{-1}(x)$, $x \in N$, is a closed sub-manifold of M.*

Corollary 2. *Under the hypotheses of Corollary* 1, *each leaf of the foliation is an ANR.*

5. Frobenius theorems for the group Diff (M)

In this section by manifold we shall mean a compact connected smooth manifold.

Let M be a manifold, and Diff(M) the group of diffeomorphisms of M. The author has shown in [5] that Diff(M) admits a differentiable structure which is locally Frechet (indeed locally nuclear) such that the multiplication and the operation of taking the inverse define smooth differentiable functions of Diff$(M) \times$ Diff(M) to Diff(M) and of Diff(M) to Diff(M) respectively.

Now let us recall the following local definition of the differential structure of Diff(M): Let $f \in$ Diff(M) and $l_f(M, TM)$ be the vector space of all liftings of f (i.e. the vector space of all functions $g: M \to TM$ such that $\pi \circ g = f$ where $\pi = TM \to M$ is the canonical projection). In order to give $l_f(M, TM)$ a Frechet topology cover M by two finite collections of trivializing (for TM) normal (for some fixed Riemannian structure) open charts $\{U_i\}_{i=1,\cdots,m}$ and $\{V_j\}_{j=1,\cdots,n}$ so that diam$(f(U_i)) < \lambda/3$ where λ is the Lebesgue number of $\{V_j\}$. Let $k_i: U_i \to U'_i \subset R^l$ and $\mathscr{I}_j: V_i \to V'_j \subset R^l$ be homeomorphisms determining the local structure on M, and suppose $f(\bar{U}_i) \subseteq V_{j(i)}$. Let $\phi_{j(i)}: \pi^{-1}(V_{j(i)}) \to V_{j(i)} \times R^l$ be a smooth diffeomorphism with $\phi_{j(i)}|\pi^{-1}(x)$, $x \in V_{j(i)}$ linear.

It is convenient to suppose that k_i extends to a homeomorphism $k_i:\bar{U}_i \to \bar{U}'_i$.

Now let $\mathscr{F}(\bar{U}'_i, R^l)$ be the Frechet space (indeed nuclear) space of smooth maps with the C^∞ topology. Set $\mathscr{F}_0 = \sum_{i=1}^{m} (\bar{U}'_i, R^l)$. Define $\gamma: l_f(M, TM) \to \mathscr{F}_0$ by $\gamma(g) = g_1(+) \cdots (+)g_m$ where $g_i \in \mathscr{F}(\bar{U}'_i, R^l)$ is the composite

$$\bar{U}'_i \xrightarrow{k_i^{-1}} \bar{U}_i \xrightarrow{g} \pi^{-1}(V_{j(i)}) \xrightarrow{\phi j} V'_{j(i)} \times R^l \longrightarrow R^l .$$

Let $\mathscr{F} = \lambda(l_f(M, TM)) \subset \mathscr{F}_0$. $\mathscr{F}$ is a closed subspace and γ is injective. By means of γ we transport the induced Frechet structure of $\mathscr{F}$ to $l_f(M, TM)$.

To fix ideas we shall suppose that the $\{U_i\}$ and $\{V_j\}$ are normal open spheres for a smooth Riemannian metric and that $\phi_{j(i)}:(V_{j(i)}) \to V'_{j(i)} \times R^l$ is given by $\phi_{j(i)}(\alpha) = (\exp_{x_0}^{-1}(\alpha), \tau_{x_0}(\alpha))$, where x_0 is the center of $V_{j(i)}$, and τ_{x_0} is the parallel translation along the unique geodesic from $\pi(\alpha)$ to x_0.

Designate by $\mathrm{Diff}_n(M)$, $D_n(M)$, and $\mathscr{D}_n(M)$ the group of C^n diffeomorphisms of M, the connected component of the identity of $\mathrm{Diff}_n(M)$, and the vector space of right invariant C^{n-1} vector fields on $D_n(M)$, respectively. It is well known that Diff_∞ is dense in Diff_n. We shall suppose $n \geq 3$.

$\mathrm{Diff}_n(M)$ is a topological group whose underlying topology is compatible with a C^n differentiable manifold structure modelled on the Banach space $\Gamma_n(M)$ of C^n vector fields on M with the C^n topology [7]. Moreover the mapping $R_\sigma:\mathrm{Diff}_n(M) \to \mathrm{Diff}_n(M)$ deffined by $R_\sigma(\tau) = \tau\sigma$ is a C^n mapping for this differentiable structure [7]. It follows that the right invariant vector fields on $\mathrm{Diff}_n(M)$ are C^{n-1} sections of the tangent bundle $T(\mathrm{Diff}_n(M)) \to \mathrm{Diff}_n(M)$. Set $T(\mathrm{Diff}_n(M)) = \tau_n(M)$.

Lemma. *Let G be a topological group whose underlying topology is compatible with a C^n differentiable manifold structure modelled on a Banach space B such that multiplication from the right $R_\sigma: G \to G$, $\sigma \in G$, defines a C^n function, and let K be a finite dimensional subspace of the vector space of C^{n-1} right invariant vector fields on G. If K is closed under the bracket operation, then K is integrable, that is, there exists a C^{n-1} submanifold of G, H, which is, in addition, a subgroup in such a way that $T_e(H)$ is canonnically isomorphic to K.*

Proof. Now suppose $\mathscr{S}$ the finite dimensional subalgebra of $L(G)$ and designate by $S(x)$ the subspace of $T_x(G)$ spanned by the vectors $\xi(x)$ for $\xi \in \mathscr{S}$. We may write $T_x(G) = S(x) + R(x)$ where $R(x)$ is a complementary subspace of $S(x)$ in $T_x(G)$. Put $\Sigma = \bigcup_{x \in G} S(x)$ and let $\pi': \Sigma \to G$ be the natural projection. We now make π' a subbundle of π. Let (U, ϕ) be a symmetric chart of G at the identity with $\phi(U) \subset E$ and put $U_a = Ua$ and let $\sigma_e:\pi'^{-1}(U) = \Sigma(U) \to U \times S(e)$ be the C^{n-1} map induced by multiplication on the right.

Define $\sigma_a:\pi'^{-1}(U_a) = \Sigma(U_a) \to U_a \times S(e)$ by $\sigma_a = (R_a \times I_{s(e)})\circ\sigma_e\circ dR_{a^{-1}}\cdot\sigma_a$ such that the following diagram

$$\begin{array}{ccc} \pi'^{-1}(U_a) & \longrightarrow & U_a \times S(e) \\ & {\scriptstyle \pi'} \searrow \quad \swarrow {\scriptstyle P_{u_a}} & \\ & U_a & \end{array}$$

is commutative where $P_{u_a} : U_a \times S(e) \to U_a$ is the canonical projection.

Now set

$$\phi_a = \phi \circ R_{a^{-1}} : U_a \to \phi(U) ,$$
$$\phi_{ab} = \phi_a \circ \phi_b^{-1} : \phi_b(U_a \cap U_n) \to \phi_a(U_a \cap U_b) .$$

Since multiplication from the right is C^n, one obtains a C^{n-1} mapping $\tau_{ba} : \phi_a(U_a \cap U_b) \times S(e) \to \phi_b(U_a \cap U_b) \times S(e)$ given by $\tau_{ba}(x, v) = (\phi_{ba}(x), D\phi_{ba}(x; v))$; under these conditions there exists a unique structure of a C^{n-1} manifold on Σ such that π' is a C^{n-1} mapping and σ_a, $a \in G$, is a C^{n-1} diffeomorphism making $\pi^1 : \Sigma \to G$ into a vector bundle with $\{(U_a, \sigma_a)\}_{a \in G}$ as a trivializing covering.

The injection of $S(x)$ into $T(x)$ shows that Σ is a subbundle of $T(x)$. As K is closed for the bracket operation in $L(G)$ it follows that Σ is closed under the bracket operation in $T(G)$ and therefore K is integrable (see [5, p. 92]). *Let H be a maximal integral manifold of G containing the identity.* As in the classical case, R_σ permutes with the maximal integral manifolds of K, and thus H is a subgroup of G. It is immediate that the Lie algebra of H is K.

Lemma [7]. *$D_m(M) \times D_n(M) \xrightarrow{\pi} D_n(M)$ given by $\pi(f, g) = f \circ g$ is C^n for $m \geq 2n$.*

Corollary. *$\alpha \in T_e(D_m(M)) \subset T_e(D_n(M))$ generates a C^n right invariant vector field on $D_n(M)$ for $m \geq 2n$.*

Theorem. *Finite dimensional and finite codimensional subalgebras of $\mathscr{D}_\infty(M)$ are integrable.*

Proof. The canonical injections $i_n^m : D_{2m}(M) \to D_{2n}(M)$, $\infty \geq m \geq n \geq 0$, are obviously C^n homorphisms. Set

$$\mathscr{I}_n^m = D(i_n^m) : \mathscr{D}_{2m}(M) \to \mathscr{D}_{2n}(M) , \qquad \infty \geq m \geq n \geq 2 .$$

It is not difficult to see that if $\mathscr{H}$ is a finite dimensional subalgebra of $D_{2m}(M)$, $m < \infty$, and H is the subgroup corresponding to it, then $i_n^m(H)$ is the subgroup corresponding to $\mathscr{I}_n^m(\mathscr{H})$.

Now suppose $\mathscr{H}$ is a finite dimensional subalgebra of $\mathscr{D}_\infty(M)$, and let H_n, $n < \infty$, be the subgroup of $D_{2n}(M)$ corresponding to $\mathscr{H}_n = \mathscr{I}_n^\infty(\mathscr{H})$. Then we have

$$H_n = i_n^m(H_m) , \quad \mathscr{H}_n = \mathscr{I}_n^m(\mathscr{H}_m) , \quad \infty \geq m \geq n \geq 2 .$$

Since

$$\varprojlim_n \mathscr{D}_{2n}(M) = \mathscr{D}_\infty(M)\,, \quad D_\infty(M) = \varprojlim_n D_{2n}(M)\,.$$

and further $\mathscr{I}_n^m$ and i_n^m are injective, we obtain that $\varprojlim_n H_n = H$ is the integral subgroup of $\mathscr{H}$ in $\mathscr{D}_\infty(M)$.

That finite codimensional subalgebras are integrable follows from Theorem 5 immediately.

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