

1-FORMS WITH THE EXTERIOR DERIVATIVE OF MAXIMAL RANK

TADASHI NAGANO

1. Introduction

On a manifold M let ω be a 1-form, whose exterior derivative $d\omega$ has the rank $= \dim M$ at every point of M . Roughly speaking, this condition on ω means that locally ω cannot be obtained by pulling back any 1-form from a lower dimensional manifold (E. Cartan [4]). The existence of ω on M sets some limits to the topology and the manifold structure of M . Clearly M must be even-dimensional and orientable. M cannot be compact (§5). In this paper we will obtain more information in this regard. In particular we will prove (Theorem 4.1) that M is the cotangent bundle of another manifold S , if M satisfies certain completeness conditions with respect to ω , and the singularities of ω have certain properties. Clearly some completeness is necessary to characterize a cotangent bundle. We would, however, like to emphasize the importance of the singularities in the study of M with ω . A point x of M is called *singular* if ω vanishes at x . Let S denote the set of all singular points on M . Locally the complement $M - S$ with ω is trivial; $M - S$ with ω is locally isomorphic to \mathbb{R}^m with $\sum_{i=1}^m x^i dx^{i+m}$ (see E. Cartan [4, p. 264], and also Arens [2]), $2m = \dim M$. But the singularities are not isomorphic to each other even locally (§2).

The study of ω is closely related to that of the vector field z on M , which is uniquely given by

$$(1.1) \quad \iota(z)d\omega = \omega,$$

where $\iota(z)$ is the inner derivation with respect to z . Let us call z *the associated vector field* of ω . The conditions on completeness of M with respect to ω and the singularities of ω will be stated in terms of z ((3.1), ..., (3.4)). In §3, these conditions will be formulated in a way more general than sufficient to show that M is a cotangent bundle. Indeed those conditions on a vector field z on a manifold M (apart from a 1-form ω) will turn out to characterize a vector bundle structure on M (Theorem 3.5).

Back to the exact symplectic form $d\omega$, a few examples would be due. Any

Stein manifold admits ω , which is not unique (§6). If M is a cotangent bundle, the form: $M \ni x \rightarrow x \circ d\pi$ is another example, where π is the projection. An ω is constructed canonically on $\mathbf{R} \times N$ if N is given a contact structure (§7).

2. Singular points

At every singular point $s \in S$, the associated vector field z gives rise to a linear operator $Z = Z_s = \partial z$ of the tangent space $T_s(M): X \rightarrow \mathcal{V}_X z$ where \mathcal{V} is the covariant differentiation with respect to any connection, i.e. in terms of a coordinate system (x^i) the operator Z has components $(\partial_i z^j)$, since z vanishes exactly on S . Z will be called the *characteristic operator* of z .

We fix a basis of $T_s(M)$ such that $d\omega$ is expressed by a skew-symmetric matrix Ω with $\Omega^2 = -1$. Differentiating (1.1), we then obtain

$$(2.0) \quad \Omega Z + {}^t Z \Omega = \Omega.$$

From this follows:

Lemma 2.1. *If c is a (real) eigenvalue of Z , then so is $(1 - c)$, and moreover $Z - c$ has the same rank as $Z - (1 - c)$.*

In fact, (2.0) implies $\Omega(Z - c)\Omega^{-1} = (1 - c) - {}^t Z$. Thus $Z - c$ has the same rank as $(1 - c) - {}^t Z$ and the transpose $(1 - c) - Z$.

As corollaries to Lemma 2.1, we have

Lemma 2.2. *The rank $(Z) \geq m$ if $2m = \dim M$.*

Lemma 2.3. *$Z^2 = Z$ when and only when $\text{rank}(Z) = m$.*

Remark. The linear operator $Z: T_0(M) \rightarrow T_0(M)$ is determined by ω only. Thus, if M' has a 1-form ω' and there exists a diffeomorphism α of M onto M' which pulls back ω' to ω , then we necessarily have $\alpha_* \circ Z = Z' \circ \alpha_*$, where Z' is the characteristic operator at $\alpha(o)$ of the associated vector field of ω' . In particular, say, the rank of Z must equal that of Z' . For the classification of ω or rather (M, ω) , which we dare not accomplish here, it will therefore be necessary to know what kind of linear operator can be the characteristic operator at a singular point of a 1-form with exterior derivative of maximal rank. Such an operator can be characterized by (2.0) or equivalently by the condition that $Z - 1/2$ belongs to the symplectic Lie algebra $\mathfrak{sp}(\Omega)$. We have already proved the necessity. Conversely if $Z - 1/2$ belongs to $\mathfrak{sp}(\Omega)$ then there exists an exact symplectic form $d\omega$ on the vector space, regarded as a manifold, on which Ω is defined, such that ω is singular at 0 and the characteristic map at 0 of the associated vector field is Z . For the proof we change the notations and put

$$\Omega = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 \\ +1 & 0 \end{bmatrix}.$$

If a matrix $Z - 1/2$ is a member of $\mathfrak{sp}(\Omega)$, then $\Omega Z - K$ is a symmetric

matrix. Let $(2s_{ij})$ be the entries of this matrix. Then the desired 1-form ω is given by

$$\omega = \sum_{i=1}^m x^i dx^{i+m} + d\left(\sum_{1 \leq i, j \leq 2m} s_{ij} x^i x^j\right).$$

Then $d\omega$ is symplectic and the associate vector field z has the given linear operator Z at the origin $\in S$.

3. A characterization of vector bundles

If M is the total space of a C^∞ or C^0 vector bundle over S , then the scalar multiplications on each fibre define a Lie transformation group of dimension one. Let z be the vector field on M which generates that group. Clearly z satisfies the following conditions (3.1), \dots , (3.4):

(3.1) z generates a global one-parameter transformation group on M .

(3.2) For each point x of M , there exists a unique $\lim_{t \rightarrow -\infty} (\exp tz)(x)$.

(3.3) The characteristic operator Z satisfies

$$Z^2 = Z$$

for each singular point of z .

(3.4) The set S of the singular points of z is a submanifold of M of codimension = rank (Z) .

Let us call z the canonical vector field of the vector bundle M .

The main purpose of this section is to prove the converse:

Theorem 3.5. *Suppose that there exists a vector field z on a manifold M satisfying the above conditions (3.1), \dots , (3.4). Then there exists a unique vector bundle structure on M such that z is the canonical vector field.*

Preceding the proof we like to intimate that the unstable manifolds of z will be the fibres of the vector bundle.

Corollary 3.6. *Two vector bundles are isomorphic if and only if there exists a diffeomorphism which transforms the canonical vector fields.*

Corollary 3.7. *The automorphism group of a vector bundle coincides with the transformation group which leaves the canonical vector field invariant.*

We are always in the C^∞ or C^0 category. The distinction between these two categories will not be specified in the proofs unless it is very substantial.

Let $N(S)$ be the set of all tangent vectors X of M at the points of S such that $Z(X) = X$. $N(S)$ is naturally the normal bundle of S in M . We are going to show that there exists a diffeomorphism of M onto $N(S)$, which transforms the vector field z into the canonical vector field of the vector bundle $N(S)$. We need the following proposition which is more or less known.

Proposition 3.8. *Assume (3.3) and (3.4). Then for small X in $N(S)$, there exists a unique curve $x(t)$ such that*

$$(3.9) \quad t(dx(t)/dt) = z(x(t)).$$

$$(3.10) \quad (dx/dt)(0) = X,$$

and $x(t)$ is differentiable or analytically dependent on (t, X) .

Three proofs are available. The first uses the techniques in the theory of ordinary differential equations (See Lefschetz [8, Chap. V, §4] or Friedrichs [5, Chap. III, §4]). The second is to show the existence of a linear connection on M with respect to which the map: $X \rightarrow x(1)$ is nothing but the exponential map of N into M , or more precisely we will construct a connection such that we have

$$(3.11) \quad \nabla_z z = z,$$

so that (3.9) is the equation of the geodesics with canonical parameter t as is easily shown by a straightforward computation. By (3.3) and (3.4), S is a regularly imbedded (locally closed) submanifold. We choose a coordinate system (x^i) , $1 \leq i \leq 2m$, such that S is given by $x^\alpha = 0$ ($1 \leq \alpha \leq q = \text{codim } S$) in the coordinate neighborhood and the matrix $(\partial_\alpha z^\beta)$ ($1 \leq \alpha, \beta \leq q$) is non-degenerate where z^β is the β -th component of z . Since the system $x^\alpha = 0$ ($1 \leq \alpha \leq q$) is equivalent to $z^\alpha = 0$ ($1 \leq \alpha \leq q$), and z vanishes on S , there exists a C^∞ or C^0 function $L_\alpha^i(x)$ ($1 \leq \alpha \leq q$; $1 \leq i \leq n = \text{dim } M$) such that $z^j \partial_j z^i - z^i = L_\alpha^i(x) z^\alpha$. Moreover, by (3.3) we have $\partial_i(z^j \partial_j z^i - z^i) = 0$ along S . Therefore there exists a C^∞ or C^0 function $\Gamma_{\alpha\beta}^i(x)$ such that

$$(3.12) \quad z^j \partial_j z^i - z^i = \Gamma_{\alpha\beta}^i(x) z^\alpha z^\beta.$$

Let $\Gamma_{jk}^i(x)$ ($1 \leq i, j, k \leq n$) be a linear connection (defined on the coordinate neighborhood) with partial components $\Gamma_{\alpha\beta}^i(x)$. Then (3.11) is satisfied. Finally the third method is to “blow up” S . We identify $M - S$ with the submanifold $z(M - S) = \{z(x) | x \in M - S\}$ of the tangent bundle $T(M)$. Let $N'(S)$ be another submanifold $\{v \in N(S) | v \neq 0\}$. In their union $z(M - S) \cup N'(S)$ we identify each $v \in N'(S)$ with all cv ($c \in \mathbb{R}, c \neq 0$). The quotient set B has a natural manifold structure. Indeed we cover each point b_0 of B not in $z(M - S)$ with the following coordinate system. Let $\pi: T(M) \rightarrow M$ and $p: z(M - S) \cup N'(S) \rightarrow B$ be the projections. Take $v_0 \in N'(S)$ with $p(v_0) = b_0$. $\pi(v_0)$ lies on S . Choose a coordinate system around $\pi(v_0)$ as used in the second proof. Assume v_0 has the first component different from zero. Now to a point $(z^i(x))$ near v_0 but not on $p(N'(S))$ we assign $(z^1(x), z^2(x)/z^1(x), \dots, z^q(x)/z^1(x), x^{q+1}(x), \dots, x^n(x))$, and to a point $v = (v^i)$ near v_0 and on $p(N'(S))$ we assign $(0, v^2/v^1, \dots, v^q/v^1, x^{q+1}, \dots, x^n)$. The manifold B thus obtained could be called the blowing up of S . Now the vector field $(z^1)^{-1}z$ is naturally transferred on a non-vanishing vector field near v_0 , which is of class C^∞ or C^0 in view of (3.12). In this way we obtain a one-dimensional linear differential system (or distribution) on the blowing up B . The integral curves of this system, restricted to

$M - S = z(M - S) \subset B$, are the required integral curves. In all the three proofs the uniqueness is obvious since the integral curves cover a neighborhood of S . Hence Proposition 3.8 is proved.

We begin with the proof of Theorem 3.5. We choose a sphere bundle $E \subset N(S)$ so that the map: $X \rightarrow x(1)$ in Proposition 3.8 is well defined on E . This map extends to $N(S)$ by (3.1) so that the canonical vector field of $N(S)$ is transferred onto z . We denote the map by \exp . By (3.2), S is not empty. Again by (3.2), \exp is surjective since $\exp(N(S))$ clearly contains a neighborhood of S . Also \exp is injective by the fundamental theorem of Cauchy on the ordinary differential equations. Finally \exp is a diffeomorphism since there are no "focal points" because of (3.1). Thus M has a vector bundle structure of $N(S)$. Surely z is the canonical vector field, and the uniqueness of the vector bundle structure comes from that of $N(S)$ for a given S in M .

Remark. Note that the existence of the singular set S is vitally important in the above theorem. The automorphisms of $M - S$ with z are far from the automorphisms of the bundle $M - S$. In fact, any diffeomorphism of the sphere bundle $E \subset M - S$ onto itself extends to an automorphism of $(M - S, z)$, which does not necessarily send fibres to fibres. In particular the automorphism group of $(\mathbb{R}^n, \sum x^i \partial_i)$ is the general linear group, while that of $(\mathbb{R}^n - \{0\}, \sum x^i \partial_i)$ is, as it were, the semidirect product of the diffeomorphism group of the unit sphere and the additive group of all the functions on the unit sphere.

Remark. We do not know if the condition (3.4) is independent of the others, namely, (3.1), (3.2) and (3.3). However, if the automorphism group of M with z and (M, z) is transitive on the non-singular domain $M - S$, and we are in the analytic category, then (3.4) follows from the other three conditions. We will state this more precisely and prove it.

Let L be the Lie algebra of all C^∞ vector fields u on M , which leave z invariant; then $[u, z] = 0$. Assume that L is transitive on $M - S$ in the sense that at each point x of M the space $L(x) = \{u(x) \mid u \in L\}$ of the values of u of L at x coincides with the tangent space $T_x(M)$ to M at x . Since L is a Lie algebra, the linear differential system: $x \rightarrow L(x)$ on M (with singularities) is completely integrable that is, for each point x of M there exists a unique maximal connected analytic submanifold $I(x)$ of M such that $x \in I(x)$ and at each point y of $I(x)$ we have $T_y(I(x)) = L(y)$. (See [9]. In this theorem the analyticity is indispensable.) Now what we want to show is

Proposition 3.13. *Under the assumptions (3.1), (3.2) and (3.3), if the infinitesimal automorphism L of M with z is transitive on $M - S$ in the above sense, then (3.4) is satisfied by S .*

Proof. Let s be a point of S , and v a vector (first small, then any) at s with $v \in N(S)$ or $Zv = v$. (So far $N(S)$ has not been known to be a normal bundle of S in M .) Then, by the technique mentioned as the first method for the proof of Proposition 3.8, it is easily seen that there exists an analytic

curve $x(t)$, $t \geq 0$, such that we have $t(dx/dt) = z(x(t))$, $x(0) = s$ and $dx/dt = v$ for $t = 0$, (since 1 is the largest eigenvalue of Z). Let $\pi: M \rightarrow S$ denote the map: $x \rightarrow \lim_{t \rightarrow -\infty} (\exp tz)(x)$. Then the above fact implies that π is surjective even if it is restricted to $M - S$; in fact clearly we have $\pi(x(t)) = s$. Surely L leaves S invariant. Moreover L is transitive on S . In fact, for any two points s_1, s_2 of S , take two points x_1 and x_2 in $M - S$ such that $\pi(x_1) = s_1$ and $\pi(x_2) = s_2$. If s_1 is sufficiently close to s_2 , then x_1 can be chosen close to x_2 since π is apparently open. L can "carry" x_1 to x_2 . This "action" commutes with π so that L can carry s_1 to s_2 . Thus L is transitive on S . Therefore S (or any one of its connected components) is the integral manifold $S(s)$ for any point s of S . In particular S is an analytic submanifold. It remains to show $\text{codim } S$ equals the rank of Z . Since S is the set of the singularities, we have $\text{codim } S \geq \text{rank}(Z)$. We have to show that $N(S)$ is a normal bundle of S in M . To any v in $N(S)$, the curve $x(t)$ defined above allows us to assign the point, say, $x(1)$ of M . This map: $N(S) \rightarrow M$ is surjective and diffeomorphic in view of the action of L . Thus $N(S)$ must be a normal bundle, and Proposition 3.13 is proved.

Remark. For the cotangent bundle of analytic manifolds the transitivity assumption in Proposition 3.13 is satisfied (see [9] for instance).

4. Characterization of cotangent bundles

Let ω be the canonical form of the cotangent bundle $T^*(S)$ of a manifold S . For a coordinate system (x^i) , $1 \leq i \leq m = \dim S$, we define a coordinate system $((p_\lambda), (x^i))$ on $T^*(S)$ so that a point x of $T^*(S)$ is a linear form $p_\lambda dx^\lambda$ on the vector space $T_{x(x)}(S)$ if $((p_\lambda), (x^i))$ is the coordinates assigned to x . Then ω is expressed by $p_\lambda dx^\lambda$ in terms of this coordinate system, and the associated vector field z with ω is expressed by $z = p_\lambda(\partial/\partial p_\lambda)$. Therefore z is the canonical vector field of the vector bundle $T^*(S)$, and satisfies the conditions (3.1), \dots , (3.4). We will prove the converse:

Theorem 4.1. *On M if ω is a 1-form, whose exterior derivative $d\omega$ is symplectic everywhere, and the associated vector field z with ω satisfies the conditions (3.1), \dots , (3.4), then M has a unique structure of the cotangent bundle $T^*(S)$ of a manifold S such that ω is the canonical form of the vector bundle $T^*(S)$.*

Here the uniqueness means the following. If F is an isomorphism of (M, ω) onto another (M', ω') , and $M' = T^*(S')$, subject to the above conditions, then there exists an (unique) isomorphism f of S' onto S such that F is the codifferential f^* of f .

Remark. In view of Lemmas 2.2 and 2.3, the condition (3.3) can be replaced by

(3.3') *the characteristic operator Z of z has the minimum rank at each singular point of z .*

Proof of Theorem 4.1. By Theorem 3.5, M is a vector bundle over S with the canonical vector field z . Indeed M is identified with the normal bundle $N(S) = \{v \in T_s(M) \mid Zv = v, s \in S\}$. $T_s(S)$ is characterized in $T_s(M)$ as the kernel of Z . Thus it follows from (2.0) that $d\omega$, evaluated at $s \in S$, is the zero bilinear form if it is restricted to $T_s(S)$ or $N_s(S)$ which is the fibre of $N(S)$ over s .

Since $d\omega$ is non-degenerate everywhere, $d\omega$, considered as a bilinear map: $N_s(S) \times T_s(S) \rightarrow \mathbf{R}$, makes $N_s(S)$ the dual space of $T_s(S)$. Thus $N(S)$ is the dual vector bundle of $T(S)$. Now to prove the uniqueness, let F be an isomorphism of (M, ω) onto (M', ω') , i.e. a diffeomorphism of M onto M' , which pulls ω' back to ω . Then $\omega = F^*\omega'$, and F sends z to z' since z (or z') is uniquely determined by ω (or ω' respectively). Again by Theorem 3.5, F is a vector bundle isomorphism. Thus F^{-1} induces a diffeomorphism f of S' onto S , and $f^* = F$ follows from the way of identifying $N(S) = M$ with $T^*(S)$. Hence Theorem 4.1 is proved.

Remark. Since the automorphism group of (M, ω) is contained in that of (M, z) where z is the associated vector field of ω , the condition (3.4) can be replaced by the condition (3.4') if we are in the analytic category.

Remark. For ω with non-degenerate $d\omega$, the conditions (3.3) and (3.4) are obviously replaced by the following condition:

$$(4.2) \quad (M, \omega) \text{ is locally isomorphic to } (\mathbf{R}^{2m}, \sum_{\lambda=1}^m x^\lambda dx^{\lambda+m}).$$

In other words, M has a (non-transitive) pseudogroup group structure defined by $(\mathbf{R}^m, \sum x^\lambda dx^{\lambda+m})$. Thus, if M has a pseudogroup group structure as defined above and it satisfies (3.2), then M is diffeomorphic to the cotangent bundle (of the singular set S).

Remark. It might be worth noting that the frame bundle of a manifold S is characterized as a $GL(m, \mathbf{R})$ -bundle over S , $m = \dim S$, with a certain \mathbf{R}^m -valued 1-form on it, called the canonical form (Kobayashi [7]), whereas the cotangent bundle of S is characterized by the \mathbf{R} -valued 1-form ω only. In particular, a diffeomorphism F between two frame bundles is prolonged from an isomorphism between their base manifolds if F pulls back the canonical form *and* is a bundle isomorphism, while a diffeomorphism between two cotangent bundles is prolonged from an isomorphism between the base manifolds if the canonical form is pulled back; this condition necessarily implies a bundle isomorphism.

5. Nonexistence on compact manifolds

A symplectic form is never exact on a compact manifold. In fact, suppose ω is a 1-form with non-degenerate $d\omega$ on a compact M . Then $\Omega = (d\omega)^m = \underbrace{d\omega \wedge \cdots \wedge d\omega}_m$ is a volume element on M , and we have $\int_M \Omega > 0$. On the other hand Ω is exact so that $\Omega = d(\omega \wedge (d\omega)^{m-1})$, which implies $\int_M \Omega = 0$. Hence we have a contradiction.

6. Existence on Stein manifolds

A complex manifold M is a Stein manifold if and only if there exists a strictly plurisubharmonic (smooth) function f on M such that $f^{-1}([-\infty, c])$ is compact for every real number (see Hörmander [6, p. 116]). If $\omega = df \circ J$, where J is the almost complex structure tensor, then $d\omega$ is symplectic, since $d\omega \circ (J \otimes 1)$ is a Kählerian metric. Moreover the associated vector field z satisfies (3.2). There exists a geometric method to find f for which both (3.1) and (3.2) are satisfied. In fact, imbed M into some \mathbb{C}^n with the usual hermitian metric (which is always possible, see [6] for instance), and put $f(x) = \text{distance}(a, x)$ where a is a (suitable) fixed point of $\mathbb{C}^n - M$. This f satisfies (3.2) also, since the length of $z(x)$ grows larger at the order of $f(x) = d(a, x)$. The point a can be so chosen that the singularity set S is discrete in M .

Remark. It was this function f to which Andreotti and Frankel [1] applied the Morse theory in order to obtain information about the homology groups Stein manifolds. In our context, since z is the gradient vector field of f in the present case, by Lemma 2.1, there are no critical points of index $> m$, and hence $H_k(M, \mathbb{R}) = 0$ for $k > m$. By using generalizations of the Morse theory such as Bott [3] and Smale [11] the theorem of Andreotti and Frankel would admit a generalization. For instance it seems plausible that the same conclusion on the homology group holds if M has an exact symplectic form $d\omega$ such that the associated vector field z satisfies (3.2), the singular set S is discrete and

(6.1) *M is the union of countable relatively compact domains with smooth boundaries to which z is transversal.*

7. The nonsingular case

Making a slight digression, we will discuss the case where

(7.0) *ω vanishes at no points.*

It is then well known that, locally, the quotient manifold of M by the orbits

of z has a natural contact structure. It will turn out that this is globally true if the automorphisms of (M, ω) are transitive. We retain the assumption (3.1): the associated vector field z generates a one-parameter group. Let L denote the Lie algebra of all vector fields u which leave ω invariant. The function $f = \iota(u)\omega = \omega(u)$ satisfies

$$(7.1) \quad zf = f.$$

In fact we have $z(\omega(u)) = (\mathcal{L}_z\omega)(u) + \omega([z, u])$, while $\mathcal{L}_z\omega = \iota(z)d\omega + d\iota(z)\omega = \iota(z)d\omega = \omega$ and $[z, u] = -\mathcal{L}_uz = 0$ since z is uniquely determined by ω . Let \mathcal{F} denote the vector space of all functions on M subject to the condition (7.1).

Lemma 7.2. *The vector space L is isomorphic to \mathcal{F} by the map $u \rightarrow \omega(u)$.*

Proof. The map is clearly linear. It is injective; if $\omega(u) = 0$ for some u in L , then $0 = \mathcal{L}_u\omega = \iota(u)d\omega + d\iota(u)\omega = \iota(u)d\omega$ and u must vanish since $d\omega$ is non-degenerate. To show the surjectivity of that map, first note that there exists a unique u such that $\iota(u)d\omega = -df$. Then

$$d\iota(u)\omega = d\iota(u)\iota(z)d\omega = -d\iota(z)\iota(u)d\omega = d\iota(z)df = d(zf).$$

Thus

$$\mathcal{L}_u\omega = \iota(u)d\omega + d\iota(u)\omega = -df + d(zf) = d((zf) - f) = 0. \quad \text{q.e.d.}$$

Hereafter we will assume that

$$(7.3) \quad L \text{ is transitive.}$$

Then the foliation given by z is regular in the sense of Palais [10], i.e.

Lemma 7.4. *Under the assumption (7.3), any neighborhood of any point p of M contains a neighborhood U of p , whose intersection with any integral curve of z is connected (if not empty).*

The proof is based on the fact (7.5) below. By (7.3) and Lemma 7.2, we can choose functions f_1, \dots, f_{n-1} from \mathcal{F} , $n = \dim M$, such that $df_1 \wedge \dots \wedge df_{n-1} \neq 0$ at p and $f_1(p) = \dots = f_{n-1}(p) = 1$. For a sufficiently small neighborhood U of p in M the set N_U of the common zero points of $f_1 - 1, \dots, f_{n-1} - 1$ in U is a regularly imbedded hypersurface of M . We may assume that z is transversal to N_U at each point since z is so at p by the definition of F . Put $\zeta(t) = \exp(tz)$ for any real number t . If U is suitably chosen, the map: $(t, x) \mapsto \zeta(t)x$ from $] -\epsilon, \epsilon[\times N_U$ into M is a diffeomorphism onto U for some positive number ϵ . If ϵ is fixed and N_U (hence U) is changed into a sufficiently smaller one, we have $f_i(x) < e^\epsilon f_i(y)$, $1 \leq i \leq n - 1$, for any two points x, y in N_U , since $f_i(a) = 1$, $1 \leq i \leq n - 1$. The definition of \mathcal{F} implies

$$(7.5) \quad f \circ \zeta(t) = e^{t\epsilon} f \quad \text{for any } f \text{ in } \mathcal{F}.$$

From this it follows that if both x and $\zeta(t)x$ are on N_U then we have $|t| < \epsilon$.

Therefore $U \cap \zeta(R)x$ coincides with $\zeta([- \epsilon, \epsilon])x$; or it is connected. q.e.d.

Obviously each leaf $\zeta(R)x$ (or the integral curve of z through x) is closed in M . The following should be known.

Proposition 7.6. *If a Lie transformation group G of a manifold M has all its orbits closed then the orbit space M/G is a Hausdorff space.*

In fact, since the projection of M onto M/G is open in our situation, it suffices to show that the equivalence relation $\{(x, gx) \in M \times M \mid x \in M, g \in G\}$ is closed in $M \times M$. Suppose a sequence $(x_\nu, g_\nu x_\nu)$, $\nu = 1, 2, \dots$, converges to (x, y) . Then also the sequence $(x, g_\nu x)$ converges to (x, y) . Since each orbit $G(x)$ is closed, y must lie on $G(x)$. Thus the equivalence relation is closed, and M/G is Hausdorff.

If ζ denotes the one-parameter transformation group generated by z then the orbit space M/ζ is naturally a differentiable manifold by Lemma 7.4, Proposition 7.6 and a theorem of Palais [10, p. 19].

In view of (7.5), z acts on M freely under the hypothesis (7.3). Thus it follows from Lemma 7.4 that each point of M has a hypersurface N_U containing p such that the map $R \times N_U \ni (t, x) \rightarrow \zeta(t)x \in M$ is a diffeomorphism onto an open subset of M . Therefore M is naturally a principal bundle over M/ζ , and the structure group is R since ζ acts freely. This bundle is thus trivial; in M there exists a hypersurface N which we can take as above N_U . Let us fix an N . Clearly z is transversal to N . Let ω_N denote the restriction of ω to N . ω_N is a contact form; we have $\omega_N \wedge (d\omega_N)^{m-1} \neq 0$ everywhere on N , $2m = \dim M$, since we have $m\omega \wedge (d\omega)^{m-1} = \iota(z)(d\omega)^m \neq 0$, $\iota(z)(\omega \wedge (d\omega)^{m-1}) = 0$. The diffeomorphism of N onto M/ζ given by the projection carries ω_N to a 1-form on M/ζ , which we denote by the same ω_N . ω_N on M/ζ depends on the choice of N by a nonzero scalar multiple, since we have

$$\mathcal{L}_z \omega = \iota(z)d\omega + d\iota(z)\omega = \iota(z)d\omega = \omega .$$

Therefore M/ζ has a well defined contact structure (α) . Clearly M/ζ is orientable.

Examining the above arguments, we obtain

Theorem 7.7. *Let \mathcal{C} be the category of the manifolds M with an exact symplectic form $d\omega$ subject to (3.1) and (7.3), and \mathcal{C}_c the category of the oriented manifolds with given contact structures (α) . Then the above correspondence which assigns $(M/\zeta, (\alpha))$ to (M, ω) is a functor of \mathcal{C} onto \mathcal{C}_c , and in that correspondence, the choices of sections N correspond to the choices of contact forms α in (α) .*

Proof. (7.3) implies that ω has no singularities ($S = \emptyset$). Thus the above process: $(M, \omega) \rightarrow (M/\zeta, (\alpha))$ is well defined. The isomorphisms between (M, ω) 's clearly give rise to the isomorphisms between $(M/\zeta, (\alpha))$'s. To prove the surjectivity of the functor, let N be an oriented contact manifold. Then there exists a contact form α subordinate to this contact structure. (Of course,

the orientation should be compatible with the volume element $\alpha \wedge (d\alpha)^{m-2}$. We put $M = \mathbf{R} \times N$, and denote by α^* the one-form on M pulled back from α by the projection. Then $\omega = e^t \alpha^*$, for $(t, x) \in M$ is a one-form with symplectic $d\omega$ and subject to (3.1) (indeed, $z = \partial/\partial t$).

We next show (7.2). Let v be the unique vector field defined by $\iota(v)\alpha = 1$ and $\iota(v)d\alpha = 0$. Then for any function f on N a unique infinitesimal automorphism u of $(N, (\alpha))$ is given by $\iota(u)\alpha = f$ and $\iota(u)d\alpha = (uf)\alpha - df$. In a natural way, u extends to $M = \mathbf{R} \times N$. Then the vector field $u - (vf)\omega$ is an infinitesimal automorphism of (M, ω) . Thus (7.2) follows, since M has sufficiently ample c^∞ or c^∞ functions. Finally, for another contact form β subordinate to the same contact structure (α) as α , there exists a strictly positive function g on N with $g\beta = \alpha$. Identify N with $\{(\log g(x), x) \in M \mid x \in N\}$, consider β as a form on this section, spread β all over M with z . Then we regain α^* . Hence we have finished the proof of the theorem except for some trivial details.

Remark. Another construction of M and ω makes the functor bijective. In fact we can define the inverse of that functor in the following way. To each coordinate system σ of M we assign a contact form α_σ on the coordinate neighborhood U_σ of σ which is subordinate to the given contact structure (α) restricted to U_σ . Strictly speaking we consider the set of all pairs (σ, α) of all charts σ of the maximal atlas of M and all contact forms α on U_σ compatible with (α) . For two pairs $(\sigma, \alpha), (\tau, \beta)$, we have a strictly positive function $g_{(\sigma, \alpha), (\tau, \beta)}$ on $U_\sigma \cap U_\tau$ such that $g_{(\sigma, \alpha), (\tau, \beta)}\beta = \alpha$. This assignment defines a principal bundle M over N with the multiplicative group $\{e^t \mid t \in \mathbf{R}\}$ as the structure group. Each coordinate system (σ, α) gives a local direct product decomposition $\mathbf{R} \times U_\sigma$ of M on which ω_σ is defined from α_σ as ω from α in the proof. All ω_σ 's patched together make the ω on M .

Remark. If it is assumed instead of (7.3) that the group of all automorphisms of (M, ω) is transitive on M , then both (7.0) and (3.1) are automatically satisfied.

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