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REMARKS ON THE FIRST MAIN THEOREM IN EQUIDISTRIBUTION THEORY. I

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1. This is the first of a series commenting on the various aspects of the First Main Theorem (FMT) in several complex variables as proved by Chern [1] and Levine [4]. Our ultimate goal will be to recast the theorem of Chern [1, p. 537] in a form which can adequately "explain" the Fatou-Bieberbach example. We note that the FMT has recently been generalizd by Stoll [6].

We shall deal exclusively with the equi-dimensional case of the FMT, i.e. the situation where the dimensions of the domain and range manifolds are the same. In [4], Levine proved the nonintegrated FMT for a holomorphic $f: D \to P_pC$ by explicitly writing down a $(2p-1)$ form Λ in P_pC, and expressed the boundary integral $\int_{\partial D} f^*\Lambda$ as the difference of the counting function and the volume of the singular chain $f(D)$. The purpose of this short note is to point out that at least for the equi-dimensional case, which we are interested in, an a priori knowledge of Λ is unnecessary; a precise statement is given in the following theorem.

Let D be an orientable compact manifold of real dimension d, and M an orientable compact riemannian manifold without boundary also of dimension d. We adopt the convention throughout that Ψ denotes the volume form of M and that $\int_M \Psi = 1$. If $f: D \to M$ is C^∞, we write $v(D)$ for $\int_D f^*\Psi$ as usual.

Theorem. *For every $a \in M$, there exists an integrable $(d-1)$-form μ_a on M such that*:

(i) *If $f: D \to M$ is C^∞, and $f^{-1}(a)$ is finite and disjoint from ∂D, then*

$$v(D) = n(D, a) + \int_{\partial D} f^*\mu_a ,$$

where $n(D, a)$ denotes the algebraic number of points in $f^{-1}(a)$.

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(ii) *μ_a is C^∞ in M-{a} with $d\mu_a = \Psi$, and its singularity at a is specified in* (12).

(iii) *If M is Kählerian and of complex dimension n $(2n = d)$, then there is an $(n-1, n-1)$ form λ_a in M-$\{a\}$ such that $d^c \lambda_a = \mu_a$.*

(iv) *If M is Kählerian, then a choice for each λ_a can be made so that λ_a depends continuously on a, and each λ_a is a positive $(n-1, n-1)$ form on M.*

We briefly explain the terms involved. First, $n(D, a)$ of (i). Since $f^{-1}(a)$ is a finite point set off ∂D, let $f^{-1}(a) = \{b_1, \cdots, b_s\}$ and around each b_i choose an open ball neighborhood U_i so that $f^{-1}(a) \cap U_i = \{b_i\}$. Let U be an open ball neighborhood of a so that $f(U_i) \subseteq U$. Then $f: U_i - \{b_i\} \to U - \{a\}$. If σ, σ_i are respectively the generators of $H_{d-1}(U - \{a\})$ and $H_{d-1}(U_i - \{b_i\})$ coherent with the orientations of M and D, then

$$f_*(\sigma_i) = n_i\sigma \tag{1}$$

for an integer n_i. By definition,

$$n(D, a) = \Sigma_i n_i \,. \tag{2}$$

Next, by *continuous dependence of λ_a on a*, we mean: if $a' \to a$, theu $\lambda_{a'} \to \lambda_a$ in the sense of currents (see de Rham [5]). To say that λ_a is *positive*, we mean that locally there exists $(n-1, 0)$ forms θ_i such that

$$\lambda_a = (\sqrt{-1})^{\text{sign}} \Sigma_i \theta_i \wedge \bar{\theta}_i \,, \tag{3}$$

where sign $= (n-1)^2$. This implies of course that λ_a induces a non-negative measure on every complex subvariety of codimension one in M. (Complex manifolds are oriented as in Weil [7, p. 33].)

2. We proceed to the proof of the theorem. (For background and motivation, see [2], [3] and [8].) We refer once and for all to de Rham [5] for notation as well as the more delicate facts about harmonic theory on compact riemannian manifolds; see particularly §§ 27–31. Let δ_a be the Dirac measure at a, i.e. if f is a continuous function on M, $\delta_a(f) = f(a)$. Since $\int_M \Psi = 1$, $\Psi - \delta_a$ is orthogonal to the harmonic n-forms (which are of course just constant multiples of Ψ) and so the equation of currents:

$$\Delta \Pi_a = \Psi - \delta_a \tag{4}$$

has a solution. Define

$$\mu_a = \delta \Pi_a \,. \tag{5}$$

Note that while Π_a is defined only up to constant multiples of Ψ, the definition of μ_a is unambiguous. Now,

$$d\mu_a = d\delta\Pi_a = \Delta\Pi_a = \Psi - \delta_a ,$$

so that in M-$\{a\}$, $d\mu_a = \Delta\Pi_a = \Psi$. (ii) is therefore clear in view of the Weyl Lemma. Let M be Kählerian. By a well-known identity: $[\Lambda, d^c] = \delta$ (Weil [7, p. 43]) so that $\delta\Pi_a = -d^c\Lambda\Pi_a$. If we define in turn:

$$(6) \qquad \lambda_a = -\Lambda\Pi_a ,$$

(iii) is immediate. To prove (iv), let us first recall some general facts about the Green operator. So M is again compact riemannian and orientable. One knows that one solution of $\Delta\Pi_a = \Psi - \delta_a$ can be chosen to be $\Pi_a = G(\Psi - \delta_a)$ where G is the Green operator. In other words, we may let

$$(7) \qquad \Pi_a = -G\delta_a ,$$

since $G\Psi = o$. Now, G is an integral operator which (when operating on functions) has kernel $g(x, y)\Psi_y$, where $g(x, y)$ is a function defined on $M \times M$ and C^∞ off the diagonal; its singularity along the diagonal will be presently specified. To make Π_a more explicit, let ϕ be a C^∞ function on M. Then Π_a as a current operates on ϕ as follows:

$$\begin{aligned} \Pi_a(\phi) &= -(G\delta_a)(\phi) = -\delta_a(G\phi) \\ &= -\delta_a\left(\int_M g(x, y)\phi(y)\Psi_y\right) \\ &= -\int_M g(a, y)\phi(y)\Psi_y . \end{aligned}$$

In other words, Π_a is a current induced by the integrable d form $\{-g(a, y)\Psi_y\}$ which is C^∞ except at a, i.e. except when $a = y$. To describe the behavior of g at (a, a), let $r_a : M \to \mathbf{R}$ be the function: $r_a(y) =$ distance from a to y in terms of the riemannian metric on M. r_a is C^∞ in a sufficiently small neighborhood of a as is well-known. Then the following holds when y is near a:

$$(8) \qquad g(a, y) = \begin{cases} \dfrac{1}{(d-2)S_d} \dfrac{1}{r_a^{d-2}} + 0\left(\dfrac{1}{r_a^{d-3}}\right), & \text{if } \ d > 2 , \\ \dfrac{1}{2\pi} \log \dfrac{1}{r_a(y)} + 0(1) , & \text{if } \ d = 2 . \end{cases}$$

Here, S_d denotes the volume of the unit $(d-1)$-sphere imbedded in $\mathbf{R}^d$ (which is $\dfrac{2\pi^{d/2}}{\Gamma(d/2)}$ to be precise), $0(1)$ a bounded function, and $0\left(\dfrac{1}{r_a^{d-3}}\right)$ a function h such that $(r_a^{d-3}h)$ is bounded. From (8) and by the compactness of the diagonal in $M \times M$, $g(x, y)$ is nonnegative in a suitable neighborhood N of the diagonal. In $M \times M - N$, let K' be the lower bound of $g(x, y)$ and let

$K > |K'|$. Then clearly $\{g(x, y) + K\}$ is strictly positive on $M \times M$ and hence, for every a, $-\{g(a, y) + K\}$ as a function of y is a strictly negative function on M. Thus far, we have only assumed M is riemannian. If now M is Kählerian, we may let

$$(9) \qquad \begin{aligned} \Pi'_a &= -(g(a,\cdot) + K)\Psi , \\ \lambda'_a &= -\Lambda\Pi'_a = (g(a,\cdot) + K)\Lambda\Psi . \end{aligned}$$

It is immediate that $\mu_a = \delta\Pi'_a$ and λ'_a satisfies all the properties thus far claimed for λ_a. Since Λ is just the interior product by the Kähler form and $(g(a,\cdot) + K) > 0$, λ'_a is obviously a positive $(n-1, n-1)$ form on M (see (3)). With this choice of λ'_a, the last part of (iv) is proved. It remains to show that with the same choice of λ'_a the other part of (iv) also holds. Now if $a' \to a$, then as currents, obviously $\delta_{a'} \to \delta_a$. Since G is continuous on currents, $G\delta_{a'} \to G\delta_a$, which means $\Pi_{a'} \to \Pi_a$ by (7), so that $\Pi'_{a'} \to \Pi'_a$ by (9). Since Λ is obviously also a continuous operator, $\lambda'_{a'} \to \lambda'_a$ again by (9). Thus we have proved (ii) to (iv) completely. The next section is devoted to the proof of (i).

3. In keeping with the notation of the discussion before (1), let $f^{-1}(a) = \{b_1, \cdots, b_s\}$ as before, and let U be a *normal* coordinate neighborhood centered at a with coordinate functions $x_1, \cdots, x_d$. We may assume $U = \{\Sigma_i x_i^2 < \varepsilon^2\}$ and that $\left\{\frac{\partial}{\partial x_i}(0)\right\}$ is an orthonormal set. We may further assume ε so small that

(α) $r_a = \{\Sigma_i x_i^2\}^{\frac{1}{2}}$, i.e. the geodesics in U emanating from a are in fact minimizing geodesics.

(β) There exists a disjoint set of open balls $\{U_i, \cdots, U_s\}$ such that $f^{-1}(a) \cap U_i = \{b_i\}$ and $f(U_i) \subseteq U$.

Now by Stokes' theorem and (ii) of our theorem,

$$\begin{aligned} v(D) = \int_D f^*\Psi &= \lim_{\varepsilon \to 0} \int_{D - \{U_1, \cdots, U_s\}} f^*\Psi \\ &= \lim_{\varepsilon \to 0} \left\{ \int_{\partial D} f^*\mu_a - \Sigma_i \int_{\partial U_i} f^*\mu_a \right\} \\ &= \int_{\partial D} f^*\mu_a - \lim_{\varepsilon \to 0} \Sigma_i \int_{\partial U_i} f^*\mu_a . \end{aligned}$$

In view of (1) and (2), to prove (i) it suffices to prove

$$(10) \qquad n_i = \lim_{\varepsilon \to 0} \int_{\partial U_i} - f^*\mu_a .$$

We pause to observe that if ζ is a *closed* $(d-1)$ form in $U - \{a\}$ such that

$\int_\sigma \zeta = 1$ (in the notation of (1)), then (1) implies that $n_i = \int_{\sigma_i} f^*\zeta$. Since we may choose ∂U_i to be σ_i and ∂U to be σ, we then have

$$(11) \qquad n_i = \int_{\partial U_i} f^*\zeta\,, \quad \text{provided } d\zeta = 0 \text{ and } \int_{\partial U} \zeta = 1\,.$$

We will in effect show that, by passing to the limit, the right side of (10) is in the form of the right side of (11). We will take up the case of $\dim M = d > 2$; because of the dichotomy of (8), the case $d = 2$ has to be treated separately, though it is simpler.

Now $\mu_a = \delta\Pi_a$, so (7) and (8) lead to the following:

$$\begin{aligned}\mu_a &= \delta(-g(\mathrm{a},\cdot)\Psi)\\ &= -{*d*}\left\{\left[\frac{-1}{(d-2)S_d}\frac{1}{r_a^{d-2}} + 0\left(\frac{1}{r_a^{d-3}}\right)\right]\Psi\right\}\\ &= {*d}\left[\frac{1}{(d-2)S_d}\frac{1}{r_a^{d-2}} + 0\left(\frac{1}{r_a^{d-3}}\right)\right],\end{aligned}$$

which simplifies to, by virtue of (α),

$$\mu_a = *\left[\frac{1}{S_d r_a^d}\Sigma_i x_i dx_i + 0\left(\frac{1}{r_a^{d-2}}\right)\right].$$

Now we know that

$$*dx_i = \Sigma_j(-1)^{j-1}\sqrt{\det(g_{mn})}\, g^{ij} dx_i \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_d\,,$$

where (g_{mn}) is the matrix of the metric tensor with respect to the coordinate system $\{x_1, \cdots, x_d\}$, and g^{ij} are the components of the matrix inverse to (g_{mn}). Since $\{x_1, \cdots, x_d\}$ is a *normal* coordinate system by choice and $\left\{\frac{\partial}{\partial x_i}(o)\right\}$ are orthonormal vectors, it follows that

$$g^{ij} \text{ and } g_{ij} = \delta_{ij} + 0(r_a^2)\,,$$

where $0(r_a^2)$ will now denote a function or a form which vanishes to at least the second order at the origin. Obviously then,

$$*dx_i = (-1)^{i-1} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_d + 0(r_a^2)\,.$$

We have therefore arrived at the following:

$$(12) \qquad \mu_a = \frac{-1}{S_d r_a^d}\Sigma_i(-1)^{i-1}x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_d + 0\left(\frac{1}{r_a^{d-2}}\right).$$

But it is clear, by way of polar coordinates, that

$$\int_{f(\partial U_i)} 0\left(\frac{1}{r_a^{d-2}}\right) \to 0 \quad \text{as} \quad \varepsilon \to 0\,.$$

Hence, the right hand side of (10) becomes

$$\lim_{\varepsilon\to 0}\int_{\partial U_i} -f^*\mu_a = \lim_{\varepsilon\to 0}\int_{\partial U_i} f^*\left(\frac{1}{S_d r_a^d}\Sigma_i(-1)^{i-1}x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_d\right)$$

$$\text{DEF} \lim_{\varepsilon\to 0}\int_{\partial U_i} f^*\theta\,.$$

By a routine calculation, $d\theta = 0$ in $U - \{a\}$ and moreover, one recognizes $\Sigma_i(-1)^{i-1}\frac{x_i}{\varepsilon}\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_d$ as the volume form of the ε-sphere imbedded in R^d. As ∂U is the ε-sphere in this coordinate system, $\int_{\partial U_i}\theta = 1$. By ($\beta$) and (11), we have $n_i = \int_{\partial U_i} f^*\theta$. Thus we have proved (10), and therewith (i). Hence the proof of the theorem is complete.

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