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A UNIQUENESS THEOREM FOR MINIMAL SUBMANIFOLDS

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1. Introduction

The following theorem is well known: There is a unique geodesic joining two points on a complete simply connected Riemannian manifold of nonpositive sectional curvature.

The main point of this paper is the following generalization.

Theorem. *Let N and B be minimal submanifolds of a Riemannian manifold M whose sectional curvature is nonpositive.* (If $\dim N = \dim M - 1$, it would suffice to know that M has nonpositive Ricci curvature.)
Suppose that:

a) *N is oriented and finite with oriented boundary $\partial N \subset B$.*
b) *B is a totally geodesic submanifold of M.*
c) *Each point p of N can be joined to B by a geodesic, which is perpendicular to B at the end-point, and varies smoothly with p.*

Conclusion: $N \subset B$.

The main tool is an integral-geometric inequality, which enables one to make various extensions of the main result, e.g., to the case where B is only a minimal submanifold of M, or where N is a manifold with singularities, e.g., a piece of an analytic subvariety of a Kähler manifold.

2. Proof of the theorem

Let M be a complete Riemannian manifold, and N and B submanifolds of M. (For notations not explained here, refer to [1] and [2].) Let exp: $T(M) \to M$ be the exponential map of the Riemannian structure, where $T(M)$ is the tangent bundle of M. Suppose there exists a vector field X on M such that:

a) For $p \in N$, $\exp(X(p)) \in B$.
b) The geodesic $t \to \exp(tX(p))$ is perpendicular to B at $t = 1$.

Let $\| \ \|$ denote the norm on tangent vectors associated with the inner product $\langle , \rangle$ defining the Riemannian metric on M, $f(p) = \|X(p)\|^2$ for $p \in N$, and Δ^N be the Laplace-Beltrami operator, relative to the induced metric on N. Our

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goal is first to find a convenient formula for $\Delta^N f$, and then to integrate it over N.

Let p be a point of N, and $s \to \sigma(s)$ a geodesic of N starting at p. Construct the homotopy $\delta(s, t) = \exp(tX(\sigma(s)))$, $0 \leq s, t \leq 1$. Then

$$
\begin{aligned}
\frac{1}{2}\frac{d}{ds}f(\sigma(s)) &= \frac{1}{2}\frac{d}{ds}\int_0^1 \langle \partial_t\delta, \partial_t\delta \rangle dt \\
&= \int_0^1 \langle \nabla_s\partial_t\delta, \partial_t\delta \rangle dt \\
&= \int_0^1 \langle \nabla_t\partial_s\delta, \partial_t\delta \rangle dt = \langle \partial_s\delta, \partial_t\delta \rangle \Big|_{t=0}^{t=1} .
\end{aligned}
\tag{2.1}
$$

Here $\partial_t\delta(s, t)$ is the tangent vector to the curve $u \to \delta(s, u)$ at $u = t$, $\partial_t\delta$ is the corresponding vector field along the homotopy δ, $\partial_s\delta$ is defined similarly, and $\nabla_t\partial_s\delta(s, t)$ is the covariant derivative (with respect to the Levi-Civita affine connection) of the vector field $u \to \partial_s\delta(s, u)$ along the curve $u \to \delta(s, u)$. The rules of this formalism are given in more detail in [1] or [2]. For example, since each curve $t \to \delta(s, t)$ is a geodesic, we have $\nabla_t\partial_t\delta(s, t) = 0$.

$$
\begin{aligned}
\frac{1}{2}\frac{d^2}{ds^2}f(\sigma(s)) &= \langle \nabla_s\partial_s\delta, \partial_t\delta \rangle - \langle \partial_s\delta, \nabla_s\partial_t\delta \rangle \Big|_{t=0}^{1} \\
&= \langle \nabla_s\partial_s\delta, \partial_t\delta \rangle \Big|_{t=0}^{1} - \int_0^1 \frac{\partial}{\partial t}\langle \partial_s\delta, \nabla_s\partial_t\delta \rangle dt \\
&= \langle \nabla_s\partial_s\delta, \partial_t\delta \rangle \Big|_{t=0}^{1} - \int_0^1 \langle \nabla_s\partial_t\delta, \nabla_s\partial_t\delta \rangle dt \\
&\quad - \int_0^1 \langle \partial_s\delta, R(\partial_t\delta, \partial_s\delta)(\partial_t\delta) \rangle dt ,
\end{aligned}
\tag{2.2}
$$

where $R(\ ,\ )(\ )$ is the curvature tensor of M. The last term can be written as

$$
\int_0^1 \|\partial_s\delta \wedge \partial_t\delta\|^2 K(\partial_t\delta, \partial_s\delta) dt ,
$$

where $K(\ ,\ )$ is the sectional curvature, and $\|\partial_s\delta \wedge \partial_t\delta(s, t)\|^2$ is the square of the area of the parallelogram spanned by $\partial_s\delta(s, t)$ and $\partial_t\delta(s, t)$. Let $S^N_{(\ )}(\ ,\ )$ and $S^B_{(\ )}(\ ,\ )$ be the second fundamental form of N and B. Write $X = X' + X''$, where X' is tangent, and X'' perpendicular to N. Then

$$
\begin{aligned}
\langle \nabla_s\partial_s\delta, \partial_t\delta \rangle(0, 1) &= S^B_{\partial_t\delta(0,1)}(\partial_s\delta(0, 1), \partial_s\delta(0, 1)) , \\
\langle \nabla_s\partial_s\delta, \partial_t\delta \rangle(0, 0) &= S^N_{X''(p)}(\sigma'(0), \sigma'(0)) .
\end{aligned}
$$

Thus,

$$\frac{1}{2}\frac{d^2}{ds^2}f(\sigma(s))|_{s=0} = S^B_{\partial_t\delta(0,1)}(\partial_s\delta(0,1), \partial_s\delta(0,1))$$

$$(2.3) \qquad - S^N_{X''(p)}(\sigma'(0), \sigma'(0)) - \int_0^1 \langle \nabla_s\partial_t\delta, \nabla_s\partial_t\delta\rangle dt$$

$$+ \int_0^1 \|\partial_s\delta \wedge \partial_t\delta\|^2 K(\partial_t\delta, \partial_s\delta)dt .$$

Let us suppose that B is totally geodesic, and the sectional curvature of M is nonpositive. Then

$$(2.4) \qquad \frac{1}{2}\frac{d^2}{ds^2}f(\sigma(s))|_{s=0} \leq S^N_{X''(p)}(\sigma'(0), \sigma'(0)) .$$

Suppose $u_1, \cdots, u_n$ form an orthonormal basis of N_p. Let $\sigma_a(s)$ be the geodesics of N beginning at p and tangent there to u_a, $a = 1, \cdots, n$. Then

$$\frac{1}{2}\sum_a \frac{d^2}{ds^2}f(\sigma_a(s))|_{s=0} \leq \sum_a S^N_{X''(p)}(u_a, u_a) .$$

The left-hand side of this inequality is just $\frac{1}{2}\Delta^N f(p)$. Let $X_1, \cdots, X_n$ be an orthonormal basis for vector fields on N so that at the boundary points, $X_1(p)$ is the inward pointing normal to ∂N. Then, we have the basic inequality

$$\frac{1}{2}\Delta^N f \leq \sum_a S^N_{X''}(X_a, X_a) .$$

The right-hand side is zero if N is a minimal submanifold of M. Integrate this over N. Green's formula gives

$$\int_N \nabla^N f = \int_{\partial N} X_1(f) ,$$

where the volume elements are assumed to be those defined by the induced Riemannian metric on N and ∂N.

(2.1) applies to calculate $X_1(f)$. In fact, $X_1(f) = \langle X_1, X\rangle$. Let us assume that $\int_{\partial N}\langle X_1, X\rangle = 0$, and N is a minimal submanifold of M, i.e., the trace of its second fundamental form is zero in every normal direction. (For example, if $\partial N \subset B$, as in the statement of Theorem 1, then $X(p) = 0$ automatically.)

Thus, we have

$$\nabla_t\partial_s\delta = 0 = \nabla_s\partial_t\delta ,$$

i.e., X has zero corvariant derivative at every point of N and in every direction tangential to N. In particular, $\langle X, X\rangle = f$ is constant along N. We also have

$$\|\partial_s\delta \wedge \partial_t\delta\|^2 K(\partial_t\delta, \partial_s\delta) = 0 .$$

If N is a hypersurface, we have either N is totally geodesic, or $X'' = 0$ on an open subset of N; that open subset is a "focal submanifold" for the family $(p, t) \to \exp(tX(p))$ of geodesics of N. At any rate, Theorem 1 is proved.

Final remarks on Theorem 1: If B is a closed submanifold of M, hypothesis b) of Theorem 1 follows from the assumption that the curvature of N is nonpositive, and, say, an assumption that M is simply connected (see [1]).

3. Weakening the hypothesis

Let $\delta_a(s, t) = \exp(tX(\sigma_a(s)))$, $a = 1, \cdots, n$. Using (2.3) again and assuming that the curvature is nonpositive give

$$\Delta(f) \leq \sum_a S^B_{\partial_t\delta_a(0,1)}(\partial_s\delta_a(0, 1), \partial_s\delta_a(0, 1)) - \sum_a S^N_{X''(p)}(\sigma'_a(0), \sigma'_a(0)) . \tag{3.1}$$

The second term on the right-hand side vanishes, of course, if N is a minimal submanifold. The first term will also vanish if B is a minimal submanifold, *providing* that $\partial_s\delta_1(0, 1), \cdots, \partial_s\delta_n(0, 1)$ is a basis for the tangent space to B. This requires

$$\dim B = \dim N . \tag{3.2}$$

Now, if (3.2) is satisfied, and each point $p \in N$ is not a focal point of B relative to the geodesic $t \to \exp(tX(p))$, then an orthonormal basis $u_1, \cdots, u_n$ of N_p can be found so that $\partial_s\delta_1(0, 1), \cdots, \partial_s\delta_n(0, 1)$ is a basis of $B_{\delta(0,1)}$. In this case the argument then goes through.

The argument also goes through if

$$S^B_{\partial_t\delta_a(0,1)}(\partial_s\delta_a(0, 1), \partial_s\delta_a(0, 1)) \geq 0 , \tag{3.3}$$

and N is a minimal submanifold. Now, (3.3) can be regarded as a "concavity" condition. The conclusion is that N cannot be completely on the "concave" side of B, if its boundary lies in B.

It is well known that complex-analytic submanifolds of Kähler manifolds are minimal submanifolds. One of the goals of minimal-submanifold theory is to understand whether or not facts known from algebraic geometry about algebraic varieties extend to general minimal submanifolds. This suggests that we investigate how singularities in N will affect the above arguments.

Suppose then that N^0 is a closed subset of N such that $N - N^0$ is a minimal submanifold, but that N^0 has no points in common with ∂N. Let us suppose that N^0 can be surrounded with "tube" T_ε, depending on a parameter ε, with boundary ∂T_ε, whose area goes to zero as $\varepsilon \to 0$. Let us apply these arguments to $N - T_\varepsilon$ instead of N. When applying Stoke's theorem to $\Delta^N f$, we will have to take into account a term of the form:

$$\int_{\partial T_\varepsilon} \langle X_1, X \rangle ,$$

where X_1 is the unit normal to the boundary ∂T_ε. Note, however, that this does not depend on the derivative of X, as one would expect a priori. It is this simple fact that gives hope that the uniqueness proofs can be extended to manifolds with singularities.

The next situation to be considered should be that where N has constant positive curvature. However, the methods used here break down in this case.

References

[1] R. Hermann, *Focal points of closed submanifolds of Riemannian spaces*, Ned. Akad. Wet. 25 (1963) 613–628.

[2] ——, *Differential geometry and the calculus of variations*, Academic Press, Now York, to appear.

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