

## ALMOST COMPLEX STRUCTURES ON TENSOR BUNDLES

A. J. LEDGER & K. YANO

### 1. Introduction

It is well known that the tangent bundle of a  $C^\infty$  manifold  $M$  admits an almost complex structure if  $M$  admits an affine connection [1], [5] or an almost complex structure [7], [8]. The main purpose of this paper is to investigate a similar problem for tensor bundles  $T_s^r M$ . We prove that if a Riemannian manifold  $M$  admits an almost complex structure then so does  $T_s^r M$  provided  $r + s$  is odd. If  $r + s$  is even a further condition is required on  $M$ . The proofs depend on some generalizations of the notions of lifting vector fields and derivations on  $M$ , which were defined previously only for tangent bundles and cotangent bundles [4], [7], [8], [9], [10].

### 2. Notations and definitions

- (i)  $M$  is a  $C^\infty$  paracompact manifold of finite dimension  $n$ .
- (ii)  $F(M)$  is the ring of real-valued  $C^\infty$  functions on  $M$ .
- (iii) For  $r + s > 0$ ,  $T_s^r M$  is the bundle over  $M$  of tensors of type  $(r, s)$ , contravariant of order  $r$  and covariant of order  $s$ .  $\pi$  is the projection of  $T_s^r M$  onto  $M$ . We write  $T_0^r M = T^r M$ ,  $T_s^0 M = T_s M$ .
- (iv)  $\mathcal{T}_s^r(M)$  is the module over  $F(M)$  of  $C^\infty$  tensor fields of type  $(r, s)$ . We write  $\mathcal{T}_0^r(M) = \mathcal{T}^r(M)$ ,  $\mathcal{T}_s^0(M) = \mathcal{T}_s(M)$ , and  $\mathcal{T}^0(M) = F(M)$ .  $\mathcal{T}(M)$  is the direct sum  $\sum_{r,s} \mathcal{T}_s^r(M)$ .  $T_p$  is the value at  $p \in M$  of a tensor field  $T$  on  $M$ , and  $\mathcal{T}_s^r(p)$  is the vector space of tensors of type  $(r, s)$  at  $p$ .
- (v) Let  $S \in \mathcal{T}_s^r(p)$  and  $T \in \mathcal{T}_s^r(p)$ . Then the real number  $S(T) = T(S)$  is defined, in the usual way, by contraction. It follows that if  $S \in \mathcal{T}_s^r(M)$  then  $S$  is a differentiable function on  $T_s^r M$ .
- (vi) A map  $D: \mathcal{T}(M) \rightarrow \mathcal{T}(M)$  is a derivation on  $M$  if
  - (a)  $D$  is linear with respect to constant coefficients,
  - (b) for all  $r, s$ ,  $D\mathcal{T}_s^r(M) \subset \mathcal{T}_s^r(M)$ ,
  - (c) for all  $C^\infty$  tensor fields  $T_1$  and  $T_2$  on  $M$ ,

$$D(T_1 \otimes T_2) = (DT_1) \otimes T_2 + T_1 \otimes DT_2,$$

(d)  $D$  commutes with contraction.

A derivation is determined by its action on  $F(M)$  and  $\mathcal{F}^1(M)$ . In particular,  $\mathcal{F}_1^1(M)$  may be identified with the set of derivations which map  $F(M)$  to zero. The set of derivations on  $M$  forms a module  $\mathcal{D}M$  over  $F(M)$ .

(vii) The notation for covariant derivatives and curvature tensors is that of [2]. The linear connections considered on  $M$  are assumed to have zero torsion.

### 3. Vector fields on $T_r^s M$

In this section we show how vector fields on  $T_r^s M$  can be induced from vector fields, tensor fields of type  $(r, s)$ , and derivations on  $M$ .

We first prove a lemma which, together with its corollary, will be of use later.

**Lemma 1.** *Let  $p \in M$  and  $S \in \pi^{-1}(p)$ . If  $W$  is a vertical vector at  $S$  (i.e. tangential to  $\pi^{-1}(p)$  at  $S$ ) and  $W(\alpha) = 0$  for all  $\alpha \in \mathcal{F}_r^s(p)$  then  $W = 0$ .*

*Proof.* The vector space  $\mathcal{F}_r^s(p)$  is dual to  $\mathcal{F}_r^s(p)$  and hence  $\alpha$  contains a system of coordinates on  $\pi^{-1}(p)$ . The result follows immediately.

**Corollary 1.** *Let  $W \in \mathcal{F}^1(T_r^s M)$ . If  $W(\alpha) = 0$  for all  $\alpha \in \mathcal{F}_r^s(M)$  then  $W = 0$ .*

*Proof.* The assumption on  $W$  implies that for  $\beta \in \mathcal{F}_{r-1}^s(M)$  and  $f \in F(M)$ ,

$$0 = \frac{1}{2}W(df^2 \otimes \beta) = W((f \circ \pi)df \otimes \beta) = W(f \circ \pi)df \otimes \beta .$$

Hence  $d\pi W = 0$ , and so  $W$  is a vertical vector field. Thus  $W = 0$  by Lemma 1, the values of  $W$  on the zero section of  $\mathcal{F}_r^s M$  being zero by continuity.

**Proposition 1.** *Let  $T \in \mathcal{F}_r^s(M)$ . Then there is a unique  $C^\infty$  vector field  $T^v$  on  $T_r^s M$  such that for  $\alpha \in \mathcal{F}_r^s(M)$ ,*

$$(1) \quad T^v(\alpha) = \alpha(T) \circ \pi .$$

*Proof.* For  $p \in M$ ,  $\pi^{-1}(p)$  is a vector space and so  $T_p$  determines a unique vertical vector field  $T_p^v$  on  $\pi^{-1}(p)$  such that for  $\alpha \in \mathcal{F}_r^s(p)$ ,  $T_p^v(\alpha) = \alpha(T_p)$ . The cross section  $T$  on  $T_r^s M$  then determines a  $C^\infty$  vertical vector field which satisfies (1).  $T^v$  will be called the *vertical lift* of  $T$ .

**Corollary 2.** *Let  $S \in \pi^{-1}(p)$ , and let  $T_S^v$  be the value of  $T^v$  at  $S$ . Then the map  $T_p \rightarrow T_S^v$  is a linear isomorphism of  $\pi^{-1}(p) \rightarrow (\pi^{-1}(p))_S$ , where  $(\pi^{-1}(p))_S$  is the tangent space to the fibre  $\pi^{-1}(p)$  at  $S$ .*

**Proposition 2.** *Let  $D$  be a derivation on  $M$ . Then there is a unique vector field  $\bar{D}$  on  $T_r^s M$  such that for  $\alpha \in \mathcal{F}_r^s(M)$*

$$(2) \quad \bar{D}\alpha = D\alpha .$$

*Proof.* Let  $\{x^i\}$  ( $i = 1, 2, \dots, n$ ) be a coordinate system on a neighbourhood  $U$  of  $p \in M$ , and  $\{\omega^\theta\}$  ( $\theta = 1, 2, \dots, n^{r+s}$ ) a basis for  $\mathcal{F}_r^s(U)$ . Then  $\{x^i \circ \pi, \omega^\theta\}$  is a coordinate system on  $\pi^{-1}(U)$ . Define  $\bar{D}$  on  $\pi^{-1}(U)$  by

$$(3) \quad \bar{D}(x^i \circ \pi) = (Dx^i) \circ \pi,$$

$$(4) \quad \bar{D}(\omega^\theta) = D(\omega^\theta).$$

Thus a  $C^\infty$  vector field  $\bar{D}$  is defined on  $\pi^{-1}(U)$ . Moreover, for  $\alpha \in \mathcal{F}_r^s(U)$  we have  $\bar{D}\alpha = D\alpha$ . Hence, using Corollary 1, it follows that  $\bar{D}$  is defined over  $T_r^sM$  as the unique solution of (2).

**Corollary 3.** *If  $f \in F(M)$  then  $\bar{D}(f \circ \pi) = (Df) \circ \pi$ .*

**Corollary 4.**  *$\bar{D}$  is a vertical vector field if and only if  $D \in \mathcal{F}_1^1(M)$ .*

**Corollary 5.** *If  $D_1, D_2$  are derivations on  $M$ , and  $f_1, f_2 \in F(M)$ , then  $f_1D_1 + f_2D_2$  is a derivation on  $M$ , and*

$$\overline{f_1D_1 + f_2D_2} = (f_1 \circ \pi)\bar{D}_1 + (f_2 \circ \pi)\bar{D}_2.$$

Thus if  $F(M)$  is identified with  $F(M) \circ \pi = \{f \circ \pi : f \in F(M)\}$  then  $D \rightarrow \bar{D}$  is a linear map of  $\mathcal{D}M \rightarrow \mathcal{F}^1(T_r^sM)$ .

**Corollary 6.** *If  $p \in M$  and  $A \in \mathcal{F}_1^1(p)$  then for  $S \in T_r^s(p)$ ,*

$$(5) \quad \bar{A}_S = -(AS)_S^s,$$

where the suffix  $S$  denotes evaluation at  $S$ .

*Proof.* Let  $\alpha \in \mathcal{F}_r^s(p)$ . Then

$$\bar{A}_S(\alpha) = (A\alpha)(S) = -(AS)_S^s(\alpha).$$

The result follows from Lemma 1.

**Corollary 7.** *Let  $X \in \mathcal{F}^1(M)$  and  $\mathcal{L}_X$  denote Lie derivation with respect to  $X$ . Then  $\bar{\mathcal{L}}_X$  is a vector field on  $T_r^sM$ . In conformity with the notation of [4], [8], [9], [10], we call  $\bar{\mathcal{L}}_X$  the complete lift of  $X$  and write  $\bar{\mathcal{L}}_X = X^c$ .*

**Remark 1.** If  $f \in F(M)$  then

$$\mathcal{L}_{fX} = f\mathcal{L}_X - X \otimes df,$$

where  $X \otimes df$  is regarded as a derivation on  $M$ . Thus

$$(6) \quad (fX)^c = (f \circ \pi)X^c - \bar{X} \otimes \overline{df}.$$

Now if  $T_r^sM$  is the tangent bundle  $T^1M$  then for  $\alpha \in \mathcal{F}_1(M)$ ,

$$\bar{X} \otimes \overline{df}(\alpha) = -\alpha(X)df.$$

Hence by Proposition 1,

$$\overline{X \otimes df} = -dfX^\circ,$$

where  $X^\circ$  is the vertical lift of  $X$  to  $T^1M$ . We then have

$$(7) \quad (fX)^\circ = fX^\circ + dfX^\circ.$$

Equation (7) was used extensively in [8] but does not appear to extend to tensor bundles of high order. Equation (6) is perhaps a useful generalization.

**Lemma 2.** *Let  $p \in M$  and  $A \in \mathcal{T}_1^1(p)$ . Suppose there exist non-negative integers  $a$  and  $b$ , not both zero, such that  $A\mathcal{T}_a^a(p) = 0$ . Then  $A = kI$  where  $k$  is some real number. If  $a \neq b$  then  $A = 0$ .*

*Proof.* We prove the lemma for the case  $a > 0$ . The proof for  $a = 0$  and  $b > 0$  is essentially the same with covariance and contravariance exchanged.

Let  $S \in \mathcal{T}_a^{a-1}(p)$  be non-zero, and let  $X \in \mathcal{T}^1(p)$ . Then

$$AS \otimes X + S \otimes AX = 0.$$

Choose  $\omega \in \mathcal{T}_{a-1}^b(p)$  such that  $\omega(S) \neq 0$ . Then  $(A - kI)X = 0$ , where  $k = -\omega(AS)/\omega(S)$ . It follows immediately that  $A = kI$ . Then for  $T \in \mathcal{T}_a^a(p)$

$$0 = AT = k(a - b)T.$$

Hence, if  $a \neq b$  then  $k = 0$  and  $A = 0$ .

**Remark 2.**  $A = kI$  for some  $k$  is a necessary and sufficient condition for  $A\mathcal{T}_a^a(p) = 0$ ,  $a \neq 0$ .

**Corollary 8.** *Let  $D \in \mathcal{D}M$  and suppose there exist non-negative integers  $a$  and  $b$ , not both zero, such that  $D\mathcal{T}_a^a(M) = 0$ . Then  $D = fI$ , where  $f \in F(M)$ . If  $a \neq b$  then  $D = 0$ .*

*Proof.* Let  $h \in F(M)$  and  $T \in \mathcal{T}_a^a(M)$ . Then

$$(Dh)T = 0.$$

It follows immediately that  $DF(M) = 0$  and hence  $D \in \mathcal{T}_1^1(M)$ . Then by Lemma 2,  $D = fI$  for some  $f \in F(M)$ , and if  $a \neq b$ , then  $f$  is zero by Lemma 2. This completes the proof.

**Remark 3.**  $D = fI$  for some  $f \in F(M)$  is a necessary and sufficient condition for  $D\mathcal{T}_a^a(M) = 0$ ,  $a \neq 0$ .

**Corollary 9.** *The map  $D \rightarrow \bar{D}$  of  $\mathcal{D}M \rightarrow \mathcal{T}^1(T_r^rM)$  is a monomorphism when  $r \neq s$  and has kernel  $\{fI : f \in F(M)\}$  when  $r = s$ .*

*Proof.* This follows from Corollaries 1, 5 and 8.

**Corollary 10.** *If  $r \neq s$  then  $T_r^rM$  admits a vertical vector field which vanishes only on the zero section of  $T_r^rM$ .*

*Proof.* The vector field  $\bar{I}$  has the required properties.

**Corollary 11.** *Let  $p \in M$ ,  $A \in \mathcal{T}_1^1(p)$  and  $T \in \mathcal{T}_s^r(p)$ ,  $r \neq s$ . Then  $\bar{A} = T^\circ$  implies  $A = 0$  and  $T = 0$ .*

*Proof.* Suppose  $\bar{A} = T^v$ . Then by Corollaries 2 and 6,  $AS = -T$  for all  $S \in \mathcal{F}_s^r(p)$ . Since  $A$  is linear it follows that  $T = 0$  and  $A\mathcal{F}_s^r(p) = 0$ . Hence  $A = 0$  by Lemma 2.

Suppose now that  $\nabla$  is a linear connection (with zero torsion) on  $M$ , and let  $X \in \mathcal{F}^1(M)$ . Then  $\nabla X \in \mathcal{F}_1^1(M)$ , and hence, by Corollary 4,  $\overline{\nabla X}$  is a  $C^\infty$  vertical vector field on  $T_s^r M$ .

Another  $C^\infty$  vector field  $\bar{V}_X$  on  $T_s^r M$  is determined by the derivation  $\nabla_X$ . In conformity with [4] we write  $\bar{V}_X = X^h$ , and call  $X^h$  the horizontal lift of  $X$ . If  $f \in F(M)$  then using Corollary 3,

$$X^h(f \circ \pi) = \bar{V}_X(f \circ \pi) = (\nabla_X f) \circ \pi = (Xf) \circ \pi .$$

Hence

$$(8) \quad d\pi X^h = X .$$

The horizontal lift clearly satisfies

$$(fX + gY)^h = (f \circ \pi)X^h + (g \circ \pi)Y^h ,$$

for  $f, g \in F(M)$  and  $X, Y \in \mathcal{F}^1(M)$ . Thus the horizontal lift is a linear map of  $\mathcal{F}^1(M) \rightarrow \mathcal{F}^1(T_s^r M)$  if, as before,  $F(M)$  and  $F(M) \circ \pi$  are identified. Since  $\bar{V}_X = 0$  if and only if  $X = 0$ , the horizontal lift is a monomorphism, and so determines a horizontal subspace  $H_S$  of dimension  $n (= \dim M)$  at each point  $S \in T_s^r M$ . Then  $C^\infty$  distribution  $H$  on  $T_s^r M$  so obtained is usually called the horizontal distribution determined by the connection  $\nabla$ .

If  $S \in T_s^r M$  then the tangent space  $(T_s^r M)_S$  is the direct sum  $V_S + H_S$ , where  $V_S$  is the subspace of vertical vectors at  $S$ . Thus, if  $W \in (T_s^r M)_S$  then

$$W = h(W) + v(W) ,$$

where  $h$  and  $v$  are the projections onto the horizontal and vertical subspaces at  $S$ . Clearly  $X^h = h(X^h)$  and  $T^v = v(T^v)$  for any vector  $X$  and tensor  $T$  of type  $(r, s)$  at  $\pi(S)$ .

#### 4. Lie brackets

We now determine, for later use, the Lie brackets of some particular types of vector fields on  $T_s^r M$ . These results generalize some of those already obtained for tangent bundles and cotangent bundles [1], [4], [7], [8], [9], [10].

**Lemma 3.** *Let  $T_1, T_2 \in \mathcal{F}_s^r(M)$  and  $X, X_1, X_2 \in \mathcal{F}^1(M)$ , and let  $D, D_1, D_2, A$  be derivations on  $M$ , where  $A \in \mathcal{F}_1^1(M)$ . Let  $R$  denote the curvature tensor field of the connection  $\nabla$ . Then*

$$(9) \quad [T_1^v, T_2^v] = 0 ,$$

$$(10) \quad [\bar{D}_1, \bar{D}_2] = \overline{[D_1, D_2]} ,$$

$$(11) \quad [\bar{D}, T^v] = (DT)^v ,$$

$$(12) \quad [X^h, T^v] = (\nabla_X T)^v ,$$

$$(13) \quad [X_1^h, X_2^h] = \overline{R(X_1, X_2)} + [X_1, X_2]^h ,$$

$$(14) \quad [X^h, \bar{A}] = \overline{\nabla_X A} ,$$

$$(15) \quad [X_1^c, X_2^c] = [X_1, X_2]^c .$$

*Proof.* Several equations can be proved by application of Corollary 1.

If  $p \in M$  then  $\pi^{-1}(p)$  is a vector space, and has the structure of an abelian Lie group. If  $S \in \mathcal{F}_s^r(M)$  then  $S^v$  is an invariant vector field on  $\pi^{-1}(p)$  and equation (9) follows immediately.

We have, from Proposition 2,

$$[\bar{D}_1, \bar{D}_2]\alpha = (\bar{D}_1\bar{D}_2 - \bar{D}_2\bar{D}_1)\alpha = [D_1, D_2]\alpha .$$

Since  $[D_1, D_2]$  is a derivation on  $M$ , from Proposition 2 we have

$$[D_1, D_2]\alpha = \overline{[D_1, D_2]}\alpha ,$$

and hence equation (10).

$$[\bar{D}, T^v]\alpha = (D(\alpha(T)) - (D\alpha)(T)) \circ \pi = (\alpha(DT)) \circ \pi = (DT)^v(\alpha) ,$$

which gives equation (11). Since  $X^h = \bar{\nabla}_X$ , equation (12) is a special case of (11).

Since  $R(X_1, X_2) \in \mathcal{F}_1^1(M)$ , we have

$$[X_1^h, X_2^h] = [\bar{\nabla}_{X_1}, \bar{\nabla}_{X_2}] = \overline{[\nabla_{X_1}, \nabla_{X_2}]} = \overline{R(X_1, X_2)} + \bar{\nabla}_{[X_1, X_2]} ,$$

from which follows immediately equation (13).

$$[X^h, \bar{A}]\alpha = \nabla_X(A\alpha) - A(\nabla_X\alpha) = (\nabla_X A)\alpha = \overline{(\nabla_X A)}\alpha ,$$

which gives equation (14). Since  $X^c = \bar{\mathcal{L}}_X$ , equation (15) is a special case of (10).

## 5. Almost complex structures

We now consider the main problem, that is, to determine a class of tensor bundles which admit almost complex structures. For this purpose it is sufficient to consider contravariant tensor bundles since a Riemannian metric tensor field induces a fibre preserving diffeomorphism of  $T_r^*M \rightarrow T^{r+s}M$ . Also

the tangent bundle  $T^rM$  of a Riemannian space always admits an almost complex structure [1], [5]. Hence we shall restrict attention to  $T^rM, r > 1$ .

**Lemma 4.** *Let  $\nabla$  and  $g$  be, respectively, a symmetric connection and a Riemannian metric tensor field on  $M$ , and  $E \in \mathcal{F}^{r-1}(M)$  be nowhere zero on  $M$ . Then  $T^rM$  admits three distributions which are mutually orthogonal with respect to a Riemannian metric tensor field  $\bar{g}$  induced on  $T^rM$  by  $\nabla$  and  $g$ .*

*Proof.* For each  $p \in M$  a scalar product  $\langle, \rangle$  is defined on the vector space  $\pi^{-1}(p)$  by  $\langle T_1, T_2 \rangle = t_1(T_2)$ , where, for any tensor  $T$  with components  $T^{i_1 i_2 \dots i_r}$ ,  $t$  is the covariant tensor associated to  $T$  by  $g$ . Thus  $t$  has components

$$t_{i_1 i_2 \dots i_r} = T^{j_1 j_2 \dots j_r} g_{i_1 j_1} g_{i_2 j_2} \dots g_{i_r j_r} ,$$

where each repeated suffix indicates summation over its range. If  $S \in T^rM$ , then a scalar product, denoted by the same symbol  $\langle, \rangle$ , is defined on the vector space  $(T^rM)_S$  by the three equations

$$(16) \quad \langle T_1^v, T_2^v \rangle = \langle T_1, T_2 \rangle \circ \pi ,$$

$$(17) \quad \langle T^v, X^h \rangle = 0 ,$$

$$(18) \quad \langle X_1^h, X_2^h \rangle = \langle X_1, X_2 \rangle \circ \pi ,$$

where  $X^h$  is the horizontal lift of  $X$  with respect to  $\nabla$ . These equations are easily seen to determine  $\bar{g}$  on  $T^rM$  with respect to which the horizontal distribution  $H$ , induced by  $\nabla$ , is orthogonal to the fibres of  $T^rM$  [3].

We now make use of  $E$ . For  $X \in \mathcal{F}^1(M)$ , define the vertical lift  $X_E^v$  of  $X$  with respect to  $E$  by

$$X_E^v = (E \otimes X)^v .$$

The map  $X \rightarrow X_E^v$  is then a monomorphism of  $\mathcal{F}^1(M) \rightarrow \mathcal{F}^1(T^rM)$ . Hence an  $n$ -dimensional  $C^\infty$  vertical distribution  $V^E$  is defined on  $T^rM$ . Let  $V^\perp$  be the distribution on  $T^rM$  which is orthogonal to  $H$  and  $V^E$ . Then  $H, V^E$  and  $V^\perp$  are the required distributions and the proof is complete.

We now give an alternative characterization of  $V^\perp$ .

**Lemma 5.** *Let  $p \in M, S \in \pi^{-1}(p)$ , and  $\mathcal{F}_E^r(p)$  be the subspace of  $\mathcal{F}^r(p)$  defined by*

$$\mathcal{F}_E^r(p) = \{T: \langle T, E \otimes X \rangle = 0 \text{ for all } X \in \mathcal{F}^1(p)\} .$$

*Then  $V_S^\perp = (\mathcal{F}_E^r(p))_S^v$ .*

*Let  $E^\perp(p)$  be the subspace of  $\mathcal{F}^{r-1}(p)$  defined by*

$$E^\perp(p) = \{T: \langle T, E \rangle = 0\} .$$

*Then  $\mathcal{F}_E^r(p) = E^\perp(p) \otimes \mathcal{F}^1(p)$ .*

*Proof.* The first part of the lemma follows from the fact that the vertical lift preserves scalar products. To prove the second part it is sufficient to note that  $E^\perp(p) \otimes \mathcal{T}^1(p) \subset \mathcal{T}_E^r(p)$ , and

$$\dim(E^\perp(p) \otimes \mathcal{T}^1(p)) = n(n^{r-1} - 1) = n^r - n = \dim \mathcal{T}_E^r(p).$$

**Theorem.** *If  $M$  admits an almost complex structure and a nowhere zero tensor field  $E \in \mathcal{T}^{r-1}(M)$ , then  $T^rM$  admits an almost complex structure.*

*Proof.* Let  $F$  be an almost complex structure on  $M$ . We define a  $C^\infty$  tensor field  $J$  of type  $(1,1)$  on  $T^rM$  by its action on the distributions  $H, V^E$  and  $V^\perp$ . Thus for  $X \in \mathcal{T}^1(M)$  and  $T \in \mathcal{T}^r(M)$  define  $J$  by

$$(19) \quad J(X^h) = X_E^v, J(X_E^v) = -X^h, J(T^v) = \bar{T}^v,$$

where  $\bar{T}$  is obtained by contracting  $T \otimes F$ , and has components  $T^{i_1 i_2 \dots i_{r-1} j} F_j^{i_r}$ , where  $T^{i_1 i_2 \dots i_r}$  and  $F_j^{i_r}$  are local components of  $T$  and  $F$  respectively. The restrictions of  $J$  to  $H + V^E$  and  $V^\perp$  are endomorphisms, and hence  $J$  is a tensor field on  $T^rM$ . It is easily seen that  $J$  is  $C^\infty$  and  $J^2 = -I$ ,  $I$  being the unit tensor. Hence  $J$  is an almost complex structure on  $T^rM$ .

**Corollary 12.** *Suppose a Riemannian manifold  $M$  admits an almost complex structure. Then  $T^rM$  admits an almost complex structure if (i)  $r$  is odd or (ii)  $r$  is even and  $M$  admits a nowhere zero vector field.*

*Proof.* (i) For  $r = 2s + 1$  choose  $E = (\otimes g^{-1})^s$ , where  $g^{-1}$  is the inverse of a metric tensor field  $g$  on  $M$ , and  $(\otimes g^{-1})^s$  is the tensor product of  $g^{-1}$  with itself  $s$  times.

(ii) For  $r = 2s, s > 1$ , choose  $E = (\otimes g^{-1})^{s-1} \otimes X$ , where  $M$  is assumed to admit a nowhere zero vector field  $X$ . For  $r = 2$  choose  $E = X$ .

### 6. Integrability of the almost complex structure $J$

We now establish necessary and sufficient conditions for the integrability of  $J$ .

Let  $e$  be the covariant tensor field of order  $r - 1$  associated to  $E$  by  $g$ ; thus, with respect to local coordinates,  $e$  has components  $e_{i_1 i_2 \dots i_{r-1}}$  given by

$$e_{i_1 i_2 \dots i_{r-1}} = g_{i_1 j_1} g_{i_2 j_2} \dots g_{i_{r-1} j_{r-1}} E^{j_1 j_2 \dots j_{r-1}}.$$

**Proposition 3.** *Suppose  $M$  admits an almost complex structure  $F$  and a nowhere zero tensor field  $E \in \mathcal{T}^{r-1}(M)$ . Then the induced almost complex structure  $J$  is integrable if and only if, for  $X, Y \in \mathcal{T}^1(M)$ ,*

$$R(X, Y) = 0, \quad \nabla_X E = 0, \quad \nabla_X F = 0, \quad \nabla_X \frac{e}{\langle E, E \rangle} = 0.$$



*Proof.* Let  $N$  be the Nijenhuis 2-form on  $T^rM$  with values in  $\mathcal{F}^1(T^rM)$ , defined by

$$N(W_1, W_2) = [W_1, W_2] + J[JW_1, W_2] + J[W_1, JW_2] - [JW_1, JW_2]$$

for  $W_1, W_2 \in \mathcal{F}^1(T^rM)$ . Then  $J$  is integrable if and only if  $N = 0$ .

Suppose  $N = 0$ . Then for  $X, Y \in \mathcal{F}^1(M)$ ,  $N(X_E^v, Y_E^v) = 0$ . Hence, putting  $W_1 = X_E^v, W_2 = Y_E^v$  we have, from (9), (12), (13), and the definition of  $J$ ,

$$\begin{aligned} \overline{R(X, Y)} &= J(\nabla_Y(E \otimes X))^v - J(\nabla_X(E \otimes Y))^v - [X, Y]^h \\ (20) \quad &= J((\nabla_Y E) \otimes X)^v - J((\nabla_X E) \otimes Y)^v - (\nabla_Y X)^h \\ &\quad + (\nabla_X Y)^h - [X, Y]^h \\ &= J((\nabla_Y E) \otimes X - (\nabla_X E) \otimes Y)^v \end{aligned}$$

since  $\nabla$  has zero torsion. Now since  $E \otimes \mathcal{F}^1(M)$  is a subspace of  $\mathcal{F}^r(M)$  there is a unique  $T \in \mathcal{F}^r(M)$  orthogonal to this subspace and a unique  $Z \in \mathcal{F}^1(M)$  such that

$$(\nabla_Y E) \otimes X - (\nabla_X E) \otimes Y = T + E \otimes Z .$$

Then from (19) and (20)

$$\overline{R(X, Y)} = \tilde{T}^v - Z^h .$$

Since  $\overline{R(X, Y)}$  is vertical,  $Z^h = 0$  and hence  $Z = 0$ . It follows from Corollary 11 that

$$(21) \quad R(X, Y) = 0 ,$$

$$(22) \quad T = 0 .$$

We thus have for all  $X, Y \in \mathcal{F}^1(M)$ ,

$$(\nabla_X E) \otimes Y = (\nabla_Y E) \otimes X .$$

Since  $M$  is assumed to admit an almost complex structure,  $\dim M \geq 2$ . Hence by choosing  $X, Y$  to be linearly independent it follows that

$$(23) \quad \nabla_X E = 0 .$$

We next consider the case  $N(X_E^v, T^v) = 0$ , where  $X_E^v \in V^E$  and  $T^v \in V^\perp$ . Then from (9), (12) and the definition of  $J$  we have

$$(24) \quad J(\nabla_X T)^v = (\nabla_X \tilde{T})^v .$$

It follows that  $(\nabla_X T)^v \in V^\perp$ . Choose  $T = S \otimes Y$  where  $S \in \mathcal{F}^{r-1}(M), Y \in \mathcal{F}^1(M)$

and  $\langle S, E \rangle = 0$  (since  $M$  is paracompact such an  $S$  exists and can be chosen to be non-zero in a neighbourhood of a point). Then by Lemma 5,  $T^v \in V^\perp$  and (24) imply that

$$(\nabla_x S) \otimes FY + S \otimes F\nabla_x Y = (\nabla_x S) \otimes FY + S \otimes \nabla_x(FY) .$$

Hence

$$S \otimes (\nabla_x F)Y = 0 ,$$

and it follows immediately that

$$(25) \quad \nabla_x F = 0 .$$

Finally, from Lemma 5 the condition  $(\nabla_x T)^v \in V^\perp$  implies that

$$(26) \quad 0 = e(\nabla_x S) = -(\nabla_x e)S .$$

But  $S$  is any tensor field which satisfies  $\langle S, E \rangle = 0$ . Hence we deduce that

$$(27) \quad \nabla_x e = \alpha(X)e ,$$

where  $\alpha \in \mathcal{F}_1(M)$ . Then  $\alpha$  is determined by

$$\alpha(X) = \frac{(\nabla_x e)(E)}{e(E)} = \frac{X(e(E))}{e(E)} = \frac{X \langle E, E \rangle}{\langle E, E \rangle} .$$

Thus

$$(28) \quad \alpha = d \log e(E) = d \log \langle E, E \rangle .$$

(If  $\nabla$  is the Riemannian connection associated with  $g$  then (23) implies (27) and  $\alpha = 0$ .) Hence, from (27) and (28), the tensor field  $\frac{e}{\langle E, E \rangle}$  has zero covariant derivative. This proves the necessity of the conditions in Proposition 3.

To prove the sufficiency we note that

$$\begin{aligned} N(X^v_E, Y^v_E) &= N(Y^h, X^h) = JN(Y^v_E, X^h) , \\ N(X^v_E, T^v) &= JN(T^v, X^h), \quad N(T^v_1, T^v_2) = 0 . \end{aligned}$$

Thus  $N = 0$  if  $N(X^v_E, Y^v_E) = N(X^v_E, T^v) = 0$ . Suppose  $\nabla_x E = 0$  and  $R(X, Y) = 0$  for all  $X, Y \in \mathcal{F}^1(M)$ . Then

$$\begin{aligned} N(X^v_E, Y^v_E) &= -J[X^h, Y^v_E] - J[X^v_E, Y^h] - [X^h, Y^h] \\ &= (\nabla_x Y)^h - (\nabla_y X)^h - [X, Y]^h = 0 . \end{aligned}$$

Suppose  $\nabla_x \frac{e}{\langle E, E \rangle} = 0$ . Then (27) follows and hence if  $T^v \in V^\perp$  then  $(\nabla_x T)^v \in V^\perp$ . If we next assume  $\nabla_x F = 0$  then we have

$$N(X_E^v, T^v) = (\nabla_x \tilde{T})^v - J(\nabla_x T)^v = 0,$$

which proves the sufficiency.

### 7. Kählerian structure on $T^rM$

We now determine necessary and sufficient conditions for the metric  $\bar{g}$  on  $T^rM$ , defined in §5, to be Kählerian with respect to  $J$ .

**Proposition 4.**  $\bar{g}$  is Hermitian with respect to  $J$  if and only if  $\langle E, E \rangle = 1$  and  $g$  is Hermitian with respect to  $F$ .

*Proof.* Suppose  $\bar{g}$  is Hermitian with respect to  $J$ . Then for  $X, Y \in \mathcal{F}^1(M)$ ,

$$\begin{aligned} \langle X, Y \rangle \circ \pi &= \langle X^h, Y^h \rangle = \langle JX_E^v, JY_E^v \rangle = \langle X_E^v, Y_E^v \rangle \\ &= \langle E \otimes X, E \otimes Y \rangle \circ \pi = \langle E, E \rangle \langle X, Y \rangle \circ \pi. \end{aligned}$$

Hence  $\langle E, E \rangle = 1$ . Now let  $p \in M$  and let  $S \in \mathcal{F}^{r-1}(p)$  be non-zero such that  $\langle S, E \rangle = 0$ . Then by Lemma 5 and the definition of  $J$  we have, for  $X, Y \in \mathcal{F}^1(p)$ ,

$$\begin{aligned} \langle S, S \rangle \langle X, Y \rangle \circ \pi &= \langle S \otimes X, S \otimes Y \rangle \circ \pi \\ &= \langle (S \otimes X)^v, (S \otimes Y)^v \rangle = \langle J(S \otimes X)^v, J(S \otimes Y)^v \rangle \\ &= \langle S \otimes FX, S \otimes FY \rangle \circ \pi = \langle S, S \rangle \langle FX, FY \rangle \circ \pi. \end{aligned}$$

Thus at  $p$ ,  $\langle X, Y \rangle = \langle FX, FY \rangle$ . Since  $p$  is arbitrary,  $g$  is Hermitian with respect to  $F$ . The sufficiency of the above conditions is easily proved by the same method.

**Proposition 5.** Suppose  $\bar{g}$  is Hermitian with respect to  $J$ . Then  $\bar{g}$  is Kählerian with respect to  $J$  if and only if  $\nabla$  is the Riemannian connection associated with  $g$ ,  $R = 0$ ,  $\nabla E = 0$  and  $\nabla F = 0$ .

*Proof.* Let  $\alpha$  be the field of 2-forms on  $T^rM$  defined for all  $W_1, W_2 \in \mathcal{F}^1(T^rM)$  by  $\alpha(W_1, W_2) = \langle W_1, JW_2 \rangle$ . Then  $\bar{g}$  is Kählerian with respect to  $J$  if and only if  $\alpha$  is closed and  $J$  is integrable [6, Chapter VII]. As usual it is sufficient to consider the action of  $\alpha$  and  $d\alpha$  on the three distributions  $H$ ,  $V^E$  and  $V^\perp$  on  $T^rM$ . Then for  $X, Y \in \mathcal{F}^1(M)$  and  $T_1^v, T_2^v \in V^\perp$  we have

$$\begin{aligned} \alpha(X_E^v, Y_E^v) &= \alpha(X^h, Y^h) = \alpha(T_1^v, X_E^v) = \alpha(T_1^v, X^h) = 0, \\ (29) \quad \alpha(X_E^v, Y^h) &= \langle E \otimes X, E \otimes Y \rangle \circ \pi = \langle X, Y \rangle \circ \pi, \\ \alpha(T_1^v, T_2^v) &= \langle T_1, \tilde{T}_2 \rangle \circ \pi. \end{aligned}$$

Suppose  $\bar{g}$  is Kählerian with respect to  $J$ . Then by Propositions 3 and 4,  $R = 0$ ,  $\nabla_X E = 0$ , and  $\nabla_X e = 0$ , for all  $X \in \mathcal{T}^1(M)$ . Let  $p \in M$ ,  $X \in \mathcal{T}^1(p)$ , and choose  $T \in \mathcal{T}^{r-1}(M)$  such that  $\langle T, E \rangle = 0$  and  $\langle T, T \rangle = 1$  on some neighbourhood  $U$  of  $p$ . Since  $R = 0$  parallel vector fields  $Y$  and  $Z$  exist on  $U$  with arbitrary initial values at  $p$ . Then using (9), (12) and Lemma 5 we have, on  $\pi^{-1}(p)$ ,

$$\begin{aligned}
 0 &= d\alpha((T \otimes Y)^v, (T \otimes X)^v, X^h) \\
 &= X \langle T \otimes Y, T \otimes FX \rangle + \langle T \otimes FY, \nabla_X(T \otimes Z) \rangle \\
 &\quad - \langle \nabla_X(T \otimes Y), T \otimes FZ \rangle \\
 (30) \quad &= X \langle Y, FZ \rangle + 2 \langle T, \nabla_X T \rangle \langle FY, Z \rangle \\
 &\quad + \langle FY, \nabla_X Z \rangle - \langle \nabla_X Y, FZ \rangle \\
 &= (\nabla_X g)(Y, FZ) - 2 \langle T, \nabla_X T \rangle \langle Y, FZ \rangle .
 \end{aligned}$$

Since  $F$  is non-singular it follows that

$$\nabla_X g = \alpha(X)g ,$$

for some  $\alpha \in \mathcal{T}_1(p)$ . Then since  $\nabla_X E = 0$  and  $\nabla_X e = 0$  it follows easily that for all  $X \in \mathcal{T}^1(p)$ ,

$$0 = \nabla_X e = (r - 1)\alpha(X)e .$$

The tensor  $e$  is non-zero and so  $\alpha = 0$ . Thus  $\nabla g = 0$  at  $p$  and hence on  $M$  since  $p$  is arbitrary. It follows that  $\nabla$ , having no torsion, is the Riemannian connection associated with  $g$ .

We now prove the sufficiency of the above conditions by showing that the 2-form  $\alpha$  is exact. Let  $X \in \mathcal{T}^1(M)$ , and  $T^v \in V^\perp$ . Define a 1-form  $\beta$  on  $T^r M$  as follows: at each point  $S \in T^r M$ ,

$$\beta(X^h) = \langle S, E \otimes X \rangle, \quad \beta(X^v) = 0, \quad \beta(T^v) = \frac{1}{2} \langle S, \bar{T} \rangle .$$

Then using (29) it follows after some calculation that  $\alpha = d\beta$ . Hence  $d\alpha = 0$ , and this together with Proposition 3 proves the sufficiency.

### 8. Integrability of $H + V^E$ and $H + V^\perp$

**Proposition 6.**  $H + V^E$  is integrable if and only if  $R = 0$  and for  $X \in \mathcal{T}^1(M)$ ,  $\nabla_X E = \alpha(X)E$ , where  $\alpha(X) = \frac{\langle E, \nabla_X E \rangle}{\langle E, E \rangle}$ .

*Proof.* It follows from (12) and (13) that  $H + V^E$  is an integrable distribution if and only if for  $X_1, X_2 \in \mathcal{T}^1(M)$ ,

$$(31) \quad (\nabla_{X_1}(E \otimes X_2))^v \in V^E ,$$

$$(32) \quad \overline{R(X_1, X_2)} \in V^E .$$

Let  $Y_1$  and  $Y_2$  be orthogonal vectors at  $p \in M$ , and let  $\langle T, E \rangle = 0$  at  $p$ . Then from (16), (32) and Corollary 6,

$$\begin{aligned} 0 &= \langle R(X_1, X_2)(T \otimes Y_1), T \otimes Y_2 \rangle \\ &= \langle T, T \rangle \langle R(X_1, X_2)Y_1, Y_2 \rangle . \end{aligned}$$

Hence  $R(X_1, X_2)Y_1 = cY_1$  where  $c$  is some real number which depends on  $X_1$  and  $X_2$ . Since  $Y_1$  is arbitrary it follows that  $R(X_1, X_2) = cI$  at  $p$ . Then at any point  $S \in \pi^{-1}(p)$  we have  $\overline{R(X_1, X_2)} = -crS^v$ , and by choosing  $S^v \in V^\perp$  it follows that  $\overline{R(X_1, X_2)} = 0$  at  $S$ ; hence  $c = 0$ . Since  $p, X_1$  and  $X_2$  are arbitrary we have  $R = 0$  on  $M$ .

Using (30) and Lemma 5 we obtain  $\nabla_X E = \alpha(X)E$  and  $\alpha$  is then uniquely determined by this equation.

The proof of the sufficiency is immediate.

**Proposition 7.**  $H + V^\perp$  is integrable if and only if  $R = 0$  and for  $X \in \mathcal{F}^1(M)$ ,  $\nabla_X e = \alpha(X)e$ , where  $\alpha = \frac{\langle e, \nabla_X e \rangle}{\langle e, e \rangle}$ .

*Proof.* The proof is similar to that of Proposition 6 and we shall use the same notation. It follows from (12), (13) and Lemma 5 that  $H + V^\perp$  is an integrable distribution if and only if for  $S^v \in V^\perp$ ,

$$(33) \quad (\nabla_{X_1}(S \otimes X_2))^v \in V^\perp ,$$

$$(34) \quad \overline{R(X_1, X_2)} \in V^\perp .$$

then from (16), (34) and Corollary 6,

$$\begin{aligned} 0 &= \langle R(X_1, X_2)(E \otimes Y_1), E \otimes Y_2 \rangle \\ &= \langle E, E \rangle \langle R(X_1, X_2)Y_1, Y_2 \rangle . \end{aligned}$$

Hence, as before,  $R = 0$ .

From (33) we obtain

$$0 = \langle \nabla_{X_1} S, E \rangle \langle X_2, Y \rangle$$

for  $Y \in \mathcal{F}^1(p)$ . Hence

$$0 = \langle \nabla_{X_1} S, E \rangle = e(\nabla_{X_1} S) = -(\nabla_{X_1} e)S .$$

It follows that  $\nabla_{X_1} e = \alpha(X_1)e$  at  $p$ . Since  $p$  and  $X_1$  are arbitrary we obtain  $\nabla_X e = \alpha(X)e$  on  $M$ , and  $\alpha$  is then uniquely determined.

The proof of the sufficiency is immediate.

**References**

- [ 1 ] P. Dombrowski, *On the geometry of tangent bundles*, J. Reine Angew. Math. **210** (1962) 73–88.
- [ 2 ] S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.
- [ 3 ] S. Sasaki, *On the differential geometry of tangent bundles of Riemannian manifolds*, Tôhoku Math. J. **10** (1958) 338–354.
- [ 4 ] A. J. Ledger & K. Yano, *The tangent bundle of a locally symmetric space*, J. London Math. Soc. **40** (1965) 487–492.
- [ 5 ] S. Tachibana & M. Okumura, *On the almost complex structure of tangent bundles of Riemannian spaces*, Tôhoku Math. J. **14** (1962) 158–161.
- [ 6 ] K. Yano, *Differential geometry on complex and almost complex spaces*, Pergamon, New York, 1965.
- [ 7 ] K. Yano & S. Ishihara, *Almost complex structures induced in tangent bundles*, Kôdai Math. Sem. Rep. **19** (1967) 1–27.
- [ 8 ] K. Yano & S. Kobayashi, *Prolongations of tensor fields and connections to tangent bundles I*, J. Math. Soc. Japan **18** (1966) 194–210.
- [ 9 ] K. Yano & A. J. Ledger, *Linear connections on tangent bundles*, J. London Math. Soc. **39** (1964) 495–500.
- [ 10 ] K. Yano & E. M. Patterson, *Vertical and complete lifts from a manifold to its co-tangent bundle*, J. Math. Soc. Japan **19** (1967) 91–113.

UNIVERSITY OF LIVERPOOL  
TOKYO INSTITUTE OF TECHNOLOGY