J. DIFFERENTIAL GEOMETRY
1 (1967) 245-256

CRITICAL POINTS AND CURVATURE FOR EMBEDDED POLYHEDRA

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Recently a new insight into the Gauss-Bonnet Theorem and other problems in global differential geometry has come about through the connection between total curvature of embedded smooth manifolds and critical point theory for non-degenerate height functions.

This paper presents an analogous program for embedded polyhedra. The methods are completely elementary, using the techniques neither of differential geometry nor of algebraic topology. As such the paper has a twofold purpose —to study global geometry of polyhedra for its own sake, and to give a deeper understanding of the theorems of global differential geometry through an elementary presentation of their finite or combinatorial content. Moreover the polyhedral theory applies to a wider class of objects, and gives a new interpretation of the relation between intrinsic and extrinsic curvature.

Although the polyhedral part of the paper is relatively self-contained, the remarks which show the connection with the differentiable theory presuppose a familiarity with the classical differentiable results. For a bibliography on these and related problems, see Kuiper [4]. This paper will contain no discussion of the possible convergence theorems relating the polyhedral and differentiable theories—for a presentation of this topic, expecially in the 2-dimensional and convex cases we refer to A. D. Alexandrow [1]. For related total curvature concepts see also Chern-Lashof [3].

A subsequent paper of the author will deal with critical points and curvature for mappings of complexes into E^l for $l > 1$, and into l-dimensional manifolds.

The author wishes to thank Professor Kuiper for his interest and advice throughout the development of this research.

1. The critical point theorem

Definition. A *convex cell complex M^k embedded in E^n* is a finite collection of cells $\{C^r\}$, where each C^0 is a point, and each C^r is a bounded convex set with interior in some affine $E^r \subset E^n$, such that the boundary ∂C^r of C^r is a union of C^s with $s < r$, and such that if $s < r$ and $C^s \cap C^r \neq \emptyset$, then $C^s \subset \partial C^r$. M^k is called *k-dimensional* if there is a C^k in M^k but no C^{k+1}.

Communicated by S. S. Chern, June 19, 1967.

Examples of convex cell complexes are embedded simplicial complexes and simplicial manifolds. In this paper the adjective "polyhedral" will refer to convex cell complexes.

Definition. A linear map $\xi\colon E^n \to E^1$ is *general for* M^k if $\xi(v) \neq \xi(w)$ whenever v and w are the vertices of a C^1 in M^k.

Definition. If ξ is general for M^k, we may define an indicator function:

$$A(C^r, v, \xi) = \begin{cases} 1 \text{ if } v \in C^r, \text{ and } \xi(v) \geq \xi(x) \text{ for all } x \text{ in } C^r, \\ 0 \text{ otherwise.} \end{cases}$$

Lemma 1. *If ξ is general for M^k, then for a fixed C^r,*

$$\sum_{v \in M} A(C^r, v, \xi) = 1 \ .$$

Proof. This expresses the fact that a general ξ on a convex C^r achieves its maximum at exactly one vertex.

Definition. If ξ is general for M^k, define the *index of v with respect to ξ* to be $a(v, \xi) = \sum_{r=0}^{k} (-1)^r \sum_{C^r \in M} A(C^r, v, \xi)$.

Definition. If $\mathscr{C}$ is a finite collection of cells, the *Euler characteristic of* $\mathscr{C}$ is $\chi(\mathscr{C}) = \sum_{r=0}^{k} (-1)^r \alpha_r(\mathscr{C})$, where $\alpha_r(\mathscr{C})$ = number of r-cells in $\mathscr{C}$.

Theorem 1 (Critical point theorem). *If ξ is general for M, then*

$$\sum_{v} a(v, \xi) = \chi(M) \ .$$

Proof.
$$\begin{aligned} \sum_{v \in M} a(v, \xi) &= \sum_{v \in M} \sum_{r=0}^{k} (-1)^r \sum_{C^r \in M} A(C^r, v, \xi) \\ &= \sum_{r=0}^{k} (-1)^r \sum_{C^r \in M} \Big[\sum_{v \in M} A(C^r, v, \xi) \Big] \\ &= \sum_{r=0}^{k} (-1)^r \alpha_r(M) \qquad \text{by Lemma 1} \\ &= \chi(M) \ . \end{aligned}$$

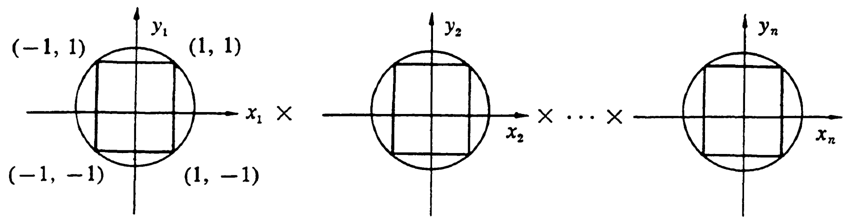
Remark 1. This is an analogue and a generalization of the classical critical point theorem for non-degenerate differentiable functions on smooth manifolds. To demonstrate this, we examine in detail an example of a polyhedral manifold and a smooth manifold in which the critical points are in exact correspondence.

Let $\tau^n \subset E^{2n}$ be the smooth n-torus embedded as a product of n copies of a circle in a plane. In coordinates,

$$\tau^n = \{(x_1, y_1, \cdots, x_n, y_n) \,|\, x_i^2 + y_i^2 = 2,\ i = 1, \cdots, n\} \ .$$

Analogously we define the polyhedral n-torus

$$T^n = \{(x_1, y_1, \cdots, x_n, y_n) \,|\, \max(|x_i|, |y_i|) = 1,\ i = 1, \cdots, n\} \ .$$



We consider the linear function $\xi : E^{2n} \to E^1$ given by the sum of coordinates: $\xi(x_1, y_1, \cdots, x_n, y_n) = x_1 + y_1 + \cdots + x_n + y_n$. On each of the manifolds, ξ achieves its maximum, $2n$, at exactly one point, with all coordinates $+1$. Similarly the minimum of $-2n$ is achieved at only one point with all coordinates -1.

The function ξ restricted to τ^n is a differentiable function with isolated non-degenerate critical points, which occur precisely at the 2^n points where $x_i = y_i = +1$ or -1 for $i = 1, \cdots, n$.

The vertices of the polyhedral manifold T^n are the 2^{2n} vertices of the $2n$-cube, i.e. the points with all coordinates $+1$ or -1. Furthermore two vertices lie in an edge of T^n if and only if they differ in precisely one coordinate. Therefore ξ is general for T^n. Moreover $a(v, \xi) = 0$ unless v has $x_i = y_i$ for all $i = 1, \cdots, n$, so the points where $a(v, \xi) \neq 0$ are precisely the critical points of ξ on τ^n.

On τ^n, a critical points is of *Morse* index r if and only if $\xi(p) = -2n + 2r$, and there are $\binom{n}{r}$ such points for each r. If a vertex v of T^n corresponds to a critical point of *Morse* index r, then the set of cells in T^n for which v is the highest vertex with respect to ξ fits together to form an open r-dimensional cell, so $a(v, \xi) = (-1)^r$.

Thus if v in a polyhedral manifold resembles a non-degenerate critical point p of a differentiable function, then $a(v, \xi) = (-1)$ to the power (Morse index of p). For example, at an absolute minimum, the Morse index is 0, and $a(v, \xi) = 1$ since $A(C^r, v, \xi) = 1$ if and only if $C^r = v$ itself.

By this correspondence, we have an exact analogy

$$\sum_{v \in M} a(v, \xi) = \sum_{r=0}^{n} (-1)^r \quad \text{(number of critical points of } \textit{Morse} \text{ index } r).$$

For the above example:

$$\chi(T^n) = \sum_{r=0}^{n} \binom{n}{r} (-1)^r = [1 + (-1)]^n = 0 ,$$

$$\chi(\tau^n) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} = [1 + (-1)]^n = 0 .$$

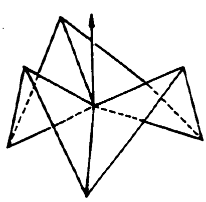
Remark 2. Theorem 1 is not only true for manifolds, but also for arbitrary convex cell complexes, with no additional structure.

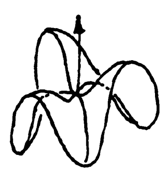
As particular examples of non-manifolds of interest, we mention the 2-dimensional pseudomanifolds and manifolds-with-boundary. Another non-manifold of interest is the complex $\bar{C}^k$ consisting of a convex cell C^k together with all of its boundary faces. For this complex $a(v, \xi) = 0$ at all v except the lowest, where $a(v, \xi) = 1$, so $\sum_{v \in \bar{C}^k} a(v, \xi) = \chi(\bar{C}^k) = 1$.

2. General mappings

In classical critical point theory, using Sard's theorem it is proved that almost every linear function on E^n, when restricted to a smoothly embedded differentiable manifold M^k, possesses only non-degenerate critical points. This is used both in the proof that non-degenerate functions exist, and in the curvature theory that follows (compare [5]). However it is not possible to prove that almost any height function on a *given* embedded polyhedral manifold has only singularities which resemble non-degenerate critical points of smooth functions (compare Kuiper [5] when the manifold is given abstractly).

For example, in the differentiable theory, the set of height functions which have a degenerate critical point of monkey-saddle type are of measure zero, while in the polyhedral theory, such maps are open.





However in the polyhedral theory, it is immediate that there are sufficiently many general maps:

Each linear function $\xi: E^n \to E^1$ corresponds to a vector ξ such that for every vector x in E^n, $\xi(x) = \xi . x =$ inner product of x and ξ. We give the set of linear functions the topology of the vectors in E^n.

Proposition 1. *The set of linear maps $\xi: E^n \to E^1$ which are general for a given convex cell complex $M^k \subset E^n$ is open and dense.*

Proof. If ξ is *not* general then $\xi(v) = \xi(w)$ for some pair of vertices v, w which span an edge of M^k. Thus ξ is in the finite union of hyperplanes $\bigcup_{C^1 \in M} \{\xi \in E^n \mid \xi(v) = \xi(w),\, v, w \text{ span } C^1\}$.

Corollary 1. *If ξ is general for M^k and v is a vertex of M, then $a(v, \xi) = a(v, \eta)$ for every η in a neighborhood of ξ.*

3. Total curvature and the Gauss-Bonnet theorem

In the case of a differentiable manifold M^k smoothly embedded in E^n, the extrinsic curvature of an open subset Ω can be defined as the integral of a curvature form, which generalizes the integral $\int_{\Omega} K dA$, where K is the Gaussian curvature of a surface in E^3. Kuiper interprets this integral as an average and deduces the Gauss-Bonnet theorem and related results [4].

We follow an analogous pattern in the polyhedral case, and proceed to apply the result to a new interpretation of Gauss' theorema egregium.

Definition. Let Ω be an open set in the convex cell complex M^k (i.e. the intersection of Ω with any C^r is a relatively open subset of C^r). The *index of Ω with respect to ξ* is $a(\Omega, \xi) = \sum_{v \in \Omega} a(v, \xi)$.

By proposition 1, $a(\Omega, \xi)$ is defined and finite for almost all ξ in E^n. Furthermore since $a(v, \xi) = a(v, \rho\xi)$ for any $\rho > 0$, the set of ξ for which $a(\Omega, \xi)$ is defined and finite is an open dense set on the unit sphere $S^{n-1} = \{\xi \in E^n \mid \xi \cdot \xi = 1\}$.

Let $d\omega^{n-1}$ be the ordinary volume element on S^{n-1}. Then

$$C_{n-1} = \int_{S^{n-1}} d\omega^{n-1} = \text{volume of } S^{n-1} .$$

Definition. The *curvature of* $\Omega = K(\Omega) = \dfrac{1}{C_{n-1}} \displaystyle\int_{S^{n-1}} a(\Omega, \xi) d\omega^{n-1}$.

Theorem 2. $$K(M^k) = \chi(M^k) .$$

Proof.

$$\begin{aligned} K(M) &= \frac{1}{C_{n-1}} \int_{S^{n-1}} a(M, \xi) d\omega^{n-1} \\ &= \frac{1}{C_{n-1}} \int_{S^{n-1}} \sum_{v \in M} a(v, \xi) d\omega^{n-1} \\ &= \frac{1}{C_{n-1}} \int_{S^{n-1}} \chi(M^k) d\omega^{n-1} \qquad \text{by Theorem 1} \\ &= \chi(M^k) \frac{1}{C_{n-1}} \int_{S^{n-1}} d\omega^{n-1} = \chi(M^k) . \end{aligned}$$

4. The theorema egregium

If M^k is an even-dimensional differentiable manifold, then it can be shown that $K(\Omega)$ depends only on intrinsic quantities associated with the manifold.

In the 2-dimensional case, the fact that the Gaussian curvature function is intrinsic is usually deduced from the fact that it can be expressed in terms of the coefficients of the first fundamental forms and their derivatives.

In the case at hand the intrinsic nature of $K(\Omega)$ can be proved by calculating it explicitly in terms of familiar intrinsic quantities. The key quantity is the normalized exterior angle of a convex cell C^k at a vertex v. This is simply the ratio of the area of the set of normals to support hyperplanes of C^k at v to the area of the entire sphere. In terms of our indicator function, we may express this as follows:

Definition. The *normalized exterior angle* of a convex cell $C^r \subset E^r$ at a vertex v is

$$\mathscr{E}(C^r, v) = \frac{1}{C_{r-1}} \int_{S^{r-1}} A(C^r, v, \xi) d\omega^{r-1} .$$

This is an intrinsic quantity associated with a convex cell and one of its points. To apply it to the problem of curvature of a k-complex in E^n we need a further lemma:

Lemma 2. *If $C^r \subset E^r \subset E^n$, then*

$$\mathscr{E}(C^r, v) = \frac{1}{C_{n-1}} \int_{S^{n-1}} A(C^r, v, \xi) d\omega^{n-1} .$$

Proof. We use induction on n, beginning with the case $n = r$. Let e_{n+1} denote a unit vector in E^{n+1} orthogonal to E^n. Then

$$S^n = \left\{ \xi \cos\theta + e_{n+1} \sin\theta \mid \xi \in S^{n-1}, \frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2} \right\} ,$$

and $d\omega^n = \alpha(\theta) d\theta d\omega^{n-1}$, where $\alpha(\theta)$ is a function only of θ. We calculate

$$C_n = \int_{S^n} d\omega^n = \int_{S^{n-1}} d\omega^{n-1} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \alpha(\theta) d\theta = C_{n-1} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \alpha(\theta) d\theta .$$

Note that $A(C^r, v, \xi\cos\theta + e_{n+1}\sin\theta) = A(C^r, v, \xi)$ for $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$ since $C^r \subset E^n$. Therefore

$$\frac{1}{C_n} \int_{S^n} A(C^r, v, \xi\cos\theta + e_{n+1}\sin\theta) d\omega^n$$

$$= \frac{1}{C_n} \int_{S^{n-1}} A(C^r, v, \xi) d\omega^{n-1} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \alpha(\theta) d\theta = \frac{1}{C_{n-1}} \int_{S^{n-1}} A(C^r, v, \xi) d\omega^{n-1} .$$

Definition. It M^k is a convex cell complex embedded in E^n and v is a vertex of M^k, then

$$K(v) = \frac{1}{C_{n-1}} \int_{S^{n-1}} a(v, \xi) d\omega^{n-1} .$$

Note that

$$\begin{aligned} K(\Omega) &= \frac{1}{C_{n-1}} \int_{S^{n-1}} a(\Omega, \xi) d\omega^{n-1} \\ &= \frac{1}{C_{n-1}} \int_{S^{n-1}} \sum_{v=\Omega} a(v, \xi) d\omega^{n-1} \\ &= \sum_{v \in \Omega} \frac{1}{C_{n-1}} \int_{S^{n-1}} a(v, \xi) d\omega^{n-1} = \sum_{v \in \Omega} K(v) . \end{aligned}$$

Theorem 3 (Theorema egregium). *$K(v)$ is intrinsic, in fact*

$$K(v) = \sum_{r=0}^{K} (-1)^r \sum_{C^r \in M} \mathscr{E}(C^r, v) .$$

Proof.
$$\begin{aligned} K(v) &= \frac{1}{C_{n-1}} \int_{S^{n-1}} a(v, \xi) d\omega^{n-1} \\ &= \frac{1}{C_{n-1}} \int_{S^{n-1}} \left[\sum_{r=0}^{K} (-1)^r \sum_{C^r \in M} A(C^r, v, \xi) \right] d\omega^{n-1} \\ &= \sum_{r=0}^{K} (-1)^r \sum_{C^r \in M} \frac{1}{C_{n-1}} \int_{S^{n-1}} A(C^r, v, \xi) d\omega^{n-1} \\ &= \sum_{r=0}^{K} (-1)^r \sum_{C^r \in M} \mathscr{E}(C^r, v) . \end{aligned}$$

Remark 3. When $r = 0$, and $n > 1$, we have $\mathscr{E}(w, v) = 0$ if $w \neq v$, and

$$\mathscr{E}(v, v) = \frac{1}{C_{n-1}} \int_{S^{n-1}} A(v, v, \xi) d\omega^{n-1} = \frac{1}{C_{n-1}} \int_{S^{n-1}} d\omega^{n-1} = 1 ,$$

and we make the convention that $\mathscr{E}(v, v) = 1$ for $n = 0, 1$ also. When $r = 1$ and $n > 1$, $\mathscr{E}(C^1, v) = 0$ if $v \not\subset C^1$ and if $C^1 = [v, w]$, then

$$A(C^1, v, \xi) = \begin{cases} 1 \text{ if } \xi \cdot v \geq \xi \cdot w, \\ 0 \text{ otherwise.} \end{cases}$$

Therefore

$$\mathscr{E}(C^1, v) = \frac{1}{C_{n-1}} \int_{S^{n-1}} A(C^1, v, \xi) d\omega^{n-1} = \frac{1}{2},$$

and by convention we set

$$\frac{1}{C_0} \int_{S^0} A(C^1, v, \xi) d\omega^0 = \frac{1}{2}.$$

This agrees with the convention of assigning $C_0 = 2$ and letting

$$\int_{S^0} A(C^1, v, \xi) d\omega^0 = A(C^1, v, \xi) + A(C^1, v, -\xi),$$

where ξ and $-\xi$ are the two points in S^0.

Remark 4. In the case that M is a 2-dimensional pseudomanifold, Theorem 3 yields

$$\begin{aligned} K(v) &= \sum_{r=0}^{k} (-1)^r \mathscr{E}(C^r, v) \\ &= 1 - \frac{1}{2} \quad \text{(number of edges)} \\ &\quad + \sum_{C^2 \text{ at } v} \frac{1}{2} \quad \text{(Exterior angle of } C^2 \text{ at } v\text{)}. \end{aligned}$$

But in a pseudo-manifold, the number of edges at a vertex is equal to the number of 2-cells there, so

$$\begin{aligned} K(v) &= \frac{2\pi}{2\pi} - \frac{1}{2\pi} \cdot \pi \quad \text{(number of 2-cells)} \\ &\quad + \sum_{C^2 \text{ at } v} \frac{1}{2\pi} \quad (\pi\text{ -angle of } C^2 \text{ at } v) \\ &= \frac{1}{2\pi}[2\pi - \sum_{C^2 \text{ at } v} \text{(angle of } C^2 \text{ at } v)], \end{aligned}$$

and this agrees with the normalized classical formula for the intrinsic curvature at a vertex of a polyhedral 2-manifold. This result may be considered as a generalization of a theorem of Pólya for convex polyhedral discs [6].

Remark 5. In the case that $M^k = \bar{C}^k$, the Gauss-Bonnet theorem states that $\sum_{v \in \bar{C}^k} K(v) = \chi(\bar{C}^k) = 1$. This can be interpreted as saying that the sum of the normalized exterior angles of a convex polyhedral cell is 1, as can be proved independently.

The sum of the normalized exterior angles of a convex cell is the ratio of the area of the set of vectors in S^{k-1}, which are normal to oriented support hyperplanes of C^k, to the area of S^{k-1}. But almost every unit vector is normal to exactly one oriented support hyperplane.

Remark 6. As in the classical 2-dimensional case, we may show, without using the extrinsic theorem, that the sum of the intrinsically defined curvatures over the set of vertices of a complex gives the Euler characteristic of the complex.

Theorem 4 (Intrinsic form of Gauss-Bonnet theorem). *Let $\kappa(v)$ be defined to be $\sum_{r=0}^{k}(-1)^r \sum_{C^r \epsilon M} \mathscr{E}(C^r, v)$.* Then $\sum_{v \epsilon M} \kappa(v) = \chi(M^k)$.

Proof.
$$\begin{aligned}\sum_{v \epsilon M} \kappa(v) &= \sum_{v \epsilon M} \sum_{r=0}^{k} (-1)^r \sum_{C^r \epsilon M} \mathscr{E}(C^r, v) \\ &= \sum_{r=0}^{k} (-1)^r \sum_{C^r \epsilon M} [\sum_{v \epsilon M} \mathscr{E}(C^r, v)] \\ &= \sum_{r=0}^{k} (-1)^r \alpha_r(M) \ .\end{aligned}$$

Note. This intrinsic theorem is related to a special case of the Allendoerfer-Weil generalization of the Gauss-Bonnet theorem for n-manifolds-with-boundary which are not embedded in a Euclidean space [2].

5. Results for manifolds

In theorem 4 we have shown that $K(v)$ is an intrinsic quantity even when the dimension of M^k is odd. In the case that M^k is an odd-dimensional *manifold*, this is a trivial statement: $K(v) \equiv 0$, which follows from the following lemma on polyhedral manifolds:

Lemma 3. *If v is a vertex of a k-dimensional polyhedral manifold M^k, and ξ is general for M^k, then $a(v, -\xi) = (-1)^k a(v, \xi)$.*

The proof of this lemma is not difficult, but it requires a new set of definitions and some technicalities, and we will not present it in this paper. Assuming this result however we may prove some corollaries:

Corollary 2. *If M^k is embedded in E^n and k is odd, then $K(v) = 0$ for all v in M^k.*

Proof.
$$\begin{aligned}K(v) &= \frac{1}{C_{n-1}} \int_{S^{n-1}} a(v, \xi) d\omega^{n-1} \\ &= \frac{1}{2} \frac{1}{C_{n-1}} \int_{S^{n-1}} [a(v, \xi) + a(v, -\xi)] d\omega^{n-1} \\ &= \frac{1}{2} \frac{1}{C_{n-1}} \int_{S^{n-1}} [a(v, \xi) + (-1)^k a(v, -\xi)] d\omega^{n-1} = 0 \ .\end{aligned}$$

Corollary 3. *If M^k is an embedded odd-dimensional manifold, then* $\chi(M^k) = 0$.

6. Generalizations

We have already pointed out that the theorems of this paper are valid for arbitrary complexes embedded in Euclidean space and linear functions on these complexes. We now make some remarks to indicate how the theorems go over to more general complexes and maps.

Remark 7. It is not necessary for Theorem 1 that the complex be embedded. To make this extension it is necessary to define an abstract convex cell complex M^k, where again the (finitely many) cells are bounded convex cells C^r in Euclidean spaces E^r, and the incidence relations that express the fact that $C^s \subset \partial C^r$ are given by isometric maps. If M^k is connected then it has a natural metric space structure. A function ξ on M^k is cell-wise linear if ξ is continuous on M^k and linear on each convex cell C^r (i.e. it coincides on C^r with the restriction to C^r of a linear map $E^r \to E^1$). Again say ξ is general if it is a homeomorphism for each $C^1 \in M$. The proof of Theorem 1 goes through without change.

To extend the curvature theorems, we introduce the concept of an isometric mapping $f\colon M^k \to E^n$ by the condition each point $p \in M$ has a neighborhood U in M such that if $g \in U$, then the $d_M(p, q) = d_{E^n}(f(p), f(q))$, where d_M denotes the (intrinsic) distance in M. Then the concept of curvature can be defined abstractly using the definition

$$K(v) = \frac{1}{C_{n-1}} \int_{S^{n-1}} a(v, \xi \circ f) d\omega^{n-1}, \text{ for } \xi \in S^{n-1} .$$

In the special case where each point p has a neighborhood which is isometrically embedded, we say that f is an isometric immersion. Isometrically immersed hypersurfaces will be treated in a later paper.

Remark 8. In a certain sense for Theorem 1 it is not necessary that the map ξ be general for M^k, but it is necessary to broaden the definition of index. If ξ is *any* cell-wise linear map on M^k, then we may define $A(C^r, v, \xi)$ and $a(v, \xi)\left(= \sum_{r=0}^{k} (-1)^r \sum_{C^r \in M} A(C^r, v, \xi)\right)$ as before if v is a vertex.

For a point x in the interior of a cell C^s, not a vertex, we may consider the smallest subdivision of M^k as a convex cell complex containing v as a vertex. One of the advantages of the concept of convex cell complex is that this is very easy to construct—simply remove the interior of C^s and replace it by the cone over ∂C^s with vertex at x. Extend the map ξ to this new complex by the identity, and define $a(x, \xi)$ as above, since now x is a vertex. By this method we assign an index to every point of M^k. In fact if x and y are both in the interior of C^s, then $a(x, \xi) = a(y, \xi)$ so we may use this common value to define an index $a(C^s, \xi)$.

If we define an indicator function

$$A(C^r, C^s, \xi) = \begin{cases} 1 & \text{if } C^s \subset \partial C^r,\ \xi(x) \geq \xi(y) \text{ for all } x \in C^s \text{ and } y \in C^r, \\ 0 & \text{otherwise,} \end{cases}$$

then we may obtain an alternate description of $a(C^s, \xi)$, which we could have used as a definition:

Definition. $$a(C^s, \xi) = \sum_{r=0}^{k} (-1)^r \sum_{C^r \in M} A(C^r, C^s, \xi) .$$

Lemma 4. *For a given C^r,*

$$\sum_{s=0}^{k} (-1)^s \sum_{C^r \in M} A(C^r, C^s, \xi) = 1 .$$

Proof. The expression on the left is the Euler characteristic of the set of points on C^r at which ξ attains its maximum value. Since C^r is a convex set, this maximum set is also a closed convex set, with Euler characteristic 1.

Theorem 5. *If ξ is any cell-wise linear map on a convex cell complex, then*

$$\sum_{s=0}^{k} (-1)^s \sum_{C^s \in M} a(C^s, \xi) = \chi(M^k) .$$

Proof.
$$\begin{aligned}\sum_{x=0}^{k} (-1)^s \sum_{C^s \in M} a(C^s, \xi) &= \sum_{s=0}^{k} (-1)^s \sum_{C^s \in M} \left[\sum_{r=0}^{k} (-1)^r \sum_{C^r \in M} A(C^r, C^s, \xi)\right] \\ &= \sum_{r=0}^{k} (-1)^r \sum_{C^r \in M} \left[\sum_{s=0}^{k} (-1)^s \sum_{C^s \in M} A(C^r, C^s, \xi)\right] \\ &= \sum_{r=0}^{k} (-1)^r \alpha_r(M) = \chi(M) .\end{aligned}$$

We may interpret this theorem by considering $a(-, \xi)$ as a function from M^k to the integers. $a(-, \xi)(x) = a(x, \xi)$. Let $\mathscr{C}$ be the set $\{C\}$ of connected components of M under the equivalence relation $x \sim y$ if and only if $a(x, \xi) = a(y, \xi)$. Let $a(C, \xi)$ be the common value of $a(x, \xi)$ for x in C. Then the theorem states that $\sum_{C \in \mathscr{C}} a(C, \xi)\chi(C) = \chi(M^k)$.

Corollary 4. *For any convex cell complex, $\sum_{s=0}^{k} (-1)^s \sum_{C^s \in M} \chi$ (open star of C^s) $= \chi(M)$ where the open star of C^s is the collection of cells C^r in M such that $C^s \subset C^r$.*

Proof. Consider the linear map ξ which sends every x to 0. Then $A(C^r, C^s, \xi) \equiv 1$ if $C^s \subset C^r$, and the corollary follows.

In particular, if M^k is a k-manifold, χ (open star of C^s) $= (-1)^k$ for every cell C^s, so $(-1)^k\chi(M) = \chi(M)$. This yields a proof of the fact that $\chi(M^k) = 0$ for an odd-dimensional manifold which uses only the property that χ (open star of x) $= (-1)^k$ for every x.

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