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HOLOMORPHIC BISECTIONAL CURVATURE

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1. Introduction

Let M be a Kähler manifold of complex dimension n and R its Riemannian curvature tensor. At each point x of M, R is a quadrilinear mapping $T_x(M) \times T_x(M) \times T_x(M) \times T_x(M) \to \boldsymbol{R}$ with well known properties.

Let σ be a plane in $T_x(M)$, i.e., a real two dimensional subspace of $T_x(M)$. Choosing an orthonormal basis X, Y for σ, we define the sectional curvature $K(\sigma)$ of σ by

$$K(\sigma) = R(X, Y, X, Y) . \tag{1}$$

We shall occasionally write $K(X, Y)$ for $K(\sigma)$. The right hand side depends only on σ, not on the choice of an orthonormal basis X, Y. The sectional curvature K is a function defined on the Grassmann bundle of (two-) planes in the tangent spaces of M. A plane σ is said to be holomorphic if it is invariant by the (almost) complex structure tensor J. The set of J-invariant planes σ is a holomorphic bundle over M with fibre $P_{n-1}(C)$ (complex projective space of dimension $n-1$). The restriction of the sectional curvature K to this complex projective bundle is called the holomorphic sectional curvature and will be denoted by H. In other words, $H(\sigma)$ is defined only when σ is invariant by J, and $H(\sigma) = K(\sigma)$. If X is a vector in σ we shall also write $H(X)$ for $H(\sigma)$.

Given two J-invariant planes σ and σ' in $T_x(M)$, we define the holomorphic bisectional curvature $H(\sigma, \sigma')$ by

$$H(\sigma, \sigma') = R(X, JX, Y, JY) , \tag{2}$$

where X is a unit vector in σ and Y a unit vector in σ'. It is a simple matter to verify that $R(X, JX, Y, JY)$ depends only on σ and σ'. Although the definition itself makes sense even for Hermitian holomorphic vector bundles (cf. Nakano [10]) as well as Hermitian manifolds we shall confine our considerations to the Kählerian case.

Since

$$H(\sigma, \sigma) = H(\sigma) , \tag{3}$$

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the holomorphic bisectional curvature carries more information than the holomorphic sectional curvature. By Bianchi's identity we have

$$R(X, JX, Y, JY) = R(X, Y, X, Y) + R(X, JY, X, JY) . \tag{4}$$

The right hand side of (4) is a sum of two sectional curvatures (up to constant factors). Hence the holomorphic bisectional curvature carries less information than the sectional curvature.

Although the concept of holomorphic bisectional curvature is new, one finds it implicity in Berger [1] and Bishop-Goldberg [5]. The purpose of this note is to give basic properties of the holomorphic bisectional curvature and to generalize geometric results on Kähler manifolds with positive sectional curvature to Kähler manifolds with positive holomorphic bisectional curvature.

2. Spaces of constant holomorphic sectional curvature

If g is a Kähler metric of constant holomorphic sectional curvature c, then

$$\begin{aligned} R(X, Y, Z, W) = \frac{c}{4}[&g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \\ &+ g(X, JZ)g(Y, JW) - g(X, JW)g(Y, JZ) \\ &+ 2g(X, JY)g(Z, JW)] . \end{aligned} \tag{5}$$

Hence

$$R(X, JX, Y, JY) = \frac{c}{2}[g(X, X)g(Y, Y) + g(X, Y)^2 + g(X, JY)^2] . \tag{6}$$

It follows that, for a Kähler manifold of constant holomorphic sectional curvature c, the holomorphic bisectional curvatures $H(\sigma, \sigma')$ lie between $c/2$ and c,

$$c/2 \leqq H(\sigma, \sigma') \leqq c \quad \text{or} \quad c \leqq H(\sigma, \sigma') \leqq c/2 ,$$

where the value $c/2$ is attained when σ is perpendicular to σ' whereas the value c is attained when $\sigma = \sigma'$.

3. Ricci tensor

For a Kähler manifold M, the Ricci tensor S may be given by

$$S(X, Y) = \textstyle\sum_{i=1}^{n} R(X_i, JX_i, X, JY) , \tag{7}$$

where $(X_1, \cdots, X_n, JX_1, \cdots, JX_n)$ is an orthonormal basis for $T_x(M)$. It is clear from (7) that if the holomorphic bisectional curvature is positive (or negative) so is the Ricci tensor.

A Riemannian manifold with metric tensor g and Ricci tensor S is called an Einstein space if $S = kg$ for some constant k.

4. Complex submanifolds

Let M be a submanifold of a Riemannian manifold N with metric tensor g. Denote by R_M and R_N the Riemannian curvature tensors of M and N and by α the second fundamental form of M in N. Then the Gauss-Codazzi equation states:

$$\begin{aligned} R_M(X, Y, Z, W) = {} & g(\alpha(X, Z), \alpha(Y, W)) \\ & - g(\alpha(X, W), \alpha(Y, Z)) + R_N(X, Y, Z, W) . \end{aligned} \tag{8}$$

(Among several possible definitions of the second fundamental form α, we have chosen the one which defines α as a symmetric bilinear mapping from $T_x(M) \times T_x(M)$ into the normal space at x.)

If N is a Kähler manifold and M a complex submanifold, then

$$\begin{aligned} R_M(X, JX, Y, JY) = {} & g(\alpha(X, Y), \alpha(JX, JY)) \\ & - g(\alpha(X, JY), \alpha(JX, Y)) + R_N(X, JX, Y, JY) . \end{aligned}$$

Hence,

$$\begin{aligned} R_M(X, JX, Y, JY) = {} & -\|\alpha(X, Y)\|^2 \\ & -\|\alpha(X, JY)\|^2 + R_N(X, JX, Y, JY) . \end{aligned} \tag{9}$$

From (9) we may conclude that the holomorphic bisectional curvature of M does not exceed that of N. In particular, if M is a complex submanifold of a complex Euclidean space, then the holomorphic bisectional curvature of M is nonpositive and hence the Ricci tensor of M is also nonpositive. (See O'Neill [11] for similar results on the holomorphic sectional curvature.)

5. Complex submanifolds of a space of positive holomorphic bisectional curvature

We prove

Theorem 1. *Let M be a compact connected Kähler manifold with positive holomorphic bisectional curvature and let V and W be compact complex submanifolds. If $\dim V + \dim W \geqq \dim M$, then V and W have a non-empty intersection.*

Theorem 1 is a slight generalization of Theorem 2 in Frankel's paper [8] in which he assumes that M is a compact Kähler manifold with positive sectional curvature. The proof we present below is a slight modification of that of Frankel.

Proof. Assume that $V \cap W$ is empty. Let $\tau(t)$, $0 \leqq t \leqq l$, be a shortest geodesic from V to W. Let $p = \tau(0)$ and $q = \tau(l)$. Let X be a parallel vector field defined along τ which is tangent to both V and W at p and q respectively. The assumption $\dim V + \dim W \geqq \dim M$ guarantees the existence of such a vector field X. Then JX is also such a vector field. Denote by T the vector field tangent to τ defined along τ. We compute the second variations of the arc-length with respect to infinitesimal variations X and JX. Then (Frankel [8]), we have

$$L''_X(0) = g(\nabla_X X, T)_q - g(\nabla_X X, T)_p - \int_0^l R(T, X, T, X)dt \,, \tag{10}$$

$$L''_{JX}(0) = g(\nabla_{JX} JX, T)_q - g(\nabla_{JX} JX, T)_p - \int_0^l R(T, JX, T, JX)dt \,. \tag{11}$$

Since $g(\nabla_X X, T)_p + g(\nabla_{JX} JX, T)_p = 0$ and $g(\nabla_X X, T)_q + g(\nabla_{JX} JX, T)_q = 0$ (cf. Frankel [8]), by adding (10) and (11) and making use of (4) we obtain

$$\begin{aligned} L''_X(0) + L''_{JX}(0) &= -\int_0^l (R(T, X, T, X) + R(T, JX, T, JX))dt \\ &= -\int_0^l R(T, JT, X, JX)dt \leqq 0 \,. \end{aligned}$$

Hence at least one of $L''_X(0)$ and $L''_{JX}(0)$ is negative. This contradicts the assumption that τ is a shortest geodesic from V to W.

Theorem 2. *A compact Kähler surface M_2 with positive holomorphic bisectional curvature is complex analytically homeomorphic to $P_2(C)$.*

The result of Andreotti-Frankel (Theorem 3 in [8]) states that a compact Kähler surface M_2 with positive sectional curvature is complex analytically homeomorphic to $P_2(C)$. The proof of Theorem 2 is the same as the proof of Theorem 3 in Frankel's paper [8]. (The only change we have to make is to use our Theorem 1 instead of Theorem 2 of [8].)

The following theorem is also a slight generalization of a result of Frankel [8].

Theorem 3. *Every holomorphic correspondence of a connected compact Kähler manifold N with positive holomorphic bisectional curvature has a fixed point.*

The statement means that every closed complex submanifold V of $N \times N$ with $\dim V = \dim N$ meets the diagonal of $N \times N$.

Proof. Setting $M = N \times N$ and $W =$ diagonal $(N \times N)$, we apply the proof of Theorem 1. Then it suffices to show that $R(T, JT, X, JX)$ is positive at some point of the geodesic τ. Since T and X are tangent vector fields of $N \times N$, they can be decomposed as follows:

$$T = T_1 + T_2\,, \qquad X = X_1 + X_2\,,$$

where T_1 and X_1 are tangent to the first factor N, and T_2 and X_2 to the second factor N. Then

$$R(T, JT, X, JX) = R_N(T_1, JT_1, X_1, JX_1) + R_N(T_2, JT_2, X_2, JX_2)\,.$$

Since T is perpendicular to the diagonal of $N \times N$ at p and q, neither T_1 nor T_2 vanishes at p and q. Since X_1 and X_2 cannot both vanish at any point, either $R_N(T_1, JT_1, X_1, JX_1)$ or $R_N(T_2, JT_2, X_2, JX_2)$ is strictly positive at p (and q). Hence, $R(T, JT, X, JX)$ is strictly positive at p.

6. The second cohomology group

A slight generalization of Theorem 1 in Bishop-Goldberg's paper [3] is given.

Theorem 4. *The second Betti number of a compact connected Kähler manifold M with positive holomorphic bisectional curvature is one.*

The following lemma is basic. It will be used also for the proof of Theorem 5.

Lemma 1. *Let ξ be a real form of bidegree* (1, 1) *on a Kähler manifold M. Then there exists a local field of orthonormal frames $X_1, \cdots, X_n, JX_1, \cdots, JX_n$ such that*

$$\xi(X_i, JX_j) = 0 \quad \text{for} \quad i \neq j\,.$$

Proof. Let $T(X, Y) = \xi(X, JY)$. The fact that ξ has bidegree (1, 1) is equivalent to $\xi(X, Y) = \xi(JX, JY)$ for all X and Y. Thus, $T(X, Y) = T(Y, X)$ and $T(JX, JY) = T(X, Y)$, that is, T is a symmetric bilinear form invariant under J. Consequently, if X_1 is a characteristic vector of T, so is JX_1. We can therefore choose an orthonormal basis $X_1, \cdots, X_n, JX_1, \cdots, JX_n$ inductively so that the only nonzero components of T are given by $T(X_i, X_i) = T(JX_i, JX_i)$, which translates into the desired statement for ξ. (If we use the complex representation for T, then T is a Hermitian form and the process above is equivalent to the diagonalization of T.)

The remainder of the proof of Theorem 4 will be given as in Berger [1], and is a standard application of a well known technique due to Bochner and Lichnerowicz. For a 2-form ξ on a compact Riemannian manifold M, we define $F(\xi)$ by the following tensor equation:

$$F(\xi) = 2R_{AB}\xi^{AC}\xi^{B}{}_{C} - R_{ABCD}\xi^{AB}\xi^{CD}.$$

It is known (cf. for instance, Bochner [6], Lichnerowicz [9, p. 6] or Yano-Bochner [12, p. 64]) that if ξ is harmonic and $F(\xi) \geqq 0$, then $F(\xi) = 0$ and ξ is parallel.

Let ξ be as in Lemma 1 and set $\xi_{ii^*} = \xi(X_i, JX_i)$. By a simple calculation we obtain

$$F(\xi) = 2 \sum_{i,j} R_{ii^*jj^*}(\xi_{ii^*} - \xi_{jj^*})^2, \tag{12}$$

where

$$R_{ii^*jj^*} = R(X_i, JX_i, X_j, JX_j) .$$

Since $R_{ii^*jj^*} > 0$ by our assumption, we conclude that $F(\xi) \geqq 0$. Assume that ξ is harmonic. Then $F(\xi) = 0$ and ξ is parallel. The equality $F(\xi) = 0$ implies $\xi_{ii^*} = \xi_{jj^*}$ at each point for $i, j = 1, \cdots, n$. Hence $\xi = f\Omega$, where f is a function on M and Ω is the Kähler form of M. Since ξ is parallel, f must be a constant function. Thus, $\dim H^{1,1}(M; C) = 1$.

Since the Ricci tensor of M is positive definite (cf. § 3), there are no nonzero holomorphic 2-forms on M (cf. Bochner [7], Lichnerowicz [9, p. 9] or Yano-Bochner [12, p. 141]). Thus $H^{2,0}(M; C) = H^{0,2}(M; C) = 0$. This completes the proof of Theorem 4.

7. Einstein-Kähler manifolds with positive holomorphic bisectional curvature

The following is a slight generalization of a result of Berger [2].

Theorem 5. *An n-dimensional compact connected Kähler manifold with an Einstein metric of positive holomorphic bisectional curvature is globally isometric to $P_n(C)$ with the Fubini-Study metric.*

Only the essential steps in the proof will be given because of its length, technical complexity and similarlity in approach to the proof of Berger's theorem. Details, however, will be provided where necessary.

Let M be an Einstein-Kähler manifold of complex dimension n and let $X_1, \cdots, X_n, JX_1, \cdots, JX_n$ be a local field of orthonormal frames. We write also $X_{1^*}, \cdots, X_{n^*}$ for $JX_1, \cdots, JX_n$ and set

$$R_{\alpha\beta\gamma\delta} = R(X_\alpha, X_\beta, X_\gamma, X_\delta) .$$

We use the convention that the indices $\alpha,\beta,\gamma,\delta$ run through $1, \cdots, n, 1^*, \cdots, n^*$ while the indices i, j, k, l run from 1 to n. Being the curvature tensor of a Kähler manifold, $R_{\alpha\beta\gamma\delta}$ satisfies in addition to the usual algebraic relations satisfied by a Riemannian curvature tensor the following relations:

$$R_{ij\alpha\beta} = R_{i^*j^*\alpha\beta} , \quad R_{i^*j\alpha\beta} = -R_{ij^*\alpha\beta} . \tag{13}$$

Lemma 2. *Let M be an Einstein-Kähler manifold such that (Ricci tensor) $=$ $k \cdot$ (metric tensor). Then*

$$\frac{1}{2} \sum_{\alpha} \nabla_\alpha \nabla_\alpha R_{11^*11^*} = \sum_{\alpha,\beta} (R_{1\alpha1^*\beta}{}^2 - R_{11^*\alpha\beta}{}^2 - R_{1\alpha1\beta}R_{1^*\alpha1^*\beta}) + k \cdot R_{11^*11^*} ,$$

where ∇ denotes the operator of covariant differentiation.

Lemma 2 is a special case of a formula of Berger in the Riemannian case (cf. Lemma (6.2) in Berger [2]); the Riemannian curvature tensors in Berger's paper differ from ours in sign.

We denote by H_1 the maximum value of the holomorphic sectional curvature of M. Since M is compact, H_1 exists and is attained by a unit vector, say X, at a point x of M. Thus, $H_1 = H(X)$. We choose a local field of orthonormal frames $X_1, \cdots, X_n, JX_1, \cdots, JX_n$ with the following properties:

$$X_1 = X \quad \text{at } x, \qquad R_{11^*i\alpha} = 0 \quad \text{for} \quad \alpha \neq i^*. \tag{14}$$

To find such a frame we apply Lemma 1 to the 2-form α_X defined by

$$\alpha_X(Y, Z) = R(X, JX, Y, Z) .$$

We denote by Q the value of $\frac{1}{2}\sum_\alpha \nabla_\alpha \nabla_\alpha R_{11^*11^*}$ at x. A straightforward calculation using Lemma 2, (13) and (14) yields

$$\begin{aligned} Q &= -H_1^2 + kH_1 - 2\textstyle\sum_{i\geqq 2} R_{11^*ii^*}{}^2 \\ &\quad + \textstyle\sum_{i,j\geqq 2}[(R_{1i1j} - R_{1i^*1j^*})^2 + (R_{1i^*1j} + R_{1i1j^*})^2] \\ &\geqq -H_1^2 + kH_1 - 2\textstyle\sum_{i\geqq 2} R_{11^*ii^*}{}^2 . \end{aligned}$$

Since $k = \sum_i R_{11^*ii^*} = R_{11^*11^*} + \sum_{i\geqq 2} R_{11^*ii^*} = H_1 + \sum_{i\geqq 2} R_{11^*ii^*}$, it follows that

$$Q \geqq \textstyle\sum_{i\geqq 2} R_{11^*ii^*}(H_1 - 2R_{11^*ii^*}) . \tag{15}$$

To prove the inequality $H_1 - 2R_{11^*ii^*} \geqq 0$, we first establish the following lemma.

Lemma 3. *Let X, JX, Y, JY be orthonormal vectors at a point of a Kähler manifold M. Let a, b be real numbers such that $a^2 + b^2 = 1$. Then*

$$\begin{aligned} &H(aX + bY) + H(aX - bY) + H(aX + bJY) + H(aX - bJY) \\ &\quad = 4[a^4H(X) + b^4H(Y) + 4a^2b^2R(X, JX, Y, JY)] . \end{aligned}$$

Proof. By a straightforward calculation we obtain

$$\begin{aligned} &H(aX + bY) + H(aX - bY) \\ &\quad = 2[a^4H(X) + b^4H(Y) + 6a^2b^2R(X, JX, Y, JY) - 4a^2b^2K(X, Y)] . \end{aligned}$$

Replacing Y by JY we obtain

$$\begin{aligned} &H(aX + bJY) + H(aX - bJY) \\ &\quad = 2[a^4H(X) + b^4H(Y) + 6a^2b^2R(X, JX, Y, JY) - 4a^2b^2K(X, JY)] . \end{aligned}$$

Lemma 3 now follows from these two identities and

$$R(X, JX, Y, JY) = K(X, Y) + K(X, JY) .$$

We apply Lemma 3 to the case $X = X_1$ and $Y = X_i$, $i \neq 1$. Since $H_1 = H(X_1)$ is the maximum holomorphic sectional curvature on M, we obtain

$$H_1 \geqq a^4 H_1 + b^4 H(X_i) + 4a^2 b^2 R_{11^* ii^*} .$$

Hence

$$(1 - a^2)(1 + a^2) H_1 \geqq b^4 H(X_i) + 4a^2 b^2 R_{11^* ii^*} .$$

Since $1 - a^2 = b^2$, dividing the inequality above by b^2 we obtain

$$(1 + a^2) H_1 \geqq b^2 H(X_i) + 4a^2 R_{11^* ii^*} .$$

Setting $a = 1$ and $b = 0$, we obtain

$$H_1 \geqq 2R_{11^* ii^*} .$$

Since, by our assumption, $R_{11^* ii^*} > 0$, we obtain from (15)

$$Q \geqq \textstyle\sum_{i \geq 2} R_{11^* ii^*} (H_1 - 2R_{11^* ii^*}) \geqq 0 .$$

On the other hand, since $R_{11^* 11^*}$ attains a (local) maximum at x, it follows that

$$Q = \tfrac{1}{2} \textstyle\sum \nabla_\alpha \nabla_\alpha R_{11^* 11^*} \leqq 0 .$$

Hence,

$$H_1 = 2R_{11^* ii^*} \quad \text{for} \quad i = 2, \cdots, n .$$

Since $k = \sum_i R_{11^* ii^*}$, we have

$$k = \tfrac{1}{2}(n + 1) H_1 . \tag{16}$$

The following lemma is also due to Berger (cf. Lemma (7.4) of [2]).

Lemma 4. *Let M be a Kähler manifold of complex dimension n. Then at any point y of M the scalar curvature $R(y)$ is given by*

$$R(y) = \frac{n(n + 1)}{\mathrm{Vol}(S^{2n-1})} \int_{S_y} H(X) dX , \quad y \in M ,$$

where $\mathrm{Vol}\,(S^{2n-1})$ *is the volume of the unit sphere of dimension $2n - 1$ and dX is the canonical measure in the unit sphere S_y in the tangent space $T_y(M)$.*

Using (16) and Lemma 4 we shall show that M is a space of constant holomorphic sectional curvature. Since M is Einsteinian, we have $R(y) = 2nk$.

By (16) we have

$$R(y) = n(n+1)H_1 . \tag{17}$$

From Lemma 4 and (17) we obtain

$$\int_{S_y} (H_1 - H(X))dX = 0 .$$

Since $H_1 \geqq H(X)$ for every unit vector X, we must have $H_1 = H(X)$. A compact Kähler manifold of constant positive holomorphic sectional curvature is necessarily simply connected and so is holomorphically isometric to $P_n(C)$.

As in Bishop-Goldberg [4], from Theorems 4 and 5 we obtain

Theorem 6. *A compact connected Kähler manifold with positive holomorphic bisectional curvature and constant scalar curvature is holomorphically isometric to $P_n(C)$.*

In fact, the Ricci 2-form of a Kähler manifold is harmonic if and only if the scalar curvature is constant. By Theorem 4, the Ricci 2-form is proportional to the Kähler 2-form. Hence the manifold is Einsteinian, and Theorem 6 follows from Theorem 5.

Corollary. *A compact, connected homogeneous Kähler manifold with positive holomorphic bisectional curvature is holomorphically isometric to $P_n(C)$.*

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