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MINIMAL IMMERSIONS OF SURFACES IN EUCLIDEAN SPHERES

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1. Introduction

The study of isolated singularities for minimal 3-dimensional varieties immersed in Euclidean n-space $\boldsymbol{E}^n$ requires as a first step a characterization of the tangent cone; the latter is the join of the origin 0 in $\boldsymbol{R}^n$ with a compact surface (the directrix) immersed in the unit Euclidean $(n-1)$-sphere S^{n-1} as a relatively minimal surface. Since comparatively little is known concerning such immersions, I propose to devote this as the first of a series of articles on the subject.

One may consider, for a start, a restricted type of singularity of a minimal 3-variety in $\boldsymbol{E}^n$, namely when this variety is topologically a manifold; in this case the directrix surface of the tangent cone is a 2-sphere immersed in S^{n-1} in a locally minimal way. This article is primarily devoted to minimal immersions of 2-spheres in Euclidean $(n-1)$-spheres. By this we mean immersions for which the total area is stationary with respect to variation, and minimal with respect to variation affecting sufficiently small portions of the surface at a time. Naturally, some of the conclusions developed here (through Lemma 5.3) apply to the minimal immersion of surfaces of positive genus as well; results pertaining to these will be collected elsewhere. In the case of minimal immersions of S^2 into the Euclidean sphere rS^{n-1} of radius r, the main result (Theorem 5.5) is that, if the image under such an immersion does not lie in any equatorial hyperplane section of rS^{n-1} then n is an odd integer and the area of the immersed S^2 is an integral multiple of $2\pi r^2$, at least equal to $\left(\dfrac{n^2-1}{8}\right)(4\pi r^2)$. There follow some discussion and examples to indicate why the above estimate is optimal.

2. Riemannian and Riemann surfaces

We denote by Σ an oriented surface, which, for the purposes of this article, may be assumed to be compact and either real analytic or differentiable. A differentiable Riemannian metric ds^2 on Σ together with the given orientation defines a covering of Σ by open domains with local (complex) isothermal parameters such as $w = u + iv$ $(i = \sqrt{-1})$ as well as its complex conjugate $\bar{w} = u - iv$. These parameters are defined up to a local holomorphic and holomorphically invertible transformation, and characterized by the following conditions.

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A complex valued function w in a domain $U \subset \Sigma$ is an isothermal parameter, if and only if

a) The map $w : U \to C$ is a topological imbedding.

b) The metric ds^2 can be expressed in U by the Hermitian differential form

$$ds^2 = 2F(w,\bar{w})(du^2 + dv^2) = 2F(w,\bar{w})|dw|^2 \tag{2.1}$$

where[1] $F(w, \bar{w})$ is a real analytic density, everywhere positive valued.

c) The real valued, exterior 2-form

$$\omega = 2F(w,\bar{w})\, du \wedge dv = iF(w,\bar{w})\, dw \wedge d\bar{w} \tag{2.2}$$

is positive with respect to the orientation of Σ. Thus Σ with its orientation and Riemannian metric is equivalent to a Riemann surface with a smoothly defined area element (2.2).

The tensor algebra bundle on Σ generated by the tangent bundle with the assigned Riemannian structure and orientation contains a complete set of irreducible representations of the structure group SO(2). In order to minimize redundancies, it is convenient to reduce this algebra as follows. First of all we consider the tensor algebra over the complex rather than the real field. Now let p, q be two rational numbers, unrestricted as to sign (more generally we could be bizarre and let p and q be complex numbers), such that $p - q$ is an integer; denote then by $E^{p,q}$ the complex line bundle whose elements are equivalence classes in the set of quadruples $(U, w, \boldsymbol{p}, v)$ where

a) U is an open domain in Σ and $\boldsymbol{p}$ a point in U;

b) w is a local isothermal parameter defined in U, and v a complex number;

c) $(U, w, \boldsymbol{p}, v)$ is equivalent to $(U', w', \boldsymbol{p}', v')$ if and only if:

$$\boldsymbol{p} = \boldsymbol{p}' \in U \cap U', \tag{i}$$

$$\begin{aligned} v' &= v \left| \frac{\partial w}{\partial w'}(\boldsymbol{p}) \right|^{2q} \left(\frac{\partial w}{\partial w'}(\boldsymbol{p}) \right)^{p-q} \\ &= v \left| \frac{\partial w}{\partial w'}(\boldsymbol{p}) \right|^{2p} \left(\overline{\frac{\partial w}{\partial w'}(\boldsymbol{p})} \right)^{q-p}, \end{aligned} \tag{ii}$$

or, if p and q are integers,

[1] We use the convention that a real- or complex-valued function such as F in a domain $U \subset C$ is written as $F(w, \bar{w})$ for $w \in U$, when it is not assumed to be holomorphic. The same convention holds in the case of vector-valued functions.

$$v' = v\left(\frac{\partial w}{\partial w'}(\boldsymbol{p})\right)^p \left(\overline{\frac{\partial w}{\partial w'}(\boldsymbol{p})}\right)^q.$$

Thus, for each (p, q), $E^{p,q}$ is a real analytic, complex line bundle over Σ; $E^{p,0}$ is a holomorphic line bundle, while $E^{0,q}$ is an antiholomorphic line bundle. The tensor product is a pairing of $E^{p,q}$ and $E^{p',q'}$ into $E^{p+p',q+q'}$, so that $E^{0,0}$ is the trivial bundle and the dual of $E^{p,q}$ can be naturally identified with $E^{-p,-q}$. Finally the weak direct sum of all $E^{p,q}$ is a bundle of commutative algebras under addition and tensor multiplication, denoted by E. This algebra is bigraded by the values of (p, q); $E^{-1,0} \oplus E^{0,-1}$ is the complex tangent bundle $\tau_C(\Sigma) = \tau(\Sigma) \otimes_{\boldsymbol{R}} \boldsymbol{C}$, where $\tau(\Sigma)$ is the real tangent bundle. The bundle E admits a $(\boldsymbol{C} : \boldsymbol{R})$-semilinear involution (conjugation) mapping $E^{p,q}$ onto $E^{q,p}$ in the obvious way, the fixed elements of $E^{p,q} \oplus E^{q,p}$ (if $p \neq q$) and the elements of $E^{p,p}$ whose fibre component v is real valued are fixed under this conjugation and are hence called the *real* elements of E. For instance the Riemannian metric (2.1) is a real analytic cross section F, positive valued everywhere, in the bundle $E^{1,1}$. It is clear that the commutative tensor algebra thus defined is, for all computational purposes, equivalent to the more cumbersome, classical tensor algebra, in the sense that its irreducible spaces under the action of the structural group $\mathrm{SO}(2) \times \boldsymbol{R}^* = \boldsymbol{C}^*$ ($\boldsymbol{R}^*$ = multiplicative group of real numbers) contain a complete set of irreducible representations of the group.

3. Connexions

The metric (2.1) on Σ enables us to define a Levi-Civita connexion, i.e. a first order differential operator ∇ on differentiable cross sections of E. More precisely, the connexion is described axiomatically in terms of a splitting $\nabla = \nabla' + \nabla''$ of the absolute derivative, where ∇' and ∇'' are of bidegree respectively (1, 0) and (0, 1) on the bigraded algebra $E = \sum_{p,q} E^{p,q}$. These operators are characterized by the following four axioms:

1) The operators ∇' and ∇'' are C-linear derivations; in other words:

(3.1) $\nabla(cf) = c\nabla f$, where $c \in C$, f a differentiable cross section in $E^{p,q}$ (this implies, here and below, the same relation in terms of either ∇' or ∇'');

(3.2) $\nabla(f + g) = \nabla f + \nabla g$, where f, g are cross sections in $E^{p,q}$;

(3.3) $\nabla(f \otimes g) = f \otimes \nabla g + (\nabla f) \otimes g$, where f and g are cross sections, respectively in $E^{p,q}$ and $E^{p',q'}$.

2) If f is a cross section in $E^{p,0}$, locally described by the complex valued, differentiable functions $v = f(w, \bar{w})$ of local isothermal parameters w, then

$$\nabla'' f = \bar{\partial} f = \frac{\partial f(w, \bar{w})}{\partial \bar{w}}. \tag{3.4}$$

3) If $\bar{f}$ is the conjugate of f, then

$$\nabla' f = (\overline{\nabla''\bar{f}}). \tag{3.5}$$

4) Finally, for the special cross section F in $E^{(1,1)}$ defining the metric (2.1), we have the identity

$$\nabla F \equiv 0. \tag{3.6}$$

It is easy to see that the six conditions (3.1) through (3.6) define ∇', ∇'' and hence $\nabla = \nabla' + \nabla''$ uniquely, as follows: if f is a differentiable cross section in $E^{p,q}$, locally defined by a differentiable function $v = f(w,\overline{w})$ in terms of a local isothermal parameter w, then

$$\nabla' f = \left(\frac{\partial f(w,\overline{w})}{\partial \overline{w}} - p\frac{\partial \log F(w,\overline{w})}{\partial w} \cdot f(w,\overline{w})\right) \tag{3.7}$$

and

$$\nabla'' f = \left(\frac{\partial f}{\partial \overline{w}} - q\frac{\partial \log F(w,\overline{w})}{\partial \overline{w}} \cdot f(w,\overline{w})\right). \tag{3.8}$$

The commutator of ∇' and ∇'' corresponds to the Ricci identity, which is expressed as follows:

If $f(w,\overline{w})$ represents locally a differentiable cross section in $E^{p,q}$, then

(3.9)

$$\begin{aligned}[\nabla', \nabla'']f &= (\nabla'\nabla'' - \nabla''\nabla')f = (p-q)f(w,\overline{w})\frac{\partial^2 \log F(w,\overline{w})}{\partial w \partial \overline{w}} \\ &= (q-p)KF(w,\overline{w}) \otimes f(w,\overline{w}),\end{aligned}$$

where $K = K(w,\overline{w}) = -\dfrac{1}{F}\dfrac{\partial^2 \log F(w,\overline{w})}{\partial w \partial \overline{w}}$ is the Gaussian curvature of the metric (2.1).

4. Isometric immersions

Consider the n-dimensional Euclidean space $\boldsymbol{E}^n$. We denote by $\boldsymbol{R}^n$ the Euclidean vector space associated with $\boldsymbol{E}^n$; thus $\boldsymbol{R}^n$ can be regarded as the group of translations of $\boldsymbol{E}^n$, or else as the fibre in the tangent bundle of $\boldsymbol{E}^n$. In conjunction with $\boldsymbol{R}^n$ we consider the complex extension, $\boldsymbol{C}^n = \boldsymbol{R}^n \otimes_{\boldsymbol{R}} \boldsymbol{C}$ of $\boldsymbol{R}^n$, regarded as the fibre of the "complexified" tangent bundle of $\boldsymbol{E}^n$. We extend the orthogonal (inner product) structure from $\boldsymbol{R}^n$ both to an orthogonal and a unitary structure in $\boldsymbol{C}^n$ in the following, natural way: if $z' = (z'_1, \cdots\cdots, z'_n)$ and $z'' = (z''_1, \cdots, z''_n)$ are two vectors in $\boldsymbol{C}^n$, then the "dot" product is the symmetric bilinear form $z' \cdot z'' = \sum_{\alpha=1}^{n} z'_\alpha z''_\alpha$; moreover we denote by $\bar{z}'$ the complex conjugate of z'

$$\bar{z}' = (\bar{z}'_1, \cdots, \bar{z}'_n);$$

we also need the absolute value (or norm) of a vector $z = (z_1, \cdots, z_n)$ to be the square root of the positive definite hermitian form $z' \cdot \bar{z}''$, so that

$$|z| = \sqrt{z \cdot \bar{z}} = (\sum_{\alpha=1}^{n} |z_\alpha|^2)^{\frac{1}{2}}.$$

Thus the dot product, conjugation and norm on $\boldsymbol{C}^n$ are the invariants defining the real orthogonal (euclidean) group $O(n)$ acting on $\boldsymbol{C}^n$.

Consider now the Grassman algebra $\Lambda(\boldsymbol{C}^n)$ generated by $\boldsymbol{C}^n$ over $\boldsymbol{C}$. If $\boldsymbol{Z} = \boldsymbol{z}_1 \wedge \cdots \wedge \boldsymbol{z}_p$ and $\boldsymbol{W} = \boldsymbol{w}_1 \wedge \cdots \boldsymbol{w}_p$ are two p-vectors in $\Lambda(\boldsymbol{C}^n)$ we can extend the $O(n)$-invariants of $\boldsymbol{C}^n$ to $\Lambda(\boldsymbol{C}^n)$ in the obvious way, by defining the dot product as the Gramian determinant

$$\boldsymbol{Z} \cdot \boldsymbol{W} = \det_{1 \le \alpha,\ \beta \le p} (\mathbf{Z}_\alpha \cdot \boldsymbol{w}_\beta),$$

the conjugation map by setting

$$\bar{\boldsymbol{Z}} = \bar{\boldsymbol{z}}_1 \wedge \cdots \wedge \bar{\boldsymbol{z}}_p,$$

and the positive valued norm $|\boldsymbol{Z}|$ of $\boldsymbol{Z}$ by means of the Hermitian product

$$|\boldsymbol{Z}|^2 = \boldsymbol{Z} \cdot \bar{\boldsymbol{Z}}.$$

With these notations, we let $X : \Sigma \to \boldsymbol{E}^n$ be a differentiable map of a surface Σ, as specified in §2, into the Euclidean n-space $\boldsymbol{E}^n$. We describe Σ locally in terms of a local isothermal parameter w with respect to the Riemannian structure induced on Σ by X; then X can be represented locally by an $\boldsymbol{E}^n$-valued differentiable function[2] $X(w, \bar{w})$ of w. The map is an isometric immersion, if and only if at each point

$$F(w, \bar{w}) = \left|\frac{\partial X}{\partial w}\right|^2 = \frac{\partial X}{\partial w} \cdot \frac{\partial X}{\partial \bar{w}}, \tag{4.1}$$

$$\frac{\partial X}{\partial w} \cdot \frac{\partial X}{\partial w} = 0 \qquad \left(\text{implying } \frac{\partial X}{\partial \bar{w}} \cdot \frac{\partial X}{\partial \bar{w}} = 0\right), \tag{4.2}$$

where $F(w, \bar{w})$ is the coefficient of the Riemannian metric (2.1). We can now form the successive derivatives of X with respect to w or $\bar{w}$ either, from the local, (or analytical) standpoint

[2]See footnote 1.

$$\partial^p \bar{\partial}^q X = \frac{\partial^{p+q} X}{\partial w^p \partial \bar{w}^q} \qquad (p,\ q = 0, 1, 2, \cdots) \tag{4.3}$$

or in the Levi-Civita (or geometrical) sense such as an ordered sequence of (noncommuting) successive derivations containing, say, p times ∇' and q times ∇'' in some given order; in this case we have a partial operator of order $p+q$, whose principal part is $\partial^p \bar{\partial}^q X$. In the case of isometric immersion, several identities hold, involving dot products of derivatives of X. For instance, considering the absolute derivatives of both members in either (4.1) or (4.2), we obtain immediately[3]

$$\nabla' X \cdot \nabla' \nabla'' X = \nabla' X \cdot {\nabla'}^2 X = \nabla'' X \cdot {\nabla'}^2 X = 0, \tag{4.4}$$

showing that all second order covariant derivatives of X are orthogonal to all first order (and thereby define the second fundamental form).

5. The main theorems

We now make the basic assumptions of the article, namely that the isometric immersion $X : \Sigma \to \boldsymbol{R}^n$ maps Σ into a Euclidean $(n-1)$-dimensional sphere rS^{n-1} of radius r and that the resulting surface in the $(n-1)$-sphere with the induced Riemannian structure is locally a solution of the Plateau problem. For this purpose we identify $\boldsymbol{E}^n$ with $\boldsymbol{R}^n$ by the choice of an origin 0; the sphere rS^{n-1} of radius r and center 0 is the set of real vectors $\boldsymbol{x} \in \boldsymbol{R}^n$ satisfying $x \cdot x = r^2$.

Assumptions. We let Σ be an oriented surface and $X : \Sigma \to \boldsymbol{R}^n$ an analytic immersion of Σ satisfying the following five conditions

1) The image lies in the sphere rS^{n-1} of radius r, *i.e.*

$$X \cdot X = r^2 \qquad \text{at every point of } \Sigma. \tag{5.1}$$

2) We carry all calculations in terms of the local isothermal coordinate systems w defined in Σ by the given orientation and the Riemannian metric induced by X.

3) The immersion of Σ in rS^{n-1} is locally minimal; this means, in terms of X as an immersion in $\boldsymbol{R}^n$, that the curvature vector $\dfrac{1}{F(w,\overline{w})} \dfrac{\partial^2 X}{\partial w \partial \overline{w}} = F^{-1} \otimes \nabla' \nabla'' X$ is everywhere orthogonal to rS^{n-1}, *i.e.* proportional to the position vector X of the image.

4) The surface Σ is homeomorphic to a 2-sphere.

[3] The operations of dot product and Grassmann multiplication are understood to be naturally extended from C^n to its tensor product with the bundle E of bigraded tensor algebras over Σ.

5) Without loss of generality, the immersion X is linearly full, *i,e.* the image $X(\Sigma)$ is not contained in any Euclidean hyperplane in $\boldsymbol{R}^n$.

Remarks. The analyticity of X can be deduced as a consequence of 3), provided only that the map be of class $\mathcal{C}^3$. The last two main assumptions will be used only later, and then the essential role that these assumptions play will be emphasized.

We draw immediately a couple of elementary conclusions from the first three of the five basic assumptions.

Lemma 5.1. *The mapping X satisfies the equation*

$$\partial\bar{\partial}X = \nabla'\nabla''X = -r^{-2}FX. \tag{5.2}$$

Proof. From Assumption 3 we have $\partial\bar{\partial}X = \lambda X$ for some real valued function λ; on the other hand we have also, from (5.1),

$$\begin{aligned} 0 &= \frac{1}{2}\partial\bar{\partial}(r^2) = \frac{1}{2}\partial\bar{\partial}(X\cdot X) = \partial\bar{\partial}X\cdot X + \bar{\partial}X\cdot\partial X \\ &= \lambda X\cdot X + F = r^2\lambda + F, \end{aligned}$$

whence $\lambda = -r^{-2}F$, as asserted.

Lemma 5.2. *The complex vector subspace of $\boldsymbol{C}^n$ spanned by the k'th order jet of X, at any point $\boldsymbol{p}\in\Sigma$ and for any positive integer k, is spanned by the $2k+1$ vectors X, ∇'^pX, ∇''^pX or equivalently X, ∂^pX and $\bar{\partial}^pX (1\le p\le k)$ evaluated at $\boldsymbol{p}$.*

Proof. The linear span over $\boldsymbol{C}$ of the k'th order jet of X can be generated equivalently by X and the $\frac{1}{2}k(k+3)$ partial derivatives of X of order $\le k$, or by X and the $2^{k+1}-2$ different Levi-Civita derivatives of X of order $\le k$; at the same time, in view of (5.2) any Levi-Civita derivative of X of order k in which each of ∇' and ∇'' appears at least once (or equivalently, any partial derivative $\dfrac{\partial^{p+q}X}{\partial w^p\partial\bar{w}^q}$ with $\min(p,q)\ge 1$ and $p+q=k$) can be expressed as a linear combination of X and derivatives of order $< k-1$; from this the conclusion follows immediately.

Until now we have not yet used assumptions 4) or 5), namely that Σ is homeomorphic to a sphere, or that X is linearly full. In the next lemma, however, Assumption 4 plays an essential role.

Lemma 5.3. *Under the basic assumptions* 1), 2), 3) *and* 4), *the complex subspace of $\boldsymbol{C}^n$ spanned by all the derivatives $\nabla'^pX (p\ge 1)$ at any point of Σ is totally isotropic with respect to the dot product, and is orthogonal to X.*

Proof. We set, formally, $\nabla'^oX = \nabla''^oX = X$; we must prove that, for any two nonnegative integers p, q with $p+q\ge 1$,

$$\nabla'^p X \cdot \nabla'^q X = 0.$$

This identity, for $1 \leq p+q \leq 3$, is true in the case of arbitrary immersions X of any surface Σ into a sphere, with no further assumptions; this follows easily from (4.2), (4.4) and (5.1). We shall prove it now for higher values of $p+q$, by induction on $p+q$. Suppose that $\nabla'^p X \cdot \nabla'^q X \equiv 0$ for all p, q with $1 \leq p+q \leq k-1$, for some integer $k \geq 2$; to prove it for $p+q=k$, we distinguish the case where k is even from that where k is odd. If k is odd, say $k = 2m+1 (m \geq 1)$, we have

$$\nabla'^m X \cdot \nabla'^{m+1} X = \frac{1}{2} \nabla' (\nabla'^m X \cdot \nabla'^m X) = 0$$

and, for any p such that $1 \leq p \leq m$

$$\begin{aligned} \nabla'^{m-p} X \cdot \nabla'^{m+p+1} X = (-1)^p \nabla' \Bigg(& \frac{1}{2} \nabla'^m X \cdot \nabla'^m X \\ & + \sum_{\mu=1}^{p} (-1)^\mu \nabla'^{m-\mu} X \cdot \nabla'^{m+\mu} X \Bigg) = 0, \end{aligned}$$

proving the induction step for k odd ; we remark that, in this case, we do not need the assumption 4) that Σ is homeomorphic to S^2.

If k is even, say $k = 2m$ $(m \geq 1)$, let $\Lambda_m = \nabla'^m X \cdot \nabla'^m X$. Then for $p+q = 2m$ and, without loss of generality, $p < q$,

$$\begin{aligned} \nabla'^p X \cdot \nabla'^q X &= (-1)^{m-p} \Lambda_m + (-1)^{m-p} \nabla' (\Sigma_{\mu=1}^{m-p} (-1)^\mu \nabla'^{m-\mu} X \cdot \nabla'^{m+\mu-1} X) \\ &= (-1)^{m-p} \Lambda_m. \end{aligned}$$

This reduces the remainder of the proof to showing that $\Lambda_m \equiv 0$.

First, we calculate $\nabla'' \Lambda_m$. From the definition of Λ_m,

$$\nabla'' \Lambda_m = 2 \nabla'' \nabla'^m X \cdot \nabla'^m X.$$

Next, we evaluate $\nabla'' \nabla'^m X$; using the Ricci-identity (3.9) $m-1$ times, we get

(5.3)
$$\begin{aligned} \nabla'' \nabla'^m X &= \textstyle\sum_{\mu=1}^{m-1} \nabla'^{\mu-1} (\nabla'' \nabla' - \nabla' \nabla'') \nabla'^{m-\mu} X + \nabla'^{m-1} (\nabla'' \nabla' X) \\ &= -r^{-2} F \nabla'^{m-1} X + \textstyle\sum_{\mu=1}^{m-1} (m-\mu) F \nabla'^{\mu-1} (K \nabla'^{m-\mu} X) \\ &= F \left(-r^{-2} + \binom{m}{2} K \right) \nabla'^{m-1} X + (\text{a linear combination of} \\ & \qquad \nabla' X, \cdots, \nabla'^{m-2} X), \end{aligned}$$

where K is the Gaussian curvature. From this, and using the induction assumption, we deduce immediately that

$$\nabla'' \Lambda_m \equiv 0.$$

This proves that, in the absence of assumption 4), if X satisfies the induction assumption $\nabla'^p X \cdot \nabla'^q X = 0$ for all $p,\ q < 2m$, then $\nabla'^m X \cdot \nabla'^m X = \Lambda_m$ is a holomorphic cross section in $E^{2m,0}$, *i.e.*, a holomorphic differential of weight $2m$. Since Σ is homeomorphic to S^2, *i.e.* a compact Riemann surface of genus zero, the only holomorphic cross section in $E^{p,0}$ for any $p > 0$ is the trivial one, so that in this case $\Lambda_m = 0$. This completes the proof of the lemma.

Remark. The above lemma and its proof are inspired by H. Hopf's analogous argument [1] in the case of closed surfaces of genus 0 immersed in $\boldsymbol{R}^3$ with constant mean curvature.

In what follows, it is useful to introduce the following differential geometric forms attached to the map X, analogous to the generalized Wronskians for differentiable curves in $\boldsymbol{E}^n$.

We denote by $T'_k X$, for any positive integer k, the following[4] cross section in $E^{k_2,0} \otimes_C \Lambda^k(\boldsymbol{C}^n)$, where $k_2 = \dfrac{1}{2}k(k+1)$,

$$T'_k X = \nabla' X \wedge \nabla'^2 X \wedge \cdots \wedge \nabla'^k X = \partial X \wedge \partial^2 X \wedge \cdots \wedge \partial^k X. \tag{5.4}$$

Clearly, T'_k vanishes at any point $\boldsymbol{p} \in \Sigma$, if and only if $\nabla' X, \nabla'^2 X, \cdots, \nabla'^k X$ at $\boldsymbol{p}$ are linearly dependent over the tensor algebra E. Since, by Lemma 5.1 the subspace of $\boldsymbol{C}^n$ spanned by the k vectors $\nabla' X, \nabla'^2 X, \cdots, \nabla'^k X$ at any point of Σ is totally isotropic with respect to the dot product, its dimensionality can not exceed $\dfrac{n}{2}$: thus $T'_k X \equiv 0$ as soon as $k > \dfrac{n}{2}$.

Lemma 5.4. *Under the basic assumptions, emphasizing here Assumption* 5), *the dimensionality* n *of the ambient space* $\boldsymbol{R}^n$ *is odd; more precisely,* $n = 2m + 1$, *where* m *is the highest value for which* $T'_m X$ *is not identically zero.*

Proof. Let m denote the highest integer for which $T'_m X = \partial X \wedge \cdots \wedge \partial^m X = \nabla' X \wedge \cdots \wedge \nabla'^m X$ is not identically zero. We shall prove first that $n \geq 2m + 1$. In fact, consider the exterior absolute value F_m of the $(2m+1)$-vector

$$X \wedge \nabla' X \wedge \nabla'^2 X \wedge \cdots \wedge \nabla'^m X \wedge \nabla'' X \wedge \cdots \wedge \nabla''^m X = X \wedge T'_m X \wedge T''_m X,$$

where $T''_m = \overline{T'_m}$. The square of F_m can be calculated from the Gramian determinant, which involves dot products such as $X \cdot X$, $X \cdot \nabla'^p X$, $X \cdot \nabla''^q X$, $\nabla'^p X \cdot \nabla'^q X$, $\nabla'^p X \cdot \nabla''^q X$, $\nabla''^p X \cdot \nabla''^q X$, etc., $(1 \leq p,\ q \leq m)$. Because of Lemma 5.3, this Gramian matrix splits into three blocks along the diagonal, namely $X \cdot X \oplus (\nabla'^p X \cdot \nabla''^q X) \oplus (\nabla''^p X \cdot \nabla'^q X)$. Thus we see that

[4]See footnote 3.

$$
\begin{aligned}
F_m &= |X| \det_{1 \leq p,\, q \leq m} (\nabla'^p X \cdot \nabla''^q X) \\
&= r(\nabla' X \wedge \nabla'^2 X \wedge \cdots \wedge \nabla'^m X) \cdot (\nabla'' X \wedge \cdots \wedge \nabla''^m X) \qquad (5.5)\\
&= r|T'_m X|^2.
\end{aligned}
$$

This shows that, wherever $T'_m X$ is not zero, F_m is strictly positive; the latter is the norm of a $(2m+1)$-vector in $\boldsymbol{C}^n$, showing that $n \geq 2m+1$.

We show now that, conversely, $n \leqq 2m+1$. Since the immersion X of Σ is real analytic and its image is not contained in any hyperplane, at each point of Σ the linear span of the jet of X of sufficiently high order is n-dimensional. By the definition of m, $T'_{m+1}X$ is identically zero, while $T'_m X$ is not; hence at any point $\boldsymbol{p} \in \Sigma$ where $T'_m X$ is not zero, $\nabla'^{m+1} X$ lies in complex m-dimensional subspace with basis $\nabla' X, \cdots, \nabla'^m X$ evaluated at $\boldsymbol{p}$. By considering inductively the vanishing of $\nabla'^k (T'_{m+1} X)(k = 1, 2, \cdots)$ it follows that each $\nabla'^{m+k} X$ at $\boldsymbol{p}$ lines in the same subspace. Therefore, for any nonnegative k, the $2m+2k+1$ vectors $X, \nabla'^p X, \nabla''^2 X$ $(1 \leq p,\, q \leqq m+k)$ are a linear combination of the $2m+1$ vectors $X, \nabla' X, \cdots, \nabla'^m X, \nabla'' X, \cdots, \nabla''^m X$. But the $2m+2k+1$ vectors span, by Lemma 5.2, the whole $(m+k)$'th order jet of X at $\boldsymbol{p}$. Therefore this jet has, for each k, a linear span of at most $2m+1$ dimensions. This shows that $n \leq 2m+1$, completing the proof.

We can state and prove now the main theorem of this article.

Theorem 5.1. *Let X: $\Sigma \to \boldsymbol{E}^n$ be an immersion of a surface Σ into $\boldsymbol{E}^n$, whose image is a locally minimal surface in a sphere rS^{n-1} of radius r, and is not contained in any hyperplane of $\boldsymbol{E}^n$. Then the following conclusions hold;*

i) *The area $A = A(X)$ of the image surface is an integer multiple of $2\pi r^2$.*

ii) *The dimension number n is odd, say $n = 2m+1$* and

$$
A \geq 4\pi r^2 \binom{m+1}{2} = 4\pi r^2 \left(\frac{n^2-1}{8}\right). \qquad (5.6)
$$

Proof. The proof of the two conclusions depend on certain calculations on the objects $T'_k X (1 \leq k \leq m)$ defined in (5.4) and on the related quantities

$$
F_k = |X|\ |T'_k X|^2 \qquad (1 \leq k \leq m),
$$

introduced in (5.5) for the special value $k = m$. Clearly $T'_k X$ is a real analytic cross section over Σ in $E^{\frac{1}{2}k(k+1),0} \otimes_C \boldsymbol{C}^{\binom{n}{k}}$, while $\mathbf{F}_k$ is a nonnegative cross section in E^{k_2, k_2}, where $k_2 = \binom{k+1}{2} = \frac{1}{2}k(k+1)$. It is convenient to replace F_k by its norm Φ_k under the Riemannian metric (2.1), since Φ_k is a scalar function. Thus we are led to consider the sequence of nonnegative real valued scalars $\Phi_k (k = 1, 2, \cdots)$, where

$$
\Phi_k = F^{-\frac{1}{2}k(k+1)} F_k. \qquad (5.7)
$$

For the sake of conformity we also set $T'_0X = T''_0X$ identically equal to 1 and $F_0 = \Phi_0$ equal to the constant r. The following differential recursion formulas now hold and are easy to verify: For each $k \geq 1$

$$\nabla' T'_k X = T'_{k-1}X \wedge \nabla'^{k+1}X \ , \tag{5.8}$$

$$\begin{aligned} \nabla' T''_k X &= \nabla'\nabla'' X \wedge \nabla''^2 X \wedge \cdots \wedge \nabla''^k X \\ &= -r^{-2} F X \wedge \nabla''^2 X \wedge \cdots \wedge \nabla''^k X \ , \end{aligned} \tag{5.9}$$

and similarly for the complex conjugates.

From the above, we can calculate formally the absolute differentials of each Φ_k (using k_2 as an abbreviation for $\binom{k+1}{2} = \frac{1}{2}k(k+1)$),

$$\begin{aligned} \nabla'\Phi_k &= rF^{-k_2}\nabla'(T'_kX \cdot T''_kX) \\ &= rF^{-k_2}(\nabla'T'_kX \cdot T''_kX + T'_kX \cdot \nabla'T''_kX). \end{aligned}$$

Because of (5.8), (5.9), and since X, by Lemma 5.3, is orthogonal to each $\nabla'^p X$, $T'_kX \cdot \nabla' T''_k X = 0$ identically; thus

$$\partial\Phi_k = \nabla'\Phi_k = rF^{-k_2}(T'_{k-1}X \wedge \nabla'^{k+1}X) \cdot T''_kX \tag{5.10}$$

and similarly $\bar\partial\Phi_k = \nabla''\Phi_k = rF^{-k_2}T'_kX \cdot (T''_{k-1}X \wedge \nabla''^{k+1}X)$. We calculate now $\bar\partial\partial\Phi_k = \nabla''\nabla'\Phi_k = \nabla'\nabla''\Phi_k$ as follows:

$$\begin{aligned} \bar\partial\partial\Phi_k = {} & rF^{-k_2}\{\nabla''(T'_{k-1}X \wedge \nabla'^{k+1}X) \cdot T''_kX + |\nabla'T'_kX|^2\} \\ = {} & rF^{-k_2}(T'_{k-1}X \ \wedge \nabla''\nabla'^{k+1}X) \cdot T''_kX + rF^{-k_2}|T'_{k-1}X \wedge \nabla'^{k+1}X|^2. \end{aligned}$$

We apply the Ricci identity (3.9) in evaluating the term with $\nabla''\nabla'^{k+1}X$, as it was done in (5.3) for the case $\nabla''\nabla'^m X$; we get then

$$T'_{k-1}X \wedge \nabla''\nabla'^{k+1}X = F(-\, r^{-2} + k_2K)T'_kX \ .$$

Thus we have

$$\begin{aligned} \bar\partial\partial\Phi_k &= rF^{1-k_2}(-\, r^{-2} + k_2K)F_k + rF^{-k_2}|\nabla'T'_kX|^2 \\ &= F(-\, r^{-2} + k_2K)\Phi_k + rF^{-k_2}|T'_{k-1}X \wedge \nabla'^{k+1}X|^2. \end{aligned} \tag{5.11}$$

We now recall the algebraic identity (Lagrange)

$$\begin{aligned} &|T'_{k-1}X \wedge \nabla'^kX|^2|T'_{k-1}X \wedge \nabla'^{k+1}X|^2 \\ &\qquad - (T'_{k-1}X \wedge \nabla'^{k+1}X) \cdot (T''_{k-1}X \wedge \nabla''^kX) \\ &\qquad\qquad \otimes (T'_{k-1}X \wedge \nabla'^kX) \cdot (T''_{k-1}X \wedge \nabla''^{k+1}X) \\ &\qquad = |T'_{k-1}X|^2|T'_{k-1}X \wedge \nabla'^kX \wedge \nabla'^{k+1}X|^2 = F_{k-1}F_{k+1} \end{aligned}$$

and apply it in conjunction with (5.10) and (5.11); one obtains the following equation

$$\begin{aligned}\Phi_k \partial\bar{\partial}\Phi_k - \partial\Phi_k \bar{\partial}\Phi_k &= F(-r^{-2} + k_2 K)\Phi_k^2 + F^{-2k_2} F_{k-1} F_{k+1} \\ &= F(-r^{-2} + k_2 K)\Phi_k^2 + F\Phi_{k-1}\Phi_{k+1}\end{aligned}$$

or, if $\Phi_k \neq 0$, we obtain the recursion formula

(5.12)
$$\partial\bar{\partial}(\log \Phi_k) = F\left(\frac{\Phi_{k-1}\Phi_{k+1}}{\Phi_k^2} - r^{-2} + k_2 K\right) \qquad (\Phi_0 = \Phi_1 = r)\ .$$

Since each of the functions Φ_k is real analytic and nonnegative, we see that either Φ_k is identically zero, or else, by considering any locally defined, real valued, analytic function ϕ whose Laplacian satisfies

$$\Delta\phi = \frac{2}{F}\partial\bar{\partial}\phi = 2(k_2 K - r^{-2})\ ,$$

we see that $\log \Phi_k - \phi$ is subharmonic; in particular, the zeros of Φ_k can be at most isolated and, analytically, of (finite) even order. Let $2j_p^{(k)}$ denote the real analytic order of the zero of Φ_k at any $p \in \Sigma$ and, if Φ_k is not identically zero, set $N_k = \Sigma_{p \,\varepsilon\, \Sigma}\, j_p^{(k)}$. We can represent N_k analytically as a sum of the residues of the logarithmic singularities of $\log \Phi_k$ by the potential theoretic formula

$$-\ 2\pi N_k = \frac{1}{2} \lim_{\varepsilon \to 0} \int_{\Sigma_\varepsilon} (\Delta \ \log\ \Phi_k)\omega\ ,$$

where $\omega = iF\ dw \wedge d\bar{w}$ is the element of area, and Σ_ε denotes the complement in Σ of an ε-neighborhood of all points where Φ_k becomes zero.

Thus we have, in view of (5.12),

$$\lim_{\varepsilon \to 0} \int_{\Sigma_\varepsilon} (r^{-2} - k_2 K)\omega - 2\pi N_k = \lim_{\varepsilon \to 0} \int_{\Sigma_\varepsilon} \frac{\Phi_{k-1}\Phi_{k+1}}{\Phi_k^2}\omega \geq 0\ .$$

According to the Gauss-Bonnet formula, $\int_\Sigma K\omega = 4\pi$, so that, evaluating the above integrals and taking the limits as $\varepsilon \to 0$, we obtain the following estimate for the area $A = \int_\Sigma \omega$ of the images $X(\Sigma)$:

$$\begin{aligned}A &= (k_2 + \frac{1}{2}N_k)4\pi r^2 + \lim_{\varepsilon \to 0} \int_{\Sigma_\varepsilon} \frac{\Phi_{k-1}\Phi_{k+1}}{\Phi_k^2}\omega \\ &\geq (k_2 + \frac{1}{2}N_k)4\pi r^2\ ,\end{aligned}$$

equality holding if and only if Φ_k is not identically zero, while Φ_{k+1} vanishes everywhere; this happens, as we showed, precisely for $k = m = \frac{n-1}{2}$, according to Lemma 5.4. Thus we have proved the formula for the area A of Σ induced by the immersion X,

$$A = \left(\frac{m(m+1)+N_m}{2}\right) 4\pi r^2 = \frac{n^2-1}{2}\pi r^2 + 2\pi N_m r^2, \tag{5.13}$$

where $2N_m$ is the total multiplicity of zeros of Φ_m. This completes the proof of the theorem.

Corollary. *Under the same assumptions of the theorem there exists a*
nonempty open set of Σ where the Gaussian curvature K satisfies

$$K \leq \frac{2}{m(m+1)r^2}. \tag{5.14}$$

Proof. Since the area A of the image of Σ is at least $4\pi m_2 r^2 = 2\pi m(m + 1)r^2$ and, by the Gauss-Bonnet formula,

$$\int_\Sigma K\omega = 4\pi ,$$

the conclusion follows immediately.

The only explicit examples known of minimal immersions of 2-sphere Σ into rS^{n-1} are those where the Gaussian curvature of the induced metric on Σ is constant.

Theorem 5.2. *Let Σ be a 2-sphere with a Riemannian metric with constant curvature K, and let $X : \Sigma \to rs^{n-1} \subset \boldsymbol{E}^n (n = 2m+1 \geq 3)$ be an isometric, minimal immersion of Σ, such that the image is not contained in any hyperplane of $\boldsymbol{E}^n$. Then*

i) *The value of K is uniquely determined at the value,*

$$K = \frac{2}{m(m+1)r^2} ; \tag{5.15}$$

ii) *The immersion X is uniquely determined up to a rigid rotation of rS^{n-1}, and the n components of the vector X are a suitably normalized basis for the spherical harmonics of order m on Σ.*

Proof. If K is constant and $X : \Sigma \to rS^{n-1}$ an isometric, minimal immersion, the functions $\Phi_k(k = 0, 1, \cdots)$ can be calculated explicitly in a simple way: from the initial data $\Phi_0 = \Phi_1 = r$ and the recursion formula (5.12), we see that each Φ_k is constant, and hence

$$\Phi_{k+1} = \frac{\Phi_k^2}{\Phi_{k-1}}\left(r^{-2} - \frac{k(k+1)}{2}K\right);$$

since each Φ_k is then a positive constant for $k \leq m = \dfrac{n-1}{2}$ and $\Phi_{m+1} \equiv 0$, we see immediately that the determination (5.15) of the value of K is both necessary and sufficient. Therefore there exists an isothermal parameter w defined on all of Σ except for one point, such that

$$F(w, \bar{w}) = m(m+1)r^2(1 + w\bar{w})^{-2}\ .$$

The immersion function $X : \Sigma \to rS^{n-1}$ satisfies the equation $\partial\bar{\partial}X = -r^{-2}FX$, or equivalently, in terms of the Laplace operator $\Delta = 2F^{-1}\partial\bar{\partial}$,

$$\Delta X = -2r^{-2}X$$

thus each component of X is an eigenfunction of Δ (spherical harmonic) corresponding to the eigenvalue $-2r^{-2}$. It is known that on the 2-sphere with constant curvature K as in (5.15) the eigenvalues of Δ are

$$\lambda_k = -\frac{k(k+1)}{K} = -\frac{2k(k+1)}{m(m+1)r^2} \qquad (k = 0, 1, 2, \cdots)$$

and the eigenspace corresponding to each λ_k is $(2k+1)$-dimensional and generated as follows: map Σ isometrically onto the Euclidean sphere $r_0S^2 \subset \boldsymbol{R}^3$ with $r_0 = K^{-\frac{1}{2}} = \dfrac{m(m+1)}{2}r$ and consider the $2k+1$ linearly independent, homogeneous polynomials of degree k in the 3 Cartesian coordinates of $\boldsymbol{R}^3$, that satisfy Laplace's equation. Their restriction to r_0S^2 are the spherical harmonics of order k. Letting now $k = m$, we pick an orthogonal basis for all of them, and verify that the sum of their squares evaluated at each point of r_0S^2 is constant; hence they can be normalized so that the sum of their squares is r^2; these are then the $2m+1$ components of the imbedding function X. This completes the proof of the theorem.

It is natural to ask whether the example given above represents the only type of minimal immersion[5] of Σ into S^{n-1}. This, however, has not yet been settled, except in the trivial case $m = 1$ $(n = 3)$. Even a simplifying assumption that the area A of the image attains its lowest possible value $2m(m+1)\cdot\pi r^2$, which is equivalent to saying $\Phi_m > 0$ everywhere, and hence also each $\Phi_k > 0$ for $k \leq m$, and $\Phi_{m+1} \equiv 0$ does not seem to help. To show the difficulty, more explicitly, we consider the apparently easiest, nontrivial case, $m = 2$ (corresponding to $n = 5$). Since $\Phi_0 = \Phi_1 = r$, we apply (5.12) to calculate each Φ_k by the induction formula

[5] *Added in proof.* Since the completion of this paper, this question has been answered negatively; the details are to appear in a sequel.

$$\Phi_{k+1} = \frac{\Phi_k^2}{\Phi_{k-1}} \left(\frac{1}{F} \partial\bar{\partial} \log \Phi_k + r^{-2} - \frac{1}{2} k(k+1) K \right).$$

Thus $\Phi_2 = r(r^{-2} - K)$ (implying, incidentally, the condition $K \leq r^{-2}$ which, for other reasons as well is necessary), and

$$\Phi_3 = r(r^{-2} - K)^2 (\frac{1}{2} \Delta \log (r^{-2} - K) + r^{-2} - 3K).$$

Thus a metric on S^2 is compatible with a linearly full, minimal immersion in rS^4, only if $K \leq r^{-2}$ everywhere, but with equality not holding identically, and

$$\Delta \log(r^{-2} - K) = 6K - 2r^{-2} \text{ wherever } K < r^{-2}; \tag{5.16}$$

the strict inequality $K < r^{-2}$ would then be satisfied everywhere, if and only if the total area of the immersed surface is $12\pi r^2$. For $m = 2$, the problem is equivalent to the question whether there are any Riemannian metrics that satisfy (5.16) other than the special case with constant curvature $K = 1/(3r^2)$.

Reference

[1] H. Hopf, *Über Flächen mit einer Relation zwischen den Hauptkrümmungen.* Math. Nachr. **4** (1950–51) 232–249.

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