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A FORMULA FOR THE BETTI NUMBERS OF COMPACT LOCALLY SYMMETRIC RIEMANNIAN MANIFOLDS

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1. Let X be a simply connected symmetric Riemannian manifold and let G be a connected Lie group acting transitively and almost effectively on X as a group of isometries. We denote by K the isotropy group of G at a point o of X. If G is compact, it is a well-known theorem of Cartan-Hodge that a differential p-form is harmonic if and only if it is G-invariant. It follows from this theorem that the p-th Betti number of X is equal to the multiplicity with which the trivial representation enters in the linear isotropic representation of K in the vector space of p-covectors at the point o.

Let us suppose now that G is a connected semi-simple Lie group with finite center all of whose simple components are non-compact. Let Γ be a discrete subgroup of G such that the quotient $\Gamma\backslash G$ is compact. We denote by $h^p(X, \Gamma)$ the vector space of all harmonic p-forms on X which are invariant by Γ. We know that the dimension of the space $h^p(X, \Gamma)$ is finite. The results obtained in the previous papers [4] shows that in several cases the dimension of $h^p(X, \Gamma)$ is also equal to the multiplicity with which the trivial representation enters in the linear isotropic representation of K in the space of p-covectors at the point o, if the number $p/dim\ X$ is small.

The purpose of the present paper is to prove a formula which relates the dimension of the space $h^p(X, \Gamma)$ with the decomposition of the unitary representation of G in the Hilbert space $L^2(\Gamma\backslash G)$ (see §2). This formula corresponds in a sense to the theorem of Cartan-Hodge and, in fact, if G is compact and Γ reduces to the identity, our formula is equivalent to Cartan-Hodge Theorem.

We shall also see as an example that, if X is the 3-dimensional hyperbolic space and if G is $SL(2, C)$ or the proper Lorentz group, the dimension of $h^1(X, \Gamma)$ is equal to the multiplicity in $L^2(\Gamma\backslash G)$ of the irreducible unitary representation $U_{2,0}$ of the principal series (see §5).

2. We retain the notations introduced in §1 so that G will denote a connected semi-simple Lie group with finite center all of whose simple components are non-compact. The group K is then a maximal compact subgroup of G. Let $\mathfrak{g}$ be the Lie algebra of left-invariant vector fields on G, and $\mathfrak{k}$ the subalgebra of $\mathfrak{g}$ corresponding to K. We denote by $\varphi(X, Y)$ $(X, Y \in \mathfrak{g})$ the Killing form of the semi-simple Lie algebra $\mathfrak{g}$ and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to φ. We

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know that

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{k}, \qquad \mathfrak{m} \cap \mathfrak{k} = (0),$$
$$[\mathfrak{m}, \mathfrak{m}] = \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{m}] = \mathfrak{m}.$$

Moreover, $\varphi(X, X)$ is positive if $X \in \mathfrak{m}$, $X \neq 0$, and negative if $X \in \mathfrak{k}$, $X \neq 0$. Let $\{X_i\}_{i=1,\dots,r}$ and $\{X_a\}_{a=r+1,\dots,n}$ be bases of $\mathfrak{m}$ and $\mathfrak{k}$ respectively such that

$$\varphi(X_i, X_j) = \delta_{ij} \qquad (1 \leq i, j \leq r),$$
$$\varphi(X_a, X_b) = -\delta_{ab} \qquad (r+1 \leq a, b \leq n).$$

In the following we shall make the convention that the indices $i, j, \dots$ will range from 1 to r, while the indices $a, b, \dots$ from $r+1$ to n.

A vector field $X \in \mathfrak{g}$ is left invariant by G and hence by Γ so that X is projectable onto $\Gamma\backslash G$. In the following we consider the elements X of $\mathfrak{g}$ as vector fields on $\Gamma\backslash G$. We denote by C the differential operator on $\Gamma\backslash G$ defined by

$$C = \sum_{i=1}^{r} X_i^2 - \sum_{a=r+1}^{n} X_a^2.$$

The operator C is called the Casimir operator of G. We may consider C as an element of the universal enveloping algebra $E(\mathfrak{g})$ of $\mathfrak{g}$. It is known that C is in the center of $E(\mathfrak{g})$.

Now let T be a unitary representation of G in a Hilbert space H. A vector $\varphi \in H$ is called a *regular* vector if the function $s \to T(s)\varphi$ is of class C^∞. We denote by W the subspace of all regular vectors of H. It is known that W is dense in H. Let $X \in \mathfrak{g}$ and let $\exp tX$ be the 1-parameter subgroup of G corresponding to X. For $\varphi \in W$, put $T(X)\varphi = \left[\frac{d}{dt} T(\exp tX)\varphi\right]_{t=0}$. Then $iT(X)$ is a self-adjoint operator with domain W. We define the self-adjoint operator C_T of H with domain W by putting

$$C_T = \sum_{i=1}^{r} T(X_i)^2 - \sum_{a=r+1}^{n} T(X_a)^2,$$

and call it the Casimir operator of the unitary representation T of G. If T is an irreducible unitary representation, there exists a real number λ_T such that $C_T\varphi = \lambda_T\varphi$ for all $\varphi \in W$.

In the following we shall denote by D_0 the set of irreducible unitary representations T of G such that $\lambda_T = 0$.

We denote by U the unitary representation of G in the Hilbert space $L^2(\Gamma\backslash G)$. The vector space $C^\infty(\Gamma\backslash G)$ of all complex valued C^∞-functions on $\Gamma\backslash G$ is a subspace of the space of regular vectors of $L_2(\Gamma\backslash G)$, and we have $Cf = -C_U f$ for all $f \in C^\infty(\Gamma\backslash G)$. The representation U decomposes into sum of a countable number of irreducible unitary representations in which each irreducible representation enters with a finite multiplicity [1]. We denote by $N(T)$ the multiplicity in U of an irreducible unitary representation T of G.

Now let $\boldsymbol{T}$ be an irreducible unitary representation of $\boldsymbol{G}$, and T_K the restriction of $\boldsymbol{T}$ onto K. It is well-known (see [2]) that the representation T_K of K decomposes into sum of a countable number of irreducible representations in which each irreducible representation enters with a finite multiplicity. We shall denote by $M(T_K;\tau)$ the multiplicity in T_K of an irreducible representation τ of K.

Let now $\mathfrak{m}^C$ be the complexification of $\mathfrak{m}$. We denote by ad^p the representation of K in the vector space $\overset{p}{\wedge}\ \mathfrak{m}^C$ induced by the adjoint action of K in $\mathfrak{m}$. Let

$$ad^p = \tau_1^p + \cdots + \tau^p{}_{s_p} \tag{2.1}$$

be the decomposition of ad^p into a sum of irreducible representations.

Theorem. *Let $\boldsymbol{G}$ be a connected semi-simple Lie group with finite center, K a maximal compact subgroup of $\boldsymbol{G}$, and Γ a discrete subgroup of $\boldsymbol{G}$ with compact quotient space $\Gamma\backslash G$. Assume that Γ acts freely on the symmetric space $X = G/K$, and let $h^p(X,\ \Gamma)$ be the vector space of all harmonic p-forms on X invariant by Γ. Let $\boldsymbol{T}$ be an irreducible unitary representation of $\boldsymbol{G}$, and T_K the restriction of $\boldsymbol{T}$ on K. Let $N(T)$ denote the multiplicity of $\boldsymbol{T}$ in the unitary representation U of $\boldsymbol{G}$ in the Hilbert space $L^2(\Gamma\backslash G)$, and $M(T_K;\tau_i^p)$ the multiplicity of the irreducible representation τ_i^p of K in T_K. Then*

$$\dim\ h^p(X,\Gamma) = \sum_{T\epsilon D_0} N(T)\Big(\sum_{i=1}^{s_p} M(T_K;\tau_i^p)\Big),$$

where D_0 denotes the set of all irreducible unitary representations of $\boldsymbol{G}$ with vanishing Casimir operator.

The following sections are devoted to proving this theorem.

3. Let η be a complex valued differential p-form in X invariant by Γ, and π_0: $G \to G/K = X$ the canonical projection of $\boldsymbol{G}$ onto X. Put $\tilde{\eta} = \eta \circ \pi_0$. Then $\tilde{\eta}$ is a p-form on $\boldsymbol{G}$ having the following properties:

$$\tilde{\eta}\circ L_\gamma = \tilde{\eta} \quad (\gamma \in \Gamma), \qquad \tilde{\eta}\circ R_K = \tilde{\eta} \quad (k \in K),$$
$$i(Y)\tilde{\eta} = 0 \quad (Y \in \mathfrak{k}),$$

where L_g (resp. R_g) denotes the left (resp. right) translation of $\boldsymbol{G}$ by $g \in G$, and $i(X)$ the operator of interior multiplication.

Now let $\omega^i (1 \leq i \leq r)$ be the left invariant 1-form on G such that $\omega^i(X_j) = \delta^i_j$. We denote by I an ordered set of p indices i_s such that $1 \leq i_1 < i_2 < \cdots < i_p \leq r$. Further put

$$\omega^I = \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}.$$

Then the p-form $\tilde{\eta}$ is written uniquely in the form

$$\tilde{\eta} = \sum_I \eta_I \omega^I,$$

where the coefficients η_I are functions on G. Now $\{\omega^I\}$ form a basis of $\overset{p}{\wedge}\ \mathfrak{m}^{*C}$, and we denote by ad^{*p} the representation of K in $\overset{p}{\wedge}\ \mathfrak{m}^{*C}$ which is contragredient to ad^p. Since the p-form ω^I is left-invariant, we have $\omega^I \circ R_k = ad^{*p}(k) \cdot \omega^I$ for all $k \in K$. Put

$$ad^{*p}(k) \cdot \omega^I = \sum_J \tau^I_J(k) \omega^J.$$

We then have $\tilde{\eta} \circ R_k = \sum_J \sum_I \tau^I_J(k)(\eta_I \circ R_k)\omega^J$ and, since $\tilde{\eta} \circ R_k = \tilde{\eta}$, we get

$$\eta_I(g \cdot k) = \sum_J \tau^J_I(k^{-1}) \eta_J(g) \qquad (g \in G,\ k \in K).$$

It follows also from $\tilde{\eta} \circ L_\gamma = \tilde{\eta}$ and $\omega^I \circ L_\gamma = \omega^I$ that

$$\eta_I(\gamma \cdot g) = \eta_I(g) \qquad (\gamma \in \Gamma).$$

Hence we may consider η_I as a function on $\Gamma \backslash G$ such that

$$\eta_I(x \cdot k) = \sum_J \tau^J_I(k^{-1}) \eta_J(x)$$

for $x \in \Gamma \backslash G$ and $k \in K$. We may also consider $\tilde{\eta}$ as a $\overset{p}{\wedge}\ \mathfrak{m}^{*C}$-valued function on $\Gamma \backslash G$ defined by

$$\tilde{\eta}(x) = \sum_I \eta_I(x) \omega^I \qquad (x \in \Gamma \backslash G).$$

We have then

$$\tilde{\eta}(x \cdot k) = ad^{*p}(k^{-1}) \tilde{\eta}(x). \tag{1}$$

Thus there corresponds to a differential p-form η on G/K invariant by Γ a $\overset{p}{\wedge}\ \mathfrak{m}^{*C}$-valued function on $\Gamma \backslash G$ satisfying the condition (1), and conversely, to each of the functions satisfying (1) corresponds a Γ-invariant p-form and this corresponds is bijective. If the form η is of class C^∞ so is the corresponding function $\tilde{\eta}$; if η is measurable (with respect to the invariant measure on G/K), so is $\tilde{\eta}$ (with respect to the invariant measure on $\Gamma \backslash G$).

Now let Ω_p be the Hilbert space of all Γ-invariant measurable p-forms on G/K such that

$$||\eta||^2 = \int_F < \eta,\ \eta > dv < +\ \infty,$$

where F denotes a compact fundamental domain for Γ, and $<,\ >$ the length of η with respect to the Riemannian metric of G/K. We can show that if η and θ are in Ω_p, and $\tilde{\eta}$ and $\tilde{\theta}$ are the corresponding $\overset{p}{\wedge}\ \mathfrak{m}^{*C}$-valued functions, then

$$(\theta, \eta) = M \sum_I \int_{\Gamma\backslash G} \theta_I \cdot \bar{\eta}_I\ dx,$$

where M is a suitable constant independent of $\eta,\ \theta$ [5].

Suppose now that η is of class C^∞, and let Δ denote the laplacian operator for the p-forms. Then we have

$$(\Delta\theta)_I = C \cdot \theta_I,$$

where C denotes the Casimir operator [5]. Therefore we get

$$(\Delta\theta,\ \eta) = M \sum_I \int_{\Gamma\backslash G} C\theta_I \cdot \bar{\eta}_I\ dx.$$

and θ is harmonic if and only if $C\theta_I = 0$ for all $I = (i_1,\ \ldots,\ i_p)$.

The Killing form φ of $\mathfrak{g}$ defines a positive definite hermitian inner product φ^* in $\overset{p}{\wedge}\ \mathfrak{m}^{*C}$ invariant by the representation ad^{*p} of K for which $\{\omega^I\}$ is an orthonormal basis. We have then

$$(\theta,\ \eta) = M \int_{\Gamma\backslash G} \varphi^*(\tilde{\theta}(x),\ \tilde{\eta}(\tilde{x}))\ dx.$$

Let

$$\overset{p}{\wedge}\ \mathfrak{m}^{*C} = F_1^* \oplus \cdots \oplus F_{s_p}^*$$

be the decomposition of $\overset{p}{\wedge}\ \mathfrak{m}^{*C}$ into the sum of mutually orthogonal irreducible K-invariant subspaces. We may assume that the irreducible representation of K in F_i^* is contragredient to τ_i^p (cf. (2.1)). Let P_i be the projection of $\overset{p}{\wedge}\ \mathfrak{m}^{*C}$ onto F_i^*, and put

$$\tilde{\eta}_i(x) = P_i\tilde{\eta}(x) \qquad (x \in \Gamma\backslash G).$$

Then $\tilde{\eta}_i$ is an F_i^*-valued function on $\Gamma\backslash G$ such that

$$\tilde{\eta}_i(xk) = \tau^{*p}_i(k^{-1})\tilde{\eta}_i(x) \qquad (x \in \Gamma\backslash G,\ k \in K).$$

Let η_i be the Γ-invariant p-form corresponding to $\tilde{\eta}_i$. We then have $\eta = \sum_i \eta_i$, and η is harmonic if and only if each η_i is harmonic (cf. [5]).

We denote by $A_{p,i}$ the vector space of all F_i^*-valued C^∞-functions f on $\Gamma\backslash G$ satisfying the conditions:

$$f(x\cdot k) = \tau_i^{*p}(k^{-1})f(x) \qquad (x \in \Gamma\backslash G,\ k \in K),$$
$$Cf = 0.$$

Then

$$\dim h^p(X,\ \Gamma) = \sum_{i=1}^{s_p} \dim A_{p,i}. \tag{3.2}$$

4. In this section we shall show that

$$\dim A_{p,i} = \sum_{T \in D_0} N(T)\cdot M(T_K;\ \tau_i^p). \tag{4.1}$$

Then the theorem follows from (3.2) and (4.1).

Let $\{\zeta^1, \cdots, \zeta^m\}$ be an orthonormal basis of F_i^*, and $\{Z_1, \cdots, Z_m\}$ the dual basis of the dual vector space F_i of F_i^*. We may consider F_i as an irreducible K-invariant subspace of $\overset{p}{\wedge}\, \mathfrak{m}^C$ such that

$$\overset{p}{\wedge}\, \mathfrak{m}^C = F_1 \oplus \cdots \oplus F_{s_p},$$

and we may assume that the representation of K in F_i is τ_i^p. To simplify the notation we write τ instead of τ_i^p. Let

$$\tau^*(k)\zeta^\lambda = \sum_\mu a_\mu^\lambda(k)\zeta^\mu.$$

Then we have

$$\tau(k)z_\lambda = \sum_\mu a_\lambda^\mu(k^{-1})z_\mu.$$

Let now

$$L^2(\Gamma\backslash G) = \sum_{a=1}^{\infty} \oplus H_a$$

be the decomposition of the Hilbert space $L^2(\Gamma\backslash G)$ into the direct sum of irreducible G-invariant closed subspaces, and U_a the irreducible unitary representation of G in H_a induced by U. Further, let

$$H_a = \sum_{b=1}^{\infty} \oplus H_{a,b}$$

be the decomposition of H_a into the direct sum of irreducible K-invariant closed subspaces. We take an index a such that $U_a \in D_0$, and suppose that the representations of K in $H_{a,1}, \cdots, H_{a,b_i} (b_i = M((U_a)_K; \tau_i^p))$ are equivalent to $\tau (= \tau_i^p)$. We fix an index b such that $1 \leq b \leq b_i$, and take a basis $\{f_\lambda\}_{\lambda=1,\cdots,m}$ of $H_{a,b}$ such that

$$U_a(k) f_\lambda = \sum_\mu a_\lambda^\mu (k^{-1}) f_\mu. \tag{4.2}$$

If $\{g_\lambda\}_{\lambda=1,\cdots,m}$ is another basis of $H_{a,b}$ which satisfies (4.2), then there exists a complex number α such that $g_\lambda = \alpha f_\lambda (\lambda = 1, \cdots, m)$ by Schur's lemma.

We define an F_i^*-valued function f on $\Gamma \backslash G$ by putting

$$f(x) = \sum_\lambda f_\lambda(x) \zeta^\lambda.$$

Then we have

$$f(x \cdot k) = \tau^*(k^{-1}) f(x).$$

Let η be the Γ-invariant p-form on G/K corresponding to the function f. We are going to show that η is harmonic. For this purpose we remark first that we have

$$(C \cdot h, \varphi) = 0 \tag{4.3}$$

for all $h \in C^\infty(\Gamma \backslash G)$ and $\varphi \in H_a$. In fact, let W_a be the space of regular vectors of H_a, and let $\varphi \in W_a$. Since C is equal to the opposite of the Casimir operator C_U of the representation U, C_U is self-adjoint, and φ is in the domain of C_U, we get $(Ch, \varphi) = -(h, C_U \varphi)$. Now $C_U \varphi = C_{U_a} \varphi = 0$, and hence $(Ch, \varphi) = 0$. Since W_a is dense in H_a, we get $(Ch, \varphi) = 0$ for all $\varphi \in H_a$.

Now let θ be a Γ-invariant p-form of class C^∞, and $\tilde{\theta}$ the corresponding $\overset{p}{\wedge} \mathfrak{m}^{*C}$-valued function on $\Gamma \backslash G$. Take an orthonormal basis $(\xi^1, \cdots, \xi^N)$ of $\overset{p}{\wedge} \mathfrak{m}^{*C}$ such that $\xi^\lambda = \zeta^\lambda (\lambda = 1, \cdots, m)$, and let $\tilde{\theta}(x) = \sum_{\lambda=1}^{N} \theta_\lambda(x) \xi^\lambda$. We have $\tilde{\eta}(x) = f(x) = \sum_{\lambda=1}^{m} f_\lambda(x) \xi^\lambda$, and

$$(\Delta \theta, \eta) = M \sum_{\lambda=1}^{m} (C \theta_\lambda, f_\lambda).$$

Since $f_\lambda \in H_a$, we get $(\Delta \theta, \eta) = 0$ by (4.3). Thus η is orthogonal to the p-forms $\Delta \theta$ and, as is well known, it follows from this that η is of class C^∞ and harmonic. Therefore the functions f_λ are of class C^∞ and satisfy the equation $C f_\lambda = 0$. It follows then that the function f belongs to $A_{p,i}$. Thus we have shown that to each $H_{a,b}$ with $U_a \in D_0, 1 \leq b \leq M((U_a)_k; \tau_i^p)$, and to each basis $\{f_\lambda\}_{\lambda=1,\cdots,m}$ of $H_{a,b}$ satisfying (4.2) there corresponds a function $f_{a,b} \in A_{p,i}$. Moreover, $f_{a,b}$ is independent of the choice of such a basis $\{f_\lambda\}$ up to a scalar multiple, and these functions $f_{a,b}$ are linearly independent. Therefore we get

$$\dim A_{p,i} \geq \sum_{T \in D_0} N(T) M(T_K; \tau_i^p).$$

Let conversely $f \in A_{p,i}$. We show that f is a linear combination of the functions $f_{a,b}$. Put

$$f(x) = \sum_\lambda f_\lambda(x) \zeta^\lambda.$$

We have then

$$U(k) f_\lambda = \sum_\mu a_\lambda^\mu(k^{-1}) f_\mu, \qquad C f_\lambda = 0. \tag{4.4}$$

Let P_a be the projection operator of $L^2(\Gamma \backslash G)$ such that $P_a \varphi = \varphi$ for $\varphi \in H_a$, and $P_a \varphi = 0$ for $\varphi \in H_b$, $b \neq a$. Then $f_\lambda = \sum_a P_a f_\lambda$. Let W (resp. W_a) be the space of regular vectors of $L^2(\Gamma \backslash G)$ (resp. H_a). Since f_λ is of class C^∞, f_λ belongs to W, and moreover $P_a f_\lambda \in W_a$ for all a. We have $P_a C_U \varphi = C_{U_a} P_a \varphi$ for $\varphi \in W$, and hence we get $C_{U_a} P_a f_\lambda = 0$, because $C_U f_\lambda = -C f_\lambda = 0$. It follows that $P_a f_\lambda = 0$ for the index a such that $U_a \notin D_0$. Now suppose that $U_a \in D_0$ and $P_a f_\lambda \neq 0$ for an index λ. We see from (4.4) that

$$U_a(k) P_a f_\lambda = \sum_\mu a_\lambda^\mu(k^{-1}) P_a f_\mu \qquad (k \in K).$$

Let F be the linear subspace of H_a spanned by the elements $P_a f_\lambda (\lambda = 1, \cdots, m)$. Then F is a K-invariant subspace of H_a, and there exists a K-module homomorphism of F_i onto F which maps Z_λ onto $P_a f_\lambda$. Since $F \neq (0)$ and F_i is an irreducible K-module, this homomorphism is an isomorphism. It follows then that $P_a f_\lambda$ are linearly independent, and F is contained in the direct sum $\sum_{b=1}^{b_i} H_{a,b} (b_i = M((U_a)_K; \tau_i^p))$. Let $\{f_{a,b;\lambda}\}_{\lambda=1,\cdots,m}$ be a basis of $H_{a,b}$ satisfying (4.2), and put

$$P_a f_\lambda = \sum_b \sum_\mu \alpha_{b,\lambda}^\mu f_{a,b;\mu}.$$

We see easily that the matrix $(\alpha_{b,\lambda}^\mu)_{\lambda,\mu=1,\cdots,m}$ commutes with the matrix $(a_\lambda^\mu(k))_{\lambda,\mu=1,\cdots,m}$ for all $k \in K$, and hence $(\alpha_{b,\lambda}^\mu)$ is a scalar matrix. Therefore $P_a f_\lambda = \sum_b \alpha_b \cdot f_{a,b;\lambda}$ with $\alpha_b \in C$, and hence $f = \sum_\alpha \sum_\lambda P_a f_\lambda \zeta^\lambda = \sum_{a,b} \alpha_b f_{a,b}$. Thus f is a linear combination of the functions $f_{a,b}$. We have thus completed the proof of (4.1) and the theorem is proved.

5. We consider now the special cases where G is the complex unimodular group $SL(2, C)$ or the proper Lorentz group.

Let $G = SL(2, C)$. A maximal compact subgroup is the special unitary group $SU(2)$, and put $K = SU(2)$. Then G/K is the 3-dimensional hyperbolic space.

The irreducible unitary representations of the compact group K are given as follows:

There is a 1-1 correspondence between the set of equivalence classes of irreducible unitary representations of K and the set of non-negative integers and non-negative half-integers. The irreducible representation ρ_k corresponding to $\frac{k}{2}$ (k: non-negative integer) is realized in the vector space of covariant symmetric tensors of order k constructed over the 2-dimensional complex vector space on which K operators (see [6]).

Now let m be the vector space of 2×2 hermitian matrices of trace 0. We then have $\mathfrak{g} = \mathfrak{m} + \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] = \mathfrak{m}$, $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{k}$, and the representation $ad_{\mathfrak{m}}$ of $\mathfrak{k}$ in $\mathfrak{m}$ is absolutely irreducible and equivalent to the representation ρ_2.

The irreducible unitary representation of $SL(2, C)$ are the following [6]:

1. Principal series $U_{m,\rho}$. These representations depend on two parameters m and ρ with $m \in \boldsymbol{Z}$ and $\rho \in \boldsymbol{R}$. $U_{m,\rho}$ is the representation in the Hilbert space $H = L^2(\boldsymbol{C})$, and the unitary operator $U_{m,\rho}(g)$ is defined by

$$(U_{m,\rho}(g)f)(z) = (bz+d)^m \, |bz+d|^{-m+i\rho-2} f\left(\frac{az+c}{bz+d}\right),$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, C).$$

The representations $U_{m,\rho}$ and $U_{n,\sigma}$ are equivalent if and only if $n = -m$ and $\sigma = -\rho$.

The Casimir operator $C_{m,\rho}$ of $U_{m,\rho}$ is:

$$C_{m,\rho} = \frac{1}{16}\left\{\left(\frac{m}{2}\right)^2 - \left(\frac{\rho}{2}\right)^2 - 1\right\} \cdot 1.$$

The irreducible representation ρ_k is contained in $U_{m,\rho}|K$ at most once, and ρ_k is actually contained in $U_{m,\rho}|K$ if and only if $\frac{m}{2}$ equals one of the numbers $\frac{k}{2}, \frac{k}{2} - 1, \frac{k}{2} - 2, \cdots$.

2. Supplementary series $U_\sigma (0 < \sigma < 2)$. The representation U_σ is realized in the Hilbert space H of complex-valued function on $\boldsymbol{C}$, the inner product (f_1, f_2) in H and the unitary operator $U_\sigma(g)$ are defined as follows:

$$(f_1, f_2) = \int_C \int_C |z_1 - z_2|^{-2+\sigma} f_1(z_1)\overline{f_2(z_2)} dz_1 dz_2,$$

$$(U_\sigma(g)f)(z) = |bz + d|^{-2-\sigma} f\left(\frac{az + c}{bz + d}\right),$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{C}).$$

The Casimir operator C_σ of U_σ is :

$$C_\sigma = \frac{1}{16}\left\{\left(\frac{\sigma}{2}\right)^2 - 1\right\} \cdot 1 \qquad (0 < \sigma < 2).$$

The representation $U_\sigma | K$ decomposes as follows:

$$U_\sigma | K = \sum_{k=0}^{\infty} \rho_{2k}.$$

Now the Casimir Operator C_σ does not vanish, and the Casimir Operator $C_{m,\rho} (m \geq 0)$ vanishes if and only if $\rho = \pm\sqrt{m^2 - 4}$. As ρ is real, we have $m \geq 2$. On the other hand, $U_{m,\rho} | K (m \geq 0)$ contains ρ_2 if and only if $m = 2$. Therefore there is one and only one irreducible unitary representation $\boldsymbol{T}$ of $SL(2, \boldsymbol{C})$ with vanishing Casimir operator such that $T|K$ contains ρ_2, that is, $T = U_{2,0}$. Moreover, the multiplicity of ρ_2 in $U_{2,0}|K$ is 1.

Let now $\boldsymbol{G}$ be the proper Lorentz group. Then $G \cong SL(2, \boldsymbol{C})/\{\pm 1\}$ and $K \cong SU(2)/\{\pm 1\}$. The irreducible unitary representations of K are $\rho_{2k} (k = 0, 1, 2 \cdots)$, and the irreducible unitary representations $\boldsymbol{T}$ of $\boldsymbol{G}$ are those of $SL(2, \boldsymbol{C})$ satisfying the condition $T(-1) = 1$, and therefore these representations are $U_{m,\rho}$ with even m and U_σ. Just as in the of $SL(2, \boldsymbol{C})$, the only irreducible unitary representation $\boldsymbol{T}$ of $\boldsymbol{G}$ with vanishing Casimir operator such that $T|K$ contains ρ_2 is the representation $U_{2,0}$. The multiplicity of ρ_2 in $U_{2,0}|K$ is 1.

From our theorem we then have the following result:

Let $\boldsymbol{G}$ be the complex unimodular group $SL(2, \boldsymbol{C})$ or the proper Lorentz group. Let Γ be a discrete subgroup of $\boldsymbol{G}$ such that $\Gamma \backslash G$ is compact. Assume that Γ acts freely on the 3-dimensional hyperbolic space G/K. Then the multiplicity of the irreducible unitary representation $U_{2,0}$ of $\boldsymbol{G}$ in the unitary representation $\boldsymbol{T}$ of $\boldsymbol{G}$ in $L^2(\Gamma \backslash G)$ equals the rank of the finitely generated abelian group Γ/Γ', Γ' being the commutator subgroup of Γ.

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