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EINSTEIN HYPERSURFACES IN A KÄHLERIAN MANIFOLD OF CONSTANT HOLOMORPHIC CURVATURE

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Introduction

In his dissertation Brian Smyth studied the complete hypersurfaces in a complex space-form whose induced metric is einsteinian and proved that these are either totally geodesic or certain hyperquadrics of the complex projective space. We wish to show in this note that the corresponding *local* theorem is true:

Theorem. *Let* V *be a kählerian manifold of dimension* ≥ 3 *with constant holomorphic sectional curvature* K. *Let* f: M $\to$ V *be a holomorphically immersed hypersurface such that the induced metric is einsteinian. Then, if* K ≤ 0, M *is totally geodesic. If* K > 0 *and* V *is identified with the complex projective space,* M *is either totally geodesic or a hypersphere* (*cf.* §3 *for definition*).

1. Preliminaries on kählerian geometry

We will summarize the basic formulas of kählerian geometry. For details cf. [1].

In order to avoid repetitions it will be agreed that our indices have the following ranges throughout this paper:

$$\begin{aligned} &1 \leq i, j, k, l \leq n, \\ &1 \leq \alpha, \beta, \gamma, \delta \leq n+1, \\ &0 \leq A, B, C, D \leq n+1. \end{aligned} \tag{1}$$

Let V be a kählerian manifold of complex dimension $n+1$. The metric defines an hermitian scalar product in the tangent spaces of V and a connection of type (1, 0) under whose parallelism the scalar product is preserved. More precisely, let $e_\alpha(x)$ be a field of unitary frames, defined for x in a neighborhood of V. Its dual coframe field consists of $n+1$ complex-valued linear differential forms θ_α of type (1, 0) such that the hermitian metric can be written

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$$ds^2 = \sum_\alpha \theta_\alpha \bar{\theta}_\alpha. \tag{2}$$

The connection forms $\theta_{\alpha\beta}$ are characterized by the conditions

$$\theta_{\alpha\beta} + \bar{\theta}_{\beta\alpha} = 0, \tag{3}$$

$$d\theta_\alpha = \sum_\beta \theta_\beta \wedge \theta_{\beta\alpha}, \tag{4}$$

and they can be interpreted geometrically as defining the covariant differential

$$De_\alpha = \sum_\beta \theta_{\alpha\beta} \otimes e_\beta. \tag{5}$$

The curvature forms $\Theta_{\alpha\beta}$ are then defined by

$$d\theta_{\alpha\beta} = \sum_\gamma \theta_{\alpha\gamma} \wedge \theta_{\gamma\beta} + \Theta_{\alpha\beta}, \tag{6}$$

and we have

$$\Theta_{\alpha\beta} = -\bar{\Theta}_{\beta\alpha} = \sum_{\gamma,\delta} R_{\alpha\beta\gamma\delta}\theta_\gamma \wedge \bar{\theta}_\delta. \tag{7}$$

The skew-hermitian symmetry of $\Theta_{\alpha\beta}$ expressed by the first equation of (7) is equivalent to the symmetry conditions

$$R_{\alpha\beta\gamma\delta} = \bar{R}_{\beta\alpha\delta\gamma}. \tag{8}$$

The Bianchi identities, which are relations obtained by exterior differentiation of (4) and (6), give the further symmetry relations

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\beta\alpha\delta} = R_{\alpha\delta\gamma\beta}, \tag{9}$$

and the equation

$$d\Theta_{\alpha\beta} + \sum_\gamma \Theta_{\alpha\gamma} \wedge \theta_{\gamma\beta} - \sum_\gamma \theta_{\alpha\gamma} \wedge \Theta_{\gamma\beta} = 0. \tag{10}$$

The metric on V is called *einsteinian*, if

$$d\left(\sum_\alpha \theta_{\alpha\alpha}\right) = \sum_\alpha \Theta_{\alpha\alpha} = \frac{R}{n+1} \sum_\alpha \theta_\alpha \wedge \bar{\theta}_\alpha, \tag{11}$$

where

$$R = \sum_{\alpha,\beta} R_{\alpha\alpha\beta\beta} = \bar{R} \tag{12}$$

is the scalar curvature.

The quantities $R_{\alpha\beta\gamma\delta}$ define the *holomorphic sectional curvature* to every tangent vector of V. In fact, let

$$\xi = \sum_\alpha \xi_\alpha e_\alpha \neq 0 \tag{13}$$

be a tangent vector at x. Then the holomorphic sectional curvature is defined to be

$$R(x,\xi) = 2 \sum_{\alpha,\cdots,\delta} R_{\alpha\beta\gamma\delta}\xi_\alpha\xi_\gamma\bar{\xi}_\beta\bar{\xi}_\delta / \left(\sum_\alpha \xi_\alpha\bar{\xi}_\alpha\right)^2. \tag{14}$$

Because of the symmetry relation (8), $R(x,\xi)$ is real.

V is said to be of *constant holomorphic sectional curvature* K if $R(x,\xi) = K$ for all (x,ξ). This is expressed by the condition

$$R_{\alpha\beta\gamma\delta} = \frac{1}{4}(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})K \tag{15}$$

or

$$\Theta_{\alpha\beta} = \frac{1}{4}K(\theta_\beta \wedge \bar{\theta}_\alpha + \delta_{\alpha\beta}\sum_\gamma \theta_\gamma \wedge \bar{\theta}_\gamma). \tag{16}$$

The above treatment depends on the choice of a frame field. As is well-known, the geometrical results which follow are independent of this choice. However, it is useful to know explicitly the effect of a change of the frame field on the various quantities. Let

$$e^*_\alpha = \sum_\beta u_{\alpha\beta}e_\beta \tag{17}$$

be a new frame field defined in the neighborhood in question, where $u_{\alpha\beta}$ are complex-valued C^∞-functions such that $(u_{\alpha\beta})$ is a unitary matrix. Let $\theta^*_\alpha, \theta^*_{\alpha\beta}$ be the forms relative to the frame field e^*_α. Then, by definition and by (5), we have

$$\theta^*_\alpha = \sum_\beta \bar{u}_{\alpha\beta}\theta_\beta \tag{18}$$

and

$$\theta^*_{\alpha\beta} = \sum_\gamma du_{\alpha\gamma}\bar{u}_{\beta\gamma} + \sum_{\gamma,\delta} u_{\alpha\gamma}\theta_{\gamma\delta}\bar{u}_{\beta\delta}. \tag{19}$$

2. Hypersurfaces in a kählerian manifold

Let $f : M \to V$ be a holomorphic immersion, with dim $M = n$, dim $V = n+1$. In a neighborhood of M we can choose a frame field in V such that $e_{n+1}(x), x \in M$, is orthogonal to the tangent hyperplane to M at x. This is expressed analytically by the condition

$$\theta_{n+1} = 0. \tag{20}$$

Since M is an immersed hypersurface, the θ_i are linearly independent. Using (4), we get

$$0 = d\theta_{n+1} = \sum_i \theta_i \wedge \theta_{i,n+1}.$$

It follows by Cartan's lemma that

$$\theta_{i,n+1} = \sum_k a_{ik}\theta_k, \tag{21}$$

where

$$a_{ik} = a_{ki}. \tag{22}$$

M is *totally geodesic* if $a_{ik} = 0$.

The e_i define a unitary frame field in the tangent bundle of M, with θ_{ij} as the connection forms. Equation (6) gives

$$d\theta_{ij} = \sum_k \theta_{ik} \wedge \theta_{kj} - \theta_{i,n+1}\bar{\theta}_{j,n+1} + \Theta_{ij},$$

so that

$$\tilde{\Theta}_{ij} = \Theta_{ij} - \theta_{i,n+1} \wedge \bar{\theta}_{j,n+1} \tag{23}$$

are the curvature forms of the induced metric on M.

Suppose now that V is of constant holomorphic sectional curvature, K, so that the equation (16) holds. Then

$$\sum_i \Theta_{ii} = \frac{1}{4}(n+1)K \sum_i \theta_i \wedge \bar{\theta}_i.$$

The condition that the induced metric on M is einsteinian can be expressed as

$$\sum_i \theta_{i,n+1} \wedge \bar{\theta}_{i,n+1} = \rho \sum_i \theta_i \wedge \bar{\theta}_i. \tag{24}$$

Using (21) this condition is equivalent to

$$\sum_i a_{ik}\bar{a}_{il} = \rho\delta_{kl}, \tag{25}$$

which gives

$$n\rho = \sum_{i,k} |a_{ik}|^2 \geq 0. \tag{26}$$

From now on suppose $n \geq 2$. We wish to show that ρ is constant. In fact, we have by (6),

$$d\left(\sum_i \theta_{i,n+1} \wedge \bar{\theta}_{i,n+1}\right) = 0,$$

so that it follows by exterior differentiation of (24) that

$$d\rho \wedge \left(\sum_i \theta_i \wedge \bar{\theta}_i\right) = 0.$$

Put

$$d\rho = \sum_k (\rho_k \theta_k + \bar{\rho}_k \bar{\theta}_k)$$

and substitute into the above; we get immediately

$$d\rho = 0.$$

If $\rho = 0$, we have by (26), $a_{ik} = 0$ and M is totally geodesic. From now on suppose that ρ *is a positive constant.*

We take the exterior derivative of the equation (21) and make use of (4) and (6). This gives

$$(27) \quad \sum_k \left(da_{ik} - \sum_j a_{ij}\theta_{kj} - \sum_j a_{jk}\theta_{ij} + a_{ik}\theta_{n+1,\, n+1} \right) \wedge \theta_k = 0.$$

It follows that we can put

$$da_{ik} - \sum_j a_{ij}\theta_{kj} - \sum_j a_{jk}\theta_{ij} + a_{ik}\theta_{n+1,\, n+1} = \sum_j a_{ikj}\theta_j,$$

where a_{ikj} are symmetric in all its indices. The complex conjugate of this equation will give a formula for $d\bar{a}_{ik}$. Differentiating (25) and substituting these expressions for da_{ik}, $d\bar{a}_{ik}$, we get

$$\sum_{i,j} (\bar{a}_{il} a_{ikj} \theta_j + a_{ik} \bar{a}_{ilj} \bar{\theta}_j) = 0,$$

from which it follows that

$$\sum_i \bar{a}_{il} a_{ikj} = 0.$$

Since $\rho > 0$, we get from the last equation

$$a_{ikj} = 0.$$

We have therefore the equation

$$(28) \qquad da_{ik} - \sum_j a_{ij}\theta_{kj} - \sum_j a_{jk}\theta_{ij} + a_{ik}\theta_{n+1,\, n+1} = 0.$$

Equation (28) is valid for a holomorphically immersed hypersurface of dimension $\geqq 2$ in a kählerian manifold of constant holomorphic sectional curvature such that its induced metric is einsteinian. Notice that (28) is still valid if M is totally geodesic, for then $a_{ik} = 0$.

We now take the exterior derivative of (28). This gives, after simplification,

$$(29) \qquad (\rho - \frac{1}{4}K)(\delta_{ij} a_{kl} + \delta_{kj} a_{il} + \delta_{jl} a_{ik}) = 0.$$

If $\rho - \dfrac{1}{4}K \neq 0$, we have

$$\delta_{ij}a_{kl} + \delta_{kj}a_{il} + \delta_{jl}a_{ik} = 0.$$

Putting $j = l$ and summing, we get $(n+2)a_{ik} = 0$, so that $a_{ik} = 0$ and M is totally geodesic. Hence M is not totally geodesic only if $\rho = K/4$, which implies $K > 0$ since $\rho > 0$. We have thus proved the first half of the theorem stated in the Introduction. Our next problem is to study the hypersurfaces satisfying the condition $\rho = \frac{K}{4} > 0$.

3. The complex projective space

For $K > 0, V$ can be realized locally as the complex projective space P_{n+1} of dimension $n+1$ with the Study-Fubini metric. We proceed to give a description of this metric.

Let V_{n+2} be the complex vector space of dimension $n+2$, whose points are the ordered ennuples of complex numbers: $Z = (z_0, \cdots, z_{n+1})$. In V_{n+2} we introduce the hermitian scalar product

$$(W, Z) = (W, Z) = \sum_A Z_A \bar{w}_A, \quad W = (w_0, \cdots, w_{n+1}). \tag{30}$$

The unitary group $U(n+2)$ in $n+2$ variables is the group of all linear homogeneous transformations on z_A leaving the scalar product (30) invariant. Let V^*_{n+2} be the subset of V_{n+2} obtained by the deletion of the zero vector. Then P_{n+1} is the orbit space of V^*_{n+2} under the action of the group $Z \to \lambda Z, \lambda$ being a complex number $\neq 0$. We have thus the projection $\pi : V^*_{n+2} \to P_{n+1}$. To a point $p \in P_{n+1}$ a vector $Z \in \pi^{-1}(p)$ is called a homogeneous coordinate vector of p, and we will frequently identify p with Z. We put

$$Z_0 = Z/(Z, Z)^{\frac{1}{2}}. \tag{31}$$

so that $(Z_0, Z_0) = 1$. Then the Study-Fubini metric is given by

$$ds^2 = (dZ_0, dZ_0) - (dZ_0, Z_0)(Z_0, dZ_0). \tag{32}$$

To study this metric let Z_A be a unitary frame in V_{n+2}, so that

$$(Z_A, Z_B) = \delta_{AB}. \tag{33}$$

In the space of all unitary frames in V_{n+2} let ω_{AB} be defined by

$$dZ_A = \sum_B \omega_{AB} Z_B, \tag{34}$$

so that we have

$$\omega_{AB} = -\bar{\omega}_{BA} = (dZ_A, Z_B). \tag{35}$$

Then ω_{AB} are the Maurer-Cartan forms of $U(n+2)$ and satisfy the structure equations

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB}. \tag{36}$$

The same equations remain valid if we restrict ourselves to a frame field defined over a submanifold of V_{n+2}. The metric (32) can then be written

$$ds^2 = \sum_\alpha \omega_{0\alpha}\bar{\omega}_{0\alpha}. \tag{37}$$

It is of the form (2) if we set

$$\theta_\alpha = \omega_{0\alpha}. \tag{38}$$

Equations (3) and (4) will be satisfied, provided that we choose

$$\theta_{\beta\alpha} = \omega_{\beta\alpha} - \delta_{\beta\alpha}\omega_{00}. \tag{39}$$

These are therefore the connection forms of the metric (32). By (36) we find the curvature forms of this metric to be

$$\Theta_{\alpha\beta} = \theta_\beta \wedge \bar{\theta}_\alpha + \delta_{\alpha\beta}\sum_\gamma \theta_\gamma \wedge \bar{\theta}_\gamma. \tag{40}$$

Comparing with (16), we see that the metric (32) has constant holomorphic sectional curvature equal to 4. From the definition of the metric it is clear that $U(n+2)$ acts on P_{n+1} as a group of isometries.

Consider in P_{n+1} a hyperquadric defined by the equation

$$\sum_{A,B} b_{AB}Z_AZ_B = 0, \quad b_{AB} = b_{BA}. \tag{41}$$

Under a unitary transformation

$$Z_A = \sum_B u_{AB}Z_B^* \tag{42}$$

this goes into the hyperquadric

$$\sum_{A,B} b_{AB}^* Z_A^* Z_B^* = 0.$$

By introducing the matrices

$$B = {}^tB = (b_{AB}), \quad B^* = {}^tB^* = (b_{AB}^*), \quad U = {}^t\overline{U}^{-1} = (u_{AB}), \tag{43}$$

we can express the relation between the coefficients b_{AB} and b_{AB}^* by the matrix equation

$$B^* = {}^tUBU. \tag{44}$$

It follows that

$$B^*\bar{B}^* = {}^tUB\bar{B}\overline{U}.$$

Thus the eigenvalues of $B\bar{B}$ are invariant under the unitary transformation. In particular, the invariance of the trace of $B\bar{B}$ gives

$$\sum_{A,B} |b_{AB}|^2 = \sum_{A,B} |b^*_{AB}|^2. \tag{45}$$

We will establish the following lemma:

Given a symmetric matrix B with complex elements, there exists a unitary matrix U such that ${}^tUBU$ is diagonal.

For $n = 0$, i.e., for a (2×2)-matrix B this can be verified by an elementary calculation, and we suppose the lemma true in this case. Let

$$\varphi(B) = \sum_{A \neq B} |b_{AB}|^2, \tag{46}$$

i.e., $\varphi(B)$ is the sum of the squares of the absolute values of the non-diagonal elements of B. Suppose $|b_{01}| \geq |b_{AB}|, A \neq B$; this can always be achieved by interchanging the rows and columns when necessary. Let U_0 be a (2×2)-unitary matrix such that ${}^tU_0 B^1 U_0$ is diagonal, where $B^1 = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$ and let

$$U = \begin{pmatrix} U_0 & 0 \\ 0 & I \end{pmatrix},$$

where I is the $(n \times n)$-unit matrix. Then we have

$$\sum_A |b^*_{AA}|^2 = \sum_A |b_{AA}|^2 + 2|b_{01}|^2$$

and

$$\varphi(B^*) = \varphi(B) - 2|b_{01}|^2.$$

Under the assumption that $|b_{01}| \geq |b_{AB}|, A \neq B$, we have

$$\varphi(B) \leq (n+1)(n+2)|b_{01}|^2,$$

from which it follows that

$$\varphi(B^*) \leq \frac{n(n+3)}{(n+1)(n+2)}\varphi(B). \tag{47}$$

Notice that the factor at the right-hand side before $\varphi(B)$ is < 1. We can therefore find a sequence of unitary matrices $U_1, \cdots, U_\nu, \cdots$, such that $\varphi({}^tU_\nu B U_\nu)$ is strictly monotone decreasing and tends to zero. Since the unitary group is compact, there exists a unitary matrix U_∞ such that $\varphi({}^tU_\infty B U_\infty) = 0$ and ${}^tU_\infty B U_\infty$ is diagonal. This proves the lemma.

It follows that the equation of the hyperquadric can be by unitary transformations brought to the normal form

$$b_0 Z_0^2 + \cdots + b_{n+1} Z_{n+1}^2 = 0. \tag{48}$$

We can further suppose that b_A are real and ≥ 0. The ratios of b_A are invariants of the hyperquadric under $U(n+2)$. In particular, the hyperquadric

$$Z_0^2 + \cdots + Z_{n+1}^2 = 0 \tag{49}$$

will be called a *hypersphere.*

A one-one mapping $T : V_{n+2} \to V_{n+2}$ is called *antilinear*, if

$$\begin{aligned} T(Z_1 + Z_2) &= T(Z_1) + T(Z_2), \\ T(\lambda Z) &= \bar{\lambda} T(Z), \quad Z,\ Z_1, Z_2 \in V_{n+2}, \end{aligned} \tag{50}$$

λ being a complex number. It induces a one-one mapping in P_{n+1}. By the properties (50) an anti-linear mapping is completely determined by its effect on a frame.

4. Completion of the proof of the theorem

We wish to prove the second part of the theorem stated in the Introduction by showing that a hypersurface in P_{n+1} (with the Study-Fubini metric) whose induced metric is einsteinian and which is not totally geodesic is necessarily a hypersphere.

Continuing the proof of §2, we have $K = 4$ and $\rho = 1$. We apply a change of the frame field as defined by (17) with

$$u_{i,\,n+1} = 0, \quad u_{n+1,\,n+1} = 1, \tag{51}$$

so that the normal vector e_{n+1} to M remains unchanged. By (18) and (19) we have respectively

$$\begin{aligned} \theta_i^* &= \sum_j \bar{u}_{ij} \theta_j, \\ \theta_{i,\,n+1}^* &= \sum_j u_{ij} \theta_{j,\,n+1}. \end{aligned}$$

If we set

$$\theta_{i,\,n+1}^* = \sum_k a_{ik}^* \theta_k^*, \tag{52}$$

we get

$$a_{ik}^* = \sum_{l,j} u_{il} a_{lj} u_{kj}. \tag{53}$$

Our lemma in §3 implies that unitary matrices (u_{ik}) can be so chosen that the matrix (a_{ik}^*) is diagonal. Moreover, since $\rho = 1$, we can even make it the unit matrix.

Suppose such a change of the frame field be already carried out. By dropping the asterisks, we have $a_{ik} = \delta_{ik}$ and

$$\theta_{i,\, n+1} = \theta_i. \tag{54}$$

Equation (28) becomes

$$\theta_{ik} + \theta_{ki} - \delta_{ik}\theta_{n+1,\, n+1} = 0. \tag{55}$$

By (39) this gives

$$\omega_{ik} + \omega_{ki} - \delta_{ik}(\omega_{00} + \omega_{n+1,\, n+1}) = 0. \tag{56}$$

We now modify Z_{n+1} by setting

$$Z^*_{n+1} = e^{i\phi} Z_{n+1}, \quad \phi \text{ real.}$$

Then

$$\omega^*_{n+1,n+1} = (dZ^*_{n+1}, Z^*_{n+1}) = id\phi + \omega_{n+1,\, n+1}.$$

and we have

$$\omega_{00} + \omega^*_{n+1,\, n+1} = id\phi + \omega_{00} + \omega_{n+1,n+1}.$$

Since $i(\omega_{00} + \omega_{n+1,n+1})$ is real-valued and

$$d(i\omega_{00} + i\omega_{n+1,n+1}) = 0,$$

we can determine ϕ so that

$$\omega_{00} + \omega^*_{n+1,n+1} = 0.$$

Dropping the asterisks again, we have

$$\omega_{00} + \omega_{n+1,n+1} = 0 \tag{57}$$

and (56) gives

$$\omega_{ik} + \omega_{ki} = 0. \tag{58}$$

Let T be an anti-linear transformation in P_{n+1}, so that

$$d(T(Z_A)) = \sum_B \bar{\omega}_{AB} T(Z_B). \tag{59}$$

By using (57), (58) and (59), we find

(60)

$$\begin{aligned}
&d(Z_0 + T(Z_{n+1})) = w_{00}(Z_0 + T(Z_{n+1})) + \sum_i \omega_{0i}(Z_i - T(Z_i)),\\
&d(T(Z_0) + Z_{n+1}) = -\ \omega_{00}(T(Z_0) + Z_{n+1}) - \sum_i \bar{\omega}_{0i}(Zi - T(Z_i)),\\
&d(Z_i - T(Z_i)) = \omega_{0i}(T(Z_0) + Z_{n+1}) - \bar{\omega}_{0i}(Z_0 + T(Z_{n+1}))\\
&\qquad\qquad + \sum_k \omega_{ik}(Z_k - T(Z_k)).
\end{aligned}$$

This is a differential system which is linear and homogeneous in the vectors $Z_0+T(Z_{n+1})$, $T(Z_0)+Z_{n+1}, Z_i-T(Z_i)$. It follows that if these vectors are zero at a point of M, they are identically zero. We choose the anti-linear transformation T so that they are zero at $p_0 \in M$ and we have

$$(61) \qquad T(Z_0) = -Z_{n+1}, \quad T(Z_{n+1}) = -Z_0, \quad T(Z_i) = Z_i$$

everywhere on M. As a consequence we get

$$(62) \qquad (Z_0,\ T(Z_0)) = 0.$$

We put

(63)

$$a_0 = (Z_0(p_0) - Z_{n+1}(p_0))/\sqrt{2},\ a_{n+1} = i(Z_0(p_0) + Z_{n+1}(p_0))/\sqrt{2},$$

$$a_j = Z_j(p_0).$$

Then a_A is a unitary frame having the property

$$(64) \qquad Ta_A = a_A.$$

Let

$$Z_0(p) = \sum_A Z_A a_A, \qquad p \in M.$$

Then

$$T(Z_0(p)) = \sum_A \bar{Z}_A a_A,$$

and equation (62) can be written

$$\sum_A Z_A^2 = 0.$$

This proves that M is a hypersphere.

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