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CHARACTERS OF SL(2) REPRESENTATIONS OF GROUPS

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Abstract

Given a compact orientable surface Σ, let $S(\Sigma)$ be the set of isotopy classes of essential unoriented simple loops in the surface. We determine a complete set of relations for a function defined on $S(\Sigma)$ to a field K to be the character of an $SL(2,K)$ representations. Furthermore, the relations are supported in the 1-holed torus and the 4-holed sphere subsurfaces. This establishes that Grothendieck's reconstruction principle is valid for $SL(2,K)$-character varieties of surface groups. As a consequence, we obtain an explicit description of the set of all characters of $SL(2,K)$ representations of a group.

1. Introduction

1.1. Given a field K and a representation ρ of a group to $SL(2,K)$, the *character* of the representation sends a group element g to the trace of the matrix $\rho(g)$. One of the result of the paper is the following,

Theorem. *Suppose K is a field so that each quadratic equation with coefficients in K has a root in K. Then a K-valued function defined on a group is the character of a $SL(2,K)$ representation of the group if and only if its restriction to each 2-generator subgroup is a $SL(2,K)$ character.*

The $SL(2,K)$ characters on 2-generator groups are well understood since the work of Fricke-Klein [7] and Vogt [33]. They are governed by the trace identity: $tr(AB) + tr(A^{-1}B) = tr(A)tr(B)$ for $SL(2,K)$ matrices A, B. In [14], Helling gave an elegant axiomatic approach to characters based on the above trace identity. Following Helling, a K-valued function f defined on a group G is called a *K-trace function* if (1) $f(xy)+f(x^{-1}y) = f(x)f(y)$ for all x, y in G and (2) $f(id) = 2$ where

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id is the identity element. Evidently an $SL(2, K)$ character is a K-trace function. Using the work of [7], [14] and [33] that K-trace functions are characters on 2-generator groups, one deduces the following result equivalent to the above theorem.

Corollary. *Suppose K is a field so that each quadratic equation with coefficients in K has a root in K. Then a K-valued function f defined on a group is the character of an $SL(2, K)$ representation of the group if and only if $f(xy) + f(x^{-1}y) = f(x)f(y)$ for all x, y in the group and $f(id) = 2$.*

This generalizes a result of Helling [14] who proved that $\mathbf{R}$-trace functions are $SL(2, \mathbf{R})$ characters under some additional assumptions.

1.2. The main result of the paper which implies Theorem 1.1 gives a characterization of $SL(2, K)$ characters defined on the fundamental groups of surfaces using subsurface groups. In contrast to Theorem 1.1 which uses the hierarchy of subgroups indexed by the number of generators to describe the characters, there exists a natural hierarchy of surfaces under inclusion indexed by the level. Recall that the *level* of a compact surface of negative Euler number is the minimal number of disjoint simple loops decomposing the surface into 3-holed spheres. It is also the complex dimension of the Teichmüller space of complex structures on the interior of the surface with punctured ends. For instance, the 3-holed sphere has level-0 and the 4-holed sphere and the 1-holed torus have level-1. This hierarchy of surfaces is prominent in Grothendieck's manuscript [10] and conformal field theory [27]. In particular, Grothendieck conjectured that the "tower of Teichmüller spaces" can be reconstructed from the Teichmüller spaces of level-1 surfaces subject to the relations supported in level-2 surfaces. Motivated by this Grothendieck's reconstruction principle, one asks if a character can be reconstructed from its restriction to the fundamental group of each level-1 subsurface. The main result of the paper gives a complete answer to this question.

To give a precise solution to the above question, we shall first note that the fundamental group is not vital with respect to the hierarchy. Indeed, if a given element in the fundamental group is "complicated" in the sense that it has no representative in any level-1 subsurface, then the condition becomes null about the class. Thus, we focus our attention to those "simple" elements in the fundamental group. This motivates the introduction of the set $S(\Sigma)$ of free homotopy classes of unoriented homotopically non-trivial simple loops on a surface Σ. The space $S(\Sigma)$ was introduced by Max Dehn [6] in his study of the mapping class groups and was independently introduced by Thurston [32] in his work on surface theory. If f is an $SL(2,K)$ character defined on the fundamental group of a surface Σ, then f induces a K-valued function on $S(\Sigma)$ which we still call an $SL(2,K)$ character. A natural property of a character f on $S(\Sigma)$ is that its restriction to each subsurface is again a character. To be more precisely, if Σ' is an essential subsurface (i.e., the inclusion map induces a monomorphism between fundamental groups), then the *restriction map* $f \circ i_*$ is again a character on $S(\Sigma')$ where $i_* : S(\Sigma') \to S(\Sigma)$ is induced by the inclusion.

We call a K-valued function f defined on the set $S(\Sigma)$ of homotopy classes of simple loops a *trace function* if the restriction of the function to each $S(\Sigma')$ is a character for each level-1 essential subsurfaces Σ'. Grothendieck's principle predicts that a trace function is a character. The main result of the paper shows that this holds except for finitely many exceptional trace functions defined on genus zero surfaces when the characteristic of the field K is not 2. All exceptional trace functions are derived from a single one defined on the 5-holed sphere which we describe as follows. Let the characteristic of the field K be not 2, and $b_1, ..., b_5$ be the boundary components of the 5-holed sphere $\Sigma_{0,5}$. Define $f_0 : S(\Sigma_{0,5}) \to K$ by sending each b_i to 2 and all other elements to -2. One checks easily (see §5.4) that f_0 is a trace function which is not the character of any representations. There are sixteen exceptional trace functions f on the 5-holed sphere all derived from f_0. Namely, an *exceptional trace function* $f : S(\Sigma_{0,5}) \to K$ satisfies the following (1) $f(S(\Sigma_{0,5})) = \{2, -2\}$, (2) $\Pi_{i=1}^5 f(b_i) = 32$, and (3) if α is a non-boundary parallel class so that α, b_i, and b_j bound a 3-holed sphere, then $f(\alpha) = -\frac{1}{2} f(b_i) f(b_j)$.

The main result of the paper is the following.

Theorem. *Suppose K is a field so that each quadratic equation with coefficients in K has a root in K. Let f be a K-valued trace function defined on the set $S(\Sigma)$ of homotopy classes of essential simple loops in a compact orientable surface Σ.*

(1) If the characteristic of the field K is 2, then the trace function f is the character of an $SL(2,K)$ representation.

(2) If the characteristic of the field K is not 2, then:

(2.1) either f is the character of an $SL(2,K)$ representation, or

(2.2) the genus of the surface Σ is zero, f takes only values $\{2, -2\}$, and there is a level-2 subsurface so that the restriction of f to the sub-

surface is one of the sixteen exceptional trace functions.

(2.3) There exist exceptional trace functions on each genus zero surface of level at least 2*. The number of exceptional trace functions on a fixed surface is finite.*

Note that surfaces in the theorem are connected and could be compact or non-compact of infinite type.

Theorem 1.2 does not cover the case where the surface Σ has level at most 1. The characterization of $SL(2,K)$ characters on the set of simple loops in level-1 surfaces (Propositions 3.4 and 3.5) is well known by the work of [7], [9], [14], [16], [26], [33] and others. It is based on the following Lemma (Lemma 2.3) well known to the experts in the field. Namely, given six elements $x_1, x_2, x_3, x_{12}, x_{23}, x_{31}$ in K, there exist three $SL(2,K)$ matrices A_1, A_2, an A_3 so that $tr(A_i) = x_i$ and $tr(A_iA_j) = x_{ij}$.

1.3. Given a group G and a field K, the set of all $SL(2,K)$ characters on G is called the *character variety* of the group. Theorem 1.1 gives an explicit algebraic description of the character variety of the group for those field K satisfying the condition in the theorem. If the group G is finitely generated, then a well known result (Proposition 2.2) shows that there exists a finite subset $F \subset G$ so that each $SL(2,K)$ character on the group is algebraically determined by its restriction to the finite set F. As a consequence of these and the Hilbert basis theorem, one obtains the following corollary which slightly generalizes a result of Culler-Shalen [5] who proved it for algebraically closed field K.

Corollary(Culler-Shalen). *Suppose G is a finitely generated group and K is a field so that each quadratic equation with coefficients in K has a root in K. Then the set of all $SL(2,K)$ characters on the group forms an affine algebraic variety defined over K. Furthermore, the defining equations are integer coefficient polynomials.*

In [11], Gonzàlez-Acuña and Montesinos-Amilibia gave a constructive proof of Culler-Shalen's result. Their proof also shows the above corollary in the case where the characteristic of K is not 2.

The assumption on the quadratic closeness of the field K can be replaced by extension fields. Namely, suppose K is any field and f is a K-trace function defined on a group generated by n elements. Then there exists an extension field F of K obtained from K by at most n quadratic extensions and a representation of the group to $SL(2,F)$ whose character is the given K-trace function.

For instance, as a consequence of Gonzál-Acuña and Montesinos-Amilibia's theorem and basic results in computational algebraic geometry, given any triangulated 3-manifold, there is an algorithm to decide if the 3-manifold group has a non-trivial representation to $SL(2, \mathbf{C})$.

1.4. There exists an interesting analogy between the hierarchy of finitely generated groups indexed by the number of generators and the hierarchy of surfaces indexed by the level. It seems that the role of level-1 surfaces is similar to that of 2-generator groups. For instance, Jorgensen [17] proved that a non-elementary subgroup of $SL(2, \mathbf{C})$ is discrete if and only if each 2-generator subgroup is discrete. A consequence of [21] shows that a faithful representation of a surface group to $SL(2, \mathbf{R})$ is discrete if and only if the restriction of the representation to each level-1 subsurface group is discrete and uniformizes the subsurface. Theorems 1.1 and 1.2 provide another comparison. Here is a third pair. Recall that a subgroup in $SL(2, K)$ is *reducible* if it leaves a 1-dimensional linear subspace in K^2 invariant. It is known [5] that a subgroup in $SL(2, K)$ is reducible if and only if each 2-generator subgroup is reducible (see §2.5). The analogous result is the following.

Theorem. *An $SL(2, K)$ representation of a surface group is reducible if and only if its restriction to each level-1 subsurface group is reducible.*

In fact, in the statement of the theorem, 3-holed sphere and 1-holed torus subgroups suffice. However, there exists an irreducible representation of a surface group to $SL(2, \mathbf{K})$ so that the restriction to each level-0 subsurface group is reducible. Such irreducible representations occur rarely (only on genus 1 surfaces) and are classified in §6 and §7.

Finally, the analogous result to the well known Proposition 2.2 is the following (see §3.9) that *there exists a finite set of homotopy classes of simple loops on each compact orientable surface so that the characters of $SL(2, K)$ representations are algebraically determined by their restrictions to the finite set.*

1.5. Since each compact 3-manifold has a Heegaard splitting, a 3-manifold group is the quotient of a surface group by a subgroup of the form $N_1 N_2$ where each N_i is normally generated by disjoint simple loops. This shows that simple loops are characteristic for 3-manifold groups (among all finitely presented group). By singling out the special feature of simple loops in Theorem 1.2, it is hoped that it will have some applications to 3-manifold groups. In particular, we are motivated by the following question. Given a Haken 3-manifold M, does there exist an irreducible $SL(2, K)$ representation of the fundamental group of the 3-manifold for some finite field K? See [15] for related topics.

1.6. As mentioned before, Theorem 1.2 may be interpreted as establishing Grothendieck's reconstruction principle for $SL(2)$ character varieties. Broadly speaking, the principle says that to study the isotopy class of a structure on a surface, one should consider the restriction of the structure to the isotopy classes of all level-1 subsurfaces and reconstruct the original isotopy class of the structure from the restrictions. Furthermore, level-2 subsurfaces should serve as the "relators" in the reconstruction process (see [10] and [24]). This reconstruction principle is shown to be valid for hyperbolic metrics and measured laminations in [21] and [22]. The proof of Theorem 1.2 is similar to the proof of [21]. Namely, first we prove the result for level-1 and level-2 surfaces and then we prove the result for all surfaces using a general gluing lemma. The main difficulty in proving Theorem 1.2 is caused by the existence of irreducible representations whose restrictions to some 2-generator subgroups are reducible. Similarly, the main difficulty in establishing Theorem 1.1 is in the case of free group on 4 generators.

1.7. The study of the algebra of characters of $SL(2, K)$ representation was started by Vogt and Fricke-Klein and is developed by many authors [3], [4], [5], [14], [16], [18], [25], [26], [29], [31] and others. It seems that there is a close relation between what we did here and those algebraic approach to the ring of $SL(2)$ characters on a group.

1.8. The organization of the paper is as follows. In §2, we recall the basic facts on traces of $SL(2, K)$ matrices and group representations. In §3, we recall the basic facts on simple loops on surfaces and the modular structure. We then use the modular structure to describe $SL(2, K)$ characters on level-1 surfaces. A multiplication of the simple loops on surfaces will also be discussed. In §4 and §5, we prove the main result for the genus zero surfaces by making extensive use of the modular structure. In §6, we prove Theorem 1.2 for the 2-holed torus. Theorem 1.2 for all surfaces is proved in §7. In §8, we prove Theorem 1.1. In the final Section §9, we discuss some questions arising from the consideration of $SL(2)$ characters.

1.9. After we finish this work, we are informed by A. Sikora that Przytycki and Sikora [28] have independently proved Theorem 1.1 for field of characteristic 0. Sikora has informed us that Theorem 1.1 for the field **C** was also proved implicitly by Bullock in [1].

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2. Preliminaries on SL(2,K) matrices

In this section, we shall introduce notation and recall basic trace identities.

2.1. We shall use the following notation and terminologies. A representation ρ of a group G to $SL(2, K)$ is called *reducible* if there exists a 1-dimensional linear subspace in K^2 invariant under the linear action of G. Otherwise the representation is called *irreducible.* Two representations ρ_1 and ρ_2 of a group G to $SL(2, K)$ are *conjugate* if there exists a matrix X in $SL(2, K)$ so that for all $g \in G$, $X\rho_1(g)X^{-1} = \rho_2(g)$. Evidently, conjugate representations have the same characters. A reducible representation is called *diagonalizable* if it is $SL(2, K)$ conjugate to a representation whose image lies in the set of diagonal matrices. The character of a reducible (resp. irreducible) representation is also called reducible (resp. irreducible). A subgroup of $SL(2, K)$ is called reducible if the inclusion map is reducible.

2.2. The following trace identities will be used frequently. They are derived from the first identity (a). The earliest source of these identities seems to be [33]. For instance, the less commonly used identity (d) is on page S11 in [33]. See [7], [9], [33] and others for a proof.

Lemma. *Let R be a commutative ring with identity. Suppose A, B, A_i, B_i are $SL(2, R)$ matrices where $i = 1, 2, 3$. Then the following identities hold:*

(a) $tr(AB) + tr(A^{-1}B) = tr(A)tr(B)$.

(b) $tr^2(A) + tr^2(B) + tr^2(AB) - tr(A)tr(B)tr(AB) = tr([A, B]) + 2$.

(c) Let $A_4 = A_1$, $A_5 = A_2$,

$$P = \sum_{i=1}^{3} tr(A_i)tr(A_{i+1}A_{i+2}) - tr(A_1)tr(A_2)tr(A_3)$$

and

$$Q = \sum_{i=1}^{3}(tr^2(A_i) + tr^2(A_iA_{i+1}) - tr(A_i)tr(A_{i+1})tr(A_iA_{i+1}))$$
$$+ tr(A_1A_2)tr(A_2A_3)tr(A_3A_1) - 4.$$

Then the two roots of the quadratic equation $x^2 - Px + Q = 0$ are $tr(A_1A_2A_3)$ and $tr(A_1^{-1}A_2^{-1}A_3^{-1})$.

(d) $tr(A_1A_3)+tr(A_1A_2A_3A_2^{-1}) = -tr(A_1A_2)tr(A_2A_3)+tr(A_1)tr(A_3) + tr(A_2)tr(A_1A_2A_3)$.

In §3, these equations will be interpreted using the modular configuration $(\hat{\mathbf{Q}}, PSL(2,\mathbf{Z}))$.

As a consequence of the lemma, one has the following useful proposition. See [5], [7], [16] and [33] for a proof.

Proposition ([5], [7], [16] and [33]). *Given a commutative ring R with identity, the trace of a word $w(A_1, ..., A_n)$ in the $SL(2,R)$ matrices $A_1, ..., A_n$ is a polynomial with integer coefficients in the traces of $A_{i_1}...A_{i_k}$ where $1 \le i_1 < ... < i_k \le n$ and $k \le n$.*

Remark. If the commutative ring R is a field K of characteristic not equal to 2, then a trace identity in [33] (page S14, line 19) shows that the trace $tr(A_1A_2A_3A_4)$ can be expressed in terms of the traces of A_i, A_iA_j, and $A_iA_jA_k$, $1 \le i,j,k \le 4$. Thus in this case, one can strengthen the proposition to $tr(A_{i_1}...A_{i_k})$ where $1 \le i_1 < ... < i_k \le n$ for $k \le 3$. This triple-trace theorem has been rediscovered independently by many mathematicians. See [2], [3] and others.

In [14], Helling proved that all trace identities in Lemma 2.2 still hold for R-trace functions. Since the proof of the above proposition uses only identity (a) in Lemma 2.2 and $tr(id) = 2$, Helling proved the following corresponding result for R-trace functions.

Corollary([14]). *Suppose G is a group generated by n elements $\{x_1, ..., x_n\}$. Then for each element $w \in G$, there exists an integer coefficient polynomial P_w in variables $t_{i_1...i_k}$, where $1 \le i_1 < ... < i_k \le n$ for $k \le n$ so that for all R-trace functions f on G, $f(w) = P_w(f(x_1), ..., f(x_{i_1}...x_{i_k}), ..., f(x_1...x_n))$.*

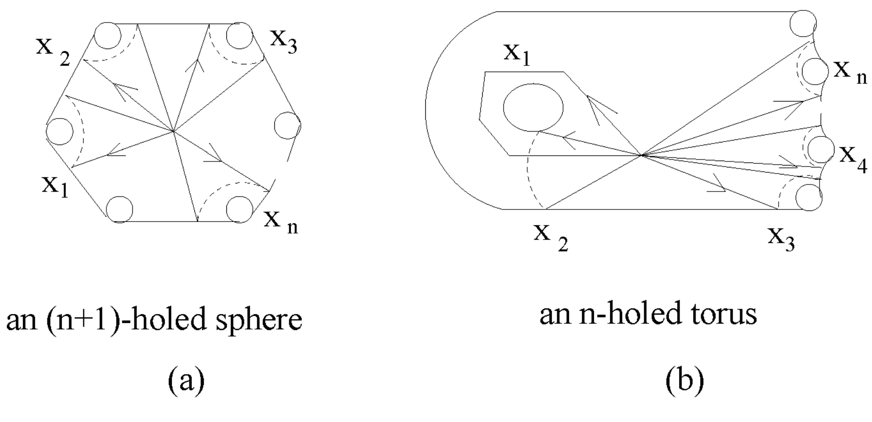


FIGURE 2.1

In particular R-trace functions on the free group F_n of n generators $< x_1, ..., x_n >$ are determined by their restrictions to the set $\{x_{i_1}...x_{i_k} | 1 \leq i_1 < ... < i_k \leq n \text{ and } k \leq n\}$. Now identify the free group F_n with the fundamental group $\pi_1(\Sigma_{0,n+1})$ of the $(n+1)$-holed sphere or $\pi_1(\Sigma_{1,n-1})$ of the $(n-1)$-holed torus. Then we may choose the set of n generators $\{x_1, ..., x_n\}$ so that each element $x_{i_1}...x_{i_k}$ is represented by a simple loop in the surface (see Figure 2.1.). This shows that R-trace functions on the fundamental groups of these surfaces are determined by their restrictions to the classes of simple loops.

For the low-rank free groups F_2, F_3 and F_4, the $(2^n - 1)$-element set $\{x_{i_1}...x_{i_k} | 1 \leq i_1 < ... < i_k \leq n\}$ are closely related to the so called modular relation and pentagon relations on the surfaces of genus zero. Indeed, take $F_2 = \pi_1(\Sigma_{0,3})$. Then the 3-element set $\{x_1, x_2, x_1x_2\}$ is represented by the three boundary components. Take $F_3 = \pi_1(\Sigma_{0,4})$. Then the 7-element set $\{x_1, x_2, x_3, x_1x_2, x_2x_3, x_1x_3, x_1x_2x_3\}$ is represented by the four boundary components and three simple loops pairwise intersecting at two points (see Figure 2.2). Take $F_4 = \pi_1(\Sigma_{0,5})$. Then the 15-element set

$$\{x_1, x_2, x_3, x_4, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_1x_2x_3, x_1x_3x_4,$$
$$x_2x_3x_4, x_1x_2x_4, x_1x_2x_3x_4\}$$

is represented by the five boundary components and 10 more simple loops closely related to the pentagon relation (see §3.2 and §4 for more discussions).

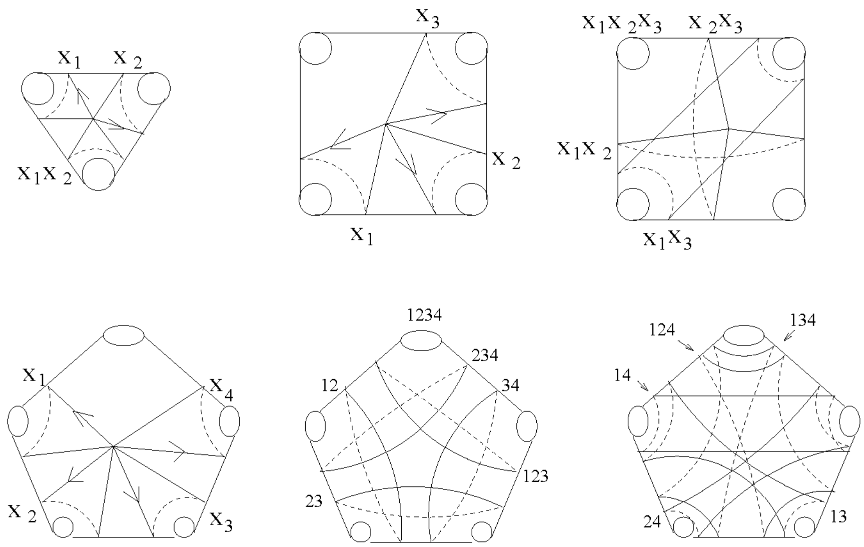


FIGURE 2.2

2.3. In the rest of the paper, we will always assume that the field K is *quadratically closed* in the sense that each quadratic equation $x^2 + ax + b = 0$, $a, b \in K$ has roots in K. Under this assumption, each $SL(2, K)$ matrix has eigenvalues in K. Furthermore, two $SL(2, K)$ matrices are $SL(2, K)$ conjugate if and only if they have the same trace.

The following lemma is known to experts in the field. Especially, in the case where the field K is algebraically closed, it follows from the combination of the work of Culler-Shalen [5] and Horowitz ([16], Theorem 4.3). Since we have not seen a written proof of the version stated below, a proof is given in the appendix for completeness.

Lemma. *Suppose K is a quadratically closed field. Given six elements $t_1, t_2, t_3, t_{12}, t_{23}$ and t_{31} in K, there exist three $SL(2, K)$ matrices A_1, A_2 and A_3 so that $tr(A_i) = t_i$ and $tr(A_iA_j) = t_{ij}$.*

Combining the lemma with Proposition 2.2 and Lemma 2.2(c), one sees that the set of all characters on the free group in three generators $< x_1, x_2, x_3 >$ is the hypersurface $\{(t_1, t_2, t_3, t_{12}, t_{23}, t_{31}, t_{123}) \in K^7|$ the equation $(*)$ holds},

$$(*) \qquad t_{123}^2 + \Big(\prod_{i=1}^{3} t_i - \sum t_i t_{jk}\Big) t_{123} + \sum_{i=1}^{3} t_i^2 + \sum t_{ij}^2 + \prod t_{ij} - \sum t_i t_j t_{ij} - 4 = 0.$$

In equation $(*)$, $t_{i_1 \ldots i_k} = tr(\rho(x_{i_1} \ldots x_{i_k}))$ and the indices i, j, k are pairwise distinct. This fact was well known after the work of Culler-Shalen [5] and Horowitz [16]. See §3.5 for more details.

2.4. It is easy to see that $(BA_1B^{-1}, BA_2B^{-1}, BA_3B^{-1})$ and $(A_1^{-1}, A_2^{-1}, A_3^{-1})$ are other solutions in Lemma 2.3. In fact, these are the set of all solutions to the the equation $tr(X_i) = t_i, tr(X_iX_j) = t_{ij}$ if and only if the group generated by $< A_1, A_2, A_3 >$ is irreducible. To derive this, let us recall the following lemma proved by Culler and Shalen ([5] Lemma 1.5.2).

Lemma (Culler-Shalen). *Suppose the field K is quadratically closed. If ρ_1 and ρ_2 are two representations of a group to $SL(2, K)$ so that ρ_1 is irreducible, then ρ_1 is conjugate to ρ_2 if and only if they have the same character functions.*

As a consequence of Culler-Shalen's lemma, Lemma 2.2 and Proposition 2.2, we see that if the group generated by $\{A_1, A_2, A_3\}$ in Lemma 2.3 is irreducible, then the solution (A_1, A_2, A_3) in Lemma 2.3 is unique up to conjugation and inverse. Evidently, if the group $< A_1, A_2, A_3 >$ in Lemma 2.3 is reducible, then the solution is not unique in the above sense.

2.5. Due to the importance of irreducible representations, we need an irreducibility criterion. The following is well known. See [5] Lemma 1.5.5, or [26] for instance.

Lemma. *Suppose the field K is quadratically closed. The group $< A, B >$ generated by two elements A, B in $SL(2, K)$ is reducible if and only if $tr([A, B]) = 2$.*

By Lemma 2.2(b), the condition in the above lemma is the same as $tr^2(A) + tr^2(B) + tr^2(AB) - tr(A)tr(B)tr(AB) - 4 = 0$. Since this expression will occur frequently, following [26], let us denote $tr^2(A) + tr^2(B) + tr^2(AB) - tr(A)tr(B)tr(AB) - 4$ by $\Delta(A, B)$.

As a consequence of the lemma, one has the following criterion for reducibility. A slightly different criterion can be found in [5], [11] (Proposition 4.4) and [26].

Corollary. *Suppose the field K is quadratically closed.*

(a). The group $< A_1, A_2, A_3 >$ generated by three elements A_1, A_2 and A_3 in $SL(2,K)$ is reducible if and only if $\Delta(A_i, A_j) = 0$ and $\Delta(A_1, A_1A_2A_3) = 0$ where $(i,j) = (1,2), (2,3), (3,1)$.

(b). The group $< A_1, ..., A_n >$ in $SL(2,K)$ generated by n elements is reducible if and only if each 3-generator subgroup $< A_i, A_j, A_k >$ is reducible, i.e., for all possible choice of indices i, j, k, $\Delta(A_i, A_j) = 0$ and $\Delta(A_i, A_iA_jA_k) = 0$.

(c) (Culler-Shalen). A subgroup of $SL(2,K)$ is reducible if and only if each 2-generator subgroup is reducible.

Proof. To see part (a), we may assume that none of A_i is $\pm id$ since otherwise it reduces to Lemma 2.5. Thus each A_i has at most two eigenspaces. By Lemma 2.5, each pair (A_i, A_j) has a common eigenspace for $i \neq j$. Now if one of A_i has exactly one eigenspace, then all A_1, A_2, and A_3 share this unique eigenspace. Thus the group is reducible. If otherwise, each A_i has two distinct eigenspaces L_j and L_k, $i \neq j \neq k \neq i$. Now suppose that the group $< A_1, A_2, A_3 >$ is irreducible. Then all three eigenspaces are pairwise distinct, i.e., $L_1 \neq L_2 \neq L_3 \neq L_1$. But by assumption, A_1 and $A_1A_2A_3$ have a common eigenspace L. Due to $A_1(L) = L$, thus L must be either L_2 or L_3, say, $L = L_2$. Then $A_1A_2A_3(L_2) = L_2$ implies that $A_2(L_2) = L_2$, i.e., L_2 is either L_1 or L_3 which contradicts the assumption.

Parts (b) and (c) follow from part (a) easily. To see (b), we first drop all generators A_i which are $\pm id$. Thus, we may assume that each A_i has at most two eigenspaces. By the assumption, any three elements A_i, A_j, and A_k have a common eigenspace. The goal is to show that all elements A_i have a common eigenspace. To this end, we form a graph whose vertices are eigenspaces of A_i's. To each element A_i, we draw an edge ending at the eigenspaces of A_i (the edge becomes a loop if A_i has only one eigenspace). Now by the assumption, any three edges of the graph has a common vertex. Thus all edges of the graph share a vertex.

To see part (c), take a subgroup G of $SL(2,K)$ which has the property that each 2-generator subgroup is reducible. By parts (a) and (b), we see that each finitely generated subgroup of G is reducible. Now by the same graph theoretical argument as in part (b), we see that the group G is reducible. q.e.d.

2.6. Given a reducible representation ρ of a group to $SL(2,K)$, we may assume after conjugate ρ by an $SL(2,K)$ matrix that the image of ρ is in the set of upper triangular matrices. Let ρ' be a new representation so that $\rho'(g)$ is the diagonal matrix whose diagonal entries are that of $\rho(g)$. Then the diagonalizable representation ρ' has the same character as that of ρ. Evidently the diagonal representation is unique up to conjugation. We call ρ' the *diagonalization* of the reducible representation ρ.

Lemma. *Suppose K is a quadratically closed field. Let ρ_1 and ρ_2 be two non-diagonalizable reducible representations of the free group $< x, y >$ on two generators to $SL(2, K)$. If ρ_1 and ρ_2 have the same character and $tr(\rho_i(x)) \neq \pm 2$, then they are conjugate.*

Indeed, under the assumption, we can conjugate the pair $(\rho_i(x), \rho_i(y))$ to the pair of matrices $(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \mu & 1 \\ 0 & \mu^{-1} \end{pmatrix})$ where $\lambda \neq \pm 1$.

3. Simple loops on surfaces and the modular configuration

We shall recall some basic facts on the set $S(\Sigma)$ of isotopy classes of simple loops on surfaces, and express the results in §2 in terms of a $(\mathbf{Q}P^1, PSL(2, \mathbf{Z}))$ modular structure on the set $S(\Sigma)$. We also establish several irreducible conditions in terms of the modular structure.

The field K is always assumed to be quadratically closed.

3.1. The following notation and terminologies will be used. Let $\Sigma = \Sigma_{g,r}$ be a compact orientable surface of genus g with r boundary components. The *level* of the surface $\Sigma_{g,r}$ is defined to be $3g+r-3$ which is the minimal number of disjoint simple loops decomposing the surface into 3-holed spheres. Recall that $S(\Sigma)$ is the set of isotopy (homotopy) classes of unoriented homotopically non-trivial simple loops on Σ. Let $S'(\Sigma)$ be the subset of $S(\Sigma)$ consisting of non-boundary parallel isotopy classes. The fundamental group of the surface is denoted by $\pi_1(\Sigma)$. The isotopy class of a loop s will be denoted by $[s]$. If b is a boundary component of the surface Σ, we usually use b to denote $[b]$. Given two isotopy classes α and β in $S(\Sigma)$, let $I(\alpha, \beta)$ be their *geometric intersection number* which is $\min\{|a \cap b| | a \in \alpha, b \in \beta\}$. If f is a function defined on $S(\Sigma)$, we define $f(a) = f([a])$. In particular, the intersection number $I([a], [b])$ is also denoted by $I(a, [b]) = I([a], b) = I(a, b)$. We use $\alpha \perp \beta$ to denote two elements $\alpha, \beta \in S(\Sigma)$ so that $I(\alpha, \beta) = 1$. And we use $\alpha \perp_0 \beta$ to denote two elements α and β so that $I(\alpha, \beta) = 2$ and their algebraic intersection number is zero. Two elements α, β are called disjoint, denoted by $\alpha \cap \beta = \emptyset$, if $I(\alpha, \beta) = 0$ and $\alpha \neq \beta$. If $I(\alpha, \beta) \neq 0$, we say that α *intersects* β. Two isotopic curves a, b will be denoted by $a \cong b$.

Let $\hat{\mathbf{Q}} = \mathbf{Q} \cup \{\infty\} = \mathbf{Q}P^1$. Two rational numbers $p/q, p'/q'$ satisfying $pq' - p'q = \pm 1$ will be denoted by $p/q \perp p'/q'$. The relation $(\hat{\mathbf{Q}}, \perp)$ is the so called *modular relation.* It is well known from elementary number theory that one may identify $\hat{\mathbf{Q}}$ with the set of cusps in Figure 3.1 so that two cusps are joint by an edge if and only if the corresponding rational numbers r, r' satisfy $r \perp r'$. We say three elements (α, β, γ) in $\hat{\mathbf{Q}}$ form a *triangle* if $\alpha \perp \beta \perp \gamma \perp \alpha$, and four distinct elements $(\alpha, \beta, \gamma; \gamma')$ form a *quadrilateral* if both (α, β, γ) and (α, β, γ') are triangles (see Figure 3.1).

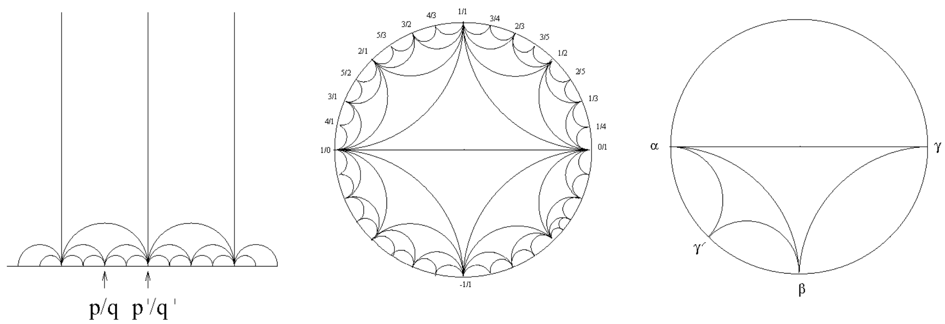


FIGURE 3.1

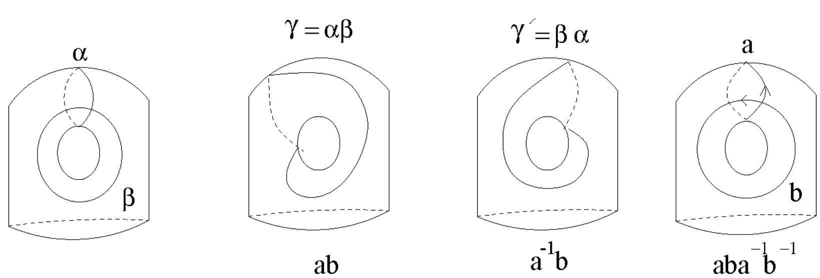
We shall always fix an orientation on $\hat{\mathbf{Q}}$ so that the triangle $(0, 1, \infty)$ is positively oriented (i.e., the right-hand orientation in Figure 3.1). A triangle is positively oriented if it determines the fixed orientation. The group of orientation preserving bijection of $(\hat{\mathbf{Q}}, \perp)$ is $PSL(2, \mathbf{Z})$ where the action of the matrices is given by the fractional linear transformations.

The importance of $(\hat{\mathbf{Q}}, SL(2, \mathbf{Z}))$ in surface theory was predicted by Grothendieck in [10] (page 11, second paragraph).

3.2. Suppose Σ is a level-1 surface $\Sigma_{1,1}$ or $\Sigma_{0,4}$. Then there exists a bijection (a slope map) $\pi : S'(\Sigma) \to \hat{\mathbf{Q}} = \mathbf{Q} \cup \{\infty\}$ so that $\pi(\alpha) = p/q$ and $\pi(\beta) = p'/q'$ satisfy $pq' - p'q = \pm 1$ if and only if $I(\alpha, \beta) = 1$ for $\Sigma = \Sigma_{1,1}$ and $I(\alpha, \beta) = 2$ for $\Sigma = \Sigma_{0,4}$. This important fact was established by M. Dehn [6] by using Dehn's coding $\begin{pmatrix} p \\ q \end{pmatrix}$ of classes in $S'(\Sigma)$.

Here is one way to construct a slope map $\pi : S'(\Sigma) \to \hat{\mathbf{Q}}$. It is well known that for the torus $\Sigma_{1,0}$, $S(\Sigma_{1,0})$ can be naturally identitied with the set of primitive elements in the first homology group $H_1(\Sigma_{1,1}, \mathbf{Z})$ modulo ± 1. Thus, by fixing a basis for the first homology group, one constructs a slope map $\pi : S(\Sigma_{1,0}) \to \hat{\mathbf{Q}}$. For the 1-holed torus $\Sigma_{1,1}$, let i be an inclusion map from $\Sigma_{1,1}$ to $\Sigma_{1,0}$. Then the induced map i_* from $S'(\Sigma_{1,1})$ to $S(\Sigma_{1,0})$ is a bijection preserving the relation $\perp$. Thus a slope map for $S'(\Sigma_{1,1})$ is the composition $\pi \circ i_*$. For the 4-holed sphere $\Sigma_{0,4}$, there exists a natural bijection $P : S'(\Sigma_{0,4}) \to S'(\Sigma_{1,1})$ so that $P(\alpha) \perp P(\beta)$ if and only if $\alpha \perp_0 \beta$. The bijection P is constructed as follows. Let $T : \Sigma_{1,1} \to \Sigma_{1,1}$ be a hyperelliptic involution. It is well known that $T(s) \cong s$ for any simple loop s. Let $\Sigma_{0,4}$ be the quotient space $\Sigma_{1,1}/x \sim T(x)$ with a regular neighborhood of the branch points removed. Then for each $[a] \in S'(\Sigma_{0,4})$ the inverse image of a in $\Sigma_{1,1}$ consists of two disjoint simple loops b and $T(b)$. Define $P([a]) = [b]$. Thus a slope map for $\Sigma_{0,4}$ is $\pi \circ i_* \circ P$.

Just like in the modular configuration, for a level-1 surface Σ, we can talk about triangles and quadrilaterals in $S'(\Sigma)$. Furthermore, when the surface Σ is oriented, by making all maps π, i and P orientation preserving, we can talk about oriented triangles in $S'(\Sigma)$.



Right-hand orientation on the front faces

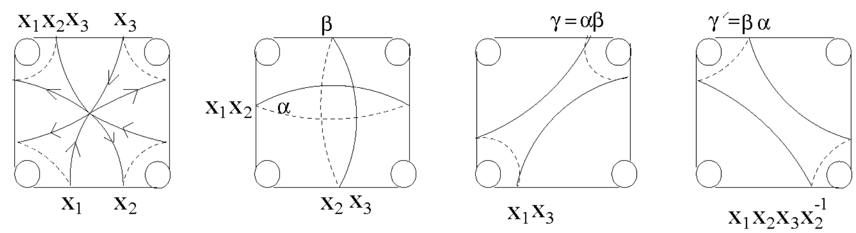
FIGURE 3.2

The relationship between the fundamental group and the modular structure on $S(\Sigma)$ can be described as follows. For the 1-holed torus $\Sigma_{1,1}$, if $(\alpha,\ \beta,\ \gamma;\ \gamma')$ is a quadrilateral in $S'(\Sigma_{1,1})$, then we can choose generators a, b in the fundamental group $\pi_1(\Sigma_{1,1})$ so that α, β, γ, and γ' are represented by a, b, ab and $a^{-1}b$ respectively (see Figure 3.2). By Proposition 2.2, the values of an $SL(2, K)$ character on $S(\Sigma_{1,1})$ is determined by its restriction on a triangle.

For the 4-holed sphere $\Sigma_{0,4}$, if $(\alpha,\ \beta,\ \gamma;\ \gamma')$ is a quadrilateral in

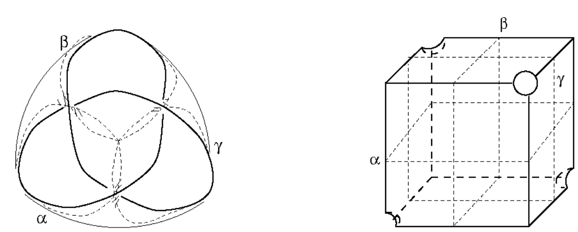
$S'(\Sigma_{0,4})$, then we can choose three generators x_1, x_2, x_3 in the fundamental group $\pi_1(\Sigma_{1,1})$ so that (1) the four boundary components of the surface are homotopic to x_1, x_2, x_3 and $x_1x_2x_3$, and (2) the classes α, β, γ and γ' are represented by x_1x_2, x_2x_3, x_1x_3 and $x_1x_2x_3x_2^{-1}$ (see Figure 3.3(a)). By Proposition 2.2 for 3-generator groups, it follows that an $SL(2,K)$ character defined on $S(\Sigma_{0,4})$ is determined by its restriction to a triangle and the four boundary components.

Thus triangles, quadrilaterals and boundary components in $S(\Sigma_{0,4})$ and $S(\Sigma_{1,1})$ are exactly the elements appeared in Lemma 2.2. One advantage of using the modular configuration is the symmetry in the modular configuration. For instance, each triangle in the modular configuration $S(\Sigma_{0,4})$ is invariant under all permutations of the four boundary components (see Figure 3.3(b)). As a consequence, there exists a 24-fold symmetry in the equation (c) in Lemma 2.2.



Right-hand orientation on the front faces

FIGURE 3.3 (a)



the Seifert surface is $\Sigma_{1,1}$ the truncated sphere is $\Sigma_{0,4}$

The three-fold symmetry in the modular configuration

FIGURE 3.3 (b)

In the following, we shall give a necessary and sufficient condition for a K-valued function defined on $S(\Sigma)$ to be a character by translating information on the fundamental group $\pi_1(\Sigma)$ to $S(\Sigma)$. These results are certainly well known (see for instance [9], [16], [26], [11] and others). The only novelty is that it is formulated in terms of the modular configuration.

3.3. For the level-0 surface $\Sigma_{0,3}$, its fundamental group is the free group on 2-generator $< x, y >$ where x, y and xy represent the three boundary components b_1, b_2 and b_3. Furthermore, we have $S(\Sigma_{0,3}) = \{b_1, b_2, b_3\}$. By Proposition 2.2, Lemmas 2.3 and 2.5, one obtains the following result (see [9]).

Proposition. *Suppose $\partial\Sigma_{0,3} = b_1 \cup b_2 \cup b_3$. Any function $f : S(\Sigma_{0,3}) \to K$ is an $SL(2,K)$ character. Furthermore, the character is reducible if and only if $\sum_{i=1}^{3} f^2(b_i) - f(b_1)f(b_2)f(b_3) = 4$.*

By a simple calculation and Lemma 2.6, one obtains the following.

Corollary. *Under the same assumption as above,*

(a) if $f : S(\Sigma_{0,3}) \to K$ satisfies $f^2(b_1) = 4$, then f is reducible if and only if $f(b_3) = f(b_1)f(b_2)/2$ when the characteristic of K is not 2 and $f(b_3) = f(b_2)$ when the characteristic of K is 2. In particular, if the characteristic of K is not 2, and $f^2(b_1) = f^2(b_2) = 4$, then f is reducible if and only if $f(b_1)f(b_2)f(b_3) = 8$.

(b) If $f : S(\Sigma_{0,3}) \to K$ is a reducible representation so that $f^2(b_1) \neq 4$, then there exist exactly two $SL(2,K)$ conjugacy classes of $SL(2,K)$ representations whose characters are f.

3.4. For the 1-holed torus, we have,

Proposition. *Let $b = \partial\Sigma_{1,1}$. A function $f : S(\Sigma_{1,1}) \to K$ is an $SL(2,K)$ character if and only if the following hold:*

$$(a) \qquad \sum_{i=1}^{3} f^2(\alpha_i) - \prod_{i=1}^{3} f(\alpha_i) - f(b) = 2 \quad \text{and}$$

$$(b) \qquad f(\alpha_3) + f(\alpha_3') = f(\alpha_1)f(\alpha_2),$$

where $(\alpha_1, \alpha_2, \alpha_3)$ and $(\alpha_1, \alpha_2, \alpha_3')$ are distinct triangles in $S'(\Sigma_{1,1})$.

The character f is reducible if and only if

$$\sum_{i=1}^{3} f^2(\alpha_i) - \prod_{i=1}^{3} f(\alpha_i) - 4 = 0.$$

Proof. The necessity of the conditions follows from the trace identities in Lemma 2.2 and the choice of generators for the fundamental group in §3.2 (see Figure 3.2). Due to the modular relation, if f_1 and f_2 are two functions satisfying equation (b) in the proposition so that they coincide when restricted to a triangle, then $f_1 = f_2$. Thus by Lemma 2.3 we obtain the sufficiency of the condition. q.e.d.

Corollary. *Suppose f is an irreducible $SL(2,K)$ character defined on the set $S(\Sigma_{1,1})$. Then either there exists a 3-holed sphere Σ' in $\Sigma_{1,1}$ so that the restriction $f|_{S(\Sigma')}$ is irreducible or the characteristic of K is not 2 and $f(\partial\Sigma_{1,1}) = -2$ and $f(\alpha) = 0$ for all $\alpha \in S'(\Sigma_{1,1})$. Furthermore, if the characteristic of K is not 2, there exists an irreducible representation ρ of $\pi_1(\Sigma_{1,1})$ so that $tr\rho(\alpha) = 0$ and $tr\rho(\partial\Sigma_{1,1}) = -2$ for all $\alpha \in S'(\Sigma_{1,1})$.*

Proof. Let the boundary of $\Sigma_{1,1}$ be b. By the irreducible assumption, $f(b) \neq 2$. Suppose otherwise that the restriction of f to each 3-holed sphere is reducible. Since each essential 3-holed sphere is bounded by b and two copies of $\alpha \in S'(\Sigma_{1,1})$, by Lemma 2.5 we have $2f^2(\alpha) + f^2(b) - f^2(\alpha)f(b) = 4$ for all $\alpha \in S'(\Sigma_{1,1})$. Since $f(b) \neq 2$, this shows that $f^2(\alpha) = f(b) + 2$ for all $\alpha \in S'(\Sigma_{1,1})$. Now take three elements $\alpha_i \in S(\Sigma_{1,1})$ forming a triangle in the modular configuration. By the above proposition and $f^2(\alpha_i) = f(b) + 2$, we obtain $f^2(\alpha_i)(3 - f(\alpha_i)) = f^2(\alpha_i)$. Thus either (1) $f(\alpha_i) = 0$ and $f(b) = -2$ or (2) $f(\alpha_i) = 2$ and $f(b) = 2$. But the case (2) and the case (1) when the characteristic of K is 2 are excluded by the irreducibility assumption.

If the characteristic of K is not 2, then one constructs an $SL(2,K)$ representation of $\pi_1(\Sigma_{1,1})$ satisfying the above condition as follows. First we note that the unit quarternion group of eight elements $\{\pm 1, \pm i, \pm j, \pm k\}$ is a subgroup of $SL(2,K)$ where $i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $j = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$. Now the character of any representation of $\pi_1(\Sigma_{1,1})$ onto the unit quarternion group takes value zero on $S'(\Sigma_{1,1})$ and -2 on the boundary component. Any two such representations are $SL(2,K)$ conjugate. q.e.d.

3.5. For the 4-holed torus, we have,

Proposition. *Let $\partial\Sigma_{0,4} = \cup_{i=1}^4 b_i$. A function $f : S(\Sigma_{0,4}) \to K$ is an $SL(2,\mathbf{K})$ character if and only if for each triangle $(\alpha_1, \alpha_2, \alpha_3)$ in*

$S'(\Sigma_{0,4})$ the following hold:

$$\text{(a)} \qquad \sum_{i=1}^{3} f^2(\alpha_i) + \prod_{i=1}^{3} f(\alpha_i) + \sum_{r=1}^{4} f^2(b_r) + \prod_{r=1}^{4} f(b_r) - \sum_{(i,r,s)\in P} f(\alpha_i)f(b_r)f(b_s) - 4 = 0,$$

where $P = \{ (i,r,s) \mid (\alpha_i, b_r, b_s)$ bounds a $\Sigma_{0,3}\}$ and,

$$(b) \qquad f(\alpha_3) + f(\alpha_3') = -f(\alpha_1)f(\alpha_2) + f(b_i)f(b_j) + f(b_k)f(b_l)$$

together with $(3,i,j)$ and $(3,k,l)$ in P.

A character is irreducible if and only if there is a 3-holed sphere so that the restriction of f to the 3-holed sphere is irreducible.

Proof. The necessity of the conditions follows from the trace identities in Lemma 2.2 and the choice of generators for the fundamental group in §3.2 (see Figure 3.3).

To show the sufficiency of the conditions, we first note that by the modular relation and the iteration equation (b), each f is determined by its restriction to a 7-element set $\{\alpha_i, b_r | i = 1,2,3; r = 1,2,3,4\}$ where α_i's form a triangle. We choose a set of generators x_i for the fundamental group $\pi_1(\Sigma_{0,4})$ as in §3.2 so that α_k is represented by $x_i x_j$, $i \neq j \neq k \neq i$. Let $t_i = f(b_i)$ and $t_{ij} = f(\alpha_k)$ where $i = 1,2,3$, $(i,j) = (1,2),(2,3),(3,1)$ and $k \neq i \neq j \neq k$. By Lemma 2.3, we find $A_i \in SL(2,K)$ so that $tr(A_i) = t_i$ and $tr(A_{ij}) = t_{ij}$. By equation (a), $f(b_4)$ is a root of the quadratic equation $x^2 - Px + Q = 0$ where P and Q are the same as in Lemma 2.2. By Lemma 2.2 (c), we may assume, after change (A_1, A_2, A_3) to $(A_1^{-1}, A_2^{-1}, A_3^{-1})$ if necessary, that $f(b_4) = tr(A_1A_2A_3)$. Define a representation of $\pi_1(\Sigma_{0,4})$ by sending the generator x_i to A_i. Then the character of this representation and the function f take the same values on the seven specific elements. Thus they are the same.

The irreducibility condition follows from Corollary 2.5(a). q.e.d..

Corollary. *(a) Suppose f is an irreducible character defined on $S(\Sigma_{0,4})$. Then for any element $\alpha \in S'(\Sigma_{0,4})$, there exists an element β with $I(\alpha,\beta) = 2$ and a 3-holed sphere Σ' bounded by β and two components of $\partial\Sigma$ so that the restriction $f|_{S(\Sigma')}$ is irreducible.*

(b) Let b be a boundary component of $\partial\Sigma_{0,4}$ and ρ be an irreducible representation of $\pi_1(\Sigma_{0,4})$. If $\rho(b) \neq \pm id$, in particular if $tr\rho(b) \neq$

± 2, then there exists a level-0 subsurface Σ' having b as a boundary component so that the restriction of ρ to Σ' is irreducible.

(c) If an $SL(2,K)$ representation of $\pi_1(\Sigma_{0,n})$ is reducible on each level-0 subsurface group, then the representation is reducible.

(d) If f is a reducible character on $S(\Sigma_{0,4})$, then for any quadrilateral $(\alpha,\beta,\gamma;\gamma')$ in $S'(\Sigma_{0,4})$, $f(\gamma)=f(\gamma')$.

Proof. To prove (a), suppose otherwise that f is reducible on each level-0 subsurface Σ' in $\Sigma_{0,4}$ bounded by a simple loop intersecting α at two points. Then take a quadrilateral with vertices $(\alpha,\beta,\alpha\beta;\beta\alpha)$. Choose three generators $\{x_1,x_2,x_3\}$ for $\pi_1(\Sigma_{0,4})$ as in Figure 3.3 so that β, $\alpha\beta$, $\beta\alpha$ are represented by x_2x_3, x_1x_3, and $x_1x_2x_3x_2^{-1}$. Let the $SL(2,K)$ representation corresponding to the character f send x_i to the matrix A_i and let $A_4=A_1A_2A_3$. Since the representation is reducible over six 3-holed spheres bounded by $\beta,\alpha\beta$ and $\beta\alpha$, we obtain the following reducible equations by Lemma 2.5; namely,

$$\Delta(A_1,A_3)=\Delta(A_2,A_3)=\Delta(A_2,A_1A_3)=\Delta(A_3,A_1A_2A_3A_2^{-1})=0.$$

It turns out that these four equations imply that the group $< A_1,A_2,A_3 >$ is reducible. Indeed, if $A_3=\pm id$, then

$$\Delta(A_2,A_1A_3)=\Delta(A_2,A_1)=0$$

shows that the group

$$< A_1,A_2,A_3 > = < A_1,A_2,\pm id >$$

is reducible by Lemma 2.5. Suppose A_3 has exactly one eigenspace. Then $\Delta(A_1,A_3)=\Delta(A_2,A_3)=0$ imply that the eigenspace is fixed by both A_1 and A_2. Thus the group is again reducible. Finally, suppose A_3 has exactly two distinct eigenspaces L_1 and L_2 so that none of L_i is fixed by both A_1 and A_2. Since $\Delta(A_3,A_i)=0$ for $i=1,2$, we may assume that $A_i(L_i)=L_i$ for $i=1,2$. By $\Delta(A_3,A_1A_2A_3A_2^{-1})=0$, one of the eigenspace L_i is fixed by $A_1A_2A_3A_2^{-1}$. If $A_1A_2A_3A_2^{-1}(L_2)=L_2$, then $A_1(L_2)=L_2$. Thus the group has a common eigenspace L_2 and is reducible. If $A_1A_2A_3A_2^{-1}(L_1)=L_1$, then $A_3(A_2^{-1}(L_1))=A_2^{-1}(L_1)$. Thus either $A_2^{-1}(L_1)=L_1$ or $A_2^{-1}(L_1)=L_2$. In the first case, L_1 is a common eigenspace for the group. The second case implies $L_1=L_2$ which is absurd. In summary, we have shown that the group is reducible which contradicts the assumption.

To prove (b), suppose otherwise that the restrictions of ρ to all level-0 subsurfaces having b as a boundary component are reducible. Let us choose a set of generators x_1, x_2, x_3 for $\pi_1(\Sigma_{0,4})$ as in §3.2 so that x_1 corresponds to b. Let $\rho(x_i)$ be the matrix A_i in $SL(2,K)$. Then by the assumption, $\Delta(A_1, A_2) = \Delta(A_1, A_3) = \Delta(A_1, A_1A_2A_3) = 0$. Since $\rho(b) \neq \pm id$, the matrix A_1 has at most two eigenspaces. If it has exactly one eigenspace, then $\Delta(A_1, A_2) = \Delta(A_1, A_3) = 0$ implies that the eigenspace is invariant under both A_2 and A_3. This contradicts the irreducible assumption. If A_1 has two distinct eigenspaces L_2 and L_3 so that $A_i(L_i) = L_i$ for $i = 2, 3$, then due to $\Delta(A_1, A_1A_2A_3) = 0$, one of L_i is invariant under $A_1A_2A_3$. But this again implies that L_i is a common eigenspace of A_1, A_2 and A_3.

To prove (c), we first note that the result holds for $n = 3$ by part (a). For $n > 3$, take generators $x_1, ..., x_{n-1}$ for the free group $\pi_1(\Sigma_{0,n})$ so that the boundary components are freely homotopic to x_i or $x_1...x_{n-1}$ as in Figure 2.1. Now each 3-generator subgroup $< x_i, x_j, x_k >$ lies in a level-1 subsurface subgroup. Thus the restriction of the representation to the 3-generator subgroup is reducible. By Corollary 2.5(b), this shows that the representation is reducible.

To see part (d), we may assume that f is the character of a diagonalizable representation ρ. Choose a set of generators $\{x_1, x_2, x_3\}$ for the fundamental group so that γ and γ' are represented by x_1x_3 and $x_1x_2x_3x_2^{-1}$ as in Figure 3.3. Then $f(\gamma) = tr\rho(x_1x_3) = tr\rho(x_1x_2x_3x_2^{-1}) = f(\gamma')$. q.e.d.

3.6. In this section, we give a different interpretation of the triangles and quadrilaterals in the set $S(\Sigma)$. This new interpretation is the basis for us in dealing with simple loops in the rest of the paper.

We begin by introducing some notation. Recall that surfaces are oriented. If a and b are two arcs intersecting transversely at a point p, then the *resolution of* $a \cup b$ *at* p *from* a *to* b is defined as follows. Fix any orientation on a and use the orientation on the surface to determine an orientation on b. Then resolve the intersection according to the orientations (see Figure 3.4). The resolution is evidently independent of the choice of the orientations on a. If $\alpha \perp \beta$ or $\alpha \perp_0 \beta$, take $a \in \alpha$, $b \in \beta$ so that $|a \cap b| = I(\alpha, \beta)$. Then the curve obtained by resolving all intersection points in $a \cap b$ from a to b is again a simple loop denoted by ab. We define $\alpha\beta$ to be the isotopy class of ab. It follows from the definition that when $\alpha \perp \beta$ then $\alpha\beta \perp \alpha, \beta$, and when $\alpha \perp_0 \beta$ then $\alpha\beta \perp_0 \alpha, \beta$. In particular, if Σ has level-1, then the positively oriented triangles in $S'(\Sigma)$ are $(\alpha, \alpha\beta, \beta)$ where $\alpha \perp \beta$ or $\alpha_0 \perp \beta$. Also the quadrilaterals are $(\alpha, \beta, \alpha\beta; \beta\alpha)$. Let $N(a)$ and $N(b)$ be two small regular neighborhoods of a and b. Then $N(a \cup b) = N(a) \cup N(b)$ is homeomorphic to $\Sigma_{1,1}$ when $\alpha \perp \beta$, and to $\Sigma_{0,4}$ when $\alpha \perp_0 \beta$. We use $\partial(\alpha, \beta)$ to denote the set of isotopy classes of the curves in $\partial N(a \cup b)$.

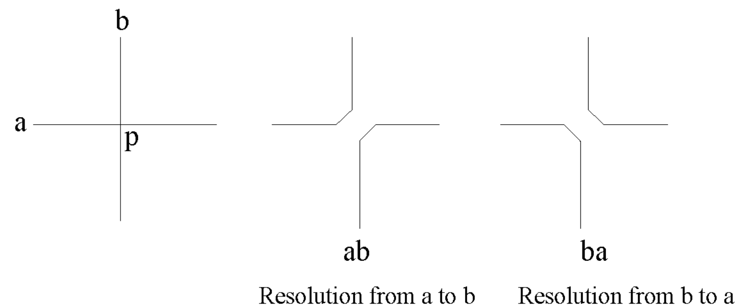


FIGURE 3.4

In terms of these notation, the equations (b) in Propositions 3.4 and 3.5 say that if $\alpha \perp \beta$, then $f(\alpha\beta)$ is determined by $f(\alpha), f(\beta)$ and $f(\beta\alpha)$, and if $\alpha \perp_0 \beta$, then $f(\alpha\beta)$ is determined by the values of f on $\{\alpha, \beta, \beta\alpha\}$ and $\partial(\alpha, \beta)$. More precisely, a function $f : S(\Sigma) \to K$ is a *trace function* if and only if it satisfies:

(1) $f(\alpha\beta) + f(\beta\alpha) = f(\alpha)f(\beta)$, for $\alpha \perp \beta$,

(2) $f^2(\alpha) + f^2(\beta) + f^2(\alpha\beta) - f(\alpha)f(\beta)f(\alpha\beta) = f(\partial(\alpha, \beta)) + 2$ for $\alpha \perp \beta$,

(3) $f^2(\alpha)+f^2(\beta)+f^2(\alpha\beta)+f(\alpha)f(\beta)f(\alpha\beta)+\Pi_{i=1}^4 f(\gamma_i)+\Sigma_{i=1}^4 f^2(\gamma_i)$ $-f(\alpha)(f(\gamma_1)f(\gamma_2) + f(\gamma_3)f(\gamma_4)) - f(\beta)(f(\gamma_1)f(\gamma_3) + f(\gamma_2)f(\gamma_4))$ $- f(\alpha\beta)(f(\gamma_1)f(\gamma_4) + f(\gamma_2)f(\gamma_3)) - 4 = 0$, for $\alpha \perp_0 \beta$, $\partial(\alpha, \beta) = \{\gamma_1, ..., \gamma_4\}$ so that $(\alpha, \gamma_1, \gamma_2)$ and $(\beta, \gamma_1, \gamma_3)$ bound level-0 subsurfaces,

(4) $f(\alpha\beta) + f(\beta\alpha) = -f(\alpha)f(\beta) + f(\gamma_1)f(\gamma_4) + f(\gamma_2)f(\gamma_3)$ under the same assumption as in (3).

3.7. One of the main reduction lemma for simple loops is the following which generalizes Lickorish's Lemma 2 in [19]. See [22] Lemma 7 for a proof.

Lemma. *Suppose $\gamma_1, ..., \gamma_m$ are pairwise disjoint classes in $S(\Sigma)$. If $\alpha \in S(\Sigma)$ intersects γ_1 and is not $\perp$ or $\perp_0$ related to γ_1, then $\alpha = \beta_1\beta_2$ with $\beta_1 \perp \beta_2$ or $\beta_1 \perp_0 \beta_2$ so that*

(1) $I(\beta_i,\gamma_1) < I(\alpha,\gamma_1)$, $I(\beta_i,\gamma_j) \le I(\alpha,\gamma_j)$, $I(\beta_2\beta_1,\gamma_1) < I(\alpha,\gamma_1)$ *and* $I(\beta_2\beta_1,\gamma_j) \le I(\alpha,\gamma_j)$ *for all* $i = 1,2$ *and* $j = 2,...,m$, *and,*

(2) if $\beta_1 \perp_0 \beta_2$, *then for each element* $\delta \in \partial(\beta_1,\beta_2)$, $I(\delta,\gamma_1) < I(\alpha,\gamma_1)$ *and* $I(\delta,\gamma_j) \le I(\alpha,\gamma_j)$ *for* $j = 2,...,m$.

The lemma says that one can "simplify" α unless $\alpha \cap \gamma_1 = \emptyset$, or $\alpha \perp \gamma_1$, or $\alpha \perp_0 \gamma_1$. In particular, if we set $\mathcal{G}_0 = \{\alpha \in S(\Sigma) |$ for each i, either $\alpha \cap \gamma_i = \emptyset$, or $\alpha \perp \gamma_i$, or $\alpha \perp_0 \gamma_i\}$, then by induction on $(I(\alpha,\gamma_1),...,I(\alpha,\gamma_m))$, we have $S(\Sigma) = \cup_{n=0}^{\infty}\mathcal{G}_n$ where $\mathcal{G}_{n+1} = \mathcal{G}_n \cup \{\alpha|$ $\alpha = \beta_1\beta_2$ either (1) $\beta_1 \perp \beta_2$ and $\{\beta_1,\beta_2,\beta_2\beta_1\} \subset \mathcal{G}_n$ or (2) $\beta_1 \perp_0 \beta_2$ and $\{\beta_1,\beta_2,\beta_2\beta_1\} \cup \partial(\beta_1,\beta_2) \subset \mathcal{G}_n\}$.

By the remark in the last two paragraphs in §3.6, we obtain,

Corollary. *(a) Let* $\gamma_1,...,\gamma_m$ *be pairwise disjoint classes in* $S(\Sigma)$ *and* f *and* g *be two trace functions on* $S(\Sigma)$. *If* $f(\alpha) = g(\alpha)$ *for each class* $\alpha \in S(\Sigma)$ *so that for each* i *either* $\alpha \cap \gamma_i = \emptyset$, *or* $\alpha \perp \gamma_i$ *or* $\alpha \perp_0 \gamma_i$, *then* $f = g$.

(b) Let γ_1 *and* γ_2 *are two disjoint classes in* $S'(\Sigma)$ *so that* γ_1 *bounds a* $\Sigma_{1,1}$ *and* γ_2 *lies in* $\Sigma_{1,1}$. *If* f *and* g *are two trace functions on* $S(\Sigma)$ *so that* $f(\alpha) = g(\alpha)$ *for all* $\alpha \perp \gamma_2$ *and* $\alpha \perp_0 \gamma_1$, *then* $f = g$.

Indeed, in part (b), if $\alpha \perp_0 \gamma_1$, then $\alpha \perp_0 \gamma_2$ cannot occur. Thus part (b) follows from part (a).

3.8. As a second consequence of the above lemma, we obtain the following result which will be used in §6. Part (a) of the corollary was known to many people [20].

Corollary. *(a) Given two non-separating classes* α *and* α', *there exists a sequence of non-separating classes* $\{\alpha_i | i = 1,...,m\}$ *starting from* α *and ending at* α' *so that* $\alpha_i \perp \alpha_{i+1}$ *for all* i.

(b) Given two essential 1-holed tori (resp. level-1 surfaces) T *and* T' *in* Σ, *there exists a sequence of essential 1-holed tori (resp. level-1 surfaces)* $\{T_i\}$ *starting from* T *and ending at* T' *so that* $S'(T_i) \cap S'(T_{i+1}) \ne \emptyset$ *for all* i.

Proof. Let us denote two classes α and α' satisfying the conclusion of (a) by $\alpha \sim \alpha'$. We use the induction on $I(\alpha,\alpha')$ to prove part (a). Clearly if $\alpha \cap \alpha' = \emptyset$ or $\alpha \perp \alpha'$, then $\alpha \sim \alpha'$. If $\alpha \perp_0 \alpha'$, since both α and α' are non-separating, one of the element $\beta \in \partial(\alpha,\alpha')$ is non-separating. Thus $\alpha \sim \beta \sim \alpha'$. In the remaining cases, by Lemma 3.7, we can write $\alpha = \beta_1\beta_2$ where either $\beta_1 \perp \beta_2$ or $\beta_1 \perp_0 \beta_2$ and $I(\beta_i,\alpha') < I(\beta_i,\alpha)$,

$i = 1, 2$. Since α is non-separating, one of β_1 or β_2, say β_1, is again non-separating. Thus by the induction hypothesis, $\beta_1 \sim \alpha'$. But $\beta_1 \perp \alpha$ or $\beta_1 \perp_0 \alpha$. Thus $\alpha \sim \alpha'$.

To see part (b), take $\alpha \in S'(T)$ and $\alpha' \in S'(T')$. Let $T_1 = T$, $T_{m+1} = T'$ and T_i be the 1-holed torus containing both α_i and α_{i+1}. Then the result follows. The result for level-1 surfaces T and T' is simpler. We omit the proof.

3.9. It is shown in Section 3 of [21] that there exists a finite set $F_0 \in S(\Sigma)$ so that $S(\Sigma) = \cup_{n=0}^{\infty} F_n$ where $F_{n+1} = F_n \cup \{\alpha| \ \alpha = \beta_1\beta_2$ either (1) $\beta_1 \perp \beta_2$ and $\{\beta_1, \beta_2, \beta_2\beta_1\} \subset F_n$ or (2) $\beta_1 \perp_0 \beta_2$ and $\{\beta_1, \beta_2, \beta_2\beta_1\} \cup \partial(\beta_1, \beta_2) \subset F_n\}$. In particular, if f is a trace function defined on $S(\Sigma)$, then f is algebraically determined by $f|_F$. This shows the following result analogous to Proposition 2.2.

Propositoin. *There exists a finite set of isotopy classes of simple loops in each compact orientable surface Σ so that $SL(2, K)$ characters and trace functions on $S(\Sigma)$ are algebraically determined by the restrictions of the characters on the finite set.*

4. $SL(2, K)$ characters on the 5-holed sphere

4.1. We shall use the following terminologies. If f is a trace function on $S(\Sigma)$ and Σ' is a subsurface of Σ, then we call $f|_{S(\Sigma')}$ the *restriction* of f to the subsurface Σ'. If $f|_{S(\Sigma')}$ is irreducible, we say Σ' is an *irreducible* subsurface with respect to f. We say a subsurface Σ' is *bounded* by $\alpha_1, ..., \alpha_k \in S'(\Sigma)$ if $\partial\Sigma' = a_1 \cup ... \cup a_k \cup b_1 \cup ... \cup b_m$ so that $a_i \in \alpha_i$ and $b_j \subset \partial\Sigma$.

The goal of this section is to prove the following case of Theorem 1.2 for the 5-holed sphere.

Theorem. *Suppose K is a quadratically closed field. If f is a K-valued trace function on $S(\Sigma_{0,5})$ so that either f is reducible on all level-0 subsurfaces or f is irreducible on a level-0 subsurface bounded by two disjoint elements in $S'(\Sigma_{0,5})$, then f is an $SL(2, K)$ character.*

In §5.2, we prove that if f is a trace function on $S(\Sigma_{0,5})$ which does not satisfy the conditions in the above theorem, then f is exceptional.

4.2. The Pentagon Relations. Given five pairwise distinct elements α_1, ..., α_5 in $S'(\Sigma_{0,5})$ so that $\alpha_i \cap \alpha_j = \emptyset$ for $i \neq j \pm 1$ mod 5, it is shown in [23] that $\alpha_i \perp_0 \alpha_{i+1}$ for all indices i mod 5 (see Figure

4.1). We say $\{\alpha_1, ..., \alpha_5\}$ forms a *pentagon* in this case. If $\{\alpha_1,..., \alpha_5\}$ forms a pentagon, then the following conditions hold:

(a) $(\alpha_i\alpha_j)\alpha_k = \alpha_i(\alpha_j\alpha_k)$.

(b) $\alpha_i\alpha_{i+1}\alpha_{i+2} = \alpha_{i+3}\alpha_{i+4}$.

(c) $(\alpha_i\alpha_j) \cap (\alpha_i\alpha_k) = \emptyset$ and $(\alpha_j\alpha_i) \cap (\alpha_k\alpha_i) = \emptyset$, $i \neq j \neq k \neq i$.

(d) $\alpha_i\alpha_j\alpha_k = \alpha_j\alpha_i\alpha_k$ and $\alpha_k\alpha_j\alpha_i = \alpha_k\alpha_i\alpha_j$, if $i \neq j \pm 1$ mod 5.

(e) $\alpha_i\alpha_j\alpha_i = \alpha_j$ if $i = j \pm 1$ mod 5.

These can be verified easily using the definition of resolution or see [22] or [23] for a proof. Note that, by (c), if $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ forms a pentagon, then $\{\alpha_1\alpha_2,\alpha_2,\alpha_3\alpha_2,\alpha_4,\alpha_5\}$ is also a pentagon (see Figure 4.1).

4.3. In this section, we prove that if f is a trace function which is irreducible on a 3-holed sphere bounded by two elements α_2 and α_5 in $S'(\Sigma_{0,5})$, then f is a character.

Let X and Y be the level-1 subsurfaces bounded by α_2 and α_5 respectively. By Proposition 3.5, we find two representations ρ_X and ρ_Y of $\pi_1(X)$ and $\pi_1(Y)$ respectively so that $\chi_{\rho_X} = f|_{S(X)}$ and $\chi_{\rho_Y} = f|_{S(Y)}$. The restrictions of ρ_X and ρ_Y to $\pi_1(X \cap Y)$ have the same character by the construction and both are irreducible. Thus by Lemma 2.4, these two restrictions are conjugate. After conjugate ρ_X, we may assume that $\rho_X|_{\pi_1(X\cap Y)} = \rho_Y|_{\pi_1(X\cap Y)}$. This defines a representation $\rho : \pi_1(\Sigma_{0,5}) \to SL(2, K)$ so that its restrictions to $\pi_1(X)$ and $\pi_1(Y)$ are ρ_X and ρ_Y respectively. Let g be the character of ρ. Then $f(\alpha) = g(\alpha)$ for all $\alpha \in S(X) \cup S(Y)$. The goal is to show that $f = g$ using the irreducibility condition.

By Corollary 3.7 applied to f and g, it suffices to prove that for each α_1 so that $\alpha_1 \perp_0 \alpha_2$ and $\alpha_1 \perp_0 \alpha_5$, we have $f(\alpha_1) = g(\alpha_1)$. To this end, we extend $\{\alpha_1, \alpha_2, \alpha_5\}$ to a set $\{\alpha_1, ..., \alpha_5\}$ forming a pentagon by setting $\alpha_3 = \partial(\alpha_1, \alpha_5) \cap S'(\Sigma_{0,5})$ and $\alpha_4 = \partial(\alpha_1, \alpha_2) \cap S'(\Sigma_{0,5})$. We shall use Proposition 3.5 to derive a system of linear equations and show that the system has a unique solution and that both $f(\alpha_1)$ and $g(\alpha_1)$ are solutions.

We begin by introducing some notation. Let h be a trace function defined on $S(\Sigma_{0,5})$. Given a set of indices $i_1, ..., i_k$, $k = 1, 2, 3$, let $x_{i_1...i_k} = h(\alpha_{i_1}...\alpha_{i_k})$ when $\alpha_{i_1}...\alpha_{i_k}$ is not in $S(X) \cup S(Y)$ and let $a_{i_1....i_k} = h(\alpha_{i_1}...\alpha_{i_k})$ when $\alpha_{i_1}...\alpha_{i_k} \in S(X) \cup S(Y)$. Let β_i be the component of $\partial\Sigma_{0,5}$ so that $\{\alpha_{i-1}, \beta_i, \alpha_{i+1}\}$ bounds a 3-holed sphere (indices mod 5) and let $b_i = h(\beta_i)$ (see Figure 4.1). Let τ be the orientation preserving involution of $\Sigma_{0,5}$ so that τ sends α_{1+i} to α_{1-i} and

β_{1+i} to β_{1-i}.

Now we derive equations for x_i, x_{ij} and x_{ijk} with coefficient in $h(\alpha)$'s where $\alpha \in S(X) \cup S(Y)$ using Proposition 3.5.

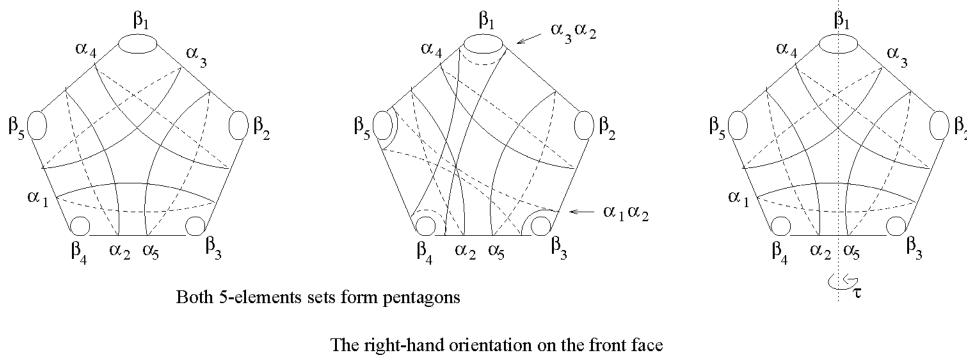


FIGURE 4.1

Since $\alpha_3 \perp_0 \alpha_4$ and $\partial(\alpha_3, \alpha_4) \cap S'(\Sigma_{0,5}) = \alpha_1$, by Proposition 3.5 (b), we obtain

$$h(\alpha_3\alpha_4) + h(\alpha_4\alpha_3) = -h(\alpha_3)h(\alpha_4) + h(\alpha_1)h(\beta_1) + h(\beta_2)h(\beta_5).$$

This is the same as,

$$x_{34} + x_{43} - b_1 x_1 = p_1. \tag{1}$$

Here and below, p_i always denotes some polynomial with integer coefficient in $h(\alpha)$'s where $\alpha \in S(X) \cup S(Y)$.

Since $\alpha_1 \perp_0 \alpha_2$ with $\partial(\alpha_1, \alpha_2) \cap S'(\Sigma_{0,5}) = \alpha_4 \in S(X)$, we obtain,

$$x_{12} + x_{21} + a_2 x_1 = p_2. \tag{2}$$

Apply the involution τ to the equation (2), we obtain

$$x_{15} + x_{51} + a_5 x_1 = p_3. \tag{3}$$

Since $\alpha_2 \perp_0 \alpha_3\alpha_4$, $\alpha_2(\alpha_3\alpha_4) = \alpha_5\alpha_1$ and $\partial(\alpha_2, \alpha_3\alpha_4) \cap S'(\Sigma_{0,5}) = \alpha_5\alpha_4 \in S(X)$, we obtain,

$$x_{51} + x_{342} + a_2 x_{34} = p_4. \tag{4}$$

Since $\alpha_4 \perp_0 \alpha_3\alpha_2$, $\alpha_4(\alpha_3\alpha_2) = \alpha_1\alpha_5$, $\alpha_3\alpha_2\alpha_4 = \alpha_3\alpha_4\alpha_2$ and $\partial(\alpha_4, \alpha_3\alpha_2) \cap S'(\Sigma_{0,5}) = \alpha_1\alpha_2$, we obtain,

$$x_{15} + x_{342} - b_1 x_{12} = p_5. \tag{5}$$

Subtracting (4) by (5), we obtain,

$$x_{51} - x_{15} + a_2 x_{34} + b_1 x_{12} = p_6. \tag{6}$$

Apply the involution τ to equation (6), we obtain,

$$x_{21} - x_{12} + a_5 x_{43} + b_1 x_{15} = p_7. \tag{7}$$

Since $\alpha_1\alpha_2 \perp_0 \alpha_5$, $\alpha_5(\alpha_1\alpha_2) = \alpha_3\alpha_4$ and $\partial(\alpha_1\alpha_2, \alpha_5) \cap S'(\Sigma_{0,5}) = \alpha_3\alpha_2 \in S(Y)$, we obtain,

$$x_{34} + x_{125} + a_5 x_{12} = p_8. \tag{8}$$

Apply the involution τ to (8) and use the fact that $x_{125} = x_{152}$ (due to §4.1(d)), we obtain,

$$x_{43} + x_{125} + a_2 x_{15} = p_9. \tag{9}$$

Subtracting (8) by (9) gives,

$$x_{34} - x_{43} - a_2 x_{15} + a_5 x_{12} = p_{10}. \tag{10}$$

Now consider the system of linear equations (1), (2), (3), (6), (7), and (10). By (1), (2) and (3), we obtain $x_{21} = -a_2 x_1 - x_{12} + p_2$, $x_{51} = -a_5 x_1 - x_{15} + p_3$ and $x_{43} = b_1 x_1 - x_{34} + p_1$. Thus, after substituting these into (6), (7) and (10), we obtain the following system of linear equations.

$$\begin{aligned} b_1 x_{12} - 2x_{15} + a_2 x_{34} - a_5 x_1 &= p_{11} \\ -2x_{12} + b_1 x_{15} - a_5 x_{34} + (a_5 b_1 - a_2) x_1 &= p_{12} \\ a_5 x_{12} - a_2 x_{15} + 2x_{34} - b_1 x_1 &= p_{13}. \end{aligned} \tag{11}$$

Let A be the 3×4 coefficient matrix of the linear system and B be the 3×3 submatrix obtained from A by removing the 4-th column. Then the determinant of B is 2Δ where $\Delta = a_2^2 + a_5^2 + b_1^2 - a_2 a_5 b_1 - 4$. Suppose B^* is the adjoint matrix of B. Then a simple calculation shows that $B^* A$ is

$$\begin{pmatrix} 2\Delta & 0 & 0 & a_2\Delta \\ 0 & 2\Delta & 0 & a_5\Delta \\ 0 & 0 & 2\Delta & -b_1\Delta \end{pmatrix}$$

Assume now that h is irreducible on the 3-holed sphere bounded by α_2 and α_5, i.e., $\Delta \neq 0$. Then we obtain a simpler system of linear equations satisfied by x_{12}, x_{15}, x_{34} and x_1.

$$\begin{aligned} 2x_{12} + a_2 x_1 &= p_{14} \\ 2x_{15} + a_5 x_1 &= p_{15} \\ 2x_{34} - b_1 x_1 &= p_{16}. \end{aligned} \tag{12}$$

By equations (1), (2), (3) and (12), we obtain

$$\begin{aligned} 2x_{21} + a_2 x_1 &= p_{17} \\ 2x_{51} + a_5 x_1 &= p_{18} \\ 2x_{43} - b_1 x_1 &= p_{19}. \end{aligned} \tag{13}$$

On the other hand, there are many different extensions of $\{\alpha_2, \alpha_5\}$ to a set forming a pentagon. For instance, $\{\alpha_1', \alpha_2, \alpha_3', \alpha_4', \alpha_5\} = \{\alpha_1\alpha_2, \alpha_2, \alpha_3\alpha_2, \alpha_4, \alpha_5\}$ is such an extension. For this extension, the same equations (12) and (13) hold. Here we have $x_{21}' = h(\alpha_2\alpha_1') = h(\alpha_2\alpha_1\alpha_2) = h(\alpha_1) = x_1$, $x_{51}' = h(\alpha_5\alpha_1') = h(\alpha_5\alpha_1\alpha_2) = h(\alpha_3\alpha_4) = x_{34}$ and $x_1' = h(\alpha_1') = x_{12}$. By (13) for the new pentagon, we obtain

$$\begin{aligned} 2x_1 + a_2 x_{12} &= p_{20} \\ 2x_{34} + a_5 x_{12} &= p_{21}. \end{aligned} \tag{14}$$

Comparing (14) with (12), we obtain

$$(4 - a_2^2)x_1 = p_{22}, \tag{15}$$

and

$$(2b_1 - a_2a_5)x_1 = p_{23}. \tag{16}$$

Apply the involution τ to (15), we obtain

$$(4 - a_5^2)x_1 = p_{24}. \tag{17}$$

Since $a_2^2 + a_5^2 + b_1^2 - a_2a_5b_1 - 4 \neq 0$, if the characteristic of the field K is not 2, then one of the coefficients $4 - a_2^2$, $4 - a_5^2$ and $2b_1 - a_2a_5$ is not zero. Thus we can solve x_1 uniquely from (15)-(17). If the characteristic of the field K is 2, then one of the coefficients a_2, a_5 or b_1 is not zero. Thus we can solve x_1 uniquely from (12).

Now take $h = f$ and $h = g$ respectively. The condition $f|_{S(X)\cup S(Y)} = g|_{S(X)\cup S(Y)}$ shows that $x_1 = f(\alpha_1)$ and $x_1 = g(\alpha_1)$ are the solutions of the *same* equations (1) -(17). Since both f and g are irreducible on the 3-holed sphere bounded by α_2 and α_5, we conclude that $f(\alpha_1) = g(\alpha_1)$. Thus by Corollary 3.7, $f = g$ follows in this case.

4.4. In this section, we prove that if f is a trace function which is reducible on each level-0 subsurface, then f is the character of a reducible representation.

We choose a 3-holed sphere decomposition of $\Sigma_{0,5}$ by α_2 and α_5 as follows. If there exists $\alpha \in S'(\Sigma_{0,5})$ so that $f^2(\alpha) \neq 4$ then choose α_2 to be one of these elements. If otherwise that $f^2(\alpha) = 4$ for all $\alpha \in S'(\Sigma_{0,5})$, choose α_2 and α_5 to be any pair of disjoint elements. We shall use the same notation introduced in §4.3. Thus X and Y are level-1 subsurfaces bounded by α_2 and α_5 respectively so that $X \cap Y$ is a level-0 subsurface. By Proposition 3.5, we find two representations ρ_X and ρ_Y of $\pi_1(X)$ and $\pi_1(Y)$ respectively so that their characters are the restrictions of f to $S(X)$ and $S(Y)$. By the reducibility criterion Lemma 2.5, both ρ_X and ρ_Y are reducible. Thus we may modify ρ_X and ρ_Y without changing their characters so that both $\rho_X(\pi_1(X \cap Y))$ and $\rho_Y(\pi_1(X \cap Y))$ consist of diagonal matrices. Now since both $\rho_X|_{\pi_1(X\cap Y)}$ and $\rho_Y|_{\pi_1(X\cap Y)}$ are diagonalizable and have the same character, thus they are conjugate. We may assume after a conjugation that $\rho_X|_{\pi_1(X\cap Y)} = \rho_Y|_{\pi_1(X\cap Y)}$. By the same argment as in §4.3, we construct a diagonalizable representation ρ of $\pi_1(\Sigma_{0,5})$ to $SL(2, K)$ extending both ρ_X and ρ_Y. Let g be the character of ρ defined on $S(\Sigma_{0,5})$. By the construction $f|_{S(X)\cup S(Y)} = g|_{S(X)\cup S(Y)}$. The goal is to show that $f = g$ under the reducible condition.

Since f is reducible on all level-0 subsurfaces, by Corollary 2.5, f is reducible on all level-1 subsurfaces. In particular, by Corollary 3.5(d), $f(\alpha\beta) = f(\beta\alpha)$ for all $\alpha \perp_0 \beta$.

We now set up the same system of linear equations in $x_{i_1 \ldots i_k}$, $1 \leq k \leq 3$ as in §4.3. Then equations (1) - (10) still hold. Due to the reducibility, $x_{12} = x_{21}$, $x_{15} = x_{51}$ and $x_{34} = x_{43}$. Thus equations (12)-(17) still hold. (Indeed, equation (12) is a consequence of (1) and $x_{12} = x_{21}$.)

Now if there is $\alpha \in S'(\Sigma_{0,5})$ so that $f^2(\alpha) \neq 4$, then $4 - f^2(\alpha_2) \neq 0$ by the choice of α_2. Thus we can solve x_1 uniquely from (15). In particular, by the same argument as in §4.3, we obtain $f(\alpha_1) = g(\alpha_1)$.

In the remaining case, $f^2(\alpha) = 4$ for all $\alpha \in S'(\Sigma_{0,5})$. First we note that Corollary 3.3(a) implies $f^2(\beta) = 4$ for all boundary component β in $S(\Sigma_{0,5})$. For each boundary component β, $f(\beta) = g(\beta)$ by the construction. We claim that that $f(\alpha) = g(\alpha)$ for all $\alpha \in S'(\Sigma_{0,5})$. Indeed, for each $\alpha \in S'(\Sigma_{0,5})$, there exists two boundary components β_1 and β_2 so that β_1, β_2 and α bound a level-0 subsurface. By the reducibility of f and g over the level-0 subsurface and Corollary 3.3(a), we conclude that $f(\alpha) = f(\beta_1)f(\beta_2)/2 = g(\beta_1)g(\beta_2)/2 = g(\alpha)$. (Here we have used the convention that if the characteristic of K is 2, then $ab/2$ is meant to be b when $a = 2$).

4.5. As a consequence of Theorem 4.1, we have,

Corollary. *Let f be a trace function defined on $S(\Sigma_{0,n})$. Suppose $\Sigma_{0,n}$ is decomposed as a union $X_1 \cup X_2$ of two incompressible subsurfaces X_1 and X_2 where $X_1 \cap X_2 \cong \Sigma_{0,3}$ is bounded by two elements in $S'(\Sigma)$. If $f|_{S(X_i)}$ is an $SL(2,K)$ character for $i = 1,2$ and either $f|_{S(X_1 \cap X_2)}$ is irreducible or f is reducible on all level-0 subsurfaces, then f is an $SL(2,K)$ character.*

Proof. Let ρ_i be an $SL(2,K)$ representation whose character is $f|_{S(X_i)}$, $i = 1,2$. If f is reducible on all level-0 subsurfaces, then by Corollary 3.5(c), both ρ_1 and ρ_2 are reducible. In this case, we may assume without changing the characters that both ρ_1 and ρ_2 are diagonalizable. Now by the same arguments as in §4.3 and §4.4, we produce a representation ρ of $\pi_1(\Sigma_{0,n})$ so that its restriction to $\pi_1(X_i)$ is conjugate to ρ_i. Let g be the character of ρ and β_1 and β_2 be two classes in $S'(\Sigma_{0,n})$ which bound $X_1 \cap X_2$. Then f and g are identical on $S(X_1) \cup S(X_2)$ $=\{\alpha \in S(\Sigma_{0,n}) | \ \alpha$ is disjoint from either β_1 or $\beta_2\}$. To show $f = g$, by Corollary 3.7, it suffices to prove $f(\alpha) = g(\alpha)$ for $\alpha \perp_0 \beta_i$, $i = 1,2$. Fix such a class α. Let Σ' be the incompressible level-2 subsurface which contains $X_1 \cap X_2$ and α. Then by the proof of Theorem 4.1 for Σ'

with respect to the decomposition $\Sigma' = (\Sigma' \cap X_1) \cup (\Sigma' \cap X_2)$, we have $f|_{S(\Sigma')} = g|_{S(\Sigma')}$. In particular, $f(\alpha) = g(\alpha)$. q.e.d.

5. Exceptional trace functions on planar surfaces

5.1. Recall that a trace function which is not the character of any representation is called *exceptional.* There are no exceptional trace functions on level-0 and level-1 surfaces. However, there exist finitely many exceptional trace functions on $\Sigma_{0,n}$ for any $n \geq 5$. The main result of the section is to identify all exceptional trace functions.

Theorem. *Suppose $f\colon S(\Sigma_{0,n}) \to K$, $n \geq 5$, is an exceptional trace function. Then the characteristic of K is not 2 and f satisfies,*

(a) $f(S(\Sigma_{0,n})) = \{2, -2\}$ and,

(b) there exists an exceptional level-2 subsurface in $\Sigma_{0,n}$.

The proof of the theorem is by induction on n. In §5.2 we prove it for $n = 5$ and in §5.3, we prove it for all $n \geq 6$.

We shall use the following notation. If $\alpha_1, ..., \alpha_m$ are disjoint classes in $S'(\Sigma_{g,n})$ so that they decompose the surface into subsurfaces Σ_{g_i,n_i} and $(g_1, n_1) \neq (g_i, n_i)$ for $i \geq 2$, then we use $\Sigma_{g_1,n_1}(\alpha_1, ..., \alpha_m)$ to denote the subsurface Σ_{g_1,n_1}. A class $\alpha \in S'(\Sigma)$ is a *boundary class* if it bounds a level-0 subsurface.

5.2. We prove a slightly stronger version of Theorem 5.1 for $n = 5$ in this section.

Let $b_1, ..., b_5$ be the boundary components of $\Sigma_{0,5}$.

Proposition. *If $f : S(\Sigma_{0,5}) \to K$ is an exceptional trace function, then*

(a) the characteristic of K is not 2,

(b) $f(S(\Sigma_{0,5})) = \{2, -2\}$ and $\Pi_{i=1}^{5} f(b_i) = 32$, and,

(c) a level-0 subsurface is irreducible if and only if it is of the form $\Sigma_{0,3}(\alpha)$ for some $\alpha \in S'(\Sigma_{0,5})$.

Proof. By Theorem 4.1, we see that $\Sigma_{0,3}(\alpha, \beta)$ is reducible for all disjoint $\alpha, \beta \in S'(\Sigma_{0,5})$ and there exists one irreducible $\Sigma_{0,3}(\gamma)$.

Lemma. *Suppose α_1, α_4 are two disjoint elements in $S'(\Sigma_{0,5})$ so that $\Sigma_{0,3}(\alpha_1)$ is irreducible. Then $f^2(\alpha_1) = f^2(\alpha_4) = f^2(b_i) = 4$ and $\Sigma_{0,3}(\alpha_4)$ is again irreducible. Furthermore, the characteristic of K is not 2.*

Proof. Since $\Sigma_{0,3}(\alpha_1) \subset \Sigma_{0,4}(\alpha_4)$ and $\Sigma_{0,3}(\alpha_1)$ is irreducible, we see that $\Sigma_{0,4}(\alpha_4)$ is irreducible. By Corollary 3.5(a) applied to α_1 in $\Sigma_{0,4}(\alpha_4)$ and by the assumption that $\Sigma_{0,3}(\alpha, \alpha_4)$ is reducible, there exists $\alpha_2 \perp_0 \alpha_1$ in $\Sigma_{0,4}(\alpha_4)$ so that $\Sigma_{0,3}(\alpha_2)$ is irreducible. Extend $\{\alpha_1, \alpha_2, \alpha_4\}$ to a pentagon set $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ so that $\alpha_i \perp_0 \alpha_{i+1}$ where indices are counted mod 5. For $i = 3, 4, 5$, $\Sigma_{0,4}(\alpha_i)$ is irreducible since it contains one of $\Sigma_{0,3}(\alpha_j)$, $j = 1$ or 2. By Corollary 3.5(b) applied to α_i in $\Sigma_{0,4}(\alpha_i)$ and by the assumption that $\Sigma_{0,3}(\alpha_i, \beta)$ is reducible, it follows that $f^2(\alpha_i) = 4$ for $i = 3, 4, 5$. Let the boundary components of $\Sigma_{0,5}$ be so labelled that $(\alpha_{i-1}, b_i, \alpha_{i+1})$ bounds a 3-holed sphere. For each $i = 1, 2, ..., 5$, $\Sigma_{0,3}(\alpha_{i-1}, \alpha_{i+1})$ is reducible and one of $f^2(\alpha_{i-1})$ or $f^2(\alpha_{i+1})$ is 4. By Corollary 3.3(a), it follows that $f(b_i) = f(\alpha_{i-1})f(\alpha_{i+1})/2$ (here $ab/2$ is meant to be b if the characteristic of K is 2 and $a = 2$). In particular $f^2(b_4) = 4$. The values of f on $\partial\Sigma_{0,3}(\alpha_1)$ are

$$\{f(\alpha_1), f(b_3), f(b_4)\} = \{f(\alpha_1), f(\alpha_2)f(\alpha_4)/2, f(\alpha_3)f(\alpha_5)/2\},$$

whose multiplication is $\frac{1}{4}\Pi_{i=1}^5 f(\alpha_i)$. For $i = 3, 4, 5$, the values of f on $\partial\Sigma_{0,3}(\alpha_i)$ are

$$\{f(\alpha_i), f(b_{i+2}), f(b_{i+3})\} = \{f(\alpha_i), f(\alpha_{i-1})f(\alpha_{i+1})/2, f(\alpha_{i+2})f(\alpha_{i+3})/2\},$$

whose multiplication is again $\frac{1}{4}\Pi_{i=1}^5 f(\alpha_i)$. Since $\Sigma_{0,3}(\alpha_1)$ is irreducible and $f^2(b_4) = f^2(\alpha_i) = 4$ for $i = 3, 4, 5$, by Corollary 3.3(a), it follows that $\Sigma_{0,3}(\alpha_i)$ is irreducible.

Now by the same argument above applied to $\{\alpha_3, \alpha_4\}$ instead of $\{\alpha_1, \alpha_2\}$, we conclude that $f^2(\alpha_1) = f^2(\alpha_2) = 4$. Thus $f^2(b_i) = 4$ for all i. In particular, this implies that the characteristic of K is not 2. Indeed, if otherwise, then all $f(\alpha_i) = f(b_i) = 0$. Thus $\Sigma_{0,3}(\alpha_1)$ would be reducible. q.e.d.

To finish the proof of the proposition, take $\alpha_1 \in S'(\Sigma_{0,5})$ so that $\Sigma_{0,3}(\alpha_1)$ is irreducible. Given any $\alpha \in S'(\Sigma_{0,5})$, by a result of Harvey [13] (see also [12]), there exists a sequence of elements $\beta_1 = \alpha_1, \beta_2, ..., \beta_m = \alpha$ in $S'(\Sigma_{0,5})$ so that $\beta_i \cap \beta_{i+1} = \emptyset$. By the lemma applied to the sequence, we conclude that $f^2(\alpha) = 4$ and $\Sigma_{0,3}(\alpha)$ is irreducible.

Suppose $\alpha \in S'(\Sigma_{0,5})$ so that b_i and b_j are in the boundary of $\Sigma_{0,3}(\alpha)$. Then by Corollary 3.3(a), $f(\alpha) = -\frac{1}{2}f(b_i)f(b_j)$. This shows that f is determined by $f|_{\partial\Sigma_{0,5}}$. Furthermore, take two disjoint α, α' in $S'(\Sigma_{0,5})$ so that $b_1 \subset \partial\Sigma_{0,3}(\alpha, \alpha')$. Then due to the reducibility and

Corollary 3.3(a), $f(b_1)f(\alpha)f(\alpha') = 8$. But also

$$f(\alpha)f(\alpha') = \frac{1}{4}f(b_2)f(b_3)f(b_4)f(b_5).$$

Thus $\Pi_{i=1}^5 f(b_i) = 32$. q.e.d.

5.3. We use the induction on n to prove Theorem 5.1. Assume that $n \geq 6$. The proof consists of several steps.

Let f be an exceptional trace function on $S(\Sigma_{0,n})$.

Claim 1. There exists an exceptional subsurface $\Sigma_{0,n-1}(\beta)$ in $\Sigma_{0,n}$.

By Corollary 4.5, there exists an irreducible level-0 subsurface Σ' in $\Sigma_{0,n}$. Now Σ' is either $\Sigma_{0,3}(\alpha_1, \alpha_2)$, or $\Sigma_{0,3}(\alpha_1)$, or $\Sigma_{0,3}(\alpha_1, \alpha_2, \alpha_3)$ for disjoint classes $\alpha_1, .., \alpha_3$ in $S'(\Sigma_{0,n})$. If $\Sigma' = \Sigma_{0,3}(\alpha_1, \alpha_2)$, by Corollary 4.5, one of the subsurface bounded by α_1 or α_2 is exceptional. Take any $\Sigma_{0,n-1}(\beta)$ which contains this exceptional subsurface, then the claim follows.

If $\Sigma' = \Sigma_{0,3}(\alpha_1)$, we use the following lemma.

Lemma. *If $\Sigma_{0,3}(\alpha)$ is irreducible and β_1, β_2 are two disjoint classes in $S'(\Sigma_{0,n})$ so that they bound a level-1 subsurface which contains $\Sigma_{0,3}(\alpha)$, then one of the subsurface bounded by β_i is exceptional.*

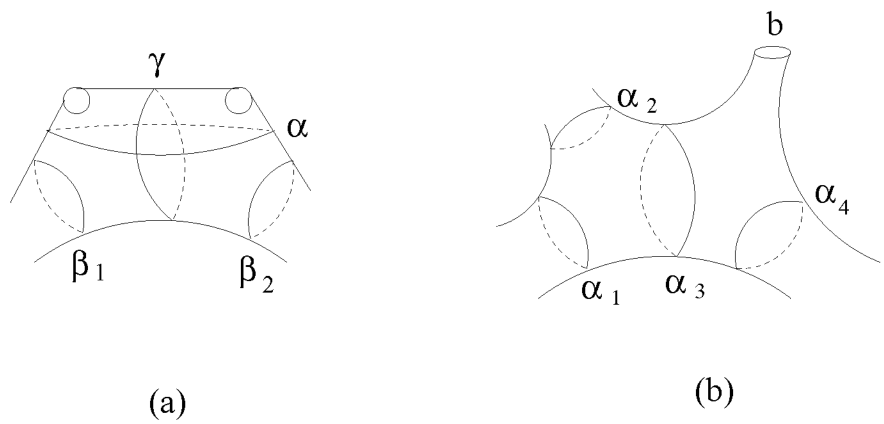


FIGURE 5.1

Indeed, by Corollary 3.5(a), there exists $\gamma \perp_0 \alpha$ in $\Sigma_{0,4}(\beta_1, \beta_2)$ so that one of $\Sigma_{0,3}(\gamma, \beta_i)$, say $\Sigma_{0,3}(\gamma, \beta_1)$, is irreducible. By Corollary 4.5, one of the subsurface bounded by γ or β_1 is exceptional. But each subsurface bounded by γ or β_1 is contained in a subsurface bounded by β_1 or β_2. Thus the lemma follows.

If $\Sigma' = \Sigma_{0,3}(\alpha_1, \alpha_2, \alpha_3)$, let Σ'' be the irreducible subsurface bounded by α_1, α_2 which contains Σ'. If Σ'' is exceptional, then Claim 1 follows. If the Σ'' is a 4-holed sphere, then by Corollary 3.5(a) applied to α_3 in Σ'', we find an irreducible $\Sigma_{0,3}(\alpha_i, \alpha')$ where $i = 1$ or 2. The claim follows by the previous argument. In the remaining case, the level of Σ'' is at least 2 and $f|_{S(\Sigma'')}$ is the character of a representation ρ of $\pi_1(\Sigma'')$. Since ρ is irreducible, there exists a boundary component $b \subset \partial\Sigma_{0,n} \cap \partial\Sigma''$ so that $\rho(b) \neq \pm id$. Consider a level-1 subsurface of Σ'' of the form $\Sigma_{0,4}(\alpha_1, \alpha_2, \alpha_4)$ so that it contains $\Sigma_{0,3}(\alpha_1, \alpha_2, \alpha_3)$ and contains b as a boundary component. By Corollary 3.5(b) applied to this level-1 subsurface, we find an irreducible $\Sigma_{0,3}(\alpha, \alpha')$ having b as a boundary component. Thus the claim follows.

Now by the induction hypothesis applied to $\Sigma_{0,n-1}(\beta)$, it follows that $\Sigma_{0,n}$ contains an exceptional level-2 subsurface, i.e., part (b) of the Theorem 5.1 follows.

Claim 2. Let b_1 and b_2 be the boundary components of $\Sigma_{0,n}$ which are not in the exceptional $\Sigma_{0,n-1}(\beta)$. Then $f^2(b_i) = 4$ for $i = 1, 2$.

Indeed, if $n \geq 7$, for each b_i, there exists an $(n-1)$-holed sphere containing both b_i and the exceptional level-2 subsurface in $\Sigma_{0,n-1}(\beta)$. Thus by the induction hypothesis, we conclude that $f(b_i)^2 = 4$ for $i = 1, 2$.

If $n = 6$, we pick a boundary class $\beta' \in S'(\Sigma_{0,6})$ so that $\Sigma_{0,3}(\beta')$ is in the exceptional $\Sigma_{0,5}(\beta)$ (see Figure 5.2). Let β_1 and β_2 be two disjoint boundary classes so that $\beta_i \cap \beta' = \emptyset$ and $\beta_i \perp_0 \beta$. Since $\Sigma_{0,5}(\beta)$ is exceptional, $\Sigma_{0,3}(\beta')$ is irreducible. By the above lemma applied to $\alpha = \beta'$, we conclude that one of $\Sigma_{0,5}(\beta_i)$, say $\Sigma_{0,5}(\beta_1)$, is exceptional. Since $b_1 \subset \Sigma_{0,5}(\beta_1)$, it follows that $f^2(b_1) = 4$. Next we assert that $\Sigma_{0,3}(\beta)$ is reducible. Assuming the assertion, by Corollary 3.3(a), we conclude that $f(b_2) = f(b_1)f(\beta)/2 = \pm 2$. To see that $\Sigma_{0,3}(\beta)$ is reducible, we construct two disjoint boundary classes γ_1, γ_2 in $\Sigma_{0,6}$ which are in $\Sigma_{0,5}(\beta)$. If $\Sigma_{0,3}(\beta)$ were irreducible, then by the above lemma applied to $\{\beta, \gamma_1, \gamma_2\}$, we conclude that one of $\Sigma_{0,5}(\gamma_i)$, say $\Sigma_{0,5}(\gamma_1)$, is exceptional. Let γ be a class disjoint from β and γ_1 and $\gamma \perp_0 \gamma_2$. Then $\Sigma_{0,3}(\beta, \gamma)$ is irreducible since it is in the exceptional $\Sigma_{0,5}(\beta)$. But

$\Sigma_{0,3}(\beta,\gamma)$ is also reducible since it is in the exceptional $\Sigma_{0,5}(\gamma_1)$. This is a contradiction.

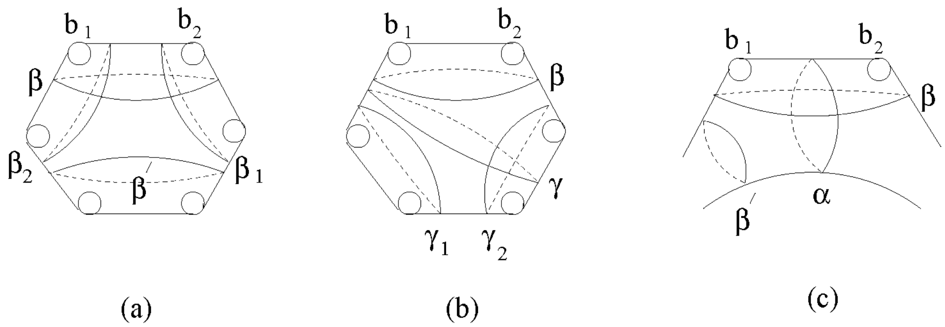


FIGURE 5.2

Claim 3. If $\alpha \perp_0 \beta$ where $\Sigma_{0,n-1}(\beta)$ is exceptional, then $f(\alpha) = \pm 2$.

Choose a class $\beta' \in S'(\Sigma_{0,n})$ disjoint from β so that β', α and one of b_1 or b_2, say b_1, bound a 3-holed sphere $\Sigma_{0,3}(\beta',\alpha)$ (see Figure 5.2(c)). By the induction hypothesis applied to $\Sigma_{0,n-1}(\beta)$, we have $f(\beta') = \pm 2$. If $\Sigma_{0,3}(\alpha,\beta')$ is reducible, then by Corollary 3.3(a), we obtain $f(\alpha) = f(\beta')f(b_1)/2 = \pm 2$. If $\Sigma_{0,3}(\alpha,\beta')$ is irreducible, then by Corollary 4.5, one of the subsurface X bounded by β' or α which contains $\Sigma_{0,3}(\alpha,\beta)$ is exceptional. Thus by the induction hypothesis applied to a subsurface containing X and α, $f(\alpha) = \pm 2$.

Claim 4. For all $\alpha \in S'(\Sigma_{0,n})$, $f(\alpha) = \pm 2$.

We use the induction on $I(\alpha,\beta)$ to prove the claim. By Claim 3, the result holds for $I(\alpha,\beta) \le 2$. If $I(\alpha,\beta) \ge 4$, by Lemma 3.7, we can express $\alpha = \alpha'\alpha''$ so that $\alpha' \perp_0 \alpha''$, $\partial(\alpha',\alpha'') = \beta_1 \cup ... \cup \beta_4$ satisfy $I(\alpha',\beta), I(\alpha'',\beta)$, $I(\alpha''\alpha',\beta), I(\beta_i,\beta) < I(\alpha,\beta)$. Thus by the induction hypothesis the values of f on the seven elements $\{\alpha',\alpha'',\alpha''\alpha',\beta_1,...,\beta_4\}$ are in $\{2,-2\}$. Now the following lemma implies that $f(\alpha) = \pm 2$.

5.4. Lemma. *Let $\partial\Sigma_{0,4} = b_1 \cup b_2 \cup b_3 \cup b_4$ and α_1, α_2, and α_3 be three classes forming a triangle in $S'(\Sigma_{0,4})$. If $f : S(\Sigma_{0,4}) \to K$ is a character so that its values on the 7-element set $\{\alpha_i, b_j\}$ are $\{2,-2\}$. Then $f(S(\Sigma_{0,4})) \subset \{2,-2\}$. Furthermore,*

(a) $2\Pi_{i=1}^{3} f(\alpha_i) = \Pi_{j=1}^{4} f(b_j)$,

(b) if the characteristic of K is not 2, then f is reducible if and only if $\Pi_{j=1}^{4} f(b_j) = 16$.

(c) If $g : \{\alpha_i, b_j\} \to \{2, -2\}$ satisfies $2\Pi_{i=1}^{3} g(\alpha_i) = \Pi_{j=1}^{4} g(b_j) = -16$, then g can be extended to an $SL(2, K)$ character on $S(\Sigma_{0,4})$.

Proof. Fix an orientation on each b_i and consider it as an element in the fundamental group. Let ρ be a representation whose character is f. Changing ρ to ρ' by $\rho'(b_i) = \pm\rho(b_i)$, $i = 1, 2, 3$, will not effect the conclusion of the lemma. Thus we may assume that $f(b_1) = f(b_2) = f(b_3) = 2$. Now if $f(b_4) = 2$, then Proposition 3.5(a) shows that $8\Sigma_{i=1}^{3} f(\alpha_i) - \Pi_{i=1}^{3} f(\alpha_i) = 40$. Since $f(\alpha_i) = \pm 2$, the only solution of the equation is $f(\alpha_i) = 2$. By Proposition 3.5(b), this implies that $f(\alpha) = 2$ for all α. If $f(b_4) = -2$, then Proposition 3.5(a) says that $\Pi_{i=1}^{3} f(\alpha_i) = -8$. Furthermore, Proposition 3.5(b) implies that $f(\alpha_3') + f(\alpha_3) = -f(\alpha_1)f(\alpha_2)$ where $(\alpha_1, \alpha_2, \alpha_3; \alpha_3')$ forms a quadrilateral. But $f(\alpha_3) = -\frac{1}{2} f(\alpha_1)f(\alpha_2)$. Thus $f(\alpha_3') = f(\alpha_3) = \pm 2$. By the modular configuration, this implies that $f(S(\Sigma_{0,4})) \subset \{2, -2\}$. The last argument also shows that for any assignments of ± 2 to α_i's so that their product is -8, there exists an extension of the assignment to a character. Thus part (c) follows. q.e.d.

The following figure 5.3 illustrates the set of all possible assignments of ± 2 to the 7-element set $\{b_i, \alpha_j\}$ in the lemma.

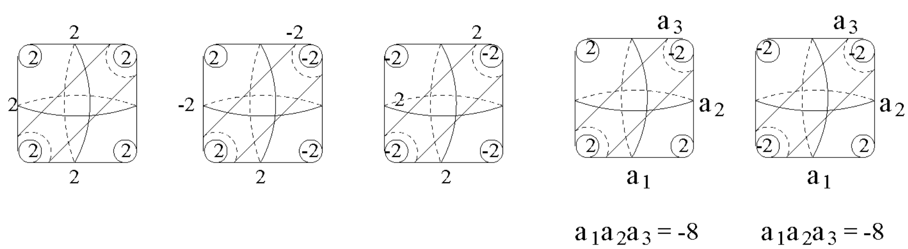


FIGURE 5.3

As a consequence of the Theorem 5.1(b) and the fact that trace functions are determined by their values on a finite subset of $S(\Sigma)$ (§3.9), we see that there are only finitely many exceptional trace functions on each planar surface.

5.5. The goal of this section is to prove that exceptional trace functions exists on each planar surface of level at least 2.

There are sixteen exceptional trace functions f on $\Sigma_{0,5}$ which we describe as follows. Let $\partial\Sigma_{0,5} = b_1 \cup ... \cup b_5$. Suppose $f : \{b_1, ..., b_5\} \to \{2, -2\}$ satisfies $\Pi_{i=1}^{5} f(b_i) = 32$. We extend f to $f : S(\Sigma_{0,5}) \to \{2, -2\}$ as follows. Given any boundary class α so that the level-0 subsurface $\Sigma_{0,3}(\alpha)$ contains b_i, b_j, then $f(\alpha) = -\frac{1}{2} f(b_i) f(b_j)$. Ones checks easily using Lemma 5.4 that f is a trace function on $S(\Sigma_{0,5})$. Furthermore, by the construction f is reducible on each $\Sigma_{0,3}(\alpha, \beta)$ and irreducible on each $\Sigma_{0,3}(\alpha)$. There is no representation whose character is the trace function f. Indeed, if $\chi_\rho = f$ for a representation ρ and $\alpha \in S'(\Sigma_{0,5})$, then by Corollary 3.5(b) applied to $\Sigma_{0,4}(\alpha)$, we have $\rho(\alpha) = \pm id$. But this implies that ρ is reducible on $\Sigma_{0,3}(\alpha)$ which contradicts the assumption. Thus these are the set of all 16 exceptional trace functions on $\Sigma_{0,5}$.

If $n \geq 6$, we construct an exceptional trace function on $\Sigma_{0,n}$ as follows. Let $b_1, ..., b_n$ be the boundary components of $\Sigma_{0,n}$. Define $f : S(\Sigma_{0,n}) \to \{2, -2\}$ as follows. Let $f(b_i) = 2$ for all i. For $\alpha \in S'(\Sigma_{0,n})$, we define $f(\alpha)$ as follows. Suppose α decomposes $\Sigma_{0,n}$ into two subsurfaces X_1 and X_2. Let s_i be the number of components of $\{b_1, ..., b_5\}$ which are in X_i. Define $f(\alpha)$ to be -2 if $(s_1, s_2) = (2, 3)$ and to be 2 otherwise. By a simple calculation using Lemma 5.4, one shows that f is a trace function and there exists an exceptional level-2 subsurface. Thus f is an exceptional trace function.

6. The characterization theorem for the z–holed torus

We show that each trace function on $S(\Sigma_{1,2})$ is a character in this section.

6.1. The pentagon relation. A five-element set $\{\alpha_1, ..., \alpha_5\}$ in $S'(\Sigma_{1,2})$ is said to form a *pentagon* if $\alpha_i \cap \alpha_{i+2} = \emptyset$ for all i mod 5. It is shown in [23] that the set is unique up to homeomorphism of the surface and that exactly two adjacent elements α_i and α_{i+1}, say α_3 and α_4, are separating classes and $\alpha_3 \perp_0 \alpha_2 \perp \alpha_1 \perp \alpha_5 \perp_0 \alpha_4$. See figure 6.1.

If $\{\alpha_1, ..., \alpha_5\}$ forms a pentagon with $I(\alpha_3, \alpha_4) = 4$, then we have,

(a) $(\alpha_i\alpha_j)\alpha_k = \alpha_i(\alpha_j\alpha_k)$ where the indices i, j, k are pairwise distinct,

(b) $\alpha_i\alpha_j\alpha_k = \alpha_j\alpha_i\alpha_k$ if the indices are pairwise distinct and $\alpha_i \cap \alpha_j = \emptyset$,

(c) $\alpha_2(\alpha_2\alpha_1\alpha_5) = \alpha_4\alpha_5\alpha_1$, $(\alpha_2\alpha_1\alpha_5)\alpha_2 = \alpha_1\alpha_5$, and $\alpha_2\alpha_1\alpha_5 \cap \alpha_1 = \emptyset$,

(d) $\alpha_1\alpha_2 \cap \alpha_1\alpha_5 = \emptyset$ and $\alpha_2\alpha_1 \cap \alpha_5\alpha_1 = \emptyset$,

(e) $\alpha_1\alpha_2\alpha_3 \cap \alpha_1\alpha_5 = \emptyset$ and $\alpha_4\alpha_5\alpha_5 \cap \alpha_1\alpha_5 = \emptyset$.

See [22] or [23] for a verification. One can also verify (a)-(e) directly. For instance $\alpha_1\alpha_5$ and $\alpha_1\alpha_2$ are obtained by applying the positive Dehn twist along α_1 to α_5 and α_2. Thus (d) holds.

By property (d), if $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ forms a pentagon, then $\{\alpha_1, \alpha_2\alpha_1, \alpha_3, \alpha_4, \alpha_5\alpha_1\}$ is also a pentagon.

6.2. In this section, we prove the following,

Proposition. *If f is a trace function on $S(\Sigma_{1,2})$ and there exist two disjoint elements α_1, α_4 in $S'(\Sigma_{1,2})$ with α_4 separating so that $f^2(\alpha_1) \neq f(\alpha_4) + 2$, then f is a character.*

Proof. Let X and Y be the level-1 subsurfaces bounded by α_4 and α_1 respectively. Then $X \cap Y$ is a level-0 subsurface bounded by $\alpha_4, \alpha_1, \alpha_1$. Since $f^2(\alpha_1) + f^2(\alpha_1) + f^2(\alpha_4) - f(\alpha_1)f(\alpha_1)f(\alpha_4) - 4 = (f(\alpha_4) - 2)(f(\alpha_4) + 2 - f^2(\alpha_1))$, f is reducible on $X \cap Y$ if and only if $f(\alpha_4) = 2$. We now construct a representation ρ of $\pi_1(\Sigma_{1,2})$ so that the restrictions of the character of ρ to X and Y are the same as $f|_{S(X)}$ and $f|_{S(Y)}$ as follows. If $f(\alpha_4) \neq 2$, due to the irreducibility of f on $X \cap Y$, the construction is the same as in §4.3. If $f(\alpha_4) = 2$, then $f(\alpha_2) \neq \pm 2$ by the assumption. By Lemma 2.6, there are exactly two conjugation classes of $SL(2, K)$ representations of $\pi_1(X)$ (respectively $\pi_1(X \cap Y)$) whose characters are $f|_{S(X)}$ (resp. $f|_{S(X\cap Y)}$). Furthermore, due to $f(\alpha_2) \neq \pm 2$, the restriction of the non-diagonalizable representation of $\pi_1(X)$ to $\pi_1(X \cap Y)$ is still non-diagonalizable. Now take a representation ρ_Y of $\pi_1(Y)$ whose character is $f|_{S(Y)}$. Then there exists a representation ρ_X of $\pi_1(X)$ whose character is $f|_{S(X)}$ so that $\rho_X|_{\pi_1(X\cap Y)} = \rho_Y|_{\pi_1(X\cap Y)}$. Let ρ be the representation of $\pi_1(\Sigma)$ whose restrictions to $\pi_1(X)$ and $\pi_1(Y)$ are ρ_X and ρ_Y and let g be its character. We have $f(\alpha) = g(\alpha)$ for $\alpha \in S(X) \cup S(Y)$. The goal is to show that $f = g$. By Corollary 3.7, it suffices to prove that for each class $\alpha_5 \perp \alpha_1$ and $\alpha_5 \perp_0 \alpha_4$, $f(\alpha_5) = g(\alpha_5)$.

Extend $\{\alpha_1, \alpha_4, \alpha_5\}$ to a 5-element set $\{\alpha_1, ..., \alpha_5\}$ forming a pentagon. The proof of $f(\alpha_5) = g(\alpha_5)$ follows the same strategy as in §4.3 by introducing a system of linear equations.

We shall use the same notation as in §4.3. Let h be a trace function on $S(\Sigma_{1,2})$ so that $h^2(\alpha_1) \neq h(\alpha_4) + 2$. Given a set of indices $i_1, ..., i_k$, $1 \leq k \leq 3$, let $x_{i_1...i_k} = h(\alpha_{i_1}...\alpha_{i_k})$ if $\alpha_{i_1}...\alpha_{i_k}$ is not in $S(X) \cup S(Y)$ and $a_{i_1...i_k} = h(\alpha_{i_1}...\alpha_{i_k})$ if $\alpha_{i_1}...\alpha_{i_k} \in S(X) \cup S(Y)$. Let β_1 and β_2 be the boundary components of $\Sigma_{1,2}$ and $b_i = h(\beta_i)$.

Using Propositions 3.4 and 3.5, we now derive a system of linear equations in $x_{i_1\ldots i_k}$ and show that the system of equation has a unique solution.

Since $\alpha_1 \perp \alpha_5$, by Proposition 3.4(b), we obtain

$$h(\alpha_1\alpha_5) + h(\alpha_5\alpha_1) = h(\alpha_1)h(\alpha_5).$$

This is the same as,

$$x_{15} + x_{51} = a_1 x_5. \tag{1}$$

Since $\alpha_4 \perp_0 \alpha_5$ so that $\partial(\alpha_4, \alpha_5) = \{\alpha_2, \alpha_2, \beta_1, \beta_2\}$, by Proposition 3.5, we obtain,

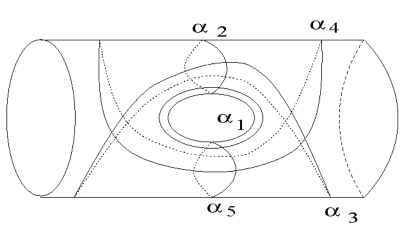
$$x_{45} + x_{54} = -a_4 x_5 + p_1, \tag{2}$$

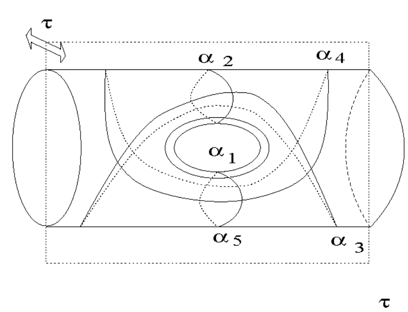
where p_1 and the p_i's below are polynomials with integer coefficients in $h(\alpha)'s$ where $\alpha \in S(X) \cup S(Y)$.

Since $\alpha_2 \perp \alpha_2\alpha_1\alpha_5$ so that $\alpha_2(\alpha_2\alpha_1\alpha_5) = \alpha_4\alpha_5\alpha_1$ and $(\alpha_2\alpha_1\alpha_5)\alpha_2 = \alpha_1\alpha_5$ and $\alpha_2\alpha_1\alpha_5 \in S(Y)$, we obtain,

$$x_{451} + x_{15} = p_2. \tag{3}$$

Let τ be the orientation reversing involution of $\Sigma_{1,2}$ fixing each α_i's (see Figure 6.1). Then $\tau(\alpha\beta) = \tau(\beta)\tau(\alpha)$ for all $\alpha \perp \beta$ or $\alpha \perp_0 \beta$.





The pentagon relation | The orientation reversing involution

FIGURE 6.1

Apply τ to (3), we obtain

$$x_{154} + x_{51} = p_3. \tag{4}$$

Since $\alpha_4 \perp_0 \alpha_1\alpha_5$ so that $\partial(\alpha_4, \alpha_1\alpha_5) = \alpha_1\alpha_2 \in S(X)$, we obtain,

$$x_{415} + x_{154} + a_4 x_{15} = p_4. \tag{5}$$

Since $\alpha_4 \cap \alpha_1 = \emptyset$, it follows that

$$x_{415} = x_{145}. \tag{6}$$

Finally, since $\alpha_1 \perp \alpha_4\alpha_5$, by (6) we obtain,

$$x_{145} + x_{451} - a_1 x_{45} = 0. \tag{7}$$

Subtracting (7) by (5) and using (6), we obtain

$$x_{451} - x_{154} - a_1 x_{45} - a_4 x_{15} = p_5. \tag{8}$$

The subtraction (4) by (3) gives,

$$x_{154} - x_{451} + x_{51} - x_{15} = p_6. \tag{9}$$

The sum of (8) and (9) gives

$$x_{51} - x_{15} - a_1 x_{45} - a_4 x_{15} = p_7. \tag{10}$$

Using (1), we simplify (10) and obtain

$$a_1 x_{45} + (2 + a_4) x_{15} - a_1 x_5 = p_8. \tag{11}$$

Since (11) holds for any 5-element set $\{\alpha_1', ..., \alpha_4', \alpha_5'\}$ forming a pentagon so that $\alpha_1' = \alpha_1$ and $\alpha_4' = \alpha_4$, it holds for the set

$$\{\alpha_1, \alpha_2\alpha_1, \alpha_3, \alpha_4, \alpha_5\alpha_1\} = \{\alpha_1', \alpha_2', \alpha_3', \alpha_4', \alpha_5'\}.$$

Now $h(\alpha_1'\alpha_5') = h(\alpha_1\alpha_5\alpha_1) = h(\alpha_5) = x_5$, $h(\alpha_5') = h(\alpha_5\alpha_1) = x_{51} = a_1x_5 - x_{15}$ and $h(\alpha_4'\alpha_5') = h(\alpha_4\alpha_5\alpha_1) = x_{451} = -x_{15}+p_2$. Thus equation (11) for this new pentagon set gives,

$$a_1(-x_{15} + p_2) + (2 + a_4)x_5 - a_1(a_1x_5 - x_{15}) = p_9,$$

which is

$$(2 + a_4 - a_1^2)x_5 = p_9.$$

Thus x_5 can be solved uniquely. Now take $h = f$ and $h = g$. We see that $f(\alpha_5) = g(\alpha_5)$. By Corollary 3.7, it follows that $f = g$.

6.3. Suppose now that f is a trace function so that $f^2(\alpha_1) = f(\alpha_4) + 2$ for all separating α_4 and non-separating α_1 with $\alpha_1 \cap \alpha_4 = \emptyset$.

We begin with the following lemma.

Lemma. *Let $\Sigma = \Sigma_{g,n}$ be a surface of level at least 2 so that $g \geq 1$ and f is a K-valued trace function on $S(\Sigma)$. Let $P(\Sigma) = \{(\alpha, \beta) \in S(\Sigma) | \alpha$ bounds a $\Sigma_{1,1}$ in Σ and β is a non-separating class lying in $\Sigma_{1,1}\}$.*

(a) If for all $(\alpha, \beta) \in P(\Sigma)$, $f^2(\beta) = f(\alpha) + 2$, then either for all $(\alpha, \beta) \in P(\Sigma)$, $(f(\alpha), f(\beta)) = (-2, 0)$ or for all $(\alpha, \beta) \in P(\Sigma)$, $(f(\alpha), f(\beta)) = (2, \pm 2)$.

(b) If there exists a pair $(\alpha, \beta) \in P(\Sigma)$ so that $f^2(\beta) \neq f(\alpha) + 2$, then there exits a pair $(\alpha', \beta') \in P(\Sigma)$ so that $f^2(\beta') \neq f(\alpha') + 2$ and one of $f^2(\alpha')$ or $f^2(\beta')$ is not 4.

Proof. To prove (a), fix $\alpha \in S'(\Sigma)$ which bounds a $\Sigma_{1,1}$ and let P_α be the set of all non-separating classes β lying in $\Sigma_{1,1}$. Take three elements β_1, β_2 and β_3 in P_α forming a triangle in the modular configuration. Let $f(\alpha) + 2$ be μ^2. Then $f^2(\beta_i) = \mu^2$. By Proposition 3.4(a), we obtain $3\mu^2 \pm \mu^3 = \mu^2$. Thus either $\mu = 0$ or $\mu^2 = 4$, i.e., either $(f(\alpha), f(\beta)) = (-2, 0)$ for all $\beta \in P_\alpha$ or $(f(\alpha), f(\beta)) = (2, \pm 2)$ for all $\beta \in P_\alpha$.

To finish the proof of (a), we need to show that the above two cases cannot occur simultaneously. The above proof shows that being $(f(\alpha), f(\beta)) = (2, \pm 2)$ or $(-2, 0)$ depends only on the 1-holed torus bounded by α. Thus part (a) follows from Corollary 3.8(b).

To prove part (b), we may assume that the characteristic of the field K is not 2 (otherwise by part (a) the result follows). Now suppose otherwise that for all $(\alpha, \beta) \in P(\Sigma)$ so that $f^2(\beta) \neq f(\alpha) + 2$, we have $f^2(\alpha) = f^2(\beta) = 4$. This implies that $f(\alpha) = -2$. Consider the level-1 subsurface $\Sigma_{1,1}$ bounded by α which contains β. Let β, β_2 and β_3 be three classes in $S'(\Sigma_{1,1})$ which form a triangle in the modular configuration. Then for $i = 2, 3$, either $f^2(\beta_i) = 4$ (if $f^2(\beta_i) \neq f(\alpha) + 2$) or $f^2(\beta_i) = 0$ (if $f^2(\beta_i) = f(\alpha) + 2$). By Proposition 3.4(a), we have $f^2(\beta) + f^2(\beta_2) + f^2(\beta_3) - f(\beta)f(\beta_2)f(\beta_3) = f(\alpha) + 2$. Thus $4 + f^2(\beta_2) + f^2(\beta_3) = \pm 2f(\beta_2)f(\beta_3)$. But this is impossible since either $f^2(\beta_i) = 4$ or 0 for $i = 2, 3$. q.e.d.

6.4. Let $\partial\Sigma_{1,2}$ be b_1 and b_2. By Proposition 6.2 and Lemma 6.3, it remains to prove the following.

Proposition. *Suppose f is a K valued trace function on $S(\Sigma_{1,2})$ so that either (a) for all $(\alpha, \beta) \in P(\Sigma_{1,2})$, $(f(\alpha), f(\beta)) = (-2, 0)$ or (b) for all $(\alpha, \beta) \in P(\Sigma_{1,2})$, $(f(\alpha), f(\beta)) = (2, \pm 2)$. Then f is a character.*

6.5. We construct a representation whose character is f satisfying condition (a) in the Proposition 6.4 in this section.

Lemma. *Let $\partial\Sigma_{1,2}$ be $b_1 \cup b_2$. Under the assumption of Proposition 6.4(a), we have $f^2(b_i) = 4$ and $f(b_1)f(b_2) = -4$.*

Proof. Take $(\alpha_1, \beta_1) \in P(\Sigma_{1,1})$ and let $\Sigma_{0,4}$ and $\Sigma_{1,1}$ be the subsurfaces bounded by β_1 and α_1. Take β_2 to be a non-separating class lying in $\Sigma_{0,4}$ so that $\beta_2 \perp_0 \alpha_1$. Then $\alpha_1\beta_2$ and $\beta_2\alpha_1$ are both non-separating. By the assumption, $f(\beta_i) = f(\beta_2\alpha_1) = f(\alpha_1\beta_2) = 0$ and $f(\alpha) = -2$. By Proposition 3.5(a) applied to $\Sigma_{0,4}$ with respect to the triangle $(\alpha_1, \beta_2, \alpha_1\beta_2)$, we obtain $f(b_1) + f(b_2) = 0$. By Proposition 3.5(b) applied to $\Sigma_{0,4}$ with respect to $\{\alpha_1 = \beta_2(\alpha_1\beta_2), (\alpha_1\beta_2)\beta_2\}$, we obtain $f(b_1)f(b_2) = -4$. Thus the result follows. q.e.d.

Here is a construction of a representation $\rho : \pi_1(\Sigma_{1,2}) \to SL(2, K)$ whose character is f. For simplicity, let $f(b_1) = 2$ and $f(b_2) = -2$. Let $\Sigma_{1,1}$ be obtained by attaching a disc to the b_1 boundary component of the surface $\Sigma_{1,2}$. By Corollary 3.4, there is a representation $\rho_0 : \pi_1(\Sigma_{1,1}) \to SL(2, K)$ so that $tr(\rho_0(\alpha)) = 0$ for all $\alpha \in S'(\Sigma_{1,1})$ and $tr(\rho_0(b_2)) = -2$. Define $\rho = \rho_0 \circ i$ where $i : \pi_1(\Sigma_{1,2}) \to \pi_1(\Sigma_{1,1})$ is the homomorphism induced by the inclusion map. Since i sends non-separating classes to non-separating classes, it follows that the character of ρ is f.

6.6. We construct a representation whose character is the trace function f satisfying condition (b) in the Proposition 6.4.

We may assume that the characteristic of the field K is not 2 in this section (otherwise it is covered by §6.5).

Lemma. *Under the assumption of Proposition 6.4 (b), we have $f(b_1) = f(b_2) = \pm 2$. In particular, f is reducible over all level-0 subsurfaces.*

Proof. Since $f(\alpha) = 2$ for all separating classes α, f is reducible on all 1-holed tori. In particular, if $\alpha_1 \perp \alpha_2$, then by Proposition 3.4(a), $f(\alpha_1\alpha_2) = \frac{1}{2}f(\alpha_1)f(\alpha_2)$. Thus $f(\alpha_1\alpha_2) = f(\alpha_2\alpha_1)$. On the other hand, if β and γ are two non-separating classes so that $\beta \perp_0 \gamma$, then there exists three non-separating classes $\delta_1 \perp \delta_2 \perp \delta_3$ so that $\beta = \delta_1\delta_2\delta_3$ and $\gamma = \delta_3\delta_2\delta_1$. (Indeed, the pair (β, γ) is unique up to the homeomorphism of the surface). Thus we have $f(\beta) = f(\gamma)$.

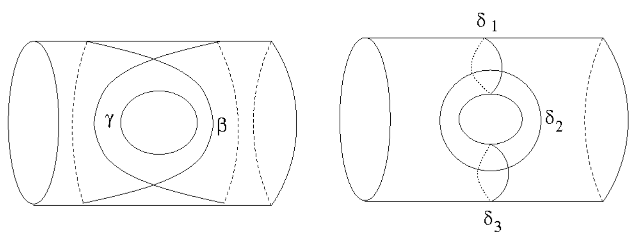


FIGURE 6.2

Now take two classes $\alpha_1 \perp \alpha_2$. Since $f(\alpha_1)f(\alpha_2)f(\alpha_1\alpha_2) = 8$, we may assume that $f(\alpha_1) = 2$. Let $\Sigma_{0,4}$ be the subsurface bounded by α_1 and let $\alpha_3 \perp_0 \alpha_4$ be two non-separating classes lying in $\Sigma_{0,4}$. Then both $\alpha_5 = \alpha_3\alpha_4$ and $\alpha_5' = \alpha_4\alpha_3$ are separating classes. By the observation above, $f(\alpha_3) = f(\alpha_4)$ and $f(\alpha_5) = f(\alpha_5') = 2$. By Proposition 3.5(a) applied to the triangle $(\alpha_3, \alpha_4, \alpha_5)$ in $\Sigma_{0,4}$, we obtain $f(b_1)+f(b_2) = \pm 4$. By Proposition 3.5(b) applied to α_5, α_5', we obtain $f(b_1)f(b_2) = 4$. Thus $f(b_1) = f(b_2) = \pm 2$. Since $f(\alpha) = 2$ for all separating classes, this shows that f is reducible on all level-0 subsurface bounded by α. Thus f is reducible on all level-0 subsurfaces. q.e.d.

We now construct a diagonalizable representation ρ of $\pi_1(\Sigma_{1,2})$ whose character is f as follows.

Take $(\alpha, \beta) \in P(\Sigma_{1,2})$ and let X and Y be the level-1 subsurfaces bounded by α and β respectively. By the same argument as in §6.2, we construct a diagonalizable $SL(2,K)$ representation of $\pi_1(\Sigma_{1,2})$ so that its character g equals f on $S(X) \cup S(Y)$. To show that $g = f$, by Corollary 3.7, it suffices to prove $f(\gamma) = g(\gamma)$ for all $\gamma \perp_0 \alpha$ and $\gamma \perp \beta$. Let δ be a class disjoint from α and γ. Then $f(\delta) = g(\delta)$. By the reducibility of f and g on the level-0 subsurface bounded by δ, γ, it follows that $f(\gamma) = \frac{1}{2}f(b_1)f(\delta) = \frac{1}{2}g(b_1)g(\delta) = g(\gamma)$. q.e.d.

7. The Proof of Theorem 1.2

The goal of this section is to prove Theorem 1.2 for surfaces of positive genus by using the induction on the level of the surface.

Let $\Sigma = \Sigma_{g,n}$ be a surface of positive genus and f a K-valued trace function defined on $S(\Sigma)$. Recall that $P(\Sigma)$ is defined to be $\{(\alpha, \beta) \in S(\Sigma_{g,n}) \times S(\Sigma_{g,n}) |\ \alpha$ bounds a $\Sigma_{1,1}$ and β is a non-separating class lying in the subsurface $\Sigma_{1,1}\}$. The proof breaks into the following two cases: (a) there exists $(\alpha, \beta) \in P(\Sigma)$ so that $f^2(\beta) \neq f(\alpha) + 2$, and (b) for all $(\alpha, \beta) \in P(\Sigma)$ $f^2(\beta) = f(\alpha) + 2$. By Lemma 6.3, case (b) is equivalent to two subcases (b1) for all $(\alpha, \beta) \in P(\Sigma)$, $(f(\alpha), f(\beta)) = (-2, 0)$ and (b2) for all $(\alpha, \beta) \in P(\Sigma)$, $(f(\alpha), f(\beta)) = (2, \pm 2)$.

We will deal with these three cases (a), (b1) and (b2) separately in the following sections.

7.1. Suppose the case (a) occurs. By Lemma 6.3(b), we may assume that one of $f^2(\alpha)$ or $f^2(\beta)$ is not 4. Let X be the level-1 subsurface bounded by α and let Y be the subsurface $\Sigma - int(N(\beta))$. Since $f|_{S(Y)}$ takes some values other than ± 2, by the induction hypothesis if $g \geq 2$ and by the result in §5 if $g = 1$, $f|_{S(Y)}$ is a character, say $f|_{S(Y)} = \chi_{\rho_Y}$ for an $SL(2, K)$ representation ρ_Y of $\pi_1(Y)$. Now if $f(\alpha) \neq 2$, then both $f|_{S(X)}$ and $f|_{S(X \cap Y)}$ are irreducible. Let $\rho_X : \pi_1(X) \to SL(2, K)$ be any representation whose character is $f|_{S(X)}$. By Lemma 2.4, we may assume after a conjugation that that $\rho_X|_{\pi_1(X \cap Y)} = \rho_Y|_{\pi_1(X \cap Y)}$. If $f(\alpha) = 2$, then both $f|_{S(X)}$ and $f|_{S(X \cap Y)}$ are reducible. Since one of $f^2(\alpha)$ or $f^2(\beta)$ is not 4, by Lemma 2.6, there exist exactly two $SL(2, K)$ conjugacy classes of representations of $\pi_1(X)$ whose characters are $f|_{S(X)}$. Thus we may choose an $SL(2, K)$ representation ρ_X of $\pi_1(X)$ so that $\rho_X|_{\pi_1(X \cap Y)} = \rho_Y|_{\pi_1(X \cap Y)}$ and $\chi_{\rho_X} = f|_{S(X)}$.

Define a representation $\rho : \pi_1(\Sigma) \to SL(2, K)$ by $\rho|_{\pi_1(X)} = \rho_X$ and $\rho|_{\pi_1(Y)} = \rho_Y$. Let g be the character of ρ. Then $g|_{S(X) \cup S(Y)} = f|_{S(X) \cup S(Y)}$.

To show that $g = f$, by Corollary 3.7, it suffices to prove $f(\gamma) = g(\gamma)$ for all $\gamma \perp_0 \alpha$ and $\gamma \perp \beta$. Given such γ, consider the level-2 subsurface $\Sigma_{1,2}$ containing both X and γ. Then by the proof of Proposition 6.2 applied to $\Sigma_{1,2}$ with respect to the decomposition X and $Y \cap \Sigma_{1,2}$, it follows that $g(\gamma) = f(\gamma)$.

7.2. To show the remaining cases, we need

Lemma. *Suppose f is a trace function on $\Sigma_{g,r}$ so that the case (b1) or (b2) holds. Then f is reducible on all level-0 subsurfaces.*

Proof. Since each level-0 subsurface is contained in a 3-holed torus subsurface, it suffices to prove the lemma for the 3-holed torus $\Sigma_{1,3}$. Each level-0 subsurface in $\Sigma_{1,3}$ is either contained in a 2-holed torus subsurface or is bounded by a boundary class. By §6.5 and §6.6, those level-0 subsurfaces contained in a 2-holed torus are reducible. It remains to show the reducibility of the level-0 subsurface $\Sigma_{0,3}(\gamma)$ bounded by a boundary class γ and two boundary components b_1 and b_2 of $\Sigma_{1,3}$. Take disjoint non-separating classes γ_1 and γ_2 so that $\gamma_i \cap \gamma = \emptyset$ for $i = 1, 2$ and take γ_3 so that $\gamma_3 \cap \gamma_i = \emptyset$ for $i = 1, 2$ and $\gamma_3 \perp_0 \gamma$. Note that γ_3, $\gamma_3\gamma$ and $\gamma\gamma_3$ are non-separating classes.

In the case (b1), $(f(\alpha), f(\beta)) = (-2, 0)$ for all $(\alpha, \beta) \in P(\Sigma_{1,3})$. By Proposition 3.5(a) applied to the level-1 subsurface bounded by γ_1, γ_2 and the triangle $(\gamma, \gamma_3, \gamma\gamma_3)$, we obtain $f^2(b_1) + f^2(b_2) + f^2(\gamma) - f(\gamma)f(b_1)f(b_2) - 4 = 0$. Thus by Proposition 3.3, $\Sigma_{0,3}(\gamma)$ is reducible.

In the case (b2), we may assume that the characteristic of K is not 2 (otherwise the result is clear). Now both $\Sigma_{0,3}(\gamma_3, \gamma_i)$, $i = 1, 2$ and $\Sigma_{0,3}(\gamma_1, \gamma_2, \gamma)$ are reducible since they lie in some 2-holed torus subsurfaces. Thus, by Corollary 3.3, $f(b_i) = \frac{1}{2} f(\gamma_3)f(\gamma_i)$, $i = 1, 2$ and $f(\gamma) = \frac{1}{2} f(\gamma_1)f(\gamma_2)$. This implies that $f(b_1)f(b_2)f(\gamma) = 8$. By Corollary 3.3, this shows that $\Sigma_{0,3}(\gamma)$ is reducible. q.e.d.

7.3. We now show that in the case (b1) or (b2), the trace function f is a character.

Take $(\alpha, \beta) \in P(\Sigma)$ and let X be the $\Sigma_{1,1}$ subsurface bounded by α and let Y be the subsurface bounded by β.

If $(f(\alpha), f(\beta)) = (2, \pm 2)$, then by Lemma 7.2, we construct a diagonalizable representation ρ of $\pi_1(\Sigma)$ so that its character $g = \chi_\rho$ satisfies $g|_{S(X)\cup S(Y)} = f|_{S(X)\cup S(Y)}$. To show that $f = g$, by Corollary 3.7, it suffices to prove $f(\gamma) = g(\gamma)$ for all $\gamma \perp_0 \alpha$ and $\gamma \perp \beta$. Consider the level-2 subsurface $\Sigma_{1,2}$ containing X and γ. Since both f and g are reducible on all level-0 subsurfaces, by the proof of Proposition 6.2, it follows that $f(\gamma) = g(\gamma)$.

If $(f(\alpha), f(\beta)) = (2, 0)$ and the characteristic of K is not 2, we note that the genus g of $\Sigma_{g,n}$ must be 1. Indeed, if $g \geq 2$, then there exists essential subsurface $\Sigma_{1,2}$ whose boundary components β_i, $i = 1, 2$ are non-separating simple loops in $\Sigma_{g,n}$. By the assumption $f(\beta_i) = 0$. But by Lemma 6.5 applied to $\Sigma_{1,2}$, we have $f(\beta_i) = \pm 2$ which is a contradiction.

We now construct a representation as follows. Let $\partial\Sigma_{1,n}$ be $b_1, ..., b_n$ and let $i : \pi_1(\Sigma_{1,n}) \to \pi_1(\Sigma_{1,1})$ be the homomorphism induced by the inclusion map $j : \Sigma_{1,n} \to \Sigma_{1,1}$ so that $j(b_n) = \partial\Sigma_{1,1}$. Let $\rho' : \Sigma_{1,1} \to SL(2, K)$ be a representation so that $\chi_{\rho'}(\alpha) = 0$ for all non-separating class α and $\chi_{\rho'}(\partial\Sigma_{1,1}) = -2$ (see Corollary 3.4). Let $\rho_0 = \rho' \circ i$ be a representation of $\pi_1(\Sigma_{1,n})$. The fundamental group $\pi_1(\Sigma_{1,n})$ is a free group on $(n+1)$ generators $x_1, ..., x_{n+1}$ where $x_1, ..., x_{n-1}$ correspond to the boundary components $b_1, ..., b_{n-1}$. Now modify ρ_0 to produce a new representation ρ of $\pi_1(\Sigma_{1,n})$ by redefining $\rho(x_i) = \pm\rho_0(x_i)$ so that $\chi_\rho(x_i) = f(b_i)$ for $i = 1, 2, ..., n-1$. Let g be the character of ρ defined on $S(\Sigma_{1,n})$. Then g satisfies $(g(\alpha), g(\beta)) = (-2, 0)$ for all $(\alpha, \beta) \in P(\Sigma_{1,n})$ (indeed each non-separating loop in $\Sigma_{1,n}$ becomes a non-separating loop in $\Sigma_{1,1}$). Furthermore, by Lemma 7.2, the character g is reducible over all level-0 subsurfaces. We prove that $f = g$ by induction on n. The result follows for $n = 1, 2$. We first claim that $f(b_n) = g(b_n)$. To see this, take a boundary class α' so that $\Sigma_{0,3}(\alpha')$ contains b_n and b_{n-1}. By the induction hypothesis applied to the subsurface $\Sigma_{1,n-1}$ bounded by α', we conclude that $f(\alpha') = g(\alpha')$. By the reducibility of f and g on $\Sigma_{0,3}(\alpha')$ and $f(b_{n-1}) = g(b_{n-1})$, it follows that $f(b_n) = g(b_n)$. Now for any separating γ, let Σ' be the planar subsurface bounded by Σ' in $\Sigma_{1,n}$. Since f and g are both reducible on all level-0 subsurfaces, f and g are reducible on Σ'. Furthermore, f and g have the same values on all but one boundary component γ of Σ'. Thus, by the reducibility, $f(\gamma) = g(\gamma)$. q.e.d.

8. Proof of Theorem 1.1

We begin with the following special case of Theorem 1.1.

8.1. Proposition. *Suppose K is a quadratically closed field and $f : G \to K$ is a K-trace function defined on a finitely generated group G. Then f is the character of an $SL(2, K)$ representation of the group.*

Proof. We first show that the result holds for $G = F_n$, the free group on n generators. Consider F_n as the fundamental group $\pi_1(\Sigma_{1,n-1})$ of the genus 1 surface with $n-1$ boundary components. Then by the work of [14], f induces a trace function, denoted by f', defined on $S(\Sigma_{1,n-1})$. By Theorem 1.2, there exists a representation ρ of the fundamental group $\pi_1(\Sigma_{1,n-1})$ to $SL(2, K)$ whose character is f'. Thus $\chi_\rho(x) = f(x)$ for each $x \in \pi_1(\Sigma_{1,n-1})$ which has a simple loop representative. Now by the remark following Corollary 2.2, f is the character of ρ on G.

For the general n-generator group G, we follow an observation of Gonzàlez-Acuña and Montesinos-Amilibia [11]. Let $\phi : F_n \to G$ be an epimorphism with $ker(\phi) = H$. Then $g = f \circ \phi$ is a K-trace function defined on F_n. Thus there exists a representation ρ of F_n whose character is g. Furthermore, by the construction $tr\rho(x) = 2$ for all $x \in H$. Now we use the following lemma of [11].

Lemma ([11]). *Suppose $\rho : F_n =< x_1, ..., x_n >\rightarrow SL(2, K)$ is a representation and $x \in F_n$ so that $tr\rho(x) = 2$ and $tr(\rho([x, x_i])) = 2$ for all i. Then either $\rho(x) = id$ or ρ is reducible.*

Indeed, if $\rho(x) \neq id$, then $\rho(x)$ has a unique eigenspace in K^2. But $tr(\rho([x, x_i]) = 2$ shows that this eigenspace is invariant under all $\rho(x_i)$. Thus ρ is reducible.

By the lemma, if ρ is irreducible, then $\rho(x) = id$ for all $x \in H$. In particular, the representation ρ induces a representation ρ' of G to $SL(2, K)$ whose character is f. If ρ is reducible, we may replace ρ by its diagonalization ρ' without changing the character. Now $tr(\rho'(x)) = 2$ if and only if $\rho'(x) = id$. Thus the same argument goes through and we construct a representation whose character is f. q.e.d.

8.2. We now prove Theorem 1.1 for any group G. Let f be a K-trace function defined on G. We shall consider the following three cases: (1) there exist $x, y \in G$ so that $f([x, y]) \neq 2$, (2) for all $x, y \in G$, $f([x, y]) = 2$ but there exists $t \in G$ so that $f(t) \neq \pm 2$, (3) for all $x, y \in G$, $f([x, y]) = 2$ and $f(x) = \pm 2$.

In the first case, consider the subgroup $< x, y >$ and restriction $f|_{<x,y>}$. By Lemma 2.3, there exists an irreducible representation ρ_0 of $< x, y >$ whose character is $f|_{<x,y>}$. Given any element $z \in G$, consider the subgroup $< x, y, z >$ and the restriction $f|_{<x,y,z>}$. By Lemma 2.3, there exists a representation $\rho :< x, y, z >\rightarrow SL(2, K)$ so that its character is $f|_{<x,y,z>}$. Both $\rho|_{<x,y>}$ and ρ_0 have the same character and both are irreducible. By Lemma 2.4, we may assume after conjugating ρ by an element in $SL(2, K)$ so that $\rho|_{<x,y>} = \rho_0$. We denote this representation by $\rho_z :< x, y, z >\rightarrow SL(2, K)$. Note that since ρ_z is irreducible, ρ_z is unique. Now define a map $\mu : G \rightarrow SL(2, K)$ by $\mu(z) = \rho_z(z)$. Clearly $tr(\mu(z)) = f(z)$ by the construction. We claim that μ is a representation. Indeed, given $z_1, z_2 \in G$, consider the 4-generator subgroup $< x, y, z_1, z_2 >$ and the restriction $f|_{<x,y,z_1,z_2>}$. By Proposition 8.1, there exists a representation $\delta :< x, y, z_1, z_2 >\rightarrow SL(2, K)$ whose character is $f|_{<x,y,z_1,z_2>}$. By Lemma 2.4, we may assume after conjugating by an element in $SL(2, K)$ that $\delta|_{<x,y>} = \rho_0$. Thus we obtain $\rho_{z_i} = \delta|_{<x,y,z_i>}$ for $i = 1, 2$. In particular this implies that $\mu(z_1 z_2) = \mu(z_1)\mu(z_2)$.

In the case (2), we consider the subgroup $< x, y >$ where $y = x$ so that $f^2(x) \neq 4$. Let ρ_0 be a diagonal representation of $< x, y >$ whose character is $f|_{<x,y>}$. Note that the assumption $f([a,b]) = 2$ implies the reducibility of the representations on all 2-generator subgroup. We go through the same argument as in the previous paragraph by taking all representations ρ_z, δ to be diagonalizable. Since $f^2(x) \neq 4$, by Lemma 2.6, these representations are unique. Thus the result follows.

Finally in the case (3), we have $f(x) = \pm 2$ and $f([x, y]) = 2$ for all $x, y \in G$. If the characteristic of K is 2, then $f = 0$ and f is the character of the trivial representation. If the characteristic of K is not 2, then by Lemma 2.2 (b), we obtain $f(xy) = f(x)f(y)/2$. Define a representation μ of G by $\mu(x) = \begin{pmatrix} f(x)/2 & 0 \\ 0 & f(x)/2 \end{pmatrix}$. Then the character of μ is f.

q.e.d.

Remark. As the proof shows, Theorem 1.1 follows as long as one establishes Theorem 1.1 for the free group on 4 generators.

9. Some questions

There are some questions arising from the above considerations concerning the finite presentations. It is known that there exists a finite set $F \subset S(\Sigma)$ so that the Teichmuller space (respectively the space of measured laminations) of Σ is defined by the restrictions of the length functions to F subject to a finite set of polynomial equations supported in level-1 subsurfaces. The analogous question for the mapping class group of the surface seems to be open. Namely, whether the mapping class group has a finite presentation whose generators are finitely many Dehn twists and whose relations (in these generators) are supported in level-1 subsurfaces. A recent work of Gervais [8] shows that one can find a finite set of Dehn twists generating the mapping class group so that the relations (in these generators) are supported in level-3 subsurfaces. Motivated by these, it is natural to ask if there exists a finite set $F \subset S(\Sigma)$ so that $SL(2, K)$ characters are determined by their restrictions to F subject to polynomial equations supported in level-1 subsurfaces. The proofs in §4 and §6 strongly suggest that the answer is affirmative. If the answer is positive, it also implies that the character variety of any finitely generated group can be defined by the restrictions of the characters to a finite set of group elements subject to polynomial equations supported in 3-generator subgroups. The work of [11] shows that the one can take the equations to be supported in 5-generator subgroups.

There are several other related problems which seem to be interesting. The first question is that given a topological group and a complex valued continuous trace function on the group, is it the character of a continuous $SL(2,\mathbf{C})$ representation of the group? The second question is whether Theorem 1.1 remains true for the characters of $GL(n,\mathbf{C})$ representations. To be more precise, suppose f is a complex valued function defined on the fundamental group of a surface so that the restriction of the function to each level-1 subsurface group is a $GL(2,\mathbf{C})$ character. Is there a $GL(n,\mathbf{C})$ representation ρ of the fundamental group so that $tr(\rho(x)) = f(x)$ for all x lying in some level-1 subsurface? The third question is motivated by Royden's theorem [30] for the Teichmüller spaces. Suppose ϕ is an algebraic automorphism of the $SL(2,\mathbf{C})$ character variety of a surface group preserving the peripheral structure. Is ϕ induced by a self-homeomorphism of the surface? Finally the analogous result to Jorgensen's discreteness criterion seems to be the following. Suppose ρ is a faithful representation of a closed surface group to $SL(2,\mathbf{C})$ so that ρ is discrete when restricted to each level-1 subsurface group. Is ρ discrete?

Appendix : A Proof of Lemma 2.3

Lemma 2.3. *Suppose K is a field in which all quadratic equations with coefficients in K have roots in K. Given six numbers x_1, x_2, x_3, x_{12}, x_{23} and x_{31} in K, there exist three matrices A_1, A_2, and A_3 in $SL(2,K)$ so that $trA_i = x_i$ and $trA_iA_j = x_{ij}$, for $i = 1,2,3$ and $(i,j) = (1,2),(2,3)$, and $(3,1)$.*

Proof. We divide the proof into three cases: in case 1, some $x_i \neq 2$, in case 2, some $x_{ij} \neq \pm 2$, and in case 3, all x_i's and x_{ij}'s are ± 2.

Case 1. Some $x_i \neq \pm 2$, say $x_1 \neq \pm 2$. Choose λ in K so that $x_1 = \lambda + \lambda^{-1}$. Clearly $\lambda \neq \pm 1$. Let $A_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $A_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $A_3 = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ be three $SL(2)$ matrices. We will find a,b,c,d,x,y,z,w in K solving the trace equations. By $trA_2 = x_2$ and $trA_1A_2 = x_{12}$, we obtain $a + d = x_2$ and $\lambda a + \lambda^{-1} d = x_{12}$. Because $\lambda \neq \pm 1$, we can solve this system of linear equations uniquely in a, d in K. Similarly, by $trA_3 = x_3$ and $trA_3A_1 = x_{31}$, we also solve x, w uniquely in K. It remains to find b, c, y, z in K so that $bc = ad - 1$, $yz = xw - 1$ and

$trA_2A_3 = x_{23}$, i.e., $cy + bz = x_{23} - ax - dw$. If $ad - 1 \neq 0$, i.e., $bc \neq 0$, choose $b = 1$. Let $c = ad-1$. Now, due to $bc \neq 0$, $cy+bz = x_{23}-ax-dw$ and $yz = xw - 1$ can be solved in terms of y, z. If $ad - 1 = 0$, there are two more subcases: $p = x_{23} - ax - dw \neq 0$ or $p = 0$. If $p = 0$, we take $b = c = 0$ and choose any pair y, z so that $yz = xw - 1$. If $p \neq 0$, choose $b = 1$, $c = 0$. Then we have $z = p \neq 0$ and $y = (xw - 1)/p$. Thus, in all cases, we find three matrices in $SL(2, K)$ satisfying the trace equations.

Case 2. Some $x_{ij} \neq \pm 2$, say $x_{12} \neq \pm 2$. Then by case 1 applied to the six ordered numbers $\{x_{12}, x_2, x_2x_3 - x_{23}, x_1, x_{31}, x_2^2x_3 - x_2x_{23} - x_3\}$, we find three $SL(2, K)$ matrices B_1, B_2, and B_3 so that $trB_1 = x_{12}$, $trB_2 = x_2$, and $trB_3 = x_2x_3 - x_{23}$, $trB_1B_2 = x_1$, $trB_1B_3 = x_{13}$ and $trB_2B_3 = x_2^2x_3-x_2x_{23}-x_3$. (Indeed, we take $B_1 = A_1A_2, B_2 = A_2^{-1}$ and $B_3 = A_2^{-1}A_3$ to find the six numbers). Now let $A_1 = B_1B_2, A_2 = B_2^{-1}$ and $A_3 = B_2^{-1}B_3$. By the basic trace identity (Lemma 2.2(a)), it follows that $trA_i = x_i$ and $trA_iA_j = x_{ij}$.

Case 3. All x_i's and x_{ij}'s are ± 2. First we note that if $trA_i = x_i$ and $trA_iA_j = x_{ij}$, then $(-A_1, A_2, A_3)$ solves the problem for the six numbers $\{-x_1, x_2, x_3, -x_{12}, x_{23}, -x_{31}\}$. Thus, by changing the signs of x_i's if necessary, we may assume that $x_1 = x_2 = x_3 = 2$. There are four cases for $(x_1, x_2, x_2, x_{12}, x_{23}, x_{31})$ up to symmetry: $(2, 2, 2, 2, 2, 2)$, $(2, 2, 2, -2, 2, 2)$, $(2, 2, 2, -2, -2, 2)$ and $(2, 2, 2, -2, -2, -2)$. The corresponding solutions are listed below.

For $(2, 2, 2, 2, 2, 2)$, a solution (A_1, A_2, A_3) is (id, id, id). For $(2, 2, 2, -2, 2, 2)$, a solution is

$$\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right).$$

For $(2, 2, 2, -2, -2, 2)$, a solution is

$$(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}).$$

Finally for $(2, 2, 2, -2, -2, -2)$, a solution is

$$(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}).$$

q.e.d.

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