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GREEN FUNCTIONS AND CONFORMAL GEOMETRY

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Abstract

We use the Green function of the Yamabe operator (conformal Laplacian) to construct a canonical metric on each locally conformally flat manifold different from the standard sphere that supports a Riemannian metric of positive scalar curvature. In dimension 3, the assumption of local conformal flatness is not needed. The construction depends on the positive mass theorem of Schoen-Yau. The resulting metric is different from those obtained earlier by other methods. In particular, it is smooth and distance nondecreasing under conformal maps. We analyze the behavior of our metric if the scalar curvature tends to 0. We demonstrate that the canonical metrics converge under surgery-type degenerations to the corresponding metric on the limit space. As a consequence, the L^2-metric on the moduli space of scalar positive locally conformally flat structures is not complete. The example of $S^1 \times S^2$ as underlying manifold is studied in detail.

Introduction

For the sake of simplicity, we assume throughout this introduction that all occurring manifolds are compact.

In understanding spaces of complex structures, it has proved to be useful to construct "canonical" metrics on complex manifolds. Such a "canonical" metric ideally is uniquely determined by the underlying complex structure, depends smoothly on that structure, and has an analyzable behavior as the underlying structure degenerates in some explicit manner. Such a metric on each complex structure then gives rise to a metric on the corresponding moduli space[1] by taking the L^2-product of tangent vectors to the moduli space — which can be expressed as harmonic sections of a certain bundle on the underlying complex manifold — w.r.t. the canonical metric.

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[1]Leaving aside the issue of smoothness of that space

The best known example is the hyperbolic metric on a compact Riemann surface of genus greater than 1 that gives rise to the Weil-Petersson metric on the moduli space. Here, the existence of the hyperbolic metric follows from the Poincaré uniformization theorem, uniqueness and smooth dependence are common knowledge of researchers, and the asymptotic behavior was analyzed by H. Masur [19]. There also exist other canonical metrics on compact Riemann surfaces that enjoy similar properties, for example the Bergman and Arakelov metrics, resp. The Arakelov metric is defined in terms of the asymptotic behavior of a Green function near its singularity, see [1], and this construction will serve as a paradigm for us below.

In higher dimensions, such canonical metrics include the Kobayashi metric which however typically lacks certain smoothness properties, as well as the Kähler-Einstein metrics constructed by Yau [31], Aubin [3], and others. In particular, Yau's solution of the Calabi conjecture and the resulting construction of Kähler-Einstein metrics on $K3$ surfaces and Calabi-Yau manifolds were decisive for understanding the moduli spaces of such complex manifolds.

In the present paper, we start to investigate moduli spaces of conformal structures on the basis of a similar principle, i.e., by introducing and studying canonical metrics associated to conformal structures. Our metric is not the first such metric associated to a conformal structure. For example, by the solution of the Yamabe problem achieved through the work of Trudinger [29], Aubin [4], and Schoen [22], each conformal structure on a compact manifold supports a metric of constant scalar curvature. In the case of positive scalar curvature, however, that metric in general is not unique. See Schoen [23] for a detailed example. Other examples that, however, only work for locally conformally flat structures, include a Kobayashi type metric introduced by Apanasov [2] and Kulkarni-Pinkall [16], as well as Nayatani's metric [20] using the Green function on S^n via the developing map for locally conformally flat structures (see Schoen-Yau [27]).

The construction of a canonical metric typically starts with some Riemannian metric in a given conformal class, but in order for the resulting metric to be canonical it should not depend on the choice of that metric in the conformal class. In this sense, the construction should be conformally invariant.

Our construction will employ the Green function of the Yamabe operator (conformal Laplacian)

$$L = 4\frac{n-1}{n-2}\Delta + S$$

where n is the dimension of the underlying manifold (supposed to be ≥ 3 here and in the sequel), Δ is the Laplace-Beltrami operator (with the sign convention that makes it a positive operator), and S is the scalar curvature of some metric g in a given conformal class. Since L is conformally invariant (up to some conformal factor), so are corresponding "harmonic" functions and Green functions. (A Green function for L exists if L is invertible which is the case if we assume that S is positive.) Such functions can therefore be used to construct conformally invariant metrics.

If g is our starting Riemannian metric, with distance function $d(\cdot,\cdot)$, and if $G(\cdot,\cdot)$ is the Green function for L, the idea is to put

$$\alpha(p) := \lim_{q\to p}(G(p,q) - \mathfrak{a}_n d(p,q)^{2-n})^{\frac{1}{n-2}}$$

(with $\mathfrak{a}_n$ some constant depending on n), and

$$\mathfrak{g} := \alpha^2 g$$

then is our new metric.

As indicated above, this construction is similar to Arakelov's construction for a metric on a compact Riemann surface. Arakelov utilized the Green function of the Bergman metric, and $d(p,q)^{2-n}$ in dimension 2 of course has to be replaced by the logarithm of the distance function. In a context different from Arakelov's setting however, such a construction arose independently. Namely, for domains in Euclidean space, Hersch [13] introduced the conformal radius by such a device, and Hersch's idea was turned into the construction of a metric by Leutwiler [18] by a similar formula as displayed above. For more recent advances in this direction, see the survey article of Bandle-Flucher [8]. It should be noted, however, that our construction goes considerably deeper than the one of Hersch and Leutwiler, because their construction is only invariant under the finite dimensional group of conformal automorphisms of Euclidean space whereas ours is invariant under conformal changes of a Riemannian metric, i.e., under an infinite dimensional group.

We should also note that the rôle of the Green function for the Yamabe problem was discovered by Bahri-Coron [7] and Schoen [22].

A crucial point in Schoen's final solution of the Yamabe problem for a conformally flat manifold [22] is to use a Green function with pole at p to convert a given conformally flat metric into an asymptotically flat one for which p corresponds to infinity. By way of constrast, our metric will be a compact one.

Our construction succeeds if either the dimension is 3 and the conformal class is arbitrary, or if the dimension is arbitrary, and we have a locally conformally flat class. This has to do with the existence and the properties of the above limit defining α, and for that purpose we need to invoke the positive mass theorem of Schoen-Yau [24],[25],[26] (a different proof of the theorem was found by Witten [30], with details developed by Parker-Taubes [21], see also the survey of Lee-Parker [17]), as does Schoen's final solution of the Yamabe problem. The positive mass theorem in its presently known form does not give sufficiently precise estimates for carrying our construction through for not locally conformally flat conformal structures in dimensions bigger than 3.

As mentioned however, at least in dimension 3, the construction is not restricted to locally conformal flat structures, in contrast to the ones of Apanasov, Kulkarni-Pinkall, and Nayatani. On the other hand, those constructions also work in certain scalar negative cases.

It turns out that the above limit α is 0 if and only if our manifold is conformally equivalent to the sphere S^n (with its standard conformal structure). Thus, the resulting metric is trivial if and only if the manifold is conformally equivalent to S^n. This is in fact needed for the Hausdorff property of the moduli space of (locally conformally flat) conformal structures, because the conformal group of S^n is noncompact, and one may locally "bubble off" a sphere from any conformal manifold.

Our metric is always smooth (of class C^∞) in the locally conformally flat case, in contrast to the Kobayashi type metrics of Apanasov and Kulkarni-Pinkall that in general are only of class $C^{1,1}$. Again in contrast to Kobayashi type metrics that are distance nonincreasing under conformal maps, our metric is (locally) distance nondecreasing under such maps. This fits together with the vanishing of the metric for S^n and the fact that there do not exist branched coverings in conformal geometry in dimension at least 3.

We already mentioned that we need a metric of positive scalar curvature in the given conformal class, in order for the Green function to exist. Of course, by the solution of the Yamabe problem, there exists a metric of constant positive scalar curvature in such a class. As mentioned however, the latter in general is not unique, not even up to a scaling factor. By way of contrast, our metric is unique in its conformal class, and therefore, in particular, it fixes a scaling factor. For a sequence of compact conformal structures with scalar curvature going to zero, our metric then tends to a noncompact metric of infinite diameter and vanishing scalar curvature. It is clear that this must be so, because in the zero curvature case, there can be no natural scaling factor.

Because of its various properties as described in the first part of the present paper, this canonical metric seems to be a good tool for investigating the moduli space of locally conformally flat structures. It is the purpose of the second part of the paper to start exploring this idea. More precisely, we investigate the behavior of the canonical metric under surgery type degenerations. Our main result (Theorem II.2.7) says that the limit of the canonical metrics yields the canonical metric of the limit space. In other words, we can follow our canonical metric through a change of topological type. In analogy with the investigations of Masur [19] about the geometry of the Mumford-Deligne compactification of the space of stable Riemann surfaces, we view our analysis as a first step towards understanding a natural compactification of the moduli space of locally conformally flat structures on a given compact differentiable manifold. We obtain asymptotics for the L^2-metric on this moduli space (see Theorem II.3.1). In particular, as in Masur's work, this L^2-metric is not complete (Corollary II.3.2).

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Part I: Definitions and elementary properties

I.1 A canonical metric for locally conformally flat manifolds with positive scalar curvature

Let (M, g) be a closed, connected, smooth Riemannian n-manifold, $n \geq 3$. The Yamabe operator L of (M, g) is defined by

$$Lu = 4\frac{n-1}{n-2}\Delta u + Su \quad \text{for} \quad u \in C^\infty(M)$$

where $\Delta = -\nabla^i \nabla_i$ is the Laplacian and S the scalar curvature of g. A basic fact about L is that for a metric $\tilde{g} = \varphi^{\frac{4}{n-2}} g$ the scalar curvature $\tilde{S}$ is given by

$$\tilde{S} = \varphi^{-\frac{n+2}{n-2}} L\varphi. \tag{I.1.1}$$

L is conformally invariant in the sense that if $\tilde{g} = \varphi^{\frac{4}{n-2}} g$, then

$$L(\varphi u) = \varphi^{\frac{n+2}{n-2}} \tilde{L}(u) \quad \text{for any} \quad u \in C^\infty(M) \tag{I.1.2}$$

where $\tilde{L}$ is the Yamabe operator w.r.t. $\tilde{g}$. A consequence of (I.1.1) and (I.1.2) is the following well-known result.

Lemma I.1.1. *Let C be a conformal class of Riemannian metrics on M. Then one and only one of the following cases holds: C contains a metric of (i) positive, (ii) negative, or (iii) identically zero scalar curvature.* *q.e.d.*

Referring to the cases (i), (ii), (iii) of Lemma I.1.1, we will say that the conformal class C is scalar positive, scalar negative, or scalar flat, resp.

If C is scalar positive, then for any metric $g \in C$ the Yamabe operator L admits a unique Green function, i.e., a function $G(p,q)$ which satisfies

$$\int_M G(p,q) Lu(q) \mathrm{d}\mu(q) = u(p) \quad \text{for any} \quad u \in C^\infty(M)$$

where $\mathrm{d}\mu = \mathrm{d}\mu(g)$ is the volume element of (M,g). This function is strictly positive. Further, if $\tilde{g} = \varphi^{\frac{4}{n-2}} g$, then

$$\tilde{G}(p,q) = \frac{1}{\varphi(p)\varphi(q)} G(p,q) \tag{I.1.3}$$

is the Green function of the Yamabe operator $\tilde{L}$ w.r.t. $\tilde{g}$.

Now we assume that the conformal class C on M is locally conformally flat and scalar positive. In the following, we use the Green function for the Yamabe operator to construct a canonical Riemannian metric in C, proceeding in a similar way as Leutwiler [18] for Euclidean domains and Arakelov [1] for Riemann surfaces.

Let $g \in C$ such that

$$g = \psi^* g_E \quad \text{on} \quad U \subset M$$

for a chart $\psi : U \to V \subset \mathbb{R}^n$, where g_E is the Euclidean metric on $\mathbb{R}^n$. By elliptic theory, $G(p,q) - \mathfrak{a}_n|\psi(p) - \psi(q)|^{2-n}$ with

$$\mathfrak{a}_n = \frac{1}{4(n-1)\omega_{n-1}}$$

where ω_{n-1} denotes the volume of the unit sphere S^{n-1}, is a smooth function on $U \times U$. For $p \in U$, we set

$$\begin{aligned} \alpha^2(p) &= \lim_{q\to p}(G(p,q) - \mathfrak{a}_n|\psi(p) - \psi(q)|^{2-n})^{\frac{2}{n-2}} \\ &= \lim_{q\to p}(G(p,q) - \mathfrak{a}_n d(p,q)^{2-n})^{\frac{2}{n-2}}. \end{aligned}$$

Here d denotes the distance function w.r.t. g. We define the symmetric rank-2 tensor field $\mathfrak{g}$ on M by setting

$$\mathfrak{g} = \alpha^2 g \quad \text{on} \quad U.$$

We have to verify that $\mathfrak{g}$ does not depend on the choice of g. Let $\tilde{g} = \varphi^{\frac{4}{n-2}} g$ be another metric in C with

$$\tilde{g} = \tilde{\psi}^* g_E \quad \text{on} \quad U$$

for a chart $\tilde{\psi} : U \to \tilde{V}$. Since $\psi \circ \tilde{\psi}^{-1}$ is the restriction of a Möbius transformation on S^n,

$$\text{(I.1.4)} \quad |\tilde{\psi}(p) - \tilde{\psi}(q)| = \varphi(p)^{\frac{1}{n-2}} \varphi(q)^{\frac{1}{n-2}} |\psi(p) - \psi(q)| \quad \text{for} \quad p, q \in U.$$

By (I.1.3) and (I.1.4),

$$\begin{aligned} \tilde{\alpha}^2(p) &= \lim_{q\to p}(\tilde{G}(p,q) - \mathfrak{a}_n|\tilde{\psi}(p) - \tilde{\psi}(q)|^{2-n})^{\frac{2}{n-2}} \\ &= \lim_{q\to p}\left[\frac{1}{\varphi(p)\varphi(q)}(G(p,q) - \mathfrak{a}_n|\psi(p) - \psi(q)|^{2-n})\right]^{\frac{2}{n-2}} \\ &= \varphi(p)^{-\frac{4}{n-2}} \alpha^2(p). \end{aligned}$$

Hence,

$$\tilde{\alpha}^2 \tilde{g} = \alpha^2 g \quad \text{on} \quad U.$$

We note that it follows from the construction that the tensor $\mathfrak{g}$ is smooth. Further, by means of (I.1.3), for an arbitrary Riemannian metric $g \in C$

$$\mathfrak{g} = \alpha^2 g$$

with

$$\alpha^2(p) = \varphi(p)^{\frac{4}{n-2}} \lim_{q\to p}\left(\frac{1}{\varphi(p)\varphi(q)}G(p,q) - \mathfrak{a}_n d_\varphi(p,q)^{2-n}\right)^{\frac{2}{n-2}} \tag{I.1.5}$$

where $d_\varphi(p,q)$ is the distance function w.r.t. the metric $\varphi^{\frac{4}{n-2}}g$ which is assumed to be Euclidean near p.

Remark I.1.2. If $n = 3$, we can choose any metric $g \in C$ for the construction of $\mathfrak{g}$, i.e., we do not have to suppose that g is Euclidean on U. Moreover, in that case we obtain a symmetric rank-2 tensor field for any scalar positive conformal class C. Namely, if $g \in C$, then elliptic theory yields that $G(p,q) - \mathfrak{a}_n d(p,q)^{-1}$ is continuous. Further, if $\tilde{g} = \varphi^4 g$, then

$$\lim_{q\to p}\left(\frac{\varphi(p)\varphi(q)}{\tilde{d}(p,q)} - \frac{1}{d(p,q)}\right) = 0$$

where $\tilde{d}$ is the distance function w.r.t. $\tilde{g}$. q.e.d.

Proposition I.1.3. *Let C be a scalar positive conformal class on M, where C is locally conformally flat or M is three-dimensional. If (M,C) is conformally equivalent to the standard sphere S^n, then $\mathfrak{g}$ vanishes identically. In all other cases, $\mathfrak{g}$ is a Riemannian metric in C.*

Proof. One has to verify that if $\alpha(p) = 0$ for some $p \in M$, then (M,C) is conformally equivalent to the standard sphere S^n. But this (and

$$\lim_{q\to p}(G(p,q) - \mathfrak{a}_n d(p,q)^{2-n}) \geq 0)$$

was shown by Schoen and Yau (cf. [28], Theorem V.3.6) by considering the Riemannian metric $G(p,\cdot)^{\frac{4}{n-2}}g$ of vanishing scalar curvature on $M \setminus \{p\}$ to which the positive mass theorem can be applied. q.e.d.

For simplicity, we will refer to $\mathfrak{g}$ in any case as the canonical metric.

I.2 Basic properties of the canonical metric

First, we state

Proposition I.2.1. *Let C_i, $i = 1, 2$, be a scalar positive conformal class on a closed, connected n-manifold M_i, $n \geq 3$, suppose C_i is locally conformally flat or $n = 3$, and let $\mathfrak{g}_i$ be the canonical metric of (M_i, C_i).*

(i) If $f : (M_1, C_1) \to (M_2, C_2)$ is a conformal diffeomorphism, then $f^*\mathfrak{g}_2 = \mathfrak{g}_1$.

(ii) A conformal covering $f : (M_1, C_1) \to (M_2, C_2)$ is length nondecreasing w.r.t. the respective canonical metrics.

Proof. (i) This is proven straightforwardly.

(ii) Let $g_2 \in C_2$. If $n > 3$, choose g_2 such that g_2 is Euclidean on an open set $U \subset M_2$. Set $g_1 = f^* g_2 \in C_1$. Let G_i denote the Green function for the Yamabe operator and d_i the distance function of (M_i, g_i). Using that the covering transformation group Γ of $f : M_1 \to M_2$ consists of isometries of (M_1, g_1), one sees that

$$G_2(f(p), f(q)) = \sum_{\gamma \in \Gamma} G_1(p, \gamma(q)).$$

It follows that for $p \in f^{-1}(U)$

$$\begin{aligned}
\alpha_2(f(p))^{n-2} &= \lim_{q \to p} (G_2(f(p), f(q)) - \mathfrak{a}_n d_2(f(p), f(q))^{2-n}) \\
&= \lim_{q \to p} (G_1(p, q) - \mathfrak{a}_n d_1(p, q)^{2-n}) + \sum_{\gamma \in \Gamma \setminus \{id\}} G_1(p, \gamma(p)) \\
&\geq \lim_{q \to p} (G_1(p, q) - \mathfrak{a}_n d_1(p, q)^{2-n}) \\
&= \alpha_1(p)^{n-2}.
\end{aligned}$$

Since $\mathfrak{g}_2 = \alpha_2^2 g_2$ on U and $\mathfrak{g}_1 = \alpha_1^2 g_1$ on $f^{-1}(U)$, the assertion follows.

q.e.d.

Remark I.2.2.

(i) In particular, Proposition I.2.1(i) yields that in the considered cases the group $\mathrm{Conf}(M, C)$ of conformal transformations of (M, C) coincides with the group $\mathrm{Isom}(M, \mathfrak{g})$ of isometries of $(M, \mathfrak{g})$, if (M, C) is not conformally equivalent to the standard sphere.

(ii) Proposition I.2.1(ii) is one of the reasons why our canonical metric differs from that of Apanasov [2] and Kulkarni-Pinkall [16]. Namely, the distance function defined by their canonical metric coincides with the distance obtained by applying Kobayashi's construction in the Möbius context. Thus, a conformal map is distance nonincreasing w.r.t. the canonical metrics defined by Apanasov and Kulkarni-Pinkall. q.e.d.

If one looks for a canonical metric in a scalar negative or scalar flat conformal class C on a closed, connected manifold, one can choose a conveniently normalized Yamabe metric, i.e., a constant scalar curvature metric. Namely, in these cases there is only one such Riemannian metric in C up to multiplication by a positive constant. If C is scalar positive, this is no longer true in general.

The next proposition shows that the choice of our canonical metric fits with the choice of the normalized Yamabe metric as the canonical metric for a scalar flat conformal class.

Proposition I.2.3. *Let g_k, $k \in \mathbb{N}$, be locally conformally flat Riemannian metrics on a closed, connected $n-$manifold M, $n \geq 3$, with constant scalar curvature $S_k > 0$ and unit volume and let g be a Riemannian metric on M with vanishing scalar curvature. Assume that*

$$g_k \to g \quad in \quad C^m(M) \quad for \quad m > \frac{3n+2}{2}.$$

Then

$$S_k^{\frac{2}{n-2}} \mathfrak{g}_k \to g \quad in \quad C^l(M) \quad for \quad 0 \leq l < m - \frac{3n+2}{2}$$

where $\mathfrak{g}_k$ denotes the canonical metric of the conformal class of g_k.

Proof. First observe that the assumptions imply that g is also locally conformally flat.

Let L_k be the Yamabe operator of (M, g_k), i.e.,

$$L_k = 4\frac{n-1}{n-2}\Delta_k + S_k \tag{I.2.1}$$

where Δ_k is the Laplacian w.r.t. g_k, and let $G_k(p,q)$ be the Green function for L_k. Let $0 = \lambda_{0,k} < \lambda_{1,k} \leq \lambda_{2,k} \leq \dots$ denote the eigenvalues of Δ_k. Because of (I.2.1), the eigenvalues of L_k are

$$4\frac{n-1}{n-2}\lambda_{j,k} + S_k, \qquad j = 0, 1, 2, \dots .$$

Hence,

$$\begin{aligned}
&\left\| G_k(p,q) - \frac{1}{S_k} \right\|^2_{L^2(M\times M, g_k \oplus g_k)} \\
&\qquad = \sum_{j=1}^{\infty} \left(4\frac{n-1}{n-2}\lambda_{j,k} + S_k \right)^{-2} \\
&\qquad \leq \left(4\frac{n-1}{n-2} \right)^{-2} \sum_{j=1}^{\infty} \lambda_{j,k}^{-2} \\
&\qquad = \left(4\frac{n-1}{n-2} \right)^{-2} \|\Gamma_k(p,q)\|^2_{L^2(M\times M, g_k \oplus g_k)}
\end{aligned}$$

where $\Gamma_k(p,q)$ is the Green function of Δ_k. Since g_k tends to g in $C^m(M)$, there exists a $c_1 > 0$ such that

$$\|\Gamma_k(p,q)\|_{L^2(M\times M, g\oplus g)} \leq c_1 \quad \text{for all} \quad k \in \mathbb{N}$$

(cf. [5]). Consequently, there exists a $c_2 > 0$ such that

$$\left\| G_k(p,q) - \frac{1}{S_k} \right\|_{L^2(M\times M, g\oplus g)} \leq c_2 \quad \text{for all} \quad k \in \mathbb{N}. \tag{I.2.2}$$

Now, let $\pi : \tilde{M} \to M$ denote the universal covering map and let $\Phi_k : \tilde{M} \to S^n$ be a developing map w.r.t. the conformal class of g_k. If Φ_k would be surjective for some $k \in \mathbb{N}$, then M would be diffeomorphic to a quotient of S^n by a finite subgroup of $\mathrm{O}(n+1)$ (cf. [27]); such a quotient, however, does not admit a scalar flat locally conformally flat structure. Therefore, we may assume that

$$\Phi_k(\tilde{M}) \subset \mathbb{R}^n \subset S^n.$$

Fix $p_0 \in M$. Let U_0 be a simply connected neighborhood of p_0 and choose a connected component $\tilde{U}_0$ of $\pi^{-1}(U_0)$. Then

$$\pi_0 = \pi_{|\tilde{U}_0} : \tilde{U}_0 \to U_0$$

is a diffeomorphism and

$$(\pi_0^{-1})^* \Phi_k^* g_E = \varphi_k^{\frac{4}{n-2}} g_k$$

for a positive function $\varphi_k \in C^\infty(M)$. We set

$$\tilde{g}_k = \varphi_k^{\frac{4}{n-2}} g_k \quad \text{on} \quad M.$$

Composing Φ_k with a scaling of $\mathbb{R}^n$ if necessary, we may assume that

$$\varphi_k(p_0) = 1 \quad \text{for all} \quad k \in \mathbb{N}. \tag{I.2.3}$$

Since $\tilde{g}_k$ is flat on U_0, by (I.1.1)

$$L_k \varphi_k = 0 \quad \text{on} \quad U_0. \tag{I.2.4}$$

Let U_0' and U_0'' be neighborhoods of p_0 such that $U_0'' \subset\subset U_0' \subset\subset U_0$ and such that U_0'' has a smooth boundary. The Harnack inequality says that there exists $c_3 > 0$ such that

$$\sup_{U_0'} \varphi_k \leq c_3 \inf_{U_0'} \varphi_k \quad \text{for all} \quad k \in \mathbb{N}. \tag{I.2.5}$$

Because of (I.2.3), it follows that

$$\sup_{U_0'} \varphi_k \leq c_3. \tag{I.2.6}$$

Let (k') be any subsequence of (k) and let $m' \in \mathbb{Z}$ with $0 \leq m' < m - \frac{n}{2}$. Using (I.2.4) and (I.2.6), elliptic theory (cf. e.g. [9]) implies that there exists a subsequence (k'') of (k') such that

$$\varphi_{k''} \to \varphi \quad \text{in} \quad C^{m'}(\overline{U_0''}). \tag{I.2.7}$$

By (I.2.3) and (I.2.5),

$$\inf_{U_0'} \varphi_k \geq \frac{1}{c_3}.$$

Thus, also

$$\frac{1}{\varphi_{k''}} \to \frac{1}{\varphi} \quad \text{in} \quad C^{m'}(\overline{U_0''}). \tag{I.2.8}$$

Let $\exp_k : T_{p_0}M \to M$ be the exponential map w.r.t. $\tilde{g}_k$. Since $\tilde{g}_k$ is flat on U_0'' and

$$\tilde{g}_{k''} \to \varphi^{\frac{4}{n-2}} g \quad \text{in} \quad C^{m'}(\overline{U_0''})$$

we may choose a neighborhood $U_1 \subset U_0''$ of p_0 such that for all k'' there exists a neighborhood $V_{k''} \subset T_{p_0}M$ of 0 such that

$$\exp_{k''|V_{k''}} : V_{k''} \to U_1$$

is a diffeomorphism. Then, setting

$$\psi_{k''} = (\exp_{k''|V_{k''}})^{-1}$$

and identifying $(T_{p_0}M, \tilde{g}_k|_{p_0})$ with $(\mathbb{R}^n, g_E)$,

$$\psi_{k''}^* g_E = \tilde{g}_{k''}.$$

Now, let $\tilde{G}_k(p,q)$ denote the Green function for the Yamabe operator $\tilde{L}_k$ of $(M, \tilde{g}_k)$. We consider

$$F_{k''}(p,q) = \tilde{G}_{k''}(p,q) - \frac{1}{\varphi_{k''}(p)\varphi_{k''}(q)S_{k''}} - \mathfrak{a}_n |\psi_{k''}(p) - \psi_{k''}(q)|^{2-n}$$

for $p, q \in U_1$. According to (I.1.1),

$$\tilde{L}_k \frac{1}{\varphi_k} = \varphi_k^{-\frac{n+2}{n-2}} S_k.$$

Thus,
(I.2.9)
$$(\tilde{L}_{k'',p} + \tilde{L}_{k'',q})F_{k''}(p,q) = -\varphi_{k''}(p)^{-\frac{n+2}{n-2}}\frac{1}{\varphi_{k''}(q)} - \varphi_{k''}(q)^{-\frac{n+2}{n-2}}\frac{1}{\varphi_{k''}(p)}.$$

Since by (I.1.3)

$$\tilde{G}_k(p,q) - \frac{1}{\varphi_k(p)\varphi_k(q)S_k} = \frac{1}{\varphi_k(p)\varphi_k(q)}\left(G_k(p,q) - \frac{1}{S_k}\right)$$

by elliptic theory we obtain from (I.2.2), (I.2.7), (I.2.8), and (I.2.9) that for any subdomain $U_2 \subset\subset U_1$ and for any $m'' \in \mathbb{Z}$ with $0 \leq m'' < m' - n$ there exists $c_4 > 0$ such that

$$\|F_{k''}(p,q)\|_{C^{m''}(\overline{U_2 \times U_2})} \leq c_4 \quad \text{for all} \quad k''. \tag{I.2.10}$$

Recall that $\mathfrak{g}_{k''} = \tilde{\alpha}^2_{k''}\tilde{g}_{k''}$ with

$$\tilde{\alpha}_{k''}(p) = \lim_{q\to p}(\tilde{G}_{k''}(p,q) - \mathfrak{a}_n|\psi_{k''}(p) - \psi_{k''}(q)|^{2-n})^{\frac{1}{n-2}}$$

for $p \in U_1$. Since by assumption

$$S_k \to 0$$

and

$$S_{k''}\tilde{\alpha}_{k''}(p)^{n-2} = S_{k''}F_{k''}(p,p) + \frac{1}{\varphi_{k''}(p)^2}$$

(I.2.8) and (I.2.10) imply that

$$S_{k''}^{\frac{1}{n-2}}\tilde{\alpha}_{k''} \to \varphi^{-\frac{2}{n-2}} \quad \text{in} \quad C^{m''}(\overline{U_2})$$

and consequently

$$S_{k''}^{\frac{2}{n-2}}\mathfrak{g}_{k''} = S_{k''}^{\frac{2}{n-2}}\tilde{\alpha}^2_{k''}\varphi_{k''}^{\frac{4}{n-2}}g_{k''} \to g \quad \text{in} \quad C^{m''}(\overline{U_2}).$$

Since the subsequence (k') may be chosen arbitrarily, this concludes the proof. q.e.d.

We mention that one can similarly verify

Proposition I.2.4. *Let g_k, $k \in \mathbb{N}$, be Riemannian metrics on a closed, connected manifold M of dimension 3 with constant scalar curvature $S_k > 0$ and unit volume and denote by $\mathfrak{g}_k$ the canonical metric* *of the conformal class of g_k. If the sequence (g_k) tends to a Riemannian metric g with vanishing scalar curvature in $C^2(M)$, then*

$$S_k^2 \mathfrak{g}_k \to g \quad in \quad C^0(M).$$

q.e.d.

A complete proof of the last proposition will be given in a forthcoming paper of the first author.

I.3 The geometry of the moduli space of locally conformally flat structures on $S^1 \times S^2$

We first recall how the choice of a canonical metric in every conformal class yields a Riemannian L^2-metric on the moduli space of conformal structures.

Let M be a closed, connected $n-$manifold. Let $\mathcal{M}(M)$ denote the space of smooth Riemannian metrics on M and $pr : \mathcal{M}(M) \to \mathcal{C}(M)$ the natural projection onto the space $\mathcal{C}(M)$ of conformal structures on M. Of course, to be precise, we have to complete $\mathcal{M}(M)$ w.r.t. a convenient Sobolev norm, but we shall suppress here the analytical details. On $\mathcal{M}(M)$ we have a natural Riemannian metric that is invariant under the action of the diffeomorphism group Diff(M) of M. This metric is given by the L^2-product $(\ ,\ )_{L^2(g)}$ on $T_g\mathcal{M}(M) = S^2(M)$ w.r.t. g, where $S^2(M)$ is the space of symmetric rank-2 tensor fields on M. We recall that for $h_1, h_2 \in S^2(M)$

$$(h_1, h_2)_{L^2(g)} = \int_M \langle h_1, h_2 \rangle_g \mathrm{d}\mu(g)$$

with

$$\langle h_1, h_2 \rangle_g = \sum_{i,j=1}^{n} h_1(e_i, e_j) h_2(e_i, e_j)$$

for an orthonormal frame $e_1, \ldots, e_n$ w.r.t. g.

Now, let $\iota : \mathcal{C}(M) \to \mathcal{M}(M)$ be a section of $pr : \mathcal{M}(M) \to \mathcal{C}(M)$ which is equivariant w.r.t. the action of Diff(M) and set $\mathcal{B}(M) = \mathcal{C}(M)/\text{Diff}(M)$. Then we can identify the tangent space $T_{[C]}\mathcal{B}(M)$ with the $L^2(\iota(C))-$orthogonal complement of

$$T_{\iota(C)}(C_+^\infty(M) \cdot \iota(C)) + T_{\iota(C)}(\text{Diff}(M) \cdot \iota(C))$$

in $S^2(M)$. Here $C_+^\infty(M)$ is the set of positive, smooth functions on M. It is clear that the $L^2(\iota(C))-$orthogonal complement of $T_{\iota(C)}(C_+^\infty(M) \cdot$

$\iota(C)) = C^\infty(M) \cdot \iota(C)$ is the space $S^2_0(M, \iota(C))$ of traceless symmetric rank−2 tensor fields on M w.r.t. $\iota(C)$. Further, since

$$T_{\iota(C)}(\mathrm{Diff}(M) \cdot \iota(C)) = \{\mathcal{L}_X \iota(C) : X \in \mathcal{X}(M)\}$$

where $\mathcal{L}$ denotes the Lie derivative, and $\mathcal{X}(M)$ denotes the space of vector fields on M, the $L^2(\iota(C))$−orthogonal complement of $T_{\iota(C)}(\mathrm{Diff}(M) \cdot \iota(C))$ is the kernel $\ker\delta_{\iota(C)}$ of the divergence operator $\delta_{\iota(C)} : S^2(M) \to \Omega^1(M)$ into the space $\Omega^1(M)$ of 1-forms on M. Thus,

$$T_{[C]}\mathcal{B}(M) \equiv \ker\delta_{\iota(C)} \cap S^2_0(M, \iota(C))$$

and the restrictions of $(\ ,\ )_{L^2(\iota(C))}$ onto $\ker\delta_{\iota(C)} \cap S^2_0(M, \iota(C))$ yield a Riemannian metric on $\mathcal{B}(M)$.

Analogously, if $\iota_0^+ : \mathcal{C}_0^+(M) \to \mathcal{M}(M)$ is an equivariant section over the space $\mathcal{C}_0^+(M)$ of scalar positive locally conformally flat structures, we get a Riemannian metric $\mathfrak{h}$ on the moduli space $\mathcal{B}_0^+(M) = \mathcal{C}_0^+(M)/\mathrm{Diff}(M)$.

In general, the spaces $\mathcal{B}(M)$ and $\mathcal{B}_0^+(M)$ have singularities.

In the following, we shall examine the Riemannian metric $\mathfrak{h}$ on $\mathcal{B}_0^+(M)$ for the case where $M = S^1 \times S^2$ and where the section ι_0^+ is given by the canonical metric defined in §I.1. For this, we identify $S^2 \equiv \mathbb{C} \cup \{\infty\}$ and denote by g_{S^2} the standard metric of S^2.

First, we describe the moduli space $\mathcal{B}_0^+(S^1 \times S^2)$.

Proposition I.3.1. *Let*

$$M_{\lambda,\theta} = (\mathbb{R} \times S^2)_{/(t,z)\sim(t+\lambda, e^{i\theta} z)} \quad \textit{for} \quad \lambda > 0 \quad \textit{and} \quad \theta \in [0, \pi]$$

be equipped with the conformal class of $g_{\lambda,\theta}$, where $g_{\lambda,\theta}$ is the Riemannian metric induced from the product metric $dt^2 \oplus g_{S^2}$ on $\mathbb{R} \times S^2$. Then $(\lambda, \theta) \in \mathbb{R}^+ \times [0, \pi] \mapsto M_{\lambda,\theta}$ yields a parametrization of $\mathcal{B}_0^+(S^1 \times S^2)$.

Proof. By [27], Theorem 4.5, for any scalar flat or scalar positive locally conformally flat structure C on $S^1 \times S^2$ there exists a Kleinian subgroup $\Gamma \subset \mathrm{Conf}(S^3)$, $\Gamma \cong \mathbb{Z}$, of the conformal group $\mathrm{Conf}(S^3)$ of the standard sphere S^3 such that $(S^1 \times S^2, C)$ is conformally equivalent to $\Omega(\Gamma)/\Gamma$, where $\Omega(\Gamma) \subset S^3$ is the domain of discontinuity of Γ. Moreover, two such structures C_1 and C_2 are equivalent iff the corresponding Kleinian groups Γ_1 and Γ_2 are conjugate in $\mathrm{Conf}(S^3)$. Observing that each Kleinian group $\Gamma \subset \mathrm{Conf}(S^3)$ for which $\Omega(\Gamma)/\Gamma$ is diffeomorphic to $S^1 \times S^2$ is generated by a hyperbolic element of $\mathrm{Conf}(S^3)$, the assertion now follows from the classification of the conjugacy classes in $\mathrm{Conf}(S^3)$ (cf. [15]). q.e.d.

Remark I.3.2. Since $\pi_1(S^1 \times S^2) = \mathbb{Z}$ is abelian and therefore amenable, by [27], Proposition 1.2, there do not exist scalar negative locally conformally flat structures on $S^1 \times S^2$. Further, the proof of the last proposition excludes the existence of scalar flat locally conformally flat structures on $S^1 \times S^2$. Hence, $\mathcal{B}_0^+(S^1 \times S^2)$ coincides with the moduli space $\mathcal{B}_0(S^1 \times S^2)$ of all locally conformally flat structures on $S^1 \times S^2$. *q.e.d.*

In the remainder of this section we want to prove

Proposition I.3.3. *For the Riemannian metric $\mathfrak{h}$ on the moduli space $\mathcal{B}_0(S^1 \times S^2)$ induced by the choice of our canonical metric, we have*

$$\mathfrak{h} = u_1(\lambda, \theta)\mathrm{d}\lambda^2 + u_2(\lambda, \theta)\mathrm{d}\theta^2$$

with

$$u_i(\lambda, \theta) \geq const.\lambda^{-4} \quad on \quad (0, 1) \times [0, \pi]$$

and

$$u_i(\lambda, \theta) \leq const.\lambda^{-1}e^{-\frac{3\lambda}{2}} \quad on \quad (1, \infty) \times [0, \pi]$$

where const. means a positive real number independent of λ and θ.

Remark I.3.4. As a consequence of the last proposition, the metric completion of $(\mathcal{B}_0(S^1 \times S^2), \mathfrak{h})$ differs from $\mathcal{B}_0(S^1 \times S^2)$ by a point which corresponds to $\lambda = \infty$. This point can be interpreted as the unique element of the moduli space of locally conformally flat structures on S^3 with two punctures. q.e.d.

For the proof of Proposition I.3.3 we shall use the following two lemmas. In the first lemma, we give explicit expressions for the canonical metrics $\mathfrak{g}_{\lambda,\theta}$ on $M_{\lambda,\theta}$; in the second one, we compare the divergence operators w.r.t. conformally equivalent Riemannian metrics.

Lemma I.3.5. *We have $\mathfrak{g}_{\lambda,\theta} = \alpha^2_{\lambda,\theta} g_{\lambda,\theta}$ with*

$$\begin{aligned}\alpha_{\lambda,\theta}(t, z) &= \alpha_{\lambda,\theta}(|z|) \\ &= \frac{\sqrt{2}}{8\omega_2} \sum_{k=1}^{\infty} \left[\cosh(k\lambda) - 1 + \frac{4|z|^2}{(1 + |z|^2)^2}(1 - \cos(k\theta))\right]^{-\frac{1}{2}}.\end{aligned}$$

Proof. Since the translations in $t-$direction and rotations on $\mathbb{C} \cup \{\infty\}$ with centre 0 are isometries of $(M_{\lambda,\theta}, g_{\lambda,\theta})$, we have

$$\alpha_{\lambda,\theta}(t, z) = \alpha_{\lambda,\theta}(|z|).$$

The Green function $G_{\lambda,\theta}(p,q)$ for the Yamabe operator of $(M_{\lambda,\theta}, g_{\lambda,\theta})$ which we identify with its pull-back onto $\mathbb{R}\times S^2$ is given by

$$G_{\lambda,\theta}((t_0,z_0),(t,z)) = \sum_{k\in\mathbb{Z}} \tilde{G}((t_0,z_0),(t+k\lambda, e^{ik\theta}z))$$

where $\tilde{G}(p,q)$ is the Green function for the Yamabe operator of $(\mathbb{R}\times S^2, dt^2\oplus g_{S^2})$. One verifies that

$$\tilde{G}((0,z_0),(t,z)) = \frac{\mathfrak{a}_3}{\sqrt{2}}(\cosh(t) - \langle z_0, z\rangle_{\mathbb{R}^3})^{-\frac{1}{2}}$$

where $\langle\ ,\ \rangle_{\mathbb{R}^3}$ is the canonical product in $\mathbb{R}^3$. Using the identification of S^2 with $\mathbb{C}\cup\{\infty\}$ via stereographic projection,

$$\langle z_0, z\rangle_{\mathbb{R}^3} = \frac{4Re(z_0\bar{z}) + (1-|z_0|^2)(1-|z|^2)}{(1+|z_0|^2)(1+|z|^2)} \quad \text{for} \quad z_0, z\in\mathbb{C}\subset S^2.$$

Since

$$\lim_{(t,z)\to(0,z_0)} (\tilde{G}((0,z_0),(t,z)) - \mathfrak{a}_3 d_{\mathbb{R}\times S^2}((0,z_0),(t,z))^{-1}) = 0$$

where $d_{\mathbb{R}\times S^2}$ is the distance function on $\mathbb{R}\times S^2$ w.r.t. $dt^2\oplus g_{S^2}$, we obtain that

$$\begin{aligned}\alpha_{\lambda,\theta}(|z_0|) &= \alpha_{\lambda,\theta}(0,z_0)\\ &= \sum_{k\in\mathbb{Z}\backslash\{0\}} \tilde{G}((0,z_0),(k\lambda, e^{ik\theta}z_0))\\ &= \frac{\sqrt{2}}{8\omega_2}\sum_{k=1}^{\infty}\left[\cosh(k\lambda) - 1 + \frac{4|z_0|^2}{(1+|z_0|^2)^2}(1-\cos(k\theta))\right]^{-\frac{1}{2}}.\end{aligned}$$

q.e.d.

Lemma I.3.6. *For the divergence operators $\delta_g, \delta_{\tilde{g}} : S^2(M) \to \Omega^1(M)$ w.r.t. conformally equivalent Riemannian metrics g and $\tilde{g} = \alpha^2 g$, $\alpha\in C^\infty_+(M)$, on M we have*

(i) $(\delta_{\tilde{g}}\alpha^2 h)(X) = (\delta_g h)(X) - n\alpha^{-1}h(\mathrm{grad}\alpha, X) + \alpha^{-1}\mathrm{d}\alpha(X)Tr_g h$ *for $h\in S^2(M)$ and $X\in\mathcal{X}(M)$, where the gradient is w.r.t. g.*

(ii) $\alpha^{2-n}\cdot(\mathrm{ker}\delta_g\cap S^2_0(M,g)) = \mathrm{ker}\delta_{\tilde{g}}\cap S^2_0(M,\tilde{g})$.

Proof. (i) Recall that

$$\begin{aligned}(\delta_g h)(X) &= -\sum_i (\nabla_{e_i} h)(e_i, X) \\ &= -\sum_i [e_i(h(e_i, X)) - h(\nabla_{e_i} e_i, X) - h(e_i, \nabla_{e_i} X)]\end{aligned}$$

for $h \in S^2(M)$ and $X \in \mathcal{X}(M)$, where $e_1, ..., e_n$ is an orthonormal frame w.r.t. g. Then (i) follows from (cf. [6], Theorem 1.159)

$$\widetilde{\nabla}_X Y = \nabla_X Y + \alpha^{-1}[\mathrm{d}\alpha(X)Y + \mathrm{d}\alpha(Y)X - g(X,Y)\mathrm{grad}\alpha]$$

for $X, Y \in \mathcal{X}(M)$, where ∇ and $\widetilde{\nabla}$ are the Levi-Civita connections w.r.t. g and $\tilde{g}$, respectively.

(ii) Let $h \in \ker\delta_g \cap S_0^2(M, g)$. It suffices to show that $\alpha^{2-n}h \in \ker\delta_{\tilde{g}} \cap S_0^2(M, \tilde{g})$. Clearly, $\alpha^{2-n}h \in S_0^2(M, \tilde{g})$. The relation $\delta_{\tilde{g}}(\alpha^{2-n}h) = 0$ is a consequence of (i). q.e.d.

Proof of Proposition I.3.3. Define $f_{s,i} : \mathbb{R} \times S^2 \to \mathbb{R} \times S^2$ by

$$f_{s,1}(t, z) = (st, z) \quad \text{and} \quad f_{s,2}(t, z) = (t, e^{ist}z).$$

Then $f_{s,1}$ and $f_{s,2}$ project to diffeomorphisms $\hat{f}_{s,1} : M_{\lambda,\theta} \to M_{s\lambda,\theta}$ and $\hat{f}_{s,2} : M_{\lambda,\theta} \to M_{\lambda,\theta+s\lambda}$. Therefore, for the canonical frame on $\mathcal{B}_0(S^1 \times S^2)$ given by the parametrization described in Proposition I.3.1, we have

$$\begin{aligned}\left.\frac{\partial}{\partial\lambda}\right|_{(\lambda,\theta)} &= \frac{1}{\lambda} P_{\mathfrak{g}_{\lambda,\theta}}\left(\left.\frac{\mathrm{d}}{\mathrm{d}s}\hat{f}_{s,1}^{*}\mathfrak{g}_{s\lambda,\theta}\right|_{s=1}\right) \\ &= \frac{1}{\lambda} P_{\mathfrak{g}_{\lambda,\theta}}\left(\alpha_{\lambda,\theta}^2 \left.\frac{\mathrm{d}}{\mathrm{d}s}\hat{f}_{s,1}^{*}g_{s\lambda,\theta}\right|_{s=1}\right)\end{aligned}$$

and

$$\begin{aligned}\left.\frac{\partial}{\partial\theta}\right|_{(\lambda,\theta)} &= \frac{1}{\lambda} P_{\mathfrak{g}_{\lambda,\theta}}\left(\left.\frac{\mathrm{d}}{\mathrm{d}s}\hat{f}_{s,2}^{*}\mathfrak{g}_{\lambda,\theta+s\lambda}\right|_{s=0}\right) \\ &= \frac{1}{\lambda} P_{\mathfrak{g}_{\lambda,\theta}}\left(\alpha_{\lambda,\theta}^2 \left.\frac{\mathrm{d}}{\mathrm{d}s}\hat{f}_{s,2}^{*}g_{\lambda,\theta+s\lambda}\right|_{s=0}\right)\end{aligned}$$

where $P_{\mathfrak{g}_{\lambda,\theta}}$ denotes the $L^2(\mathfrak{g}_{\lambda,\theta})$−orthogonal projection onto $\ker\delta_{\mathfrak{g}_{\lambda,\theta}} \cap S_0^2(M, \mathfrak{g}_{\lambda,\theta})$. One computes that

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\hat{f}_{s,1}^{*}g_{s\lambda,\theta}\right|_{s=1} = 2\mathrm{d}t^2$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}\hat{f}^*_{s,2}g_{\lambda,\theta+s\lambda}\bigg|_{s=0} = \frac{8r^2}{(1+r^2)^2}\mathrm{d}\varphi\mathrm{d}t$$

using polar coordinates (r,φ) on $\mathbb{C}\subset S^2$. Hence,

$$\text{(I.3.1)} \qquad \frac{\partial}{\partial\lambda}\bigg|_{(\lambda,\theta)} = \frac{2}{\lambda}P_{\mathfrak{g}_{\lambda,\theta}}(\alpha^2_{\lambda,\theta}\mathrm{d}t^2).$$

On the other hand, observe that

$$\frac{8r^2}{(1+r^2)^2}\mathrm{d}\varphi\mathrm{d}t \in S^2_0(M_{\lambda,\theta},\mathfrak{g}_{\lambda,\theta}).$$

Moreover, since

$$\delta_{g_{\lambda,\theta}}(\beta(r)\mathrm{d}\varphi\mathrm{d}t) = \beta(r)\delta_{g_{\lambda,\theta}}(\mathrm{d}\varphi\mathrm{d}t) = 0$$

for any smooth function $\beta(r)$, applying Lemma I.3.6(i) and the fact that $\alpha_{\lambda,\theta}$ depends only on $r=|z|$, we get

$$\begin{aligned}\delta_{\mathfrak{g}_{\lambda,\theta}}\left(\alpha^2_{\lambda,\theta}\frac{r^2}{(1+r^2)^2}\mathrm{d}\varphi\mathrm{d}t\right) &= \delta_{g_{\lambda,\theta}}\left(\frac{r^2}{(1+r^2)^2}\mathrm{d}\varphi\mathrm{d}t\right)\\ &= 0.\end{aligned}$$

Thus,

$$\text{(I.3.2)} \qquad \frac{\partial}{\partial\theta}\bigg|_{(\lambda,\theta)} = 8\frac{\alpha^2_{\lambda,\theta}}{\lambda}\frac{r^2}{(1+r^2)^2}\mathrm{d}\varphi\mathrm{d}t.$$

Since

$$\left(\alpha^2_{\lambda,\theta}\mathrm{d}t^2, \frac{\partial}{\partial\theta}\bigg|_{(\lambda,\theta)}\right)_{L^2(\mathfrak{g}_{\lambda,\theta})} = 0,$$

also

$$\text{(I.3.3)} \qquad \mathfrak{h}\left(\frac{\partial}{\partial\lambda}\bigg|_{(\lambda,\theta)}, \frac{\partial}{\partial\theta}\bigg|_{(\lambda,\theta)}\right) = \left(\frac{\partial}{\partial\lambda}\bigg|_{(\lambda,\theta)}, \frac{\partial}{\partial\theta}\bigg|_{(\lambda,\theta)}\right)_{L^2(\mathfrak{g}_{\lambda,\theta})} = 0.$$

From (I.3.2), we further get that

$$\begin{aligned}\mathfrak{h}\left(\frac{\partial}{\partial\theta}\bigg|_{(\lambda,\theta)}, \frac{\partial}{\partial\theta}\bigg|_{(\lambda,\theta)}\right) &= \left\| \frac{\partial}{\partial\theta}\bigg|_{(\lambda,\theta)}\right\|^2_{L^2(\mathfrak{g}_{\lambda,\theta})}\\ \text{(I.3.4)} \qquad &= \int_{M_{\lambda,\theta}}\frac{8}{\lambda^2}\frac{r^2}{(1+r^2)^2}\mathrm{d}\mu(\mathfrak{g}_{\lambda,\theta})\\ &= \frac{8}{\lambda}\int_{S^2}\alpha_{\lambda,\theta}(r)^3\frac{r^2}{(1+r^2)^2}\mathrm{d}\mu(g_{S^2}).\end{aligned}$$

(I.3.1) implies that

$$\mathfrak{h}\left(\frac{\partial}{\partial\lambda}\Big|_{(\lambda,\theta)},\frac{\partial}{\partial\lambda}\Big|_{(\lambda,\theta)}\right) \leq \frac{4}{\lambda^2}\left\|\alpha_{\lambda,\theta}^2(\mathrm{d}t^2-\frac{1}{3}g_{\lambda,\theta})\right\|^2_{L^2(\mathfrak{g}_{\lambda,\theta})}$$

$$= \frac{8}{3\lambda}\int_{S^2}\alpha_{\lambda,\theta}(r)^3\mathrm{d}\mu(g_{S^2}). \tag{I.3.5}$$

Now, set

$$e_{\lambda,\theta}=\frac{1}{\|u_{\lambda,\theta}\|_{L^2(\mathfrak{g}_{\lambda,\theta})}}u_{\lambda,\theta} \quad \text{with} \quad u_{\lambda,\theta}=\alpha_{\lambda,\theta}^{-1}\left(\mathrm{d}t^2-\frac{1}{3}g_{\lambda,\theta}\right).$$

From Lemma I.3.6(ii), we know that

$$e_{\lambda,\theta}\in \ \ker\delta_{\mathfrak{g}_{\lambda,\theta}}\cap S_0^2(M_{\lambda,\theta},\mathfrak{g}_{\lambda,\theta}).$$

By means of (I.3.1), it follows that

$$\begin{aligned}\left\|\frac{\partial}{\partial\lambda}\Big|_{(\lambda,\theta)}\right\|_{L^2(\mathfrak{g}_{\lambda,\theta})} &\geq \left\|\left(\frac{2}{\lambda}\alpha_{\lambda,\theta}^2\mathrm{d}t^2,e_{\lambda,\theta}\right)_{L^2(\mathfrak{g}_{\lambda,\theta})}e_{\lambda,\theta}\right\|_{L^2(\mathfrak{g}_{\lambda,\theta})}\\ &= \left|\left(\frac{2}{\lambda}\alpha_{\lambda,\theta}^2\mathrm{d}t^2,e_{\lambda,\theta}\right)_{L^2(\mathfrak{g}_{\lambda,\theta})}\right|\\ &= \frac{2}{\lambda}\frac{|(\alpha_{\lambda,\theta}^2\mathrm{d}t^2,u_{\lambda,\theta})_{L^2(\mathfrak{g}_{\lambda,\theta})}|}{\|u_{\lambda,\theta}\|_{L^2(\mathfrak{g}_{\lambda,\theta})}}.\end{aligned}$$

Since

$$\begin{aligned}(\alpha_{\lambda,\theta}^2\mathrm{d}t^2,u_{\lambda,\theta})_{L^2(\mathfrak{g}_{\lambda,\theta})} &= \int_{M_{\lambda,\theta}}\left\langle\mathrm{d}t^2,\mathrm{d}t^2-\frac{1}{3}g_{\lambda,\theta}\right\rangle_{g_{\lambda,\theta}}\mathrm{d}\mu(g_{\lambda,\theta})\\ &= \frac{2}{3}\lambda\omega_2\end{aligned}$$

and

$$\begin{aligned}\|u_{\lambda,\theta}\|^2_{L^2(\mathfrak{g}_{\lambda,\theta})} &= \int_{M_{\lambda,\theta}}\left\langle\mathrm{d}t^2,\mathrm{d}t^2-\frac{1}{3}g_{\lambda,\theta}\right\rangle_{g_{\lambda,\theta}}\alpha_{\lambda,\theta}^{-3}\mathrm{d}\mu(g_{\lambda,\theta})\\ &= \frac{2\lambda}{3}\int_{S^2}\alpha_{\lambda,\theta}(r)^{-3}\mathrm{d}\mu(g_{S^2})\end{aligned}$$

we arrive at

$$\mathfrak{h}\left(\frac{\partial}{\partial\lambda}\Big|_{(\lambda,\theta)},\frac{\partial}{\partial\lambda}\Big|_{(\lambda,\theta)}\right)\geq\frac{8\omega_2}{3\lambda}\left(\int_{S^2}\alpha_{\lambda,\theta}(r)^{-3}\mathrm{d}\mu(g_{S^2})\right)^{-1}. \tag{I.3.6}$$

Because of (I.3.3), (I.3.4), (I.3.5), and (I.3.6), it remains to estimate $\alpha_{\lambda,\theta}(r)$. Since

$$\lim_{\lambda\to\infty} \frac{\sum_{k=1}^{\infty}[\cosh(k\lambda)-1]^{-\frac{1}{2}}}{e^{-\frac{\lambda}{2}}} = \sqrt{2} \tag{I.3.7}$$

we have

$$\begin{aligned}\alpha_{\lambda,\theta}(r) &\leq \frac{\sqrt{2}}{8\omega_2}\sum_{k=1}^{\infty}[\cosh(k\lambda)-1]^{-\frac{1}{2}}\\ &\leq const.\ e^{-\frac{\lambda}{2}}\end{aligned}$$

for $\lambda \in (1,\infty)$. We conclude the proof by the observation that

$$\begin{aligned}\alpha_{\lambda,\theta}(r) &\geq \frac{\sqrt{2}}{8\omega_2}\sum_{k=1}^{\infty}[\cosh(k\lambda)+1]^{-\frac{1}{2}}\\ &\geq \frac{\sqrt{2}}{8\omega_2}\lambda^{-1}\int_{\lambda}^{\infty}[\cosh(s)+1]^{-\frac{1}{2}}\mathrm{d}s\\ &\geq const.\lambda^{-1}\end{aligned}$$

for $\lambda \in (0,1)$. q.e.d.

Remark I.3.7. Using the above considerations, in particular (I.3.5), (I.3.6), and (I.3.7), one can show that $\lambda^{-1}e^{-\frac{3\lambda}{2}}K(\lambda,0)$ tends to a negative constant as $\lambda \to \infty$ and

$$\lim_{\lambda\to 0} K(\lambda,0) = 0$$

where K is the curvature of $\mathfrak{h}$. *q.e.d.*

Part II: Surgery type degenerations of conformally structures

II.1 Preparations

Let M_0 be a closed and connected $n-$manifold with two punctures p_1, p_2 or the disjoint union of two closed and connected $n-$manifolds M^1, M^2 with one puncture $p_i \in M^i$ each, $n \geq 3$, and let C_0 be a scalar positive locally conformally flat structure on M_0. Let U_1, U_2 be disjoint neighborhoods of the punctures p_1, p_2 with local coordinates $x_i : U_i \to B(2) = \{x \in \mathbb{R}^n : |x| < 2\}$ such that the pull-back $g_{E,i}$ of the Euclidean metric g_E via x_i is contained in the restriction of C_0 onto U_i. We set

$$V_{t,i} := \{p \in M_0 : t < |x_i(p)| < 1\} \quad \text{for} \quad i = 1, 2$$

and

$$\hat{M}_0 = M_0 \setminus \{p_1, p_2\}.$$

For $t \in (0,1)$ and $A \in \mathrm{SO}(n)$, we form an $n-$manifold $M_{t,A}$ by removing the balls $\{|x_i| \le t\}$, $i = 1, 2$, from M_0 and identifying $V_{t,1}$ with $V_{t,2}$ via $f_{t,A} : V_{t,1} \to V_{t,2}$ given by

$$x_2 \circ f_{t,A}(p) = \frac{t}{|x_1(p)|^2} A x_1(p).$$

If $K \subset \hat{M}_0$ is compact, for small t we shall consider K as a compact subset of $M_{t,A}$ via the canonical inclusion of $M_{t,A} \setminus \{|x_i| = t^{\frac{1}{2}}\}$ into $\hat{M}_0$. Further, x_i will also serve as local coordinates on $M_{t,A}$.

The conformal class C_0 induces a locally conformally flat structure $C_{t,A}$ on $M_{t,A}$. As shown by O. Kobayashi [14] (cf. also [10]), $C_{t,A}$ is scalar positive at least for small t. For our purposes, it will be useful to proceed more explicitly. To do so, we first prove the following lemma. It says that we may locally interpolate between any locally conformally flat metric of positive scalar curvature and a cylindrical one within the class of locally conformally flat metrics with positive scalar curvature. (Note that in contrast to this result, in the interpolation lemma of Gromov-Lawson [11], the conformal class may change.)

Lemma II.1.1. *Let φ be a positive, smooth function on $B = \{x \in \mathbb{R}^n : |x| < 1\}$ such that the Riemannian metric $\varphi^{\frac{4}{n-2}} g_E$ has positive scalar curvature. Then for each $\varepsilon_0 \in (0,1)$, there exist real numbers $a > 0$ and $\varepsilon_1 \in (0, \varepsilon_0)$ and a positive, smooth function ψ on $B \setminus \{0\}$ such that*

(i) $\psi(x) = a\varphi(x)$ *for* $|x| \ge \varepsilon_0$

(ii) $\psi(x) = |x|^{\frac{2-n}{2}}$ *for* $|x| \le \varepsilon_1$

(iii) The Riemannian metric $\psi^{\frac{4}{n-2}} g_E$ has positive scalar curvature on $B \setminus \{0\}$.

Proof. Let η_1, η_2 be nonnegative, smooth functions on $(0,1)$ such that for $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$,

(a) $\eta_1(t) = 1$ for $t \leq \varepsilon_2$ and $\eta_1(t) = 0$ for $t \geq \varepsilon_0$.

(b) $\eta_2(t) = 0$ for $t \leq \varepsilon_1$ and $\eta_2(t) = 1$ for $t \geq \varepsilon_2$.

We set

$$\psi(x) = \eta_1(|x|)|x|^{\frac{2-n}{2}} + a\eta_2(|x|)\varphi(x). \tag{II.1.1}$$

Now we fix ε_2 and η_1 and show that we can choose a, ε_1, and η_2 such that $\psi^{\frac{4}{n-2}} g_E$ has positive scalar curvature on $B \setminus \{0\}$, i.e., by (I.1.1)

$$\Delta_E \psi > 0 \qquad \text{on} \quad B \setminus \{0\}$$

where Δ_E denotes the Laplacian w.r.t. the Euclidean metric g_E.

With the ansatz (II.1.1), we get

$$\Delta_E \psi = \Delta_E(\eta_1(|x|)|x|^{\frac{2-n}{2}}) + a\Delta_E \varphi \qquad \text{on} \quad \{\varepsilon_2 \leq |x| < 1\}.$$

Hence, since $\varphi^{\frac{4}{n-2}} g_E$ has positive scalar curvature, i.e.,

$$\Delta_E \varphi > 0 \qquad \text{on} \quad B \tag{II.1.2}$$

we can choose $a > 0$ such that $\Delta_E \psi > 0$ on $\{\varepsilon_2 \leq |x| < 1\}$.

Recall that

$$\Delta_E = -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{n-1}}$$

with $r = |x|$, where $\Delta_{S^{n-1}}$ is the Laplacian on the unit sphere S^{n-1}. Setting $\rho = \log r$, we obtain

$$\Delta_E = e^{-2\rho}\left(-\frac{\partial^2}{\partial \rho^2} - (n-2)\frac{\partial}{\partial \rho} + \Delta_{S^{n-1}}\right).$$

Because of (II.1.2) and

$$\psi = e^{\frac{2-n}{2}\rho} + a\tilde{\eta}_2(\rho)\varphi \quad \text{on} \quad \{0 < |x| \leq \varepsilon_2\}$$

with $\tilde{\eta}_2(\rho) = \eta_2(e^\rho)$, we obtain

$$\begin{aligned} \Delta_E \psi \quad &\geq \quad e^{-2\rho} a \left(\frac{(2-n)^2}{4a} e^{\frac{2-n}{2}\rho} - \frac{\partial^2 \tilde{\eta}_2}{\partial \rho^2}\varphi \right. \\ &\qquad \left. -2\frac{\partial \tilde{\eta}_2}{\partial \rho}\frac{\partial \varphi}{\partial \rho} - (n-2)\frac{\partial \tilde{\eta}_2}{\partial \rho}\varphi \right) \quad \text{on} \quad \{0 < |x| \leq \varepsilon_2\}. \end{aligned}$$

Thus, since the functions φ and $\frac{\partial\varphi}{\partial\rho} = r\frac{\partial\varphi}{\partial r}$ are bounded, we can choose ε_1 and η_2 such that $\Delta_E\psi > 0$ on $\{0 < |x| \leq \varepsilon_2\}$. q.e.d.

Let g_0 be a Riemannian metric in the conformal class C_0 with positive scalar curvature S_0. By the lemma, we may assume without loss of generality that there exists a positive function $\varphi_0 \in C^\infty(\hat{M}_0)$ such that the Riemannian metric $\hat{g}_0 = \varphi_0^{\frac{4}{n-2}} g_0$ on $\hat{M}_0$ has positive scalar curvature, and

$$\hat{g}_0 = \frac{1}{|x_i|^2} g_{E,i} \quad \text{on} \quad \{0 < |x_i| < \frac{3}{2}\} \qquad \text{for} \quad i = 1, 2$$

i.e., $(\hat{M}_0, \hat{g}_0)$ has cylindrical ends. Since the gluing map $f_{t,A}$ is isometric w.r.t. $\hat{g}_0$, the metric $\hat{g}_0$ induces a Riemannian metric $g_{t,A}$ on $M_{t,A}$, which is the cylinder metric $\frac{1}{|x_i|^2} g_{E,i}$ on $\{t < |x_i| < \frac{3}{2}\}$ and coincides with $\hat{g}_0$ on some compact $K \subset \hat{M}_0$ for small t. In particular, there exists a positive real number c_0 such that

$$\text{(II.1.3)} \qquad \inf_{M_{t,A}} S_{t,A} \geq c_0 \quad \text{for all} \quad t \in (0, 1) \quad \text{and} \quad A \in \mathrm{SO}(n)$$

where $S_{t,A}$ is the scalar curvature of $g_{t,A}$.

Before ending this section, we need to introduce some further notations. For $0 < t < 1$, we set

$$M_t = M_{t,\mathbb{I}}, \quad g_t = g_{t,\mathbb{I}}, \quad S_t = S_{t,\mathbb{I}} \quad \text{and} \quad C_t = C_{t,\mathbb{I}}$$

where $\mathbb{I}$ denotes the unit element in $\mathrm{SO}(n)$. Let $\mathfrak{g}_t$ for $0 \leq t < 1$ and $\mathfrak{g}_{t,A}$ for $0 < t < 1$ and $A \in \mathrm{SO}(n)$ be the canonical metrics of C_t and $C_{t,A}$, respectively. We write

$$\mathfrak{g}_t = \alpha_t^2 g_t$$

with a positive function $\alpha_t \in C^\infty(M_t)$. The volume elements of g_t and $\hat{g}_0$ are denoted by $\mathrm{d}\mu_t$ and $\mathrm{d}\hat{\mu}_0$, respectively. Let $G_t(p,q)$ for $0 \leq t < 1$ denote the Green function for the Yamabe operator L_t of (M_t, g_t). Finally, we set

$$\hat{G}_0(p,q) = \frac{1}{\varphi_0(p)\varphi_0(q)} G_0(p,q)$$

which is the Green function of the Yamabe operator $\hat{L}_0$ of $(\hat{M}_0, \hat{g}_0)$.

II.2 The asymptotic behavior of the Green functions and canonical metrics

In this section, we shall investigate the asymptotic behavior of the Green functions G_t as t tends to 0. From that, we shall deduce the behavior of the canonical metrics $\mathfrak{g}_t$.

We start with

Proposition II.2.1. *For any domain $U \subset\subset \hat{M}_0$,*

$$G_t - \hat{G}_0 \to 0 \quad in \quad C^\infty(U \times U) \quad as \quad t \to 0.$$

Proof. We consider domains

$$U \subset\subset U^{(1)} \subset\subset U^{(2)} \subset\subset U^{(3)} \subset\subset \hat{M}_0.$$

Recall that $U^{(3)}$ is embedded in M_t for small t. By construction,

$$g_t = \hat{g}_0 \quad \text{and} \quad L_t = \hat{L}_0 \quad \text{on} \quad U^{(3)}.$$

We denote the Green function of the Dirichlet problem for $\hat{L}_0$ on $U^{(3)}$ by $G_{D,U^{(3)}}(p,q)$ and define $F_t, \hat{F}_t \in C^\infty(U^{(3)} \times U^{(3)})$ by

$$F_t(p,q) = G_t(p,q) - G_{D,U^{(3)}}(p,q)$$

and

$$\hat{F}_t(p,q) = G_t(p,q) - \hat{G}_0(p,q).$$

Then

$$(\hat{L}_{0,p} + \hat{L}_{0,q})F_t = 0 \tag{II.2.1}$$

where $\hat{L}_{0,p}$ and $\hat{L}_{0,q}$ are the actions of $\hat{L}_0$ w.r.t. the first and second argument, respectively. Since the scalar curvature $\hat{S}_0$ of $\hat{g}_0$ is positive, we can apply the maximum principle to (II.2.1) to obtain that

$$F_t \geq 0. \tag{II.2.2}$$

Using (II.1.3), we see that

$$\begin{aligned} 1 &= \int_{M_t} G_t(p,q) L_t 1(q) \mathrm{d}\mu_t(q) \\ &= \int_{M_t} G_t(p,q) S_t(q) \mathrm{d}\mu_t(q) \\ &\geq c_0 \int_{M_t} G_t(p,q) \mathrm{d}\mu_t(q). \end{aligned} \tag{II.2.3}$$

It follows that there exists some $c_1 > 0$ such that

$$\int_{U^{(2)}} \int_{U^{(2)}} F_t(p,q) \mathrm{d}\hat{\mu}_0(p) \mathrm{d}\hat{\mu}_0(q) \leq c_1 \quad \text{for all} \quad t \in (0,\varepsilon).$$

Hence,

$$\inf_{U^{(2)} \times U^{(2)}} F_t \leq \frac{c_1}{\mathrm{vol}(U^{(2)}, \hat{g}_0)^2} \quad \text{for all} \quad t \in (0,\varepsilon). \tag{II.2.4}$$

By means of the Harnack inequality (cf. [9], Corollary 8.21), we get from (II.2.1) and (II.2.2) that there exists a $c_2 > 0$ such that

$$\sup_{U^{(2)} \times U^{(2)}} F_t \leq c_2 \inf_{U^{(2)} \times U^{(2)}} F_t \quad \text{for all} \quad t \in (0,\varepsilon). \tag{II.2.5}$$

Combining (II.2.4) and (II.2.5),

$$\sup_{U^{(2)} \times U^{(2)}} F_t \leq \frac{c_1 c_2}{\mathrm{vol}(U^{(2)}, \hat{g}_0)^2}. \tag{II.2.6}$$

By standard arguments, (II.2.1), (II.2.2) and (II.2.6) imply that the family $(F_t)_{t \in (0,1)}$ is bounded in the Sobolev space $L_1^p(U^{(1)} \times U^{(1)}, \hat{g}_0 \oplus \hat{g}_0)$ for each $p \geq 1$. Now elliptic theory yields that for each sequence $t_k \to 0$ in $(0,1)$ and for each $m \in \mathbb{N}$, there exists a subsequence $(t_{k'})$ of (t_k) such that $(F_{t_{k'}})$ converges in $C^m(U \times U)$ and therefore

$$\hat{F}_{t_{k'}} \to \hat{F}_0 \quad \text{in} \quad C^m(U \times U).$$

Since $U \subset\subset \hat{M}_0$ is chosen arbitrarily, $\hat{F}_0$ can be extended to a smooth function on $\hat{M}_0 \times \hat{M}_0$ with

$$(\hat{L}_{0,p} + \hat{L}_{0,q})\hat{F}_0 = 0. \tag{II.2.7}$$

It remains to show that $\hat{F}_0$ vanishes identically. For this, we fix

$$q \in M_0 \setminus \bigcup_{i=1}^{2} \{|x_i| \leq a_0\}, \qquad 0 < a_0 \leq 1$$

and set

$$v_t(p) = G_t(p,q) \qquad \text{for small} \quad t$$

and

$$\hat{u}_0(p) = \hat{F}_0(p,q).$$

Since $L_t v_t = 0$ on $M_t \setminus \{q\}$ and since g_t is the cylinder metric on $\{t < |x_i| < 1\}$, by the Harnack inequality, there exists some $c_3 > 0$ such that

$$\sup\{v_t(p) : \xi < |x_i(p)| < a_1\xi\} \leq c_3 \inf\{v_t(p) : \xi < |x_i(p)| < a_1\xi\}$$

for all t and all $\xi \in \left[\frac{t}{a_0}, \frac{a_0}{a_1}\right]$. On the other hand, because of (II.2.3) and

$$\mathrm{vol}(\{\xi < |x_i| < a_1\xi\}, g_t) = \omega_{n-1} \log a_1$$

we have

$$\begin{aligned} \inf\{v_t(p) : \xi < |x_i(p)| < a_1\xi\} &\leq \frac{1}{\omega_{n-1}\log a_1} \int\limits_{\{\xi<|x_i(p)|<a_1\xi\}} v_t(p)\mathrm{d}\mu_t \\ &\leq \frac{1}{c_0\omega_{n-1}\log a_1}. \end{aligned}$$

Consequently,

$$\sup\{v_t(p) : \frac{t}{a_0} < |x_i(p)| < a_0\} \leq \frac{c_3}{c_0\omega_{n-1}\log a_1} \quad \text{for small} \quad t.$$

Recalling that

$$\hat{G}_0(p,q) \to 0 \quad \text{as} \quad p \to p_i \quad , \quad i = 1, 2$$

the above considerations show that the function $\hat{u}_0 \in C^\infty(\hat{M}_0)$ is bounded.

Now observe that by (I.1.2) and (II.2.7)

$$L_0(\varphi_0 \hat{u}_0) = 0 \quad \text{on} \quad \hat{M}_0.$$

Further, since $\hat{u}_0$ is bounded and

$$\varphi_0 = O(|x_i|^{\frac{2-n}{2}}) \quad \text{as} \quad p \to p_i$$

we have

$$\varphi_0 \hat{u}_0 \in L^{\frac{n}{n-2}}(M_0).$$

Applying a result of Harvey and Polking (see [12], Theorem 4.1), it follows that $L_0(\varphi_0\hat{u}_0) = 0$ in the weak sense on M_0. Then elliptic theory implies $\varphi_0\hat{u}_0 \in C^\infty(M_0)$. Finally, since the operator L_0 is invertible, $\varphi_0\hat{u}_0$ and therefore $\hat{u}_0$ have to vanish identically. q.e.d.

The following lemma says that the family (G_t) is monotonically increasing in t. More precisely, setting

$$\hat{M}_t = M_0 \setminus \bigcup_{i=1}^{2} \{|x_i| \leq t^{\frac{1}{2}}\}$$

and using the canonical identification of $\hat{M}_t$ with $M_t \setminus \{|x_i| = t^{\frac{1}{2}}\}$, we have

Lemma II.2.2. *For $p, q \in \hat{M}_t$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} G_t(p,q) \geq 0.$$

Proof. We fix $t \in (0,1)$, $p \in \hat{M}_t$, and $u_t \in C^\infty(M_t)$. For $-t < \tau < 0$, let $u_{t+\tau} \in C^\infty(M_{t+\tau})$ such that

$$u_{t+\tau} = u_t \quad \text{on} \quad \hat{M}_t.$$

Because of $g_{t+\tau} = g_t$ on $\hat{M}_t$,

$$\begin{aligned} u_t(p) &= \int\limits_{M_{t+\tau}} G_{t+\tau}(p,q) L_{t+\tau} u_{t+\tau}(q) \mathrm{d}\mu_{t+\tau}(q) \\ &= \int\limits_{\hat{M}_t} G_{t+\tau}(p,q) L_t u_t(q) \mathrm{d}\mu_t(q) \\ &\quad + \sum_{i=1}^{2} \int\limits_{\{(t+\tau)^{\frac{1}{2}} < |x_i| < t^{\frac{1}{2}}\}} G_{t+\tau}(p,q) L_{t+\tau} u_{t+\tau}(q) \mathrm{d}\mu_{t+\tau}(q). \end{aligned}$$

Further,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\tau} \int\limits_{\{(t+\tau)^{\frac{1}{2}} < |x_i| < t^{\frac{1}{2}}\}} G_{t+\tau}(p,q) L_{t+\tau} u_{t+\tau}(q) \mathrm{d}\mu_{t+\tau}(q) \Bigg|_{\tau=0} \\ &= -\frac{1}{2t} \int\limits_{\{|x_i| = t^{\frac{1}{2}}\}} G_t(p,q) L_t u_t(q) \mathrm{d}\nu_t(q) \end{aligned}$$

where $\mathrm{d}\nu_t$ is the volume form induced from the restriction of g_t onto $\{|x_i| = t^{\frac{1}{2}}\}$. Thus,

$$\int\limits_{\hat{M}_t} \frac{\mathrm{d}G_t}{\mathrm{d}t}(p,q) L_t u_t(q) \mathrm{d}\mu_t(q) = \frac{1}{t} \int\limits_{\{|x_i| = t^{\frac{1}{2}}\}} G_t(p,q) L_t u_t(q) \mathrm{d}\nu_t(q).$$

Since $L_t : C^\infty(M_t) \to C^\infty(M_t)$ is surjectiv, we obtain that

$$\int\limits_{\hat{M}_t} \frac{\mathrm{d}G_t}{\mathrm{d}t}(p,q)u(q)\mathrm{d}\mu_t(q) = \frac{1}{t} \int\limits_{\{|x_i|=t^{\frac{1}{2}}\}} G_t(p,q)u(q)\mathrm{d}\nu_t(q)$$

for each $u \in C^\infty(M_t)$. Since $G_t > 0$, this yields the assertion. q.e.d.

As a consequence of Proposition II.2.1 and Lemma II.2.2, we get

Corollary II.2.3. *For all $t \in (0,1)$ and for all $p, q \in \hat{M}_t$,*

$$G_t(p,q) \geq \hat{G}_0(p,q).$$

q.e.d.

In the next step, we want to derive uniform estimates from above for the Green functions G_t near $\{|x_i| = t^{\frac{1}{2}}\}$. For this, we need some preparations.

Let $Z_{\lambda,b} = \{x \in \mathbb{R}^n : \frac{\lambda}{b} \leq |x| \leq \lambda b\}$ and $S^{n-1}_\lambda = \{x \in \mathbb{R}^n : |x| = \lambda\}$ for $b > 1$ and $\lambda > 0$. Let $G_{N,\lambda,b}(x,y)$ denote the Green function of the Neumann problem on $Z_{\lambda,b}$ for the Yamabe operator L_Z w.r.t. the cylinder metric $g_Z = \frac{1}{|x|^2} g_E$. We extend $G_{N,\lambda,b}(x,y)$ by reflections to $(\mathbb{R}^n \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\})$ and set

$$m_{b,+}(\sigma) = \sup\{G_{N,\lambda,b}(x_0,y) : |y| = \lambda\sigma\}$$

and

$$m_{b,-}(\sigma) = \inf\{G_{N,\lambda,b}(x_0,y) : |y| = \lambda\sigma\}$$

for $\sigma > 0$, where $x_0 \in S^{n-1}_\lambda$. Clearly, $m_{b,+}(\sigma)$ and $m_{b,-}(\sigma)$ do not depend on λ and the choice of x_0, and

$$m_{b,\pm}(\frac{1}{\sigma}) = m_{b,\pm}(\sigma). \tag{II.2.8}$$

Lemma II.2.4. *For each $c > 1$, there exists a $\delta_0 > 1$ such that*

$$m_{b,+}(b) < m_{b,-}(cb) \quad \textit{for all} \quad b > \delta_0.$$

Proof. One verifies that the Green function $G_Z(x,y)$ for the Yamabe operator L_Z of $(\mathbb{R}^n \setminus \{0\}, g_Z)$ is given by

$$G_Z(x,y) = \mathfrak{a}_n \left(\frac{|x-y|^2}{|x||y|} \right)^{\frac{2-n}{2}}$$

and that for $x_0 \in S_\lambda^{n-1}$,

$$G_{N,\lambda,b}(x_0, y) = \sum_{k\in\mathbb{Z}} G_Z(x_0, b^{2k}y).$$

It follows that

$$m_{b,+}(\sigma) = G_{N,\lambda,b}(x_0, \sigma x_0) = \mathfrak{a}_n \sum_{k\in\mathbb{Z}} \left(\frac{(1-b^{2k}\sigma)^2}{b^{2k}\sigma}\right)^{\frac{2-n}{2}}$$

and

$$m_{b,-}(\sigma) = G_{N,\lambda,b}(x_0, -\sigma x_0) = \mathfrak{a}_n \sum_{k\in\mathbb{Z}} \left(\frac{(1+b^{2k}\sigma)^2}{b^{2k}\sigma}\right)^{\frac{2-n}{2}}.$$

Therefore,

$$m_{b,+}(b) = 2\mathfrak{a}_n \sum_{k=0}^{\infty} \left(\frac{(1-b^{2k+1})^2}{b^{2k+1}}\right)^{\frac{2-n}{2}}$$

and

$$m_{b,-}(cb) = \mathfrak{a}_n \sum_{k=0}^{\infty} \left[\left(\frac{(1+cb^{2k+1})^2}{cb^{2k+1}}\right)^{\frac{2-n}{2}} + \left(\frac{(1+c^{-1}b^{2k+1})^2}{c^{-1}b^{2k+1}}\right)^{\frac{2-n}{2}}\right].$$

One concludes the proof by verifying that for each $c > 1$, there exists a $\delta_0 > 1$ such that

$$\left(\frac{(1+c\tilde{b})^2}{c\tilde{b}}\right)^{\frac{2-n}{2}} + \left(\frac{(1+c^{-1}\tilde{b})^2}{c^{-1}\tilde{b}}\right)^{\frac{2-n}{2}} \geq 2\left(\frac{(1-\tilde{b})^2}{\tilde{b}}\right)^{\frac{2-n}{2}}$$

for all $\tilde{b} > \delta_0$. q.e.d.

Lemma II.2.5. *Let $G(p,q)$ be the Green function for the Yamabe operator L on a closed Riemannian $n-$manifold (M,g), $n \geq 3$, with positive scalar curvature S. Suppose that a cylinder $(Z_{\lambda,cb}, g_Z)$ for $c, b > 1$, $\lambda > 0$ is isometrically embedded in (M,g) and that $m_{b,+}(b) < m_{b,-}(cb)$. Then*

$$G_{N,\lambda,b}(p,q) > G(p,q)$$

for each $p \in S_\lambda^{n-1} \subset Z_{\lambda,cb}$ and for all $q \in Z_{\lambda,cb}$.

Proof. Fix $p \in S_\lambda^{n-1}$ and assume that

$$\min\{G_{N,\lambda,b}(p,q) - G(p,q) : q \in Z_{\lambda,cb}\} \leq 0.$$

By the maximum principle for L, this minimum is attained at a boundary point q_0 of $Z_{\lambda,cb}$. Because of (II.2.8) and $m_{b,+}(b) < m_{b,-}(cb)$, we obtain that

$$\begin{aligned} G(p,q_0) &\geq G_{N,\lambda,b}(p,q_0) - G_{N,\lambda,b}(p,q) + G(p,q) \\ &\geq m_{b,-}(cb) - m_{b,+}(b) + G(p,q) \\ &\geq G(p,q) \end{aligned} \tag{II.2.9}$$

for each q in the boundary of $Z_{\lambda,b}$. Consequently, $G(p,\cdot)$ has a maximum in the open set $M \setminus Z_{\lambda,b}$, which contradicts the maximum principle.

q.e.d.

Now we are able to estimate G_t in the following way:

Proposition II.2.6. *There exists a $\delta_1 \in (0,1)$ such that for all $t \in (0,\delta_1^2)$ and for all $p,q \in U_i$, $i=1,2$, satisfying $t^{\frac{1}{2}} \leq |x_i(p)| \leq \delta_1$ and $|x_i(p)|^2 \leq |x_i(q)| \leq 1$,*

$$G_t(p,q) < G_{N,\lambda,b}(x_i(p),x_i(q)) \quad \textit{with} \quad \lambda = |x_i(p)| \quad \textit{and} \quad b = \frac{1}{|x_i(p)|}.$$

Proof. By construction, (M_t,g_t) contains the cylinder $\{t < |x_1| \leq \frac{3}{2}\} \cup \{t < |x_2| \leq \frac{3}{2}\}$. Thus, the statement follows from Lemma II.2.4 and Lemma II.2.5. q.e.d.

We now study the behavior of the canonical metrics $\mathfrak{g}_t$. Let the positive functions $\tilde{\alpha}_{t,i}$ for $0 \leq t < 1$ and $i=1,2$ be defined by

$$\mathfrak{g}_t = \tilde{\alpha}_{t,i}^2 g_{E,i} \quad \text{on} \quad W_{t,i} = \{t^{\frac{1}{2}} < |x_i| < 1\}.$$

Since $\mathfrak{g}_t = \alpha_t^2 g_t$,

$$\tilde{\alpha}_{t,i} = \frac{\alpha_t}{|x_i|} \quad \text{for} \quad 0 < t < 1.$$

Theorem II.2.7.

(i) For any domain $U \subset\subset \hat{M}_0$,

$$\mathfrak{g}_t \longrightarrow \mathfrak{g}_0 \quad \textit{in} \quad C^\infty(U) \quad \textit{as} \quad t \to 0.$$

(ii) There exists a constant $c_b > 0$ such that for each $t \in (0,1)$,

$$\tilde{\alpha}_{0,i} \leq \tilde{\alpha}_{t,i} \leq c_b \quad on \quad W_{t,i}.$$

Proof (i) We have to show that for $U \subset\subset \hat{M}_0$

$$\alpha_t \longrightarrow \alpha_0 \quad \text{in} \quad C^\infty(U) \quad \text{as} \quad t \to 0.$$

Let φ be a positive, smooth function on M_0 such that the metric $\varphi^{\frac{4}{n-2}} g_0$ is Euclidean on a neighborhood $U' \subset U$ and let $p \in U'$. Then, by (I.1.5),

$$\begin{aligned} \alpha_0(p) &= \varphi(p)^{\frac{2}{n-2}} \lim_{q \to p} \left(\frac{1}{\varphi(p)\varphi(q)} G_0(p,q) - \mathfrak{a}_n d_{0,\varphi}(p,q)^{2-n} \right)^{\frac{1}{n-2}} \\ &= \varphi(p)^{\frac{2}{n-2}} \lim_{q \to p} \left(\frac{\varphi_0(p)\varphi_0(q)}{\varphi(p)\varphi(q)} \hat{G}_0(p,q) - \mathfrak{a}_n d_{0,\varphi}(p,q)^{2-n} \right)^{\frac{1}{n-2}} \end{aligned}$$

and, for small $t > 0$,

$$\alpha_t(p) = \varphi(p)^{\frac{2}{n-2}} \lim_{q \to p} \left(\frac{\varphi_0(p)\varphi_0(q)}{\varphi(p)\varphi(q)} G_t(p,q) - \mathfrak{a}_n d_{0,\varphi}(p,q)^{2-n} \right)^{\frac{1}{n-2}}$$

where $d_{0,\varphi}(p,q)$ is the distance w.r.t. $\varphi^{\frac{4}{n-2}} g_0$. Therefore,

$$\alpha_t(p) - \alpha_0(p) = \varphi_0(p)^{\frac{2}{n-2}} \lim_{q \to p} (G_t(p,q) - \hat{G}_0(p,q))^{\frac{1}{n-2}}.$$

Now Proposition II.2.1 implies the claim.

(ii) Since

$$g_{E,i} = |x_i|^2 g_t \qquad \text{on} \quad W_{t,i}$$

for $t > 0$ and

$$g_{E,i} = |x_i|^2 \varphi_0^{\frac{4}{n-2}} g_0 \qquad \text{on} \quad W_{0,i}$$

by (I.1.5),

$$\tilde{\alpha}_{t,i}(p) = \lim_{q \to p} \left(|x_i(p)|^{\frac{2-n}{2}} |x_i(q)|^{\frac{2-n}{2}} G_t(p,q) - \mathfrak{a}_n |x_i(p) - x_i(q)|^{2-n} \right)^{\frac{1}{n-2}}$$

for $t > 0$ and

$$\tilde{\alpha}_{0,i}(p) = \lim_{q \to p} \left(|x_i(p)|^{\frac{2-n}{2}} |x_i(q)|^{\frac{2-n}{2}} \hat{G}_0(p,q) - \mathfrak{a}_n |x_i(p) - x_i(q)|^{2-n} \right)^{\frac{1}{n-2}}.$$

Hence, the first inequality in (ii) is a consequence of Corollary II.2.3.

To prove the second inequality, we make use of Proposition II.2.6. Then, with $\lambda = |x_i(p)| \le \delta_1$ and $b = \frac{1}{|x_i(p)|}$, we have

$$\begin{aligned}\tilde{\alpha}_{t,i}(p) \quad &\le \quad \lim_{q\to p}\big(|x_i(p)|^{\frac{2-n}{2}}|x_i(q)|^{\frac{2-n}{2}}G_{N,\lambda,b}(x_i(p),x_i(q)) \\ &\qquad -\mathfrak{a}_n|x_i(p)-x_i(q)|^{2-n}\big)^{\frac{1}{n-2}}.\end{aligned}$$

Since, for λ and b as above,

$$G_{N,\lambda,b}(x_i(p),x_i(q)) = \mathfrak{a}_n \sum_{k\in\mathbb{Z}}\left(\frac{|x_i(p)-|x_i(p)|^{2k}x_i(q)|^2}{|x_i(p)|^{2k+1}|x_i(q)|}\right)^{\frac{2-n}{2}}$$

(cp. the proof of Lemma II.2.4), we arrive at

$$\tilde{\alpha}_{t,i}(p) \le 2\mathfrak{a}_n|x_i(p)|^{2-n}\sum_{k=1}^{n}(|x_i(p)|^{-k}-|x_i(p)|^{k})^{2-n}$$

for $t^{\frac{1}{2}} < |x_i(p)| \le \delta_1$. Observing that for $|x_i(p)| \le \delta_1$,

$$\begin{aligned}&|x_i(p)|^{2-n}\sum_{k=1}^{n}(|x_i(p)|^{-k}-|x_i(p)|^{k})^{2-n} \\ &\le \quad |x_i(p)|^{2-n}\sum_{k=1}^{n}(|x_i(p)|^{-k}-\delta_1|x_i(p)|^{-k})^{2-n} \\ &= \quad (1-\delta_1)^{2-n}|x_i(p)|^{2-n}\sum_{k=1}^{n}|x_i(p)|^{k(n-2)} \\ &= \quad \frac{(1-\delta_1)^{2-n}}{1-|x_i(p)|^{n-2}} \\ &\le \quad \frac{(1-\delta_1)^{2-n}}{1-\delta_1^{n-2}}\end{aligned}$$

and using that by (i),

$$\tilde{\alpha}_{t,i} \to \tilde{\alpha}_{0,i} \quad \text{in} \quad C^\infty(\{\delta_1 < |x_i| < 1\}) \quad \text{as} \quad t \to 0$$

the second inequality of (ii) follows. q.e.d.

II.3 Consequences for the L^2–geometry of the moduli space $\mathcal{B}_0^+$

In this section, we shall study implications of the asymptotic behavior of the canonical metric $\mathfrak{g}_t$ for the geometry of the moduli space $\mathcal{B}_0^+$ of scalar positive locally conformally flat structures w.r.t. the L^2–metric $\mathfrak{h}$ induced by our canonical metrics (cf. §I.3).

We consider the locally conformally flat structures C_t and $C_{t,A}$ given in §II.1 and define the tangent vectors X_t and $X_{t,Y}$ at $[C_t]$ on $\mathcal{B}_0^+$ for $t \in (0,1)$ and $Y \in \mathrm{so}(n)$, where $\mathrm{so}(n)$ is the Lie algebra of $\mathrm{SO}(n)$, by

$$X_t = \frac{\mathrm{d}}{\mathrm{d}t}[C_t] \quad \text{and} \quad X_{t,Y} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left[C_{t,\exp(\tau Y)}\right]\Big|_{\tau=0}.$$

We are going to estimate the length of the vectors X_t and $X_{t,Y}$ w.r.t. $\mathfrak{h}$.

Theorem II.3.1. *As $t \to 0$,*

(i) $\mathfrak{h}(X_t, X_t) = O(\frac{1}{t})$ *and*

(ii) $\mathfrak{h}(X_{t,Y}, X_{t,Y}) = O(\frac{1}{|\log t|^2})$ *for each* $Y \in \mathrm{so}(n)$.

Proof. (i) Let the homeomorphisms $\Phi_{t,\tau} : M_t \to M_{t+\tau}$ be given by

$$x_i \circ \Phi_{t,\tau}(p) = \frac{[1-(t+\tau)^{\frac{1}{2}}]|x_i(p)| + [(t+\tau)^{\frac{1}{2}} - t^{\frac{1}{2}}]}{(1-t^{\frac{1}{2}})|x_i(p)|} x_i(p)$$

for $p \in W_{t,i} \subset M_t$, $i = 1, 2$, and

$$\Phi_{t,\tau}(p) = p \quad \text{for} \quad p \in M_0 \setminus \bigcup_{i=1}^{2} \{|x_i| < 1\}.$$

We set

$$\chi_t := \frac{\mathrm{d}}{\mathrm{d}\tau}\Phi^*_{t,\tau} g_{t+\tau}\Big|_{\tau=0}.$$

One checks that on $W_{t,i}$,

$$\chi_t = \frac{1}{t^{\frac{1}{2}}(t^{\frac{1}{2}}-1)r_i^3}\mathrm{d}r_i^2$$

and therefore

$$< \alpha_t^2 \chi_t, \alpha_t^2 \chi_t >_{\mathfrak{g}_t} = < \chi_t, \chi_t >_{g_t} = \frac{1}{t(1-t^{\frac{1}{2}})^2 r_i^2}$$

with $r_i = |x_i|$. Clearly,

$$\chi_t = 0 \quad \text{on} \quad M_0 \setminus \bigcup_{i=1}^{2} \{|x_i| < 1\}.$$

Using Theorem II.2.7(ii), it follows that

$$\begin{aligned} \|\alpha_t^2 \chi_t\|^2_{L^2(\mathfrak{g}_t)} &= \frac{1}{t(1-t^{\frac{1}{2}})^2} \sum_{i=1}^{2} \int_{W_{t,i}} \frac{1}{r_i^2} \tilde{\alpha}^n_{t,i} \mathrm{d}\mu_{E,i} \\ &\leq 2\frac{c_b^n \omega_{n-1}}{t(1-t^{\frac{1}{2}})^2} \int_{t^{\frac{1}{2}}}^{1} r^{n-3} \mathrm{d}r \\ &= O(\frac{1}{t}) \end{aligned}$$

as $t \to 0$, where $\mathrm{d}\mu_{E,i}$ is the volume element of $g_{E,i}$. Now, choosing convenient diffeomorphism $\tilde{\Phi}_{t,\tau} : M_t \to M_{t+\tau}$, near $\Phi_{t,\tau}$, we conclude that

$$\begin{aligned} \mathfrak{h}(X_t, X_t) &= \left\| P_{\mathfrak{g}_t} \left(\frac{\mathrm{d}}{\mathrm{d}\tau} \tilde{\Phi}^*_{t,\tau} \mathfrak{g}_{t+\tau} \Big|_{\tau=0} \right) \right\|^2_{L^2(\mathfrak{g}_t)} \\ &= \left\| P_{\mathfrak{g}_t} \left(\alpha_t^2 \, \frac{\mathrm{d}}{\mathrm{d}\tau} \tilde{\Phi}^*_{t,\tau} g_{t+\tau} \Big|_{\tau=0} \right) \right\|^2_{L^2(\mathfrak{g}_t)} \\ &\leq \left\| \alpha_t^2 \, \frac{\mathrm{d}}{\mathrm{d}\tau} \tilde{\Phi}^*_{t,\tau} g_{t+\tau} \Big|_{\tau=0} \right\|^2_{L^2(\mathfrak{g}_t)} \\ &\leq 2\|\alpha_t^2 \chi_t\|^2_{L^2(\mathfrak{g}_t)} \\ &= O(\frac{1}{t}) \end{aligned}$$

as $t \to 0$, where $P_{\mathfrak{g}_t} : S^2(M) \to \ker\delta_{\mathfrak{g}_t} \cap S^2_0(M, \mathfrak{g}_t)$ denotes the $L^2(\mathfrak{g}_t)-$orthogonal projection.

(ii) We fix $Y \in \mathrm{so}(n)$ and define the homeomorphisms $\Psi_{t,\tau} : M_t \to M_{t,\exp(\tau Y)}$ by

$$x_1 \circ \Psi_{t,\tau}(p) = \exp\left(\tau \frac{\log|x_1(p)|}{|\log t|} Y \right) x_1(p)$$

for $p \in \{t < |x_i| < 1\} \subset M_t$ and

$$\Psi_{t,\tau}(p) = p \quad \text{for} \quad p \in M_0 \setminus \bigcup_{i=1}^{2} \{|x_i| < 1\}.$$

One checks that

$$x_2 \circ \Psi_{t,\tau}(p) = \exp\left(\tau \frac{\log|x_2(p)|}{\log t} Y\right) x_2(p)$$

for $p \in \{t < |x_i| < 1\}$. Thus, $\Psi_{t,\tau}$ is indeed a homeomorphism. We set

$$\chi_{t,Y} := \frac{\mathrm{d}}{\mathrm{d}\tau} \Psi^*_{t,\tau} g_{t,\exp(\tau Y)}\bigg|_{\tau=0}$$

and assume without loss of generality that

$$Y = (Y_{ij}) \quad \text{with} \quad Y_{ij} = \delta_{2i}\delta_{1j} - \delta_{1i}\delta_{2j}.$$

Then, on $\{t < |x_i| < 1\}$,

$$\chi_{t,Y} = \frac{2\xi^2}{|\log t| r_1} \mathrm{d}r_1 \mathrm{d}\theta$$

where $r_1 = |x_1|$ and $x_1 = (x_{11}, \dots, x_{1n})$ with $x_{11} = \xi\cos\theta$ and $x_{12} = \xi\sin\theta$, and

$$\begin{aligned} < \alpha_t^2 \chi_{t,Y}, \alpha_t^2 \chi_{t,Y} >_{\mathfrak{g}_t} &= < \chi_{t,Y}, \chi_{t,Y} >_{g_t} \\ &= \frac{2\xi^2}{|\log t|^2 r_1^2} \\ &\leq \frac{2}{|\log t|^2}. \end{aligned}$$

Using again Theorem II.2.7(ii), we arrive at

$$\|\alpha_t^2 \chi_{t,Y}\|^2_{L^2(\mathfrak{g}_t)} = O\left(\frac{1}{|\log t|^2}\right)$$

as $t \to 0$, from which we derive the claim as in the proof of (i). q.e.d.

An immediate consequence of the last theorem is

Corollary II.3.2. *The curve $t \in (0,1) \longmapsto [C_t] \in \mathcal{B}_0^+$ on the moduli space $\mathcal{B}_0^+$ of scalar positive locally conformally flat structures has finite length w.r.t. $\mathfrak{h}$. In particular, $(\mathcal{B}_0^+, \mathfrak{h})$ is not complete.* q.e.d.

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