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THE SYMPLECTIC GLOBAL COORDINATES ON THE MODULI SPACE OF REAL PROJECTIVE STRUCTURES

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A *convex* real projective structure on a smooth surface M is a representation of M as a quotient of a convex domain $\Omega \subset \mathbb{RP}^2$ by a discrete group $\Gamma \subset \mathbf{PGL}(3,\mathbb{R})$ acting properly and freely on Ω. If $\chi(M) < 0$, then the equivalence classes of convex real projective structures form a moduli space $\mathfrak{P}(M)$ which is an extension of the Teichmüller space $\mathfrak{T}(M)$.

Wolpert [17] proved that the Weil-Petersson Kähler form of the Teichmüller space $\mathfrak{T}(M)$ of a closed surface $\Sigma(g,0)$ with $\chi(M) < 0$ is expressed by

$$\omega = \sum_{i=1}^{g} d\ell_i \wedge d\theta_i,$$

where ℓ_i, θ_i are Fenchel-Nielsen coordinates on $\mathfrak{T}(M)$. In this paper, I will prove $\mathfrak{P}(M)$ has analogous properties.

In Section 1, we study the set of positive hyperbolic elements $\mathbf{Hyp}_+$ of $\mathbf{PGL}(3,\mathbb{R})$ since the holonomy group Γ of a convex real projective structure lies in $\mathbf{Hyp}_+$. In Section 2, we show the parameters (ℓ, m) on $\mathfrak{P}(M)$ extend Fenchel-Nielsen's length parameter ℓ. Let π be the fundamental group of M and G a connected algebraic Lie group. In Section 3, we study local properties of $\mathrm{Hom}(\pi, G)/G$ since $\mathfrak{P}(M)$ embeds onto an open subset of $\mathrm{Hom}(\pi, \mathbf{PGL}(3,\mathbb{R}))/\mathbf{PGL}(3,\mathbb{R})$. In Section 4,

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we induce a symplectic form ω on $\mathfrak{P}(M)$ for a *closed* surface M by Fox's calculus and the fundamental cycle. In Section 5, we generalize $\mathfrak{P}(M)$ for a compact oriented surface M with *boundary*. Restriction to boundary components defines the symplectic leaves of the *parabolic* foliation of a Poisson structure on $\mathfrak{P}(M)$. We show the modified form $\tilde{\omega}$ is a symplectic form on each parabolic leaf. We actually calculate the symplectic forms $\tilde{\omega}$ for the pair of pants with using Mathematica. In last Section 6, we induce generalized twist parameters θ, β and prove that $\mathfrak{P}(M)$ is symplectomorphic to $\mathbb{R}^{16g-16}$.

Theorem 0.1. *Let $M = \Sigma(g, 0)$ be a closed smooth surface. Then the symplectic form on the moduli space $\mathfrak{P}(M)$ of convex real projective structures is*

$$\omega = \sum_{i=1}^{3g-3} d\ell_i \wedge d\theta_i + \sum_{i=1}^{3g-3} dm_i \wedge d\beta_i + \sum_{j=1}^{2g-2} dt_j \wedge ds_j,$$

where ℓ_i, m_i are length parameters, θ_i, β_i are twisting parameters, and s_j, t_j are internal parameters on $\mathfrak{P}(M)$.

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1. Real projective structures on a compact oriented surface

1.1. The real projective plane $\mathbb{RP}^2$ is the space of all lines through the origin in $\mathbb{R}^3$. Let A be an element of $\mathbf{GL}(3, \mathbb{R})$, the group of linear transformations of $\mathbb{R}^3$. Then A preserves lines through the origin and induces a *projective transformation* of $\mathbb{RP}^2$. The group of projective transformations of $\mathbb{RP}^2$ is denoted by $\mathbf{PGL}(3, \mathbb{R})$. Since the scalar matrices $\mathbb{R}^*$ in $\mathbf{GL}(3, \mathbb{R})$ act trivially on $\mathbb{RP}^2$, we have an exact sequence

$$\{1\} \to \mathbb{R}^* \to \mathbf{GL}(3, \mathbb{R}) \to \mathbf{PGL}(3, \mathbb{R}) \to \{1\}.$$

The homomorphism $\mathbf{GL}(3, \mathbb{R}) \to \mathbf{SL}(3, \mathbb{R})$ defined by

$$A \mapsto (\det A)^{-1/3} A$$

induces an isomorphism $\mathbf{PGL}(3,\mathbb{R}) \to \mathbf{SL}(3,\mathbb{R})$ as *analytic* groups. Thus from now on we shall identify the groups $\mathbf{PGL}(3,\mathbb{R})$ and $\mathbf{SL}(3,\mathbb{R})$.

1.2. An element $A \in \mathbf{SL}(3,\mathbb{R})$ is called *positive hyperbolic* if it has three distinct positive real eigenvalues. The set of positive hyperbolic elements of $\mathbf{SL}(3,\mathbb{R})$ is denoted by $\mathbf{Hyp}_+$. If A is positive hyperbolic, then it can be represented by the diagonal matrix

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix} \tag{1}$$

with

$$\lambda\mu\nu = 1, \quad 0 < \lambda < \mu < \nu \tag{2}$$

via an $\mathbf{SL}(3,\mathbb{R})$-conjugation. We define

$$(\lambda, \tau) : \mathbf{Hyp}_+ \to \mathbb{R}^2$$

by $\lambda(A)$ to be the smallest eigenvalue of A, and $\tau(A)$ the sum of the other two eigenvalues. For the A above,

$$\lambda(A) = \lambda, \quad \tau(A) = \mu + \nu.$$

As you see the pair $(\lambda(A), \tau(A))$ is invariant under $\mathbf{SL}(3,\mathbb{R})$-conjugation.

The following two Propositions 1.1 and 1.2 are works from Goldman [8].

Proposition 1.1. *Consider the action of* $\mathbf{SL}(3,\mathbb{R})$ *on* $\mathbf{Hyp}_+$ *by conjugation. Then* $(\lambda, \tau) : \mathbf{Hyp}_+ \to \mathbb{R}^2$ *is an* $\mathbf{SL}(3,\mathbb{R})$*-invariant fibration with the image*

$$\mathcal{I} = \{(\lambda, \tau) \in \mathbb{R}^2 \mid 0 < \lambda < 1,\ 2/\sqrt{\lambda} < \tau < \lambda + \lambda^{-2}\}.$$

If $(\lambda, \tau) \in \mathcal{I}$, then $A \in \mathbf{Hyp}_+$ is determined by

$$\begin{aligned} \lambda &= \lambda(A), \\ \mu &= \frac{1}{2}\left\{\tau(A) - \sqrt{\tau(A)^2 - 4/\lambda(A)}\right\}, \\ \nu &= \frac{1}{2}\left\{\tau(A) + \sqrt{\tau(A)^2 - 4/\lambda(A)}\right\} \end{aligned}$$

up to $\mathbf{SL}(3,\mathbb{R})$-conjugation.

Another pair of invariants of a positive hyperbolic element is more geometric. They are related to the translation lengths of A. I will explain this in Section 2. We define

$$\ell(A) = \log\left(\frac{\nu}{\lambda}\right), \quad m(A) = \frac{3}{2}\log(\mu),$$

where A is a positive hyperbolic projective transformation represented by the diagonal matrix (1) with $\lambda\mu\nu = 1$ and $0 < \lambda < \mu < \nu$.

Proposition 1.2. *Consider the action of* $\mathbf{SL}(3,\mathbb{R})$ *on* $\mathbf{Hyp}_+$ *by conjugation. Then* $(\ell, m) : \mathbf{Hyp}_+ \to \mathbb{R}^2$ *is an* $\mathbf{SL}(3,\mathbb{R})$*-invariant fibration with the image*

$$\mathcal{I}' = \{(\ell, m) \in \mathbb{R}^2 \mid \ell > 0,\ -\frac{1}{2}\,\ell < m < \frac{1}{2}\,\ell\ \}.$$

If $(\ell, m) \in \mathcal{I}'$, then $A \in \mathbf{Hyp}_+$ is determined by

$$\begin{aligned} \lambda &= \exp\left(-\frac{\ell(A)}{2} - \frac{m(A)}{3}\right), \\ \mu &= \exp\left(\frac{2\,m(A)}{3}\right), \\ \nu &= \exp\left(\frac{\ell(A)}{2} - \frac{m(A)}{3}\right) \end{aligned}$$

up to $\mathbf{SL}(3,\mathbb{R})$-conjugation. Therefore (λ, τ) and (ℓ, m) are two sets of coordinates on the conjugacy classes in $\mathbf{Hyp}_+$.

1.3. Let Ω be an open subset of $\mathbb{RP}^2$. A map $\phi : \Omega \to \mathbb{RP}^2$ is called *locally projective* if for each component $W \subset \Omega$, there exists a projective transformation $g \in \mathbf{PGL}(3,\mathbb{R})$ such that $\phi|_W = g|_W$. Clearly a locally projective map is a local diffeomorphism. Let M be a connected smooth surface. A *real projective structure* or $\mathbb{RP}^2$*-structure* on M is a maximal collection $\{(U_\alpha, \psi_\alpha)\}$ such that:

1. $\{U_\alpha\}$ is an open covering of M.

2. For each α, $\psi_\alpha : U_\alpha \to \mathbb{RP}^2$ is a diffeomorphism onto its image.

3. The change of coordinates is locally projective ; If (U_α, ψ_α) and (U_β, ψ_β) are two coordinate charts with $U_\alpha \cap U_\beta \neq \emptyset$, then $\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta)$ is locally projective.

A smooth surface with an $\mathbb{RP}^2$-structure is called an $\mathbb{RP}^2$-manifold. Let M, N be smooth surfaces. If $f : M \to N$ is a smooth map which is a local diffeomorphism and N is an $\mathbb{RP}^2$-manifold, then we can give an $\mathbb{RP}^2$-structure on M via f. In particular every covering space of an $\mathbb{RP}^2$-manifold has the canonically induced $\mathbb{RP}^2$-structure.

The following *Development Theorem* is the fundamental fact about $\mathbb{RP}^2$-structures.

Theorem 1.3 (Ch. Ehresmann [5]). *Let $p : \tilde{M} \to M$ denote a universal covering map of an $\mathbb{RP}^2$-manifold M, and π the corresponding group of covering transformations.*

1. *There exist a projective map $\mathbf{dev} : \tilde{M} \to \mathbb{RP}^2$ and homomorphism $h : \pi \to \mathbf{PGL}(3, \mathbb{R})$ such that for each $\gamma \in \pi$ the following diagram commutes:*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\mathbf{dev}} & \mathbb{RP}^2 \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle h(\gamma)} \\ \tilde{M} & \xrightarrow[\mathbf{dev}]{} & \mathbb{RP}^2 \end{array}$$

2. *Suppose $(\mathbf{dev}', h')$ is another pair satisfying above conditions. Then there exists a projective transformation $g \in \mathbf{PGL}(3, \mathbb{R})$ such that $\mathbf{dev}' = g \circ \mathbf{dev}$ and $h' = \iota_g \circ h$ where $\iota_g : \mathbf{PGL}(3, \mathbb{R}) \to \mathbf{PGL}(3, \mathbb{R})$ denotes the inner automorphism defined by g ; that is, $h'(\gamma) = (\iota_g \circ h)(\gamma) = g \circ h(\gamma) \circ g^{-1}$:*

$$\begin{array}{ccccc} \tilde{M} & \xrightarrow{\mathbf{dev}} & \mathbb{RP}^2 & \xrightarrow{g} & \mathbb{RP}^2 \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle h(\gamma)} & & \downarrow{\scriptstyle h'(\gamma)} \\ \tilde{M} & \xrightarrow[\mathbf{dev}]{} & \mathbb{RP}^2 & \xrightarrow[g]{} & \mathbb{RP}^2 \end{array}$$

The projective map $\mathbf{dev} : \tilde{M} \to \mathbb{RP}^2$ is called a *developing map*. The homomorphism $h : \pi \to \mathbf{PGL}(3, \mathbb{R})$ is called the *holonomy homomorphism*. The image $\Gamma = h(\pi) \subset \mathbf{PGL}(3, \mathbb{R})$ is called the *holonomy group*.

The developing image $\mathbf{dev}(\tilde{M}) \subset \mathbb{RP}^2$ is a Γ-invariant open set. In many cases the developing map is a diffeomorphism from $\tilde{M}$ onto $\mathbf{dev}(\tilde{M})$. In this case the holonomy homomorphism is an isomorphism from π onto a discrete subgroup of $\mathbf{PGL}(3, \mathbb{R})$ which acts properly and freely on $\mathbf{dev}(\tilde{M})$. In general it is not; we can find examples in Sullivan and Thurston's paper [15].

A domain $\Omega \subset \mathbb{RP}^2$ is called *convex* if there exist a projective line $l \subset \mathbb{RP}^2$ such that $\Omega \subset (\mathbb{RP}^2 - l)$, and Ω is a convex subset of the affine plane $\mathbb{RP}^2 - l$; that is if $x, y \in \Omega$, then the line segment $\bar{xy}$ lies in Ω. By definition, $\mathbb{RP}^2$ itself is not convex. A convex domain Ω is called *strictly* convex if Ω is a proper subset of $\mathbb{RP}^2 - l$. A *geodesic* on M is a curve $g \subset M$ such that for each component $\tilde{g}_0 \subset p^{-1}(g) \subset \tilde{M}$, the developing map takes $\tilde{g}_0$ into a line in $\mathbb{RP}^2$.

An $\mathbb{RP}^2$-structure on M is *convex* if **dev** is a diffeomorphism onto a convex domain in $\mathbb{RP}^2$. If M is a convex $\mathbb{RP}^2$-manifold with boundary, then we always assume each boundary component is geodesic. The following fundamental facts are from Goldman's paper [8].

Proposition 1.4. *Let M be an $\mathbb{RP}^2$-manifold. Then the following statements are equivalent:*

1. *M has a convex $\mathbb{RP}^2$-structure.*

2. *M is projectively isomorphic to a quotient Ω/Γ where $\Omega \subset \mathbb{RP}^2$ is a convex domain and $\Gamma \subset \mathbf{PGL}(3, \mathbb{R})$ is a discrete group acting properly and freely on Ω.*

3. *Every path in M is homotopic (relative end points) to a unique geodesic path.*

If M is a convex $\mathbb{RP}^2$-manifold, then we can identify $M = \Omega/\Gamma$ where Ω is the developing image and Γ the holonomy group.

Theorem 1.5 (N. Kuiper [12]). *Let $M = \Omega/\Gamma$ be a compact oriented surface with a convex $\mathbb{RP}^2$-structure. Suppose that $\chi(M) < 0$. Then the following statements hold.*

1. *$\Omega \subset \mathbb{RP}^2$ is a strictly convex domain with C^1 boundary and therefore contains no affine line.*

2. *Either $\partial\Omega$ is a conic in $\mathbb{RP}^2$ or is not $C^{1+\varepsilon}$ for any $0 < \varepsilon < 1$.*

3. *If $A \in \Gamma$ is nontrivial, then $A \in \mathbf{Hyp}_+$. Furthermore every homotopically nontrivial closed curve on M is freely homotopic to a unique closed geodesic.*

2. The geometric meaning of invariants (ℓ, m)

In this section we will show the invariants (ℓ, m) extend Fenchel-Nielsen's length parameter. See Chapter 2 of Abikoff's book [1] about Fenchel-Nielsen coordinates on the Teichmüller space.

2.1. For any $A \in \mathbf{Hyp}_+$ there exist three non-collinear fixed points and an A-invariant line in $\mathbb{RP}^2$. We define:

- $\mathrm{Fix}_-(A) \Leftrightarrow$ the *repelling* fixed point of $A \Leftrightarrow$ the fixed point corresponding to an eigenvector in $\mathbb{R}^3$ for the smallest eigenvalue λ,
- $\mathrm{Fix}_0(A) \Leftrightarrow$ the *saddle* fixed point of $A \Leftrightarrow$ the fixed point corresponding to an eigenvector in $\mathbb{R}^3$ for the middle eigenvalue μ,
- $\mathrm{Fix}_+(A) \Leftrightarrow$ the *attracting* fixed point of $A \Leftrightarrow$ the fixed point corresponding to an eigenvector in $\mathbb{R}^3$ for the largest eigenvalue ν,
- $\sigma(A) \Leftrightarrow$ the *principal line* of $A \Leftrightarrow$ the line joining the repelling and attracting fixed points of A.

Before we discuss the geometric meaning of the invariants (ℓ, m), we need some knowledge about the cross-ratio and Hilbert distance.

Let $\bar{\mathbb{C}}$ denote $\mathbb{C} \cup \{\infty\}$ the extended complex numbers, and $\mathcal{D}_4(\bar{\mathbb{C}})$ the subset of $\bar{\mathbb{C}} \times \bar{\mathbb{C}} \times \bar{\mathbb{C}} \times \bar{\mathbb{C}}$ consisting of distinct four points. The *cross-ratio* is the mapping $\mathbf{X} : \mathcal{D}_4(\bar{\mathbb{C}}) \to \bar{\mathbb{C}}$ defined by

$$\mathbf{X}\{w_1, w_2, w_3, w_4\} = \frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_2)(w_3 - w_4)},$$

and is invariant under the $\mathbf{GL}(2, \mathbb{C})$-action on $\mathcal{D}_4(\bar{\mathbb{C}})$. Let $a_i, 1 \leq i \leq 4$, be *collinear* distinct four points of $\mathbb{RP}^2$. Then there exists $B \in \mathbf{SL}(3, \mathbb{R})$ such that the second homogeneous coordinate of each $B(a_i)$ is zero. Through the identification

$$\left[\begin{array}{c} x \\ 0 \\ z \end{array}\right] = \begin{cases} x/z & \text{if } z \neq 0, \\ \infty & \text{if } z = 0\ , \end{cases} \tag{3}$$

we can think of $B(a_i) \in \mathbb{R} \cup \{\infty\}$ the extended real line. Furthermore they are distinct. Let $\mathcal{CD}_4(\mathbb{RP}^2)$ denote the subset of $\mathbb{RP}^2 \times \mathbb{RP}^2 \times \mathbb{RP}^2 \times \mathbb{RP}^2$ consisting of all collinear distinct four points. The cross-ratio $\mathbf{CR} : \mathcal{CD}_4(\mathbb{RP}^2) \to \mathbb{R}$ is defined by

$$\mathbf{CR}\{a_1, a_2, a_3, a_4\} = \mathbf{X}\{B(a_1), B(a_2), B(a_3), B(a_4)\},$$

where $B \in \mathbf{SL}(3,\mathbb{R})$ satisfying $B(a_i)$ lies in the extended real line $\mathbb{R} \cup \{\infty\}$ through the identification (3) for each i. Suppose B' is another such element in $\mathbf{SL}(3,\mathbb{R})$; then we can show $B^{-1}B' \in \mathbf{SL}(2,\mathbb{R})$ via the identification

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \leftrightarrow \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{bmatrix}. \tag{4}$$

Therefore the cross-ratio $\mathbf{CR}$ on $\mathcal{CD}_4(\mathbb{RP}^2)$ is independent of the choice of $B \in \mathbf{SL}(3,\mathbb{R})$.

The *Hilbert distance* $h : \mathbb{RP}^2 \times \mathbb{RP}^2 \to \mathbb{R}^+$ is defined by

$$h(a,b) = \inf_{\vec{xy}} \left(\log \mathbf{CR}\{x,a,b,y\}\right),$$

where the points a,b lie on the oriented segment $\vec{xy}$ with a is the first, b the second point and inf runs over all such $\vec{xy}$.

FIGURE 1. The Hilbert distance

The Hilbert distance defines a metric on a *strictly convex* subset Ω of $\mathbb{RP}^2$ and it is called the *Hilbert metric*. From Theorem 1.5, if $\partial\Omega$ is a conic in the Klein model of unit disc hyperbolic space, then the Hilbert metric is the hyperbolic metric.

2.2. Convex real projective structures are an extension of hyperbolic structures. We are going to adapt basic properties of hyperbolic structures to convex real projective structures. For a hyperbolic manifold M, let Ω be the developing image in $\mathbb{H}^2$ and A be an element of the holonomy group $\Gamma \subset \mathbf{PSL}(2,\mathbb{R})$. Then A is a hyperbolic element. The translation length t_A of A is defined by

$$t_A = \inf_{z\in\Omega} \rho(z, A(z)),$$

where ρ is the hyperbolic metric on Ω. Then the translation length t_A of A is achieved if and only if z lies on the principal line of A. We can extend the concept of translation length to real projective structures.

Proposition 2.1. *Let $a \in \mathbb{RP}^2$ and $A \in \mathbf{Hyp}_+$. Then the Hilbert distance*

$$h(a, A(a)) = \log \mathbf{CR}\{\mathrm{Fix}_-(A), a, A(a), \mathrm{Fix}_+(A)\}.$$

Proof. Since $A^{-n}(a) \to \mathrm{Fix}_-(A)$ and $A^m(a) \to \mathrm{Fix}_+(A)$ as $n, m \to \infty$, it is enough to show that

$$\mathbf{CR}\{A^{-n}(a), a, A(a), A^m(a)\} > \mathbf{CR}\{A^{-n-1}(a), a, A(a), A^{m+1}(a)\}.$$

Without loss of generality we may let

$$a = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}, \quad A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix},$$

since there exists $B \in \mathbf{SL}(3, \mathbb{R})$ such that $B(a) = [x', 0, z']^t$ and $\mathbf{CR}$ is invariant under $\mathbf{SL}(3, \mathbb{R})$-conjugation. Then

$$\begin{aligned} &\mathbf{X}\{A^{-n}(a), a, A(a), A^m(a)\} \\ &= \mathbf{X}\left\{[\lambda^{-n}x, 0, \nu^{-n}z]^t, [x, 0, z]^t, [\lambda x, 0, \nu z]^t, [\lambda^m x, 0, \nu^m z]^t\right\} \\ &= \frac{(\alpha^{-n}w - \alpha w)(w - \alpha^m w)}{(\alpha^{-n}w - w)(\alpha w - \alpha^m w)} = \frac{(\alpha^{-n} - \alpha)(1 - \alpha^m)}{(\alpha^{-n} - 1)(\alpha - \alpha^m)}, \end{aligned}$$

where $\alpha = \lambda/\nu$ and $w = x/z$. By an easy computation we can show for $0 < \alpha < 1$

$$\frac{(\alpha^{-n} - \alpha)(1 - \alpha^m)}{(\alpha^{-n} - 1)(\alpha - \alpha^m)} > \frac{(\alpha^{-n-1} - \alpha)(1 - \alpha^{m+1})}{(\alpha^{-n-1} - 1)(\alpha - \alpha^{m+1})},$$

and the proposition is proved. q.e.d.

For any $A \in \mathbf{Hyp}_+$, A can be uniquely decomposed as HV up to $\mathbf{SL}(3, \mathbb{R})$-conjugation where

$$H = \begin{bmatrix} \lambda\sqrt{\mu} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu\sqrt{\mu} \end{bmatrix}, \qquad V = \begin{bmatrix} 1/\sqrt{\mu} & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1/\sqrt{\mu} \end{bmatrix}. \tag{5}$$

We call H the *horizontal* factor and V the *vertical* factor of the decomposition of a positive hyperbolic transformation. Sometimes H will be called the *pure hyperbolic* factor.

Consider H. By Proposition 2.1, the Hilbert distance between a and $H(a)$ for any $a = [1-s, 0, s]^t \in \sigma(A) = \sigma(H)$ is

$$\begin{aligned} &h(a, H(a)) \\ &\quad = \log \mathbf{CR} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1-s \\ 0 \\ s \end{bmatrix}, \begin{bmatrix} \lambda\sqrt{\mu}\,(1-s) \\ 0 \\ \nu\sqrt{\mu}\,s \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \\ &\quad = \log \frac{\left(\infty - \frac{\lambda}{\nu}\frac{1-s}{s}\right)\left(\frac{1-s}{s} - 0\right)}{\left(\infty - \frac{1-s}{s}\right)\left(\frac{\lambda}{\nu}\frac{1-s}{s} - 0\right)} = \log\left(\frac{\nu}{\lambda}\right) = \ell(A). \end{aligned}$$

We call $\ell(A) = h(a, H(a))$ the *horizontal translation length*, and it is the length of the boundary component represented by A.

FIGURE 2. The horizontal translation length

Consider V. Then the stationary set is the line joining $[1,0,0]^t$ and $[0,0,1]^t$ and the fixed point $[0,1,0]^t$. Without loss of generality we assume $\mu > 1$. Thus for any $a = [1-s, y, s]^t$ in the segment joining $[1-s, 0, s]^t$ and $[0,1,0]^t$, the point $V(a)$ goes toward $[0,1,0]^t$ since $\mu > 1$.

Hence the Hilbert distance between a and $V(a)$ is

$$
\begin{aligned}
&h(a, V(a)) \\
&= \log \mathbf{CR} \left\{ \begin{bmatrix} 1-s \\ 0 \\ s \end{bmatrix}, \begin{bmatrix} 1-s \\ y \\ s \end{bmatrix}, \begin{bmatrix} \frac{1-s}{\sqrt{\mu}} \\ y\mu \\ \frac{1-s}{\sqrt{\mu}} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \\
&= \log \mathbf{X} \left\{ B \begin{bmatrix} 1-s \\ 0 \\ s \end{bmatrix}, B \begin{bmatrix} 1-s \\ y \\ s \end{bmatrix}, B \begin{bmatrix} \frac{1-s}{\sqrt{\mu}} \\ y\mu \\ \frac{1-s}{\sqrt{\mu}} \end{bmatrix}, B \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \\
&\qquad \text{where } B = \begin{bmatrix} (1-s)^{-1} & 0 & 0 \\ s & \mu & s-1 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \log \mathbf{X} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ y \end{bmatrix}, \begin{bmatrix} 1/\sqrt{\mu} \\ 0 \\ y\mu \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \\
&= \log \frac{\left(\infty - y^{-1}\mu^{-\frac{3}{2}}\right)\left(y^{-1} - 0\right)}{\left(\infty - y^{-1}\right)\left(y^{-1}\mu^{-\frac{3}{2}} - 0\right)} = \log\left(\mu^{\frac{3}{2}}\right) = \frac{3}{2}\log(\mu) = m(A).
\end{aligned}
$$

We call $m(A) = h(a, V(a))$ the *vertical translation length.*

FIGURE 3. The vertical translation length

We can easily compute the relations among H, V and A.

$$\begin{aligned}
\ell(H) &= \log\left(\frac{\nu\sqrt{\mu}}{\lambda\sqrt{\mu}}\right) = \log\left(\frac{\nu}{\lambda}\right) = \ell(A),\\
m(H) &= \frac{3}{2}\log(1) = 0,\\
\ell(V) &= \log\left(\frac{1/\sqrt{\mu}}{1/\sqrt{\mu}}\right) = \log(1) = 0,\\
m(V) &= \frac{3}{2}\log(\mu) = m(A),
\end{aligned}$$

which imply that the horizontal translation lengths of A and H are the same, and the vertical translation lengths of A and V are the same. Therefore H is the pure hyperbolic structure of A. If $V = I$, then A is derived from the hyperbolic structures.

3. Local properties of $\mathbf{Hom}(\pi, \mathbf{SL}(3,\mathbb{R}))/\mathbf{SL}(3,\mathbb{R})$

3.1. Let M be a compact oriented smooth surface, $\tilde{M}$ a fixed universal covering space, and $\pi = \pi_1(M)$ the corresponding group of covering transformations. Consider (f, N) where N is an $\mathbb{RP}^2$-manifold and $f : M \to N$ a diffeomorphism. Such a pair is equivalent to a developing pair $(\mathbf{dev}, h)$ where $\mathbf{dev} : \tilde{M} \to \mathbb{RP}^2$ and $h : \pi \to \mathbf{SL}(3,\mathbb{R})$. We say two pairs (f, N) and (f', N') are *equivalent* if there exists a projective isomorphism $g : N \to N'$ such that $g \circ f$ is isotopic to f' ; that is there exists a diffeomorphism $g' : M \to M$ such that g' is isotopic to the identity map I_M and the following diagram commutes :

$$\begin{array}{ccc}
M & \xrightarrow{f} & N \\
{\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\
M & \xrightarrow[f']{} & N'
\end{array}$$

The set of equivalence classes is denoted by $\mathbb{RP}^2(M)$ and is called the *deformation space* or *moduli space* of $\mathbb{RP}^2$-structures on M. The deformation space $\mathbb{RP}^2(M)$ has the natural topology that is locally equivalent to the quotient space $\mathrm{Hom}(\pi, \mathbf{SL}(3,\mathbb{R}))/\mathbf{SL}(3,\mathbb{R})$.

Let $\mathfrak{P}(M)$ denote the subset of $\mathbb{RP}^2(M)$ corresponding to the *convex* $\mathbb{RP}^2$-structures. Goldman showed $\mathfrak{P}(M)$ is homeomorphic to $\mathbb{R}^{-8\cdot\chi(M)}$.

Theorem 3.1 (W. Goldman [8]). *Let M be a compact oriented surface with $\chi(M) < 0$. Then $\mathfrak{P}(M)$ is an open subset of $\mathbb{RP}^2(M)$ and the restriction of* $\mathbf{hol} : \mathbb{RP}^2(M) \to \mathrm{Hom}(\pi, \mathbf{SL}(3,\mathbb{R}))/\mathbf{SL}(3,\mathbb{R})$ *to $\mathfrak{P}(M)$ is an embedding of $\mathfrak{P}(M)$ onto a Hausdorff real analytic manifold of dimension $-8 \cdot \chi(M)$.*

3.2. Since $\mathfrak{P}(M)$ embeds into $\mathrm{Hom}(\pi, \mathbf{SL}(3,\mathbb{R}))/\mathbf{SL}(3,\mathbb{R})$, we need to know the basic properties of $\mathrm{Hom}(\pi, G)/G$ where π denotes the fundamental group of a compact oriented smooth surface M, and G is a connected Lie group. Suppose $M = \Sigma(g, n)$ is a smooth surface with genus g, n boundary components and $\chi(M) = 2 - 2g - n < 0$. Then π admits $2g + n$ generators $A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_n$ with a single relation

$$R = A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} C_1 \cdots C_n.$$

Thus $\mathrm{Hom}(\pi, G)$ may be identified with the collection of all $(2g+n)$-tuples

$$(A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_n) \subset G^{2g+n}$$

elements of G satisfying

$$R(A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_n) = I.$$

Since $R : G^{2g+n} \to G$ is a polynomial equation and

$$\mathrm{Hom}(\pi, G) = R^{-1}(I) \subset G^{2g+n},$$

if G is an algebraic Lie group, then $\mathrm{Hom}(\pi, G)$ is an *algebraic variety.*

In general $\mathrm{Hom}(\pi, G)$ is not smooth. Suppose $\phi \in \mathrm{Hom}(\pi, G)$ and $Z(\phi)$ is the centralizer of $\phi(\pi)$ in G. Goldman [7] showed ϕ is a nonsingular point of $\mathrm{Hom}(\pi, G)$ if and only if dim $Z(\phi)/Z(G) = 0$ where $Z(G)$ denotes the center of G. Let $\mathrm{Hom}(\pi, G)^-$ be the set of nonsingular points of $\mathrm{Hom}(\pi, G)$. Then G acts freely on the smooth Zariski open subset $\mathrm{Hom}(\pi, G)^-$. But unfortunately $\mathrm{Hom}(\pi, G)^-/G$ is generally not Hausdorff. Let $\mathrm{Hom}(\pi, G)^{--}$ be the subset of $\mathrm{Hom}(\pi, G)^-$ consisting of homomorphisms whose image does not lie in a parabolic subgroup of G. Then $\mathrm{Hom}(\pi, G)^{--}$ is a Zariski open subset of $\mathrm{Hom}(\pi, G)^-$, and $\mathrm{Hom}(\pi, G)^{--}/G$ is a Hausdorff smooth manifold of dimension $-$dim $G \cdot \chi(M)$. For more detail see Goldman's paper [7].

Suppose $\phi \in \mathrm{Hom}(\pi, G)$ is an irreducible representation. Then $\phi \in \mathrm{Hom}(\pi, G)^{--}$. Since the holonomy representation of a real projective structure from the fundamental group π of a compact oriented surface to an algebraic Lie group G is irreducible, we restrict our interest to $\mathrm{Hom}(\pi, G)^{--}$.

3.3. $\mathrm{Hom}(\pi, G)^{--}/G$ is the quotient space of $\mathrm{Hom}(\pi, G)^{--}$ with $\phi' \sim \phi$ if and only if there exists $g \in G$ such that

$$\phi'(\gamma) = g \circ \phi(\gamma) \circ g^{-1}$$

for all $\gamma \in \pi$. Let $\phi \in \mathrm{Hom}(\pi, G)^{--}$ and $[\phi] \in \mathrm{Hom}(\pi, G)^{--}/G$ its equivalence class. We want to show the tangent space to $\mathrm{Hom}(\pi, G)^{--}/G$ at $[\phi]$ is isomorphic to the first group cohomology $H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$. To do this we need some knowledge about group cohomology [14].

Suppose π is a group and V is a π-module. Then there exists a linear representation $\rho : \pi \to \mathrm{End}(V)$. We define $C^0(\pi; V_\rho) = V$, and $C^p(\pi; V_\rho)$ consists of all maps $\pi^p \to V$ for $p > 0$. The coboundary operator $d : C^p(\pi; V_\rho) \to C^{p+1}(\pi; V_\rho)$ is defined by

$$\begin{aligned} df(g_1, \ldots, g_{p+1}) &= \rho(g_1) f(g_2, \ldots, g_{p+1}) \\ &\quad + \sum_{i=1}^{p} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{p+1}) \\ &\quad + (-1)^{p+1} f(g_1, \ldots, g_p), \end{aligned}$$

where $f \in C^p(\pi; V_\rho)$, $g_1, \ldots, g_{p+1} \in G$. Since $d \circ d = 0$, we can construct the cohomology space called the *group cohomology*:

$$H^p(\pi; V_\rho) = \frac{Z^p(\pi; V_\rho)}{B^p(\pi; V_\rho)} = \frac{\text{kernel } (d : C^p(\pi; V_\rho) \to C^{p+1}(\pi; V_\rho))}{\text{image } (d : C^{p-1}(\pi; V_\rho) \to C^p(\pi; V_\rho))}.$$

Consider coboundary operators for $p = 0$ and 1. For any

$$v \in C^0(\pi; V_\rho) = V, \; f \in C^1(\pi; V_\rho)$$

and $x, y \in \pi$, we have

$$dv(x) = \rho(x) v - v \tag{6}$$

$$df(x, y) = \rho(x) f(y) - f(xy) + f(x). \tag{7}$$

Therefore $f \in Z^1(\pi; V_\rho)$ if and only if $f(xy) = f(x) + \rho(x) f(y)$ for all $x, y \in \pi$, and $f \in B^1(\pi; V_\rho)$ if and only if there exists $v \in V$ such that $f(x) = dv(x) = \rho(x) v - v$ for all $x \in \pi$.

Let $\mathfrak{g}$ be the Lie algebra of G. Then $\mathfrak{g}$ is a π-module $\mathfrak{g}_{\mathrm{Ad}\phi}$ by the composition

$$\pi \xrightarrow{\phi} G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g}).$$

Recall $\mathrm{Ad}(g)(X) = (r_{g^{-1}})_* X$ where $g \in G, X \in \mathfrak{g}$, and $r_{g^{-1}}$ is the right translation.

Proposition 3.2 (A. Weil [16]). *Let π be the fundamental group of a compact oriented surface, G a connected algebraic Lie group, and $\phi \in \mathrm{Hom}(\pi, G)^{--}$. Then the tangent space to $\mathrm{Hom}(\pi, G)^{--}/G$ at $[\phi]$ is isomorphic to the first group cohomology $H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$.*

4. The symplectic structure on Hom$(\pi, G)^{--}/G$

Atiyah and Bott [2] initiated a new approach to the study of moduli spaces of homomorphisms from fundamental groups of surfaces to compact connected Lie groups by methods of the gauge theory. In particular they found natural symplectic structures on certain smooth open subsets of these moduli spaces.

In this paper I will extend the symplectic form to the compact oriented surface with boundary. Before doing this, we first study the symplectic form on the moduli space of a closed surface.

4.1. Fox [6] developed a noncommutative differential calculus for words in a free group. It turns out Fox's calculus is a very useful tool for calculations in spaces of representations $\mathrm{Hom}(\pi, G)$. The goal of this section is to determine the explicit formula for the symplectic 2-form on $\mathrm{Hom}(\pi, G)^{--}/G$ using Fox's calculus. We start with the summary of Fox's calculus.

Let Π be a free group with generators $x_1, \dots, x_n$, and $\mathbb{Z}\Pi$ its integral group ring. For every element $\sum n_i \sigma_i \in \mathbb{Z}\Pi$, we assume $n_i = 0$ for all but finitely many. The *augmentation* homomorphism $\varepsilon : \mathbb{Z}\Pi \to \mathbb{Z}$ is a ring homomorphism defined by

$$\varepsilon\left(\sum n_i \sigma_i\right) = \sum n_i.$$

The *Fox derivation* of $\mathbb{Z}\Pi$ is a $\mathbb{Z}$-linear map $D : \mathbb{Z}\Pi \to \mathbb{Z}\Pi$ satisfying

$$D(m_1 m_2) = D(m_1)\varepsilon(m_2) + m_1 D(m_2),$$

where $m_1, m_2 \in \mathbb{Z}\Pi$. The integral group ring $\mathbb{Z}\Pi$ is a Π-bimodule where Π acts on the left by ordinary left-multiplication and on the right by trivial. Since for any $x, y \in \Pi$,

$$D(xy) = D(x)\varepsilon(y) + xD(y) \quad \text{and} \quad \varepsilon(y) = 1,$$

the Fox derivation is a 1-cocycle on Π with coefficients in $\mathbb{Z}\Pi$.

Proposition 4.1 (R. Fox [6]). *Suppose $x_1, \dots, x_n$ are the generators for group Π and $\mathrm{Der}(\Pi)$ is the set of all Fox derivations. We define $(D \circ m)(x) = D(x)\varepsilon(m)$ for $D \in \mathrm{Der}(\Pi)$, $m \in \mathbb{Z}\Pi$ and $x \in \Pi$. Then $\mathrm{Der}(\Pi)$ is freely generated as a right $\mathbb{Z}\Pi$-module by n element $\partial_i = \partial/\partial x_i, i = 1, \dots, n$ such that $(\partial/\partial x_i)(x_j) = \delta_{ij} I$ where I is the identity element of Π.*

For any word $w \in \Pi$, computing the Fox derivation ∂w is quite mechanical. The following are examples which we need later. For any generators $A, B \in \Pi$, we have

- $\partial_A(AB) = \partial_A(A)\varepsilon(B) + A\partial_A(B) = I$;
- $\partial_B(AB) = \partial_B(A)\varepsilon(B) + A\partial_B(B) = A$;
- $\partial_A(A^{-1}) = -A^{-1}$;
- $\partial_A(ABA^{-1}) = I - ABA^{-1}$;
- $\partial_B(ABA^{-1}) = A$;
- $\partial_A(ABA^{-1}B^{-1}) = I - ABA^{-1}$;
- $\partial_B(ABA^{-1}B^{-1}) = A - ABA^{-1}B^{-1}$.

From the above examples we can see for any word $w \in \Pi$, $\varepsilon(\partial_i w)$ equals the total exponent sum of the letter x_i in the word w. Fox proves that these free derivations satisfy a very useful rule of differential calculus ; the Mean Value Theorem.

Proposition 4.2 (Fox[6]). *Suppose $x_1, \dots, x_n$ are the generators for a group Π. Then for any $m \in \mathbb{Z}\Pi$,*

$$\sum_{i=1}^{n} (\partial_i m)(x_i - I) = m - \varepsilon(m)I.$$

4.2. To write the explicit formula for the symplectic structure ω on $\mathrm{Hom}(\pi, G)^{--}/G$, we need more knowledge about the group homology theory [4]. Let Π be a group and $\mathbb{Z}\Pi$ the integral group ring. Let $C_n(\Pi)$ be the $\mathbb{Z}$-module freely generated by $\Pi^n = \Pi \times \cdots \times \Pi$ and $C_0(\Pi) = \mathbb{Z}$. We define the boundary operator $\partial_n : C_n(\Pi) \to C_{n-1}(\Pi)$ by for $n \geq 2$

$$\begin{aligned} \partial_n(u_1, \dots, u_n) &= \varepsilon(u_1)(u_2, \dots, u_n) \\ &\quad + \sum_{i=1}^{n-1} (-1)^i (u_1, \dots, u_i u_{i+1}, \dots, u_n) \\ &\quad + (-1)^n (u_1, \dots, u_{n-1})\varepsilon(u_n), \end{aligned}$$

and $\partial_1(u) = 0$. Since $\partial_n \circ \partial_{n+1} = 0$, we can construct the homology space called the *group homology*:

$$H_n(\Pi) = \frac{Z_n(\Pi)}{B_n(\Pi)} = \frac{\text{kernel } (\partial_n : C_n(\Pi) \to C_{n-1}(\Pi))}{\text{image } (\partial_{n+1} : C_{n+1}(\Pi) \to C_n(\Pi))}.$$

For $n = 2$, $\partial_2 : C_2(\Pi) \to C_1(\Pi)$ is defined by

$$\partial_2(u, v) = \varepsilon(u)v - uv + u\varepsilon(v). \tag{8}$$

Our interesting case is a closed surface group $\pi = \Pi/R$ where π is a group generated by $2g$ generators $A_1, B_1, \ldots, A_g, B_g$ and with one relation

$$R = A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1}. \tag{9}$$

Consider a 2-chain on π

$$Z_R = \sum_{i=1}^{g} \left((\frac{\partial R}{\partial A_i}, A_i) + (\frac{\partial R}{\partial B_i}, B_i) \right) = \sum_i n_i(x_i, y_i) \in \mathbb{Z}(\pi \times \pi). \tag{10}$$

Then its boundary is

$$\begin{aligned} \partial Z_R &= \sum_{i=1}^{g} \left\{ \left(\varepsilon(\frac{\partial R}{\partial A_i})A_i - \frac{\partial R}{\partial A_i}A_i + (\frac{\partial R}{\partial A_i})\varepsilon(A_i) \right) \right. \\ &\qquad \left. + \left(\varepsilon(\frac{\partial R}{\partial B_i})B_i - \frac{\partial R}{\partial B_i}B_i + (\frac{\partial R}{\partial B_i})\varepsilon(B_i) \right) \right\} \\ &= \sum_{i=1}^{g} \left\{ \left(0 - \frac{\partial R}{\partial A_i}(A_i - I) \right) + \left(0 - \frac{\partial R}{\partial B_i}(B_i - I) \right) \right\} \\ &= -\sum_{i=1}^{g} \left\{ \frac{\partial R}{\partial A_i}(A_i - I) + \frac{\partial R}{\partial B_i}(B_i - I) \right\} \\ &= -(R - I) = 0 \end{aligned}$$

by the Mean Value Theorem 4.2 and the fact $\varepsilon(\partial R/\partial x)$ is the total exponent sum of x in R. The following Proposition 4.3 is due to Goldman [7] and Lyndon [13].

Proposition 4.3. *Let R be the relation (9) for the fundamental group π of a closed surface of genus g. Let Z_R be the 2-cycle (10) on π. Then its homology class $[Z_R]$ generates $H_2(\pi)$.*

We call Z_R the *fundamental cycle* of the fundamental group π. In next section I will extend the concept of the fundamental cycle Z_R to compact oriented surfaces with boundary.

4.3. Finally we can formulate the explicit formula of the symplectic form on $\mathrm{Hom}(\pi, G)^{--}/G$ where π is the fundamental group of a *closed* surface and G is a connected algebraic Lie group. Let $\mathfrak{g}$ be the Lie algebra of G with an Ad-invariant nondegenerate symmetric bilinear form $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. If $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$, then we could take β is just the trace form. Let $u, v \in Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$. We define a $\mathbb{Z}$-linear map

$$\beta_*(u, v) : \mathbb{Z}(\pi \times \pi) \to \mathbb{R}$$

by

$$\beta_*(u,v)\left(\sum_{i=1}^{k} n_i(x_i, y_i)\right) = \sum_{i=1}^{k} n_i\{\beta(u(x_i), x_i.v(y_i))\}, \tag{11}$$

where $x_i.v(y_i)$ means $\mathrm{Ad}\phi(x_i)v(y_i)$ for $(x_i, y_i) \in \pi \times \pi \subset C_2(\pi)$. Then $\beta_*(u, v) \in Z^2(\pi; \mathbb{R})$.

Theorem 4.4. *Let π be the fundamental group of a* closed *surface. Define*

$$\omega : H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \times H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \to \mathbb{R} \ \text{ by}$$

$$\omega([u], [v]) = \beta_*(u, v) Z_R, \tag{12}$$

where Z_R is the fundamental cycle (10) of π. Then ω is a symplectic form on $\mathrm{Hom}(\pi, G)^{--}/G$.

5. A symplectic leaf on the moduli space of a compact oriented surface with boundary

In this section I give an explicit formula for the symplectic structure on a symplectic leaf on the moduli space of real projective structures of a compact oriented surface with boundary.

5.1. First I recall some basic definitions and properties. Let M be a smooth m-dimensional manifold and $C^\infty(M)$ the set of all smooth functions on M. A *Poisson manifold* M is a smooth manifold endowed with a *Poisson bracket* $\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ such that

1. $\{f,g\} = -\{g,f\}$ (skew-symmetry),
2. $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ (the Jacobi identity),
3. $\{h,fg\} = \{h,f\}g + f\{h,g\}$ (the Leibniz rule).

If M is a Poisson manifold, then $C^\infty(M)$ has the Lie algebra structure, and $\{f,\cdot\}$ is a vector field, called the *Hamiltonian vector field* of f and denoted by X_f. The Poisson bracket is determined by a skew-symmetric bilinear form on T^*M ; that is there exists a 2-vector (or bivector) field $P \in \wedge^2 TM$ such that

$$\{f,g\} = P(df \wedge dg).$$

P will be called the *Poisson bivector* of M and locally expressed by

$$P = \sum_{1\le i<j\le m} a^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j},$$

where a^{ij}'s are locally defined smooth functions on M.

A *symplectic manifold* is a smooth manifold M endowed with a nondegenerate closed 2-form ω. For any $f \in C^\infty(M)$, ω allows us to associate a vector field X_f dual to the differential df ; that is

$$\omega(X_f,\cdot) = df(\cdot).$$

Then we can define $\{,\}$ on $C^\infty(M)$ by

$$\{f,g\} = X_f \cdot g = dg(X_f) = \omega(X_g, X_f).$$

Thus $\{,\}$ is skew-symmetric since ω is alternating and satisfies the Jacobi identity and the Leibniz rule since ω is closed and X_f is dual to the differential df.

It turns out that symplectic manifolds are Poisson manifolds but there are many Poisson manifolds that are not symplectic. The moduli space of real projective structures on a compact oriented surface with boundary is an example. A function $f \in C^\infty(M)$ is a *Casimir function* if $\{f,g\} = 0$ for all $g \in C^\infty(M)$. If the only Casimir functions are constant, then $\{,\}$ comes from a symplectic structure. But the above moduli space has nonconstant Casimir functions. Let $f : G \to \mathbb{R}$ be an Ad-invariant function (e.g. trace function). For any $\gamma \in \pi$ we can define an associated function $f_\gamma : \mathrm{Hom}(\pi, G)^{--}/G \to \mathbb{R}$ by

$$f_\gamma([\phi]) = f \circ \phi(\gamma).$$

Then f_γ does not depend on the choice of ϕ since f is Ad-invariant. We call f_γ the *Goldman function*. From Audin [3], $f_{C_1}, \dots, f_{C_n}$ are Casimir functions where $C_1, \dots, C_n$ are the boundary generators of π.

Let N be a submanifold of a Poisson manifold M. Generally $\{f, g\}|_N$ is not related to $f|_N$ and $g|_N$. If $\{f, g\}|_N$ depends only on $f|_N$ and $g|_N$ for all $f, g \in C^\infty(M)$, then N is called a *Poisson submanifold*. Moreover if $\{,\}|_N$ can be defined from a symplectic form, then N is called a *symplectic submanifold*. Since any Poisson manifold has a *symplectic foliation,* a foliation whose leaves are the maximal symplectic submanifolds, we will find a symplectic foliation of the moduli space $\mathfrak{P}(M)$ and formulate the explicit symplectic form on its leaf.

5.2. Let M be a compact oriented surface with genus g, n boundary components and $\chi(M) = 2-2g-n < 0$. Let π be the fundamental group of M. Then π has $2g + n$ generators

$$A_1, \dots, A_g, B_1, \dots, B_g, C_1, \dots, C_n$$

with a single relation

$$R = \prod_{i=1}^{g} [A_i, B_i] \prod_{j=1}^{n} C_j = I,$$

where $[A_i, B_i] = A_i B_i A_i^{-1} B_i^{-1}$. We extend the concept of fundamental cycle Z_R of the group π with boundary generators $C_1, \dots, C_n$. Let

$$Z_R = \sum_{k=1}^{g} \left(\frac{\partial R}{\partial A_k}, A_k \right) + \sum_{k=1}^{g} \left(\frac{\partial R}{\partial B_k}, B_k \right) + \sum_{h=1}^{n} \left(\frac{\partial R}{\partial C_h}, C_h \right). \tag{13}$$

Then the boundary of Z_R is

$$\begin{aligned}
\partial Z_R &= \sum_{i=1}^{g}\left\{\left(\varepsilon(\frac{\partial R}{\partial A_i})A_i - \frac{\partial R}{\partial A_i}A_i + (\frac{\partial R}{\partial A_i})\varepsilon(A_i)\right)\right\} \\
&\quad + \sum_{i=1}^{g}\left\{\left(\varepsilon(\frac{\partial R}{\partial B_i})B_i - \frac{\partial R}{\partial B_i}B_i + (\frac{\partial R}{\partial B_i})\varepsilon(B_i)\right)\right\} \\
&\quad + \sum_{j=1}^{n}\left\{\left(\varepsilon(\frac{\partial R}{\partial C_j})C_j - \frac{\partial R}{\partial C_j}C_j + (\frac{\partial R}{\partial C_j})\varepsilon(C_j)\right)\right\} \\
&= \sum_{i=1}^{g}\left\{\left(0 - \frac{\partial R}{\partial A_i}(A_i - I)\right) + \left(0 - \frac{\partial R}{\partial B_i}(B_i - I)\right)\right\} \\
&\quad + \sum_{j=1}^{n}\left\{\left(C_j - \frac{\partial R}{\partial C_j}(C_j - I)\right)\right\} = \sum_{j=1}^{n} C_j \\
&\quad - \sum_{i=1}^{g}\left\{\frac{\partial R}{\partial A_i}(A_i - I) + \frac{\partial R}{\partial B_i}(B_i - I)\right\} - \sum_{j=1}^{n}\left\{\frac{\partial R}{\partial C_j}(C_j - I)\right\} \\
&= \sum_{j=1}^{n} C_j - (R - I) = \sum_{j=1}^{n} C_j
\end{aligned}$$

by the Mean Value Theorem 4.2 and the fact $\varepsilon(\partial R/\partial x)$ is the total exponent sum of x in R. So the boundary of Z_R is the sum of boundary generators. Hence we will call Z_R the *fundamental relative cycle* of π.

We define a bilinear form $\omega : Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \times Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \to \mathbb{R}$ by

$$\omega(u, v) = \beta_*(u, v) Z_R, \tag{14}$$

where Z_R is the fundamental relative cycle (13) of π, and β_* is defined as before in (11).

In the following proposition 5.2, we first meet an obstruction for ω to be symplectic. To do this we need a lemma.

Lemma 5.1. *For any* $u, v \in Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ *and* $x, y \in \pi$,

$$\begin{aligned}
&\beta(u(x), x.v(y)) + \beta(v(x), x.u(y)) \\
= \ &\beta(u(xy), v(xy)) - \beta(u(x), v(x)) - \beta(u(y), v(y)).
\end{aligned}$$

Proof. Since u is a 1-cocycle, $u(xy) = u(x) + x.u(y)$. So we have

$$\begin{aligned}
&\beta(u(xy), v(xy)) = \beta(u(x) + x.u(y), v(x) + x.v(y)) \\
&= \beta(u(x), v(x)) + \beta(u(x), x.v(y)) + \beta(v(y), x.u(y)) + \beta(u(y), v(y)),
\end{aligned}$$

because β is symmetric and $\beta(x.u(y), x.v(y)) = \beta(u(y), v(y))$. It proves the lemma. q.e.d.

In the following proposition we know ω is skew-symmetric unless π has boundary generators $C_1, \dots, C_n$.

Proposition 5.2. *For any* $u, v \in Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$, *we have*

$$\omega(u,v) + \omega(v,u) + \sum_{j=1}^{n} \beta(u(C_j), v(C_j)) = 0, \tag{15}$$

where $C_1, \dots, C_n$ *are the boundary generators of* π.

Proof. From the examples of Fox's calculus, the fundamental relative cycle has the following expression:

$$\begin{aligned} Z_R \;=\; & \sum_{k=1}^{g} \left(\prod_{i=1}^{k-1} [A_i, B_i](1 - A_k B_k A_k^{-1}),\ A_k \right) \\ & + \sum_{k=1}^{g} \left(\prod_{i=1}^{k-1} [A_i, B_i](A_k - A_k B_k A_k^{-1} B_k^{-1}),\ B_k \right) \\ & + \sum_{h=1}^{n} \left(\prod_{i=1}^{g} [A_i, B_i] \prod_{j=1}^{h-1} C_j,\ C_h \right). \end{aligned}$$

For the convenience of notation, $\prod_{i=1}^{0} F(i)$ means the identity element of π where $F(i) \in \pi$. Then

$$\begin{aligned} & \omega(u,v) = \beta_*(u,v) Z_R \\ & = \sum_{k=1}^{g} \beta \left(u(\prod_{i=1}^{k-1} [A_i, B_i]), \prod_{i=1}^{k-1} [A_i, B_i].v(A_k) \right) \\ & - \sum_{k=1}^{g} \beta \left(u(\prod_{i=1}^{k-1} [A_i, B_i] A_k B_k A_k^{-1}), \prod_{i=1}^{k-1} [A_i, B_i] A_k B_k A_k^{-1}.v(A_k) \right) \\ & + \sum_{k=1}^{g} \beta \left(u(\prod_{i=1}^{k-1} [A_i, B_i] A_k), \prod_{i=1}^{k-1} [A_i, B_i] A_k.v(B_k) \right) \\ & - \sum_{k=1}^{g} \beta \left(u(\prod_{i=1}^{k-1} [A_i, B_i][A_k, B_k]), \prod_{i=1}^{k-1} [A_i, B_i][A_k, B_k].v(B_k) \right) \\ & + \sum_{h=1}^{n} \beta \left(u(\prod_{i=1}^{g} [A_i, B_i] \prod_{j=1}^{h-1} C_h), \prod_{i=1}^{g} [A_i, B_i] \prod_{j=1}^{h-1} C_j.v(C_h) \right). \end{aligned}$$

Let $\bar{\beta}(x) = \beta(u(x), v(x))$. Then by Lemma 5.1,

$$\begin{aligned}
&\omega(u,v) + \omega(v,u) \\
&= \sum_{k=1}^{g} \bar{\beta}\left(\prod_{i=1}^{k-1}[A_i,B_i]A_k\right) - \sum_{k=1}^{g} \bar{\beta}\left(\prod_{i=1}^{k-1}[A_i,B_i]\right) - \sum_{k=1}^{g} \bar{\beta}\left(A_k\right) \\
&\quad - \sum_{k=1}^{g} \bar{\beta}\left(\prod_{i=1}^{k-1}[A_i,B_i]A_kB_k\right) + \sum_{k=1}^{g} \bar{\beta}\left(\prod_{i=1}^{k}[A_i,B_i]B_k\right) + \sum_{k=1}^{g} \bar{\beta}\left(A_k\right) \\
&\quad + \sum_{k=1}^{g} \bar{\beta}\left(\prod_{i=1}^{k-1}[A_i,B_i]A_kB_k\right) - \sum_{k=1}^{g} \bar{\beta}\left(\prod_{i=1}^{k-1}[A_i,B_i]A_k\right) - \sum_{k=1}^{g} \bar{\beta}\left(B_k\right) \\
&\quad - \sum_{k=1}^{g} \bar{\beta}\left(\prod_{i=1}^{k}[A_i,B_i]B_k\right) + \sum_{k=1}^{g} \bar{\beta}\left(\prod_{i=1}^{k}[A_i,B_i]\right) + \sum_{k=1}^{g} \bar{\beta}\left(B_k\right) \\
&\quad + \sum_{h=1}^{n} \bar{\beta}\left(\prod_{i=1}^{g}[A_i,B_i]\prod_{j=1}^{h}C_j\right) - \sum_{h=1}^{n} \bar{\beta}\left(\prod_{i=1}^{g}[A_i,B_i]\prod_{j=1}^{h-1}C_j\right) \\
&\quad - \sum_{h=1}^{n} \bar{\beta}\left(C_h\right) = -\bar{\beta}(I) + \bar{\beta}(R) - \sum_{h=1}^{n} \bar{\beta}\left(C_h\right) \\
&= -\sum_{h=1}^{n} \beta\left(u(C_h), v(C_h)\right). \qquad \text{q.e.d.}
\end{aligned}$$

5.3. The boundary ∂M of M is a disjoint union of $(\partial M)_j$, $j = 1, \ldots, n$ such that each boundary component is diffeomorphic to the unit circle S^1. Let π_j be the fundamental group of $(\partial M)_j$. Then for each j, π_j is the cyclic group $< C_j >$ generated by C_j.

Let $\phi \in \mathrm{Hom}(\pi, G)^{--}$ and $\psi_j : \pi_j \to \pi$ the inclusion. Then $\phi_j \in \mathrm{Hom}(\pi_j, G)^{--}$ where $\phi_j = \phi \circ \psi_j$. Define

$$\begin{aligned}
C^p(\{\pi_j\}; \mathfrak{g}_{\mathrm{Ad}\phi}) &= \bigoplus_{j=1}^{n} C^p(\pi_j; \mathfrak{g}_{\mathrm{Ad}\phi_j}) \\
&= \{(f_1, \ldots, f_n) : \pi_1^p \times \cdots \times \pi_n^p \to \mathfrak{g}_{\mathrm{Ad}\phi_1} \times \cdots \times \mathfrak{g}_{\mathrm{Ad}\phi_n}\},
\end{aligned}$$

and the boundary map $d : C^p(\{\pi_j\}; \mathfrak{g}_{\mathrm{Ad}\phi}) \to C^{p+1}(\{\pi_j\}; \mathfrak{g}_{\mathrm{Ad}\phi})$ by

$$d(f_1, \ldots, f_n) = (df_1, \ldots, df_n).$$

Then by definition

$$B^p(\{\pi_j\};\mathfrak{g}_{\mathrm{Ad}\phi}) = \bigoplus_{j=1}^{n} B^p(\pi_j;\mathfrak{g}_{\mathrm{Ad}\phi_j}).$$

We confuse $\mathfrak{g}_{\mathrm{Ad}\phi_j}$ with $\mathfrak{g}_{\mathrm{Ad}\phi}$ since ϕ_j is the restriction ϕ to π_j. Because $d \circ d = 0$, $\{C^p(\{\pi_j\};\mathfrak{g}_{\mathrm{Ad}\phi}), d\}$ forms a cochain complex.

We define $\alpha : C^p(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}) \to C^p(\{\pi_j\};\mathfrak{g}_{\mathrm{Ad}\phi})$ by $\alpha(f) = (f_1, \dots, f_n)$ where $f_j = f|_{\pi_j} = f \circ \psi_j : \pi_j^p \to \mathfrak{g}_{\mathrm{Ad}\phi}$. Then the following diagram commutes:

$$\begin{array}{ccc} C^{p-1}(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}) & \xrightarrow{\alpha} & C^{p-1}(\{\pi_j\};\mathfrak{g}_{\mathrm{Ad}\phi}) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ C^{p}(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}) & \xrightarrow[\alpha]{} & C^{p}(\{\pi_j\};\mathfrak{g}_{\mathrm{Ad}\phi}) \end{array}$$

Define

$$Z^p_{\mathrm{par}}(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}) = \{f \in Z^p(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}) \mid \alpha(f) \in B^p(\{\pi_j\};\mathfrak{g}_{\mathrm{Ad}\phi})\}.$$

Then we have the following relation.

Proposition 5.3.

$$B^p(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}) \subseteq Z^p_{\mathrm{par}}(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}) \subseteq Z^p(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}). \tag{16}$$

Proof. By definition $Z^p_{\mathrm{par}}(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}) \subseteq Z^p(\pi;\mathfrak{g}_{\mathrm{Ad}\phi})$. Suppose $g \in B^p(\pi;\mathfrak{g}_{\mathrm{Ad}\phi})$. Then there exists $f \in C^{p-1}(\pi;\mathfrak{g}_{\mathrm{Ad}\phi})$ such that $df = g$. Since the above diagram commutes,

$$\alpha(g) = \alpha(d(f)) = (\alpha \circ d)(f) = (d \circ \alpha)(f) = d(\alpha(f)).$$

Therefore $\alpha(g) \in B^p(\{\pi_j\};\mathfrak{g}_{\mathrm{Ad}\phi})$. Hence $g \in Z^p_{\mathrm{par}}(\pi;\mathfrak{g}_{\mathrm{Ad}\phi})$. q.e.d.

We define

$$H^p_{\mathrm{par}}(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}) = Z^p_{\mathrm{par}}(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}) / B^p(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}),$$

and call the p-th *parabolic cohomology* with coefficients $\mathfrak{g}_{\mathrm{Ad}\phi}$. Then we have a long exact sequence

$$\begin{aligned} \cdots \to H^{p-1}_{\mathrm{par}}(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}) &\to H^{p-1}(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}) \to H^{p-1}(\{\pi_j\};\mathfrak{g}_{\mathrm{Ad}\phi}) \\ \to H^{p}_{\mathrm{par}}(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}) &\to H^{p}(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}) \to H^{p}(\{\pi_j\};\mathfrak{g}_{\mathrm{Ad}\phi}) \to \cdots , \end{aligned}$$

and therefore we get a relation $H^1_{\mathrm{par}}(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}) \subseteq H^1(\pi;\mathfrak{g}_{\mathrm{Ad}\phi})$ since $H^0(\{\pi_j\};\mathfrak{g}_{\mathrm{Ad}\phi}) = 0$. Suppose M is closed. Then $H^p(\{\pi_j\};\mathfrak{g}_{\mathrm{Ad}\phi}) = 0$ for all p. That means $H^p_{\mathrm{par}}(\pi;\mathfrak{g}_{\mathrm{Ad}\phi})$ is the same as $H^p(\pi;\mathfrak{g}_{\mathrm{Ad}\phi})$ if M is a closed surface. For more detail about the parabolic cohomology, see Guruprasad, Huebschmann, Jeffrey, and Weinstein's joint paper [10].

Consider $p = 1$ since we are interested in $H^1(\pi;\mathfrak{g}_{\mathrm{Ad}\phi})$ the tangent space of $\mathrm{Hom}(\pi, G)^{--}/G$. Let $u \in Z^1_{\mathrm{par}}(\pi;\mathfrak{g}_{\mathrm{Ad}\phi})$. Then $u \in Z^1(\pi;\mathfrak{g}_{\mathrm{Ad}\phi})$ and $\alpha(u) \in B^1(\{\pi_j\};\mathfrak{g}_{\mathrm{Ad}\phi})$; that means: $u(xy) = u(x)+x.u(y)$ for $x, y \in \pi$ and there exist $D_1, \ldots, D_n \in C^0(\pi_j;\mathfrak{g}_{\mathrm{Ad}\phi}) = \mathfrak{g}$ such that $u_j = dD_j$ where $u_j = u|_{\pi_j}$. Since $\pi_j =< C_j >$,

$$u(C_j) = u_j(C_j) = dD_j(C_j) = C_j.D_j - D_j. \tag{17}$$

Weil [16] called the 1-cocycle satisfying the condition (17) a *parabolic element*. We will use his terminology.

5.4. Let D be an element of the Lie algebra $\mathfrak{g} = C^0(\pi;\mathfrak{g}_{\mathrm{Ad}\phi})$. Then $dD \in B^1(\pi;\mathfrak{g}_{\mathrm{Ad}\phi})$ such that $dD(x) = x.D - D$ for any $x \in \pi$.

Proposition 5.4. *For any $u, v \in Z^1(\pi;\mathfrak{g}_{\mathrm{Ad}\phi})$ and $D \in \mathfrak{g}$, we have*

$$\omega(dD, v) = \sum_{j=1}^{n} \beta(D, v(C_j)) \tag{18}$$

$$\omega(u, dD) = \sum_{j=1}^{n} \beta(u(C_j^{-1}), D). \tag{19}$$

Proof. Since for any $x, y \in \pi$,

$$\begin{aligned} \beta_*(dD, v)(x, y) &= \beta(dD(x), x.v(y)) \\ &= \beta(x.D - D, x.v(y)) \\ &= \beta(x.D, x.v(y)) - \beta(D, x.v(y)) \\ &= \beta(D, v(y)) - \beta(D, x.v(y)) \end{aligned}$$

and $u(x^{-1}) = -x^{-1}.u(x)$, we can compute the following:

$$\begin{aligned}
\omega(dD, v) &= \beta_*(dD, v)Z_R \\
&= \sum_{k=1}^{g}\left(\beta(D, v(A_k)) - \beta(D, \prod_{i=1}^{k-1}[A_i, B_i].v(A_k))\right) \\
&\quad - \sum_{k=1}^{g}\left(\beta(D, v(A_k)) - \beta(D, \prod_{i=1}^{k-1}[A_i, B_i]A_kB_kA_k^{-1}.v(A_k))\right) \\
&\quad + \sum_{k=1}^{g}\left(\beta(D, v(B_k)) - \beta(D, \prod_{i=1}^{k-1}[A_i, B_i]A_k.v(B_k))\right) \\
&\quad - \sum_{k=1}^{g}\left(\beta(D, v(B_k)) - \beta(D, \prod_{i=1}^{k-1}[A_i, B_i]A_kB_kA_k^{-1}B_k^{-1}.v(B_k))\right) \\
&\quad + \sum_{h=1}^{n}\left(\beta(D, v(C_h)) - \beta(D, \prod_{i=1}^{g}[A_i, B_i]\prod_{j=1}^{h-1}C_j.v(C_h))\right) \\
&= -\sum_{k=1}^{g}\beta(D, \prod_{i=1}^{k-1}[A_i, B_i].v(A_k)) \\
&\quad - \sum_{k=1}^{g}\beta(D, \prod_{i=1}^{k-1}[A_i, B_i]A_k.v(B_k)) \\
&\quad - \sum_{k=1}^{g}\beta(D, \prod_{i=1}^{k-1}[A_i, B_i]A_kB_k.v(A_k^{-1})) \\
&\quad - \sum_{k=1}^{g}\beta(D, \prod_{i=1}^{k-1}[A_i, B_i]A_kB_kA_k^{-1}.v(B_k^{-1})) \\
&\quad - \sum_{h=1}^{n}\beta(D, \prod_{i=1}^{g}[A_i, B_i]\prod_{j=1}^{h-1}C_j.v(C_h)) + \sum_{h=1}^{n}\beta(D, v(C_h)) \\
&= -\beta(D, v(R)) + \sum_{h=1}^{n}\beta(D, v(C_h)) \\
&= \sum_{h=1}^{n}\beta(D, v(C_h)).
\end{aligned}$$

From Equation (18) and Proposition 5.2,

$$\begin{aligned}\omega(u,dD) &= -\omega(dD,u) - \sum_{j=1}^{n} \beta(dD(C_j),u(C_j)) \\ &= -\sum_{j=1}^{n} \beta(D,u(C_j)) - \sum_{j=1}^{n} \beta(C_j.D - D, u(C_j)) \\ &= -\sum_{j=1}^{n} \beta(D,u(C_j)) - \sum_{j=1}^{n} \beta(C_j.D, u(C_j)) \\ &\quad + \sum_{j=1}^{n} \beta(D,u(C_j)) \\ &= \sum_{j=1}^{n} \beta(D, -C_j^{-1}.u(C_j)) \\ &= \sum_{j=1}^{n} \beta(D, u(C_j^{-1})). \end{aligned}$$

q.e.d.

5.5. We know ω is not skew-symmetric on $Z^1(\pi;\mathfrak{g}_{\mathrm{Ad}\phi})$ if M has nonempty boundary. To make ω skew-symmetric, we restrict our domain to the subspace $Z^1_{\mathrm{par}}(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}) \subseteq Z^1(\pi;\mathfrak{g}_{\mathrm{Ad}\phi})$ and define a bilinear form $\tilde{\omega} : Z^1_{\mathrm{par}}(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}) \times Z^1_{\mathrm{par}}(\pi;\mathfrak{g}_{\mathrm{Ad}\phi}) \to \mathbb{R}$ by

$$\tilde{\omega}(u,v) = \omega(u,v) - \sum_{j=1}^{n} \beta(D_j, v(C_j)), \tag{20}$$

where $u|_{\pi_j} = u_j = dD_j$.

Suppose we have another $\bar{D}_j \in \mathfrak{g}$ such that $u|_{\pi_j} = dD_j = d\bar{D}_j$. Then

$$C_j.\bar{D}_j - \bar{D}_j = d\bar{D}_j(C_j) = dD_j(C_j) = C_j.D_j - D_j.$$

Hence $C_j.(\bar{D}_j - D_j) = (\bar{D}_j - D_j)$, and therefore there exists $F_j \in \mathfrak{g}$ such that $\bar{D}_j = D_j + F_j$ and $C_j.F_j = F_j$ for each $j = 1,\dots,n$. Thus

$$C_j^{-1}.F_j = C_j^{-1}.(C_j.F_j) = (C_j^{-1}C_j).F_j = F_j.$$

Let $v|_{\pi_j} = dD'_j$. Then

$$\begin{aligned}\beta(\bar{D}_j, v(C_j)) &= \beta(D_j + F_j, v(C_j))\\ &= \beta(D_j, v(C_j)) + \beta(F_j, v(C_j))\\ &= \beta(D_j, v(C_j)) + \beta(F_j, dD'_j(C_j))\\ &= \beta(D_j, v(C_j)) + \beta(F_j, C_j.D'_j - D'_j)\\ &= \beta(D_j, v(C_j)) + \beta(F_j, C_j.D'_j) - \beta(F_j, D'_j))\\ &= \beta(D_j, v(C_j)) + \beta(C_j^{-1}.F_j, D'_j) - \beta(F_j, D'_j)\\ &= \beta(D_j, v(C_j)).\end{aligned}$$

So $\sum_{j=1}^{n} \beta(D_j, v(C_j))$ does not depend on the choices of D_j's. Therefore $\tilde{\omega}$ is well-defined on $Z^1_{\text{par}}(\pi; \mathfrak{g}_{\text{Ad}\phi})$. If M is closed, then $\tilde{\omega} = \omega$.

Proposition 5.5 . *$\tilde{\omega}$ is skew-symmetric on $Z^1_{\text{par}}(\pi; \mathfrak{g}_{\text{Ad}\phi})$.*

Proof. Recall Proposition 5.2. Then

$$\begin{aligned}\tilde{\omega}(u,v) + \tilde{\omega}(v,u) &= \omega(u,v) - \sum_{j=1}^{n} \beta(D_j, v(C_j))\\ &\quad + \omega(v,u) - \sum_{j=1}^{n} \beta(D'_j, u(C_j))\\ &= -\sum_{j=1}^{n}\{\beta(u(C_j), v(C_j)) + \beta(D_j, v(C_j)) + \beta(D'_j, u(C_j))\}\\ &= -\sum_{j=1}^{n}\{\beta(C_j.D_j - D_j, v(C_j)) + \beta(D_j, v(C_j))\\ &\qquad + \beta(D'_j, u(C_j))\}\\ &= -\sum_{j=1}^{n}\{\beta(C_j.D_j, v(C_j)) + \beta(D'_j, u(C_j))\}\\ &= -\sum_{j=1}^{n}\{\beta(C_j.D_j, C_j.D'_j - D'_j) + \beta(D'_j, C_j.D_j - D_j)\}\\ &= -\sum_{j=1}^{n}\{\beta(C_j.D_j, C_j.D'_j) - \beta(C_j.D_j, D'_j)\\ &\qquad + \beta(D'_j, C_j.D_j) - \beta(D'_j, D_j)\}\\ &= 0.\end{aligned}$$

Therefore $\tilde{\omega}$ is skew-symmetric. q.e.d.

Consider $dD \in B^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \subseteq Z^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$. Then $dD|_{\pi_j} = dD$ for all j. By Equation (18) and skew-symmetry, for any $v \in Z^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ we get

$$\tilde{\omega}(dD, v) = \omega(dD, v) - \sum_{j=1}^{n} \beta(D, v(C_j)) = 0, \tag{21}$$

$$\tilde{\omega}(v, dD) = -\tilde{\omega}(dD, v) = 0. \tag{22}$$

Therefore we can induce $\tilde{\omega}$ on $H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$.

Theorem 5.6. *Define* $\tilde{\omega} : H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \times H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \to \mathbb{R}$ *by*

$$\tilde{\omega}([u], [v]) = \tilde{\omega}(u, v) = \omega(u, v) - \sum_{j=1}^{n} \beta(D_j, v(C_j)), \tag{23}$$

where $u, v \in Z^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ *representing* $[u], [v] \in H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ *and* $u|_{\pi_j} = dD_j$. *Then* $\tilde{\omega}$ *is a symplectic form on* $H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$.

Proof. (Well-definedness on $H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$) Suppose $u, \bar{u}$ are two representatives of $[u]$. Then there exists $dD \in B^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ such that $\bar{u} - u = dD$. Similarly there exist $v, \bar{v} \in Z^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ and $dD' \in B^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ such that $\bar{v} - v = dD'$. Then

$$\begin{aligned} \tilde{\omega}(\bar{u}, \bar{v}) &= \tilde{\omega}(u + dD, v + dD') \\ &= \tilde{\omega}(u, v) + \tilde{\omega}(dD, v) + \tilde{\omega}(u, dD') + \tilde{\omega}(dD, dD') \\ &= \tilde{\omega}(u, v). \end{aligned}$$

Therefore $\tilde{\omega}$ is well-defined on $H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$.

(Bilinearity and skew-symmetry) $\tilde{\omega}$ is clearly bilinear since ω and β are bilinear. By Proposition 5.5 and well-definedness, $\tilde{\omega}$ is skew-symmetric on $H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$.

(Nondegeneracy) Let $[u] \in H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ such that $\tilde{\omega}([u], \cdot) = 0$. Let $u \in Z^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ be a representative of $[u]$. For any integer $1 \leq k \leq n-1$, we define $v_{C_{k,k+1}} \in Z^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ by

$$v_{C_{k,k+1}}(A_i) = v_{C_{k,k+1}}(B_i) = v_{C_{k,k+1}}(C_j) = 0$$

for all i, j, but $v_{C_{k,k+1}}(C_k) \neq 0$ and $v_{C_{k,k+1}}(C_{k+1}) \neq 0$. Recall that

$$R = \prod_{i=1}^{g} [A_i, B_i] \prod_{j=1}^{n} C_j = I.$$

By the cocycle condition we have

$$0 = v_{C_{k,k+1}}(R) = v_{C_{k,k+1}}(C_k) + C_k.v_{C_{k,k+1}}(C_{k+1}),$$

and the relation $v_{C_{k,k+1}}(C_{k+1}) = -C_k^{-1}.v_{C_{k,k+1}}(C_k)$. Since $\tilde{\omega}(u,\cdot)$ vanishes,

$$\begin{aligned} 0 &= \tilde{\omega}(u, v_{C_{k,k+1}}) \\ &= \omega(u, v_{C_{k,k+1}}) - \beta(D_k, v_{C_{k,k+1}}(C_k)) - \beta(D_{k+1}, v_{C_{k,k+1}}(C_{k+1})) \\ &= \omega(u, v_{C_{k,k+1}}) - \beta(D_k, v_{C_{k,k+1}}(C_k)) - \beta(D_{k+1}, -C_k^{-1}.v_{C_{k,k+1}}(C_k)) \\ &= \omega(u, v_{C_{k,k+1}}) - \beta(D_k - C_k.D_{k+1}, v_{C_{k,k+1}}(C_k)). \end{aligned}$$

Therefore

$$\begin{aligned} &\beta(D_k - C_k.D_{k+1}, v_{C_{k,k+1}}(C_k)) = \omega(u, v_{C_{k,k+1}}) \\ &= \beta\left(u(\prod_{i=1}^{g}[A_i,B_i]C_1\cdots C_{k-1}), (\prod_{i=1}^{g}[A_i,B_i]C_1\cdots C_{k-1}).v_{C_{k,k+1}}(C_k)\right) \\ &\quad +\beta\left(u(\prod_{i=1}^{g}[A_i,B_i]C_1\cdots C_k), (\prod_{i=1}^{g}[A_i,B_i]C_1\cdots C_k).v_{C_{k,k+1}}(C_{k+1})\right) \\ &= \beta\left(-u((\prod_{i=1}^{g}[A_i,B_i]C_1\cdots C_{k-1})^{-1}), v_{C_{k,k+1}}(C_k)\right) \\ &\quad +\beta\left(-u((\prod_{i=1}^{g}[A_i,B_i]C_1\cdots C_k)^{-1}), v_{C_{k,k+1}}(C_{k+1})\right) \\ &= \beta\left(-u(C_k\cdots C_n), v_{C_{k,k+1}}(C_k)\right) \\ &\quad +\beta\left(-u(C_{k+1}\cdots C_n), -C_k^{-1}.v_{C_{k,k+1}}(C_k))\right) \\ &= \beta\left(-u(C_k\cdots C_n) + C_k.u(C_{k+1}\cdots C_n), v_{C_{k,k+1}}(C_k)\right) \\ &= \beta\left(-u(C_k), v_{C_{k,k+1}}(C_k)\right). \end{aligned}$$

Since β is nondegenerate and $v_{C_{k,k+1}}(C_k) \neq 0$, for each k, we have a relation

$$D_k - C_k.D_{k+1} = -u(C_k). \tag{24}$$

Since $u(C_k) = dD(C_k) = C_k.D_k - D_k$, by Equation (24) we have $D_k = D_{k+1}$ for all k.

Let $D = D_1 = \cdots = D_n$ and $\bar{u} = u - dD$. Then $\bar{u}$ is a parabolic element. I claim that $\bar{u} = 0$. From the definition of $\bar{u}$, it follows that

$\bar{u}(C_j) = u(C_j) - dD(C_j) = 0$ for all j. Define $v_{A_l} \in Z^1_{\text{par}}(\pi; \mathfrak{g}_{\text{Ad}\phi})$ by $v_{A_l}(A_i) = v_{A_l}(B_i) = v_{A_l}(C_j) = 0$ for all i, j but $v_{A_l}(A_l) \neq 0 \in \mathfrak{g}$. Similarly, define v_{B_l} for $l = 1, \ldots, g$. Since $\tilde{\omega}(\bar{u}, v_{B_g}) = \tilde{\omega}(u, v_{B_g}) - \tilde{\omega}(dD, v_{B_g}) = 0$, we have $\omega(\bar{u}, v_{B_g}) = \sum_{j=1}^{n} \beta(\bar{D}_j, v_{B_g}(C_j)) = 0$. Hence

$$\begin{aligned} 0 = \omega(\bar{u}, v_{B_g}) &= -\beta\left(\bar{u}((\prod_{i=1}^{g-1}[A_i, B_i]A_g)^{-1}), v_{B_g}(B_g)\right) \\ &\quad + \beta\left(\bar{u}((\prod_{i=1}^{g-1}[A_i, B_i][A_g, B_g])^{-1}), v_{B_g}(B_g)\right) \\ &= -\beta\left(\bar{u}(B_g A_g^{-1} B_g^{-1} C_1 \cdots C_n), v_{B_g}(B_g)\right) \\ &\quad + \beta\left(\bar{u}(C_1 \cdots C_n), v_{B_g}(B_g)\right). \end{aligned}$$

Since $\bar{u}(C_1 \cdots C_n) = 0$ and $v_{B_g}(B_g) \neq 0$, it follows that

$$\bar{u}(B_g A_g^{-1} B_g^{-1}) = 0. \tag{25}$$

Similarly, $\tilde{\omega}(\bar{u}, v_{A_g}) = 0$ induces

$$\begin{aligned} 0 = \omega(\bar{u}, v_{A_g}) &= -\beta\left(\bar{u}(A_g B_g A_g^{-1} B_g^{-1} C_1 \cdots C_n), v_{A_g}(A_g)\right) \\ &\quad + \beta\left(\bar{u}(B_g^{-1} C_1 \cdots C_n), v_{A_g}(A_g)\right). \end{aligned}$$

Since $\bar{u}(C_1 \cdots C_n) = 0$ and $v_{A_g}(A_g) \neq 0$, we have

$$\bar{u}(B_g^{-1}) = \bar{u}(A_g B_g A_g^{-1} B_g^{-1}). \tag{26}$$

From Equations (25) and (26), it follows that

$$\bar{u}(B_g^{-1}) = \bar{u}(A_g) + A_g.\bar{u}(B_g A_g^{-1} B_g^{-1}) = \bar{u}(A_g).$$

By Equation (25), we have

$$\begin{aligned} 0 &= \bar{u}(B_g A_g^{-1} B_g^{-1}) \\ &= \bar{u}(B_g) + B_g.\bar{u}(A_g^{-1}) + B_g A_g^{-1}.\bar{u}(B_g^{-1}) \\ &= \bar{u}(B_g) - B_g A_g^{-1}.\bar{u}(A_g) + B_g A_g^{-1}.\bar{u}(B_g^{-1}) \\ &= \bar{u}(B_g) - B_g A_g^{-1}.\bar{u}(B_g^{-1}) + B_g A_g^{-1}.\bar{u}(B_g^{-1}) \\ &= \bar{u}(B_g). \end{aligned}$$

Therefore $\bar{u}(B_g) = 0$ and $\bar{u}(A_g) = \bar{u}(B_g^{-1}) = -B_g^{-1}.\bar{u}(B_g) = 0$. By induction $\bar{u}(A_i) = \bar{u}(B_i) = 0$ for all $i = 1, \ldots, g$. Therefore $\bar{u} = u - dD = 0$. That is $[u] = 0 \in H^1_{\text{par}}(\pi; \mathfrak{g}_{\text{Ad}\phi})$.

(Closedness) To do this we represent $H^1_{\text{par}}(\pi; \mathfrak{g}_{\text{Ad}\phi})$ as the cohomology $H^1_{\text{par}}(M; \text{ad}P_\phi)$ with coefficients in the flat vector bundle. Then $H^1(M; \text{ad}P)$ is a tangent space of $\mathcal{F}(P)/\mathcal{G}(P)$ flat connections modulo gauge transformations that is identified with $\text{Hom}(\pi, G)/G$, and isomorphic to the de Rham cohomology $H^1_{\text{DR}}(M; \mathfrak{g})$. For more detail see Goldman [7] and Millson [9]. Let ∂M be the disjoint union $\cup_{j=1}^n \partial M_j$. Define $[\sigma \otimes \xi] \in H^1_{\text{par}}(M; \text{ad}P_\phi)$ if $\sigma \otimes \xi \in Z^1(M; \text{ad}P_\phi)$ and for each $j = 1, \ldots, n$ there exists $f_j \in C^\infty(\partial M_j)$ such that

$$(\sigma \otimes \xi)|_{\partial M_j} = df_j \otimes \xi,$$

where σ is a 1-cocycle and ξ is a $\mathfrak{g}$-valued section on M. Define

$$\tilde{\omega} : H^1_{\text{par}}(M; \text{ad}P) \times H^1_{\text{par}}(M; \text{ad}P) \to \mathbb{R} \quad \text{by}$$

$$\tilde{\omega}([\sigma \otimes \xi], [\sigma' \otimes \xi']) = \int_M (\sigma \wedge \sigma')\ B(\xi, \xi') - \sum_{j=1}^n \int_{\partial M_j} (f_j \wedge \sigma')\ B(\xi, \xi').$$

Then the above 2-form $\tilde{\omega}$ does not involve the connection A explicitly, so it is invariant under the translations of $\mathcal{F}(P)$. Therefore $\tilde{\omega}$ is parallel and hence closed. q.e.d.

5.6. Consider the moduli space $\mathfrak{P}(M)$ of convex real projective structures on $M = \Sigma(g, n)$. Since $\mathfrak{P}(M)$ embeds onto an open subset of $\text{Hom}(\pi, \mathbf{SL}(3, \mathbb{R}))/\mathbf{SL}(3, \mathbb{R})$, $\dim \mathfrak{P}(M) = -\dim \mathbf{SL}(3, \mathbb{R}) \cdot \chi(M) = 16g - 16 + 8n$.

Let $\{z_1, \ldots, z_N\}$ be a coordinate of $\mathfrak{P}(M)$ where $N = 16g - 16 + 8n$. Since the holonomy homomorphism $h : \pi \to G = \mathbf{SL}(3, \mathbb{R})$ is isomorphic to its image holonomy group Γ, we identify the elements of π and those image in G up to conjugation. Therefore each element of π can be expressed by $\{z_1, \ldots, z_N\}$ and $A.X = AXA^{-1}$ for $A \in \pi$ and $X \in \mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$. For each k, define $u_{z_k} : \pi \to \mathfrak{g}$ by

$$u_{z_k}(A) = (\partial_{z_k} A)A^{-1}, \tag{27}$$

since $\text{tr}\{(\partial_{z_k} A)A^{-1}\} = \partial_{z_k}\{\log \det A\} = 0$. Then for any $A, B \in \pi$,

$$\begin{aligned} u_{z_k}(AB) &= (\partial_{z_k} AB)(AB)^{-1} = \{(\partial_{z_k} A)B + A(\partial_{z_k} B)\}B^{-1}A^{-1} \\ &= (\partial_{z_k} A)A^{-1} + A(\partial_{z_k} B)B^{-1}A^{-1} = u_{z_k}(A) + A.u_{z_k}(B). \end{aligned}$$

Therefore $u_{z_k} \in Z^1(\pi; \mathfrak{g})$ for each k.

We find a necessary and sufficient condition for u_{z_k} to be parabolic. Since $\mathfrak{P}(M)$ corresponds convex real projective structures, every element of π has three distinct positive real eigenvalues. For each boundary generator C_j there exists an invertible matrix $P_j \in G$ such that $P_j^{-1}C_jP_j = E_j$ the diagonal matrix (1). Even though C_j has repeated eigenvalues there exists $P_j \in G$ such that $P_j^{-1}C_jP_j = J_j$ the Jordan canonical form. Then we still have a sufficient condition for u_{z_k} to be parabolic.

Proposition 5.7. *Suppose z_k is a coordinate in $\{z_1, \dots, z_N\}$. Then u_{z_k} is a parabolic element if and only if $\partial_{z_k}(P_j^{-1}C_jP_j) = \partial_{z_k}(E_j) = 0$ for each boundary generator C_j where $j = 1, \dots, n$.*

Proof. Suppose $u_{z_k}(C) = dD(C)$. Then

$$(\partial_{z_k}C)C^{-1} = u_{z_k}(C) = dD(C) = C.D - D = CDC^{-1} - D.$$

Therefore $(\partial_{z_k}C) = CD - DC$. That means u_{z_k} is a parabolic element if and only if there exist $D_1, \dots, D_n \in \mathfrak{g}$ such that $\partial_{z_k}C_j = C_jD_j - D_jC_j$ for each $j = 1, \dots, n$. For the simplicity put $C_j = C$, $E_j = E$, and $P_j = P$. Since $E = P^{-1}CP$,

$\partial_{z_k}C = \partial_{z_k}(PEP^{-1}) = (\partial_{z_k}P)EP^{-1} + P(\partial_{z_k}E)P^{-1} + PE(\partial_{z_k}P^{-1})$
$= (\partial_{z_k}P)(P^{-1}CP)P^{-1} + P(\partial_{z_k}E)P^{-1} + P(P^{-1}CP)(-P^{-1}(\partial_{z_k}P)P^{-1})$
$= (\partial_{z_k}P)P^{-1}C + P(\partial_{z_k}E)P^{-1} - C(\partial_{z_k}P)P^{-1}$.

($\Leftarrow$) Suppose $\partial_{z_k}E = 0$. Let $D = -(\partial_{z_k}P)P^{-1}$. Then $\partial_{z_k}C = CD - DC$. Therefore u_{z_k} is a parabolic element.

($\Rightarrow$) Suppose u_{z_k} is a parabolic element. Then there exists $\bar{D} \in \mathfrak{g}$ such that $\partial_{z_k}C = C\bar{D} - \bar{D}C$. Let $X = \bar{D} - D$ where $D = -(\partial_{z_k}P)P^{-1}$. Then $CX - XC = (C\bar{D} - \bar{D}C) - (CD - DC) = P(\partial_{z_k}E)P^{-1}$. So

$$\partial_{z_k}E = P^{-1}CXP - P^{-1}XCP. \tag{28}$$

Since $C = PEP^{-1}$, Equation (28) becomes

$$\partial_{z_k}E = E\bar{X} - \bar{X}E. \tag{29}$$

where $\bar{X} = P^{-1}XP$. Since E is a diagonal matrix, each diagonal entry of $E\bar{X} - \bar{X}E$ is zero. Therefore $\partial_{z_k}E = 0$ since $\partial_{z_k}E$ is a diagonal matrix whose diagonal entries are all zero. q.e.d.

Now we calculate the dimension of $H^1_{\text{par}}(\pi; \mathfrak{g}_{\text{Ad}\phi})$. Suppose $[u] \in H^1_{\text{par}}(\pi; \mathfrak{g}_{\text{Ad}\phi})$. Then for each j, $u|_{\pi_j} \in B^1(\pi_j; \mathfrak{g}_{\text{Ad}\phi})$; that is $[u|_{\pi_j}] =$

$0 \in H^1(\pi_j; \mathfrak{g}_{\mathrm{Ad}\phi})$. Thus $u(C_j)$ lies in a fixed conjugacy class in G. Hence we do not have any choice to select $[u(C_j)]$ in $\mathrm{Hom}(\pi, G)^{--}/G$. Recall π has n boundary generators that are positive hyperbolic. Since for each positive hyperbolic element has 2-dimensional coordinate parameters (λ, τ) or (ℓ, m),

$$\dim H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) = \dim H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) - 2n = 16g - 16 + 6n. \tag{30}$$

5.7. Example : A pair of pants $\Sigma(0,3)$. Let M be a pair of pants $\Sigma(0,3)$; that is, a surface homeomorphic to S^2 minus three open discs. Then π admits a representation $< C_1, C_2, C_3 \mid R = C_1C_2C_3 = I >$ and dim $\mathfrak{P}(M) = 16g - 16 + 8n = 8$. Since each positive hyperbolic element C_j has two parameters (λ_j, τ_j) and π has three generators, we need two more parameters s, t. We call s, t the *internal parameters* of $\mathfrak{P}(M)$. Goldman [8] expressed C_1, C_2, C_3 using parameters $\{\lambda_1,\ \lambda_2,\ \lambda_3,\ \tau_1,\ \tau_2,\ \tau_3,\ s,\ t\}$ of $\mathfrak{P}(M)$ where $0 < \lambda_j < 1,\ 2/\sqrt{\lambda_j} < \tau_j < \lambda_j + \lambda_j^{-2},\ s > 0$ and $t > 0$. Since s and t are positive, we replace the coordinates s and t by e^s and e^t. Then the matrices C_1, C_2 and C_3 are represented by

$$C_1 = \begin{pmatrix} \frac{e^{-s}\sqrt{\lambda_3}}{\sqrt{\lambda_2}\sqrt{\lambda_1}} + \tau_1 & \frac{e^{-s+t}\sqrt{\lambda_3}}{\sqrt{\lambda_2}\sqrt{\lambda_1}} & 0 \\ -\frac{e^{-s-t}\sqrt{\lambda_3}}{\sqrt{\lambda_2}\sqrt{\lambda_1}} - \frac{e^{s-t}\sqrt{\lambda_2}}{\sqrt{\lambda_3}\sqrt{\lambda_1}} - e^{-t}\tau_1 & -\frac{e^{-s}\sqrt{\lambda_3}}{\sqrt{\lambda_2}\sqrt{\lambda_1}} & 0 \\ a_{31} & a_{32} & \lambda_1 \end{pmatrix}, \tag{31}$$

where

$$\begin{aligned} a_{31} =& \frac{1}{2}e^{2s}\lambda_3 + \frac{e^{-s-t}\sqrt{\lambda_3}}{2\sqrt{\lambda_2}\sqrt{\lambda_1}} + \frac{e^{s-t}\sqrt{\lambda_2}}{2\sqrt{\lambda_3}\sqrt{\lambda_1}} + \frac{e^{s-t}\sqrt{\lambda_3}\sqrt{\lambda_1}}{2\ \lambda_2^{3/2}} + \frac{e^{3s-t}\sqrt{\lambda_1}}{2\sqrt{\lambda_3}\sqrt{\lambda_2}} \\ &+ \frac{\lambda_1}{2} + \frac{1}{2}e^{2s-t}\tau_3 + \frac{e^{-t}\lambda_3\tau_3}{2\lambda_2} + \frac{1}{2}e^{s}\sqrt{\lambda_3}\sqrt{\lambda_2}\sqrt{\lambda_1}\tau_2 + \frac{1}{2}e^{-t}\tau_1 \\ &+ \frac{e^{2s-t}\lambda_1\tau_1}{2\lambda_2} + \frac{e^{s-t}\sqrt{\lambda_3}\sqrt{\lambda_1}\tau_3\tau_1}{2\sqrt{\lambda_2}}, \end{aligned}$$

$$a_{32} = \frac{e^{-s}\sqrt{\lambda_3}}{2\sqrt{\lambda_2}\sqrt{\lambda_1}} + \frac{e^{s}\sqrt{\lambda_3}\sqrt{\lambda_1}}{2\ \lambda_2^{3/2}} + \frac{\lambda_3\tau_3}{2\lambda_2},$$

$$C_2 = \begin{pmatrix} -\frac{e^{-s}\sqrt{\lambda_1}}{\sqrt{\lambda_3}\sqrt{\lambda_2}} & 0 & -\frac{2e^{-s}\sqrt{\lambda_1}}{\sqrt{\lambda_3}\sqrt{\lambda_2}} \\ \frac{e^{-s-t}\sqrt{\lambda_1}}{\sqrt{\lambda_3}\sqrt{\lambda_2}} + \frac{\sqrt{\lambda_2}\sqrt{\lambda_1}}{e^{t-s}\lambda_3^{3/2}} + \frac{e^{-t}\lambda_1\tau_1}{\lambda_3} & \lambda_2 & b_{23} \\ \frac{e^{s}\sqrt{\lambda_3}}{2\sqrt{\lambda_2}\sqrt{\lambda_1}} + \frac{e^{-s}\sqrt{\lambda_1}}{2\sqrt{\lambda_3}\sqrt{\lambda_2}} + \frac{\tau_2}{2} & 0 & \frac{e^{-s}\sqrt{\lambda_1}}{\sqrt{\lambda_3}\sqrt{\lambda_2}} + \tau_2 \end{pmatrix}, \tag{32}$$

where

$$b_{23} = 2\lambda_2 + \frac{2e^{-s-t}\sqrt{\lambda_1}}{\sqrt{\lambda_3}\sqrt{\lambda_2}} + \frac{2\sqrt{\lambda_2}\sqrt{\lambda_1}}{e^{t-s}\lambda_3^{3/2}} + \frac{2\lambda_1\tau_1}{e^t\lambda_3},$$

$$C_3 = \begin{pmatrix} \lambda_3 & e^t\lambda_3 + \frac{e^{-s}\sqrt{\lambda_2}}{\sqrt{\lambda_3}\sqrt{\lambda_1}} + \frac{e^{s}\sqrt{\lambda_1}}{\sqrt{\lambda_3}\sqrt{\lambda_2}} + \tau_3 & \frac{2e^{-s}\sqrt{\lambda_2}}{\sqrt{\lambda_3}\sqrt{\lambda_1}} \\ 0 & \frac{e^{-s}\sqrt{\lambda_2}}{\sqrt{\lambda_3}\sqrt{\lambda_1}} + \tau_3 & \frac{2e^{-s}\sqrt{\lambda_2}}{\sqrt{\lambda_3}\sqrt{\lambda_1}} \\ 0 & -\frac{e^{-s}\sqrt{\lambda_2}}{2\sqrt{\lambda_3}\sqrt{\lambda_1}} - \frac{e^{s}\sqrt{\lambda_1}}{2\sqrt{\lambda_3}\sqrt{\lambda_2}} - \frac{\tau_3}{2} & -\frac{e^{-s}\sqrt{\lambda_2}}{\sqrt{\lambda_3}\sqrt{\lambda_1}} \end{pmatrix} \tag{33}$$

up to conjugation. For more detail see Goldman [8]. The eigenvalues λ_j, μ_j, ν_j of C_j are as follow:

$$\left\{\lambda_1,\ \mu_1 = \frac{\sqrt{\lambda_1}\ \tau_1 - \sqrt{-4+\lambda_1\ \tau_1^2}}{2\ \sqrt{\lambda_1}},\ \nu_1 = \frac{\sqrt{\lambda_1}\ \tau_1 + \sqrt{-4+\lambda_1\ \tau_1^2}}{2\ \sqrt{\lambda_1}}\right\},$$

$$\left\{\lambda_2,\ \mu_2 = \frac{\sqrt{\lambda_2}\ \tau_2 - \sqrt{-4+\lambda_2\ \tau_2^2}}{2\ \sqrt{\lambda_2}},\ \nu_2 = \frac{\sqrt{\lambda_2}\ \tau_2 + \sqrt{-4+\lambda_2\ \tau_2^2}}{2\ \sqrt{\lambda_2}}\right\},$$

$$\left\{\lambda_3,\ \mu_3 = \frac{\sqrt{\lambda_3}\ \tau_3 - \sqrt{-4+\lambda_3\ \tau_3^2}}{2\ \sqrt{\lambda_3}},\ \nu_3 = \frac{\sqrt{\lambda_3}\ \tau_3 + \sqrt{-4+\lambda_3\ \tau_3^2}}{2\ \sqrt{\lambda_3}}\right\}.$$

From (30), dim $H^1_{\mathrm{par}}(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) = 8 - 6 = 2$. By Proposition 5.7, u_s and u_t are parabolic elements since eigenvalues of C_1, C_2 and C_3 do not involve s and t.

Now we calculate the symplectic form $\tilde{\omega}$. Let us compute $\tilde{\omega}(u_s, u_t)$. Suppose $u_s|_{\pi_1} = dD_1, u_s|_{\pi_2} = dD_2$ and $u_s|_{\pi_3} = dD_3$. Since

$$\begin{aligned} Z_R =&(\partial_{C_1}R, C_1) + (\partial_{C_2}R, C_2) + (\partial_{C_3}R, C_3) \\ =&(I, C_1) + (C_1, C_2) + (C_1C_2, C_3), \end{aligned}$$

we have

$$\begin{aligned} \omega(u_s, u_t) &= \beta_*(u_s, u_t)Z_R \\ &= \beta(u_s(I), I.u_t(C_1)) \\ &\quad +\beta(u_s(C_1), C_1.u_t(C_2)) + \beta(u_s(C_1C_2), C_1C_2.u_t(C_3)) \\ &= 0 + \beta(u_s(C_1), C_1.u_t(C_2)) + \beta(u_s(C_3^{-1}), C_3^{-1}.u_t(C_3)) \\ &= \beta(u_s(C_1), C_1.u_t(C_2)) + \beta(C_3.u_s(C_3^{-1}), u_t(C_3)) \\ &= \beta(u_s(C_1), C_1u_t(C_2)C_1^{-1}) - \beta(u_s(C_3)), u_t(C_3)) \\ &= \mathrm{tr}\{(\partial_s C_1)C_1^{-1}C_1(\partial_t C_2)C_2^{-1}C_1^{-1}\} \\ &\quad -\mathrm{tr}\{(\partial_s C_3)C_3^{-1}(\partial_t C_3)C_3^{-1}\} \\ &= \mathrm{tr}\{(\partial_s C_1)(\partial_t C_2)C_3 + (\partial_s C_3)(\partial_t C_3^{-1})\}. \end{aligned}$$

So $\tilde{\omega}(u_s, u_t)$

$$\begin{aligned}
&= \omega(u_s,u_t) - \beta(D_1, u_t(C_1)) - \beta(D_2, u_t(C_2)) - \beta(D_3, u_t(C_3)) \\
&= \operatorname{tr}\{(\partial_s C_1)(\partial_t C_2)C_3 + (\partial_s C_3)(\partial_t C_3^{-1}) \\
&\qquad - D_1(\partial_t C_1)C_1^{-1} - D_2(\partial_t C_2)C_2^{-1} - D_3(\partial_t C_3)C_3^{-1}\}.
\end{aligned}$$

Let P_1 be a matrix such that $P_1^{-1}C_1P_1$ is diagonal. Then by the proof of Proposition 5.7, $-(\partial_s P_1)P_1^{-1}$ is one of D_1. Therefore we have the explicit formula for symplectic form $\tilde{\omega}(u_s, u_t)$

$$\begin{aligned}
&= \operatorname{tr}\{(\partial_s C_1)(\partial_t C_2)C_3 + (\partial_s C_3)(\partial_t C_3^{-1}) + (\partial_s P_1)P_1^{-1}(\partial_t C_1)C_1^{-1} \\
&\qquad + (\partial_s P_2)P_2^{-1}(\partial_t C_2)C_2^{-1} + (\partial_s P_3)P_3^{-1}(\partial_t C_3)C_3^{-1}\}.
\end{aligned}$$

Put P_j to be the transpose of an eigenvector matrix of C_j for each j. Then we get an amazing result calculated by Mathematica.

$$\tilde{\omega}(u_s, u_t) = -1.$$

Theorem 5.8. *Suppose M is a pair of pants $\Sigma(0,3)$, and $\mathfrak{P}(M)$ is the moduli space of convex real projective structures on M. Then the symplectic form on each parabolic leaf of $\mathfrak{P}(M)$ is*

$$\tilde{\omega} = dt \wedge ds.$$

Proof. See the detailed Mathematica computation in the appendix of Hong C. Kim's dissertation [11]. q.e.d.

6. The global symplectic coordinates on $\mathfrak{P}(M)$

Let M be a closed oriented surface $\Sigma(g,0)$ with $\chi(M) = 2-2g < 0$, and $\mathfrak{P}(M)$ the moduli space of convex real projective structures on M. For a simply closed geodesic γ on M, we want to associate the twisting parameters $\theta_\gamma, \beta_\gamma \in \mathbb{R}$ on γ. Let $A \in \mathbf{SL}(3,\mathbb{R})$ be the diagonal representation (1) of γ. For any $u, v \in \mathbb{R}$, we define $g_{(u,v)} \in \mathbf{SL}(3,\mathbb{R})$ by

$$g_{(u,v)} = \begin{bmatrix} e^{-\frac{u}{2}-\frac{v}{3}} & 0 & 0 \\ 0 & e^{\frac{2}{3}v} & 0 \\ 0 & 0 & e^{\frac{u}{2}-\frac{v}{3}} \end{bmatrix}, \tag{34}$$

then $\ell(g_{(u,v)}A) = |u + \ell(A)|$ and $m(g_{(u,v)}A) = v + m(A)$.

For $u, v \in \mathbb{R}$ and γ a simply closed geodesic on M, we define

$$\Psi_{\gamma(u,v)} : \mathfrak{P}(M) \to \mathfrak{P}(M)$$

by $\Psi_{\gamma(u,v)}([h]) = [h']$ where $[h], [h'] \in \mathrm{Hom}(\pi, G)/G$ such that $h' \equiv h$ except $h'(x_\gamma) = g_{(u,v)} h(x_\gamma) g_{(u,v)}^{-1}$ for $x_\gamma \in \pi = \pi_1(M)$ corresponding to γ.

The flows $\Psi_{\gamma(u,0)}$ and $\Psi_{\gamma(0,v)}$ on $\mathfrak{P}(M)$ are called the *generalized twisting flows*. Vector fields $\partial/\partial\theta_\gamma, \partial/\partial\beta_\gamma$ generated by $\Psi_{\gamma(u,0)}, \Psi_{\gamma(0,v)}$ are called the *generalized twisting vector fields* whose potential functions are $\ell(\gamma)$ and $m(\gamma)$ respectively.

Let $\Gamma = \{\gamma_i\}_{i=1,\dots,3g-3}$ be a set of nontrivial homotopically distinct disjoint simply closed geodesics on M such that Γ decompose M as the disjoint union of $2g-2$ pairs of pants. At each γ_i, length parameters ℓ_i, m_i and twisting parameters θ_i, β_i are defined where $\ell_i \in \mathbb{R}_+$ and $m_i, \theta_i, \beta_i \in \mathbb{R}$. Then the coordinate vector fields $\partial/\partial\theta_i, \partial/\partial\beta_i$ are the generalized twisting vector fields generated by the flows $\Psi_{\gamma_i(u,0)}, \Psi_{\gamma_i(0,v)}$ and their potential functions are ℓ_i, m_i; that is,

$$\omega(\partial/\partial\theta_i,\ \ ) = -d\ell_i, \quad \omega(\partial/\partial\beta_i,\ \ ) = -dm_i. \tag{35}$$

A pair of pants P has a unique reflection ρ fixing (set wise) each boundary component. The fixed point set of ρ consists of the three geodesics connecting pairs of boundary components of P. The restriction of ρ to a boundary component γ of P is completely determined by its fixed point. Consider pairs of pants P_1, P_2 with boundaries γ_1, γ_2 and unique reflections ρ_1, ρ_2 respectively. Assume $\ell_{\gamma_1} = \ell_{\gamma_2}$ and $m_{\gamma_1} = -m_{\gamma_2}$; that is, C_{γ_1} and $C_{\gamma_2}^{-1}$ are conjugate where $C_{\gamma_i} \in \mathbf{SL}(3, \mathbb{R})$ is the holonomy representation of γ_i. Then we can form the (geometric) sum $P_1 \vee P_2$ by gluing γ_1 and γ_2 such that the real projective structures on P_1 and P_2 extend to $P_1 \vee P_2$. The pair (ρ_1, ρ_2) extends to a reflection ρ on $P_1 \vee P_2$. Hence given a surface M with a pants decomposition $\{P_j\}$, by varying the twisting parameters we may arrange that the reflection maps ρ_j of P_j piece together. Therefore the surface M has a reflection ρ fixing each boundary of $\{P_j\}$.

If we only allow the integer times of Dehn twist on the gluing processor of the boundaries of the partition $\{P_j\}$, then ρ is an orientation reversing map which preserving the real projective structure on M. The map ρ induces a map $\hat{\rho} : \mathfrak{P}(M) \to \mathfrak{P}(M)$ defined by $\hat{\rho}([h]) = [h \circ \rho_\#]$ where $\rho_\# : \pi_1(M) \to \pi_1(M)$. Let ω be the symplectic form on $\mathfrak{P}(M)$. Then $\hat{\rho}^*\omega = -\omega$ since $\hat{\rho}$ is orientation reversing.

Lemma 6.1. *For the above* $\hat{\rho}$,

1. $\hat{\rho}^* d\ell_i = d\ell_i, \quad \hat{\rho}^* dm_i = -dm_i$,
2. $\hat{\rho}^* d\theta_i = -d\theta_i + a_i d\ell_i, \quad \hat{\rho}^* d\beta_i = -d\beta_i + b_i dm_i$ *where* $a_i, b_i \in \mathbb{Z}$.

Proof. Suppose the eigenvalues of $h(\gamma_i)$ are $\lambda_i < \mu_i < \nu_i$. Then the eigenvalues of $(h \circ \rho_\#)(\gamma_i)$ are $\nu_i^{-1} < \mu_i^{-1} < \lambda_i^{-1}$ since $[(h \circ \rho_\#)(\gamma_i)] = [h(\gamma_i)^{-1}]$. Therefore 1 holds. Since $[h]$ and $[h \circ \rho_\#]$ have the opposite orientations, the twisting parameters θ_i, β_i are measured with opposite senses. Furthermore there is an ambiguity of the integer times of Dehn twist for extending ρ_j to ρ. Therefore 2 holds. q.e.d.

From the duality formula (35), we can determine

$$\omega(\partial/\partial\theta_k, X) = -d\ell_k(X) = 0, \quad \omega(\partial/\partial\beta_k, Y) = -dm_k(Y) = 0,$$

where X and Y are vector fields such that

$$X \in \mathrm{Vect}(\mathfrak{P}(M)) - < \partial/\partial\ell_k >$$

and

$$Y \in \mathrm{Vect}(\mathfrak{P}(M)) - < \partial/\partial m_k >,$$

where $1 \leq i \leq 3g-3$ and $1 \leq j \leq 2g-2$.

Proposition 6.2. *Let* M *be a closed surface* $\Sigma(g,0)$ *having an orientation reversing map* ρ *fixing the elements of a partition* $\Gamma = \{\gamma_i\}$ *and preserving the real projective structure on* M. *Then for any* i *and* j,

$$\omega(\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}) = \omega(\frac{\partial}{\partial\ell_i}, \frac{\partial}{\partial\ell_j}) = \omega(\frac{\partial}{\partial\beta_i}, \frac{\partial}{\partial\beta_j}) = \omega(\frac{\partial}{\partial m_i}, \frac{\partial}{\partial m_j}) = 0. \tag{36}$$

Proof. The proof is based on the analogous properties of Fenchel-Nielsen coordinates on the Teichmüller space due to Wolpert [17]. First let us obtain a description for $\hat{\rho}_*$ the adjoint of $\hat{\rho}^*$. From Lemma 6.1 we find that

$$\hat{\rho}^*(\partial/\partial\theta_i) = -\partial/\partial\theta_i, \quad \hat{\rho}^*(\partial/\partial\ell_i) = \partial/\partial\ell_i + a_i\partial/\partial\theta_i, \tag{37}$$

$$\hat{\rho}^*(\partial/\partial\beta_i) = \partial/\partial\beta_i, \quad \hat{\rho}^*(\partial/\partial m_i) = -\partial/\partial m_i + b_i\partial/\partial\beta_i. \tag{38}$$

Now we make the first calculation:

$$-\omega(\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}) = \hat{\rho}^*\omega(\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}) = \omega(\hat{\rho}_*\frac{\partial}{\partial\theta_i}, \hat{\rho}_*\frac{\partial}{\partial\theta_j}) = \omega(-\frac{\partial}{\partial\theta_i}, -\frac{\partial}{\partial\theta_j}).$$

Thus $\omega(\partial/\partial\theta_i, \partial/\partial\theta_j) = 0$. Similarly $\omega(\partial/\partial\beta_i, \partial/\partial\beta_j) = 0$.

The second argument for $\omega(\partial/\partial\beta_i, \partial/\partial\beta_j)$ is as follow. Since ω is a skew-symmetry, without loss of generality we assume $i \neq j$. Then

$$\begin{aligned}
-\omega(\partial/\partial\ell_i, \partial/\partial\ell_j) &= \hat{\rho}^*\omega(\partial/\partial\ell_i, \partial/\partial\ell_j) \\
&= \omega(\hat{\rho}_*\partial/\partial\ell_i, \hat{\rho}_*\partial/\partial\ell_j) \\
&= \omega(\partial/\partial\ell_i + a_i\partial/\partial\theta_i, \partial/\partial\ell_j + a_j\partial/\partial\theta_j) \\
&= \omega(\partial/\partial\ell_i, \partial/\partial\ell_j) \\
&\quad + a_j\omega(\partial/\partial\ell_i, \partial/\partial\theta_j) + a_i\omega(\partial/\partial\theta_i, \partial/\partial\ell_j) \\
&\quad + a_i a_j\omega(\partial/\partial\theta_i, \partial/\partial\theta_j) \\
&= \omega(\partial/\partial\ell_i, \partial/\partial\ell_j) - a_j\omega(\partial/\partial\theta_j, \partial/\partial\ell_i) \\
&\quad + a_i\omega(\partial/\partial\theta_i, \partial/\partial\ell_j) + 0 \\
&= \omega(\partial/\partial\ell_i, \partial/\partial\ell_j) + a_j d\ell_j(\partial/\partial\ell_i) - a_i d\ell_i(\partial/\partial\ell_j) \\
&= \omega(\partial/\partial\ell_i, \partial/\partial\ell_j) + a_j\delta_{ji} - a_i\delta_{ij} \\
&= \omega(\partial/\partial\ell_i, \partial/\partial\ell_j).
\end{aligned}$$

Thus $\omega(\partial/\partial\ell_i, \partial/\partial\ell_j) = 0$. Similarly $\omega(\partial/\partial m_i, \partial/\partial m_j) = 0$. q.e.d.

To show $\omega(\partial/\partial s_j, \partial/\partial s_k) = 0$ and $\omega(\partial/\partial t_j, \partial/\partial t_k) = 0$, we need a Mathematica computation. Now consider a surface M with genus 2. It can be formed by gluing boundaries of two punctured tori. A punctured torus can be obtained by gluing two boundaries γ_1, γ_2 of a pair of pants such that $\ell_1 = \ell_2$ and $m_1 = -m_2$. Let $C_1, C_2 \in \mathbf{SL}(3, \mathbb{R})$ be the holonomy representations of γ_1, γ_2 respectively. Then there exists an invertible matrix Q such that $C_2 = QC_1^{-1}Q^{-1}$.

Suppose that $P_j \in \mathbf{SL}(3, \mathbb{R})$ such that $P_j^{-1}C_jP_j = E_j$ the diagonal matrix (1). Let $\mathfrak{Z}$ be the following subset of $\mathbf{SL}(3, \mathbb{R})$;

$$\left\{ \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}.$$

Since C_2 and C_1^{-1} are conjugate, we have $E_2 = ZE_1^{-1}Z^{-1}$ for any element $Z \in \mathfrak{Z}$. Put $Q = P_2ZP_1^{-1}$. Then we get

$$QC_1^{-1}Q^{-1} = (P_2ZP_1^{-1})C_1^{-1}(P_1Z^{-1}P_2^{-1}) = P_2(ZE_1^{-1}Z^{-1})P_2^{-1} = C_2.$$

Suppose we have another $\bar{P}_j$ such that $\bar{P}_j^{-1}C_j\bar{P}_j = E_j$ for each j. Then there exists a diagonal matrix D_j that $\bar{P}_j = P_jD_j$. Therefore in general

$$Q = (P_2D_2)Z(P_1D_1)^{-1} = P_2(D_2ZD_1^{-1})P_1^{-1} = P_2ZDP_1^{-1},$$

where D is a diagonal matrix and $Z \in \mathfrak{Z}$.

Proposition 6.3. *For any diagonal matrix $D \in \mathbf{SL}(3,\mathbb{R})$ with nonnegative entries, there exists a unique element Z of $\mathfrak{Z}$ such that $Q = P_2ZDP_1^{-1}$ is positive hyperbolic.*

Proof. By Goldman [8], Q is positive hyperbolic if and only if $x > 0, y > 0$ and

$$f(x,y) = x^2y^2 - 4(x^3 + y^3) + 18xy - 27 > 0,$$

where $x = \operatorname{tr}(Q)$ and $y = \operatorname{tr}(Q^{-1})$. Suppose $x > 4, y > 4$ and $\sqrt{4x} < y < \frac{x^2}{4}$. Then $y^2 - 4x > 0$ and $x^2 - 4y > 0$. Hence

$$f(x,y) = (x^2 - 4y)(y^2 - 4x) + 2(xy - 16) + 5 > 0.$$

Through the Mathematica computation there exists a unique $Z \in \mathfrak{Z}$ such that $x > 0$ and $y > 0$. At this case we can compute $x > 4,\ y > 4$ and $\sqrt{4x} < y < \frac{x^2}{4}$ by using Mathematica. q.e.d.

Consider the geodesic γ connecting the boundary components γ_1 and γ_2 of a pair of pants. Then there exists a diagonal matrix $D_0 \in \mathbf{SL}(3,\mathbb{R})$ such that $\ell(P_2ZD_0P_1^{-1})$ equals the length of the geodesic γ and $m(P_2ZD_0P_1^{-1})$ vanishes. Therefore $Q = P_2ZD_0DP_1^{-1}$ where

$$D = \begin{bmatrix} e^{-\frac{\theta}{2}-\frac{\beta}{3}} & 0 & 0 \\ 0 & e^{\frac{2}{3}\beta} & 0 \\ 0 & 0 & e^{\frac{\theta}{2}-\frac{\beta}{3}} \end{bmatrix}.$$

Now we have a representation of a punctured torus $\Sigma(1,1)$:

$$A_1 = C_1, \quad B_1 = Q = P_2ZD_0DP_1^{-1}, \quad F_1 = C_3$$

since

$$A_1B_1A_1^{-1}B_1^{-1}F_1 = C_1QC_1^{-1}Q^{-1}C_3 = C_1C_2C_3 = I.$$

Similarly we can glue two punctured tori M_1 and M_2 via two boundaries such that $\ell_1 = \ell_2$ and $m_1 = -m_2$. Suppose M_i has a relation $A_iB_iA_i^{-1}B_i^{-1}F_i = I$. Let $Q = P_6ZDP_3^{-1}$ where $P_3^{-1}F_1P_3 = E_3$ and $P_6^{-1}F_2P_6 = E_6$ the diagonal matrices (1). Then $F_2 = QF_1^{-1}Q^{-1}$. Therefore the (geometric) sum $M = M_1 \vee M_2$ is a closed surface with genus 2 having a relation $\bar{A}_1\bar{B}_1\bar{A}_1^{-1}\bar{B}_1^{-1}\bar{A}_2\bar{B}_2\bar{A}_2^{-1}\bar{B}_2^{-1} = I$ where $\bar{A}_1 = A_1, \bar{B}_1 = B_1, \bar{A}_2 = Q^{-1}A_2Q$ and $\bar{B}_2 = Q^{-1}B_2Q$.

Suppose that a closed surface M with genus 2 has a relation

$$R = A_1B_1A_1^{-1}B_1^{-1}A_2B_2A_2^{-1}B_2^{-1} = I.$$

Then its fundamental cycle is

$$\begin{aligned} Z_R &= (I - A_1B_1A_1^{-1}, A_1) + (A_1 - A_1B_1A_1^{-1}B_1^{-1}, B_1) \\ &\quad +(A_1B_1A_1^{-1}B_1^{-1} - RB_2, A_2) \\ &\quad +(A_1B_1A_1^{-1}B_1^{-1}A_2 - R, B_2). \end{aligned}$$

Hence the coefficient of its symplectic form ω is

$$\begin{aligned} \omega(u,v) &= \beta(u(A_1B_1^{-1}A_1^{-1}), v(A_1)) - \beta(u(A_1^{-1}), v(B_1)) \\ &\quad +\beta(u([A_1,B_1]^{-1}), v(B_1)) - \beta(u([A_2,B_2]), v(A_2)) \\ &\quad +\beta(u(B_2^{-1}), v(A_2)) - \beta(u(B_2A_2B_2^{-1}), v(B_2)). \end{aligned}$$

Proposition 6.4. *For each j and k,*

$$\omega(\partial/\partial s_j, \partial/\partial s_k) = 0, \quad \omega(\partial/\partial t_j, \partial/\partial t_k) = 0. \tag{39}$$

Proof. By the definition of u_{z_k} in (27), $\omega(u_{s_j}, u_{s_k})$ becomes

$$\begin{aligned} \mathrm{tr}\Bigg\{ &\frac{\partial(A_1B_1^{-1}A_1^{-1})}{\partial s_j}(A_1B_1A_1^{-1})\frac{\partial A_1}{\partial s_k}A_1^{-1} - \frac{\partial A_1^{-1}}{\partial s_j}A_1\frac{\partial B_1}{\partial s_k}B_1^{-1} \\ &+\frac{\partial F_1}{\partial s_j}F_1^{-1}\frac{\partial B_1}{\partial s_k}B_1^{-1} - \frac{\partial F_2^{-1}}{\partial s_j}F_2\frac{\partial A_2^{-1}}{\partial s_k}A_2 \\ &+\frac{\partial B_2^{-1}}{\partial s_j}B_2\frac{\partial A_2}{\partial s_k}A_2^{-1} - \frac{\partial(B_2A_2B_2^{-1})}{\partial s_j}(B_2A_2^{-1}B_2^{-1})\frac{\partial B_2}{\partial s_k}B_2^{-1}\Bigg\}. \end{aligned}$$

Through the Mathematica calculation, we have the desired result. q.e.d.

Let $\{P_j\}_{j=1,\ldots,2g-2}$ be the disjoint union of pairs of pants decomposed by a partition $\Gamma = \{\gamma_i\}_{i=1,\ldots,3g-3}$. Let s_j, t_j be the internal parameters of each pair of pants P_j where $s_j, t_j \in \mathbb{R}$. Then $\{\ell_i, m_i, \theta_i, \beta_i, s_j, t_j\}$ is a global coordinate on $\mathfrak{P}(M)$ for $1 \le i \le 3g-3$ and $1 \le j \le 2g-2$. By Proposition 6.2, 6.4, Equation (35) and Theorem 5.8, the symplectic form ω on $\mathfrak{P}(M)$ is represented by

$$\omega = \sum_{i=1}^{3g-3} d\ell_i \wedge d\theta_i + \sum_{i=1}^{3g-3} dm_i \wedge d\beta_i + \sum_{j=1}^{2g-2} f_j ds_j \wedge dt_j,$$

where f_j is a smooth function on $\mathfrak{P}(M)$ such that $f_j|_{\mathfrak{L}(M)} \equiv -1$ for each parabolic leaf $\mathfrak{L}(M)$ fixing $\ell_i, m_i, \theta_i, \beta_i$. So $f_j = f_j(\ell_i, m_i, \theta_i, \beta_i)$; that is, f_j is independent of parameters s_j, t_j.

Theorem 6.5. *Let M be a closed smooth surface $\Sigma(g,0)$. Then the symplectic form on the moduli space $\mathfrak{P}(M)$ of convex real projective structures is*

$$\omega = \sum_{i=1}^{3g-3} d\ell_i \wedge d\theta_i + \sum_{i=1}^{3g-3} dm_i \wedge d\beta_i + \sum_{j=1}^{2g-2} dt_j \wedge ds_j,$$

where ℓ_i, m_i are length parameters, θ_i, β_i are twisting parameters, and s_j, t_j are internal parameters on $\mathfrak{P}(M)$. Therefore $\mathfrak{P}(M)$ is symplectomorphic to $\mathbb{R}^{16g-16}$.

Proof. Suppose the symplectic form is

$$\omega = \sum_{i=1}^{3g-3} d\ell_i \wedge d\theta_i + \sum_{i=1}^{3g-3} dm_i \wedge d\beta_i + \sum_{j=1}^{2g-2} f_j \; ds_j \wedge dt_j,$$

where $f_j = f_j(\ell_i, m_i, \theta_i, \beta_i) \in C^\infty(\mathfrak{P}(M))$. Then

$$\omega(\partial/\partial s_j, \cdot) = f_j \; dt_j.$$

Let X_{t_j} be the Hamiltonian vector fields of t_j. Then $\partial/\partial s_j = f_j X_{t_j}$. Since the symplectic form ω is invariant under Hamiltonian vector fields, $L_{X_{t_j}}\omega = 0$. Hence

$$L_{\partial/\partial s_j}\omega = L_{f_j X_{t_j}}\omega = f_j L_{X_{t_j}}\omega = 0.$$

So

$$\begin{aligned} 0 &= L_{\partial/\partial s_j}\omega = \iota_{\partial/\partial s_j} d\omega + d\iota_{\partial/\partial s_j}\omega = 0 + d(\omega(\partial/\partial s_j, \cdot)) \\ &= d(f_j \; dt_j) = df_j \wedge dt_j + f_j \; d^2 t_j = df_j \wedge dt_j. \end{aligned}$$

Therefore f_j is independent of all parameters except t_j. Since f_j is independent of t_j, f_j is independent of all parameters. We conclude $f_j = -1$ since $f_j|_{\mathfrak{L}(M)} = -1$. q.e.d.

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