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COUNTING MASSIVE SETS AND DIMENSIONS OF HARMONIC FUNCTIONS

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0. Introduction

In a previous article of the authors [18], they introduced the notion of d-massive sets. Using this, they proved a structural theorem for polynomial growth harmonic maps into a Cartan-Hadamard manifold with strongly negative sectional curvature. When $d = 0$, the maximum number of disjoint 0-massive sets $m_0(M)$ admissible on a complete manifold M is known [6] to be the same as the dimension $h_0(M)$ of bounded harmonic functions on the manifold. This relationship is not established and it is unclear if it is at all true for $d > 0$. However, the theory of estimating $m_d(M)$ has an analogous counterpart in the theory of estimating $h_d(M)$. This was first indicated in [18]. The purpose of this article is to give sharp estimates on both of these quantities in various situations. Before we outline the main results in this article, let us recall some definitions and set up the appropriate notation.

Throughout this article, M^n is assumed to be an n-dimensional, complete, non-compact manifold without boundary. The operator Δ denotes the Laplacian with respect to the given Riemannian metric. Let $p \in M$ be a fixed point in M and $r_p(x)$ the geodesic distance function from $x \in M$ to the point p.

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Definition 0.1. For each non-negative number d we denote by

$$\mathcal{H}_d(M) = \left\{ f \,\middle|\, \Delta f = 0 \text{ and } |f(x)| = O(r_p^d(x)) \right\}$$

the *space of polynomial growth harmonic functions of degree at most d*. We also denote the dimension of $\mathcal{H}_d(M)$ by

$$h_d(M) = \dim \mathcal{H}_d(M).$$

Definition 0.2. A set $\Xi \subset M$ is a *d-massive set* if it supports a non-negative, nonconstant subharmonic function which is of at most polynomial growth of degree d. In particular, there exists a nonconstant function $v \geq 0$ on Ξ such that v satisfies

$$v = 0 \qquad \text{on } \partial\Xi,$$

$$\Delta v \geq 0 \qquad \text{on } \Xi,$$

and

$$|v(x)| = O(r_p^d(x)) \qquad \text{for all } x \in \Xi.$$

We say that v is a *potential function* of Ξ.

Definition 0.3. We denote m_d to be the maximum number of disjoint d-massive sets admissible on M.

To motivate some of our results, we would like to give a brief history on some of these problems. A more detailed account on the subject can be found in a survey paper of the first author [14].

In 1975, Yau [25] proved a strong Liouville theorem for complete manifolds with non-negative Ricci curvature. In particular, his theorem implies that $h_0(M) = 1$ if M has non-negative Ricci curvature. Cheng, in his 1980 article [3], generalized Yau's argument and showed that if M has non-negative Ricci curvature, then $h_d(M) = 1$ for any $d < 1$. Observing that on $\mathbb{R}^n$, the spaces $\mathcal{H}_d(\mathbb{R}^n)$ are spanned by homogeneous harmonic polynomials of degree at most d, hence one can compute directly that

$$\begin{aligned} h_d(\mathbb{R}^n) &= \binom{n+d-1}{d} + \binom{n+d-2}{d-1} \\ &\sim \frac{2}{(n-1)!}\, d^{n-1} \qquad \text{as } d \to \infty. \end{aligned} \tag{0.1}$$

In view of these, Yau conjectured that a manifold with non-negative Ricci curvature should have $h_d(M) < \infty$. This conjecture was partially confirmed for the case when $d = 1$ by Tam and the first author [15]. In fact, they proved that if M has non-negative Ricci curvature, and its volume growth satisfies

$$V_p(r) = O(r^k)$$

for some $k > 0$, then

$$\begin{aligned} h_d(M) &\leq k + 1 \\ &= h_d(\mathbb{R}^k). \end{aligned}$$

Note that such k must exist and satisfy $1 \leq k \leq n$ by the Bishop volume comparison theorem and a theorem of Yau and Cheeger-Gromov-Taylor. In particular, this implies the estimate

$$h_d(M) \leq h_d(\mathbb{R}^n).$$

The theorem of Li-Tam naturally motivated some conjectures.

Conjecture 0.4. *Let M be a complete manifold with non-negative Ricci curvature. Suppose M has volume growth satisfying*

$$V_p(r) = O(r^k)$$

for some $1 \leq k \leq n$. Then

$$h_d(M) \leq h_d(\mathbb{R}^k)$$

for all $d \geq 0$.

A more conservative version of this can be stated as:

Conjecture 0.5. *Let M be a complete manifold with non-negative Ricci curvature. Then*

$$h_d(M) \leq h_d(\mathbb{R}^n)$$

for all $d \geq 0$. Moreover, equality holds for some $d \geq 1$ if and only if $M = \mathbb{R}^n$.

It turns out that Conjecture 0.4 is valid if the manifold is of dimension 2. This was verified by Li-Tam [16] and Kasue [8]. In fact, they gave sharp upper bounds on $h_d(M)$ for surfaces with finite total curvature. Furthermore, Li-Tam [16] also gave sharp lower bounds for $h_d(M)$ for surfaces with finite total curvature that has quadratic area growth.

The equality case of Conjecture 0.5 for $d = 1$ was also proved. In [12], the first author proved that if M^m is a complex m-dimensional, complete, Kähler manifold with non-negative Ricci curvature, and

$$h_1(M) = h_1(\mathbb{R}^{2m}),$$

then $M = \mathbb{C}^m$. This was generalized to the real case in [2], which they proved the equality statement of Conjecture 0.5 for $d = 1$.

Recently, Yau's conjecture of finite dimensionality was proved in [5]. In fact, they proved that if M has non-negative Ricci curvature, then there exists $C > 0$ depending only on n, such that

$$h_d(M) \le C\, d^{n-1}.$$

In view of the formula (0.1) for $\mathbb{R}^n$, this estimate is sharp in the order of d as $d \to \infty$. The authors also proved that if a manifold satisfies a Poincaré inequality and a volume doubling property, then $h_d(M)$ is finite and can be estimated in terms of a constant depending on the manifold and d. However, in this case, the order in d is not sharp.

Shortly after the announcement of [5], the first author [13] proved a much general estimate with a substantially simpler proof. He proved that if M satisfies a mean value inequality and a volume comparison condition, then

$$h_d(M) \le C\, d^{n-1}.$$

Later on in [19], the authors showed that the finiteness of $h_d(M)$ is actually valid in a much bigger class of manifolds. In particular, they proved that if M satisfies a weak mean value inequality and it has polynomial volume growth, then $h_d(M)$ must be finite for all $d \ge 1$. However, in this case, the estimate on $h_d(M)$ is exponential in d as $d \to \infty$.

The aim of the first section is to show that (Theorem 1.6) if M satisfies a Sobolev inequality $\mathcal{S}(B, \nu)$ (see Definition 1.4), then

$$h_d(M) \le C(B, \nu)\, d^{\nu}.$$

As it was previously noted in [18], a slight modification of the same argument also shows (Theorem 1.7) that

$$m_d(M) \le C(B, \nu)\, d^{\nu}.$$

In most known cases, such as, manifolds that are quasi-isometric to one with non-negative Ricci curvature, and minimal submanifolds with

Euclidean volume growth in $\mathbb{R}^N$, they all satisfy the Sobolev inequality $\mathcal{S}(B,n)$. Hence the estimates given by Theorem 1.6 and Theorem 1.7 are of the order d^n. Therefore in these situations we gain in generality by a substantial amount but sacrifice the sharpness in the order of d.

In §2, we consider the case where M has non-negative sectional curvature. While we cannot confirm Conjecture 0.5, we are able to prove that if M also has maximal volume growth with

$$\liminf_{r\to\infty} r^{-n}\, V_p(r) = \alpha_0$$

for some $0 < \alpha_0 \leq \omega_n$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$, then

$$\begin{aligned}\limsup_{d\to\infty} d^{-n} \sum_{i=0}^{d} h_i(M) &\leq \frac{2\,\alpha_0}{n!\,\omega_n} \\ &\leq \frac{2}{n!}.\end{aligned}$$

Moreover, the equality

$$\limsup_{d\to\infty} d^{-n} \sum_{i=0}^{d} h_i(M) = \frac{2}{n!}.$$

holds if and only if $M = \mathbb{R}^n$.

In §3, we will give sharp pointwise estimates on the maximum number of d-massive sets, $m_d(M)$, for a large class of surfaces. First we will show that if M is a complete surface with finite total curvature, then $m_d(M)$ has a sharp upper bound (Theorem 3.3) for all values of d. The bound depends only on the area growth at each end of M and d. Recall that in [18], the authors proved that if $u : M \to N$ is a harmonic map of at most polynomial growth of degree d into a Cartan-Hadamard manifold with strongly negative curvature, then the image of u must contained in an ideal polygon of at most m_d vertices. Combining the sharp estimate of m_d in Theorem 3.3, this gives a sharp bound on the number of vertices in the structural theorem for harmonic maps. In particular, this recovers the estimate in a formula given by Han-Tam-Treibergs-Wan [7] for the special case of $u : \mathbb{R}^2 \to \mathbb{H}^2$. It turns out that for complete surfaces, one only needs to assume that M has at most quadratic volume growth to give an estimate (Theorem 3.4) on m_d which is sharp in the order of d as $d \to \infty$. However, in this case, the constant in the estimate is not sharp.

Finally, in the last section §4, we study the space of bounded harmonic functions in the setting of a discrete group, Γ, acting discontinuously, isometrically on a complete manifold. This point of view was first taken up by Lyons-Sullivan in [20] for the case where M is a regular cover of a compact manifold. Using the notion of 0-massive sets, we will generalize some of the results in [20] and results by subsequent authors. One of the directions of generalization is that we do not require the quotient space M/Γ to be compact.

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1. Sobolev inequality and dimension estimates

In this section, we will assume that the complete manifold M satisfies a Sobolev inequality of a form which is scaling invariant.

Definition 1.1. A complete manifold M is said to satisfy a *Sobolev inequality* $\mathcal{S}(B,\nu)$ if there exist a point $p \in M$, and constants $B > 0$ and $\nu > 2$, such that for all $r > 0$, and all $f \in C_c^\infty(B_p(r))$, we have

$$\left(\int_{B_p(r)} |f|^{\frac{2\nu}{\nu-2}}\right)^{\frac{\nu-2}{\nu}} \leq B\, r^2\, V_p^{-\frac{2}{\nu}}(r) \int_{B_p(r)} \left(|\nabla f|^2 + r^{-2} f^2\right).$$

We will show that for any manifold satisfying $\mathcal{S}(B,\nu)$ the space $\mathcal{H}_d(M)$ must be finite dimensional and its dimension is bounded from above by $C\,d^\nu$. Note that when apply to manifolds with non-negative Ricci curvature, which satisfies $\mathcal{S}(B,n)$, the estimate takes the form

$$h_d(M) \leq C\, d^n$$

rather than the sharp order d^{n-1}. However, the assumption $\mathcal{S}(B,\nu)$ is more general then non-negative Ricci curvature. We will first establish some preliminary lemmas before we prove this estimate.

Lemma 1.2. *Let V be a k-dimensional subspace of a vector space W. Assume that W is endowed with an inner product L and a bilinear form Φ. Then for any given linearly independent set of vectors $\{w_1, \dots, w_{k-1}\} \subset W$, there exists an orthonormal basis $\{v_1, \dots, v_k\}$ of V with respect to L such that $\Phi(v_i, w_j) = 0$ for all $1 \leq j < i \leq k$.*

Proof. Since the space spanned by the set $\{w_1, \dots, w_{k-1}\}$ has dimension $k-1$, there exists $v_k \in V$ such that v_k has unit length with respect to L and $\Phi(v_k, w_j) = 0$ for all $1 \leq j < k$. Let us denote by V_1 to be the subspace of V which is orthogonal to v_k with respect to L. Obviously, $\dim V_1 = k - 1$. One can then choose $v_{k-1} \in V_1$ such that v_{k-1} has unit length with respect to L and $\Phi(v_{k-1}, w_j) = 0$ for all $1 \leq j < k - 1$. Since $v_{k-1} \in V_1$ we have $L(v_k, v_{k-1}) = 0$. Inductively, one can choose $\{v_k, \dots, v_1\}$, an orthonormal basis of V with respect to L such that $\Phi(v_i, w_j) = 0$ for all $1 \leq j < i \leq k$. q.e.d.

Let ϕ be a positive function defined on a fixed geodesic ball $B_p(r)$. Let us introduce two inner products L and Φ on the space

$$W = L^2(B_p(r), dx) \cap L^2(B_p(r), \phi\, dx)$$

by

$$L_r(f, g) = \int_{B_p(r)} f(x)\, g(x)\, dx$$

and

$$\Phi_r(f, g) = \int_{B_p(r)} f(x)\, g(x)\, \phi(x)\, dx.$$

For each $i = 1, 2, \dots$, let $\lambda_i(r)$ be the i-th Dirichlet eigenvalue of $B_p(r)$ arranged in non-decreasing order. We have the following lemma.

Lemma 1.3. *Let V be a k-dimensional subspace of $\mathcal{H}_d(M)$. For any fixed number $\beta > 1$ and any subspace Y of V, let $tr_{L_{\beta r}} L_r(Y)$ be the trace of the bilinear form L_r with respect to the inner product $L_{\beta r}$ on Y. Then for any fixed integer m with $0 \leq m \leq k$, we have*

$$\min_{\dim Y = k-m} tr_{L_{\beta r}} L_r(Y) \leq \sum_{i=m+1}^{k} \frac{4}{(\beta - 1)^2 r^2 \lambda_i(\beta r)},$$

where the minimum is taken over all subspaces Y in V with $\dim Y = k - m$. In particular,

$$tr_{L_{\beta r}} L_r(V) \leq \sum_{i=1}^{k} \frac{4}{(\beta - 1)^2\, r^2\, \lambda_i(\beta r)}.$$

Proof. Let ϕ be non-negative function defined on $B_p(\beta r)$ satisfying $\phi = 1$ on $B_p(r)$, $0 \leq \phi \leq 1$ on $B_p(\beta r)$, $\phi = 0$ on $\partial B_p(\beta r)$, and

$$|\nabla \phi| \leq \frac{2}{(\beta - 1)\, r}.$$

Observe that by unique continuation,

$$V \subset L^2(B_p(\beta r), dx) \cap L^2(B_p(r), \phi\, dx)$$

is a k-dimensional subspace. Applying Lemma 1.2 with $\{w_1, \dots, w_k\}$ being the Dirichlet eigenfunctions of $B_p(\beta r)$ corresponding to the eigenvalues $\{\lambda_1(\beta r), \dots, \lambda_k(\beta r)\}$, we get an orthonormal basis $\{v_1, \dots, v_k\}$ of V with respect to the inner product $L_{\beta r}$ and

$$\Phi_{\beta r}(v_i, w_j) = \int_{B_p(\beta r)} v_i(x)\, w_j(x)\, \phi(x)\, dx = 0$$

for $1 \leq j < i \leq k$. Thus, for any $1 \leq i \leq k$, the variation principle implies that

$$\lambda_i(\beta r) \int_{B_p(\beta r)} (\phi\, v_i)^2 \leq \int_{B_p(\beta r)} |\nabla(\phi\, v_i)|^2. \tag{1.1}$$

On the other hand, we have

$$\begin{aligned} 0 &= \int_{B_p(\beta r)} \phi^2\, v_i\, \Delta v_i \\ &= -\int_{B_p(\beta r)} \phi^2\, |\nabla v_i|^2 - 2 \int_{B_p(\beta r)} \phi\, v_i\, \langle \nabla\phi, \nabla v_i \rangle \\ &= -\int_{B_p(\beta r)} |\nabla(\phi\, v_i)|^2 + \int_{B_p(\beta r)} |\nabla\phi|^2\, v_i^2. \end{aligned} \tag{1.2}$$

Combining (1.1) and (1.2), we conclude that

$$\begin{aligned} \int_{B_p(r)} v_i^2 &\leq \int_{B_p(\beta r)} (\phi\, v_i)^2 \\ &\leq \lambda_i^{-1}(\beta r) \int_{B_p(\beta r)} |\nabla(\phi\, v_i)|^2 \\ &\leq \lambda_i^{-1}(\beta r) \int_{B_p(\beta r)} |\nabla\phi|^2\, v_i^2 \\ &\leq \frac{4}{(\beta - 1)^2\, r^2\, \lambda_i(\beta r)}. \end{aligned}$$

Hence, if we let Y to be the space spanned by $\{v_{m+1}, \dots, v_k\}$, then $\dim Y = k - m$ and

$$tr_{L_{\beta r}} L_r(Y) = \sum_{i=m+1}^{k} \int_{B_p(r)} v_i^2 \leq \sum_{i=m+1}^{k} \frac{4}{(\beta - 1)^2\, r^2\, \lambda_i(\beta r)}.$$

This completes the proof. q.e.d.

The following result was established by the first author in [13].

Lemma 1.4. *Let K be a k-dimensional linear space of functions defined on a manifold N with polynomial volume growth of order ν. Suppose each function $u \in K$ is of polynomial growth of at most degree d. For any $\beta > 1$, $\delta > 0$, and $r_0 > 0$, there exists $r > r_0$ such that if $\{u_i\}_{i=1}^k$ is an orthonormal basis of K with respect to the inner product $L_{\beta r}(u,v) = \int_{B_p(\beta r)} u\, v$, then*

$$\sum_{i=1}^k \int_{B_p(r)} u_i^2 \geq k\, \beta^{-(2d+\nu+\delta)}.$$

Lemma 1.5. *Suppose that M is a complete manifold satisfying the Sobolev inequality $\mathcal{S}(B,\nu)$. If $\lambda_k(r)$ is the k-th Dirichlet eigenvalue of $B_p(r)$, then there exists a positive integer k_0 depending only on B and ν such that*

$$\lambda_k(r) \geq C(B,\nu)\, r^{-2}\, k^{\frac{2}{\nu}}$$

for all $k \geq k_0$ and $r > 0$.

Proof. We follow an argument due to S. Y. Cheng and the first author [4]. Let $H(x,y,t)$ be the Dirichlet heat kernel of $B_p(r)$. Then by the semigroup property of $H(x,y,t)$, the Sobolev inequality, and the fact that

$$\int_{B_p(r)} H(x,y,t)\, dy \leq 1,$$

we have

$$\begin{aligned}
&H^{\frac{\nu+2}{\nu}}(x,x,2t)\\
&\quad= \left(\int_{B_p(r)} H^2(x,y,t)\,dy\right)^{\frac{\nu+2}{\nu}}\\
&\quad\leq \left(\int_{B_p(r)} H^{\frac{2\nu}{\nu-2}}(x,y,t)\,dy\right)^{\frac{\nu-2}{\nu}} \left(\int_{B_p(r)} H(x,y,t)\,dy\right)^{\frac{4}{\nu}}\\
&\quad\leq B\,r^2\,V_p^{-\frac{2}{\nu}}(r)\int_{B_p(r)} \left(|\nabla H|^2(x,y,t) + r^{-2}H^2(x,y,t)\right)\,dy\\
&\quad= -B\,r^2\,V_p^{-\frac{2}{\nu}}(r)\Bigg(\int_{B_p(r)} H(x,y,t)\,\Delta H(x,y,t)\,dy\\
&\qquad\qquad - r^{-2}H(x,x,2t)\Bigg)\\
&\quad= -B\,r^2\,V_p^{-\frac{2}{\nu}}(r)\frac{\partial}{\partial t}H(x,x,2t) + B\,V_p^{-\frac{2}{\nu}}(r)H(x,x,2t).
\end{aligned}$$

We may rewrite this into the form

$$\frac{\partial}{\partial t}H^{-\frac{2}{\nu}}(x,x,2t) + \frac{2}{\nu}\,r^{-2}\left(H^{-\frac{2}{\nu}}(x,x,2t)\right) \geq B^{-1}\,r^{-2}\,V_p^{\frac{2}{\nu}}(r),$$

which is equivalent to

$$\frac{\partial}{\partial t}\left(\exp\left(\frac{2t}{\nu\,r^2}\right)\,H^{-\frac{2}{\nu}}(x,x,2t)\right) \geq B^{-1}\,r^{-2}\,V_p^{\frac{2}{\nu}}(r)\,\exp\left(\frac{2t}{\nu\,r^2}\right).$$

Integrating this inequality with respect to t and noting that

$$\lim_{\epsilon\to 0} H^{-\frac{2}{\nu}}(x,x,2\epsilon) = 0$$

by the heat kernel short time asymptotic behavior, we conclude that

$$\begin{aligned}
\exp\left(\frac{2t}{\nu r^2}\right)\,H^{-\frac{2}{\nu}}(x,x,2t) &\geq B^{-1}\,r^{-2}\,V_p^{\frac{2}{\nu}}(r)\int_0^t \exp\left(\frac{2s}{\nu\,r^2}\right)\,ds\\
&= \frac{\nu}{2B}\,V_p^{\frac{2}{\nu}}(r)\,\left(\exp\left(\frac{2t}{\nu\,r^2}\right) - 1\right).
\end{aligned}$$

This implies that

$$H(x,x,2t) \leq C\,V_p^{-1}(r)\,\left(1 - \exp(-\frac{2t}{\nu\,r^2})\right)^{-\frac{\nu}{2}},$$

where $C > 0$ is a constant depending only on B and ν. Integrating this inequality over the ball $B_p(r)$, we arrived at the estimate

$$\sum_{i=1}^{\infty} e^{-2\lambda_i t} \leq C \left(1 - \exp\left(-\frac{2t}{\nu\, r^2}\right)\right)^{-\frac{\nu}{2}}. \tag{1.3}$$

For any $k \geq 1$, setting $t = \frac{1}{\lambda_k}$ in (1.3) gives

$$C\, k \leq \left(1 - \exp\left(-\frac{2}{\nu\, \lambda_k\, r^2}\right)\right)^{-\frac{\nu}{2}}.$$

Hence,

$$\frac{1}{\lambda_k} \leq C r^2 \ln \frac{1}{1 - C k^{-\frac{2}{\nu}}}. \tag{1.4}$$

By choosing k_0 such that $1 - C k_0^{\frac{-2}{\nu}} \geq \frac{1}{2}$, it is easy to conclude from (1.4) that

$$\lambda_k \geq C(B, \nu)\, k^{\frac{2}{\nu}}\, r^{-2}$$

for $k \geq k_0$, where k_0 depends only on B and ν. q.e.d.

Theorem 1.6. *Let M be a complete manifold satisfying the Sobolev inequality $\mathcal{S}(B, \nu)$. Then the space $\mathcal{H}_d(M)$ is of finite dimensional and its dimension h_d satisfies the estimate*

$$h_d \leq C(B, \nu)\, d^{\nu}$$

for all $d \geq 1$.

Proof. It has been verified in [19] that if M satisfies the Sobolev inequality $\mathcal{S}(B, \nu)$, then it must have polynomial volume growth of at most order ν. Lemma 1.5 also implies that

$$\lambda_k(r) \geq C(B, \nu)\, r^{-2}\, k^{\frac{2}{\nu}}$$

for all $k \geq k_0$ and $r > 0$, where k_0 is a constant depending on B and ν. In particular,

$$\sum_{i=k_0+1}^{k} \lambda_i^{-1}(\beta r) \leq C(B, \nu)\, \beta^2\, r^2\, k^{1-\frac{2}{\nu}}.$$

Lemma 1.4 yields that for any k-dimensional subspace V of $\mathcal{H}_d(M)$ and any $\beta > 1$, there exists $R > 0$ such that

$$tr_{L_{\beta R}} L_R(V) \geq k\,\beta^{-(2d+\nu+1)}$$

as M has polynomial volume growth of order ν. Applying Lemma 1.3 with $m = k_0$, we conclude that there exists a subspace Y of V with $\dim Y = k - k_0$ such that

$$\begin{aligned} k\,\beta^{-(2d+\nu+1)} - k_0 &\leq tr_{L_{\beta R}} L_R(V) - k_0 \\ &\leq tr_{L_{\beta R}} L_R(Y) \\ &\leq 4(\beta-1)^{-2}\,R^{-2} \sum_{i=k_0+1}^{k} \lambda_i^{-1}(\beta R) \\ &\leq C(B,\nu)\,\beta^2\,(\beta-1)^{-2}\,k^{1-\frac{2}{\nu}}. \end{aligned}$$

Choosing $\beta = 1 + d^{-1}$, we obtain that $k \leq C(B,\nu)\,d^{\nu}$. This shows that $h_d \leq C(B,\nu)\,d^{\nu}$, for all $d \geq 1$. q.e.d.

For any k disjoint d-massive sets $\Xi_1, \dots, \Xi_k$ in M, let $v_1, \dots, v_k$ be the corresponding potential functions, and V the k-dimensional space spanned by $v_1, \dots, v_k$. Then it is not difficult to see Lemma 1.3 also holds in this case. In particular, one can conclude the same estimate for k as in Theorem 1.6.

Theorem 1.7. *Let M be a complete manifold satisfying the Sobolev inequality $\mathcal{S}(B,\nu)$. Then for all $d \geq 1$, the maximum number of disjoint d-massive sets m_d of M satisfies $m_d \leq C(B,\nu)\,d^{\nu}$.*

2. Sharp asymptotic estimates on the dimensions of harmonic functions

In this section, we will assume that M has non-negative sectional curvature. In this case, we will show that if $h_d(M) = \dim \mathcal{H}_d(M)$, then $\sum_{i=1}^{d} h_i(M)$ has an upper bound which is asymptotically sharp as $d \to \infty$.

For a fixed point $p \in M$, let $\mathcal{H}'_d(M) \subset \mathcal{H}_d(M)$ be the subspace of harmonic functions f on M of growth order at most d satisfying $f(p) = 0$. It is more convenient for us to work with the spaces $\mathcal{H}'_d(M)$ instead. Obviously, if we denote $h'_d(M) = \dim \mathcal{H}'_d(M)$, then $h'_d(M) = h_d(M) - 1$.

Using polar coordinates in the tangent space T_pM and the exponential map at p, we can write any point $x \in M$ in the form $x = (\theta, r)$

where $\theta \in S_pM$, the unit sphere of T_pM, and $r = r(p,x)$ is the distance from p to x. Let us define subsets of S_pM by

$$\Gamma(\infty) = \left\{ \theta \in S_pM \;\middle|\; J_\infty(\theta) = \liminf_{t\to\infty} \frac{J(\theta,t)}{t^{n-1}} > 0 \right\}$$

and

$$\Gamma_\tau(\infty) = \left\{ \theta \in S_pM \;\middle|\; J_\infty(\theta) \geq \tau \right\},$$

where $J(\theta,t)$ is the Jacobian of the exponential map $\exp_p$ at the point (θ,t). Here we are taking the convention that $J(\theta,t)$ is defined to be 0 if (θ,t) lies beyond the cut locus of p, which is equivalent to $t > r(p,\exp_p(\theta,t))$. It is clear that the set $\Gamma(\infty)$ is nonempty if M has maximal volume growth. Moreover, if we let

$$\Gamma(t) = \left\{ \theta \in S_pM \;\middle|\; J(\theta,t) > 0 \right\}$$

and

$$\Gamma_\tau(t) = \left\{ \theta \in S_pM \;\middle|\; J(\theta,t) > \tau t^{n-1} \right\},$$

then $\Gamma_\tau(t)$ monotonically decreases to $\Gamma_\tau(\infty)$ as the function $t^{-(n-1)}\, J(\theta,t)$ is non-increasing in t. One easily verifies that the set $\Gamma_\tau(\infty)$ is compact for any $\tau > 0$. In particular, for any given $\delta > 0$, there exist $\delta \geq \tau(\delta) > 0$ and $R_\delta > 0$, such that, for any $t \geq R_\delta$, in terms of the standard spherical measure on S_pM we have

$$|\Gamma_\tau(t) \setminus \Gamma_\tau(\infty)| < \delta$$

and

$$|\Gamma(\infty) \setminus \Gamma_\tau(\infty)| < \delta.$$

Moreover, for any $\theta \in \Gamma_\tau(\infty)$,

$$J(\theta,t) \leq (1+\delta)\, J_\infty(\theta) t^{n-1}.$$

Let us fix a r_0 with $r_0 \geq R_\delta$. Since the set $\Gamma_\tau(r_0)$ is open in S_pM, we may choose an open subset $\Gamma'(r_0) \subset \Gamma_\tau(r_0)$ such that $\Gamma'(r_0)$ has only finitely many components and the boundary $\partial\Gamma'(r_0)$ is smooth. Also, we may arrange to have the spherical measure

$$|\Gamma_\tau(r_0) \setminus \Gamma'(r_0)| < \delta.$$

Let us define

$$\Omega'(r_0) = \{(\theta, r_0) \in \partial B_p(r_0) \,|\, \theta \in \Gamma'(r_0)\}$$

and

$$\Omega(t) = \{(\theta, t) \in \partial B_p(t) \,|\, \theta \in \Gamma(t)\}.$$

Also, if we set

$$\tilde{\Gamma}_\delta(t) = \{\theta \in S_pM \,|\, r_0^{-(n-1)} J(\theta, r_0) \leq (1+\delta) t^{-(n-1)} J(\theta, t)\} \cap \Gamma'(r_0),$$

then we define

$$\tilde{\Omega}_\delta(t) = \{(\theta, t) \in \partial B_p(t) \,|\, \theta \in \tilde{\Gamma}_\delta(t)\}.$$

Let us denote the induced metric on $\Omega(t)$ by $d\bar{s}^2(\theta, t)$ and list the Neumann eigenvalues of $\Omega'(r_0)$ with respect to the metric $d\bar{s}^2(\theta, r_0)$ in the nondecreasing order by $0 = \eta_1(r_0) \leq \eta_2(r_0) \leq \cdots \leq \eta_i(r_0) \leq \ldots$. First, we prove the following lemma.

Lemma 2.1. *Let M be a complete manifold with non-negative sectional curvature of $\dim M \geq 3$. Assume that M has maximal volume growth. Let H be a subspace of $\mathcal{H}'_d(M)$ of dimension k. Then for $r \geq r_0 \geq R_\delta$ and any D_r-orthonormal basis $\{v_i\}_{i=1}^k$ of H with respect to the bilinear form*

$$D_r(f, g) = \int_{B_p(r)} \langle \nabla f, \nabla g \rangle,$$

and for any $\epsilon > 0$, we have

$$2r_0 \sum_{i=1}^k \eta_i^{\frac{1}{2}}(r_0) \leq (1+\delta)\, r \sum_{i=1}^k \int_{\partial B_p(r)} |\nabla v_i|^2 + C\,\delta\, \left(r_0^2\, \eta_k(r_0)\, \epsilon^{2-n} + \epsilon^{-n}\right) \sup_{u \in S(H)} D_{((1+\epsilon)r)}(u, u),$$

where $S(H)$ denotes the unit sphere of H with respect to the norm D_r.

Proof. Let us first remark that the inequality is independent of the choice of orthonormal basis $\{v_i\}_{i=1}^k$. By [9], there exists a Lipschitz map Φ_r from $\partial B_p(r_0)$ onto $\partial B_p(r)$ with Lipschitz constant at most r/r_0 and $\Phi_r(\theta, r_0) = (\theta, r)$ for $\theta \in \Gamma(r)$. So the set of functions

$$\{v_1 \circ \Phi_r, \ldots, v_k \circ \Phi_r\}$$

are defined on $\Omega'(r_0)$. Let $\{w_1, \dots, w_{k-1}\}$ be the first $(k-1)$ Neumann eigenfunctions on $\Omega'(r_0)$ with respect to the eigenvalues $\{\eta_1(r_0), \dots, \eta_{k-1}(r_0)\}$. Then after arguing as in Lemma 1.2, we get a D_r-orthonormal basis $\{u_1, \dots, u_k\}$ of the space spanned by $\{v_i\}$, such that

$$\int_{\Omega'(r_0)} f_i\, w_j = 0$$

for all $1 \leq j < i \leq k$, where $f_i = u_i \circ \Phi_r$. Hence by the variational principle, we conclude that

$$\eta_i(r_0) \int_{\Omega'(r_0)} f_i^2 \leq \int_{\Omega'(r_0)} |\bar{\nabla} f_i|^2, \tag{2.1}$$

where we denote $\bar{\nabla}$ to be the tangential gradient of $\Omega'(r_0)$. Note that we did not assume that the functions $\{f_i\}$ are non-zero. In fact, when that becomes the case, inequality (2.1) is still valid. Summing up over i and using the harmonicity of u_i, (2.1) gives

$$\begin{aligned}
2\frac{r_0}{r} \sum_{i=1}^{k} \eta_i^{\frac{1}{2}}(r_0) &= 2\frac{r_0}{r} \sum_{i=1}^{k} \eta_i^{\frac{1}{2}}(r_0) \int_{B_p(r)} |\nabla u_i|^2 \\
&= 2\frac{r_0}{r} \sum_{i=1}^{k} \eta_i^{\frac{1}{2}}(r_0) \int_{\partial B_p(r)} u_i \frac{\partial u_i}{\partial r} \\
&\leq \left(\frac{r_0}{r}\right)^2 \sum_{i=1}^{k} \eta_i(r_0) \int_{\partial B_p(r)} u_i^2 \\
&\quad + \sum_{i=1}^{k} \int_{\partial B_p(r)} \left(\frac{\partial u_i}{\partial r}\right)^2.
\end{aligned}$$

The first term on the right-hand side can be written as

$$\begin{aligned}
\left(\frac{r_0}{r}\right)^2 \sum_{i=1}^{k} \eta_i(r_0) \int_{\partial B_p(r)} u_i^2 &= \left(\frac{r_0}{r}\right)^2 \sum_{i=1}^{k} \eta_i(r_0) \int_{\tilde{\Omega}_\delta(r)} u_i^2 \\
&\quad + \left(\frac{r_0}{r}\right)^2 \sum_{i=1}^{k} \eta_i(r_0) \int_{\partial B_p(r)\setminus\tilde{\Omega}_\delta(r)} u_i^2 \\
&\leq \left(\frac{r_0}{r}\right)^{3-n} \sum_{i=1}^{k} \eta_i(r_0) \int_{\Omega'(r_0)} f_i^2 \\
&\quad + \left(\frac{r_0}{r}\right)^2 \sum_{i=1}^{k} \eta_i(r_0) \int_{\partial B_p(r)\setminus\tilde{\Omega}_\delta(r)} u_i^2,
\end{aligned}$$

where we have used the fact that $r^{-(n-1)}\, J(\theta, r)$ is monotonically non-increasing. Hence we have

$$
\begin{aligned}
2\frac{r_0}{r} \sum_{i=1}^{k} \eta_i^{\frac{1}{2}}(r_0) \leq \left(\frac{r_0}{r}\right)^{3-n} \sum_{i=1}^{k} \eta_i(r_0) \int_{\Omega'(r_0)} f_i^2 & \\
+ \left(\frac{r_0}{r}\right)^{2} \sum_{i=1}^{k} \eta_i(r_0) \int_{\partial B_p(r) \setminus \tilde{\Omega}_\delta(r)} u_i^2 & \\
+ \sum_{i=1}^{k} \int_{\partial B_p(r)} \left(\frac{\partial u_i}{\partial r}\right)^2 . &
\end{aligned}
\tag{2.2}
$$

Applying (2.1) to the first term on the right-hand side of (2.2), we have

$$
\begin{aligned}
\left(\frac{r_0}{r}\right)^{3-n} \sum_{i=1}^{k} \eta_i(r_0) \int_{\Omega'(r_0)} f_i^2 &\leq \left(\frac{r_0}{r}\right)^{3-n} \sum_{i=1}^{k} \int_{\Omega'(r_0)} |\bar{\nabla} f_i|^2 \\
&= \left(\frac{r_0}{r}\right)^{3-n} \sum_{i=1}^{k} \int_{\Gamma'(r_0) \setminus \tilde{\Gamma}_\delta(r)} |\bar{\nabla} f_i|^2\, J(\theta, r_0)\, d\theta \\
&\quad + \left(\frac{r_0}{r}\right)^{3-n} \sum_{i=1}^{k} \int_{\tilde{\Gamma}_\delta(r)} |\bar{\nabla} f_i|^2\, J(\theta, r_0)\, d\theta.
\end{aligned}
$$

Using the facts that $r^{-(n-1)}\, J(\theta, r) \leq 1$ and $\frac{r_0}{r}\, |\bar{\nabla} f_i| \leq |\bar{\nabla} u_i|$, the first term on the right-hand side can be estimated by

$$
\begin{aligned}
&\left(\frac{r_0}{r}\right)^{3-n} \sum_{i=1}^{k} \int_{\Gamma'(r_0) \setminus \tilde{\Gamma}_\delta(r)} |\bar{\nabla} f_i|^2\, J(\theta, r_0)\, d\theta \\
&\qquad \leq r^{n-1}\, |\Gamma'(r_0) \setminus \tilde{\Gamma}_\delta(r)| \max_{\partial B_p(r)} \sum_{i=1}^{k} |\bar{\nabla} u_i|^2 \\
&\qquad \leq r^{n-1}\, |\Gamma_\tau(r_0) \setminus \Gamma_\tau(\infty)| \max_{\partial B_p(r)} \sum_{i=1}^{k} |\nabla u_i|^2 \\
&\qquad \leq \delta\, r^{n-1} \max_{\partial B_p(r)} \sum_{i=1}^{k} |\nabla u_i|^2,
\end{aligned}
$$

and the second term on the right-hand side can be estimated by

$$\begin{aligned}\left(\frac{r_0}{r}\right)^{3-n} \sum_{i=1}^k \int_{\tilde{\Gamma}_\delta(r)} |\bar{\nabla} f_i|^2 \, J(\theta, r_0)\, d\theta &\leq (1+\delta) \sum_{i=1}^k \int_{\tilde{\Omega}_\delta(r)} |\bar{\nabla} u_i|^2 \\ &\leq (1+\delta) \sum_{i=1}^k \int_{\partial B_p(r)} |\bar{\nabla} u_i|^2.\end{aligned}$$

Putting all these into (2.2), we conclude that

$$\begin{aligned}2\frac{r_0}{r} \sum_{i=1}^k \eta_i^{\frac{1}{2}}(r_0) \leq (1+\delta) &\sum_{i=1}^k \int_{\partial B_p(r)} |\nabla u_i|^2 \\ &+ \left(\frac{r_0}{r}\right)^2 \eta_k(r_0) \sum_{i=1}^k \int_{\partial B_p(r)\setminus\tilde{\Omega}_\delta(r)} u_i^2 \\ &+ \delta\, r^{n-1} \max_{\partial B_p(r)} \sum_{i=1}^k |\nabla u_i|^2.\end{aligned} \tag{2.3}$$

To estimate the second term on the right-hand side of (2.3), we observe that

$$\begin{aligned}\sum_{i=1}^k \int_{\partial B_p(r)\setminus\tilde{\Omega}_\delta(r)} u_i^2 &\leq A(\partial B_p(r) \setminus \tilde{\Omega}_\delta(r)) \max_{\partial B_p(r)} \sum_{i=1}^k u_i^2 \\ &\leq C\,\delta\, r^{n-1} \max_{\partial B_p(r)} \sum_{i=1}^k u_i^2.\end{aligned} \tag{2.4}$$

Since the term $\sum_{i=1}^k u_i^2$ is independent of the choice of orthonormal basis, we may assume that for a fixed point $x \in \partial B_p(r)$ we have $u_i(x) = 0$ for all $i > 1$, and hence

$$\sum_{i=1}^k u_i^2(x) = u_1^2(x). \tag{2.5}$$

On the other hand, since $u_1(p) = 0$, if γ is a minimizing geodesic joining x to p, then

$$|u_1(x)| \leq \int_\gamma |\nabla u_1|.$$

To estimate the line integral, we use the fact that $|\nabla u_1|^2$ is subharmonic because M has non-negative sectional curvature. The mean value inequality implies that

$$\begin{aligned}\int_\gamma |\nabla u_1| &\leq C \int_0^r \left(V_{\gamma(t)}^{-1}((1+\epsilon)r - t) \int_{B_{\gamma(t)}((1+\epsilon)r-t)} |\nabla u_1|^2 \right)^{\frac{1}{2}} dt \\ &\leq C \int_0^r V_{\gamma(t)}^{-\frac{1}{2}}((1+\epsilon)r - t)\, dt \left(\int_{B_p((1+\epsilon)r)} |\nabla u_1|^2 \right)^{\frac{1}{2}}.\end{aligned}$$

Using the assumption that M has maximal volume growth, we have

$$\begin{aligned}&\int_0^r V_{\gamma(t)}^{-\frac{1}{2}}((1+\epsilon)r - t)\, dt \\ &\qquad \leq C \int_0^r ((1+\epsilon)r - t)^{-\frac{n}{2}}\, dt \\ &\qquad = C \int_{\epsilon r}^{(1+\epsilon)r} s^{-\frac{n}{2}}\, ds \\ &\qquad = \frac{2C}{n-2} \left(\epsilon^{1-\frac{n}{2}} - (1+\epsilon)^{1-\frac{n}{2}} \right) r^{1-\frac{n}{2}}.\end{aligned}$$

We now conclude that

$$u_1^2(x) \leq C\, (\epsilon\, r)^{2-n} \int_{B_p((1+\epsilon)r)} |\nabla u_1|^2.$$

Combining with (2.4) and (2.5), we have

$$\sum_{i=1}^k \int_{\partial B_p(r) \setminus \tilde{\Omega}_\delta(r)} u_i^2 \leq C\, \delta\, r\, \epsilon^{2-n} \sup_{u \in S(H)} \int_{B_p((1+\epsilon)r)} |\nabla u|^2, \tag{2.6}$$

where $S(H)$ is the unit sphere of H with respect to the inner product D_r.

Similarly, by an orthonormal transformation if necessary, we have

$$\max_{\partial B_p(r)} \sum_{i=1}^k |\nabla u_i|^2 = \sum_{i=1}^n |\nabla u_i|^2(x)$$

for some point $x \in \partial B_p(r)$. Indeed, if $x \in \partial B_p(r)$ is the maximum point for the function $\sum_{i=1}^k |\nabla u_i|^2$, then we may consider the subspace

$H_x = \{u \in H \mid \nabla u(x) = 0\}$. Obviously H_x is of at most codimension n because $\nabla u(x) \in T_x M$ which is of dimension n.

Again, by the subharmonicity of $|\nabla u_i|^2$ from the mean value inequality we conclude that

$$\begin{aligned} \sum_{i=1}^{n} |\nabla u_i|^2(x) &\le C\, V_x^{-1}\left(\frac{\epsilon}{2} r\right) \sum_{i=1}^{n} \int_{B_x(\frac{\epsilon}{2} r)} |\nabla u_i|^2 \\ &\le C\,(\epsilon\, r)^{-n} \sum_{i=1}^{n} \int_{B_p((1+\epsilon)r)} |\nabla u_i|^2 \\ &\le C\,(\epsilon\, r)^{-n}\, n \sup_{u\in S(H)} \int_{B_p((1+\epsilon)r)} |\nabla u|^2. \end{aligned} \tag{2.7}$$

Substituting the estimates (2.6) and (2.7) into (2.3), we have

$$\begin{aligned} 2\frac{r_0}{r} \sum_{i=1}^{k} \eta_i^{\frac{1}{2}}(r_0) \le& (1+\delta) \sum_{i=1}^{k} \int_{\partial B_p(r)} |\nabla u_i|^2 \\ &+ C\,\delta\, r^{-1} \left(r_0^2\, \eta_k(r_0)\, \epsilon^{2-n} + \epsilon^{-n}\right) \sup_{u\in S(H)} D_{(1+\epsilon)r}(u,u), \end{aligned}$$

and our proof is complete. q.e.d.

Theorem 2.2. *Let M^n be a complete manifold with non-negative sectional curvature. Let $0 \le \alpha_0 \le \omega_n$ be a constant such that*

$$\liminf_{r\to\infty} r^{-n}\, V_p(r) = \alpha_0.$$

If $h_d = \dim \mathcal{H}_d(M)$ denotes the dimension of the space of polynomial growth harmonic functions of at most degree d, then for any sequence of positive numbers

$$\{0 = a_0 < a_1 < \cdots < a_i < \cdots\}$$

satisfying $a_i \to \infty$, h_d satisfies

$$\limsup_{j\to\infty} a_j^{-n} \sum_{i=1}^{j} (a_i - a_{i-1})\, h_{a_{i-1}} \le \frac{2\alpha_0}{n!\,\omega_n}.$$

In particular, by taking $a_i = i$, we have

$$\begin{aligned} \limsup_{d\to\infty} d^{-n} \sum_{i=1}^{d} h_{i-1} &\le \frac{2\alpha_0}{n!\,\omega_n} \\ &\le \frac{2}{n!}. \end{aligned}$$

Moreover, the equality

$$\limsup_{d\to\infty} d^{-n} \sum_{i=1}^{d} h_{i-1} = \frac{2}{n!}$$

holds if and only if $M = \mathbb{R}^n$.

Proof. The case $n = 1$ is trivial, and the result was proved in [16] for $n = 2$. Hence we may assume that $n \geq 3$. The theorem clearly holds if

$$\limsup_{j\to\infty} a_j^{-n} \sum_{i=1}^{j} (a_i - a_{i-1})\, h_{a_{i-1}} = 0.$$

So we assume that there exists a sequence of $b_i \to \infty$ such that

$$\limsup_{j\to\infty} b_j^{-n} \sum_{i=1}^{j} (b_i - b_{i-1})\, h_{b_{i-1}} > 0,$$

which implies

$$\limsup_{d\to\infty} d^{-(n-1)}\, h_d > 0.$$

Hence, M has maximal volume growth by [5] and $\alpha_0 > 0$.

Using the sequence $0 = a_0 < a_1 < a_2 \cdots < a_j = d$, we can decompose the space $\mathcal{H}'_d(M)$ with respect to the inner product D_1 into a direct sum

$$\mathcal{H}'_d(M) = H_1 \oplus \cdots \oplus H_j,$$

where each $H_i \subset \mathcal{H}'_d(M)$ is the subspace consisting of harmonic functions of growth order between a_{i-1} and a_i. Let us denote the dimension of H_i by $\dim H_i = k_i$. According to the growth assumption and the definition of H_i, we have

$$\det{}_{D_1} D_r \leq C\, r^s,$$

where $s = \sum_{i=1}^{j} (2(a_i - 1) + n)\, k_i$. Let us set $\beta = 1 + d^{-1}$. Then for any $r_0 > 0$, there exists a $R > r_0$ such that

$$\det{}_{D_R} D_{\beta R} \leq \beta^{(s+1)}. \tag{2.8}$$

For any $0 < \tau < 1$, since

$$\begin{aligned}\det{}_{D_R} D_{(1+d^{-1}(1-\tau))R} &\leq \det{}_{D_R} D_{\beta R} \\ &\leq \beta^{(s+1)},\end{aligned}$$

which can be written as

$$\det_{D_1} D_{(1+d^{-1}(1-\tau))R} \leq \beta^{(s+1)} \det_{D_1} D_R,$$

the mean value theorem implies that there exists a r satisfying

$$R \leq r \leq (1 + (1-\tau)d^{-1})R$$

such that

$$(\ln \det_{D_1} D_r)' \leq (1 + 2\tau)\,(s+1)\, r^{-1}$$

for d sufficiently large. Note that in terms of a D_r-orthonormal basis $\{v_i\}$, this inequality can be written as

$$\sum_{i=1}^{h'_d} \int_{\partial B_p(r)} |\nabla v_i|^2 \leq (1+2\tau)\,(s+1)\, r^{-1}. \tag{2.9}$$

On the other hand, if we choose $\{u_i\}_{i=1}^{h'_d}$ to be a D_R-orthonormal basis of $\mathcal{H}'_d(M)$ that diagonalizes $D_{\beta R}$, then (2.8) implies that

$$\beta^{s+1} \geq \prod_{i=1}^{h'_d} D_{\beta R}(u_i, u_i). \tag{2.10}$$

For a fixed constant $A \geq 1$, let

$$I = \left\{ i \;\middle|\; \beta^{\frac{A(s+1)}{h'_d}} \geq D_{\beta R}(u_i, u_i) \right\}$$

and k be the cardinality of I. We now claim that $k \geq \frac{(h'_d)(A-1)}{A}$. Indeed, since $D_{\beta R}(u_i\, u_i) \geq 1$ for all i, we have

$$\prod_{i=1}^{h'_d} D_{\beta R}(u_i, u_i) \geq \beta^{\frac{A(s+1)(h'_d - k)}{h'_d}}.$$

Combining with (2.10), we conclude that

$$(s+1) \geq \frac{A(s+1)(h'_d - k)}{h'_d},$$

which yields the claim.

Let H be the subspace of $\mathcal{H}'_d(M)$ spanned by the set $\{u_i\}$ with $i \in I$. Without loss of generality, we may assume that $I = \{1, 2, \ldots, k\}$. For any $u \in H$, we can write $u = \sum_{i=1}^k b_i u_i$ for some set of constants $\{b_i\}$. Using the assumption that $u \in H$, we have

$$\begin{aligned} D_r(u,u) &\geq D_R(u,u) \\ &= \sum_{i=1}^k b_i^2\, D_R(u_i,u_i) \\ &\geq \beta^{-\frac{A(s+1)}{h'_d}} \sum_{i=1}^k b_i^2\, D_{\beta R}(u_i,u_i) \\ &= \beta^{-\frac{A(s+1)}{h'_d}}\, D_{\beta R}(u,u). \end{aligned}$$

In particular, this implies that

$$\sup_{u\in S(H)} D_{\beta R}(u,u) \leq \beta^{\frac{A(s+1)}{h'_d}}, \tag{2.11}$$

where $S(H)$ is the unit sphere of H with respect to D_r. For a small $\delta > 0$, choose $r_0 \geq R_\delta$, where $R_\delta > 0$ is as in Lemma 2.1. We now apply Lemma 2.1 to H with this particular choice of r with ϵ chosen to satisfy $(1+\epsilon)r = \beta R$. Then using (2.9) and (2.11), we obtain

$$\begin{aligned} 2r_0 \sum_{i=1}^k \eta_i^{\frac{1}{2}}(r_0) &\leq (1+\delta)\, r \sum_{i=1}^k \int_{\partial B_p(r)} |\nabla v_i|^2 \\ &\quad + C\,\delta\, \left(r_0^2\, \eta_k(r_0)\, \epsilon^{2-n} + \epsilon^{-n}\right) \sup_{u\in S(H)} D_{(1+\epsilon)r}(u,u) \\ &\leq (1+\delta)\,(1+2\tau)\,(s+1) \\ &\quad + C\,\delta\, \left(r_0^2\, \eta_k(r_0)\, \epsilon^{2-n} + \epsilon^{-n}\right)\, \beta^{\frac{A(s+1)}{h'_d}} \\ &\leq (1+2\tau+2\delta)\,(s+1) \\ &\quad + C\,\delta\, \tau^{-n} d^n \left(r_0^2\, \eta_k(r_0)\, \tau^2\, d^{-2} + 1\right)\, \left(1+d^{-1}\right)^{\frac{A(s+1)}{h'_d}}. \end{aligned} \tag{2.12}$$

On the other hand, the Weyl's asymptotic behavior of the eigenvalues of $\Omega'(r_0)$ gives

$$\eta_i^{\frac{n-1}{2}}(r_0) \sim C_{n-1}\, i\, A^{-1}(\Omega'(r_0)),$$

where

$$C_{n-1} = \frac{(2\pi)^{n-1}}{\omega_{n-1}} = \frac{n!\,\omega_n}{2}.$$

Hence combining with (2.12), we conclude that for $\xi > 0$ sufficiently small, there exists $k_0(\delta, \xi)$ such that for $i \geq k_0$,

$$\begin{aligned}\eta_i(r_0) &\geq (1-\xi)\, C_{n-1}^{\frac{2}{n-1}}\, i^{\frac{2}{n-1}}\, A^{-\frac{2}{n-1}}(\Omega'(r_0)) \\ &\geq (1-\xi)\, C_{n-1}^{\frac{2}{n-1}}\, i^{\frac{2}{n-1}}\, r_0^{-2}(n\,\alpha(\delta, r_0))^{-\frac{2}{n-1}},\end{aligned}$$

where $\alpha(\delta, r_0) \to \alpha_0$ as $\delta \to 0$ and $r_0 \to \infty$ since $\liminf_{r\to\infty} r^{n-1}\, A_p(r) = n\,\alpha_0$ according to the hypothesis on the volume growth. Similarly, we have

$$\begin{aligned}\eta_k(r_0) &\leq (1+\xi)\, C_{n-1}^{\frac{2}{n-1}}\, k^{\frac{2}{n-1}}\, r_0^{-2}\,(n\,\alpha(\delta, r_0))^{-\frac{2}{n-1}} \\ &\leq C\,(h'_d)^{\frac{2}{n-1}}\, r_0^{-2}.\end{aligned}$$

In particular, this implies that

$$\begin{aligned}r_0 \sum_{i=1}^{k} \eta_i^{\frac{1}{2}}(r_0) &\geq (1-\xi)^{\frac{1}{2}}\, C_{n-1}^{\frac{1}{n-1}}\,(n\,\alpha(\delta, r_0))^{-\frac{1}{n-1}} \sum_{i=k_0}^{k} i^{\frac{1}{n-1}} \\ &\geq (1-\xi)^{\frac{1}{2}}\, C_{n-1}^{\frac{1}{n-1}}\,(n\,\alpha(\delta, r_0))^{-\frac{1}{n-1}} \\ &\qquad \cdot \frac{n-1}{n}\left(k^{\frac{n}{n-1}} - k_0^{\frac{n}{n-1}}\right).\end{aligned} \tag{2.13}$$

Note that since

$$\begin{aligned}s &= \sum_{i=1}^{j}(2(a_i-1)+n)\,k_i \\ &\leq (2d-2+n)\sum_{i=1}^{j} k_i \\ &\leq h'_d\,(2d-2+n),\end{aligned}$$

it follows that

$$\frac{s+1}{h'_d} \leq 2d-2+n \leq 3d$$

for d sufficiently large. Also note that, we already have the estimate $h_d \leq C\, d^{n-1}$. Combining (2.12) and (2.13) yields

$$\begin{aligned}&(1-\xi)^{\frac{1}{2}}\, C_{n-1}^{\frac{1}{n-1}}\,(n\,\alpha(\delta, r_0))^{-\frac{1}{n-1}}\,\frac{n-1}{n}\left(k^{\frac{n}{n-1}} - k_0^{\frac{n}{n-1}}\right) \\ &\quad\leq (1+2\tau+2\delta)\sum_{i=1}^{j}\left(a_i - 1 + \frac{n}{2}\right)k_i \\ &\qquad + \frac{1}{2} + \tau + C\,\delta\,\tau^{-n}\,d^n\,(1+d^{-1})^{3Ad}.\end{aligned}$$

Using the fact that $k_i = h'_{a_i} - h'_{a_{i-1}}$ and the estimate $k \geq \frac{h'_d\,(A-1)}{A}$, we can rewrite this inequality into the form

$$
\begin{aligned}
(1+2\tau+2\delta) &\sum_{i=1}^{j} (a_i - a_{i-1})\, h'_{a_{i-1}} \\
&\leq (1+2\tau+2\delta)\left(d-1+\frac{n}{2}\right) h'_d \\
&\quad + C_1 + C\,\delta\,\tau^{-n}\, d^n\, (1+d^{-1})^{3Ad} \\
&\quad - (1-\xi)^{\frac{1}{2}}\, C_{n-1}^{\frac{1}{n-1}}\, (n\,\alpha(\delta, r_0))^{-\frac{1}{n-1}} \\
&\qquad \cdot \frac{n-1}{n} \left(\frac{A-1}{A}\right)^{\frac{n}{n-1}} (h'_d)^{\frac{n}{n-1}},
\end{aligned}
\tag{2.14}
$$

where C_1 is a constant independent of d. As a function of h'_d, the right-hand side of (2.14) achieves its maximum when h'_d satisfies

$$
\begin{aligned}
(1-\xi)^{\frac{1}{2}}\, C_{n-1}^{\frac{1}{n-1}}\, (n\,\alpha(\delta, r_0))^{-\frac{1}{n-1}} &\left(\frac{A-1}{A}\right)^{\frac{n}{n-1}} (h'_d)^{\frac{1}{n-1}} \\
&= (1+2\tau+2\delta)\left(d-1+\frac{n}{2}\right).
\end{aligned}
$$

Plugging this value of h'_d into (2.14) and simplifying we obtain

$$
\begin{aligned}
(1+2\tau+2\delta) &\sum_{i=1}^{j} (a_i - a_{i-1})\, h'_{a_{i-1}} \\
&\leq \left((1+2\tau+2\delta)\left(d-1+\frac{n}{2}\right)\right)^n \\
&\qquad \cdot (1-\xi)^{-\frac{n-1}{2}}\, C_{n-1}^{-1}\, \alpha(\delta, r_0) \left(\frac{A-1}{A}\right)^{-n} \\
&\quad + C_1 + C\,\delta\,\tau^{-n}\, d^n\, (1+d^{-1})^{3Ad}.
\end{aligned}
$$

Dividing both sides by d^n and letting $d \to \infty$, we conclude that

$$
\begin{aligned}
(1+2\tau+2\delta) &\limsup_{d\to\infty} d^{-n} \sum_{i=1}^{j} (a_i - a_{i-1})\, h'_{a_{i-1}} \\
&\leq (1+2\tau+2\delta)^n\, (1-\xi)^{-\frac{n-1}{2}}\, C_{n-1}^{-1}\, \alpha(\delta, r_0) \left(\frac{A-1}{A}\right)^{-n} \\
&\quad + C\,\delta\,\tau^{-n}\, e^{3A}.
\end{aligned}
$$

Now letting $\delta \to 0$, $r_0 \to \infty$, and then $A \to \infty$, we are led to

$$\begin{aligned}(1+2\tau)\limsup_{d\to\infty} d^{-n} &\sum_{i=1}^{j}(a_i - a_{i-1})\, h'_{a_{i-1}} \\ &\leq (1+2\tau)^n\,(1-\xi)^{-\frac{n-1}{2}}\, C_{n-1}^{-1}\,\alpha_0.\end{aligned}$$

Finally, we let ξ and τ both tend to 0 and obtain that

$$\limsup_{d\to\infty} d^{-n} \sum_{i=1}^{j}(a_i - a_{i-1})\, h'_{a_{i-1}} \leq C_{n-1}^{-1}\,\alpha_0.$$

Substituting the value of C_{n-1} implies

$$\limsup_{d\to\infty} d^{-n} \sum_{i=1}^{j}(a_i - a_{i-1})\, h_{a_{i-1}} \leq \frac{2\alpha_0}{n!\,\omega_n},$$

as to be proved. q.e.d.

Corollary 2.3. *Let M^n be a complete manifold with non-negative sectional curvature. Let $0 \leq \alpha_0 \leq \omega_n$ be a constant such that*

$$\liminf_{r\to\infty} r^{-n}\, V_p(r) = \alpha_0.$$

If $h_d = \dim \mathcal{H}_d(M)$ denotes the dimension of the space of polynomial growth harmonic functions of at most degree d, then

$$\liminf_{d\to\infty} d^{-(n-1)}\, h_d \leq \frac{2\alpha_0}{(n-1)!\,\omega_n}.$$

Moreover, the equality

$$\liminf_{d\to\infty} d^{-(n-1)}\, h_d = \frac{2}{(n-1)!}$$

holds if and only if $M = \mathbb{R}^n$.

Proof. Assume the contrary that

$$\liminf_{d\to\infty} d^{-(n-1)}\, h_d > \frac{2\alpha_0}{(n-1)!\,\omega_n}.$$

In particular, there exists an $\epsilon > 0$ such that

$$h_d \geq \left(\frac{2\alpha_0}{(n-1)!\,\omega_n} + \epsilon\right) d^{n-1}$$

for d sufficiently large. This implies that

$$\sum_{j-1}^{d} h_j \geq \left(\frac{2\alpha_0}{n!\,\omega_n} + \epsilon\right) d^n.$$

for d sufficiently large, which contradicts with Theorem 2.2.

The equality case occurs when $\alpha_0 = \omega_n$ and the conclusion that $M = \mathbb{R}^n$ follows from the equality case of the Bishop comparison theorem.

q.e.d.

Theorem 2.4. *Let M be a two-dimensional complete manifold with finite topological type. Assume that M has at most quadratic volume growth and there exists a constant $A \geq 0$ satisfying*

$$r^{-2} V_p(r) \leq A\pi$$

as $r \to \infty$. Then the space $\mathcal{H}_d(M)$ is finite dimensional. Moreover, if ℓ denotes the number of the ends of M, then

$$h_d \leq \ell\,(8Ad + 1) + 1.$$

Proof. The finite topological type assumption implies that M has finitely many ends and the first Betti number of M is finite. Let $E_1, \ldots, E_\ell$ be the ends of M. Define for each E_i the space $\mathcal{H}_d(E_i)$ consisting of all the harmonic functions on E_i of growth order at most d and vanishing identically on ∂E_i. We will show that $\dim \mathcal{H}_d(E_i) \leq 8Ad + 1$ for each i. The theorem then follows from a theorem in [21]. We will write E as a generic end from now on. For r sufficiently large, denote $E(r)$ to be the unbounded component of $E \setminus B_p(r)$. Then the boundary $\partial E(r)$ is connected by a result in [17]. Let $\eta_i(r)$ be the i-th Neumann eigenvalue of $\partial E(r)$ and w_i the corresponding eigenfunction. Let us denote $k = \dim \mathcal{H}_d(E)$, and by Lemma 1.2 then we can choose an orthonormal basis $\{u_1, \ldots, u_k\}$ of the space $\mathcal{H}_d(E)$ with respect to the inner product

$$D_r(f, g) = \int_{B_p(r)\cap E} \langle \nabla f, \nabla g\rangle$$

such that

$$\int_{\partial E(r)} u_i\, w_j = 0$$

for all $1 \leq j < i \leq k$. Hence by the variational principle, we conclude that

$$\eta_i(r) \int_{\partial E(r)} u_i^2 \leq \int_{\partial E(r)} |\bar{\nabla} u_i|^2,$$

where $\bar{\nabla}$ is the tangential gradient of $\partial E(r)$. Thus we have

$$\begin{aligned}
2\sum_{i=1}^{k} \eta_i^{\frac{1}{2}}(r) &= 2\sum_{i=1}^{k} \eta_i^{\frac{1}{2}}(r) \int_{B_p(r)\cap E} |\nabla u_i|^2 \\
&= 2\sum_{i=1}^{k} \eta_i^{\frac{1}{2}}(r) \int_{\partial E(r)} u_i \frac{\partial u_i}{\partial r} \\
&\leq 2\sum_{i=1}^{k} \left(\eta_i(r) \int_{\partial E(r)} u_i^2\right)^{\frac{1}{2}} \left(\int_{\partial E(r)} \left(\frac{\partial u_i}{\partial r}\right)^2\right)^{\frac{1}{2}} \\
&\leq \sum_{i=1}^{k} \left(\eta_i(r) \int_{\partial E(r)} u_i^2 + \int_{\partial E(r)} \left(\frac{\partial u_i}{\partial r}\right)^2\right) \\
&\leq \sum_{i=1}^{k} \int_{\partial E(r)} |\nabla u_i|^2 \\
&= (\ln \det_{D_1} D_r)'.
\end{aligned} \tag{2.15}$$

If $L(r)$ is the length of $\partial E(r)$, then $\eta_1(r) = 0$ and

$$\eta_{2i}(r) = \eta_{2i+1}(r) = \left(\frac{2\pi\, i}{L(r)}\right)^2$$

for $i \geq 1$. Substituting into (2.15) gives that

$$\frac{\pi\,(k-1)\,(k+1)}{L(r)} \leq (\ln \det_{D_1} D_r)'.$$

Integrating both sides and noting that $\det_{D_1} D_r \leq C\, r^{2kd}$ as M has at most quadratic volume growth, we obtain

$$\begin{aligned}
\pi\,(k-1)\,(k+1) \int_1^R \frac{dr}{L(r)} &\leq \ln \det_{D_1} D_R \\
&\leq 2dk \ln R + C.
\end{aligned} \tag{2.16}$$

If $V(R)$ denotes the volume of the set $B_p(R) \cap E$, then

$$\begin{aligned}\left(R - \frac{R}{2}\right)^2 &\leq \int_{\frac{R}{2}}^{R} L(t)\, dt \int_{\frac{R}{2}}^{R} \frac{dt}{L(t)}\\ &\leq V(R) \int_{\frac{R}{2}}^{R} \frac{dt}{L(t)}.\end{aligned}$$

Using the volume growth assumption, for any $\epsilon > 0$ and for R sufficiently large, this inequality implies

$$\int_{\frac{R}{2}}^{R} \frac{dt}{L(t)} \geq \frac{1}{4\pi\,(A+\epsilon)},$$

and hence

$$\int_{1}^{R} \frac{dt}{L(t)} \geq \frac{\log_2 R}{4\pi\,(A+\epsilon)}.$$

Substituting into (2.16) and letting $R \to \infty$ after dividing both sides by $\ln R$, we conclude that $k \leq 8(A+\epsilon)\, d + 1$. Since ϵ is arbitrary this completes the proof. q.e.d.

When M has only one end and has subquadratic volume growth, we have the following corollary.

Corollary 2.5. *Let M be a two-dimensional complete Riemannian surface with finite topological type. If M has only one end and the volume growth of M is subquadratic, that is, $V_p(r) = o(r^2)$, then for all $d > 0$, $\dim \mathcal{H}_d(M) = 1$.*

3. Sharp estimates on the number of massive sets

In this section, we will assume that M is a complete manifold with finite first Betti number. Let $K \subset M$ be a compact subset of M, and $\{E_1, \ldots, E_i, \ldots\}$ the set of unbounded components of $M \setminus K$. We say that each E_i is an end of M with respect to K. Let Ξ be a connected d-massive set of M, and v the corresponding potential function which is non-constant. Let $a = \sup_K v$. Then the set

$$\Xi(a) = \{x \in \Xi \mid v(x) > a\}$$

is disjoint from K. Also, the sets $\Xi(a) \cap E_i$ are mutually disjoint with each other. Since v is subharmonic, the maximum principle implies that

$\sup_{\Xi} v > a$. Hence at least one $\Xi(a) \cap E_i \neq \emptyset$. Obviously $\Xi(a) \cap E_i \subset E_i$ is also a d-massive set because $v - a$ is the corresponding potential function. This observation implies that for the purpose of getting an upper bound of the maximum number of disjoint d-massive sets on M, we may assume that each of the d-massive sets is contained in some end E_i. Hence the maximum number of disjoint d-massive sets on M is given by the sum of the maximum number of disjoint d-massive sets on each end.

Without loss of generality, we assume now M has finitely many ends. For a fixed end E of M, if we denote $E(r)$ to be the unbounded component of $E \setminus B_p(r)$, then by the result of the first author and Tam [17], the assumption that M has finite first Betti number implies that the set $\partial E(r)$ is connected for r sufficiently large. In this case, if $\Xi_1, \ldots, \Xi_k$ in E, $k \geq 2$, are disjoint d-massive sets with $v_1, \ldots, v_k$ being the corresponding potential functions, then the set $S_i(r) = \Xi_i \cap \partial E(r)$ must be nonempty and open in $\partial E(r)$ for sufficiently large r. Note that since the sets $S_i(r)$, $i = 1, \ldots, k$, are disjoint and $\partial E(r)$ is connected, the boundary $\partial S_i(r) \neq \emptyset$. Let $\lambda_1(S_i(r))$ be the first nonzero Dirichlet eigenvalue of $S_i(r)$ with respect to the Laplacian on $\partial B_p(r)$. Denote

$$D_r(v_i, v_i) = \int_{\Xi_i \cap B_p(r)} |\nabla v_i|^2.$$

Then we have the following lemma.

Lemma 3.1. *Under the preceding notation, we have*

$$\sum_{i=1}^{k} (\ln D_r(v_i, v_i))' \geq 2 \sum_{i=1}^{k} \lambda_1^{\frac{1}{2}}(S_i(r)).$$

Proof. Let $\{S_{ij}(r)\}_{j \in J}$ be the connected components of $S_i(r)$. Then $v_i = 0$ on $\partial S_{ij}(r)$ for $j \in J$. So we have for $j \in J$,

$$\lambda_1(S_{ij}(r)) \int_{S_{ij}(r)} v_i^2 \leq \int_{S_{ij}(r)} |\bar{\nabla} v_i|^2,$$

where $\bar{\nabla}$ denotes the gradient on the sphere $\partial B_p(r)$. Note that for each $j \in J$,

$$\lambda_1(S_i(r)) \leq \lambda_1(S_{ij}(r)).$$

Hence,

$$\lambda_1(S_i(r)) \int_{S_{ij}(r)} v_i^2 \leq \int_{S_{ij}(r)} |\bar{\nabla} v_i|^2.$$

Summing over $j \in J$, we conclude that

$$\lambda_1(S_i(r)) \int_{S_i(r)} v_i^2 \leq \int_{S_i(r)} |\bar{\nabla} v_i|^2.$$

So for each i, using the fact that v_i is nonnegative and subharmonic, we get

$$\begin{aligned}
2\lambda_1^{\frac{1}{2}}(S_i(r))\, D_r(v_i, v_i) &\leq 2\lambda_1^{\frac{1}{2}}(S_i(r)) \int_{\Xi_i \cap (M \setminus E(r))} |\nabla v_i|^2 \\
&\leq 2\lambda_1^{\frac{1}{2}}(S_i(r)) \int_{S_i(r)} v_i \frac{\partial v_i}{\partial r} \\
&\leq 2 \left(\lambda_1(S_i(r)) \int_{S_i(r)} v_i^2 \right)^{\frac{1}{2}} \left(\int_{S_i(r)} \left(\frac{\partial v_i}{\partial r} \right)^2 \right)^{\frac{1}{2}} \\
&\leq 2 \left(\int_{S_i(r)} |\bar{\nabla} v_i|^2 \right)^{\frac{1}{2}} \left(\int_{S_i(r)} \left(\frac{\partial v_i}{\partial r} \right)^2 \right)^{\frac{1}{2}} \\
&\leq \int_{S_i(r)} |\bar{\nabla} v_i|^2 + \int_{S_i(r)} \left(\frac{\partial v_i}{\partial r} \right)^2 \\
&\leq \int_{\partial B_p(r)} |\nabla v_i|^2 \\
&= D_r'(v_i, v_i).
\end{aligned}$$

So we have

$$(\ln D_r(v_i, v_i))' \geq 2\lambda_1^{\frac{1}{2}}(S_i(r))$$

for all $1 \leq i \leq k$, and the proof is complete. q.e.d.

Lemma 3.2. *Let M be a two-dimensional complete manifold of finite topological type. Suppose that M has finitely many ends given by $\{E_1, \ldots, E_\ell\}$ and is of at most quadratic volume growth. Then for any fixed point $p \in M$, the number of disjoint d-massive sets m_d of M is bounded by*

$$m_d \leq \sum_{i=1}^{\ell} \max\{1, \liminf_{r \to \infty} \frac{d \ln r}{\pi \int_1^r \frac{dt}{L_j(t)}}\},$$

where $L_j(t)$ is the length of the curve given by $\partial B_p(t) \cap E_j$ for each $1 \leq j \leq \ell$.

Proof. By our previous observation, it suffices to prove the formula for $\ell = 1$. Hence, let us assume E is the only end of M. We may also assume that M has at least two disjoint d-massive sets. Since M has finite topological type, the first Betti number of M is finite. In this case, the sets $S_i(r)$ will have $\partial S_i(r) \neq \emptyset$ for all sufficiently large r. For any collection of disjoint d-massive sets $\{\Xi_1, \ldots, \Xi_k\}$ of M with corresponding potential functions $\{v_1, \ldots, v_k\}$, we apply Lemma 3.1 and obtain

$$\sum_{i=1}^{k} (\ln D_r(v_i, v_i))' \geq 2 \sum_{i=1}^{k} \lambda_1^{\frac{1}{2}}(S_i(r)), \tag{3.1}$$

where $S_i(r) = \Xi_i \cap \partial E(r)$, and $E(r)$ is the unbounded component of $M \backslash B_p(r)$. Since M is two dimensional, each set $S_i(r)$ consists of disjoint open segments in $\partial B_p(r)$. Let us denote the length of the sets $\partial B_p(r)$ and $S_i(r)$ by $L(r)$ and $s_i(r)$ respectively. Then clearly,

$$\sum_{i=1}^{k} s_i(r) \leq L(r),$$

as $\{S_i(r)\}$ are mutually disjoint. Applying the fact that

$$\lambda_1(S_i(r)) \geq \frac{\pi^2}{(s_i(r))^2}$$

to (3.1) and the Schwarz inequality, we have

$$\begin{aligned} \sum_{i=1}^{k} (\ln D_r(v_i, v_i))' &\geq 2\pi \sum_{i=1}^{k} \frac{1}{s_i(r)} \\ &\geq \frac{2\pi\, k^2}{L(r)}. \end{aligned} \tag{3.2}$$

For a fixed $r_0 > 0$ such that each $D_{r_0}(v_i, v_i)$ is positive, we integrate the inequality (3.2) from r_0 to a sufficiently large r. Notice that for each i,

$$D_r(v_i, v_i) \leq C\, r^{2d}$$

as v_i is a nonnegative subharmonic function of polynomial growth of order at most d and M has at most quadratic volume growth. Hence,

$$\begin{aligned} 2d\, k\, \ln r + C &\geq \sum_{i=1}^{k} (\ln D_r(v_i, v_i) - \ln D_{r_0}(v_i, v_i)) \\ &\geq 2\pi\, k^2 \int_{r_0}^{r} \frac{dt}{L(t)}. \end{aligned}$$

Thus,

$$k \leq \liminf_{r\to\infty} \frac{d \ln r}{\pi \int_{r_0}^{r} \frac{dt}{L(t)}},$$

and the lemma is proved. q.e.d.

Now we can prove the following results concerning the number of disjoint d-massive sets in a two-dimensional manifold. First, recall that a complete surface M with finite total curvature must have finite topological type with finitely many ends $\{E_1, \ldots, E_\ell\}$. Moreover, if we denote $V_j(r)$ for $1 \leq j \leq \ell$ to be the volume of the set $E_j \cap B_p(r)$, then the limit

$$\lim_{r\to\infty} \frac{V_j(r)}{\pi r^2} = \alpha_j$$

exists.

Theorem 3.3. *Let M be a complete surface of finite total curvature. Then the number of disjoint d-massive subsets m_d in M is bounded above by*

$$m_d \leq \sum_{j=1}^{\ell} \max\{1, 2d\,\alpha_j\}.$$

In particular, for the case where $M = \mathbb{R}^2$, we have $m_d \leq \max\{1, 2d\}$.

Proof. Since M has finite total curvature, if we denote the length of the set $\partial B_p(r) \cap E_j$ by $L_j(r)$, then it is known that [16]

$$\begin{aligned} \lim_{r\to\infty} \frac{L_j(r)}{\pi\, r} &= 2 \lim_{r\to\infty} \frac{V_j(r)}{\pi\, r^2} \\ &= 2\alpha_j. \end{aligned}$$

This implies that for any $\epsilon > 0$, and for sufficiently large r,

$$L_j(r) \leq 2\pi\,(\alpha_j + \epsilon)\, r.$$

Applying Lemma 3.2, we conclude that m_d satisfies

$$m_d \leq \sum_{j=1}^{\ell} \max\{1, 2d\,(\alpha_j + \epsilon)\}.$$

Since $\epsilon > 0$ is arbitrary, this proves the first part of the theorem. For the case where M is the Euclidean plane, there is only one end and $\alpha = 1$. So

$$m_d \leq \max\{1, 2d\},$$

and the proof is complete. q.e.d.

We would like to remark that this estimate on m_d is sharp. In fact, for the case of $\mathbb{R}^2$, $m_d = 2d$ for any positive integer d by counting the number of nodal domains of a degree d homogeneous harmonic polynomial of the type $r^d \cos(d\theta)$.

Theorem 3.4. *Let M be a two-dimensional complete manifold with finite topological type and of at most quadratic volume growth. Let $A \geq 0$ be a constant satisfying*

$$r^{-2} V_p(r) \leq A\,\pi$$

as $r \to \infty$. First of all, the number of disjoint d-massive sets m_d of M must be finite. In fact, if ℓ is the number of ends of M, then

$$m_d \leq \ell \max\{4A\,d, 1\}.$$

In particular, $m_d \leq \ell$ if M has subquadratic volume growth.

Proof. Since M has finite topological type, M has finitely many ends and the first Betti number of M is finite. Again, it suffices to prove the estimate for the case where M has only one end and that the number of d-massive sets is at least 2. For any set of disjoint d-massive sets $\{\Xi_1, \ldots, \Xi_k\}$ in M with $\{v_1, \ldots, v_k\}$ being their corresponding potential functions, Lemma 3.2 implies that

$$k \leq \liminf_{r \to \infty} \frac{d \ln r}{\pi \int_1^r \frac{dt}{L(t)}}, \tag{3.3}$$

where $L(r)$ is the length of $\partial B_p(r)$. Since $V_p(r) \leq A\pi\, r^2$, the argument in Theorem 2.4 implies

$$\int_1^r \frac{dt}{L(t)} \geq \frac{\log_2 r}{4A\,\pi}.$$

Substituting into (3.3), we conclude

$$k \leq 4A\,d \ln 2 \leq 4A\,d.$$

Thus, in general, if M has ℓ number of ends, then

$$m_d \leq \ell \max\{4A\,d, 1\},$$

and the proof is complete. q.e.d.

Combining Theorems 3.3 and 3.4 with a result in [18], we have the following structural result concerning the image of harmonic maps.

Corollary 3.5. *Let M be a complete surface and N either a strongly negatively curved Cartan Hadamard manifold or a two-dimensional visibility manifold. Suppose $u : M \to N$ is a harmonic map from M into N of at most polynomial growth of degree $d \geq 1$. If M has finite topological type with at most quadratic volume growth, then the image $u(M)$ is contained in an ideal polygon in N with finitely many vertices at infinity. In fact, if $E_1, \dots, E_\ell$ are the ends of M and*

$$\limsup_{r\to\infty} r^{-2}\, V_p(r) \leq A,$$

then the number of vertices are at most $\ell \max\{4A\,d, 1\}$. In particular, if M has only one end and is of subquadratic area growth, then u must be constant.

If, in addition, M has finite total curvature and

$$\lim_{r\to\infty} \frac{V_j(r)}{\pi\, r^2} = \alpha_j$$

where $V_j(r)$ is the volume of $E_j \cap B_p(r)$ for $1 \leq j \leq \ell$, then the number of vertices are at most $\sum_{j=1}^{\ell} \max\{1, 2d\,\alpha_j\}$. In particular, if M has nonnegative curvature and $u(M)$ is non-constant and of linear growth, then $u(M)$ must be a geodesic line in N.

Note that by a result of S. Y. Cheng, for the case $d < 1$, the harmonic map is actually constant if M has nonnegative curvature. Also, for the case that the domain manifold is the Euclidean plane, let $\phi\, dz^2$ be the Hopf differential of an orientation preserving harmonic diffeomorphism u from $\mathbb{R}^2$ into N, where N is a strongly negatively curved surface. Then ϕ being a polynomial of degree m implies that u is actually of polynomial growth of order $\frac{m}{2} + 1$ provided that the pull-back metric by u is complete on $\mathbb{R}^2$. We refer to [24] and [22] for the details. With this in mind, Corollary 3.6 gives a generalization of a result in [7].

4. Bounded harmonic functions on a covering space

In this section, we will study the space of bounded harmonic functions on a manifold M with a group Γ acting as isometries. In particular, as a confirmation to the belief that there are many bounded harmonic functions on the manifold if the group is large in a suitable sense, we shall show that the dimension of the space of bounded harmonic functions must be infinite on any regular covering space with the covering group being nonamenable. This was first studied by T. Lyons and D. Sullivan [20] for the case when M/Γ is compact where they show that $\mathcal{H}_0(M) \geq 2$ if Γ is nonamenable. Later, Kifer [10] and Toledo [23] showed that $\mathcal{H}_0(M) = \infty$. Our approach give a similar type result but without the assumption that M/Γ is compact. We will begin with the following lemma of Grigor'yan [6].

Lemma 4.1. *Suppose* $\dim \mathcal{H}_0(M) = k < \infty$. *Then there exists a basis* $f_1, \dots, f_k$ *for* $\mathcal{H}_0(M)$ *such that* $0 \leq f_i \leq 1$, $\sup_M f_i = 1$, *and* $\sum_{i=1}^k f_i = 1$.

The following lemma was established in [18]. We include a sketch of proof here for convenience.

Lemma 4.2. *Let* M *be a complete Riemannian manifold with* $\dim \mathcal{H}_0(M) = k < \infty$. *Let* $\hat{M}$ *be the Stone-Cěch compactification of* M. *Then there exist disjoint subsets* S_i, $i = 1, \dots, k$, *in* $\hat{M}$ *such that for every bounded continuous subharmonic function* v *on* M *with* $S = \{\hat{x} \in \hat{M} \mid v(\hat{x}) = \sup_M v\}$, $S_i \subset S$ *for some* i, *and either* $S_j \cap S = \emptyset$ *or* $S_j \subset S$ *for each* $1 \leq j \leq k$.

Proof. A result in [6] implies that $\dim \mathcal{H}_0(M) = k$ if and only if M has exactly k disjoint massive subsets $\Xi_1, \dots, \Xi_k$. For each $i \in \{1, \dots, k\}$, define the set

$$S_i = \cap \{\hat{x} \in \hat{M} \mid f(\hat{x}) = \sup f\},$$

where the intersection is taken over all the potential functions f of Ξ_i. Then S_i is nonempty. See [18].

For a bounded continuous subharmonic function v on M and the set

$$S = \{\hat{x} \mid v(\hat{x}) = \sup v\},$$

we check that S must contain some S_i. Moreover, for each j, either $S \cap S_j = \emptyset$ or $S_j \subset S$. First observe that for an arbitrary open set U in $\hat{M}$ such that $S_i \subset U$, there exists a potential function g_U of Ξ_i satisfying $\{\hat{x} \mid g_U(\hat{x}) = \sup g_U\} \subset U$. In fact, note that

$$\hat{M} \setminus U \subset \hat{M} \setminus S_i = \cup \{\hat{x} \in \hat{M} \mid f(\hat{x}) < \sup f\},$$

where the union is over all potential functions f of Ξ_i. The compactness of $\hat{M} \setminus U$ implies that there exists finitely many potential functions

$f_1, \ldots, f_m$ of Ξ_i such that

$$\hat{M} \setminus U \subset \cup_{j=1}^m \{\hat{x} \,|\, f_j(\hat{x}) < \sup f_j\}.$$

Let us define $g_U = f_1 + \cdots + f_m$. Then it is clear that $\{\hat{x} \,|\, g_U(\hat{x}) = \sup g_U\} \subset U$.

Now we argue that $S \cap S_i \neq \emptyset$ for some i. If this is not the case, then there are disjoint open sets U_i in $\hat{M}$ with $S_i \subset U_i$ such that for $\epsilon > 0$ sufficiently small the set

$$\Xi = \{x \in M \,|\, v(x) > \sup v - \epsilon\}$$

must satisfy $\Xi \cap U_i = \emptyset$. Then

$$\{\Xi, U_1 \cap M, \ldots, U_k \cap M\}$$

are $k+1$ disjoint massive sets of M, which is impossible. Therefore, $S \cap S_i \neq \emptyset$ for some i. To see that $S_i \subset S$, for any open set U with $S_i \subset U$ and $U \cap M \subset \Xi_i$, let g_U be the corresponding function g_U constructed above, and define the function $w = g_U + v$. Note that

$$\{\hat{x} \,|\, w(\hat{x}) = \sup w\} \subset U \cap S.$$

Thus, for sufficiently small $\epsilon > 0$, the set

$$W = \{x \,|\, w(x) > \sup w - \epsilon\} \subset \Xi_i,$$

and $f = w - \sup w + \epsilon$ is a potential function of this massive set W. In particular, by extending f to be zero outside W, f is a potential function of Ξ_i with

$$S_i \subset \{\hat{x} \,|\, f(\hat{x}) = \sup f\} = \{\hat{x} \,|\, w(\hat{x}) = \sup w\} \subset U \cap S.$$

Since U is arbitrary, we conclude that $S_i \subset S$. This argument also shows that for any j, either $S \cap S_j = \emptyset$ or $S_j \subset S$. This completes the proof. q.e.d.

Lemma 4.3. *Suppose* $\dim \mathcal{H}_0(M) = k < \infty$ *and* $f_1, \ldots, f_k$ *form a basis for* $\mathcal{H}_0(M)$ *such that* $0 \leq f_i \leq 1$, $\sup_M f_i = 1$, *and* $\sum_{i=1}^k f_i = 1$. *Then for any* γ *an isometry of* M, *there exists a permutation* σ *of* $\{1, 2, \ldots, k\}$ *such that* $\gamma(f_i) = f_{\sigma(i)}$ *for* $1 \leq i \leq k$, *where* $\gamma(f_i)(x) = f_i(\gamma(x))$.

Proof. Let $\hat{M}$ be the Stone-Cěch compactification of M. Then Lemma 4.2 implies that there exist subsets $S_i \subset \hat{M}$, $1 \leq i \leq k$, such that $f_i(\hat{x}) = \sup f_i = 1$ for $\hat{x} \in S_i$, and $f_i(\hat{x}) = 0$ for $\hat{x} \in S_j$ with $i \neq j$. Also, for any bounded continuous subharmonic function f on M, the set $\{\hat{x} : f(\hat{x}) = \sup f\}$ contains some S_i. In particular, there is a permutation σ of $\{1, \dots, k\}$ such that

$$\gamma(f_i)(\hat{x}) = f_{\sigma(i)}(\hat{x}) = 1$$

for $\hat{x} \in S_{\sigma(i)}$. Notice that we also have $\sum_{i=1}^k \gamma(f_i) = 1$ on M. Therefore, $\gamma(f_i)(\hat{x}) = 0$ for $\hat{x} \in S_j$ with $j \neq \sigma(i)$. Now the function $g_i = \gamma(f_i) - f_{\sigma(i)}$ is bounded harmonic and $g_i(\hat{x}) = 0$ for all $\hat{x} \in S_j$ and $1 \leq j \leq k$. One concludes then that $\inf g_i = \sup g_i = 0$, or $g_i = 0$. Thus, $\gamma(f_i) = f_{\sigma(i)}$.

q.e.d.

Theorem 4.4. *Let Γ be a group acting discontinuously and isometrically on a complete manifold M. If $\dim \mathcal{H}_0(M) < \infty$, then there exists a bounded linear functional F on $L^\infty(M)$ such that $F(1) = 1$, $F(f) \geq 0$ for $f \geq 0$, and F is Γ invariant. That is, for any $f \in L^\infty(M)$ and $\gamma \in \Gamma$, $F(f) = F(\gamma(f))$, where $\gamma(f)(x) = f(\gamma(x))$. In particular, Γ must be amenable.*

Proof. Let $\{f_1, \dots, f_k\}$ be the basis of $\mathcal{H}_0(M)$ as in Lemma 4.3. For any $f \in \mathcal{H}_0(M)$, write $f = \sum_{i=1}^k c_i f_i$, where c_i are constants. It is clear that for $f \geq 0$, $c_i \geq 0$, $1 \leq i \leq k$. Define $L(f) = k^{-1} \sum_{i=1}^k c_i$. Then $L(1) = 1$ and $L(f) \geq 0$ for $f \geq 0$. Note that

$$\begin{aligned} \gamma(f) &= \sum_{i=1}^k c_i \, \gamma(f_i) \\ &= \sum_{i=1}^k c_i \, f_{\sigma(i)} \\ &= \sum_{i=1}^k d_i \, f_i \end{aligned}$$

with $c_i = d_{\sigma(i)}$. In particular, we have

$$\sum_{i=1}^k c_i = \sum_{j=1}^k d_j$$

and

$$L(f) = L(\gamma(f)) = k^{-1} \sum_{i=1}^k c_i.$$

On the other hand, by a result of Lyons and Sullivan [20], there exists a bounded linear map P from $L^\infty(M)$ into $\mathcal{H}_0(M)$ such that $P(1) = 1$ and $P(f) \geq 0$ for $f \geq 0$. Moreover, $P(\gamma(f)) = \gamma(P(f))$ for any $\gamma \in \Gamma$. We now define a linear functional F on $L^\infty(M)$ by $F(f) = L(P(f))$. It is clear that F is bounded and $F(1) = 1$. Also, for $f \geq 0$, $F(f) \geq 0$. For any $\gamma \in \Gamma$,

$$F(\gamma(f)) = L(P(\gamma(f))) = L(\gamma(P(f))) = L(P(f)) = F(f).$$

This completes the proof. q.e.d.

Corollary 4.5. *Under the same notation as above, if Γ is a nonamenable group, then $\dim \mathcal{H}_0(M) = \infty$. In particular, the dimension of the space of bounded harmonic functions on a nonamenable covering of any Riemannian manifold must be infinite.*

Corollary 4.6. *Let $\tilde{M}$ be the universal covering space of complete manifold M. If $\dim \mathcal{H}_0(\tilde{M})$ is finite, then the fundamental group of M is amenable.*

We remark that R. Brooks [1] has shown that in the case where M is compact, then the fundamental group of M is nonamenable if and only if the L^2 spectrum of the Laplacian on $\tilde{M}$ does not contain 0. Thus, we can conclude that on the universal covering space of a compact manifold the finite dimensionality of the space of bounded harmonic functions implies that 0 is in the L^2 spectrum of the Laplacian.

Theorem 4.7. *Let M be a Riemannian covering over N, where N is a parabolic manifold. Then either $\dim \mathcal{H}_0(M) = 1$ or $\dim \mathcal{H}_0(M) = \infty$.*

Proof. Denote by Γ the covering group. Then Γ acts on M as isometries. If $\dim \mathcal{H}_0(M) = k < \infty$, let $\{f_1, \ldots, f_k\}$ be the basis of $\mathcal{H}_0(M)$ as in Lemma 4.1. By Lemma 4.3, Γ acts on $\{f_1, \ldots, f_k\}$ as permutations. In particular, the function $F(x) = \sum_{i=1}^k f_i^2(x)$ is Γ invariant, and F may be viewed as a function on N. However,

$$\Delta F = 2 \sum_{i=1}^k |\nabla f_i|^2 \geq 0 \tag{4.1}$$

and F is bounded. Since N is assumed to be parabolic, F must be a constant function. In particular, by (4.1), $\nabla f_i = 0$ and f_i is constant. Then we must have $k = 1$. This finishes the proof. q.e.d.

H. Donnelly has pointed out to us that the preceeding theorem was first proved by Larusson [11] using a different method. However, our argument has the advantage of unifying all these results using the point of view of massive sets.

Theorem 4.8. *Let $I(M)$ be the group of the isometries on a complete manifold M. If the quotient space $M/I(M)$ is compact, then either* $\dim \mathcal{H}_0(M) = 1$ *or* $\dim \mathcal{H}_0(M) = \infty$.

Proof. Suppose that $\dim \mathcal{H}_0(M) = k < \infty$. Then as in the proof of Theorem 4.7, for the basis $\{f_1, \ldots, f_k\}$ of $\mathcal{H}_0(M)$ specified by Lemma 4.1, the function $F(x) = \sum_{i=1}^{k} f_i^2$ is $I(M)$ invariant. Hence, $F(x)$ can be viewed as a function on $M/I(M)$. In particular, F must attain its maximum at some point on the compact set $M/I(M)$. That is, $F(x)$ as a function on M attains its maximum at some interior point. Since F is subharmonic, F must be constant and $k = 1$ as before. q.e.d.

Second Proof. Suppose that $\dim \mathcal{H}_0(M) = k$ and $1 < k < \infty$. Let f be a nonconstant bounded harmonic function such that $\sup f = 1$. Then there exists a sequence of points $\{x_m\}$ such that $\lim_{m\to\infty} f(x_m) = 1$. Since $M/I(M)$ is compact, one finds a sequence of isometries γ_m and points $y_m \in B_p(R)$ with $\gamma_m(y_m) = x_m$, where $B_p(R)$ is a fixed geodesic ball. Consider the sequence of bounded harmonic functions $h_m(x) = f(\gamma_m(x))$. Since $\dim \mathcal{H}_0(M) < \infty$, there exists a subsequence, denoted by $\{h_m\}$ again, such that h_m converges uniformly on M to a bounded harmonic function h. By choosing a subsequence if necessary, we may assume y_m converges to some point $y \in M$. Now

$$\begin{aligned} h(y) &= \lim_{m\to\infty} h(y_m) \\ &= \lim_{m\to\infty} \{h_m(y_m) + h(y_m) - h_m(y_m)\} \\ &= \lim_{m\to\infty} f(x_m) \\ &= 1. \end{aligned}$$

Thus, $h = 1$ on M as obviously $\sup h = 1$. So for any given $\epsilon > 0$ and sufficiently large m,

$$\sup |h_m - h| = \sup |f(\gamma_m(x)) - 1| < \epsilon.$$

This implies that $\inf f > 1 - \epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude that $f = 1$. However, this is a contradiction. So $k = 1$ or $k = \infty$. q.e.d.

The preceding theorem generalizes a result by Kifer [10] and Toledo [23]. The argument here also seems simpler.

Finally, we prove a result concerning the dimension of the space of bounded harmonic functions with finite Dirichlet energy.

Theorem 4.9. *Let M be a complete Riemannian manifold. Suppose that an infinite group Γ acts discontinuously and isometrically on M. Then $\dim \mathcal{H}_{0D}(M)$ is either 1 or ∞, where $\mathcal{H}_{0D}(M)$ is the space of bounded harmonic functions on M with finite Dirichlet energy.*

Proof. Let V be the space of all bounded nonconstant harmonic functions with finite Dirichlet energy on M endowed with the inner product $D(f,g) = \int_M \langle \nabla f, \nabla g\rangle$. Choose an orthonormal basis $\{f_1, \dots f_k\}$ of V. For any element $\gamma \in \Gamma$, let $h_i(x) = f_i(\gamma(x))$. Then $\{h_1, \dots, h_k\}$ is also an orthonormal basis of V. In particular, there exists an orthogonal $k \times k$ matrix $A = (a_{ij})$ such that $h_i = \sum_{j=1}^k a_{ij} f_j$. One verifies that

$$\sum_{i=1}^k |\nabla h_i(x)|^2 = \sum_{i=1}^k |\nabla f_i(x)|^2.$$

Now if we let $N = M/\Gamma$, then

$$\begin{aligned} k &= \sum_{i=1}^k \int_M |\nabla f_i(x)|^2 \\ &= \sum_{i=1}^k \sum_{\gamma\in\Gamma} \int_{\gamma(N)} |\nabla f_i(x)|^2 \\ &= \sum_{\gamma\in\Gamma} \int_N \sum_{i=1}^k |\nabla h_i(x)|^2 \\ &= \sum_{\gamma\in\Gamma} \int_N \sum_{i=1}^k |\nabla f_i(x)|^2 \\ &= |\Gamma| \int_N \sum_{i=1}^k |\nabla f_i(x)|^2. \end{aligned}$$

Since $|\Gamma| = \infty$, this implies that either $k = \infty$ or $k = 0$. The theorem is proved by observing that in the latter case, the space $\mathcal{H}_{0D}(M)$ only contains the constant function. q.e.d.

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