J. DIFFERENTIAL GEOMETRY
52 (1999) 453-609

GR = SW : COUNTING CURVES AND CONNECTIONS

CLIFFORD HENRY TAUBES

The purpose of this article is to present a proof of the assertion that a compact, symplectic 4-manifold has its Seiberg-Witten invariants equal to its Gromov invariants. In this regard, remark that the original Seiberg-Witten invariants are defined for any smooth, compact, oriented 4-manifold; and they are determined by the underlying differentiable structure when the Betti number b^{2+} is larger than 1. After the choice of orientation for the real line $\det^+ = H^0 \otimes \det(H^1) \otimes \det(H^{2+})$, the Seiberg-Witten invariants constitute a map from the set, $\mathcal{S}$, of $\mathrm{Spin}^{\mathbb{C}}$ structures on the 4-manifold to the integers. There is also an extension of SW in the case where the Betti number b^1 is positive to a map SW: $\mathcal{S} \to \Lambda^* H^1(X;\mathbb{Z})$. Here,

$$\Lambda^* H^1(X;\mathbb{Z}) = \mathbb{Z} \oplus H^1 \oplus \Lambda^2 H^1 \oplus \cdots \oplus \Lambda^{b_1} H^1.$$

Note that the projection of the image of SW on the summand $\mathbb{Z}$ reproduces the original map as defined from $\mathcal{S}$ to $\mathbb{Z}$. In either guise, this map, SW, is computed by an algebraic count of solutions to a certain non-linear system of differential equations on the manifold.

As remarked in [25], a symplectic manifold has a natural orientation as does the line $\det^+$. Furthermore, there is a canonical identification of the set $\mathcal{S}$ with $H^2(X;\mathbb{Z})$. Thus, on a symplectic 4-manifold, SW can be viewed as a map from $H^2(X;\mathbb{Z})$ to $\mathbb{Z}$, or, more generally, from $H^2(X;\mathbb{Z})$ to $\Lambda^* H^1(X;\mathbb{Z})$.

Meanwhile, a compact symplectic 4-manifold has a second natural map from $H^2(X;\mathbb{Z})$ to $\mathbb{Z}$, its Gromov invariant, Gr. The map Gr also

Received April 7, 1998. Supported in part by the National Science Foundation.

extends on a $b^1 > 0$ symplectic 4-manifold to a map from $H^2(X;\mathbb{Z})$ into $\Lambda^* H^1(X;\mathbb{Z})$; the extension is sometimes called the Gromov-Witten invariant, but it will be denoted here by Gr as well. In either guise, Gr, assigns to a class e a certain weighted count of compact, symplectic submanifolds whose fundamental class is Poincaré dual to e.

The invariant SW was introduced to the mathematical community by Witten [31] after his ground breaking work with Seiberg in [21], [22]. See also [9], [16], [8] and [15]. The Gromov invariant was introduced initially by Gromov in [5] and then generalized by Witten [32] and Ruan [20]. The version of Gr used here comes from [26]. (Note that Gr here does not count maps from a fixed complex curve. It differs in this fundamental sense from the Gromov-Witten invariant introduced in [32].) For the uninitiated, the precise definition of SW and Gr are provided in the first section of this paper.

Here is the main theorem:

Theorem 1. *Let X be a compact, symplectic manifold with $b^{2+} > 1$. Use the symplectic structure to orient X and the line* $\det^+$; *and use the symplectic structure to define SW as a map from $H^2(X;\mathbb{Z})$ to $\Lambda^* H^1(X;\mathbb{Z})$. Also, use the symplectic structure to define*

$$Gr\ : H^2(X;\mathbb{Z}) \to \Lambda^* H^1(X;\mathbb{Z}).$$

Then $SW = Gr$.

The equivalence between the Gromov invariant and the original SW map into $\mathbb{Z}$ was announced by the author in [25].

The proof of Theorem 1 can be divided into three main parts. The first part explains how a non-zero Seiberg-Witten invariant implies the existence of symplectic submanifolds. The second part explains how a symplectic submanifolds can be used to construct a solution to a version of the Seiberg-Witten equations. The third part compares the counting procedures for the two invariants. The first and second parts of the proof can be found in [27] and [28], respectively. Of necessity, this article will draw heavily from constructions in the latter two references. This article will also refer to the discussion in [26] which gives a complete definition of the Gromov invariant.

Some of the early applications of Theorem 1 are described in [10].

A restricted version of Theorem 1 holds in the case where $b^{2+} = 1$. Here, a fundamental complication is that the Seiberg-Witten invariant depends on more than the differentiable structure. This is to say that there is a dependence on a so called choice of chamber. However, the symplectic form selects out a unique chamber, and with this understood, one has:

Theorem 2. *Let X be a compact, oriented 4-manifold with $b^{2+} = 1$ and a symplectic form. Then the symplectic form canonically defines a chamber in which the equivalence SW=Gr holds for classes $e \in H^2(X;\mathbb{Z})$ which obey $\langle e, s\rangle \geq -1$ whenever $s \in H_2(X;\mathbb{Z})$ is represented by an embedded, symplectic sphere with self-intersection number -1.*

Here, $\langle , \rangle$ denotes the pairing between cohomology and homology. Note that McDuff [12] has suggested a generalization of the definition of Gr on $b^{2+} = 1$ manifolds to make SW = Gr hold on all classes.

The remainder of this article is divided up into sections. The first section below summarizes the definitions of both the Seiberg-Witten invariant and the Gromov-Witten invariant. The second section reduces the proofs of Theorems 1 and 2 to several key propositions. Those key propositions which are not contained already in [26], [27] or [28] are proved in the Sections 3–7.

1. The Seiberg-Witten and the Gromov-Witten invariants

The purpose of this first section is to give a precise definition of the Seiberg-Witten invariants and also the Gromov invariants for a symplectic 4-manifold. The former is considered in Subsections 1a-c, and the latter in Subsections 1d,e. A final subsection returns to the milieu of the Seiberg-Witten invariants to consider some of the special circumstances which arise when the 4-manifold X has $b^{2+} = 1$.

a) The Seiberg-Witten equations

The Seiberg-Witten equations were first introduce by Seiberg and Witten in [21], and [22], [31]. A purely mathematical approach to these equations was first taken in [9]. The book by Morgan [15] is a more complete reference (see also [8]).

In this subsection, X is a compact, connected, oriented, 4-dimensional manifold. Let $b^1 = \dim(H^1(X))$ denote the first Betti number of X and let b^{2+} denote the dimension of a maximal subspace,

$$H^{2+}(X;\mathbb{R}) \subset H^2(X)$$

where the cup product form is positive.

Fix a smooth Riemannian metric on X. The metric defines the principal SO(4) bundle of orthornormal frames, $\mathrm{Fr} \to X$. Of the various associated bundles to this frame bundle, two in particular play central roles. These are the bundles Λ_+ of self-dual 2-forms and Λ_- of anti-self dual 2-forms. Note that $\Lambda^2 TX \approx \Lambda_+ \oplus \Lambda_-$.

By definition, a Spin$^{\mathbb{C}}$ structure on X is an equivalence class of lifts of Fr to a principal Spin$^{\mathbb{C}}(4)$ bundle $F \to X$. In this regard, recall that the group Spin$^{\mathbb{C}}(4)$ is the group $(SU(2) \times SU(2) \times U(1))/\{\pm 1\}$, this being a central extension of $SO(4) = (SU(2) \times SU(2))/\{\pm 1\}$ by the circle $U(1)$. (The homomorphism Spin$^{\mathbb{C}} \to (SU(2) \times SU(2))/\{\pm 1\}$ simply forgets the factor of $U(1)$.)

A Spin$^{\mathbb{C}}$ lift F of Fr has two canoncial associated $\mathbb{C}^2$ bundles, $S_\pm \to X$ which are defined using the two evident homomorphisms of Spin$^{\mathbb{C}}$ to $U(2) = (SU(2) \times U(1))/\{\pm 1\}$. Note that S_+ is distinguished by the fact that the projective bundle is the unit 2-sphere bundle in Λ_+. (There is, of course, an analogous relationship between S_- and Λ_-.)

With the preceding understood, the original version of Seiberg and Witten's equations can now be defined. These are equations for a pair (A, ψ), where A is a connection on $\det(S_+)$, and ψ is a section of S_+. The equations read:

$$
\begin{aligned}
&D_A \psi = 0, \\
(1.1) \qquad &P_+ F_A = \frac{1}{4}\tau(\psi \otimes \psi^*) + \mu.
\end{aligned}
$$

In the first line above, D_A is the Dirac operator, a first order differential operator which maps sections of S_+ to sections of S_-. This D_A is defined as the composition of Clifford multiplication (a homomorphism from $S_+ \otimes T^*X$ to S_-) with covariant differentiation using the connection on S_+ which comes from the Levi-Civita connection on Fr and the connection A on $\det(S_+)$. In the second line of (1.1), P_+ denotes the orthogonal projection from $\Lambda^2 T^*X$ to Λ_+, and F_A denotes the curvature 2-form of A. Meanwhile, τ is the adjoint of the Clifford multiplication endomorphism from $\Lambda_+ \otimes \mathbb{C}$ into $\mathrm{End}\,(S_+)$, and μ is a fixed, imaginary valued, anti-self dual 2-form on X. (Any choice for μ will do.)

There is a natural action of the group of smooth maps from X to $U(1)$ on the set of solutions to (1.1). The action sends a map g and a pair (A, ψ) to $(A + 2g dg^{-1}, g\psi)$. Use $\mathcal{M}$ to denote the set of orbits under this group action. (Typically, notational distinctions will not be made between a pair (A, ψ) and its orbit in $\mathcal{M}$.)

Topologize $\mathcal{M}$ as follows: First, introduce the manifold Conn $(\det(S_+))$ of Hermitian connections on $\det(S_+)$. This is an affine Frechet manifold modelled on $i{\cdot}\Omega^1$. (Here, Ω^1 denotes the vector space of smooth 1-forms on X.) With Conn $(\det(S_+))$ understood, introduce the space Conn $(\det(S_+)) \times C^\infty(S_+)$. The group $C^\infty(X; S^1)$ acts smoothly on the latter (as indicated above), and the space of orbits of this group action, $(\mathrm{Conn}\,(\det(S_+)) \times C^\infty(S_+))/C^\infty(X; S^1)$, is given the quotient topology. The space $\mathcal{M}$ sits in this quotient, and the implicit topology on $\mathcal{M}$ is the subspace topology inherited from the orbit space $(\mathrm{Conn}\,(\det(S_+)) \times C^\infty(S_+))/C^\infty(X; S^1)$.

Here are some basic properties of $\mathcal{M}$ (see, [31] or [9], [15], [8]):

- $\mathcal{M}$ is always compact.
- If $b^2_+ > 0$, then there is a Baire set of $\mathcal{U} \subset C^\infty(X; i\Lambda_+)$ of choices for μ in (1.1) whose corresponding $\mathcal{M}$ has the structure of a smooth, manifold of dimension

$$2d = -\frac{1}{4}(2\chi + 3\tau) + \frac{1}{4}c_1 \bullet c_1.$$

Here, χ is the Euler characteristic of X and τ is the signature of X. Also, "$\bullet$" signifies the pairing on $H^2(X; \mathbb{Z})$ which is cup product composed with evaluation in the fundamental class. Furthermore, when $\mu \in \mathcal{U}$, the following hold:

a) There are no points in $\mathcal{M}$ where the corresponding ψ is zero.

b) $\mathcal{M}$ is orientable, and an orientation of $\det^+$ canonically orients $\mathcal{M}$.

c) The subspace of orbits

$$(p, (A, \psi)) \in (X \times \mathrm{Conn}(\det(S_+)) \times C^\infty(S_+))/ \{\phi \in C^\infty(X; S^1) : \phi(p) = 1\},$$

where $(A, \psi) \in \mathcal{M}$ naturally defines a smooth, principal S^1 bundle $\mathcal{E} \to X \times \mathcal{M}$.

(1.2)

(A Baire set is a countable intersection of open and dense sets and so is dense. The Baire set in question is characterized by the condition that a certain family of first order, elliptic differential operators that is parameterized by the points in $\mathcal{M}$ has, at each point in $\mathcal{M}$, trivial cokernel.)

Here are some additional comments about (1.2):

- The number 2d in (1.2) can be even or odd. Its parity is the same as that of $\frac{1}{2}(\chi + \tau) = 1 - b^1 + b^2_+$.

- Equation (1.2) implies the following: When $d < 0$ and $\mu \in \mathcal{U}$, then $\mathcal{M} = \varnothing$, since there are no negative dimensional manifolds.

- In the case $d = 0$ and $\mu \in \mathcal{U}$, $\mathcal{M}$ consists of a finite set of points. In this case, an orientation on $\mathcal{M}$ simply assigns either $+1$ or -1 to each point. (This is because $H_0(\text{point}; \mathbb{Z})$ already has a canonical generator, which is the point itself. The orientation assigns a fundamental class which is either the point, or $-$ the point.)

- Let $c_1(\mathcal{E})$ denote the first Chern class of the principal S^1 bundle $\mathcal{E} \to X \times \mathcal{M}$. Then slant product with $c_1(\mathcal{E})$ defines a map, $\phi : H_*(X; \mathbb{Z}) \to H^{2-*}(\mathcal{M}; \mathbb{Z})$.

(1.3)

b) The Seiberg-Witten invariant

Let $\mathcal{S}$ denote the set of Spin$^{\mathbb{C}}$ structures on X. Although $\mathcal{S}$ requires a choice of Riemannian metric for its definition, there is a natural identification between such sets defined by any two metrics. (Remember that the space of metrics on X is convex.) Thus, one can speak unambiguously about $\mathcal{S}$ without reference to a particular metric. (Note that $\mathcal{S}$ is an affine space modelled on $H^2(X; \mathbb{Z})$.) Likewise, the definition of SW requires a choice of Riemannian metric; and it also requires a choice of perturbing form μ in the set $\mathcal{U}$ of (1.2). Here is the definition of SW:

Definition 1.1. Fix the following: an orientation for the line $\det^+$, a Riemannian metric on X, a Spin$^{\mathbb{C}}$ structure in $\mathcal{S}$, and also $\mu \in \mathcal{U}$ so that the conclusions of (1.2) and (1.3) are valid. Let d be as defined in (1.2). Then, the value of SW $\in \Lambda^* H^1(X; \mathbb{Z})$ on the given Spin$^{\mathbb{C}}$ structure is defined as follows:

- If $d < 0$, then SW$= 0$.

- If $d = 0$, then $\mathcal{M}$ is a finite set of points and the chosen orientation for $\det^+$ defines a map $\varepsilon : \mathcal{M} \to \{\pm 1\}$. With this understood, then

$$\text{SW} \equiv \sum_{\Xi \in \mathcal{M}} \varepsilon(\Xi) \tag{1.4}$$

which is an element in the $\mathbb{Z}$ summand of $\Lambda^* H^1$.

- In general, $\mathrm{SW} \in \mathbb{Z} \oplus H^1 \oplus \cdots \oplus \Lambda^{2d} H^1$ with non-zero projection in $\Lambda^p H^1$ only if p has the same parity $1 - b^1 + b^{2+}$. In this case, SW is defined by its values on the set of decomposable elements in $\Lambda^p(H_1(X;\mathbb{Z})/\,\mathrm{Torsion})$; and

$$\mathrm{SW}(\gamma_1 \wedge \cdots \wedge \gamma_p) = \int_{\mathcal{M}} \phi(\gamma_1) \wedge \cdots \wedge \phi(\gamma_p) \wedge \phi(*)^{d-p/2}, \tag{1.5}$$

where $*$ is the class of a point generating $H_0(X)$.

The next proposition asserts that the apparent dependence of SW on the choice of metric and μ is spurious:

Proposition 1.2. *Let X be a compact, connected, oriented 4-manifold with $b^{2+} > 1$. Then the value of SW is independent of the choice of Riemannian metric and form μ. In fact, SW depends only on the diffeomorphism type of X. Furthermore, SW pulls back naturally under orientation preserving diffeomorphisms. This is to say that if $\varphi : X \to X'$ is a smooth, orientation preserving diffeomorphism, then $SW_X(\varphi^*\eta) = \varphi^* SW_{X'}(\eta)$. Finally, SW changes sign when the orientation of the line* $\det^+$ *is switched.*

(Note that $\mathrm{Spin}^{\mathbb{C}}$ structures pull back because metrics do.) See, e.g. [15] or [8] for a proof of this proposition.

The preceding proposition does not hold in general in the case where the 4-manifold X has $b^{2+} = 1$. However, the failure of this proposition can be readily analyzed, and the results are summarized in Proposition 1.3, below. To state the proposition precisely, it is convenient to make a short digression to consider some special features of $b^{2+} = 1$ manifolds.

To begin the digression, introduce $\mathrm{Met}\,(X)$ to denote the Frechet space of smooth, Riemannian metrics on X. Given a metric g on X, let ω_g denote the unique (up to multiplication by $\mathbb{R}^*$), non-trivial, self-dual, harmonic 2-form on X. With ω_g understood, then each $c \in H^2(X;\mathbb{Z})$ defines a "wall" in $\mathrm{Met}\,(X) \times i \cdot \Omega^{2+}$ whose elements consist of pairs (g, μ) where $2\pi \cdot [\omega_g] \bullet c = i \cdot \int_X \omega_g \wedge \mu$. The wall divides $\mathrm{Met}\,(X) \times i \cdot \Omega^{2+}$ into two open sets, each of which is called a "c-chamber".

Given the preceding, then Proposition 1.2 has the following $b^{2+} = 1$ version:

Proposition 1.3. *Let X be a compact, connected, oriented 4-manifold with $b^{2+} = 1$. Let s be a* $\mathrm{Spin}^{\mathbb{C}}$ *structure on X. Then the value* *of* $SW(s) \in \Lambda^* H^1(X;\mathbb{Z})$ *is constant on any* $c = c_1(\det(S_+))$ *chamber in* $\text{Met}(X) \times i \cdot \Omega^{2+}$.

See, e.g. [15] or [8] for a proof. However, the point is that the arguments for Proposition 1.2 work on an open set in Met $(X) \times i \cdot \Omega^{2+}$ where the corresponding $\mathcal{M}$ contains no elements where the corresponding ψ vanishes identically. Indeed, the count for SW can change along a path in Met $(X) \times i \cdot \Omega^{2+}$ only when the path intersects elements in $\mathcal{M}$ where the corresponding ψ vanishes identically. And, such elements occur if and only if (g, μ) lies in the wall. (The change in SW as the wall is crossed can be computed. See [9], [11], [18].)

c) The Seiberg-Witten invariants on symplectic manifolds

As remarked in the introduction, a symplectic 4-manifold has a natural orientation, a natural orientation for the line $\det^+$ and a natural identification between $\mathcal{S}$ and $H^2(X;\mathbb{Z})$. The introduction also asserted that a symplectic manifold with $b^{2+} = 1$ also has a natural chamber. The purpose of this subsection is to explain these assertions.

The orientation of X. A symplectic 4-manifold is, by definition, a pair (X, ω), where X is a smooth 4-manifold, and ω is a closed 2-form on X with $\omega \wedge \omega$ nowhere zero. (The characteristic number $\frac{1}{2}(\chi + \tau) = 1 - b^1 + b^{2+}$ must be even for X to admit a symplectic form.) Because $\omega \wedge \omega$ is nowhere zero, this form orients X, and is the orientation that the introduction referred to. It will be assumed throughout.

The orientation of $\det^+$. The description of the orientation for the line $\det^+$ is conveniently divided into five steps.

Step 1. A choice of orientation for $\det^+$ is equivalent to a choice of orientation for the virtual vector space $H^1(X;\mathbb{R}) - (H^0(X;\mathbb{R}) \oplus H^{2+}(X;\mathbb{R})$. After a metric on X is chosen, the latter can be viewed using Hodge theory as the formal difference between the kernel and the cokernel of the operator $\delta_0 = (P_+ d, d^*) : \Omega^1 \to \Omega^0 \oplus \Omega^{2+}$. Here, Ω^1 is the space of smooth 1-forms, Ω^0 is the space of smooth functions and Ω^{2+} is the space of smooth, self dual 2-forms.

Step 2. Every symplectic manifold admits almost complex structures, endomorphisms J of TX with square -1. As noted by Gromov [5], one can find almost complex structures with the property that the bilinear form

$$g = \omega(\cdot, J(\cdot)) \tag{1.6}$$

defines a Riemannian metric on TX. Such a J will be called *ω-compatible.*

The almost complex structure J decomposes $TX \otimes \mathbb{C} \equiv T_{1,0} \oplus T_{0,1}$ into a sum of complex 2-plane bundles such that J has eigenvalue i on the former and $-i$ on the latter. The complexified cotangent bundle decomposes analogously as $T^{1,0} \oplus T^{0,1}$.

Thus, the endomorphism J acts, by definition on the domain of the operator δ_0.

Step 3. If the metric g is chosen as in (1.6), then there is also a natural almost complex structure (call it J_R) which acts on the range of δ_0. The latter is induced from a square -1 endomorphism (also called J_R) on the vector bundle $\varepsilon_{\mathbb{R}} \oplus \Lambda_+$ whose sections define δ_0's range. Here, $\varepsilon_{\mathbb{R}} \to X$ denotes the product bundle $X \times \mathbb{R}$. Likewise, $\varepsilon_{\mathbb{C}}$, below, will denote the product complex line bundle.

To define J_R, remark first that the metric in (1.6) splits $\Lambda^2 T^* X$ as $\Lambda_+ \oplus \Lambda_-$. The form ω is self dual with respect to this splitting and has norm $\sqrt{2}$ everywhere. Conversely, if g is any metric for which ω is self-dual and has norm $\sqrt{2}$, then $J \equiv g^{-1}\omega$ defines an almost complex structure J on TX such that (1.6) holds. Note that J induces an endomorphism of $\Lambda^2 T^* X$ with square 1 which preserves Λ_+. The $+1$ eigenspace of this endomorphism on Λ_+ is the span of ω. The orthogonal compliment is the -1 eigenspace. The latter is an oriented, 2-plane bundle over X which is the underlying real bundle of the complex line bundle $K^{-1} = \Lambda^2 T^{0,1}$.

With the preceding understood, view $\varepsilon_{\mathbb{R}} \oplus \Lambda_+$ as a complex 2-plane bundle by writing the latter as $\varepsilon_{\mathbb{C}} \oplus K^{-1}$, where $x + y \cdot \omega \in \varepsilon_{\mathbb{R}} \oplus \Lambda_+$ is identified with $x + \sqrt{-1} \cdot y \in \varepsilon_{\mathbb{C}}$. Multiplication by $\sqrt{-1}$ on $\varepsilon_{\mathbb{C}} \oplus K^{-1}$ defines the endomophism J_R on $\varepsilon_{\mathbb{R}} \oplus \Lambda_+$.

Step 4. In general, $\delta_0 J - J_R \delta_0 \neq 0$. However, this difference is always a zero'th order operator. The symbol of δ_0 intertwines J with J_R; and δ_0 itself intertwines J with J_R when J is an integrable almost complex structure.

The fact that $\delta_0 J - J_R \delta_0$ is zero'th order implies that there is a relatively compact perturbation of δ_0 which does intertwine J and J_R. For example, $\delta_1 = 2^{-1} \cdot (\delta_0 - J_R \cdot \delta_0 \cdot J)$ has this property.

Since δ_1 differs from δ_0 by a zero'th order operator, both its kernel and cokernel are finite dimensional. Furthermore, because δ_1 intertwines J with J_R, its kernel and cokernel have natural structures as complex vector spaces. And, since complex vector spaces have canonical orientations, the virtual vector space kernel $(\delta_1)-$cokernel (δ_1) has a canonical orientation.

Step 5. The complex orientation for kernel (δ_1)−cokernel (δ_1) canonically orients the line $\det^+ = \text{kernel}(\delta_0) - \text{cokernel}(\delta_0)$. The argument here is standard K-theory since the family of operator

$$\{\delta_t = t \cdot \delta_0 + (1-t) \cdot \delta_1\}_{t \in [0,1]}$$

defines a continuous map of Fredholm operators with respect to appropriate Hilbert space completions of Ω^1 and $\Omega^0 \oplus \Omega^{2+}$. (The Sobolev spaces L_1^2 for the range and L^2 for the domain will suffice.) The point is that the association of the virtual vector space kernel (δ_t)−cokernel (δ_t) to $t \in [0,1]$ defines an element in the real K-theory of the interval (see the Appendix in [2].) Since the interval is contractible, this element is trivial. In particular, it has vanishing first Stieffel-Whitney class, so it is orientable and an orientation at $t = 1$ induces one at $t = 0$.

Note that the purpose of orienting the line $\det^+$ is to obtain a reasonably canonical orientation for the moduli space $\mathcal{M}$. In this regard, the symplectic orientation of $\det^+$ induces an orientation on $\mathcal{M}$ which is described directly in Section 4.

The identification of $\mathcal{S}$ with $H^2(X;\mathbb{Z})$. As remarked, the set $\mathcal{S}$ has the natural structure of an affine space modelled on $H^2(X;\mathbb{Z})$. This implies that the identification in question arises immediately with the specification of a "canonical" $\text{Spin}^{\mathbb{C}}$ structure. And, as observed in [29], there is a canonical $\text{Spin}^{\mathbb{C}}$ structure on a symplectic manifold.

With the metric chosen from an ω-compatible J, the canonical $\text{Spin}^{\mathbb{C}}$ structure is characterized by the identifications

$$S_+ = \mathbb{I} \oplus K^{-1} \quad \text{and} \quad S_- = T^{0,1}, \tag{1.7}$$

where $K^{-1} = \Lambda^2 T^{0,1}$ again. Indeed, this splitting of S_+ is defined as follows: Clifford multiplication defines an endomorphism from Λ_+ into the bundle of skew hermitian endomorphisms of S_+. With the preceding understood, the splitting of S_+ in (1.7) is the decomposition of S_+ into eigenbundles for the action of ω; here ω acts with eigenvalue $-2i$ on the trivial summand $\mathbb{I}$, and it acts with eigenvalue $+2i$ on the K^{-1} summand.

As just remarked, the identification in (1.5) of a canonical element in $\mathcal{S}$ identifies

$$\mathcal{S} \approx H^2(X;\mathbb{Z}). \tag{1.8}$$

Under this identification, a class $e \in H^2(X;\mathbb{Z})$ is sent to the Spin$^{\mathbb{C}}$ structure whose $S_\pm$ bundles are given by

$$S_+ = E \oplus (K^{-1} \otimes E) \quad \text{and} \quad S_- = T^{0,1} \otimes E, \tag{1.9}$$

where E is a complex line bundle whose first Chern class is isomorphic to e. Once again, this splitting of S_+ is into eigenbundles for the action of ω on S_+; and the convention is that the bundle where ω acts as $-2i$ is written first.

By the way, after the identification in (1.8), the dimension $2d$ of the Seiberg-Witten moduli space (as given in (1.2)) can be rewritten as follows: If $e \in H^2(X;\mathbb{Z})$ and if e is used to determine the Spin$^{\mathbb{C}}$ structure as in (1.9), then the formal dimension of the moduli space $\mathcal{M}$ is

$$2 \cdot d = e \bullet e - c \bullet e \tag{1.10}$$

where $c = c_1(K)$ with $K = \Lambda^2 T^{1,0}$. (The number $e \bullet e - c \bullet e$ is even because the class c is characteristic: Its mod 2 reduction is the second Stieffel-Whitney class of X.)

The natural chamber when $b^{2+} = 1$. Suppose now that X is a compact, oriented 4-manifold with $b^{2+} = 1$ and a symplectic form ω. The latter defines a canonical c-chamber for each $c \in H^2(X;\mathbb{Z})$ by requiring μ in (1.1) to obey $i : \int_X \omega \wedge \mu > 2\pi \cdot [\omega] \bullet c$. This last chamber will be called the "symplectic chamber".

Note, by the way, that two symplectic forms ω and ω' on X define the same chamber when $[\omega] \bullet [\omega'] > 0$. Thus, the symplectic chamber depends only on the form ω up to continuous deformations through closed forms ν with $[\nu] \bullet [\nu'] > 0$.

In the subsequent discussions, the Seiberg-Witten invariant for such a pair (X, ω) will always denote the map SW from Proposition 1.3 as defined in the symplectic chamber. This $b^{2+} = 1$ definition of SW will be implicit in the subsequent discussions.

d) Pseudo-holomorphic submanifolds

As noted in the introduction, the Gromov-Witten invariant is defined by counting (in a suitable sense) pseudo-holomorphic submanifolds on the symplectic manifold X. Thus, a more complete description of this invariant must start with a digression to discuss pseudo-holomorphic submanifolds. There are four parts to this discussion.

Part 1. The complex line bundle $K = \Lambda^2 T^{1,0}$ is called the canonical bundle. Note that the isomorphism class of K, and thus its first Chern class $c \in H^2(X;\mathbb{Z})$, are independent of the choice of ω-compatible almost complex structure J. Furthermore, this isomorphism class and also c are both unchanged if ω is changed through a continuous family of symplectic forms. Note the sign convention here: $c \bullet [\omega] < 0$ when $X = \mathbb{CP}^2$.

Part 2. A submanifold Σ in X is called pseudo-holomorphic when J preserves $T\Sigma$. It follows from the non-degeneracy of (1.6) that ω is non-degenerate on $T\Sigma$ and so orients Σ. Infact, J induces the structure of a complex curve on Σ. Then, the inclusion map of Σ into X is pseudo-holomorphic in the sense of Gromov [5].

If Σ is a connected and compact pseudo-holomorphic submanifold, then the genus of Σ is constrained by the adjunction formula to equal

$$\text{genus} = 1 + \frac{1}{2}(e \bullet e + c \bullet e), \tag{1.11}$$

where e is the Poincaré dual to the fundamental class $[\Sigma]$ of Σ.

Henceforth, all pseudo-holomorphic submanifolds in this article should be assumed to be compact unless stated to the contrary.

Part 3. Fix a pseudo-holomorphic submanifold Σ. Since J preserves $T\Sigma$, it must also preserve the orthogonal compliment in TX of $T\Sigma$. The latter is the normal bundle, N, of Σ. Thus, N has a natural structure as a complex line bundle over Σ. The metric from TX defines a connection on $N \to \Sigma$, and thus endows N with a holomorphic structure as a bundle over the complex curve Σ. With this understood, one can introduce the associated d-bar operator, $\bar\partial$, to map sections of N to sections of $N \otimes T^{0,1}C$. Here, $T^{0,1}C$ is the usual anti-holomorphic summand of $T^*C \otimes_{\mathbb{R}} \mathbb{C}$.

One's first guess is that the kernel of $\bar\partial$ corresponds to the vector space of deformations of Σ in X which are pseudo-holomorphic to first order. However, this guess is wrong, in general. Rather, this vector space corresponds to the kernel of certain canonical, zero'th order deformation of $\bar\partial$. This deformation is an $\mathbb{R}$ linear operator, D, which also maps sections of N to sections of $N \otimes T^{0,1}C$, and which is defined as follows: The 1-jet off of Σ of the almost complex structure defines a pair (ν, μ) of section of $T^{0,1}C$ and $N^{\otimes 2} \otimes T^{0,1}C$. (See (2.3) in [28].) Then

$$Dh = \bar\partial h + \nu \cdot h + \mu \cdot \bar h. \tag{1.12}$$

Part 4. Note that the index of D is given by the Riemann-Roch formula, which is to say that it equals $2d$ in (1.10) in the case where $e \in H^2(X;\mathbb{Z})$ is Poincaré dual to $[\Sigma]$. As the index is, by definition, the difference between the dimensions (over $\mathbb{R}$) of the kernel and the cokernel of D, a necessary condition for the triviality of cokernel(D) is that $2 \cdot d$ be non-negative. In general, this condition is not sufficient. However, all pseudo-holomorphic submanifolds have trivial cokernel if the almost complex structure is chosen from a certain Baire subset of ω compatible almost complex structures. (This fact is proved in, e.g. [14].)

e) The Gromov-Witten type invariants

Fix $e \in H^2(X;\mathbb{Z})$. This subsection defines $\mathrm{Gr}(e) \in \Lambda^* H^1(X;\mathbb{Z})$. (The reader is referred to [26] for the proofs of the assertions below.) The discussion here is broken into seven parts.

Part 1. Introduce the integer $d = d(e)$ as defined by (1.10). Then $\mathrm{Gr}(e)$ lies in the direct sum $\mathbb{Z} \oplus \Lambda^2 H^1 \oplus \cdots \oplus \Lambda^{2d} H^1$. Its projection into $\Lambda^{2p} H^1$ (for $0 \leq p \leq d$) can be determined by evaluating $\mathrm{Gr}(e)$ on a decomposable element in $\Lambda^{2p}(H_1(X;\mathbb{Z})/\,\mathrm{Torsion})$. Of course, when $p = 0$, the corresponding component of $\mathrm{Gr}(e)$ is simply an integer. With the preceding understood, make the following choices when $d > 0$: First, choose $p \in \{0, \ldots, 2d\}$ and if $p > 0$, choose an element

$$\gamma_1 \wedge \cdots \wedge \gamma_{2p} \in \Lambda^{2p} H_1/\,\mathrm{Torsion}\,.$$

Then, for each $j \in \{1, \ldots, 2p\}$, choose an oriented, embedded circle in X to represent the class γ_j. To simplify notation, the chosen circle will be denoted by γ_j also. Make these choices of $\Gamma = \{\gamma_j\}_{1 \leq j \leq 2p}$ so that the distinct circles are disjoint. With Γ chosen, choose a set $\Omega \subset X$ of $d - p$ distinct points which miss each circle in Γ.

Part 2. Let $\mathcal{H} = \mathcal{H}(e, J, \Gamma, \Omega)$ denote the set whose typical element is an unordered set, h, of pairs $\{(C_k, m_k)\}$, where each C_k is a compact, oriented, pseudo-holomorphic submanifold in X, and the corresponding m_k is a positive integer. The elements in h should be constrained as follows:

1. For each k, introduce e_k to denote the Poincaré dual to $[C_k]$, and $d_k = e_k \bullet e_k - c \bullet e_k$. Require $d_k \geq 0$.
2. Require that $m_k = 1$ unless $d_k = 0$ and the genus of C_k is also 0. Thus, C_k is a torus with trivial normal bundle.

3. Require that $\Sigma_k m_k e_k = e$.

4. There is a partition $\Gamma = \cup_k \Gamma_k$, where each Γ_k contains some even number $2 \cdot p_k$ elements with $0 \leq p_k \leq d_k$. Furthermore, C_k intersects precisely once each $\gamma \in \Gamma_k$; and no $\gamma \in \Gamma_k$ is tangent to C_k at their intersection point. Moreover, C_k has empty intersection with the elements of $\Gamma - \Gamma_k$.

5. Each C_k contains precisely $d_k - p_k$ points of Ω.

6. Require that the $C_k \cap C_{k'} = \varnothing$ when $k \neq k'$.

(1.13)

(The final condition implies that $e_k \bullet e'_k = 0$ when $k \neq k'$. And this implies that $\mathcal{H}$ is empty whenever d (from (1.10)) is negative.)

Part 3. Suppose that $h \in \mathcal{H}$, and that $(C_k, m_k) \in h$ is such that $d_k > 0$. This data can be used to define a real vector space V_k of dimension $2 \cdot d_k$ as follows: First of all, each $z \in C_k \cap \Omega$ contributes a summand $N|_z$ to V_k, where $N \to C_k$ is the normal bundle to C_k in X. Meanwhile, each $\gamma \in \Gamma_k$ contributes a real line summand to V_k; the latter being the line $N|_z / p(T\gamma|_z)$, where $z = \gamma \cap C_k$, and $p : TX|_z \to N|_z$ is the tautological projection.

Note that N is naturally oriented, as is each $\gamma \in \Gamma_k$. This means that each of the summands of V_k has a natural orientation. Thus, V_k inherits an orientation with the choice of an ordering for the set Γ_k. For, this ordering gives the order of the oriented real line summands in V_k. The summands which are indexed by the points in $C_k \cap \Omega$ are each naturally complex, and so their order in V_k is immaterial.

With V_k understood, note that any section a of the normal bundle N defines a tautological element in V_k by restricting a to the points in $C_k \cap \Omega$ and to the points where the elements of Γ_k intersect Ω. The preceding defines a tautological map from $C^\infty(N)$ to V_k whose restriction to the kernel of the operator D in (1.12) will be denoted by G_k.

Part 4. Here are some salient properties of the set $\mathcal{H}$:

- Each $h \in \mathcal{H}$ is a finite set.

- There is a Baire set $\mathcal{W}$ of triples (J, Γ, Ω), for which J is ω compatible and the corresponding set $\mathcal{H}$ is finite. Furthermore when $h \in \mathcal{H}$ and $(C_k, m_k) \in h$, then:

a) The operator D in (1.12) has trivial cokernel.

b) If $d_k > 0$, the homomorphism G_k : kernel$(D) \to V_k$ is an isomorphism.

c) If $d_k = 0$ and $m_k > 1$, then the pull-back of D to any finite cover of the torus C_k also has trivial cokernel (and kernel).

(1.14)

These facts are proved in [26]; see also [12].

Part 5. Assume now that the data (J, Γ, Ω) is chosen from the set $\mathcal{W}$ in (1.14). Let $h \in \mathcal{H}$ and $(C, m) = (C_k, m_k) \in h$. The purpose of this part of the discussion is to associate to such a pair an integer, $r(C, m)$. There are three cases to consider.

If m = 1 and $\mathbf{d_k = 0}$. Here, $r(C, 1) \in \{\pm 1\}$ and it counts (mod 2) the spectral flow for a path of zero'th order deformations of D which starts with D and ends with a $\mathbb{C}$-linear operator

$$D_1 = \bar{\partial} + \nu' : C^\infty(C; N) \to C^\infty(C; N \otimes T^{0,1}C)$$

whose kernel and cokernel are also trivial. The path $t \to D_t$ can be chosen so that:

- The set of t where cokernel $(D_t) \neq \{0\}$ is a finite number, N.
- At such t where cokernel $(D_t) \neq \{0\}$, the dimension of this cokernel is 1.
- At such t where cokernel $(D_t) \neq \{0\}$, the restriction of the t-derivative of D_t to kernel (D_t) composes with projection onto cokernel (D_t) as an isomorphism.

(1.15)

Because D' is $\mathbb{C}$-linear, the set of ν' where kernel $(D') \neq \{0\}$ is a codimension 2 variety in $C^\infty(C; T^{0,1}C)$. This insures that $r(C, 1)$ depends only C and, in particular, not on the details of D's deformation.

If m = 1 and $\mathbf{d_k > 0}$. The integer $r(C, 1) \in \{\pm 1\}$ again. However, the definition in this case requires the choice of an ordering of the elements of Γ_k. As remarked above, the latter serves to orient the vector space V_k. Next, choose a continuous path $t \to D_t$ so that:

- For each $t \in [0, 1]$, D_t is a zero'th order deformation of D.

- $D_0 = D$.
- D_t has trivial cokernel for all t.
- $D_1 = \bar\partial + \nu'$ is $\mathbb{C}$-linear.

(1.16)

With the preceding understood, the association of the kernel (D_t) to $t \in [0,1]$ defines a $2 \cdot d_k$ dimensional, real vector bundle over $[0,\ 1]$. The fiber of this vector bundle over $t = 1$ is complex, so naturally oriented; and the latter induces an orientation of the kernel (D), the fiber over $t = 0$. With this orientation for kernel (D), the linear map G_k is then an isomorphism between two oriented vector spaces. Now define $r(C, 1) = +1$ if G_k preserves orientation, and otherwise define $r(C, 1) = -1$.

Note that $r(C, 1)$ is independent of the choice of the path $\{D_t\}$, but it will change sign if the ordering of Γ_k is changed by a permutation with odd parity.

If $\mathbf{m > 1}$. As noted above, this requires C to be a torus (and N to be topologically trivial.) There are three distinct isomorphism classes of non-trivial real line bundles over C, and by tensoring (over $\mathbb{R}$) the range and domain of D with any one of these, one obtains a suite of 3 new operators. Agree to call any one of these a "twisted version" of D. Note that the index of D and any of its twisted versions is zero. This is because d_k is zero when C is a torus with trivial normal bundle.

With the preceding understood, the value of $r(C, m)$ depends only on the various possibilities for the mod(2) spectral flow for the operator D and its twisted versions. (Once again, the genericity assumptions on J are such as to insure that these spectral flows are well defined.) This is to say, that $r(C, m)$ depends only on the mod 2 spectral flow for D and on the number of D's twisted version which have non-trivial spectral flow. (And, of course, it depends on m.) In this regard, once m is fixed, there are eight possibilities for $r(C, m)$; and it is convenient to label the possibilities with a tag, $\pm k$, where the $\pm$ indicates whether the spectral flow for D is $+1$ or -1, and where $k \in \{0, 1, 2, 3\}$ indicates the number of the twisted versions of D which have non-trivial spectral flow.

For a fixed tag, $\pm k$, it proves convenient to present the data $\{r(C, m)\}_{m=1,2,\ldots}$ with the help of a "generating function", $f_{\pm k}(t)$. This is to say that $f_{\pm k}$ is, by definition, that formal power series for which the coefficient of t^m is $r(C, m)$. This sort of presentation is convenient here only because $f_{\pm k}(t)$ is, in all cases, a fairly simple function of t. Here are the eight generating functions:

- $f_{+0}(t) = \dfrac{1}{1-t}$.
- $f_{+1}(t) = 1 + t$.
- $f_{+2}(t) = \dfrac{1+t}{1+t^2}$.
- $f_{+3}(t) = \dfrac{(1+t)(1-t^2)}{1+t^2}$.
- $f_{-0}(t) = 1 - t$.
- $f_{-1}(t) = \dfrac{1}{1+t}$.
- $f_{-2}(t) = \dfrac{1+t^2}{1+t}$.
- $f_{-3}(t) = \dfrac{1+t^2}{(1+t)(1-t^2)}$.

(1.17)

End the digression.

Part 6. Suppose that $e \in H^2(X;\mathbb{Z})$ has been chosen, and that $d = e \bullet e - c \bullet e \geq 0$. Let $p \in \{0, \dots, d\}$ and

$$\gamma_1 \wedge \cdots \wedge \gamma_{2p} \in \Lambda^{2p}(H_1(X;\mathbb{Z})/\,\text{Torsion}).$$

Fix $(J, \Gamma, \Omega) \in \mathcal{W}$ so that the conclusions of (1.14) hold. Then, let $h = \{(C_k, m_k)\} \in \mathcal{H}$. The preceding step defined an integer weight $r(C_k, m_k)$ for each $(C_k, m_k) \in h$ from the given data and the choice of an ordering on the corresponding Γ_k. The purpose of this step is to use the data $\{r(C_k, m_k)\}$ to define an integer weight, $q(h)$, to the set h. The definition of $w(h)$ is simplest when $p = 0$, whence

$$q(h) = \prod_k r(C_k, m_k). \tag{1.18}$$

In the case where $p > 0$, each $r(C_k, m_k)$ depends on the choice of an ordering for the corresponding Γ_k. This dependence is compensated for in the definition of $q(h)$ as follows: The chosen orderings of the Γ_k

also induce an ordering of Γ which differs from the given labeling by a permutation, σ, of the set $\{1, \ldots, 2p\}$. The latter has a parity, which will be denoted by $\varepsilon(\sigma) \in \{\pm 1\}$. Note that $\varepsilon(\sigma)$ is insensitive to the choice of ordering for $\{(C_k, m_k)\}$ as each Γ_k has an even number of elements.

With the preceding understood, associate the weight

$$q(h) = \varepsilon(\sigma) \cdot \prod_k r(C_k, m_k) \tag{1.19}$$

to each $h \in \mathcal{H}$ in the case when $p > 0$. Note that $q(h)$ in (1.19) is insensitive to the chosen orderings of each of the Γ_k's.

Part 7. Here is the definition of Gr:

Definition 1.4. Define $\mathrm{Gr}\colon H^2(X;\mathbb{Z}) \to \Lambda^* H^1(X;\mathbb{Z})$ as follows: Set Gr(0)= 1. For $e \in H^2(X;\mathbb{Z}) - \{0\}$, set $d = e \bullet e - c \bullet e$. Then

- $\mathrm{Gr}(e) = 0$ if $d < 0$.
- If $d \geq 0$,
  a) Fix $J \in \mathcal{W}$, and use J to define $\mathcal{H} = \mathcal{H}(e, J, \varnothing, \varnothing)$. With $\mathcal{H}$ understood, define the projection of $\mathrm{Gr}(e)$ in the $\mathbb{Z}$ summand of $\Lambda^* H^1(X;\mathbb{Z})$ to equal $\Sigma_{h\in\mathcal{H}} q(h)$ where $q(h)$ is given by (1.18).
  b) Fix $p \in \{1, \ldots, d\}$ and then fix
  $$\gamma_1 \wedge \cdots \wedge \gamma_{2p} \in \Lambda^{2p}(H_1(X;\mathbb{Z})/\,\mathrm{Torsion}).$$
  Then, choose $(J, \Gamma, \Omega) \in \mathcal{W}$ and use this data to define $\mathcal{H} = \mathcal{H}(e, J, \Gamma, \Omega)$. With $\mathcal{H}$ understood, define
  $$\mathrm{Gr}\ (e)(\gamma_1 \wedge \cdots \wedge \gamma_{2p}) \equiv \Sigma_{h\in\mathcal{H}} q(h),$$
  where $q(h)$ is given by (1.19).

The following proposition describes the salient properties of the preceding definition:

Proposition 1.5. *Let (X, ω) be a pair consisting of a smooth, compact, connected 4-manifold X with a symplectic form ω. If $e \in H^2(X;\mathbb{Z})$, then the value of* $\mathrm{Gr}(e)$ *as given in Definition 1.4 is independent of the precise choice for the data (J, Γ, Ω) and thus depends only on the symplectic form ω. Furthermore,* $\mathrm{Gr}(\cdot)$ *is constant if ω is* *changed through a continuous path of symplectic forms. Finally,* Gr *behaves naturally with respect to diffeomorphisms of X in the following sense: Let $\varphi : X \to X$ be a diffeomorphism, and let Gr_ω and $\mathrm{Gr}_{\varphi^*\omega}$ denote the respective Gromov invariants as defined by ω and $\varphi^*\omega$. Then* $\mathrm{Gr}_{\varphi^*\omega}(\varphi^* e) = \varphi^*(\mathrm{Gr}_\omega(e))$.

A proof of this proposition can be found in [26].

2. The proof of Theorem 1

The purpose of this section is to reduce the proofs of Theorems 1 and 2 to a few key propositions. In this regard, take X henceforth to be a compact, connected, oriented 4-manifold with symplectic form ω. When a metric is required for X, it will be assumed implicitly to come from an ω-compatible almost complex structure J via (1.6).

a) Some special cases

The proof that SW = Gr treats certain classes $e \in H^2(X;\mathbb{Z})$ as special cases. This subsection constitutes a digression of sorts to handle these special cases.

The first of the special cases concerns the class $0 \in H^2(X;\mathbb{Z})$.

Proposition 2.1. *Let X be a compact, oriented, symplectic 4-manifold. Then* SW(0) *and* Gr(0) *are both* +1.

Proof of Proposition 2.1. First, Gr(0) = 1 by definition. For SW, Proposition 4.1 and Theorem 1.3 of [27] imply that there is a unique gauge orbit of solution to the $e = 0$, $\mu_0 = 0$ and $r \gg 1$ version (2.4). (See also [29].) Section 7a proves that its sign is +1.

The next proposition considers the remaining special cases. The statement of this proposition requires the introduction of the set $S \in H^2(X;\mathbb{Z})$ of classes with square -1 which can be represented by the Poincaré dual of an embedded, symplectic 2-sphere.

Proposition 2.2. *Let X be a compact, oriented, symplectic 4-manifold with $b^{2+} > 1$. Then* SW(e) *and* Gr(e) *vanish unless $s \bullet e \in \{-1, 0\}$ for all $s \in S$.*

Proof of Proposition 2.2. Consider first the case for Gr. (Note that the argument for Gr does not require the $b^{2+} > 1$ assumption.) To begin, suppose that $s \in S$. Then it is a consequence of Gromov's work [5] that s can be represented by an embedded, pseudo-holomorphic 2-sphere. (According to the main theorem in [27], any embedded sphere with self intersection number -1 in a $b^{2+} > 1$, compact, symplectic 4-manifold is homologous to a psuedo-holomorphic sphere.) Now, suppose $e \in H^2(X;\mathbb{Z})$. Write $e = \Sigma_k m_k \cdot e_k$, where each class e_k is represented by the Poincaré dual of a pseudo-holomorphic submanifold, and $e_k \bullet e_{k'} = 0$ unless $k = k'$. Suppose also that $m_k = 1$ whenever $e_k \in S$. Since pseudo-holomorphic submanifolds intersect with locally positive intersection number (see, e.g. [13]), it follows that $s \bullet e_k \geq 0$ unless $e_k = s$. $s \bullet s = -1$, because at most one such e_k can equal s and therefore $s \bullet e \geq -1$. And, any other e_k's must have zero intersection number with s by assumption. Thus, $s \bullet e \in \{-1, 0\}$.

The case for SW follows from a "blow up" formula for SW which describes the Seiberg-Witten invariant for a connect sum $Y \# \underline{\mathbb{CP}^2}$ in terms of those for Y. This formula is reproduced in [3]. To apply this blow-up formula, remark first that if a class $s \in H^2(X;\mathbb{Z})$ is represented by an embedded, pseudo-holomorphic sphere of square -1, then X is a connect sum, $Y \# \underline{\mathbb{CP}^2}$, where Y is a symplectic manifold, and s is Poincaré dual to the image in X of the generator of $H_2(\underline{\mathbb{CP}^2})$. The blow-up formula in [3] can now be used to prove the desired vanishing theorem for SW(e) given the apriori knowledge that the manifold Y has simple type in the sense that the Y version of SW vanishes on any class e' where $d(e') = e' \bullet e' - c_Y \cdot e' > 0$. The latter assertion is proved as Assertion 6 of Theorem 0.1 in [27] for the original version SW($\cdot$) which maps $H^2(X;\mathbb{Z})$ into $\mathbb{Z}$. However, the same argument (in Section 7c of [27]) also proves that the extended SW($\cdot$) $\in \Lambda^* H^1(X;\mathbb{Z})$ vanishes on classes e' with $d(e') > 0$. (The preceding argument uses the $b^{2+} > 1$ assumption in the proof of the blow-up formula, and in an application of the adjunction formula from [9] and [16].)

b) The Seiberg-Witten equations with parameter r

The proof that SW = Gr procedes from here through a step by step reinterpretation of the defining formula (1.5) for the Seiberg-Witten invariant. This process starts in this subsection where the Seiberg-Witten equations are rewritten in a convenient form.

It turns out that there is a useful way to rewrite (1.1) on a symplectic manifold X which exploits the decomposition in (1.9). This rewriting of (1.1) requires a preliminary, two part digression. Part 1 of the digression observes that the bundle K^{-1} comes equipped with a canonical connection (up to the action of $C^\infty(X;U(1))$ (see, e.g. [29]). To define this canonical connection, remember first that for any fixed Spin$^{\mathbb{C}}$ structure, the choice of a connection on $\det(S_+)$ and the Levi-Civita connection on the bundle Fr defines a connection on the Spin$^{\mathbb{C}}$ lift F. Thus, the choice of a connection (say A) on $\det(S_+)$ gives a covariant derivative, ∇_A, on sections of S_+. Now consider the canonical Spin$^{\mathbb{C}}$ structure in (1.7). Restriction of ∇_A to a section of the trivial summand $\mathbb{I}$ and projection of the resulting covariant derivative onto $\mathbb{I} \otimes T^*X$ define a covariant derivative ∇_A on the trivial complex line bundle. With the preceding understood, remark that there is a unique choice of connection A_0 (up to the afore-mentioned gauge equivalence) on $\det(S_+) = K^{-1}$ for which the corresponding covariant derivative on the trivial line bundle admits a non-trivial, covariantly constant section.

For Part 2 of the digression, consider the general Spin$^{\mathbb{C}}$ structure in (1.7). Since $\det(S_+) = E^2 \otimes K^{-1}$, the choice of the connection A_0 on K^{-1} allows any connection A on $\det(S_+)$ to be written uniquely as

$$A = A_0 + 2a, \tag{2.1}$$

where a is a connection on the complex line bundle E. Thus, with A_0 chosen, the Seiberg-Witten equations in (1.1) can be thought of as equations for a pair (a, ψ), where a is a connection on E, and ψ is a section of S_+ in (1.9).

End the digression. With this reinterpretation of (1.1) understood, note now that it proves useful to "renormalize" the form μ in (1.1) by writing

$$\mu = -\frac{ir}{4}\omega + P_+F_{A_0} + i\mu_0. \tag{2.2}$$

Here, r can be any non-negative number and μ_0 can be any section of Λ_+. (In practice, think of μ_0 as being close to 0.) Furthermore, in the case where $r > 0$, it also proves useful to write

$$\psi = r^{1/2}(\alpha, \beta) \tag{2.3}$$

to correspond with the splitting in (1.9). Then, with the preceding understood, the Seiberg-Witten equations in (1.1) read

$$\begin{aligned} &D_A(\alpha, \beta) = 0, \\ &P_+F_a + \frac{ir}{8}(1 - |\alpha|^2 + |\beta|^2)\omega - \frac{r}{4}(\alpha\beta^* - \alpha^*\beta) - i\mu_0 = 0. \end{aligned} \tag{2.4}$$

Here, $\alpha\beta^*$ and $\alpha^*\beta$ are sections of K and K^{-1}, where the latter are naturally identified as the orthogonal compliment of the span of ω in

$\Lambda_+ \otimes \mathbb{C}$. Note that this last equation differs from the analogous equations in [25], [27] and [29] in that the β used here is $-i$ times that used in the previous papers. The insertion of this factor of $-i$ here avoids numerous factors of i later on.

Rewriting (1.1) as in (2.4) realizes the Seiberg-Witten equations as equations for data $(a, (\alpha, \beta)) \in \mathrm{Conn}(E) \times C^\infty(S_+)$, where $\mathrm{Conn}(E)$ is the space of smooth connections on the complex line bundle E. With the preceding understood, introduce, for $r > 0$, the moduli space

$$\mathcal{M}^{(r)} \subset (\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1)$$

of equivalence classes of solutions to (2.4) for some previously chosen form μ_0.

c) The appearance of G and Ω in the count for SW

The purpose of this subsection is to reinterpret the integral in (1.5) for the $d > 0$ case of SW as a weighted count of certain preferred elements of the space $\mathcal{M}^{(r)}$. This reinterpretation of SW explains how Γ and Ω, which appear in the definition of Gr, enter the definition of SW.

To reinterpret (1.5), first choose a set $\Gamma = \{\gamma_j\}_{1 \le j \le 2p}$ of embedded, oriented, 1-dimensional submanifolds of X as in the definition of Gr in Section 1e. Remember that this set is to be pairwise disjoint, and that for each j, the corresponding γ_j generates the class of the same name which appears in (1.5). With Γ chosen, select a set $\Omega \subset (X - \cup_j \gamma_j)$ of $d - p$ distinct points.

With Γ and Ω chosen, let

$$\mathcal{M}^{(r)}_{\Gamma,\Omega} = \{(a, (\alpha, \beta)) \in \mathcal{M}^{(r)} : \alpha^{-1}(0) \text{ intersects each point in } \Omega \text{ and each } \gamma \in \Gamma\}. \tag{2.5}$$

Under favorable circumstances, the number in (1.5) is obtained from $\mathcal{M}^{(r)}_{\Gamma,\Omega}$ by summing certain signs (± 1) which are associated to its points. Needless to say, such a computational scheme requires some regularity from $\mathcal{M}^{(r)}_{\Gamma,\Omega}$. The precise statement of these requirements uses a certain family of differential operator that points in $\mathrm{Conn}(E) \times C^\infty(S_+)$ parameterize. The following digression serves to introduce this family of operators.

The operator associated to a given $\Xi = (a, (\alpha, \beta))$ will be denoted by L_Ξ or, simply, L when there is minimal chance of confusion. This operator L_Ξ maps $i\cdot\Omega^1\oplus C^\infty(S_+)$ to $i\cdot(\Omega^0\oplus\Omega^{2+})\oplus C^\infty(S_-)$ by sending $(a',(\alpha',\beta'))$ to the element whose three components are:

- $*d*a'+i\cdot\frac{\sqrt{r}}{\sqrt{2}}im(\bar\alpha\alpha'+\bar\beta\beta')$
- $P_+da'-i\frac{\sqrt{r}}{2\sqrt{2}}re(\bar\alpha\alpha'-\bar\beta\beta')\cdot\omega+\frac{\sqrt{r}}{\sqrt{2}}(\bar\alpha\beta'+\bar\alpha'\beta-\alpha\bar\beta'-\alpha'\bar\beta)$
- $\bar\partial_a\alpha'-(\bar\partial_{A_0+a})^*\beta'+\frac{\sqrt{r}}{2\sqrt{2}}\alpha a'_{01}+\frac{\sqrt{r}}{2\sqrt{2}}\beta\overline{a'_{01}}$.

(2.6)

Here, $\bar\partial_a$ is the projection of the covariant derivative onto $T^{0,1}X$, and $(\bar\partial_{A_0+a})^*$ is the formal L^2-adjoint of the projection of the covariant derivative onto $\Lambda^2T^{0,1}X=K^{-1}$. Also, $a'_{0,1}$ is the projection of a' onto $T^{0,1}X$. Note that the assignment of Ξ to L_Ξ is naturally equivariant with respect to the action of $C^\infty(X;S^1)$ on the spaces involved. This is to say that when the latter group is allowed to act on $C^\infty(S_\pm)$ by multiplication, on $i\cdot\Omega^{1,0,2+}$ trivially and on $\mathrm{Conn}(E)\times C^\infty(S_+)$ as defined earlier, then $L_{\phi\cdot\Xi}(\phi\cdot\xi)=\phi\cdot L_\Xi\xi$.

As asserted in Proposition 6.2 of [28], the space $\mathcal{M}^{(r)}$ (as defined with some previously chosen form μ_0 in (2.4)) has a natural smooth manifold structure near those points Ξ where cokernel $L_\Xi=\{0\}$. A point Ξ where L_Ξ has such a trivial cokernel will be called a smooth point. Note that $\mathcal{M}^{(r)}$ can be assumed to consist solely of smooth points if μ_0 is chosen from a suitable Baire set.

End the digression.

Here are the precise requirements for $\mathcal{M}^{(r)}_{\Gamma,\Omega}$:

- $\mathcal{M}^{(r)}_{\Gamma,\Omega}$ is a finite set of points; and each such point is a smooth point of $\mathcal{M}^{(r)}$. (There is no need to assume that $\mathcal{M}^{(r)}$ consists solely of smooth points.)
- Let $(a,(\alpha,\beta))\in\mathcal{M}^{(r)}_{\Gamma,\Omega}$ and let $\gamma\in\Gamma$. Then $\alpha^{-1}(0)\cap\gamma$ is a single point, q, and the covariant derivative of α along γ at q is non-zero.
- Let $(a,(\alpha,\beta))\in\mathcal{M}^{(r)}_{\Gamma,\Omega}$. Define

  $$G:\mathrm{kernel}(L)\to(\oplus_{x\in\Omega}E|_x)\bigoplus(\oplus_{\gamma\in\Gamma}(E|_q/\nabla\alpha(T\gamma|_q)))$$

  as follows: Assign to $(a',(\alpha',\beta'))$ in kernel(L) the vector whose component in $E|_x$ is $\alpha'(x)$ and whose component in $E|_q/\alpha\nabla(T\gamma|_q)$ is the projection of $\alpha'(q)$. Then require that G be an isomorphism.

(2.7)

Say that $\mathcal{M}^{(r)}_{\Gamma,\Omega}$ is regular when (2.7) holds. In general, there is no apriori reason for $\mathcal{M}^{(r)}_{\Gamma,\Omega}$ to be regular. However, with Γ and Ω fixed, there will be an open and dense set in $\Omega^{2+}(X)$ of choices of μ_0 for which the resulting $\mathcal{M}^{(r)}_{\Gamma,\Omega}$ is regular.

When $\mathcal{M}^{(r)}_{\Gamma,\Omega}$ is regular, then a sign can be associated to each of its points. The definition of this sign involves the following three steps:

Step 1. The bundle E, being complex, is naturally oriented. Also, γ is, by assumption, oriented, so $E/\nabla\alpha(T\gamma)$ is oriented at the point where g intersects $\alpha^{-1}(0)$.

Step 2. With the preceding understood, the bundle

$$V = (\oplus_{x\in\Omega} E|_x) \bigoplus (\oplus_{\gamma\in\Gamma}(E|_q/\nabla\alpha(T\gamma|_q))) \tag{2.8}$$

is oriented with an ordering (up to even permutations) of the set $\{\gamma_j\}_{1\leq j\leq 2p}$. The latter is ordered as $\{\gamma_1, \gamma_2, \dots, \gamma_{2p}\}$.

Step 3. The endomorphism G in (2.7) maps the oriented space kernel(L) to the oriented space V. Assign $+1$ to $(a, (\alpha, \beta)) \in \mathcal{M}^{(r)}_{\Gamma,\Omega}$ when G is orientation preserving. Otherwise, assign -1 to $(a, (\alpha, \beta))$.

Given the preceding, consider:

Proposition 2.3. *Suppose that $d > 0$ and that Γ and Ω are defined as above. Suppose, as well, that $\mathcal{M}^{(r)}_{\Gamma,\Omega}$ is regular. Then the integeral in (1.5) is equal to the sum of the numbers (± 1) which are assigned to the points of $\mathcal{M}^{(r)}_{\Gamma,\Omega}$.*

Proof of Proposition 2.3. The basis for this reinterpretation rests on the observation that the assignment to a point

$$((a, (\alpha, \beta)), x) \in \mathcal{M}^{(r)} \times X$$

of $\alpha(x)$ defines a section, s, of the complex line bundle $\mathcal{E}$ of equation (1.2). Let $\underline{s}$ be a small perturbation of s which vanishes transversely. Then, this zero set, $\underline{s}^{-1}(0) \subset \mathcal{M}^{(r)} \times X$, is a smooth, codimension 2 submanifold with a natural orientation. The latter endows $\underline{s}^{-1}(0)$ with a fundamental class which represents the Poincaré dual to the first Chern class c of $\mathcal{E}$. With this understood, the assertion in Proposition 2.3 follows in a straightforward manner by reinterpreting cup products of the cohomology classes in terms of intersections of submanifolds which represent their Poincaré duals.

d) The appearance of pseudo-holomorphic curves in the count for SW

A fundamental input to the proof of Gr = SW is the fact that solutions to (2.4) for large r (and $\mu_0 = 0$) determine pseudo-holomorphic curves in X with fundamental class Poincaré dual to e. Here is the precise statement:

Proposition 2.4. *Let X be a compact, oriented, symplectic manifold. Fix an ω-compatible almost complex structure on X, and use the resulting metric to define the Seiberg-Witten equations. Fix $e \in H^2(X;\mathbb{Z})$ and use e to define the Spin$^{\mathbb{C}}$ structure in (1.9). Also, fix a finite (maybe empty) collection $\{\varsigma_k\}$ of closed subsets of X. Given $\varepsilon > 0$, and then given r sufficiently large, the following is true: If $(a, (\alpha, \beta)) \in \mathcal{M}^{(r)}$ has $\alpha^{-1}(0)$ intersect each ς_k, then there is a compact (not necessarily connected), complex curve C with a pseudo-holomorphic map $\varphi : C \to X$ with*

1. *$\varphi_*[C]$ equal to the Poincaré dual of e;*
2. *$\varphi(C) \cap \varsigma_k \neq 0$ for each k;*
3. *$\sup_{x:\alpha(x)=0} \operatorname{dist}(x, \varphi(C)) + \sup_{x\in\varphi(C)} \operatorname{dist}(x, \alpha^{-1}(0)) < \varepsilon$.*

Proof of Proposition 2.4. This follows immediately from Theorem 1.3 in [27].

Needless to say, Proposition 2.4 plays a premier role in the proof that SW = Gr.

The next result considers the images of the curves C which appear in the previous proposition for certain special choices of the set $\{\varsigma_k\}$. However, the statement of the proposition requires the introduction of the set $S \subset H^2(X;\mathbb{Z})$ whose elements have square -1 and can be represented as the Poincaré dual of a symplectically embedded 2-sphere. The statement of the proposition also reintroduces the set $\mathcal{W}$ of (1.14).

Proposition 2.5. *Let X be a compact, oriented, symplectic manifold. Fix an ω-compatible almost complex structure on X, and use the resulting metric to define the Seiberg-Witten equations. Fix a class $e \in H^2(X;\mathbb{Z})$ to define the Spin$^{\mathbb{C}}$ structure in (1.9) with the property that $s \bullet e \geq -1$ for all elements $s \in S$. Let $d = 2^{-1}(e \bullet e - c \bullet e)$ and when $d > 0$, fix $p \in \{0, \ldots, d\}$. When $p > 0$, fix an ordered set $\underline{\Gamma} \subset H_1(X;\mathbb{Z})$ of $2p$ classes. Then, there exists a Baire set of data $(J, \Gamma, \Omega) \in \mathcal{W}$ where:*

- *J is a smooth, ω-compatible almost complex structure on X and*
  a) *If $d > 0$ and $p > 0$, then $\Gamma = \{\gamma_1, \ldots, \gamma_{2p}\}$ is a set of $2p$ pair-wise disjoint, embedded loops in X which generate the elements in $\underline{\Gamma}$. Otherwise, $\Gamma = \varnothing$.*
  b) *If $d - p > 0$, then Ω is a set of distinct, $d - p$ points in X which miss all loops in Γ. Otherwise, $\Omega = \varnothing$.*

- *Use J to define the metric in (1.6). For $r > 0$, let $\mathcal{M}^{(r)}$ denote here the μ_0 version of the moduli space of solutions to (2.4) for the Spin$^{\mathbb{C}}$ structure in (1.9). Given $\varepsilon > 0$, then for all sufficiently large r,*
  a) *$\mathcal{M}^{(r)} = \varnothing$ if $d < 0$.*
  b) *If $d = 0$, let $(a, (\alpha, \beta)) \in \mathcal{M}^{(r)}$; and if $d > 0$ suppose that $(a, (\alpha, \beta)) \in \mathcal{M}^{(r)}_{\Gamma,\Omega}$. In either case, there exists*

$$h = \{(C_k, m_k)\} \in \mathcal{H} = \mathcal{H}(e, J, \Gamma, \Omega)$$

such that

$$\sup_{x:\alpha(x)=0} \operatorname{dist}(x, \cup_k C_k) + \sup_{x\in\cup_k C_k} \operatorname{dist}(x, \alpha^{-1}(0)) < \varepsilon. \tag{2.9}$$

Proof of Proposition 2.5. Suppose that there exists a class $e \in H^2(X; \mathbb{Z})$ as described, plus a triple $(J, \Gamma, \Omega) \in \mathcal{W}$ and some $\varepsilon > 0$ such that the conclusions of the proposition failed to hold. The argument below will show that (J, Γ, Ω) lies in the compliment of a certain Baire set. There are five steps to the argument.

Step 1. Under the assumptions, one can find an increasing, unbounded sequence $\{r_n\}$ of positive numbers and, with $r = r_n$, a point $(a_n, (\alpha_n, \beta_n)) \in \mathcal{M}^{(r)}$ which violates (2.9). On the other hand, it follows from Theorem 1.3 in [27] that there is a compact, complex curve C with a pseudo-holomorphic map $\varphi : C \to X$ which obeys:

- $\varphi_\bullet[C]$ is Poincaré dual to e.
- $\varphi(C)$ intersects each point in Ω and each loop in Γ.
- $\lim_{n\to\infty}[\sup_{x:\alpha_n(x)=0} \operatorname{dist}(x, \varphi(C)) + \sup_{x\in\varphi(C)} \operatorname{dist}(x, \alpha_n^{-1}(0))] = 0.$

(2.10)

The goal in the subsequent steps will be to prove that $\varphi(C) = \cup_k C_k$ if (J, Γ, Ω) have been chosen in a suitably generic fashion.

Step 2. Consider the set $\varphi(C)$ in (2.10). The compliment of a finite set of points, Λ, in $\varphi(C)$ is an embedded, pseudo-holomorphic submanifold. (See, e.g [33], [19].) Let $\Sigma^0 = \varphi(C) - \Lambda$. Then Σ^0 is the compliment of a finite set of points in some compact, complex curve, Σ. Furthermore, the tautological embedding of Σ^0 in X extends as a pseudo-holomorphic map from, $\psi : \Sigma \to X$.

With the preceding understood, let $\{\Sigma_k\}$ denote the components of Σ, and, for each k, let e_k denote the Poincaré dual to $\psi_*[\Sigma_k]$. Note that $e = \Sigma_k m_k \cdot e_k$ where the m_k are positive integers.

Consider now the possibilities for the set $\{\Sigma_k\}$. The first observation stems from the characterization of ψ given in [33], [19] and [14]. In particular, the map ψ can be perturbed to a map ψ' which is an immersion, with locally positive intersection numbers, and which pulls back ω as a strictly positive form. Let n_k denote the number of double points of ψ''s restriction to Σ_k. Then the genus of Σ_k is $1 + 2^{-1} \cdot (e_k \bullet e_k - n_k + c \bullet e_k)$. Furthermore, for generic J, there will be no pseudo-holomorphic maps which represent e_k from a surface of this genus unless $d_k = 2^{-1} \cdot (e_k \bullet e_k - n_k - c \bullet e_k) \geq 0$. (This can be proved using the analysis for the proof of Proposition 5.2 in [26].)

Thus, if J is chosen from a suitable Baire set, one can suppose without loss of generality that $d_k \geq 0$ for each k.

Step 3. Next, note that the analysis for the proof of Proposition 5.2 in [26] can also be used to prove that there is a Baire set of choices for (J, Γ, Ω) for which the following conditions hold:

- For each k, there exist $p_k \in \{0, \ldots, d_k\}$ and a subset $\Gamma_k \subset \Gamma$ such that Σ_k interects each member of Γ_k exactly once, and no member of $\Gamma - \Gamma_k$.
- Each Σ_k contains exactly $d_k - p_k$ points of Ω.
- Each Σ_k is immersed by ψ.
- When Σ_k intersects a member $\gamma \in \Gamma_k$, the tangent line to γ is not in $T\Sigma_k$.

(2.11)

Note that the first two lines of (2.11) imply that $d = 2^{-1}\cdot(e\bullet e - c\bullet e)$ equals $\Sigma_k d_k$.

Step 4. Agree now to change notation and use $\{s_\sigma\}$ to label those e_k for which the corresponding Σ_k is a sphere with square -1. Use $\{t_k\}$ to label the remaining classes. Then

$$(2.12)\quad \begin{aligned} d = \Sigma_k d_k = \Sigma_k m_k(d_k + n_k) &+ 2^{-1}\Sigma_k m_k(m_k - 1)\cdot t_k \bullet t_k \\ &+ \Sigma_{k\neq k'} m_k m_{k'} t_k \bullet t_{k'} + \Sigma_{k,\sigma} m_k m_\sigma t_k \bullet s_\sigma \\ &+ 2^{-1}\Sigma_\sigma(m_\sigma - m_\sigma^2). \end{aligned}$$

To view the ramifications of (2.12), introduce $e\bullet s_\sigma = \Sigma_k m_k t_k \bullet s_\sigma - m_\sigma$. Then,

$$(2.13)\quad 2^{-1}(m_\sigma - m_\sigma^2) = 2^{-1} m_\sigma(1 + e\bullet s_\sigma) - 2^{-1}\Sigma_k m_k m_\sigma t_k \bullet s_\sigma,$$

and it follows from (2.12) that

$$(2.14)\quad \begin{aligned} \Sigma_k d_k = \Sigma_k m_k(d_k + n_k) &+ 2^{-1}\cdot\Sigma_k m_k(m_k - 1)\cdot t_k \bullet t_k \\ &+ \Sigma_{k\neq k'} m_k m_{k'} t_k \bullet t_{k'} + 2^{-1}\cdot\Sigma_k m_k m_\sigma t_k \bullet s_\sigma \\ &+ 2^{-1}\cdot\Sigma_\sigma m_\sigma(1 + e\bullet s_\sigma). \end{aligned}$$

In the case where $e\bullet s_\sigma \geq -1$, all terms on the right side of (2.14) are positive and so the equality holds if and only if:

- $n_k = 0$.
- $m_k = 1$ unless $d_k = t_k \bullet t_k = 0$.
- $t_k \bullet t_{k'} = 0$ unless $k = k'$.
- $t_k \bullet s_\alpha = 0$.

(2.15)

Step 5. Given the constraints in the last two lines of (2.11) and in (2.15), it follows that $\{(\Sigma_k, m_k)\} \in \mathcal{H}$ when the triple (J, Ω, Γ) are chosen from a Baire subset of elements in $\mathcal{W}$. But, this conclusion contradicts the assumed violation of (2.5) by the original sequence $\{(a_n, (\alpha_n, \beta_n))$, and thus verifies the proposition.

e) The appearance of SW solutions in the count for Gr

While the points of the $\mu_0 = 0$ version of $\mathcal{M}^{(r)}_{\Gamma,\Omega}$ determine elements in $\mathcal{H}$, the points in the latter set also determine elements in this same $\mu_0 = 0$ version of $\mathcal{M}^{(r)}_{\Gamma,\Omega}$. Indeed, each point $h \in \mathcal{H}$ will typically determine a subset $\Phi(h) \in \mathcal{M}^{(r)}_{\Gamma,\Omega}$ when r is large. The set in question comes from Proposition 5.2 in [28] and is described in more detail below in (2.20), (2.21) and (2.22) plus Proposition 2.10.

By the way, under certain circumstances, $\Phi(h)$ will contain precisely one element, in which case the assignment $\Phi(\cdot)$ defines a map from $\mathcal{H}$ to $\mathcal{M}^{(r)}_{\Gamma,\Omega}$ when r is large. In particular, consider:

Proposition 2.6. *Let X be a compact, oriented, symplectic manifold. Fix an ω-compatible almost complex structure on X, and use the resulting metric to define the Seiberg-Witten equations. Fix $e \in H^2(X;\mathbb{Z})$ and use e to define the Spin$^{\mathbb{C}}$ structure in* (1.9). *Assume that $s \bullet e \geq -1$ for all elements $s \in S$. Let $d = 2^{-1} \cdot (e \bullet e - c \bullet e)$ and when $d > 0$, fix $p \in \{0, \ldots, d\}$. When $p > 0$, fix an ordered set $\underline{\Gamma} \subset H_1(X;\mathbb{Z})$ of $2p$ classes. Suppose that there exists a triple (J, Γ, Ω) in the Baire subset of Proposition* 2.5 *where the corresponding set $\mathcal{H}$ contains only elements of the form $\{(C_k, m_k = 1)\}$. Let $\mathcal{M}^{(r)}_{\Gamma,\Omega}$ denote the $r > 0$ and $\mu_0 = 0$ version of* (2.5). *When r is large, then*

- *$\mathcal{M}^{(r)}_{\Gamma,\Omega}$ satisfies the conditions in* (2.7).
- *There is a $1-1$ map $\Phi^{(r)} : \mathcal{H} \to \mathcal{M}^{(r)}_{\Gamma,\Omega}$.*
- *The map $\Phi^{(r)}$ is also onto.*
- *When $(a, (\alpha, \beta)) = \Phi^{(r)}(\{(C_k, 1)\})$, then $\alpha^{-1}(0)$ is an embedded, symplectic submanifold of X which is isotopic to $\cup_k C_k$ and obeys*

$$\sup_{x:\alpha(x)=0} \operatorname{dist}(x, \cup_k C_k) + \sup_{x \in \cup_k C_k} \operatorname{dist}(x, \alpha^{-1}(0)) < \zeta \cdot r^{-1/2},$$

where ζ is independent of r.

(2.16)

The first three assertions of the proposition constitute a special case of Proposition 2.10, below. However, note here that the existence of a 1-1 map $\Phi^{(r)}$ which obeys (2.16) is a direct consequence of Propositions 5.2, 6.1 and 6.3 in [28]. The bulk of the remaining arguments for the first three assertions concerns the assertion that $\Phi^{(r)}$ is *onto* $\mathcal{M}^{(r)}_{\Gamma,\Omega}$. (These arguments are given here in Sections 5 and 6.) Equation (2.16) is also asserted in Proposition 2.10, but follows directly from Proposition 2.5. The assertion that $\alpha^{-1}(0)$ is symplectic follows from the analysis in Section 4 of [27].

Consider the nature of the assignment $\Phi(\cdot)$ in the cases where Proposition 2.6's assumptions fail. The discussion of this more general case is broken into six parts.

Part 1. Let C be a complex curve, let $N \to C$ be a holomorphic line bundle with a hermitian metric and let m be a positive integer. Then, let ν be a section over C of $T^{0,1}C$ and let μ be one of $N^2 \otimes T^{0,1}C$. Note that this data appears for free whenever (C, m) comes as an element of some h in some $\mathcal{H}$. Indeed, in this case, take N to be the normal bundle to C and take (ν, μ) as in (1.11). (A precise definition of (ν, μ) in this case is given in Section 2a of [28].)

Now, introduce the subspace $\mathcal{Z}_0 \subset C^\infty(\oplus_{1\leq q\leq m} N^q)$ as in Section 3 and Proposition 3.2 of [28]. In this regard, recall that the elements of $\mathcal{Z}_0$ can be viewed as solutions of a certain non-linear, ellipitic equation which has the schematic form

$$\bar{\partial} y + \nu \aleph \cdot y + \mu \mathbb{F}(y) = 0. \tag{2.17}$$

Here, $\aleph$ acts as multiplication by q on the N^q summand of $\oplus_{1\leq q\leq m} N^q$; and $\mathbb{F}$ is a certain smooth, fiber preserving map from $\oplus_{1\leq q\leq m} N^q$ to $\oplus_{1\leq q\leq m} N^{q-2}$. (Note that $\mathbb{F}$ is not holomorphic in the fiber coordinates.)

In the case where $m = 1$, the map $\mathbb{F}$ sends y to its complex conjugate and so in the $m = 1$ case, $\mathcal{Z}_0 = \text{kernel}(D)$. In the case where $m > 1$, the map $\mathbb{F}$ is not $\mathbb{R}$-linear, as can be seen from the fact that $\mathbb{F}(0) \neq 0$. (See Proposition 3.4 in [28].) In particular, in the $m > 0$ case, the author has no explicit description of the set $\mathcal{Z}_0$.

Part 2. Fortunately, an explicit picture of $\mathcal{Z}_0$ in the $m > 1$ case is not required here. Indeed, over and above that which is established in [28], the proof that SW = Gr requires only a certain compactness criteria for $\mathcal{Z}_0$. The statement of this criteria requires the introduction of a certain variant of the operator in (1.11). The variant is defined on any holomorphic curve C' which comes equipped with a holomorphic map f to the given curve C. The variant of (1.11) on C' will be denoted by D' and it is defined as in (1.11) but with C' replacing C, $f^*\nu$ replacing ν, and $f^*\mu$ replacing μ.

Proposition 2.7. *Suppose that C is a complex torus and suppose that N is a topologically trivial, holomorphic line bundle over C. Fix a* *pair* (ν, μ) *as above and define the corresponding set*

$$\mathcal{Z}_0 \subset C^\infty(\oplus_{1\leq q\leq m} N^q)$$

as in Proposition 3.2 *of* [28]. *Then* $\mathcal{Z}_0$ *is compact when the following is true: The operator* D' *corresponding to any connected, holomorphic covering* $f : C' \to C$ *of degree* $m' \leq m$ *has trivial kernel and cokernel.*

(This proposition is proved in Section 3) The preceding proposition plus Lemma 5.4 in [26] have the following corollary:

Proposition 2.8. *Fix a class* $e \in H^2(X; \mathbb{Z})$ *and let*

$$d = 2^{-1} \cdot (e \bullet e - c \bullet e).$$

Assume that $d \geq 0$ *and when* $d > 0$, *fix* $p \in \{0, \dots, d\}$. *When* $p > 0$, *fix an ordered set* $\underline{\Gamma} \subset H_1(X; \mathbb{Z})$ *of* $2p$ *classes. There is a Baire subset of the set* $\mathcal{W}$ *in Proposition* 2.5 *which has the following additional property: If* (J, Γ, Ω) *is taken from this set, then the corresponding set* $\mathcal{H}$ *contains only elements of the form* $\{(C_k, m_k)\}$ *where the corresponding* $\mathcal{Z}_0$ *for the* $m_k > 1$ *pairs is compact. Indeed, when* $m_k > 1$, *then the operator* D' *corresponding to any connected, holomorphic covering* $f : C' \to C_k$ *of degree* m_k *or less has trivial cokernel.*

Part 3. Part 1, above, introduced the space $\mathcal{Z}_0$. For each $y \in \mathcal{Z}_0$, introduce the linear operator

$$\Delta_y : C^\infty(\oplus_{1\leq q\leq m} N^q) \to C^\infty((\oplus_{1\leq q\leq m} N^q) \otimes T^{0,1}C)$$

which sends y' to

$$\Delta_y \cdot y' = \bar{\partial} y' + \nu \aleph \cdot y' + \mu \mathbb{F}_{*y} \cdot y', \tag{2.18}$$

where $\mathbb{F}_{*y}$ denotes the differential of $\mathbb{F}$ at y. Say that $\mathcal{Z}_0$ is regular when Δ_y has trivial cokernel at each $y \in \mathcal{Z}_0$. When $\mathcal{Z}_0$ is regular and also compact, then it consists of a finite set of points. (See Proposition 3.2 in [28].)

With the notion of "regular" understood, the next proposition generalizes Proposition 2.6. However, before stating the proposition, introduce the following notation: First, when $m_k > 1$ and (C_k, m_k) comes from some $h \in \mathcal{H}$, use $\mathcal{Z}_0^{(k)}$ to denote the (C_k, m_k) version of $\mathcal{Z}_0$. Second, when $h = \{(C_k, m_k)\} \in \mathcal{H}$, let Y_h denote the one point space when all $m_k = 1$, and otherwise set $Y_h = \times_{k:m_k>1} \mathcal{Z}_0^{(k)}$.

Proposition 2.9. *Fix a class $e \in H^2(X;\mathbb{Z})$ with the property that $s \bullet e \geq -1$ for all elements $s \in S$. Let $d = 2^{-1} \cdot (e \bullet e - c \bullet e)$ and when $d > 0$, fix $p \in \{0, \ldots, d\}$. When $p > 0$, fix an ordered set $\underline{\Gamma} \subset H_1(X;\mathbb{Z})$ of $2p$ classes. Suppose that there exists a triple (J, Γ, Ω) in the Baire subset of Propositions 2.5 and 2.8 where the corresponding set h has the following property: Each $\mathcal{Z}_0^{(k)}$ corresponding to each $m_k > 1$ pair (C_k, m_k) from any $h = \mathcal{H}$ is regular. Let $\mathcal{M}^{(r)}_{\Gamma,\Omega}$ denote the $r > 0$ and $\mu_0 = 0$ version of (2.5). When r is large, then:*

- *$\mathcal{M}^{(r)}_{\Gamma,\Omega}$ satisfies the conditions in (2.7).*
- *There is a $1-1$ map $\Phi^{(r)} : \cup_{h\in\mathcal{H}} Y_h \to \mathcal{M}^{(r)}_{\Gamma,\Omega}$.*
- *The map $\Phi^{(r)}$ is also onto.*
- *When $(a, (\alpha, \beta)) = \Phi^{(r)}(\{(C_k, m_k)\})$, then $\alpha^{-1}(0)$ obeys (2.16).*

This proposition is also a special case of Proposition 2.10. As with Proposition 2.6, all but the third point follow more or less directly from Propositions 5.2, 6.1 and 6.3 plus Lemma 6.6 in [28] so the brunt of the proof focuses on the third point.

Part 4. Unfortunately, the author knows no way to guarantee that the assumptions of Proposition 2.9 hold. However, in the case where some $\mathcal{Z}_0^{(k)}$ is not regular, one can still proceed, although the discussion is somewhat more complicated.

To begin, take (J, Γ, Ω) from the Baire set in Propositions 2.5 and 2.8. Take $h = \{(C_k, m_k)\}$ from the corresponding set $\mathcal{H}$. The discussion in this Part 4 describes a certain finite dimensional manifold, $\mathcal{K}^{(k)}_\Lambda$, which can be associated to each pair (C_k, m_k).

In the case where $m_k = 1$, set $\Lambda (\equiv \Lambda_k) = \{0\}$ and let $\mathcal{K}^{(k)}_\Lambda$ denote an open neighborhood of the origin in the kernel of the (C_k, m_k) version of the operator D in (1.11).

In the case where $m_k > 1$, let N denote the normal bundle to C_k and choose a finite dimensional vector subspace

$$\Lambda = \Lambda_k \subset C^\infty((\oplus_{1\leq q\leq m} N^q) \otimes T^{0,1}C_k)$$

so that the following is true: For each $y \in \mathcal{Z}_0$, the vector space Λ should project surjectively onto the cokernel of Δ_y. The existence of such a Λ is guaranteed by Lemma 5.1 of [28] because $\mathcal{Z}_0$ is compact.

With $\Lambda = \Lambda_k$ understood, let $\mathcal{K}_\Lambda^{(k)} \subset C^\infty(\oplus_{1\le q\le m} N^q)$ denote a certain open neighborhood of $\mathcal{Z}_0$ in the subspace of $y \in C^\infty(\oplus_{1\le q\le m} N^q)$ which obey the equation

$$\bar{\partial} y + \nu\aleph \cdot y + \mu\mathbb{F}(y) \in \Lambda. \tag{2.19}$$

In particular, it follows from Lemma 5.1 in [28] that one can choose the neighborhood in question so that the resulting $\mathcal{K}_\Lambda^{(k)}$ is a smooth submanifold in $C^\infty(\oplus_{1\le q\le m} N^q)$ whose dimension is that of Λ. One can also choose $\mathcal{K}_\Lambda^{(k)}$ to have compact closure. These constraints will henceforth be assumed implicitly. Further requirements on $\mathcal{K}_\Lambda^{(k)}$ may also be made, but it turns out that all of these can be met by "shrinking" any given version by replacing it by its intersection with some smaller neighborhood of $\mathcal{Z}_0$.

Part 5. Once again, take the data (J, Γ, Ω) from the Baire set described in Propositions 2.5 and 2.8. Let $h = \{(C_k, m_k)\}$ be in the corresponding $\mathcal{H}$. Propositions 5.2, 6.1 and 6.3 in [28] assert that the data $\{\mathcal{K}_\Lambda^{(k)}\}$ can be chosen, as described above, so that for all r sufficiently large, there exists an embedding

$$\Psi_{h,r}(\equiv \Psi_r) : \times_k \mathcal{K}_\Lambda^{(k)} \to (\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1), \tag{2.20}$$

and a smooth map

$$\psi_{h,r}(\equiv \psi_r) : \times_k \mathcal{K}_\Lambda^{(k)} \to \times_k \Lambda_k \tag{2.21}$$

with the following properties:

- $\psi_{h,r}^{-1}(0)$ is embedded by $\psi_{h.r}$ onto an open set in the $\mu_0 = 0$ version of $\mathcal{M}^{(r)}$.
- For each k, introduce ψ_Λ^k to denote the map from $\mathcal{K}_\Lambda^{(k)}$ to Λ_k which associates the left-hand side of (2.19) to each point y. Then $|\psi_{h,r} - \times_k \psi_\Lambda^k| \le \zeta \cdot r^{-1/2}$, where ζ is independent of r.
- If $\Xi \in \mathcal{M}^{(r)} \cap \mathrm{Image}(\Psi_{h,r})$, then $\mathrm{kernel}(L_\Xi)$ is in the image of the differential of $\Psi_{h,r}$.
- Let $y \in \times_k \mathcal{K}_\Lambda^{(k)}$ and write $\Psi_{h,r}(y) = (a, (\alpha, \beta))$. Then

$$\sup_{x:\alpha(x)=0} \mathrm{dist}(x, \cup_k C_k) + \sup_{x\in\cup_k C_k} \mathrm{dist}(x, \alpha^{-1}(0)) < \zeta \cdot r^{-1/2},$$

where ζ is independent of y and r.

(2.22)

Here are two remarks concerning (2.22). First, with regard to the third point, note that the tangent space to

$$(\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X;S^1)$$

at Ξ has a natural identification with the subspace of

$$(a', (\alpha', \beta')) \in i \cdot \Omega^1 \oplus C^\infty(S_+)$$

for which the first line of (2.6) gives zero.

For the second remark, note that the fourth line of (2.22) insures that the images of $\{\Psi_{h,r} : h \in \mathcal{H}\}$ are disjoint when r is large.

Part 6. The analog of Proposition 2.9 in the general case is given below. However, the statement requires the following short digression to introduce new notation.

To begin the digression, suppose that the assumptions in Propositions 2.8 and 2.5 are satisfied. When $h = \{(C_k, m_k)\} \in \mathcal{H}$, introduce $Y_h \subset \times_k \mathcal{K}_\Lambda^{(k)}$ to denote the $\Psi_{h,r}$-inverse image of the set of $(a, (\alpha, \beta))$ where $\alpha^{-1}(0)$ intersects each $\gamma \in \Gamma$ and each $x \in \Omega$. When $y \in Y_h$ and the corresponding $\alpha^{-1}(0)$ intersects each γ exactly once, and at a point where $\nabla\alpha|_{T\gamma} \neq 0$, introduce the vector bundle V as in (2.8). In this case, define a map $G_y : T(\times_k \mathcal{K}_\Lambda^{(k)})|_y \to V$ as follows: First, associate to $y' \in T(\times_k \mathcal{K}_\Lambda^{(k)})$ its push-forward, $(a', (\alpha', \beta'))$ via the differential of $\Psi_{h,r}$. Then, G_y is obtained by restricting α' to the points of Ω and to those of $\{\gamma \cap \alpha^{-1}(0) : \gamma \in \Gamma\}$. End the digression.

Proposition 2.10. *Fix a class $e \in H^2(X;\mathbb{Z})$ with $s \bullet e \geq -1$ for all $s \in S$ and then let $d = 2^{-1} \cdot (e \bullet e - c \bullet e)$. Assume that $d \geq 0$ and when $d > 0$, fix $p \in \{0, \ldots, d\}$. When $p > 0$, fix an ordered set $\underline{\Gamma} \subset H_1(X;\mathbb{Z})$ of $2p$ classes. The Baire subset of $\mathcal{W}$ in Propositions 2.5 and 2.8 can be assumed to have the following additional property: Take (J, Γ, Ω) from this set to define $\mathcal{H}$. For each $h = \{(C_k, m_k)\} \in \mathcal{H}$, the corresponding data $\{\mathcal{K}_\Lambda^{(k)}\}$ can be chosen so that when r is large, then the following hold:*

- *For each $h = \{(C_k, m_k)\} \in \mathcal{H}$, the corresponding $Y_h \subset \times_k \mathcal{K}_\Lambda^{(k)}$ is a smooth submanifold which is diffeomorphic to $\times_{k:m_k>1} \mathcal{K}_\Lambda^{(k)}$.*

- *For each $y \in Y_h$, the corresponding $\alpha^{-1}(0)$ intersects each $\gamma \in \Gamma$ only once, and at a point where $\nabla\alpha|_{T\gamma} \neq 0$.*

- *For each $y \in Y_h$, the corresponding map G_y is a surjection.*

- *The $\Psi_{h,r}$ inverse image of $\mathcal{M}^{(r)}_{\Gamma,\Omega}$ is $\psi^{-1}_{h,r}(0) \cap Y_h$. Here, $\psi_{h,r}$ comes from* (2.21).

- *Each $\Xi \in \mathcal{M}^{(r)}_{\Gamma,\Omega}$ lies in the image of some $\Psi_{h,r}$.*

Proof of the first four assertions of Proposition 2.10. Write the map $\Psi_{h,r}$ as in Proposition 5.2 and (5.2) of [28]. Were the term α' missing, then the first four assertions would follow directly from (1.14) using the definition of $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r)$ in Sections 2 and 3b of [28]. The fact that the assertions still hold with the α' term present follows directly from the estimates in Lemma 6.6 and (4.1) in [28].

The proof of the final assertion of Proposition 2.10 occupies Sections 5 and 6 here.

f) The appearance of a curve count in the computation of SW

The Propositions 2.9 and 2.10 suggest that (1.5) can be reinterpreted as a weighted count of elements of $\mathcal{H}$ where the weight for each h is obtained by creatively counting points in the appropriate version of Y_h. Such a reinterpretation of (1.5) is possible, and is presented in this subsection. The results are summarized precisely in Propositions 2.11, below, and Proposition 2.13 which appears in the subsequent subsection. (These correspond, respectively, to the special case of Proposition 2.9 and the general case of Proposition 2.10.) However, the answer is schematically as follows: The weight for h is a product of factors. Here, each (C_k, m_k) with $m_k = 1$ contributes the factor $r(C_k, 1)$ which is described in Part 5 of Section 1e. Meanwhile, each (C_k, m_k) with $m_k > 1$ contributes a factor, $r'(C_k, m_k)$, which is obtained as an algebraic count of the points in the (C_k, m_k) version of the space $\mathcal{Z}_0$. Finally there is an overall factor of $\varepsilon(\sigma) = \pm 1$ just as in (1.19).

The remainder of the discussion in this subsection is relevant to the special case where the assumptions of Proposition 2.9 hold. The discussion is divided into two parts.

Part 1. This part describes how to obtain the factor $r'(C, m)$ when the corresponding $\mathcal{Z}_0$ is regular. (Remember that this means that the operator Δ_y in (2.18) has trivial cokernel for each $y \in \mathcal{Z}_0$.) For this purpose, return to the milieu of Proposition 2.7 where C is a complex torus, $N \to C$ is a topologically trivial, holomorphic line bundle and m is a positive integer. Also, specify (ν, μ) so that the operator D' corresponding to any holomorphic covering $f : C' \to C$ of degree m or less has trivial kernel. In this case, $\mathcal{Z}_0$ is compact.

If the operator Δ_y in (2.18) has trivial cokernel for each $y \in \mathcal{Z}_0$, then $\mathcal{Z}_0$ consists of a finite set of points (see Proposition 3.2 in [28]). In this case, the weight $r'(C, m)$ can be obtained by summing ± 1 weights associated to the points of $\mathcal{Z}_0$. These weights are obtained as follows: Consider a 1-parameter family of operators of the form

$$\{\Delta_y + n_t\}_{t \in [0,1]}, \tag{2.23}$$

where

$$\{n_t : L^2_1(\oplus_{1 \leq q \leq m} N^q) \to L^2((\oplus_{1 \leq q \leq m} N^q) \otimes T^{0,1}C)\}_{t \in [0,1]}$$

is a smooth family of bounded operator which obeys the following:

- $n_0 = 0$.
- $\Delta_y + n_1$ is $\mathbb{C}$-linear and invertible.
- $\|n_t(y')\|_2 \leq 8^{-1} \cdot \|y'\|_{1,2} + \zeta \cdot \|y'\|_2$.
- The set of $t \in [0, 1]$ where cokernel$(\Delta_y + n_t) \neq \{0\}$ is some finite number N.
- At such t where cokernel$(\Delta_y + n_t) \neq \{0\}$ this cokernel is 1-dimensional, and the restriction of the derivative of n_t to kernel$(\Delta_y + n_t)$ projects surjectively onto cokernel$(\Delta_y + n_t)$.

(2.24)

Here, $|| \cdot ||_2 = (\int_C | \cdot |_2)^{1/2}$ is the standard L^2-norm. The norm $|| \cdot ||_{1,2}$ denotes the L^2_1 norm where the derivative portion is computed using any metric compatible, covariant derivative on N. That is,

$$|| \cdot ||_{1,2} = (||\nabla(\cdot)||_2^2 + || \cdot ||_2^2)^{1/2}.$$

(Note that the constraint in the third line insures that each $\Delta_y + n_t$ is a Fredholm operator.) In (2.24), ζ can be any t-independent constant.

Standard perturbative arguments show that there are paths $\{n_t\}$ which obey (2.24); in fact, one can take $\{n_t\}$ to be zero'th order operators coming from a path of homomorphisms from $\oplus_{1\leq q\leq m}N^q$ to $(\oplus_{1\leq q\leq m}N^q)\otimes T^{0,1}C$. (See, e.g. [7].)

The integer N depends on the precise path $\{n_t\}$ chosen. However, $(-1)^N$ is path independent. With this understood, associate to each $y\in\mathcal{Z}_0$ the weight $\varepsilon_y\equiv(-1)^N$ and associate to $\mathcal{Z}_0$ the sum of the weights in the set $\{\varepsilon_y : y\in\mathcal{Z}_0\}$.

Part 2. This part considers the count for (1.5) in the fortunate case where the assumptions of Proposition 2.9 hold.

Proposition 2.11. *Make the same assumptions as in Proposition* 2.9. *When* $h\in\mathcal{H}$, *define the weight*

$$q'(h)=\varepsilon(\sigma)\cdot\Pi_{k:m_k=1}r(C_k,1)\Pi_{k:m_k>1}r'(C_k,m_k).$$

Here, $r(C_k,1)$ *is defined as in Part* 5 *of Section* 1*e, while* $\varepsilon(\sigma)$ *is defined as in Part* 6 *of Section* 1*e, and* $r'(C_k,m_k)$ *is defined in Part* 1 *of this subsection. Then the integeral in* (1.5) *is equal to* $\Sigma_{h\in\mathcal{H}}q'(h)$.

This proposition is a special case of Proposition 2.13; although a special case of it is proved directly in Section 7. In any case, note that Proposition 2.9 guarantees that the indexing set for the count is correct. Thus, the issue here is soley that of getting the weight assignments correct.

g) Curve counting for SW in the general case

The discussion here for the general case has five parts. The first three parts consider the count for points in $\mathcal{Z}_0$, and the last two consider the relevance of the latter count to the computation of (1.5).

Part 1. This part and Part 2 define the weight $r'(C,m)$ in the case where $\mathcal{Z}_0$ is not assumed to be regular. Thus, return again to the milieu of Proposition 2.7 where C is a complex torus, $N\to C$ is a topologically trivial, holomorphic line bundle and m is a positive integer. Also, specify (ν,μ) so that the operator D' corresponding to any holomorphic covering $f: C'\to C$ of degree m or less has trivial kernel. As before, this last assumption insures that $\mathcal{Z}_0$ is compact.

In the case where Δ_y has non-trivial cokernel for some $y\in\mathcal{Z}_0$, one assigns a weight using a two step procedure. The first step reintroduces the manifold $\mathcal{K}_\Lambda$ as in (2.19) and defines an orientation for the virtual vector bundle $T\mathcal{K}_\Lambda-(\mathcal{K}_\Lambda\times\Lambda)$. The orientation in question is explained in this part of the subsection. Given such an orientation, Part 4 of this subsection defines the weight for $\mathcal{Z}_0$.

To orient the virtual bundle $T\mathcal{K}_\Lambda - (\mathcal{K}_\Lambda \times \Lambda)$ first introduce Q_Λ, the L^2 orthogonal projection in $L^2((\oplus_{1\le q\le m} N^q)\otimes T^{0,1}C)$ onto Λ. Now, fix $y \in \mathcal{K}_\Lambda$ and consider $(1 - Q_\Lambda)\cdot\Delta_y$. Since Λ maps surjective onto the cokernel of Δ_y (by assumption), this operator maps $L^2_1(\oplus_{1\le q\le m} N^q)$ onto $(1-Q_\Lambda)\cdot L^2((\oplus_{1\le q\le m} N^q)\otimes T^{0,1}C)$. (This is to say that the latter map is surjective.) Furthermore, the kernel of $(1 - Q_\Lambda)\cdot\Delta_y$ is equal to the tangent space at y of $\mathcal{K}_\Lambda$.

With the preceding understood, choose a smooth family $\{n_t\}_{t\in[0,1]}$ of operators from $L^2_1(\oplus_{1\le q\le m} N^q)$ to $L^2((\oplus_{1\le q\le m} N^q)\otimes T^{0,1}C)$ which satisfies the first three points of (2.24). As Λ has, by assumption, positive dimension, one can choose $\{n_t\}$ so that for each t, the operator $(1-Q_\Lambda)\cdot(\Delta_y+n_t)$ maps onto $(1-Q_\Lambda)\cdot L^2((\oplus_{1\le q\le m} N^q)\otimes T^{0,1}C)$. With this understood, then the association of the kernel of $(1-Q_\Lambda)\cdot(\Delta_y+n_t)$ to $t \in [0, 1]$ defines a vector bundle over $[0, 1]$. Since such vector bundles are necessarily trivial, an orientation for the virtual vector space $\text{kernel}((1 - Q_\Lambda)\cdot(\Delta_y + n_1)) - \Lambda$ serves to orient $T\mathcal{K}_\Lambda|_y - \Lambda$.

To orient $\text{kernel}((1-Q_\Lambda)\cdot(\Delta_y+n_1))-\Lambda$, remark that the assumed invertibility of the operator $\Delta_y + n_1$ insures that $Q_\Lambda\cdot(\Delta_y + n_1)$ maps the kernel of $(1-Q_\Lambda)\cdot(\Delta_y+n_1)$ isomorphically onto Λ. An orientation of $\text{kernel}((1 - Q_\Lambda)\cdot(\Delta_y + n_1)) - \Lambda$ is then obtained by demanding that $Q_\Lambda\cdot(\Delta_y + n_1)$ preserve orientation.

With regard to the previous definition, note that the assumed $\mathbb{C}$-linearity of Δ_y+n_1 insures that the orientation so defined is independent of the precise path $\{n_t\}$.

Part 2. The association to $y \in \mathcal{K}_\Lambda$ of the left-hand side of (2.19) defines a smooth map $\psi_\Lambda : \mathcal{K}_\Lambda \to \Lambda$ whose zero set is $\mathcal{Z}_0$. Let w be any smooth map from $\mathcal{K}_\Lambda$ to Λ which obeys

- $\psi_\Lambda + w$ has only non-degenerate zeros.
- $|w| < |\psi_\Lambda|$ on the compliment of a compact set which contains $\mathcal{Z}_0$.

(2.25)

Associate to each zero of $\psi_\Lambda + w$ the sign of the determinant of the differential of the map $\psi_\Lambda+w$. Note that this sign is well defined because of the first point in (2.25) and $T\mathcal{K}_\Lambda - (\mathcal{K}_\Lambda \times \Lambda)$ has been oriented.

The second point in (2.25) and the fact that $\mathcal{Z}_0$ is compact insure that there are a finite number of zeros of $\psi_\Lambda + w$. With this understood, define the weight $r'(C,m)$ for $\mathcal{Z}_0$ to equal the sum of the signs associated to each zero of $\psi_\Lambda + w$.

Because the set of maps to Λ which obey the second point of (2.25) is convex, the usual argument for the invariance of the Euler class works here to prove that this $r'(C,m)$ is independent of the choice for w. Similar reasoning shows that $r'(C,m)$ is independent of the choice of Λ provided that projection maps Λ surjectively onto cokernel(Δ_y) for all $y \in \mathcal{Z}_0$. (Indeed, to compare the weight assignment for different Λ's, it is enough to consider the assignments for the case where $\Lambda \subset \Lambda'$. In this case, $\mathcal{K}_\Lambda$ sits in $\mathcal{K}_{\Lambda'}$ as a submanifold. Furthermore, the composition of $(1-Q_\Lambda)$ with the differential of $\psi_{\Lambda'}$ defines an isomorphism, $p_{\Lambda,\Lambda'}$, from the normal bundle of $\mathcal{K}_\Lambda$ to Λ'/Λ. This implies that the Euler class as computed using $\mathcal{K}_{\Lambda'}$ is the product of that using $\mathcal{K}_\Lambda$ with ± 1; where the sign is determined by whether $p_{\Lambda,\Lambda'}$ preserves or reverses orientations. It is left as an exercise to verify from the definition in Part 3 that $p_{\Lambda,\Lambda'}$ preserves orientation.)

To summarize, the preceding has defined a weight assignment, $r'(C,m)$ for the case where the data (C,N,m,ν,μ) satisfy the following property: When $f : C' \to C$ is a holomorphic covering map for degree m or less, then the corresponding operator D' has trivial cokernel.

Part 3. This part summarizes the properties of the weight assignment that was just defined. For this purpose, consider

Proposition 2.12. *Let $\mathcal{Y}$ denote the Frechet space of 5-tuples (C,N,m,ν,μ) where C is a torus with a complex structure; N is a topologically trivial, holomorphic line bundle over C; m is a positive integer; ν is a section over C of $T^{0,1}C$; and μ is a section over C of $N^2 \otimes T^{0,1}C$. Let $\mathcal{Y}' \subset \mathcal{Y}$ denote the subset with the following property: The operator D' on any holomorphic, 2-sheeted covering $f : C' \to C$ has trivial kernel. Then, for any positive integer m, the weight assignment $r'(C,m)$ given in Parts 1-3, above, for $\mathcal{Z}_0$ extends to data points in $\mathcal{Y}'$ to define a locally constant map from $\mathcal{Y}'$ to $\mathbb{Z}$.*

Proof of Proposition 2.12. It is left as an exercise using perturbation theory to prove that $r'(C,m)$ given in Steps 1-3 defines a locally constant, $\mathbb{Z}$-valued function on the subspace $\mathcal{Y}'' \subset \mathcal{Y}$ of the 4-tuples (C,N,ν,μ) with the property that the operator D' has trivial cokernel for any holomorphic cover $f : C' \to C$ of degree m or less. However, $\mathcal{Y}'' \subset \mathcal{Y}'$ and Lemma 5.13 in [26] imply that this inclusion identifies the path components of the two spaces.

Part 4. Proposition 2.10 insures that the count for (1.5) can be interpreted as a count of elements in $\mathcal{H}$ with appropriate weights. The following proposition summarizes:

Proposition 2.13. *The conclusions of Proposition* 2.10 *can be strengthened to include the following: When* $h \in \mathcal{H}$, *define the weight*

$$q'(h) = \varepsilon(\sigma) \cdot \Pi_{k:m_k=1} r(C_k, 1)\, \Pi_{k:m_k>1} r'(C_k, m_k).$$

Here, $r(C_k, 1)$ *is defined as in Part* 5 *of Section* 1*e, while* $\varepsilon(\sigma)$ *is defined as in Part* 6 *of Section* 1*e, and* $r'(C_k, m_k)$ *is defined in Parts* 1-3 *of this subsection. Then the integeral in* (1.5) *is equal to* $\Sigma_{h\in\mathcal{H}} q'(h)$.

This proposition is proved in Section 7. However, it is timely here to sketch how the proof goes.

Part 5. The sketch of the proof of Proposition 2.13 requires a digression to present a rather formal reinterpretation of the computation in (1.5) as a computation for an Euler class of a vector bundle over a finite dimensional manifold. Here is the idea: A finite dimensional manifold $Y \subset (\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1)$ will be called a "Kuranishi model" for the $\mu_0 = 0$ version of $\mathcal{M}^{(r)}_{\Gamma,\Omega}$ when the following conditions are satisfied:

- Y has compact closure.
- A neighborhood of $\mathcal{M}^{(r)}_{\Gamma,\Omega}$ in $\mathcal{M}^{(r)}$ is contained in Y.
- If $\Xi \in \mathcal{M}^{(r)} \cap Y$, then the kernel of L_Ξ is a subspace of $TY|_\Xi$.

(2.26)

With regard to the third point, remember that the tangent space to $(\mathrm{Conn}(E)\times C^\infty(S_+))//C^\infty(X; S^1)$ at Ξ has a natural identification with the vector space of $(a', (\alpha', \beta')) \in i \cdot \Omega^1 \oplus C^\infty(S_+)$ which give zero in the top line of (2.6). Thus, kernel(L_Ξ) is naturally a subvector space in this tangent space.

Next, introduce the subspace $Y_{\Gamma,\Omega} \subset Y$ of $(a, (\alpha, \beta))$ for which $\alpha^{-1}(0)$ contains each point in Ω and intersects each $\gamma \in \Gamma$. Say that Y is a "regular Kuranishi model for $\mathcal{M}^{(r)}_{\Gamma,\Omega}$" when

- If $\Xi = (a, (\alpha, \beta)) \in Y_{\Gamma,\Omega}$, then $\alpha^{-1}(0)$ intersects each $\gamma \in \Gamma$ exactly once, and at a point where $\nabla\alpha|_{T\gamma} \neq 0$.

- If $\Xi = (a, (\alpha, \beta)) \in Y_{\Gamma,\Omega}$, introduce the vector space V as in (2.8) and define the homomorphism $G : TY|_\Xi \to V$ as follows: Assign to $(a', (\alpha', \beta')) \in TY|_\Xi$ the vector whose component in $E|_x$ is $\alpha'(x)$ and whose component in $E|_q/\nabla\alpha(T\gamma|_q)$ is the projection of $\alpha'(q)$. Then, require that G be a surjective.

- $Y_{\Gamma,\Omega}$ is a smooth, codimension $2d$ submanifold of Y.

(2.27)

Note that the third point above follows from the first two using the implicit function theorem. In fact, $TY_{\Gamma,\Omega}|_\Xi$ is equal to the kernel of the homomorphisms G.

The relevance of the Kuranishi model to the computation of (1.5) stems from the following proposition:

Proposition 2.14. *Let Y be a regular Kuranishi model for $\mathcal{M}^{(r)}_{\Gamma,\Omega}$. Then there exists:*

- *A canonical,* $\dim(Y_{\Gamma,\Omega})$*-dimensional vector bundle $W \to Y_{\Gamma,\Omega}$ with fiber metric.*

- *A canonical orientation for the virtual vector bundle $TY_{\Gamma,\Omega} - W$.*

- *A canonical section w of $W \to Y_{\Gamma,\Omega}$ such that $w^{-1}(0) = \mathcal{M}^{(r)}_{\Gamma,\Omega}$.*

Furthermore, this data has the following significance: Let w' be any section of W such that:

- *$w + w'$ have only transversal zeros,*

- *$|w'| < |w|$ on the compliment of a compact subset which contains $\mathcal{M}^{(r)}_{\Gamma,\Omega}$.*

Then $w + w'$ has finitely many zeros; each zero has a weight in $\{\pm 1\}$ which is $+1$ if the differential of $w + w'$ is orientation preserving and -1 otherwise; and the sum of these weights is equal to the integral in (1.5).

(Note that the standard arguments for the invariance of the Euler class apply here to prove that the just described weighted count of the zeros of $w + w'$ is independent of w' subject to the given constraints.)

This last proposition is proved in Section 4.

End the digression.

The notion of a Kuranishi model for the $\mu_0 = 0$ version of $\mathcal{M}^{(r)}_{\Gamma,\Omega}$ is relevant precisely because a neighborhood

$$Y \subset \cup_{h\in\mathcal{H}} \Psi_{h,r}(\times_k \mathcal{K}^{(k)}_\Lambda) \tag{2.28}$$

of $\cup_h \psi^{-1}_{h,r}(0)$ is just such a regular Kuranishi model when r is large. This is guaranteed by (2.22) and Proposition 2.10. Thus, to prove Proposition 2.13, it suffices to prove the following is true when r is large: For each $h \in \mathcal{H}$, there is an open neighborhood $\mathcal{O}_h \subset Y_h$ of $\psi^{-1}_{h,r}(0)$ and

- There is a bundle isomorphism $\Phi : \Psi^*_{h,r} W \to \mathcal{O}_h \times (\times_k \Lambda_k)$.
- A certain pair of orientations of $T\mathcal{O}_h - \mathcal{O}_h \times (\times_k \Lambda_k)$ agree. The first is induced via Φ using the orientation given in Proposition 2.14 for $TY_h - \Psi^*_{h,r}W$. The second is defined as follows: First, write $Y_h = \times_{k:m_k>1} \mathcal{K}^{(k)}_\Lambda$ as in Proposition 2.10. Then, orient each of the $m_k > 1$ versions of $T\mathcal{K}^{(k)}_\Lambda - \mathcal{K}^{(k)}_\Lambda \times \Lambda_k$ with the orientation given in Part 1 of Subsection g, above. Finally, take the induced, product orientation for the virtual bundle $T\mathcal{O}_h - \mathcal{O}_h \times (\times_k \Lambda_k)$ and multiply by $e(\sigma) \cdot \Pi_{k:m_k=1} r(C_k, 1)$, where each $r(C_k, 1) = \pm 1$ is defined in Part 5 of Section 1e using $\Gamma_k = \{\gamma \in \Gamma : \gamma \cap C_k \neq \varnothing\}$, and $\varepsilon(\sigma)$ is defined in Part 6 of Section 1e.
- The Euler number computed from a perturbation of $\Phi \cdot \Psi^*_{h,r} w$ is identical to that which is computed from a perturbation of the map ψ_r of (2.21). (Note that both maps to $\times_k \Lambda_k$ have the same zero set.)
- The Euler number which is computed by ψ_r in (2.21) gives the weight for h in Proposition 2.13.

(2.29)

h) The final arguments

Given now Proposition 2.13 (or, if one is fortunate, Proposition 2.11), the equality between SW and Gr follows from the next proposition:

Proposition 2.15. *Let C be a holomorphic torus, m a positive integer, N a topologically trivial, holomorphic line bundle over C, ν a section* *of $T^{0,1}C$ and m a section of $N^2 \otimes T^{0,1}C$. Suppose that (C, N, m, ν, μ) are such that the operator D' has trivial cokernel in the case where $f : C' \to C$ is a holomorphic covering of degree m or less. Define $r'(C, m)$ as in Proposition 2.12. Reintroduce $r(C, m)$ as defined in the $m > 1$ case of Part 5 of Section 1e. Then $r(C, m) = r'(C, m)$.*

Proof of Proposition 2.15. The proof is arranged into four steps.

Step 1. Suppose that C is a complex torus, $N \to C$ is a holomorphic line bundle, ν is a section of $T^{0,1}C$, and μ is one of $N^2 \otimes T^{0,1}C$. Suppose that the data (C, N, ν, μ) lies in the space $\mathcal{Y}'$ of Proposition 2.12. Introduce the formal power series

$$f(C; z) = 1 + \Sigma_{m \geq 1} r'(C, m) \cdot z^m. \tag{2.30}$$

Apriori, $f(C; z)$ depends on the complex structure of C, the holomorphic structure on C's normal bundle N and the pair (ν, μ). However, according to Proposition 2.12, this power series is locally constant on the components of the space $\mathcal{Y}'$. This is to say that (2.30) defines a locally constant function from $\mathcal{Y}'$ into the space of formal power series.

Step 2. Consider the case where $X = T^4$, the 4-torus, with its symplectic structure which comes by writing $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ and then pushing forward the form $\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$ from a fundamental domain in $\mathbb{R}^4$. Fix an ω-compatible almost complex structure J. Let e be the cohomology class on T^4 which is Poincaré dual to the 2-torus which is defined setting x^3 and x^4 to zero. When m is a positive integer, let $\mathcal{J}[m]$ denote the set of connected, pseudo-holomorphic submanifolds whose fundamental class is Poincaré dual to $m \cdot e$. (Since T^4 has trivial canonical bundle, the adjunction formula guarantees that each such manifold is a torus.) It follows from Proposition 5.2 in [26] that the each $\mathcal{J}[m]$ is finite when J is chosen from an appropriate Baire subset of ω-compatible, almost complex structures. With the preceding understood, introduce the formal power series

$$P(z) = \Pi_{m \geq 1} \Pi_{C \in \mathcal{J}[m]} f(C; z^m). \tag{2.31}$$

Step 3. The Seiberg-Witten invariants for T^4 are all known; the only non-zero invariant is the class $0 \in H^2(X; \mathbb{Z})$. This implies, via Proposition 2.13, that $P(z) = 1$. This is to say that $P(z)$ is, infact, independent of the precise choice for the almost complex structure J (as long as the latter is suitably generic).

On the other hand, as J is changed, elements will appear in $\mathcal{J}[m]$ and also disappear. A limited example of this phenomena is described in Section 6 of [26]. In particular it is not hard to prove that one can obtain, by varying J, a torus C which realizes any given component of the space $\mathcal{Y}'$ in Proposition 2.12.

Now, according to Lemma 3.1 in [26], the space $\mathcal{Y}'$ is the compliment in $\mathcal{Y}$ of a codimension 1 variety, $\mathcal{D}$. Then, the equality $P(z) = 1$ forces relations on the polynomials $f(\cdot, z)$ which are assigned to components of $\mathcal{Y}'$ which sit on opposite sides of a codimension 1 stratum of $\mathcal{D}$. These relations can be discerned by repeating (word for word) the analysis in Section 5e of [26]; they are precisely those in (5.26), (5.29), (5.30), and (5.31) in [26].

Step 4. As remarked in [26], the relations in (5.26), and (5.29-31) of [26] determine $P(\cdot, z)$ on $\mathcal{Y}'$ uniquely in terms of its value on the component $\mathcal{Y}_0 \subset \mathcal{Y}'$ where $\mu_0 = 0$. (Lemma 3.1 in [26] asserts in part that the condition $\mu_0 = 0$ singles out a unique component of $\mathcal{Y}'$.)

Fortunately, the value for $f(\cdot, z)$ on this component can be computed explicitly. Indeed, on such a component, the space $\mathcal{Z}_0$ for a given m is, according to (2.17), the vector space

$$\oplus_{1 \leq q \leq m} kernel(\bar{\partial} + q \cdot \nu : C^{\infty}(N^q) \to C^{\infty}(N^q \otimes T^{0,1}C).$$

If ν is chosen in a suitably generic fashion, then each of the kernels in question will be trivial and so, for each m, the space $\mathcal{Z}_0$ will consist of a single point. Furthermore, the operator in (2.18) is invertible for the generic ν when $\mu = 0$, so the single point in $\mathcal{Z}_0$ will be a regular point. Also, the operator in (2.18) in this case is already $\mathbb{C}$-linear, so Part 1 of Section f, above, asserts that $r'(C, m) = 1$ for all m. Therefore, $f(\cdot, z)$ on $\mathcal{Y}_0$ is equal to $(1 - z)^{-1}$. According to the discussion surrounding (5.32) in [26], this implies that $r(\cdot, m) = r'(\cdot, m)$ for all m on all components of $\mathcal{Y}'$.

i) Summary

To summarize: The proof of Theorem 1 is completed with proofs of Proposition 2.1 in Section 7, Proposition 2.7 in Section 3, Proposition 2.10 (just the final assertion) in Sections 5 and 6, Proposition 2.13 in Section 7, and Proposition 2.14 in Section 4.

3. $\mathcal{Z}_0$ and compactness: The proof of Proposition 2.7

This section serves as a digression of sorts to prove Proposition 2.7.

If the reader is willing to take Proposition 2.7 on faith, then the reader can skip ahead to Section 4. For the reader who is continuing in this section, the milieu is primarily that of Proposition 2.7.

The proof of Proposition 2.7 is reduced in the first subsection below to a corollary of a pair of auxiliary propositions. The proofs of these two propositions then occupy the remainder of this section.

a) Noncompactness in $\mathcal{Z}_0$

Suppose that C is a connected, complex curve; $\pi : N \to C$ is a holomorphic line bundle with hermitian metric; ν is a section of $T^{0,1}C$ and μ is a section of $N^2 \otimes T^{0,1}C$. The data C, N, ν, and μ, plus a positive integer m, is sufficient to define the subspace $\mathcal{Z}_0 \subset C^\infty(\oplus_{1\leq q\leq m} N^q)$ as described in Section 3 of [28]. In this regard, remember that $\mathcal{Z}_0$ consists of sections $y = (y_1, \dots, y_m)$ which obey an equation of the form given in (2.7).

The first lemma below describes, in part the behavior of non-compact subsets of $\mathcal{Z}_0$. The statement of this lemma requires a short, four part digression. Part 1 of the digression introduces the tautological section s of $\pi^*N \to N$ which assigns each point to itself. Part 2 of the digression is to introduces a certain almost complex structure J on the total space of N. This J is defined by the condition that $T^{1,0}N$ is locally spanned by the π-pull-back of forms in $T^{1,0}C$ and by $\nabla_\theta s + \pi^*\nu \cdot s + \pi^*\mu \cdot \bar{s}$.. Here, ∇_θ is the unitary connection on N which is defined by the holomorphic and hermitian structures.

Part 3 of the digression introduces the function, R, on $C^\infty(\oplus_{1\leq q\leq m} N^q)$ which sends y to

$$R[y] = \sup_C \sup_{1\leq q\leq m} |y_q|^{1/q}. \tag{3.1}$$

Part 4 of the digression introduces the map

$$p : C^\infty(C; \oplus_{1\leq q\leq m} N^q) \to C^\infty(N; \pi^* N^m)$$

which assigns to $y = (y_1, \dots, y_m)$ the section

$$p[y] = s^m + \pi^* y_1 \cdot s^{m-1} + \cdots + \pi^* y_m. \tag{3.2}$$

Proposition 3.1. *Let C be a connected, compact complex curve and let $m \geq 0$ be an integer. Let $N \to C$ be a holomorphic, hermitian line bundle and let $(\nu, \mu) \in C^\infty(T^{0,1}C \oplus (N^2 \otimes T^{0,1}C))$. Use this data* *to define the space* $\mathcal{Z}_0$. *Let* $\{y_j = (y_{j1}, \dots, y_{jm})\}_{j=1,2,\dots} \subset \mathcal{Z}_0$ *be a sequence with the property that the set* $\{R_j \equiv R[y_j]\}_{j=1,2,\dots}$ *is increasing and unbounded. For each* j, *define* $\Sigma_j \subset N$ *to be the set of points* $\eta \in N$ *where* $p[y_j](R_j \cdot \eta) = 0$. *Then there is a compact, complex curve* C', *a* J*-pseudo-holomorphic map* $\varphi : C' \to N$ *which does not factor through* C, *and an infinite subsequence (hence renumbered consecutively) of* $\{\Sigma_j\}$ *with the property that the (subsequence)* $\{\Sigma_j\}$ *converges as* $j \to \infty$ *as integral currents in* N *to the current* $\int_{C'} \varphi^*(\cdot)$. *Furthermore, this convergence is pointwise in the sense that*

$$\{\sup\{\operatorname{dist}(\eta, \varphi(C')) + \operatorname{dist}(\Sigma_j, \eta') : \eta \in \Sigma_j \text{ and } \eta' \in \varphi(C')\}\}_{j=1,2,\dots} \tag{3.3}$$

converges to zero as $j \to \infty$.

The next proposition considers the possibilities for C' and φ when C is a torus and where N is the trivial bundle.

Proposition 3.2. *Let* C *be a torus and suppose that* N *is topologically trivial. Fix the data* $(\nu, \mu) \in C^\infty(T^{0,1}C \oplus N^2 \otimes T^{0,1}C)$. *Let* $\Sigma \subset N$ *be the image of a compact, complex curve by a non-constant, pseudo-holomorphic map. Then* Σ *is an embedded,* J*-pseudo-holomorphic torus in* N *which represents a non- zero multiple of* C *in* $H_2(N;\mathbb{Z}) = \mathbb{Z}$. *Furthermore,*

- *The embedding* $\Sigma \subset N$ *gives* Σ *the structure of a complex curve,* C', *for which the tautological inclusion into* N *is pseudo- holomorphic. Then, the compostion of the projection* $\pi : N \to C$ *with the tautological inclusion defines a holomorphic covering map* $f : C' \to C$.
- *There is a not identically zero section* h *of the bundle* $f^*N \to C'$ *which obeys* $\bar\partial h + (f^*\nu) \cdot h + (f^*\mu) \cdot \bar h = 0$.
- *The tautological inclusion of* C' *into* N *is also obtained by composing* h *with the tautological map from* f^*N *to* N.

Proof of Proposition 2.7. The proposition is an immediate corollary to the preceding two propositions.

The next six subsections contain the proof of Proposition 3.1. The subsequent subsection contains the proof of Proposition 3.2.

b) A vortex digression

Of necessity, the proof must exploit the properties of (2.7) whose solutions comprise $\mathcal{Z}_0$. In this regard, remark that the complications in the proof stem from the $\mathbb{F}$ term in (2.7) and the fact that the latter is not known (for $m > 1$) explicitly. However, the $\mathbb{F}$ term is obtained by considering the original characterization of $\mathcal{Z}_0$ in Section 3 of [26] as a space of sections over C of a fiber bundle whose typical fiber is the moduli space of solutions to the vortex equations on $\mathbb{C}$. Thus, the proof of Proposition 3.1 requires an unavoidable return to the vortex story of Sections 2b,c and 3 of [26]. In particular, each solution to (2.7) determines (via Proposition 3.2 in [26]) a solution to (3.5) in [26] and vice-versa. The proof of Lemma 2.7 will exploit, for the most part, the version of $\mathcal{Z}_0$ as the space of solutions to (3.5) in [26]. The reader may wish to consult Sections 2b,c and 3 of [26] to review this whole story. Also, the notation below is borrowed freely from these same sections of [26].

The proof of Proposition 3.1 comprises seven steps. This subsection contains the first step, which is a digression to establish some useful technical estimates about vortices and points on $\mathbb{C}$. The first lemma below summarizes facts about vortex solutions on $\mathbb{C}$. The second lemma summarizes a clustering property of points in $\mathbb{C}$.

Lemma 3.3, below, introduces the moduli space $\mathfrak{C}_m$ of solutions to (2.4) in [26] on $\mathbb{C}$ with vortex number m. Given such a solution $c = (v, \tau)$, the lemma also introduces the operator Θ_c as described in (2.12) of [26]. In the statement of the lemma, the vector space kernel(Θ_c) consists of elements which are anihilated by Θ_c and are square integrable on $\mathbb{C}$.

Lemma 3.3. *Fix $m \geq 1$ and there is a constant ζ such that the following is true: Let $c = (v, \tau) \in \mathfrak{C}_m$. At each $\eta \in \mathbb{C}$, one has*

- $(1 - |\tau|^2) \leq \zeta \cdot \Sigma_{\lambda:\tau(\lambda)=0}\, e^{-|\eta-\lambda|/\zeta}$.
- $|\nabla_v \tau| \leq \zeta \cdot (1 - |\tau|^2)$.
- *If* $(a, \alpha) \in \text{kernel}(\Theta_c)$, *then* $|(a, \alpha)| \leq \zeta \cdot ||(a, \alpha)||_2 \Sigma_{\lambda:\tau(\lambda)=0}\, e^{-|\eta-\lambda|/\zeta}$

(3.4)

Furthermore, suppose that $\lambda \in \mathbb{C}$ and $r \geq 1$ have the property that $|\eta - \lambda|$ is either less than r or greater than $5r$ whenever η is a zero of τ. Let $\chi_{\lambda,r}$ denote the function on $\mathbb{C}$ whose value at η is $\chi(|\eta - \lambda|/2r)$. Then *there exists an element* $w \in \text{kernel}(\Theta_c)$ *which has the form*

$$w = \chi_{\lambda,r}((2\sqrt{2})^{-1}(1-|\tau|^2)d\bar{\eta},\ [\partial_v\tau])) + w', \tag{3.5}$$

where w' *obeys* $||w'||_2 \leq \zeta e^{-r/\zeta}$.

The next lemma will be applied to the points in the set $\tau^{-1}(0)$.

Lemma 3.4. *Fix* $m \geq 1$ *and* $n \geq 1$. *Then, there exists* $\zeta \geq 1$ *with the following property. Let* $\Lambda \subset \mathbb{C}$ *be a set of* m *points. Let* $r > 0$ *be given. Then there exist* $\zeta' \in (1, \zeta)$ *and a decomposition of* Λ *as the disjoint union of non-empty subsets* $\{\Lambda_j\}$ *with the property that:*

- $\text{diam}(\Lambda_j) < \zeta' r$,
- $\text{dist}(\Lambda_j, \Lambda_k) > n\zeta' r$.

The remainder of this step is occupied with the proofs of these two propositions.

Proof of Lemma 3.3. For the first assertion in (3.4), note that Lemma 4.9 in [27] finds a constant ζ such that $(1-|\tau|^2)$ is bounded by $\delta > 0$ at points where the distance to any zero of τ is greater than ζ/δ. It then follows from (8.1) and (8.2b) of [6] that there is a constant $\zeta \geq 1$ with the property that the function $u = 2^{-1}\cdot(1-|\tau|^2)$ obeys the equation $d^*du + 8^{-1}\cdot u \leq 0$ at points with distance greater than ζ from $\tau^{-1}(0)$. With this last point understood, deduce the first line of (3.4) using the comparison principle with the Greens function for the operator $d^*d + 8^{-1}$ with poles at the zeros of τ. (Remember that when $\kappa > 0$, the Green's function with pole at λ for the operator $d^*d + \kappa^2$ on $\mathbb{C}$ is bounded at by $\zeta \cdot e^{-\kappa|\eta-\lambda|/\zeta}$ at points $\eta \in \mathbb{C}$ with $|\eta - \lambda| \geq 1$.)

The second assertion in (3.4) is (8.2c) in [6]. To prove the third assertion in (3.4), note first that standard elliptic regularity theorems provide a c-independent bound for $\sup_{\mathbb{C}}|(a,\alpha)|$ by a c-independent multiple of $||(a,\alpha)||_2$. To obtain a pointwise bound, the Bochner-Weitzenbock formula for $\Theta_c^\dagger\Theta_c$ will be employed to obtain a differential inequality for $|(a,\alpha)|$. It follows from this Bochner-Weitzenbock formula, (3.4.2), and Lemma 4.9 in [27] that there exists a c-independent constant ζ which is such that $u = |(a,\alpha)|$ obeys $d^*du + 4^{-1}\cdot u < 0$ at points with distance ζ or more from $\tau^{-1}(0)$. Given the preceding, use the comparison principle with the Green's function for the operator $d^*d + 4^{-1}$ with poles at the zeros of τ.

To prove (3.5), first use (3.4) to conclude that

$$\left|\Theta_c(\chi_\bullet((2\sqrt{2})^{-1}(1-|\tau|^2)d\bar{\eta},\,[\partial_v\tau]))\right| \le \zeta e^{-r/\zeta}. \tag{3.6}$$

Meanwhile, L^2-orthogonal projection finds $w \in \text{kernel}(\Theta_c)$ and a unique, L^2_1 element w'' such that

$$\chi_\bullet((2\sqrt{2})^{-1}(1-|\tau|^2)d\bar{\eta},\,[\partial_v\tau]) = w + \Theta_c^\dagger w''. \tag{3.7}$$

Take the L^2 inner product of (3.7) with $\Theta_c^\dagger w''$ and integrate by parts where appropriate to conclude (using (3.6)) that

$$||\Theta_c^\dagger w''||_2^2 \le \zeta \cdot e^{-r/\zeta_\bullet}||w''||_2.$$

Meanwhile, the Bochner-Weitzenboch formula in (2.13) of [28] implies with (3.4) that $||\Theta_c^\dagger w''||_2 \ge \zeta^{-1}\cdot ||w''||_2$ where $\zeta \ge 1$ is independent of c. These last two inequalities imply (3.5).

Proof of Lemma 3.4. The construction of the subsets $\{\Lambda_j\}$ involves iterating the "basic clustering construction". The basic clustering construction takes as input a number r_1 and gives as output a partition of Λ into non-empty subsets $\{\Lambda_j(r_1)\}$ with the property that the diameter of each $\Lambda_j(r_1)$ is less than $m\cdot r_1$, and with the property that when $j \ne k$, then the distance between $\Lambda_j(r_1)$ and $\Lambda_k(r_1)$ is greater than r_1. Given that such a "basic clustering construction" exists (see below), here is an iterative algorithm which constructs the data for Lemma 3.4: The starting value for r_1 on the first run through is r. The iteration for the algorithm increases the value of r_1 to $2\cdot n\cdot m\cdot r_1$ on the subsequent run. Anyway, here is the generic run: Given $r_1 \ge r$, invoke the basic clustering construction to create $\{\Lambda_j(r_1)\}$. If it is the case that the distance between $\Lambda_j(r)$ and $\Lambda_k(r)$ is greater than $n\cdot m\cdot r_1$ for all $j \ne k$, then stop and set $\zeta' = m\cdot r_1/r$ and $\{\Lambda_j\} = \{\Lambda_j(r_1)\}$ for use in Lemma 3.4. Otherwise, rerun the "basic clustering construction" using $2n\cdot m\cdot r_1$ for the new value of r_1.

This algorithm must terminate after m steps because the number of elements in the partition of Λ as $\cup_j\Lambda_j(\cdot)$ for a given run is at least one less then that of the previous run. Here is the basic clustering construction: Define an equivalence relation on the points of Λ by asserting that $\eta \sim \eta'$ when there is an ordered subset $\{\eta_1, \dots, \eta_{p+1}\} \subset \Lambda$ such that $\eta_1 = \eta$, $\eta_{p+1} = \eta'$, and $|\eta_{j+1} - \eta_j| < r_1$ for all $j \in \{1, \dots, p\}$. Let $\{\Lambda_j(r_1)\}$ denote the set of equivalence classes. It follows that these sets partition Λ. Furthermore, $\text{diam}(\Lambda_j(r_1)) \le m\cdot r_1$.

c) "Renormalized" sections of N^q.

This subsection constitutes Steps 2 and 3 of the proof of Proposition 3.1. Suppose that $y \in \mathcal{Z}_0$, and set $R = R[y]$. Given $z \in C$, let $\Lambda(z)$ denote the zeros of the restriction of $p[y]$ to $\pi^{-1}(z)$. Given $\lambda \in \Lambda(z)$, let $m(\lambda)$ denote the multiplicity of λ as a zero of $p[y]$ on $\pi^{-1}(z)$. Given the preceding, introduce, for each $q \in \{1, \dots, m\}$, the section $h_q[y]$ of N^q whose value at z is

$$h_q[y] = \Sigma_{\lambda \in \Lambda(z)} m(\lambda) \cdot (\lambda/R)^q. \tag{3.8}$$

Note that $h_q[y]$ is a smooth section with smooth dependence on y. (For example, $h_1 = y_1/R$, $h_2 = R^{-2} \cdot ((y_1)^2 - 2 \cdot y_2)$, and in general, h_q is a polynomial function of y.)

Lemma 3.5. *Given $q \geq 1$, there is a constant $\zeta \geq 1$ with the following significance: Let $y \in \mathcal{Z}_0$, construct $h_q = h_q[y]$ as in (3.8) and then the following are true:*

- $|\bar{\partial} h_q + \Sigma_{\lambda \in \Lambda(z)} q m(\lambda)(\lambda/R)^{q-1}(\nu\lambda/R + \mu\bar{\lambda}/R)| \leq \zeta R^{-1/2}$.
- $\displaystyle\int_C |\nabla_\theta h_q|^2 \leq \zeta$.
- *Let $z, z' \in C$ and let $P_{z,z'} : N^q|_{z'} \to N^q|_z$ denote the parallel transport along some shortest geodesic. Then*
$$|h_q(z) - P_{z,z'} h_q(z')| \leq \zeta \cdot \operatorname{dist}(z, z')^{1/2}.$$

(3.9)

The remainder of this step and Step 3 are occupied with the

Proof of Lemma 3.5. Given the first line in (3.9), it follows that $|\bar{\partial} h_q| \leq \zeta$. The second line follows from this last inequality after integrating $|\bar{\partial} h_q|^2$ over C and then integrating by parts. The third line also follows from the bound $|\bar{\partial} h_q| + |h_q| \leq \zeta$ using standard regularity estimates for the $\bar{\partial}$-operator. Thus, it remains only to verify the first line of (3.9). This occupies Step 3 of the proof of Proposition 3.1.

Step 3. The verification of the first line of (3.9), proceeds as follows: To begin, use y to parameterize a solution, $c(y)$, to the vortex equations in each fiber of N as in Proposition 3.2 of [28]. Indeed, after a choice of $\mathbb{C}$-linear identification of a fiber with $\mathbb{C}$, the vortex $c(y) = (v, \tau)$ is determined from $p[y]$ by the formula

$$(v, \tau) = \left(\bar{\partial}u - \partial\bar{u},\ p[y]e^{-u}\right). \tag{3.10}$$

Here, u is a complex valued function on $\mathbb{C}$ whose real part is the unique, real valued function on $\mathbb{C}$ which solves the equation

$$i \cdot \partial\bar{\partial}\,\mathrm{Re}(u) = 8^{-1} * (1 - |p|^2 e^{-2\,\mathrm{Re}(u)}) \tag{3.11}$$

on $\mathbb{C}$ with the asymptotic condition

$$\mathrm{Re}(u) = m \cdot \ln|\eta| + o(1)$$

as $|\eta| \to \infty$. Meanwhile, the imaginary part of u is chosen (in part) so that at points $\eta \in \mathbb{C}$ with $|\eta| \gg R$, then $\tau = f \cdot \eta^m$ where f is a positive, real valued function. (In (3.10) and (3.11), $p[y]$ is viewed as the m'th order polynomial $\eta^m + y_1 \cdot \eta^{m-1} + \cdots + y_m$ on $\mathbb{C}$.)

Note that (3.11) guarantees that (v, τ) obeys the vortex equation (see [30])

- $i \cdot dv = 4^{-1} * (1 - |\tau|^2)$,
- $\bar{\partial}\tau + v_{0,1}\tau = 0$,

(3.12)

where $v_{0,1} = \bar{\partial}u$ is the $(0,1)$ part of the 1-form v.

Next, remark that h_q can be obtained directly from τ using the identity in (2.6.3) of [28]. To be precise,

$$h_q[y]|_z = (8\pi)^{-1} R^{-q} \int_{N|_z} s^q (1 - |\tau|^2). \tag{3.13}$$

With (3.13) understood, compute dh_q by passing the derivative under the integral. The result is

$$dh_q = (8\pi)^{-1} R^{-q} \int_{N|_z} s^q d^H (1 - |\tau|^2). \tag{3.14}$$

Here, d^H is defined as a certain horizontal derivative which is defined by the hermitian connection θ on N. To be precise here, consider a local trivialization for N with fiber coordinates $\eta \in \mathbb{C}$. Then d^H sends a function f on N to

$$d^H f \equiv d^C f + \theta \cdot \left(\bar{\eta}\frac{\partial}{\partial \eta} - \eta\frac{\partial}{\partial \bar{\eta}}\right) f, \tag{3.15}$$

where d^c is the usual exterior derivative along C. Meanwhile, because (v, τ) obeys (3.12) on each fiber, the expression in (3.14) is equal to

$$dh_q = (2\pi)^{-1} i R^{-q} \int_{N|_z} s^q d^V d^H v. \tag{3.16}$$

Here, d^V is the exterior derivative along the fibers of N. And, as (v, τ) obeys (3.12) on each fiber, there is a section φ on N of $i \cdot \pi^*(T^*C)$ such that $(v^0, \tau^0) = (d^H v + d^V \varphi,\, d^H \tau - \varphi\tau)$ has the form $((\frac{1}{2\sqrt{2}}(a - \bar{a}), \alpha)$, where (a, α) is a π^*TC-valued element in the kernel of Θ_c on each fiber. Note that φ is found as the solution of a certain inhomogeneous Laplace equation on each fiber. The equation in question is:

$$(d^V)^* d^V \varphi + 4^{-1}|\tau|^2 \varphi - *d*(d^H v) - i4^{-1} im(\bar{\tau} d^H \tau) = 0. \tag{3.17}$$

(With regard to the proof of existence for (3.16), keep in mind that $(d^H v, d^H \tau)$ decays to zero exponentially fast along each fiber of N.)

Since $(d^V)^2 = 0$, the expression in (3.16) is valid with v^0 replacing $d^H v$. Then, integration by parts (with the decay estimates from (3.4)) finds (3.16) equivalent to

$$dh_q = -(4 \cdot \sqrt{2}\pi)^{-1} i R^{-q} \int_{N|_z} q s^{q-1} d^V s \wedge a. \tag{3.18}$$

Now, (a, α) are π^*TC-valued; so the decomposition

$$T^*C \otimes \mathbb{C} = T^{0,1}C \oplus T^{1,0}C$$

induces the decomposition $(a, \alpha) = (a_{0,1}, \alpha_{0,1}) + (a_{1,0}, \alpha_{1,0})$. With this understood, $\bar{\partial} h_q$ is given by (3.18) but with $a_{0,1}$ replacing a. The fact that $a_{0,1}$ determines $\bar{\partial} h_q$ is fortunate because, $(a_{0,1}, \alpha_{0,1})$ is determined directly by the defining equation for $\mathcal{Z}_0$. More precisely, $(a_{0,1}, \alpha_{0,1})$ is determined by the vortex picture version of (2.7) which is equation (3.5) in [28]. In particular, according to (3.5) in [28] and (3.4.3), at a point $\eta \in \pi^{-1}(z)$, the norm of $(a_{0,1}, \alpha_{0,1})$ is no greater than

$$|(a_{0,1}, \alpha_{0,1})| \leq \zeta R \Sigma_{\lambda \in N|_z : \tau(\lambda) = 0}\, e^{-|\eta - \lambda|/\zeta}. \tag{3.19}$$

It thus follows from (3.18) that

$$|\bar{\partial} h_q| \leq \zeta. \tag{3.20}$$

To proceed further, fix $z \in C$ and invoke Proposition 3.2 using $r = \sqrt{R}$, $n = 10$ and $\Lambda = \Lambda(z) = p[y]^{-1}(0) \cap N|_z$. Let $\{\Lambda_j\}$ denote the resulting partition of $\Lambda[z]$. And, for each j, let d_j denote the diameter of Λ_j. Let ξ_j denote the center of mass of Λ_j, and let m_j denote the sum of the multiplicities of the points in Λ_j.

Reintroduce a standard bump function, $\chi : [0,\infty) \to [0,1]$ which is non- increasing and which equals one on $[0,1]$ and zero on $[2,\infty)$. Then, use d_j to promote χ to the function χ_j on $N|_z$ whose value at a point η is $\chi(|\eta - \xi_j|/2 \cdot d_j)$.

With the preceding accomplished, Lemma 3.4, (3.18), (3.19) and Taylor's theorem with remainder imply that

$$\left|\bar{\partial} h_q + \Sigma_j (4\sqrt{2\pi})^{-1} i R^{-q} \int_{N|_z} (\chi_j q s^{q-1} d^V s \wedge a_{0,1})\right| \leq \zeta e^{-\sqrt{R}/\zeta}. \tag{3.21}$$

It then follows from (3.19), (3.21) and the bound on $\operatorname{diam}(\Lambda_j)$ by $\zeta \cdot R^{1/2}$ that

$$|\bar{\partial} h_q - \Sigma_j q(\xi_j/R)^{q-1} b_j| \leq \zeta R^{-1/2}, \tag{3.22}$$

where

$$b_j = -(4 \cdot \sqrt{2\pi})^{-1} i R^{-1} \int_{N|_z} \chi_j d^V s \wedge a_{0,1}. \tag{3.23}$$

It remains as yet to identify b_j. For this purpose, it is convenient to rewrite (3.23). The first step in this process only simplifies the notation. For this step, choose a $\mathbb{C}$-linear, hermitian identification of $N|_z$ with $\mathbb{C}$ so that s can be replaced by the complex coordinate $\eta \in \mathbb{C}$. And, all derivatives will henceforth denote derivatives along $\mathbb{C}$ unless explicitly noted otherwise.

The second step in rewriting (3.23) inserts the identity

$$1 = 1 - |\tau|^2 + |\tau|^2$$

in front of $a_{0,1}$. The resulting equation for b_j is:

(3.24)

$$b_j = -R^{-1}(2\pi)^{-1} i \int_{\mathbb{C}} \chi_j \left(\frac{1}{2\sqrt{2}} (1 - |\tau|^2) d\eta \wedge a_{0,1} + |\tau|^2 d\eta \wedge a_{0,1} \right).$$

This last equation is further modified by using the fact that $(a_{0,1}, \alpha_{0,1})$ are annihilated by Θ_c to write $\tau \cdot a_{0,1} = -2 \cdot \sqrt{2} \cdot \bar{\partial}_v \alpha_{0,1}$. With this understood, (3.24) becomes

(3.25)
$$b_j = -R^{-1}(2\pi)^{-1} i \int_{\mathbb{C}} \chi_j \left(\frac{1}{2\sqrt{2}}(1 - |\tau|^2) d\eta \wedge a_{0,1} - \bar{\tau} d\eta \wedge \bar{\partial}_v \alpha_{0,1} \right).$$

Next, integrate by parts on the second term under the integral in (3.25) to write the latter as

$$\begin{aligned} b_j = &- R^{-1}(2\pi)^{-1} i \int_{\mathbb{C}} \chi_j \Big(\frac{1}{2\sqrt{2}}(1 - |\tau|^2) d\eta \wedge a_{0,1} \\ &+ \left[\bar{\partial}_v \bar{\tau}\right] \alpha_{0,1} d\eta \wedge d\bar{\eta} \Big) + \varepsilon_1, \end{aligned} \tag{3.26}$$

where $|\varepsilon_1| \leq \zeta \cdot e^{-\sqrt{R}/\zeta}$.

The next step in the rewriting of (3.23) invokes (3.5) to conclude that there exists an element $w \in \text{kernel}(\Theta_c)$ which differs from

$$\chi_j \cdot (\frac{1}{2\sqrt{2}} \cdot (1 - |\tau|^2), [\partial_v \tau])$$

by some w' with L^2 norm bounded by $\zeta \cdot e^{-\sqrt{R}/\zeta}$. Thus, b_j differs from

$$-R^{-1}\pi^{-1} \int_{\mathbb{C}} \langle w, (a_{0,1}, \alpha_{0,1}) \rangle \tag{3.27}$$

by no more than $\zeta \cdot e^{-\sqrt{R}/\zeta}$. Here, $\langle , \rangle$ denotes the Hermitian inner product on $T^{0,1}\mathbb{C} \oplus \mathbb{C}$.

To procede, introduce $(v^1_{0,2}, \tau^1_{0,1})$ as in (3.5) of [28]. Then (3.27) is equal to

$$-R^{-1}\pi^{-1} \int_{\mathbb{C}} \langle w, (v^1_{0,2}, \tau^1_{0,2}) \rangle, \tag{3.28}$$

since $(v^1_{0,2}, \tau^1_{0,1})$ differs from $(a_{0,1}, \alpha_{0,1})$ by an element in the image of Θ_c, and w is L^2-orthogonal to all such elements.

Now, according to (3.5) of [28], the expression in (3.28) is equal to

$$-R^{-1}\pi^{-1} \int_{\mathbb{C}} \left\langle w, (\nu\eta + \mu\bar{\eta}) \left(\frac{1}{2\sqrt{2}}(1 - |\tau|^2, [\partial_v \tau]) \right) \right\rangle. \tag{3.29}$$

And, the latter differs from

$$-R^{-1}\pi^{-1} \int_{\mathbb{C}} \chi_j (\nu\eta + \mu\bar{\eta})(8^{-1}(1 - |\tau|^2) + |\partial_v \tau|^2) \tag{3.30}$$

by no more than $\zeta \cdot e^{-\sqrt{R}/\zeta}$. (Use (3.5) here.) Then, it follows from (3.4) that (3.30) differs from

$$(3.31) \qquad -\left(\nu(\xi_j/R) + \mu\left(\bar{\xi}_j/R\right)\right)\pi^{-1}\int_{\mathbb{C}} \chi_j(8^{-1}(1-|\tau|^2), |\partial_v \tau|^2)$$

by no more than $\zeta \cdot R^{-1/2}$. Finally, integration by parts (using (3.12)) plus (3.4) establishes that the integral in (3.31) (with the factor of π^{-1}) differs from

$$(3.32) \qquad (8\pi)^{-1}\int_{N|_z} \chi_j(1-|\tau|^2)$$

by less than $\zeta e^{-\sqrt{R}/\zeta}$. This last integral is within $\zeta R^{-1/2}$ of m_j. Indeed, this follows from (3.12) using (3.4.1) and (3.4.2) plus the fact that $\tau/|\tau|$ has winding number m_j on any circle with center ξ_j and radius between $2 \cdot d_j$ and $4 \cdot d_j$.

Thus, (3.31) implies that $b_j = m_j \cdot (\nu \cdot (\xi_j/R) + \mu \cdot (\bar{\xi}_j/R))$ plus a term whose norm is no greater than $\zeta R^{-1/2}$. In particular, this means that

$$(3.33) \qquad \bar{\partial} h_q = -\Sigma_j q(\xi_j/R)^{q-1} m_j\left(\nu(\xi_j/R) + \mu\left(\bar{\xi}_j/R\right)\right) + \varepsilon,$$

where $|\varepsilon| \leq \zeta R^{-1/2}$. The first assertion in (3.9) then follows from (3.32) by invoking the definition of ξ_j as the center of mass of the points in Λ_j.

d) The set Σ

This subsection defines the set S and constitutes Step 4 of the proof of Proposition 3.1. To begin, let $\{y_j\} \subset \mathcal{Z}_0$ be as described in Proposition 3.1. For each index j, construct the m-tuple $h[y_j] = (h_1[y_j], \ldots, h_m[y_j])$, where $h_q[\cdot]$ is given by (3.8). It follows from the final assertion of Lemma 3.5 that the sequence $\{h[y_j]\}$ is equicontinuous, and thus there is a subsequence (hence relabled consecutively) such that $\{h[y_j]\}$ converges pointwise and uniformly on C to a Hölder continuous (with exponent 1/2), Sobolev class L^2_1 section $h = (h_1, \ldots, h_m)$ of $\oplus_{1\leq q\leq m} N^q$.

With h understood, introduce

$$y = (y_1, \ldots, y_m)$$

of $\oplus_{1\leq q\leq m} N^q$ where y is such that the zeros, $\{\lambda_j\}_{1\leq j\leq m} \in \mathrm{Sym}^m(N)$, of

$$p[y] = s^m + y_1 s^{m-1} + \cdots + y_m$$

satisfy $\Sigma_j \lambda_j^p = h_p$. (Note that

$$y_1 = h_1, y_2 = 2^{-1}(h_1^2 - h_2), \ldots$$

and, in general y_p is a polynomial function of $\{h\}_{p' \leq p}$.) Note that y is Hölder continuous also, and Sobolev class L_1^2.

It follows from the first assertion of Lemma 3.5 that the relabled sequence $\{\Sigma_j\}$ converges pointwise to $\Sigma \equiv p[y]^{-1}(0)$ as described by Proposition 3.1. Remark that Σ cannot coincide with C because of the pointwise convergence of $\{h[y_j]\}$ to h. It follows from the definition of $R[y_j]$ that h cannot be the zero section. With regard to Proposition 3.1, it remains as yet to prove that Σ is the image by a pseudo-holomorphic map of a compact, complex curve. This task occupies the remaining steps of the proof of Proposition 3.1.

e) The regular points of Σ

Define the order function (a map from Σ to $\{1, \ldots\}$) by declaring the order of a point $x \in \Sigma$ to be the order of vanishing of the section p at x. With this understood, call a point in Σ *regular* when the order function is constant on some open neighborhood of the point. A non-regular point will also be called singular. Note that the set of regular points is open (by definition) and dense. (If $U \subset \Sigma$ is an open set, let $o(U)$ denote the minimum of the order function on U. As the order function is integer valued, this $o(U)$ is achieved by points in U. Such points are regular because $p[y]$ is continuous.)

This step considers the pseudo-holomorphicity of Σ near its regular points.

Lemma 3.6. *The set of regular points in Σ defines a (possibly non-compact) pseudo-holomorphic submanifold of N.*

Proof of Lemma 3.6. Let x be a regular point of Σ of some order b. Then x has a neighborhood in Σ which is the graph of a Hölder continuous, Sobolev class L_1^2 section, t, of N over an open set in C. Also, $\bar{\partial}t$ has bounded norm. (This all follows directly from the assertions in Lemma 3.5.) With this last point understood, Lemma 3.6 is reached by proving that $Dt = \bar{\partial}t + \nu \cdot t + \mu \cdot \bar{t} = 0$. Thus the proof of this assertion follows.

Let B be a small, open, convex ball with center $\pi(x)$ on which t is defined. By shrinking B if necessary, one can assume that there exists $\delta > 0$ such that at each $z \in B$, the distance between $t(z)$ and its compliment in $\Sigma \cap N|_z$ is greater than δ. Furthermore, one can assume that the distance between $t(z)$ and the parallel transport of x along a short geodesic in B is no more than $\delta/16$. (Remember that t is Hölder continuous.)

The polynomial $p[y](s) = s^m + y_1 \cdot s^{m-1} + \cdots + y_m$ factors as $(s-t)^b \cdot p_1$ where $p_1(t(z)) \neq 0$ for all $z \in B$. Note that p_1 is also Hölder continuous; and $\bar{\partial}^H p_1$ has bounded norm.

With the preceding understood, consider the section q of $\pi^* N$ over $N|_B$ which is given by

$$q = 3s \left(\frac{p_1(s)^2}{p_1(t)^2} - \frac{2}{3} \frac{p_1(s)^3}{p_1(t)^3} \right). \tag{3.34}$$

Here, t is not distinguished from its pull-back by π to N. Note that $q(t) = t$ and $q(\xi) = 0$ if $\xi \in \Sigma - \text{Image}(t)$. Also, if $z \in B$ is fixed, then q defines a polynomial function on $N|_z$ of the tautological section s of $\pi^* N$ over $N|_z$. In this regard, the derivative, q', of q with respect to s obeys $q'(t) = 1$ and $q'(\xi) = 0$ if $\xi \in \Sigma - \text{Image}(t)$.

From the definition of Σ and the first assertion of Lemma 3.5 it follows that t is the $C^{0,1/2}$ and L^2_1 limit of the sequence of sections $\{t_j\}$ of $N|_U$ where

$$t_j(z) = (8\pi)^{-1} R^{-1} \int_{N|_z} q(R[c_j]^{-1} s)(1 - |\tau_j|^2). \tag{3.35}$$

Thus by the second assertion of Lemma 3.5 and the stated properties of q', $|\bar{\partial} t_j + \nu t_j + \mu \bar{t}_j|$ converges to zero as j tends to infinity. This last fact proves the lemma.

f) The singular points of Σ

This subsection argues that Σ contains only finitely many singular points. The argument here constitutes Step 6 of the proof of Proposition 3.1.

It is sufficient to prove that there are finitely many singular points in the restriction of Σ to $\pi^{-1}(B)$ where B is any small disk in C. To begin, choose such a disk, and let $B_{\text{reg}} \subset B$ denote set of points z for which $\Sigma \cap \pi^{-1}(z)$ is regular. Note that B_{reg} is open and dense. Let $B' \subset B_{\text{reg}}$ be a path component. Then $\pi : \Sigma|_{B'} \to B'$ is a covering map. This means that there is a positive integer $k \leq m$ such that locally on B', Σ is the image of sections $\{t_\alpha\}_{\alpha \leq m}$ with associated multiplicities $\{m[\alpha]\}$. (This is to say that the polynomial p over B' factors as $\Pi_\alpha (s - t_\alpha)^{m[\alpha]}$.) Each t_α is, locally, a section of N which is annihilated by the operator

D. Globally, the set $\{t_\alpha\}$ defines a section of the tensor product of N with a representation of $\pi_1(B')$ in a group of permutations.

In any event, the section $q' = \Pi_{\alpha\neq\beta}(t_\alpha - t_\beta)^{m[\alpha]m[\beta]}$ defines a smooth section over B' of the appropriate power of the line bundle N. The significance of this q' is that it is non-vanishing on B', and that it extends continously as the zero section to $B - B'$. This extension of q' to the whole of B will be denoted by q. However, note that q' has a second extension to the whole of B in that q' is the restriction to B' of a section of a power of N which is obtained as products from the set $\{h_p\}_{1\leq p\leq m}$. This means, in particular, that q' has bounded L^2_1 Sobolev norm over B', and thus the extension, q, of q' by zero on $B - B'$ has bounded L^2_1 norm over the whole of B. This observation will be relevant momentarily.

Since each t_α is annihilated by the operator D, the section q' obeys a differential equation of the form

$$\bar\partial_\theta q' = w \cdot q', \tag{3.36}$$

where w is uniformly bounded on the closure of B' and smooth inside. It is a straightforward matter to find a continuous, Sobolev class L^2_1 function u on B which obeys the equation

$$\bar\partial u = w. \tag{3.37}$$

With u understood, consider that $e^{-u} \cdot q$ is a continuous, Sobolev class L^2_1 section of N over B which satisfies

$$\bar\partial(e^{-u}q) = 0 \tag{3.38}$$

on B' and is zero on $B - B'$. It follows that $e^{-u} \cdot q$ is holomorphic in B, and thus the set $B - B'$ is finite.

g) Proof of Proposition 3.1

This subsection completes the proof of Proposition 3.1 with an argument that Σ is the image of a compact, complex curve by a pseudo-holomorphic map. To begin, let $\Sigma_{\mathrm{reg}} \subset \Sigma$ denote the set of regular points. It follows from the previous step that $\Sigma - \Sigma_{\mathrm{reg}}$ is a finite set, and that each end of Σ_{reg} has the topology of a punctured disk.

Let $C'_0 = \Sigma_{\mathrm{reg}}$. Then C'_0 inherits the structure of a complex curve with finitely many ends, with each being a punctured disk. As there is no obstruction to extending a complex structure from a punctured disk in $\mathbb{C}$ to the whole disk, C_0' is naturally the compliment of a finite set of points in a compact, complex curve C'. Furthermore, C' admits a tautological map, φ, into N which is continuous and whose restriction to C_0' is its embedding as Σ_{reg}. (The map φ sends $C' - C_0'$ to the singular points of Σ.) Thus, φ is pseudo-holomorphic except possibly at the finite number of points in $C' - C_0'$, where it is, at worst, continuous. However, a standard removable singularity theorem implies that φ is everywhere pseudo-holomorphic.

Here is the removable singularities argument: Take a local complex coordinate for C' near a point in $C' - C_0'$, and take local coordinates for N near the φ image of the given point. Then φ becomes a continuous map from a disk in $\mathbb{C}$ to $\mathbb{C}^2$ which obeys an elliptic equation on the punctured disk. Elliptic regularity theorems from, e.g. [17], can be used to prove that φ is smooth over the whole disk, and thus obeys the elliptic equation everywhere. The point is that the hard removable singularity theorem of Sacks and Uhlenbeck [24] is not necessary here — the hard part of the theorem in [24] is the proof that the map in question is continous. Here, φ is given as continuous.

h) Proof of Proposition 3.2

There are five steps to the proof. The first four prove that Σ is an embedded torus.

Step 1. There is a compact, complex curve Σ' and a pseudo-holomorphic map (which will be called φ) from Σ' to N whose image is Σ and which is an embedding on the compliment of a finite set of points in Σ'. The push-forward of the fundamental class of Σ' is necessarily a positive multiple, say q, of the fundamental class of Σ. Because the images of pseudo-holomorphic maps have locally positive intersection number, this implies that Σ and C are disjoint. (Remember that $[C] \bullet [C] = 0$.) This local positivity of intersections also yields that the local intersection numbers between Σ and the fibers of N are all positive. Thus, counting multiplicity, Σ intersects any fiber exactly q times. However, as φ is mostly an embedding, the local intersections with all but finitely many fibers of N have multiplicity 1.

Step 2. This step describes the local form for the map φ. For this purpose, remark that the composition of φ with the projection π gives a holomorphic map between Σ' and C since π is pseudo-holomorphic. Thus, Σ' is a ramified cover of C. As remarked, the points where the differential of φ is zero are mapped by φ to the multiplicity greater than

1 intersection points of Σ with a fiber of π. The structure of j near such a point in Σ' can be seen as follows: First, there is a complex coordinate w on a neighborhood in Σ' centered at such a point, and there is a complex coordinate z on a neighborhood in C of its $\pi \cdot \varphi$-image such that the composition of φ with π is given by $z = w^b$ where $b \in \{2, \dots, q\}$.

With the preceding understood, let η denote the fiber coordinate with respect to a local, $\bar{\partial}$-holomorphic trivialization for N near $\pi(\varphi(p))$. Then φ near p has the following form:

Lemma 3.7. *Let $p \in \Sigma'$ be a point where the differential of $\pi \cdot \varphi$ is zero. Then, there is:*

- *A ball $B \subset \Sigma'$ with center p with a complex coordinate w (where 0 corresponds to p).*
- *A complex coordinate z for C near $\pi \cdot \varphi(p)$ such that $\pi \cdot \varphi$ sends w to $z = w^b$ for some $b \in \{2, \dots, q\}$.*
- *An extension of z to a coordinate system (z, η) for N near $\varphi(p)$ such that φ sends w to*

$$(z, \eta) = (w^b,\ g(w)), \tag{3.39}$$

where g is a complex valued function on a neighborhood of $w = 0$. Furthermore, if a divisor $k \in \{2, \dots, b\}$ of b has been chosen, then g can be written as $g = g_0 + g_1$ where

a) $g_0 = a_0 + \wp_0$ where $a_0 \in \mathbb{C} - 0$; and $\wp_0$ is a function which is invariant under the multiplicative action on $\mathbb{C}$ of the group of k'th roots of unity. Also, $|\wp_0| \leq \zeta \cdot |w|^k$.

b) $g_1 = a_1 \cdot w^\lambda + \wp_1$, where $|\wp_1| \leq \zeta \cdot |w|^{\lambda+1}$, $a_1 \in \mathbb{C} - 0$, and $\lambda \in \mathbb{Z}_+$ is not divisable by k.

Proof of Lemma 3.7. The existence of w and z satisfying the first two points above follows, as remarked, from the fact that $\pi \cdot \varphi$ is holomorphic. To explain the third point, first introduce the fiber coordinate η for N near $\pi^{-1}(0)$ so that $T^{0,1}N$ is locally spanned by $d\bar{z}$ and $d\bar{\eta} + f \cdot dz$; here, $f = f(z, \eta)$ is a smooth function of z and η with linear dependence on η and its complex conjugate. Since φ is supposed to be pseudo-holomorphic, both of these forms must annihilate the push forward by φ of $\partial/\partial w$. To understand the implications of this requirement, introduce a function g of w by writing $\varphi(w) = (w^b, g(w))$. With g understood, then the push-forward of $\partial/\partial w$ is the vector field $b \cdot w^{b-1}\partial/\partial z + g_w \cdot \partial/\partial\eta + \bar{g}_w \cdot \partial/\partial\bar{\eta}$. Thus, the condition that φ is pseudo-holomorphic translates to the following condition on the function g :

$$\bar{g}_w + \varphi^*(f) b w^{b-1} = 0. \tag{3.40}$$

In general, this last equation is an $\mathbb{R}$-linear equation for the complex function g since $\varphi^*(f)$ has the form $f(w^b, g)$, which is a linear functional of g and its complex conjugate. It follows from the preceding remarks that the space of solutions to (3.40) in a small ball B about 0 is a vector space over $\mathbb{R}$ with an action of the group $\mathbb{Z}_b$ of b'th roots of unity. In particular, if $k \in \{2, \ldots, b\}$ is a factor of b, then $\mathbb{Z}_k$ is a subgroup of $\mathbb{Z}_b$. Thus, any solution g has the form $g_0 + g_1$ where g_0 is the $\mathbb{Z}_k$-invariant part, and g_1 averages to zero under the $\mathbb{Z}_k$ action.

In the situation at hand, the first observation is that $g_1(0) = 0$ because evaluation at the origin is a $\mathbb{Z}_k$ invariant map from functions to $\mathbb{C}$. However, Aronszajn's unique continuation principle [1] implies that g_1 can not vanish to infinite order at 0. Thus, there exists an integer λ which is not divisable by p and is such that

$$g_1 = a_\lambda w^\lambda + \mathcal{O}(|w|^{\lambda+1}), \tag{3.41}$$

where $a_\lambda \in \mathbb{C}$. (In general, one might expect $\Sigma_{0\leq k\leq\lambda}\alpha_k \cdot w^{\lambda-k} \cdot \bar{w}^k$ for the $\mathcal{O}(|w|^\lambda)$ part of g_1; but only (3.41) is consistent with (3.40).) Furthermore, $a_\lambda \neq 0$, for otherwise Σ would not intersect the generic fiber of π in the correct number of points.

As for g_0, because Σ is disjoint from C, the function g_0 can not vanish at the origin. The estimate for the size of $\wp_0$ follows using Taylor's theorem with remainder in conjunction with (3.40).

Step 3. This step uses Lemma 3.7 to perturb φ so that the resulting map, φ_1, is an immersion with some special properties. The following lemma summarizes:

Lemma 3.8. *Given a positive integer k and $\varepsilon > 0$, there is a smooth map $\varphi_1 : \Sigma' \to N$ with the following properties:*

- *φ_1 is homotopic to φ and agrees with the latter on the compliment of radius ε balls about the singular points. In general, φ_1 is ε close to φ in the C^k topology.*
- *φ_1 is an immersion where pairs of sheets intersect transversely with only positive double points.*

- *J followed by orthogonal projection defines a homomorphism J_1 on $T\Sigma'$ whose square is ε-close to -1.*

Proof of Lemma 3.8. Introduce a bump function χ on $[0, \infty)$ which is non- increasing, 1 on $[0, 1]$ and 0 on $[2, \infty)$. Given ε as in the lemma, and given $\varepsilon_1 \in \mathbb{C} - 0$ with $|\varepsilon_1|$ small, perturb φ near $w = 0$ to

$$\varphi_1(w) = (w^b, \varepsilon_1 \chi(2|w|/\varepsilon)w + g_0 + g_1). \tag{3.42}$$

This perturbation has support where $|w| < \varepsilon$. A homotopy between φ_1 and φ is obtained by replacing ε_1 in (3.42) by $t \cdot \varepsilon_1$ and letting t run between 0 and 1. Also, if $|\varepsilon_1| \ll \varepsilon^{k+1}$, then the map φ_1 will be ε-close to φ in the C^k topology. Furthermore, it follows from Lemma 3.7 that φ_1 is an immersion. To check the assertion about the endomorphism J, consider that

$$\begin{aligned} \varphi_{1*}(\partial/\partial w) =& bw^{b-1}\partial/\partial z + g_w \partial/\partial \eta \\ &+ \bar{g}_w \partial/\partial \bar{\eta} + \varepsilon_1 \chi \partial/\partial \eta \\ &+ 2\varepsilon^{-1}\chi'|w|(\varepsilon_1 \partial/\partial \eta + \bar{\varepsilon}_1 \partial/\partial \bar{\eta}). \end{aligned} \tag{3.43}$$

The action of $(-i) \cdot J$ on the latter simply changes the sign of the term with $\bar{\varepsilon}_1$. Thus, φ_1 is J-pseudo-holomorphic except where $\varepsilon/2 \le |w| \le \varepsilon$. And, here, the ratio of the norm of the anti-holomorphic to that of the holomorphic part of (3.43) is no greater than $\zeta|\varepsilon_1|\varepsilon^{1-b}$. This can be made smaller than ε by choosing $|\varepsilon_1| \ll \varepsilon^b$.

Now consider the possiblities for the self intersections of $\varphi_1(\Sigma)$. (This is the last issue to check for the proof of Lemma 3.8.) In this regard, note that φ_1 can take w and w' to the same point only if $w' = q \cdot w$ where q is a non-trivial b'th root of unity. To obtain more information, fix such a root q, and let $k \in \{2, \ldots, b\}$ be the smallest integer for which $q^k = 1$. Use this choice of k in Lemma 3.7 to define g_0 and g_1. According to Lemma 3.7, the points w and $q \cdot w$ are sent to the same point in N by φ_1 if and only if

$$\varepsilon_1 \chi \cdot (q-1)w = g_1(w) - g_1(qw). \tag{3.44}$$

In light of the final assertion of Lemma 3.7, the latter equation reads

$$\varepsilon_1 \chi (q-1)w = (1 - q^\lambda) a_\lambda w^\lambda + \mathcal{O}(|w|^{\lambda+1}). \tag{3.45}$$

(Note that $1 - q^\lambda \neq 0$ because $\lambda \neq 0 \bmod(k)$.)

When ε is small and $|\varepsilon_1|$ much smaller, there will be precisely $\lambda - 1$ non-trivial solutions w to (3.45) for each choice of q. (To see this, note that when $|\varepsilon_1| \ll \varepsilon^\lambda$, the solutions to (3.45) which occur where $\chi(2 \cdot |w|/\varepsilon) > 0$ all occur where $\chi(\cdot) = 1$. And where $\chi(\cdot) = 1$, the fixed point equation in (3.45) is a perturbation of the holomorphic fixed point equation $\varepsilon_1 \cdot (q-1) \cdot w = (1-q^\lambda) \cdot a_\lambda \cdot w^\lambda$.) It is left as an exercise for the reader using (3.44) to check that all pairwise intersections of sheets are transverse and have positive local intersection numbers. (The reason for this is that when $|\varepsilon_1|$ is small, then (3.45) is a perturbation of a holomorphic fixed point equation, and the sheets for the corresponding holomorphic immersion have the aforementioned properties.)

Step 4. This step uses Lemma 3.8 and a version of the adjunction formula to conclude that Σ' is a torus. Indeed,

$$2^{-1}(\varphi_{1*} - i \cdot J \cdot \varphi_{1*}) : T_{1,0}\Sigma' \to \varphi_{1*}(T_{1,0}N)$$

defines a $\mathbb{C}$-linear homomorphism between complex vector bundles over Σ'. The third point of Lemma 3.8 insures that the map is injective. The cokernel bundle is a complex line bundle $E \to \Sigma_1$ whose underlying real bundle is the normal bundle to the immersion φ_1. Since C is a torus and N is topologically trivial, it follows that $c_1(T_{0,1}N) = 0$, and thus

$$c_1(T_{1,0}\Sigma_1) = -c_1(E). \tag{3.46}$$

On the otherhand $c_1(E)$ can be computed by counting with signs the zero's of a suitably generic section. Because of the geometric interpretation of E as the normal bundle to the immersion, such a section defines a deformation of φ_1 to a second immersion, φ_1'. Then, the intersection number between $\varphi_1(\Sigma')$ and $\varphi_1'(\Sigma')$ is equal to $c_1(E) + 2 \cdot n$, where n is the number of double points for the immersion φ_1. (According to the second point of Lemma 3.8, all such points count positively.) On the otherhand, according to Step 1, the intersection number between $\varphi_1(\Sigma')$ and $\varphi_1'(\Sigma')$ computes a multiple $[C] \bullet [C]$, and the latter is zero. Thus, $c_1(E) = -2 \cdot n \leq 0$ with equality if and only if φ_1 is an embedding.

Now return to (3.46). The unavoidable conclusion is that $c_1(T_{1,0}\Sigma') \leq 0$ with equality if and only if φ_1 is an embedding. This means that Σ' is either a torus (whence φ_1 is an embedding), or else it is a sphere. The latter case can be ruled out because there are no holomorphic maps from $\mathbb{CP}^1$ to a torus that are not constant, and $\pi \cdot \varphi'$ is a non-constant, holomorphic map.

Step 5. Because Σ' is a torus and $\pi \cdot \varphi'$ is holomorphic, it follows that the latter is a covering map. This requires Σ to be a submanifold which intersects each fiber of π exactly q times, all with multiplicity one.

The fact that each point in $\Sigma \cap \pi^{-1}(z)$ has multiplicity one implies that Σ is locally the image of sections $(h_1, \ldots, h_q)$ of N. Here, $h_i \neq h_j$ when $i = j$. Furthermore, because Σ is holomorphic, each h_i is annihilated by the operator D in (1.11). (This last condition makes Σ pseudo-holomorphic.) As demonstrated in Section 5h of [26], the data $(h_1, \ldots, h_q)$ can be viewed as follows: There exists a section h of $\pi^* N$ over Σ which is unique up to deck transformations for the covering map π, obeys $\bar{\partial} h + (f^*\nu) \cdot h + (f^*\mu) \cdot \bar{h} = 0$, and whose push-forward via π gives $(h_1, \ldots, h_q)$. This last fact implies the remaining assertions of Proposition 3.2.

4. Orientations and other constructions for $\mathcal{M}^{(r)}$

The section serves as a digression of sorts to set up some background concerning $\mathcal{M}^{(r)}$. In particular, the discussion below concerns, first, the manner in which an orientation for the line $\det^+ = H^0 \otimes \det(H^1) \otimes \det(H^{2+})$ induces one on $\mathcal{M}^{(r)}$. The second concern is Proposition 2.14 and the computation of the integral in (1.5) from a Kuranishi model. In particular, Proposition 2.14 is proved in Subsection 4c, below.

a) Orienting $\mathcal{M}^{(r)}$

The purpose of this subsection is to review the method by which an orientation of the line $\det^+$ induces one on $\mathcal{M}^{(r)}$. For this purpose, fix a class $e \in H^2(X; \mathbb{Z})$ and consider the $\mathrm{Spin}^{\mathbb{C}}$ structure whose $S_\pm$ is given by (1.9) where $E \to X$ is a complex line bundle with first Chern class e. Fix $r \geq 1$ and let $\mathcal{M}^{(r)} \subset (\mathrm{Conn}(E) \times C^\infty(\Sigma_+))/C^\infty(X; S^1)$ denote the moduli space of solutions to (2.4) for some choice of form μ_0. In this subsection no notational distinction is made between the different versions of $\mathcal{M}^{(r)}$ as defined by different choices for μ_0 in (2.4). In a subsequent subsection, the moduli space of solutions to (2.4) for a given $\mu_0 \neq 0$ will be denoted by $\mathcal{M}^{(r)}[\mu_0]$.

Let $\Xi = (a, (\alpha, \beta)) \in \mathcal{M}^{(r)}$. Say that Ξ is a *smooth point* when the operator L in (2.6) has trivial cokernel. It follows from Proposition 6.2 in [28] that $\mathcal{M}^{(r)}$ has the structure of a smooth manifold of dimension $2 \cdot d$ (as in (1.10)) near a smooth point Ξ. What follows is a definition of the orientation on $\mathcal{M}^{(r)}$ in a neighborhood of a smooth point. (The

Sard-Smale Theorem [23] can be used as in the proof of Theorem 3.17 of [4] to prove that $\mathcal{M}^{(r)}$ consists entirely of smooth points when μ_0 is chosen from an appropriate open and dense subset of $i \cdot \Omega^{2+}$. See, e.g. [15] or [8].)

The definition for the orientation for the Seiberg-Witten invariant near a smooth point is simplest when the integer

$$d = d(e) = 2^{-1} \cdot (e \bullet e - c \bullet e)$$

vanishes, so this case will be considered first. (Note that $d \geq 0$ apriori because $\mathcal{M}^{(r)}$ has a smooth point.)

The case $\mathbf{d = 0}$. Since the zero'th homology of a point has a canonical generator, an orientation for a zero dimensional space is an association of ± 1 weight to each element. With this understood, the association of $a \pm 1$ weight to a smooth point Ξ procedes as follows: First define an almost complex structure J_D on $i \cdot TX \oplus S_+$ by using J on the TX summand and by using multiplication by i on the S_+ summand. And, define an almost complex structure, J_R on $i \cdot (\varepsilon_{\mathbb{R}} \oplus \Lambda_+) \oplus S_-$ by using the endomorphism of the same name from Step 3 of Section 1c on the $i \cdot (\varepsilon_{\mathbb{R}} \oplus \Lambda_+)$ summand, and by using multiplication by i on the S_- summand. Next, introduce the operator L as in (2.6), and consider a smooth path of linear operators of the form $\{L + n_t\}_{t \in [0,1]}$, where:

- $n_t : L^2(i \cdot T^* \oplus S_+) \to L^2(i \cdot (\varepsilon_{\mathbb{R}} \oplus \Lambda_+) \oplus S_-)$ is bounded for each t.
- $n_0 = 0$.
- $L + n_1$ is surjective.
- $L + n_1$ is also complex linear with respect to the complex structures on J_R on $i \cdot T^* \oplus S_+$ and J_D on $i \cdot (\varepsilon_{\mathbb{R}} \oplus \Lambda_+)$.
- The set of $t \in [0, 1]$ where $\text{cokernel}(L + n_t) \neq \{0\}$ is finite, say with N elements.
- If $\text{cokernel}(L + n_t) \neq \{0\}$, then this cokernel is 1-dimensional.
- For each t, the t-derivative of n_t at t restricts to $\text{kernel}(L + n_t)$ to map the latter isomorphically onto $\text{cokernel}(L + n_t)$.

(4.1)

(Note that $\dim(\text{kernel}(L+n_t)) = \dim(\text{cokernel}(L+n_t))$ in this case because the index of L is the integer $2 \cdot d$.)

Straightforward arguments from analytic perturbation theory (as in [7]) can be used to prove that such a family $\{n_t\}$ exists. In fact, one can take n_t to be a zero'th order local operator which comes from a section of $\text{Hom}(i \cdot T^* \oplus S_+; i \cdot (\varepsilon_{\mathbb{R}} \oplus \Lambda_+) \oplus S_-)$.

With (4.1) understood, the sign for $(a, (\alpha, \beta)) \in \mathcal{M}^{(r)}$ is defined to equal $(-1)^N$, where N is given in (4.1). This can be shown to be independent of the choice of the path $\{n_t\}$.

By the way, the operator $L + n_1$ can be chosen to have the form

$$L + n_1 = 2^{-1}(L - J_R \cdot L \cdot J) - 2^{-1}(\sigma - J_R \cdot \sigma \cdot J), \tag{4.2}$$

where σ is an endomorphism from $i \cdot T^* \oplus S_+$ to $i \cdot (\varepsilon_{\mathbb{R}} \oplus L_+) \oplus S_-$. In this example, n_1 is a zero order, local operator. This follows from the fact that the symbol of L intertwines J with J_R. It is left to the reader to verify that $\text{cokernel}(L + n_1) = \{0\}$ for a suitably generic choice of σ. (Use perturbation theory from [7] to prove this.)

The case $\mathbf{d > 0}$. To orient $\mathcal{M}^{(r)}$ in this case, first recall that the tangent space to $\mathcal{M}^{(r)}$ at the orbit of $\Xi = (a, (\alpha, \beta))$ is canonically identified with the kernel of the operator L. This means that it is sufficient to coherently orient the kernel of L. For this purpose, choose a smooth path $\{n_t : L^2(i \cdot T^* \oplus S_+) \to L^2(i \cdot (\varepsilon_{\mathbb{R}} \oplus \Lambda_+) \oplus S_-)\}_{t \in [0,1]}$ which obeys the first four points in (4.1). Because $2 \cdot d > 0$ now, such a path can be found where $L + n_t$ has trivial cokernel for all t. (Use analytic perturbation theory to prove this assertion.)

As $L + n_t$ has trivial kernel, and as the index of $L + n_t$ is the integer $2 \cdot d$, it follows that the assignment to $t \in [0, 1]$ of the vector space $\text{kernel}(L + n_t)$ defines a smooth, $2 \cdot d$ dimensional vector bundle over $[0, 1]$. The fiber of this bundle at $t = 0$ is $T\mathcal{M}^{(r)}|_{\Xi}$, and the fiber at $t = 1$ is the vector space $\text{kernel}(L + n_1)$. However, as the operator $L + n_1$ is $\mathbb{C}$-linear, its kernel has the structure of a complex vector space, and so is naturally oriented. Then, the orientation of the fiber of the vector bundle over $\{0\} \in [0, 1]$ induces one on the fiber over any other point, and over $\{1\}$ in particular.

b) Kuranishi models

The discussion in [27] relates pseudo-holomorphic submanifolds to $\mu_0 = 0$ solutions to $\mathcal{M}^{(r)}$. For this reason, it is necessary to have a computational scheme for the Seiberg-Witten invariants which involves only the $\mu_0 = 0$ version of (2.4) even in the case where the resulting space $\mathcal{M}^{(r)}$ or $\mathcal{M}^{(r)}_{\Gamma,\Omega}$ has non-smooth points. The scheme used below involves the notion of a Kuranishi model (see (2.26)) for subspaces of $\mathcal{M}^{(r)}$. This subsection considers the Kuranishi model in an abstract setting.

The discussion in this subsection about Kuranishi models is broken into three parts.

Part 1. Suppose that

$$Y \subset (\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1)$$

is a finite dimensional submanifold with compact closure which contains $\mathcal{M}^{(r)}$ and has the additional property that the kernel of the operator L at points $\Xi \in \mathcal{M}^{(r)}$ is contained in $TY|_\Xi$. Such a submanifold Y will be called a "Kuranishi model" for $\mathcal{M}^{(r)}$. More generally, if $\mathcal{N} \in \mathcal{M}^{(r)}$ is a compact subset, then a submanifold

$$Y \subset (\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1)$$

will be called a Kuranishi model for $\mathcal{N}$ if the following two conditions are met:

- Y has compact closure.
- Y contains an open neighborhood $\mathcal{N}_1 \subset \mathcal{M}^{(r)}$ of $\mathcal{N}$.
- If $\Xi \in \mathcal{N}_1$, then $\mathrm{kernel}(L_\Xi) \subset TY|_\Xi$.

(4.3)

The following lemma asserts that Kuranishi models exist:

Lemma 4.1. *Let $\mathcal{N} \in \mathcal{M}^{(r)}$ be a compact set. Then $\mathcal{N}$ has a Kuranishi model.*

Part 2. This part and Part 3 contain the

Proof of Lemma 4.1. When

$$\Xi = (a, (\alpha, \beta)) \in (\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1),$$

introduce the vector subspace, $\mathcal{T}_\Xi$, of elements $(a', (\alpha', \beta'))$ in

$$i \cdot \Omega^1(X) \oplus C^\infty(S_+),$$

which gives 0 for the first line in (2.6). A ball in this vector space $\mathcal{T}_\Xi$ about the origin provides a local chart around Ξ for

$$(\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1)$$

via the map which sends a point $(b, (\eta, \lambda)) \in \mathcal{T}_\Xi$ to the orbit of the configuration $(a + \frac{\sqrt{r}}{2\sqrt{2}}b,\ (a + \eta,\ \beta + \lambda))$ in

$$(\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1).$$

(See, e.g. [15] or [8].) Here and below, it is assumed that (α, β) is not identically zero.

Next, let $C^\infty(X; S^1)$ act on the vector space $i \cdot \Omega^{2+} \oplus C^\infty(S_-)$ via its standard multiplication action on $C^\infty(S_-)$. With this understood, the quotient space

$$(\mathrm{Conn}(E) \times C^\infty(S_+)) \times_{C^\infty(X;S^1)} (i \cdot \Omega^{2+} \oplus C^\infty(S_-)) \tag{4.4}$$

defines a smooth vector bundle over the smooth part of $(\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1)$. Indeed, the chart for the latter given by a small ball in any $\mathcal{T}_\Xi$ gives a trivialization of (4.4) as a vector bundle.

The claim now is that there exists a finite dimensional, smooth subbundle W over a neighborhood of $\mathcal{N}$ in $(\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1)$ of the vector bundle in (4.4) with the following property: At each $\Xi \in \mathcal{N}$, the projection of W onto cokernel(L_Ξ) is surjective.

To prove this claim, first consider a point $\Xi = (a, (\alpha, \beta)) \in \mathcal{N}$. Because the operator L varies continuously with movement in $\mathcal{N}$, there is an open, coordinate neighborhood, U_Ξ, of Ξ in

$$(\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1),$$

which has the property that for all Ξ' in a slightly larger open set, the projection of the kernel of $L_\Xi^\dagger$ onto the cokernel of $L_{\Xi'}$ is surjective. (Take U_Ξ to be a small radius ball in the space $\mathcal{T}_\Xi$. The slightly large open set can be a concentric ball with slightly larger radius.)

Next, use the fact that $\mathcal{N}$ is compact, to find a finite set Γ of points in $\mathcal{N}$ with the property that the corresponding set of open sets $\{U_\Xi\}_{\Xi \in \Gamma}$ covers $\mathcal{N}$.

With Γ understood, here is the remaining task: For each $\Xi \in \Gamma$, extend the vector space kernel$(L_\Xi^\dagger)$ over $\cup_{\Xi'} U_{\Xi'}$ as a vector sub-bundle, W_Ξ of (4.4). Furthermore, the sum of these extensions (not direct sum) should define a vector subbundle of (4.4) over $\cup_\Xi U_\Xi$. That is, a vector subbundle $W \to \cup_\Xi U_\Xi$ of (4.4) should result from the assignment to each point the vector space of linear combinations of elements from the corresponding fibers of $\{W_\Xi\}_{\Xi\in\Gamma}$. (Here, each W_Ξ is a subbundle of (4.4) by assumption.)

The existence of such extensions of $\{\text{kernel}(L^\dagger_\Xi)\}_{\Xi\in\Gamma}$ is a straightforward exercise, using a partition of unity for the (finite) cover $\{U_\Xi\}_{\Xi\in\Gamma}$ and the fact that at the image of L at any point is infinite dimensional. The details of this last part of the argument are left to the reader.

For each Ξ where W is defined, use Π_Ξ to denote the L^2-orthogonal compliment in the fiber of (4.4) over Ξ onto the subspace $W|_\Xi$.

Part 3. Note that the following expression defines a section, $\mathbb{H}$, of (4.4):

- $\frac{1}{\sqrt{r}}\left(P_+F_a + \frac{ir}{8}(1-|\alpha|^2+|\beta|^2\omega - \frac{r}{4}(\alpha\bar{\beta}-\bar{\alpha}\beta)\right)$
- $D_A(\alpha,\beta)$

(4.5)

With W and $\mathbb{H}$ understood, let Y_1 denote the set of points Ξ for which W is defined and which obey the constraint

$$(1-\Pi_\Xi)\cdot\mathbb{H}(\Xi)=0. \tag{4.6}$$

It follows from the definition of W that the differential of (4.6) is surjective along $\mathcal{N}$, and thus the implicit function theorem insures that there is a neighborhood, Y, of $\mathcal{N}$ in Y_1 which has the structure of a smooth, finite dimensional manifold. Since $\mathbb{H}$ vanishes along $\mathcal{N}$, the tangent space to Y_1 along $\mathcal{N}$ is the kernel of the operator $(1-\Pi_\Xi)\cdot L_\Xi$. The latter contains the kernel of L_Ξ. Thus, Y is a Kuranishi model for $\mathcal{N}$.

Note that any Sobolev class L^2_1 solution to (4.6) will consist of C^∞ data. This follows from standard elliptic arguments, since elements in $W|_\Xi$ are smooth sections of $i\cdot\Omega^{2+}\oplus C^\infty(S_-)$, and those with unit L^2 norm obey Ξ-independent bounds on derivatives to all orders. See, e.g. [17].

c) Proof of Proposition 2.14

The proof of Proposition 2.14 is broken into eight steps.

Step 1. Let Y be a Kuranishi model for $\mathcal{M}^{(r)}_{\Gamma,\Omega}$. Introduce the L^2-orthogonal compliment, N_Ξ, of $TY|_\Xi$ in $\mathcal{T}_\Xi$. Because the kernel of L is tangent to Y at points in $\mathcal{M}^{(r)}$, no generality is lost by assuming that at each $\Xi = (a, (\alpha, \beta)) \in Y$, the operator L maps N_Ξ injectively into $i \cdot \Omega^{2+} \oplus C^\infty(S_-)$. One can also assume without loss of generality that Y has compact closure in $(\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1)$.

Note that these last two assumptions can be achieved by "shrinking Y" in the following manner: Take the given Kuranishi model Y and restrict attention to an open neighborhood $Y' \subset Y$ of $\mathcal{M}^{(r)}$. Then rename Y' as Y.

Step 2. At each $\Xi \in Y$, let $W|_\Xi$ denote the quotient of

$$i \cdot \Omega^{2+} \oplus C^\infty(S_-)$$

by $L(N_\Xi)$. This is a finite dimensional vector space, and as Ξ varies in Y, these spaces fit together to define a vector bundle, $W \to Y$. Note that the real K-theory class of the formal difference $TY - W$ has a natural orientation. Indeed, this K-theory class is isomorphic to the index class of the operator L. The latter is represented by the formal difference $\mathrm{kernel}(L) - \mathrm{cokernel}(L)$.) And, the orientation for the index class of L is obtained by deforming L to a $\mathbb{C}$-linear operator as in the previous subsections.

To be more precise about this orientation, suppose, for the sake of argument that $\dim(Y) > 0$. Fix $\Xi \in Y$, and consider Π_Ξ in this step as a projection onto $W|_\Xi$ as a subvector space in $L^2(i \cdot (e_\mathbb{R} \oplus \Lambda_+) \oplus S_-)$. Now, one can choose a family of operators $\{n_t\}_{t\in[0,1]}$ to satisfy the first four lines of (4.1); and so that for each $t \in [0,1]$, the operator $(1 - \Pi_\Xi) \cdot (L_\Xi + n_t)$ has trivial cokernel when mapping from $L^2_1(i \cdot T^* \oplus S_+)$ to the Hilbert space $(1 - \Pi_\Xi) \cdot L^2(i \cdot (\varepsilon_\mathbb{R} \oplus \Lambda_+) \oplus S_-)$.

With the preceding understood, the assignment of $\mathrm{kernel}((1 - \Pi_\Xi) \cdot (L_\Xi + n_t))$ to $t \in [0,1]$ defines a smooth vector bundle $H \to [0,1]$ with the property that $H_0 = TY$. Meanwhile, the association to $v \in H_1$ of $\Pi_\Xi \cdot (L_\Xi - n_1) \cdot v$ defines a map, $K_1 : H_1 \to W|_\Xi$, and one can also require of $\{n_t\}_{t\in[0,1]}$ that K_1 be surjective.

Meanwhile, the kernel of K_1 is the kernel of $L_\Xi + n_1$, and so a complex vector space in a natural way. With this understood, the orientation between H_1 and $W|_\Xi$ is defined by the condition that K_1 define an orientation preserving isomorphism from $H_1/\mathrm{kernel}(K_1)$ to $W|_\Xi$. The choice for an orientation for $H_1 - W|_\Xi$ then induces one for $H_0 - W|_\Xi = TY|_\Xi - W_\Xi$ (since an orientation on H_0 induces one on H_1).

Step 3. Introduce $Y_{\Gamma,\Omega}$ as in (2.27) and the bundle $V = (\oplus_{x\in\Omega} E|_x) \oplus (\oplus_{\gamma\in\Gamma}(E|_q/\nabla\alpha(T\gamma|_q)))$ as in (2.8). Note that the kernel of the homomorphism G in (2.27) is the same as $TY_{\Gamma,\Omega}$. Thus, G defines an isomorphism between the normal bundle of $Y_{\Gamma,\Omega}$ in Y and the vector bundle V. And, since V has a natural orientation, the normal bundle of $Y_{\Gamma,\Omega}$ in Y can be oriented by declaring that G preserve orientation. With this last point understood, the orientation for $TY - W$ in the previous step induce an orientation of the virtual bundle $TY_{\Gamma,\Omega} - W$.

Note that the existence of a regular Kuranishi model for $\mathcal{M}^{(r)}_{\Gamma,\Omega}$ can be assumed by choosing Γ and Ω in a sufficiently generic way. Indeed, the first point in (2.27) can be achieved at Ξ in $\mathcal{M}^{(r)}_{\Gamma,\Omega}$ by making a generic choice of Γ, Ω. With this understood, suppose that Y is a Kuranishi model for $\mathcal{M}^{(r)}_{\Gamma,\Omega}$. If the second point of (2.27) holds at all $\Xi \in \mathcal{M}^{(r)}_{\Gamma,\Omega}$, then by shrinking Y if necessary, the first two points of (2.27) can be assumed to hold at all points of $Y_{\Gamma,\Omega}$. This then guarantees that third point of (2.27). On the other hand, if the second point of (2.27) does not hold at all points $\Xi \in \mathcal{M}^{(r)}_{\Gamma,\Omega}$, then Y can be "enlarged" to insure that it does. This enlarging process simply replaces Y by a neighborhood of the zero section in a finite dimensional subbundle of the normal bundle of Y in $(\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1)$.

Step 4. This step constructs a canonical section over Y of the bundle W whose zero set is homeomorphic to $\mathcal{M}^{(r)} \cap Y$. Thus, the restriction, w, of this section to $Y_{\Gamma,\Omega}$ will have $w^{-1}(0) = \mathcal{M}^{(r)}_{\Gamma,\Omega}$.

The construction is as follows: At $\Xi \in Y$, let Π_Ξ denote the L^2 orthogonal projection onto the L^2 orthogonal compliment of $L(N_\Xi)$. Solve for $x = x(\Xi) \in N_\Xi$ with the property that $(1-\Pi_\Xi)\cdot\mathbb{H}(\Xi+x) = 0$. Here, $\mathbb{H}(\cdot) \in i\cdot\Omega^{2+} \oplus C^\infty(S_-)$ assigns to $(a,(\alpha,\beta))$ the expression in (4.5). The implicit function theorem insures that there is a unique small solution $x = x(\Xi)$ on some neighborhood of $\mathcal{M}^{(r)}$ in Y which varies smoothly as a function of the point Ξ. By shrinking Y if necessary, one can assume that $x(\cdot)$ is defined on the whole of Y. Now, the section w of W is defined by the assignment of $\Pi_\Xi \cdot \mathbb{H}(\Xi + x(\Xi)) \in W|_\Xi$ to $\Xi \in Y$.

Because the assignment of $\Xi \in Y$ to $x(\Xi)$ is a section of Y's normal bundle, and Y is assumed to contain $\mathcal{M}^{(r)}$, one can assume (by shrinking Y if necessary) that $\mathcal{M}^{(r)} = w^{-1}(0)$.

Step 5. Put the results from the preceding steps away for the time being to consider the computation of (1.5). In particular, compute (1.5) using Proposition 2.3 with the moduli space $\mathcal{M}^{(r)}[\mu_0]$ of equivalence classes of solutions to (2.4) as defined with $\mu_0 \neq 0$ very small and r very large.

This step considers this space $\mathcal{M}^{(r)}[\mu_0]$. In particular, note that $\mathcal{M}^{(r)}[\mu_0]$ must be very close to $\mathcal{M}^{(r)} \equiv \mathcal{M}^{(r)}[0]$ since both are compact and the defining equation gives uniform apriori estimates of solutions which depend continuously on the choice for μ_0. The closeness of $\mathcal{M}^{(r)}[\mu_0]$ to $\mathcal{M}^{(r)}$ allows for the following construction of the former: At each $\Xi \in Y$, solve for small $x = x(\Xi) \in N_\Xi$ which makes

$$(1 - \Pi_\Xi) \cdot (\mathbb{H}(\Xi + x) - ir^{-1/2}(\mu_0, 0)) = 0. \tag{4.7}$$

When μ_0 has small norm, there will be a unique, small solution $x(\Xi)$ to this equation for each $\Xi \in Y$. Furthermore, by shrinking Y if necessary, one can be sure that when μ_0 is small, then the assignment to $\Xi = (a, (\alpha, \beta))$ of $\Xi + x(\Xi)$ in $\text{Conn}(E) \times C^\infty(S_+)$ defines a smooth embedding, $\Phi : Y \to (\text{Conn}(E) \times C^\infty(S_+))/C^\infty(\Xi; S^1)$ which is evidently isotopic to the identity. (The isotopy sends $t \in [0,1]$ and Ξ to $\Xi + t \cdot x(\Xi)$.) In addition, the assignment to Ξ of $\Pi_\Xi \cdot (\mathbb{H}(\Xi + x) - i \cdot r^{-1/2} \cdot (\mu_0, 0))$ can be assumed to define a smooth section, $w[\mu_0]$, of the bundle W over Y.

(The meaning of the term "small" as used above can be made precise as follows: There exists $\xi > 0$ such that if the L^2 norm of μ_0 is less than ξ, then there will be a unique L^2_1 solution $x(\Xi)$ with L^2_1 norm less than ξ^{-1}. Furthermore, for each $k \geq 0$, there exists $\xi_k \geq 1$ such that the L^2_{k+1} norm of $x(\Xi)$ will be bounded by $\xi_k \cdot ||\mu_0||_{2,k}$.)

With the preceding understood, it is a straightforward exercise to verify that

$$\mathcal{M}^{(r)}[\mu_0] = \Phi(w[\mu_0]^{-1}(0)). \tag{4.8}$$

Step 6. Now consider the operator L for $\Phi(\Xi)$ when $\Xi \subset w[\mu_0]^{-1}(0)$. Let $w[\mu_0]_*$ denote the differential of $w[\mu_0]$ at a zero of $w[\mu_0]$, understood as a linear map from the fiber of TY to that of W. The claim is that Φ_* intertwines $\text{kernel}(w[\mu_0]_*|_\Xi)$ with $\text{kernel}(L)$. Here is why: If $v \in TY|_\Xi$, then the component of $L_{\Phi(\Xi)}(v + x_* v)$ in $i \cdot \Omega^{2+} \oplus C^\infty(S_+)$ is annihilated by $(1 - \Pi_\Xi)$. This follows from the definition of Φ and because $\Xi \in w[\mu_0]^{-1}(0)$. Furthermore, if v is annihilated by the differential of $w[\mu_0]$, then this same component is also annihilated by Π_Ξ. Thus, $L_{\Phi(\Xi)}(v + x_* v)$ lies in the $i \cdot \Omega^0$ summand of the range. But then there is unique tangent vector $u(v) \in i \cdot \Omega^1 \otimes C^\infty(S_+)$ to the $C^\infty(X; S^1)$ orbit through the point $\Xi + x(\Xi)$ which, when added to $v + x_* v$, puts the result in the kernel of $L_{\Phi(\Xi)}$. In this regard, note that the element $v + x_* v + u(v)$ can not vanish when μ_0 is small because the L^2_3 norm of $u(v)$ is bounded by $\xi \cdot ||v||_{3,2} \cdot ||\mu_0||_{2,2}$ where ξ can be assumed independent of Ξ.

Conversely, if v' is annihilated by $L_{\Phi(\Xi)}$, then there is a unique choice of tangent $u(v')$ to the orbit of $C^\infty(X; S^1)$ through $\Xi + x(\Xi)$ so that $v' + u(v)$ is in the space $\mathcal{T}_\Xi$. With the preceding understood, write $v' + u(v) = v + v_1$, where v is tangent to Y at Ξ, and $v_1 \in N_\Xi$. Then, the vanishing of the $(1 - \Pi_\Xi)$ projection of the $i \cdot \Omega^{2+} \oplus C^\infty(S_-)$ part of $L_{\Phi(\Xi)}(v' + u(v))$ implies that $v_1 = x_* v$. (Use the implicit function theorem here.) And, given this last fact, the vanishing of the Π_Ξ projection of the part of $L_{\Phi(\Xi)}(v' + u(v))$ in $i \cdot \Omega^{2+} \oplus C^\infty(S_-)$ implies that v is in the kernel of $w[\mu_0]_*$.

Step 7. The fact that Φ_* intertwines the kernels of $w[\mu_0]_*$ and L shows that $\mathcal{M}^{(r)}[\mu_0]$ consists of smooth points precisely when $w[\mu_0]$ has transverse zeros. Furthermore, the implicit function theorem can be used to show that when μ_0 is small, then $\mathcal{M}^{(r)}[\mu_0]_{\Gamma,\Omega}$ obeys (2.7) precisely when $w[\mu_0]$ has transverse zeros when restricted to $Y_{\Gamma,\Omega}$.

Step 8. According to Proposition 2.3, a count with ± 1 weights of the points in $\mathcal{M}^{(r)}[\mu_0]_{\Gamma,\Omega}$ gives (1.5). On the otherhand, a count of the points in $w[\mu_0]^{-1}(0) \cap Y_{\Gamma,\Omega}$ with ± 1 weights yields the Euler class computation in Proposition 2.14. Thus, Proposition 2.14 follows by demonstrating that the ± 1 weights in the two counts are the same. The latter task is accomplished below in the $d = 0$ case. The general case is left to the reader.

To begin, reinterpret the projection Π_Ξ as the L^2-orthogonal projection onto $W|_\Xi$ in $i \cdot \Omega^0 \oplus i \cdot \Omega^{2+} \oplus C^\infty(S_-)$. Remember that $W|_\Xi$ has zero projection into the first summand. Now consider the family of operators $L_{\Phi(\Xi)} + n_t$ as in (4.1). Assuming that W has positive fiber dimension, one can use analytic perturbation theory from [7] to find n_t as in (4.1) with the property that for each $t \in [0, 1]$, the operator $L_{\Phi(\Xi)} + n_t$ maps $L^2_1(i \cdot T^* \oplus S_+)$ surjectively onto

$$(1 - \Pi_\Xi) \cdot L^2(i \cdot (\varepsilon_{\mathbb{R}} \oplus \Lambda_+) \oplus S_-).$$

Then, for each t, let $H_t \subset L^2_1(i \cdot T^* \oplus S_+)$ denote the kernel of $(1 - \Pi_\Xi) \cdot (L_{\Phi(\Xi)} + n_t)$. This is a finite dimensional vector space, and as t varies through $[0, 1]$, the latter define a vector bundle $H \to [0, 1]$. Note that the argument which showed that Φ_* intertwines kernel$(w[\mu_0]_*)$ with kernel(L) proved that $H_0 = TY|_\Xi$.

Now, consider that the association of $v \in H_t$ to

$$\Pi_{\Xi} \cdot ((L_{\Phi(\Xi)} + n_t)(v)) \in W_{\Xi}$$

defines a homomorphism, K_t; and as t varies in $[0, 1]$, the latter fit together to define a homomorphism of vector bundles

$$K : H \to [0, 1] \times W|_{\Xi}.$$

Furthermore, $K_0 = w[\mu_0]_*$. Note that K_t has a kernel precisely when $L_{\Phi(\Xi)} + n_t$ has a kernel, so the spectral flows for the family $\{L_{\Phi(\Xi)} + n_t\}$ and for the family $\{K_t\}$ agree. With this understood, the sign comparison is completed with the remark that the definition of the orientation for $TY|_{\Xi} - W|_{\Xi}$ sets $\det(K_1) = 1$. To be precise, the orientation on $TY|_{\Xi} - W|_{\Xi}$ was defined by considering the corresponding K_1 for a family which took L_{Ξ} to a $\mathbb{C}$-linear operator. However the latter family could have been chosen so that its endpoint also equaled $L_{\Phi(\Xi)} + n_1$.

5. The proof of Proposition 2.10 and the image of $\Psi_{h,r}$

The purpose of this section and Section 6 is to prove Proposition 2.10. Since all but the final assertion are discussed in Section 2, the focus here is on the final assertion. In this regard, Proposition 2.5 will be used to reduce the final assertion of Proposition 2.10 to a special case of Proposition 5.1, below.

Proposition 5.1 addresses the issue of whether or not the constructions from Section 5 of [28] capture all of the large r and $\mu_0 = 0$ solutions to (2.4) which lie in $\mathcal{M}^{(r)}_{\Gamma,\Omega}$. The statement of said proposition involves some complicated preconditions, and so a preliminary, digression is required to set the stage. (These complicated preconditions are due to the fact that singular pseudo-holomorphic curves can at times arise from limits of sequences of solutions to (2.4). Per force, such sequences do not come from the gluing construction of the previous subsections because that construction starts by choosing a pseudo-holomorphic submanifold. Most probably, there is an extension of the gluing construction to allow singular pseudo-holomorphic curves in X, but any such extension would lengthen an already lengthy story.) The digression below has five parts.

Part 1. Let $C \subset X$ be a compact, pseudo-holomorphic submanifold. Let $N \to C$ denote the normal bundle to C in X. Specify an almost complex structure J_∞ on TN by the condition that the J_∞ version of the (1, 0) part of T^*N be spanned locally by forms $\lambda_0 \in \pi^* T^{1,0} C$

and $\lambda_1 = \nabla_\theta s + \pi^*\nu \cdot s + \pi^*\mu \cdot \bar{s}$. Here, ν is the section of $T^{0,1}C$ and μ is the section of $N^2 \otimes T^{0,1}C$ which appear in (1.11). Also, s is the tautological section of π^*N over N.

Part 2. Introduce the notion of a *constraint set* for C. This is a finite (possibly empty) set, K, of disjoint subsets of N; where each element is either a point in C or else a real line through the origin in a fiber of $\pi : N \to C$.

Part 3. Let $m \geq 1$ be an integer and let K be a constraint set of C. A submanifold C will be called (m, K)-rigid when every J_∞- pseudo-holomorphic map into N which satisfies the following three conditions factors through C (the zero section of N):

1. The domain is a compact, complex curve.
2. The push forward of the fundamental class of the domain curve is a multiple of $[C]$ which divides m.
3. The image of the map intersects all members of K.

(5.1)

For example, if $m = 1$ and the operator from $C^\infty(C; N)$ to $C^\infty(C; N \otimes T^{0,1}C)$ which sends a section h to

$$Dh = \bar{\partial}_\theta h + \nu h + \mu \bar{h} \tag{5.2}$$

has trivial kernel, then C is $(1, \varnothing)$-rigid.

Part 4. Let $m \geq 1$ and let $y = (y_1, \dots, y_m)$ be a section of $\oplus_{1 \leq q \leq m} N^q$. Let K be a constraint set for C. Then y will be said to intersect all members of the constraint set K when the zero set of the section $p(y) = s^m + \pi^* y_1 \cdot s^{m-1} + \cdots + \pi^* y_m$ of $\pi^* N^m$ contains all points of K and also intersects all lines from K. (Remember that s is the tautological section of $\pi^* N \to N$.)

Part 5. Suppose K is a constraint set for C. An *extension* of K is the set of subsets of X which consists of the points in K (as elements in $C \subset X$) and a collection of properly embedded, open arcs in some tubular neighborhood of C. Here, the arc components of the extension are indexed by the line components of K as follows: The arc which corresponds to the line $\gamma \in K$ intersects C in a single point, the point where γ intersects C. Furthermore, the projection to N of the tangent line of the arc at its intersection point with C should give γ.

End the digression.

Proposition 5.1. *Let $\{(C_k, m_k)\}$ be a finite set of pairs consisting of a compact, pseudo-holomorphic submanifold C_k and a positive integer m_k with the property that the submanifolds in the set $\{C_k\}$ are pairwise disjoint. Fix a constraint set K_k for each C_k and suppose that each C_k is (m_k, K_k)-rigid. For each k, let $e_k \in H^2(X; \mathbb{Z})$ denote the Poincaré dual to $[C_k]$, and let $e = \Sigma_k m_k \cdot e_k$. Let $E \to X$ be a complex line bundle with first Chern class e, and use E to define the Spin$^{\mathbb{C}}$ structure as in (1.9). Let $\{r_j\}$ be an unbounded, increasing sequence of positive, real number and, for each j, let $\{(a_j, (\alpha_j, \beta_j))\}$ be a sequence of solutions to (2.4) using the Spin$^{\mathbb{C}}$ structure in (1.9) and using $r = r_j$ and $\mu_0 = 0$. Suppose that for each k, an extension of K_k has been chosen, and suppose that for each j, the set $\alpha_j^{-1}(0)$ intersects each element of the chosen extension. Suppose, in addition that*

$$\lim_{j\to\infty}\{\sup_{x:\alpha_j(x)=0} \operatorname{dist}(x, C) + \sup_{x\in C} \operatorname{dist}(x, \alpha_j^{-1}(0))\} \tag{5.3}$$

exists and has limit zero. When j is large, then $(a_j, (\alpha_j, \beta_j))$ is described by Proposition 5.3 in [28]. To be more precise, the following hold:

- *There is, for each k, a compact subset $\mathcal{K}^{(k)}$ in the (C_k, m_k) version of $\mathcal{Z}_0$. With $\mathcal{K}^{(k)}$ understood, choose a subspace Λ_k in the (C_k, m_k) version of $C^\infty((\oplus_{1\le q\le m} N^q) \otimes T^{0,1}C)$ with the property that the projection of Λ_k to cokernel(Δ_y) is surjective for each $y \in \mathcal{K}^{(k)}$. Here, Δ_y is the operator in (2.18).*

- *With $\mathcal{K}^{(k)}$ and Λ_k understood, there is a finite dimensional submanifold $\mathcal{K}^{(k)}_\Lambda \subset C^\infty(\oplus_{1\le q\le m} N^q)$ as described in Lemma 5.1 and (5.1) of [28] in the case $\Lambda = \Lambda_k$.*

- *When j is large, $(a_j, (\alpha_j, \beta_j))$ is contained in the image of the $r = r_j$, map*

$$\Psi_r : \times_k \mathcal{K}^{(k)}_\Lambda \to (\operatorname{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1), \tag{5.4}$$

which is described in Proposition 5.2 of [28].

Proof of Proposition 2.10. As remarked, only the final assertion needs to be proved. To start the proof, note that when $h = \{(C_k, m_k)\} \in \mathcal{H}$, then each (C_k, m_k) comes with a natural constraint set, K_k. This constraint set can be described as follows: Return to (1.13) and introduce the subset Γ_k of Γ with its $2 \cdot p_k$ elements. Each $\gamma \in \Gamma_k$ intersects C_k precisely once, and at a point where its tangent line is not tangent to C_k. Said tangent line then projects to a line throught the origin in the normal bundle to C_k at the intersection point. With this understood, K_k consists of these $2p_k$ lines and the $d_k - p_k$ points in Ω which lie on C_k. The claim now is that under the given assumptions, each C_k is (m_k, K_k)-rigid. Given the claim, then the final assertion of Proposition 2.10 follows from Propositions 2.5 and 5.1.

To prove the claim, consider first the case where $m_k = 1$. In the case where $d_k = 0$, then C_k is $(1, \varnothing)$ rigid when D in (5.2) has no kernel, and this is guaranteed by the choice of (J, Γ, Ω) from the appropriate Baire set. Likewise, when $m_k = 1$ and $d_k > 1$, then C_k is $(1, K_k)$ rigid when the linear homorphism G_k in (1.14) is an isomorphism. Once again, this is guaranteed by the choice of (J, Γ, Ω).

Note that a pseudo-holomorphic map to N which pushes forward the fundamental class to equal $[C]$ must have domain C. This is because the composition of the map with the projection to C is holomorphic and degree 1. The adjunction formula then guarantees that the map in question is an embedding. Because the fibers of N are also pseudo-holomorphic, the image of the map must coincide with the image of a section h of N. The condition that the latter be pseudo-holomorphic is the same as the condition that $Dh = 0$.

In the case where $m_k > 1$, the curve C_k is then a torus and the normal bundle N is topologically trivial. In this case, $K_k = \varnothing$ and the choice of (J, Γ, Ω) guarantees that C_k is $(m_k, \varnothing)$ rigid. This follows from Propositions 2.8 and 3.2.

a) Proof of Proposition 5.1

Suppose that a sequence $\{(a_j, (\alpha_j, \beta_j))\}$ of solutions to (2.4) satisfies (5.3). Then, there is a fundamental distinction between two different sorts of behavior. The distinction is based on the rate at which the sequence of sets $\{\alpha_j^{-1}(0)\}$ converges to $\cup_k C_k$. In order to make this distinction, introduce, for each j, the number

$$\delta_j = \sup_{x:\alpha_j(x)=0} \operatorname{dist}(x, \cup_k C_k). \tag{5.5}$$

According to (5.3), the sequence $\{\delta_j\}$ limits to zero as j tends to ∞.

With the preceding understood, consider the sequence $\{\delta_j \cdot \sqrt{r_j}\}$. Distinguish between the case where

$$\limsup_{j\to\infty} \delta_j \sqrt{r_j} \tag{5.6}$$

is finite, and where it is not finite. The next two propositions explain the significance of this distinction.

Proposition 5.2. *Let $\{(C_k, m_k)\}$ be a finite set of pairs consisting of a compact, pseudo-holomorphic submanifold C_k and a positive integer m_k with the property that the submanifolds in the set $\{C_k\}$ are pairwise disjoint. For each k, let $e_k \in H^2(X;\mathbb{Z})$ denote the Poincaré dual to $[C_k]$, and let $e = \Sigma_k m_k \cdot e_k$. Let $E \to X$ be a complex line bundle with first Chern class e, and use E to define the Spin$^{\mathbb{C}}$ structure as in (1.9). Let $\{r_j\}$ be an unbounded, increasing sequence of positive, real numbers and, for each j, let $\{(a_j, (\alpha_j, \beta_j))\}$ be a sequence of solutions to (2.4) using the Spin$^{\mathbb{C}}$ structure in (1.9) and using $r = r_j$ and $\mu_0 = 0$. Suppose that the limit in (5.3) exists and is zero and that the limit in (5.6) is finite. Then the conclusions of Proposition 5.1 hold for $\{(a_j, (\alpha_j, \beta_j))\}$.*

Take particular notice of the fact that this last proposition makes no mention of any constraint assignments. However, constraint assignments assumptions are required in the proof of Proposition 5.1 when ruling out the case where (5.6) is infinite. To see this, suppose first that (5.6) is infinite. By passing to a subsequence if necessary, one can arrange that there exists a fixed $C \subset \cup_k C_k$ such that for all j, the number δ_j is equal to the distance from some point in $\alpha_j^{-1}(0)$ to C.

Introduce the normal bundle $N \to C$ and its disk subbundle N^0 with its identification (as in Lemmas 2.1 and 2.2 of [28]) with a tubular neighborhood of C in X. (This identification will be implicit in the subsequent discussions.) For each j, consider the fiberwise multiplication by δ_j^{-1} as a map from N to itself. This map sends the disk bundle of radius δ_j into the disk bundle of radius 1. These maps induce (from the given almost complex structure J on N_0) a sequence $\{J_j\}$ of almost complex structures on N which converge in the C^∞ topology on compact subsets to the homogeneous almost complex structure J_∞.

For each j, use multiplication by δ_j^{-1} to push the set $\alpha_j^{-1}(0)$ forward to give a subset of N with at least one point having distance 1 from C. Let $\Sigma_j \subset N$ denote this new set. Note that $[\Sigma_j] = m \cdot [C]$ in $H_2(N;\mathbb{Z})$ where $m \geq 1$ is the integer which is paired with C.

Now, consider

Proposition 5.3. *Make the same assumptions as in Proposition 5.2 except suppose now that (5.6) is infinite. Then there is a compact, complex curve C^0, a J_∞-pseudo-holomorphic map $\psi : C^0 \to N$, a positive integer m^0, and a subsequence of $\{\Sigma_j\}$ (hence relabled consecutively) with the following properties:*

1. *$[C^0]$ is homologous to $m^0 \cdot [C]$.*
2. *$lim_{j\to\infty}\{\sup_{x\in\Sigma_j} \operatorname{dist}(x, \psi(C^0)) + \sup_{x\in\psi(C^0)} \operatorname{dist}(x, \Sigma_j)\}$ exists and is zero.*
3. *The integer m^0 divides m.*

Furthermore, if, for all j sufficiently large, each $\alpha_j^{-1}(0)$ intersects all members of the extension of some given constraint K, then $\psi(C^0)$ intersects all members of K.

Note that these two propositions together imply Proposition 5.1. Here is why: If each C_k is (m_k, K_k)-rigid, then the conclusions of Proposition 5.3 can not be met; and therefore the assumption in Proposition 5.2 that (5.6) is finite must hold. Hence, Proposition 5.1 follows from the conclusions of Proposition 5.2.

It is convenient to prove Propositions 5.2 and 5.3 in reverse order. The remainder of this section is occupied with the proof of Proposition 5.3, while the next section considers Proposition 5.2.

b) Key estimates

There are two sorts of key estimates for solutions $(a, (\alpha, \beta))$ of (2.4) which are used in the proof of Proposition 5.2. These are provided in this subsection. The first estimate describes $\alpha^{-1}(0)$ in small balls, and the second describes the curvature of the connection a in the same sorts of balls. Note that these estimates use the apriori information that $\alpha^{-1}(0)$ is close everywhere to a pseudo-holomorphic submanifold to obtain regularity information. (Here is an analogy: Let $C \subset \mathbb{C}^2$ be a complex analytic subvariety whose points are all within an apriori bounded distance from a complex line L. Then C is a union of complex lines parallel to L.)

To set the stage here, suppose that $\cup_k C_k \subset X$ is a compact, pseudo-holomorphic submanifold, and that $(a, (\alpha, \beta))$ is a large r solution to (2.4) with the property that each component of $\alpha^{-1}(0)$ is contained in a tubular neighborhood of some component of $\cup_k C_k$. Furthermore, assume that for each component $C \subset \cup_k C_k$,

$$\delta = \max\{r^{-1/2}, \sup_{x \in N^0 : \alpha(x)=0} \operatorname{dist}(x, C)\} \tag{5.7}$$

is much less than the radius of a disk bundle N^0 for C. Here, the radius of N^0 should be chosen so that the exponential map from Lemmas 2.1 and 2.2 in [28] can be used to implicity identify N^0 with a tubular neighborhood of C in X. Also, this neighborhood should be disjoint from $\cup_k C_k - C$.

With the preceding understood, focus attention on a single component $C \subset \cup_k C_k$. Note that the homology class carried by $\alpha^{-1}(0) \cap N^0$ is some multiple of $[C]$, and let m denote this multiple.

The first key lemma is stated below. In the statement of the lemma, θ denotes the Hermitian connection on N which is induced by the metric connection on TX.

Lemma 5.4. *Suppose that $E \to X$ is a complex line bundle. Let C be as described above. Given $\varepsilon > 0$, there exists a constant $\zeta_\varepsilon \geq 1$ with the following significance: Suppose that $r \geq \zeta_\varepsilon$ and that $(a, (\alpha, \beta))$ is a solution to (2.4) as described above for the* $\mathrm{Spin}^{\mathbb{C}}$ *structure which is defined by E. Suppose also that δ is given by (5.7) with $\delta < 1/\zeta_\varepsilon$. Let $z \in C$ and let $B \subset C$ be the disk of radius δ and center z. Identify $N|_B$ with $B \times \mathbb{C}$ by parallel transport using the connection θ along the radial geodesics out from z. Then, there is a set $\Lambda \subset \mathbb{C}$ of m or less points such that for any $z' \in B$, every point in $\pi^{-1}(z') \cap \alpha^{-1}(0)$ has distance $\varepsilon \cdot \delta$ or less from a point in Λ, and vice versa.*

As remarked above, the second key lemma concerns the curvature of the connection a. To state the lemma, consider a disk $B \subset C$, and introduce, on $N^0|_B$, an orthonormal basis $\{\kappa_0, \kappa_1\}$ for the J-version of $T^{1,0}C$ with the following properties: First, κ_0 is a section of the $\pi^* T^* C$ summand which is in $T^{1,0}C$ on C. Second, $\kappa_1 = \varsigma \cdot \nabla_\theta s + \sigma$ where σ is a section of $\pi^* T^* C \otimes \pi^* N$ which vanishes along C, and where ζ is a real valued function which behaves near C as $\varsigma = 1 + \mathcal{O}(|s|^2)$. (See Section 2a of [28].)

Use the basis $\{\kappa_0, \kappa_1\}$ to expand the curvature of the connection a as

$$\begin{aligned} F_a = & f_0 \kappa_0 \wedge \bar{\kappa}_0 + f_1 \kappa_1 \wedge \bar{\kappa}_1 + f_+ \kappa_0 \wedge \kappa_1 \\ & - \bar{f}_+ \bar{\kappa}_0 \wedge \bar{\kappa}_1 + f_- \kappa_0 \wedge \bar{\kappa}_1 - \bar{f}_- \bar{\kappa}_0 \wedge \kappa_1. \end{aligned} \tag{5.8}$$

Here, f_0 and f_1 are real valued functions on N^0, while $f_\pm$ are sections of appropriate line bundles over N^0. The goal is to estimate the L^1 norms of f_0 and $f_\pm$ and f_1 over $\pi^{-1}(B)$.

Lemma 5.5. *There is a constant $\zeta \geq 1$ with the following significance. Suppose that $E \to X$ is a complex line bundle, that $r \geq \zeta$ and that $(a,(\alpha,\beta))$ is a solution to (2.4) as described above for the $Spin^{\mathbb{C}}$ structure which is defined by E. Suppose that δ is given by (5.7), and that this number is much less than the radius of the disk bundle N^0. Let $B \subset C$ be a disk of radius δ. Then:*

1. $|\int_{\pi^{-1}(B)} f_1 - \pi^2 m\delta^2| \leq \zeta\delta^2 e^{-\sqrt{r}/\zeta}$.

2. $\int_{\pi^{-1}(B)}(|f_1| - f_1) \leq \zeta\delta^2 r^{-1}$.

3. $\int_{\pi^{-1}(B)} |f_+| \leq \zeta r^{-1/2}\delta^2$.

4. $\int_{\pi^{-1}(B)}(|f_0| + \delta^{1/2}|f_-|) \leq \zeta\delta^3$.

(5.9)

c) Proof of Lemmas 5.4 and 5.5

This subsection is occupied with the proofs of the preceding two lemmas.

Proof of Lemma 5.4. The proof is by contradiction. Suppose, given $\varepsilon > 0$, no such constant ζ_ε exists. Then, one can find an unbounded sequence of values for r, and a sequence of values for δ tending to zero; and for each such pair of (r,δ), one could find a corresponding solution $(a,(\alpha,\beta))$ to (2.4) as described in the lemma with a point $z_0 \in C$ which violated the conclusions of the lemma. To obtain a contradiction, for each element in the sequence, take the corresponding point z_0 and fix complex, Gaussian coordinates (z,η) for a neighborhood of z_0 in N^0. Here, the $\eta = 0$ surface should be tangent to C at the origin (which is z_0). Now, dilate this coordinate system by δ^{-1} so that the radius δ ball centered at 0 becomes the radius 1 ball centered at z_0.

Corresponding to each pair of (r,δ) values is the corresponding solution $(a,(\alpha,\beta))$ to (2.4). Pull this solution back to the dilated ball (as in Section 4 of [27]) and denote the result as $(a',(\alpha',\beta'))$.

Now, there are two cases to consider. In the first case, the sequence of (r,δ) values is such that $\sqrt{r}\cdot\delta$ is bounded. Here, the analysis of

Section 4 in [27] can be applied directly to argue that the sequence of sets $\{\alpha'^{-1}(0)\}$ has a subsequence (hence relabled consecutively) which converges nicely on compact domains to a 2-dimensional, complex algebraic variety $S \subset \mathbb{C}^2$ having m or less components. Furthermore, no point in S can have distance more than 1 from the plane $\eta = 0$ since no point in $\alpha^{-1}(0)$ has distance more than δ from C. It follows then that S is a set of m or less planes, each parallel to the plane h $= 0$. Thus, the nature of the convergence of the sequence $\{\alpha'^{-1}(0)\}$ to S as described in Section 4 of [27] contradicts the assumption that the sequence was obtained from a violation of the conclusions of Lemma 5.4.

The second case has the sequence of values for $\sqrt{r} \cdot \delta$ unbounded. Here, the dilated fields $(a', (\alpha', \beta'))$ solve the Seiberg-Witten equations on compact subsets of $\mathbb{C}$ with r replaced by $r' = r \cdot \delta^2$. These solutions obey estimates as in (1.25) of [27] using this value of r', since one can simply rescale the estimates for $(a, (\alpha, \beta))$ from X. With this understood, one can then repeat the arguments in Sections 5 and 6 of [27] to conclude that, again, the corresponding sequence of sets $\{\alpha'^{-1}(0)\}$ has a subsequence (hence relabled consecutively) which converges on compact domains in $\mathbb{C}^2$ to a complex, algebraic, dimension 2 subvariety $S \subset \mathbb{C}^2$ with m or less components, and whose points all lie at distance 1 or less from the plane $\eta = 0$. Thus, S is, again, a finite set of m or less planes, all parallel to $\eta = 0$. The nature of this convergence (as detailed in Section 6 of [27]) contradicts the assumption that the sequence was obtained from a violation of the conclusions of Lemma 5.4.

Proof of Lemma 5.5. The proof has nine steps. Before starting, remark that in the course of this proof and in subsequent proofs through out this section, the symbol ζ will represent the "generic" constant, that is, a number which is larger than 1 and whose value is independent of r, $(a, (\alpha, \beta))$ and of δ. Thus, ζ depends only on the metric and symplectic form near C, and on the first Chern class of the line bundle E. Furthermore, the precise value of ζ is allowed to change from line to line. This convention obviates the need to label such constants with subscripts. (Imagine labeling the lines in this article and then implicitly labeling each occurrence of ζ by the line on which it appears.)

Step 1. This first step estimates f_1. For this purpose, remark that the integral of f_1 over a fiber $\pi^{-1}(z)$ of N^0 is equal to $i/2$ times that of F_a over $\pi^{-1}(z)$. It follows from the estimates in (1.24) of [27] that the latter is equal (up to an error of size $\zeta \cdot e^{-\sqrt{r}/\zeta}$) to π times the evaluation of $c_1(E)$ on a class in $H_2(X; \mathbb{Q})$ which has intersection number 1 with C

and intersection number zero with the remaining components of $\cup_k C_k$. The value of $c_1(E)$ on the latter class is equal to the integer m.

Step 2. This step estimates $|f_1| - f_1$. The estimate uses the following two results:

Lemma 5.6. *Fix a complex line bundle $E \to X$ and there is a constant $\zeta \geq 1$ which depends only on $c_1(E)$ and on the Riemannian metric and which has the following significance: Let $r > \zeta$ and suppose that $(a, (\alpha, \beta))$ is a solution to (2.4) with the* $\mathrm{Spin}^{\mathbb{C}}$ *structure as in (1.9). Then,*

$$|P_- F_a| - \frac{r}{4\sqrt{2}}(1 - |\alpha|^2) \leq \zeta \exp[-\zeta^{-1/2} r^{1/2} \operatorname{dist}(x, \alpha^{-1}(0))]. \tag{5.10}$$

Lemma 5.7. *Fix a complex line bundle $E \to X$ and there is a constant $\zeta \geq 1$ which depends only on $c_1(E)$ and on the Riemannian metric and which has the following significance: Let $r > \zeta$ and suppose that $(a, (\alpha, \beta))$ is a solution to (2.4) with the* $\mathrm{Spin}^{\mathbb{C}}$ *structure as in (1.9). Let $B \subset C$ be a ball of radius $t \in (r^{-1/2}, z^{-1})$ and let $s \in (r^{-1/2}, t)$. Then, there exists a set Ω of less than $\zeta \cdot t^2/s^2$ balls of radius s with the following properties:*

1. *Each ball has center on $\alpha^{-1}(0) \cap B$.*
2. *For each ball in Ω, construct the concentric ball with radius $s/2$. Then, the resulting set consists of pairwise disjoint balls.*
3. *Every point in B with distance $\zeta^{-1} \cdot s$ or less from $\alpha^{-1}(0)$ is contained in a ball from Ω.*

Assume these last two results momentarily, to continue with the proof of (5.9.2). First of all, it follows from the definition of $\{\kappa_0, \kappa_1)$ that $\omega = i \cdot 2^{-1} \cdot (\kappa_0 \wedge \bar{\kappa}_0 + \kappa_1 \wedge \bar{\kappa}_1)$. Thus, (2.4) implies that

$$f_0 + f_1 = 8^{-1} r(1 - |\alpha|^2 + |\beta|^2), \tag{5.11}$$

and that $|f_0 - f_1| \leq 2^{-1/2} \cdot |P_- F_a|$. With the preceding understood, it follows from (5.10) and (5.11) that

$$|f_0 - f_1| - (f_0 + f_1) \leq \zeta \cdot \exp[-\zeta^{-1/2} r^{1/2} \operatorname{dist}(x, \alpha^{-1}(0))]. \tag{5.12}$$

Thus,

- $f_0 \geq -\zeta \exp[-\zeta^{-1} r^{1/2} \operatorname{dist}(x, \alpha^{-1}(0))]$.
- $f_1 \geq -\zeta \exp[-\zeta^{-1} r^{1/2} \operatorname{dist}(x, \alpha^{-1}(0))]$.

(5.13)

It follows from (5.13) that $|f_1| - f_1 \leq \zeta \cdot \exp[-\zeta^{-1} r^{1/2} \operatorname{dist}(x, \alpha^{-1}(0))]$.

The integral over B of $\exp[-\zeta^{-1} r^{1/2} \operatorname{dist}(x, \alpha^{-1}(0))]$ can be estimated by considering, for each integer $n \geq 1$, the integral of the latter over the region in B consisting of points with distance from $\alpha^{-1}(0)$ in the range from $(n-1) \cdot r^{-1/2}$ to $n \cdot r^{-1/2}$. Using Lemma 5.7, the contribution from this last set is no greater than

$$\zeta e^{-n/\zeta} (n^4 r^{-2})(\delta^2/(n^2 r^{-1})) \leq \zeta \delta^2 r^{-1} n^2 e^{-n/\zeta}. \tag{5.14}$$

Thus, the sum over all non-negative, integer n of (5.14) yields

$$\int_B \exp[-\zeta^{-1} r^{1/2} \operatorname{dist}(x, \alpha^{-1}(0))] \leq \zeta \delta^2 r^{-1}. \tag{5.15}$$

Equation (5.9.2) follows directly from (5.15).

Proof of Lemma 5.6. This assertion is proved by mimicking the proof in [27] of the assertion for $(1 - |\alpha|^2)$ in Proposition 4.4 in [27]. Indeed, agree to denote the left-hand side of (5.10) by y, and then this function obeys equation (4.22) in [27] with appropriate constants. Given Proposition 3.4 in [27], the argument subsequent to this equation gives the bound in (5.10).

Proof of Lemma 5.7. Mimic the proof of Lemma 3.6 in [27]. The point is that each ball of radius $s/2$ with center on $\alpha^{-1}(0)$ contributes some $\zeta^{-1} \cdot s^2$ to the integral $r^{-1} \cdot (1 - |\alpha|^2)$ over B (see Proposition 3.1 of [27]). On the other hand, said integral can be no bigger than $\zeta \cdot \delta^2$ (see Proposition 3.1 in [27]).

Step 3. This step considers the estimate for f_+ in (5.9). For this, note that $|f_+|$ is bounded by a fixed multiple of $r \cdot |\beta|$, and the latter is bounded, curtesy of (1.24) in [27], by

$$\zeta r^{1/2} (\exp[-\zeta^{-1} r^{1/2} \operatorname{dist}(x, \alpha^{-1}(0))] + r^{-1}).$$

The integral over B of the latter can be estimated using (5.15).

Step 4. This step estimates $|f_-|$ in terms of f_0 and f_1. To begin, let p and q be complex numbers, and consider the expression

$$f_0 \cdot |p|^2 + f_1 \cdot |q|^2 - 2 \cdot \operatorname{Re}(f_- \cdot q\bar{p}). \tag{5.16}$$

(Let $\kappa = p \cdot \kappa_0 + q \cdot \kappa_1$, which is (1,0) form on N^0. Then, (5.16) is $- * (\kappa \wedge \bar{\kappa} \wedge F_a)$.) Since $2^{-1}(f_0 + f_1)$ is the component of $i \cdot F_a$ along ω, one finds that (5.16) is no smaller than

$$(5.17) \qquad (2\sqrt{2})^{-1}(|p|^2 + |q|^2) \cdot \left(\frac{r}{4\sqrt{2}}(1 - |\alpha|^2 + |\beta|^2) - |P_- F_a| \right).$$

Now use (5.10) and (5.17) to conclude that (5.16) is no smaller than

$$(5.18) \qquad -\zeta \cdot (|p|^2 + |q|^2) \cdot \exp[-\zeta^{-1/2} r^{1/2} \operatorname{dist}(x, \alpha^{-1}(0))].$$

This last equation is true for any choice of p and q. In particular, take $p = \pm R \cdot f_-/|f_-|$, where $R > 0$ is arbitrary, and take $q = \delta$. With these choices understood, it follows that

$$(5.19) \qquad \begin{aligned} R \cdot \delta \cdot |f_-| \leq \zeta \cdot \big((\delta^2 + R^2) \cdot \exp[-\zeta^{-1/2} r^{1/2} \operatorname{dist}(x, \alpha^{-1}(0))] \\ + \delta^2 \cdot f_1 + R^2 \cdot f_0 \big). \end{aligned}$$

This last expression gives the desired bound for $|f_-|$.

Step 5. This step begins the task of estimating the integral of $|f_0|$ over B. For this purpose, introduce a standard bump function χ on $[0, \infty)$ which is non-increasing and obeys $\chi(t) = 1$ where t is less than 1, and $\chi(t) = 0$ where t is greater than 2. Promote χ to a function, χ_C, on C by setting $\chi_C(z) = \chi(\operatorname{dist}(z, z_0)/\delta)$.

Now, introduce the 1-form

$$(5.20) \qquad \gamma = i2^{-1} \chi_C \cdot (s \cdot \nabla_\theta \bar{s} - \bar{s} \nabla_\theta s).$$

Note that the exterior derivative of γ is

$$(5.21) \quad d\gamma = i \chi_C \nabla_\theta s \wedge \nabla_\theta \bar{s} - i \cdot |s|^2 \chi_C F_\theta + i2^{-1} d\chi_C \cdot (s \cdot \nabla_\theta \bar{s} - \bar{s} \nabla_\theta s).$$

Step 6. Consider the integral over N^0 of the form $d\gamma \wedge i \cdot F_a$. Use integration by parts to express this as an integral over ∂N^0, and use (1.24.5) in [27] and the fact that δ is much smaller than the distance from $\alpha^{-1}(0)$ to the boundary of $X - N^0$ to conclude that

$$(5.22) \qquad \left| \int_{N^0} d\gamma \wedge i \cdot F_a \right| \leq \zeta \, e^{-\sqrt{r} \cdot \delta / \zeta}.$$

Step 7. With (5.21), this last equation can be used to estimate the relative sizes of the various components of the curvature of F_a. Indeed, the immediate implication is

$$\left|\int_{\pi^{-1}(B)} f_0\right| \leq \zeta \int_{\pi^{-1}(B')} (\delta^2|f_1| + |f_+| + |f_-|). \tag{5.23}$$

Here, $B' \subset C$ is the ball of radius $2 \cdot \delta$ having the same center as B.

To exploit (5.23), invoke the first line of (5.13) and (5.15) to obtain an estimate for the integral over $\pi^{-1}(B)$ of $|f_0|$. Next, invoke (5.19) with $R = \varepsilon' \cdot \delta$ and $\varepsilon' \leq 1$. Here, choose $\varepsilon' > 0$ so as to be independent of r, δ, and $(a, (\alpha, \beta))$; and so that (5.19) and (5.23) (plus (5.8.1) and (5.8.2)) imply the estimate

$$\int_{\pi^{-1}(B)} |f_0| \leq \zeta\delta^3 + (100)^{-1} \int_{\pi^{-1}(B')} |f_0|. \tag{5.24}$$

Step 8. Now, the assignment to z_0 in B of the number that is given by the left-hand side of (5.24) defines a continuous function on C, which thus has a maximum. Consider z_0 now where this maximum occurs. If δ is small (less than some ζ^{-1}), then the ball B' can be covered by less than 50 balls of radius δ. This means that the integral term on the right-hand side of (5.24) (with the factor of 1/100 in front) is no greater than 1/2 of the number on the left-hand side of (5.24). Thus, the left side of (5.24) is no greater than $\zeta \cdot \delta^3$ where z_0 is chosen for the left side to be its largest. Hence, for any choice of z_0,

$$\int_{\pi^{-1}(B)} |f_0| \leq \zeta\delta^3. \tag{5.25}$$

Step 9. The estimate for the integral of $\delta^{1/2} \cdot |f_-|$ in (5.8.4) is obtained by taking $R = \delta^{1/2}$ in (5.19) and then invoking (5.15), (5.8.1) and (5.24) to bound the integral of $\delta^{3/2} \cdot |f_-|$ by a uniform muliple of δ^4.

d) The proof of Proposition 5.3

The proof requires nine steps. For the first three steps, suppose, as in the previous section, that $\cup_k C_k \subset X$ is a compact, pseudo-holomorphic submanifold, and that $(a, (\alpha, \beta))$ is a large r solution to (2.4) with the property that each component of $\alpha^{-1}(0)$ is contained in a tubular neighborhood of some component of $\cup_k C_k$. Furthermore, assume that (5.7) holds for each component $C \subset \cup_k C_k$.

Until further notice, focus attention on a given component $C \subset \cup_k C_k$ and reintroduce the integer m which is the multiple of $[C]$ that is carried by the homology class of $\alpha^{-1}(0) \cap N_0$.

Step 1. Given $z \in C$, introduce the function $\chi_{C,z}$ on C whose value at a point z' is equal to $\chi(\text{dist}(z, z')/\delta)$. Here, again, χ is a standard, non-increasing bump function on $[0, \infty)$ which is one on $[0, 1]$ and zero on $[2, \infty)$. Thus, $\chi_{C,z}$ has support in the radius $2 \cdot \delta$ ball in C with center z. Agree to identify the fiber of N over z' in the support of $\chi_{C,z}$ with $N|_z$ via parallel transport out from z along the short geodesic between z and z' using the connection θ. This identification is implicitly assumed in what follows.

With the preceding understood, define, for $p = 1, 2, \ldots$, the section h_p of $N^{\otimes p}$ over C whose value at z is

$$h_p(z) = (m2\pi^2\delta^2)^{-1} \int_{N^0} \delta^{-p} s^p iF \wedge \pi^*(\chi_{C,z} \cdot \omega_C). \tag{5.26}$$

Here, s denotes the tautological section of $\pi^* N \to N$, and ω_C denotes the volume form on C.

Here is the first fundamental lemma:

Lemma 5.8. *For each $p = 1, 2, \ldots$, there exists $\zeta_p \geq 1$ such that when $r > \zeta_p$ and $\delta < \zeta_p^{-1}$, then $|h_p| \leq \zeta_p$ and also $|\bar{\partial}_\theta h_p| \leq \zeta_p$. Furthermore, in the case where $p = 1$, one has $|Dh_1| \leq \zeta_1 \cdot (\delta^{-1/2} + (\sqrt{r}\delta)^{-1})$, where D is the operator in (5.2).*

Step 2. This step consists of the

Proof of Lemma 5.8. The fact that $|h_p|$ is uniformly bounded follows from the definition of δ, from the exponential decay estimates in (1.24) of [27], and from Lemma 5.5. To estimate $\bar{\partial}_\theta h_p$, note first that h_p can be written as a weighted average of a push-forward via the map $\pi : N^0 \to C$. That is,

$$h_p = (m2\pi^2\delta^2)^{-1} \int_C \pi_*(s^p iF_a) \cdot \chi_{C,z} \cdot \omega_C. \tag{5.27}$$

Now, agree to identify the fiber of T^*C at any point z' in the support of $\chi_{C,z}$ with the fiber of T^*C at z by using the Levi-Civita connection on C to parallel transport along the short geodesic from z to z'. With this understood, the covariant derivative of h_p is equal to

$$\nabla_\theta h_p|_z = (m2\pi^2\delta^2)^{-1} \int_C d_\theta \pi_*(s^p iF_a) \otimes \chi_{C,z} \cdot \omega_C + \text{error}, \tag{5.28}$$

where the error term is bounded by $\zeta \cdot \delta$. There are two sources for this error term. The first source is the fact that when z' is in the support of $\chi_{C,z}$, then $|\theta_{z'} - \theta_z| \le \zeta \cdot \delta$. The second source of error arises as follows: The assignment of a pair of points (z, z') in C to the number $\chi_{C,z}(z')$ defines a smooth function on $C \times C$ whose z-derivative is minus its z' derivative up to an error which is bounded by a uniform multiple of δ. (Remember here that $TC|_{z'}$ and $TC|_z$ have been implicitly identified when z' and z are both in the support of $\chi_{C,z}$.) With the preceding understood, the bound on the size of the error term follows using the same argument which proves that $|h_p|$ is bounded.

Given now (5.28), the next task is to interchange the order of the push-forward and the exterior derivative. These two operations would commute were the fiber of N^0 a compact manifold with boundary. As it is, the failure to commute can be expressed in terms of the push-forward of $s^p \cdot i \cdot F_a$ from the boundary of the closure of N^0. This last term is bounded by $\zeta \cdot e^{-\sqrt{r}/\zeta}$, and so can be ignored as far as the proof of Lemma 5.8 is concerned. Thus,

$$\begin{aligned} \nabla_\theta h_p|_z =&(m2\pi^2\delta^2)^{-1} p \cdot \delta^{-p} \\ &\cdot \int_C \pi_*(s^{p-1} \cdot \nabla_\theta s \wedge i \cdot F_a) \otimes \chi_{C,z} \cdot \omega_C + \text{error}, \end{aligned} \tag{5.29}$$

where the error term here is still bounded by $\zeta \cdot \delta$ in norm.

Introduce ς and σ by writing κ_1 from the basis $\{\kappa_0, \kappa_1\}$ of $T^{0,1}N^0$ as $\kappa_1 = \varsigma \cdot \nabla_\theta s + \sigma$ where σ is a section of $\pi^* T^* C \otimes \pi^* N$ which vanishes along C and where ς is a positive function on N^0 which behaves as $\varsigma = 1 + \mathcal{O}(|s|^2)$ near C. (See Section 2a and (2.2) of [28].) Then, this last expression can be written as

$$\begin{aligned} \nabla_\theta h_p|_z =&(m2\pi^2\delta^2)^{-1} p \delta^{-p} \\ &\cdot \int_C \pi_*(s^{p-1} \cdot \varsigma^{-1}(\kappa_1 - \sigma) \wedge i \cdot F_a) \otimes \chi_{C,z} \cdot \omega_C \\ &+ \text{error}. \end{aligned} \tag{5.30}$$

Now, one can project this onto the (0,1) part ot T^*C. And, at the risk of adding a factor of $\zeta \cdot \delta$ to the error term, this projection gives

$$\bar\partial_\theta h_p|_z = (m2\pi^2\delta^2)^{-1} p \delta^{-p} \int_C \pi_*(s^{p-1} q) \chi_{C,z} \omega_C + \text{error}, \tag{5.31}$$

where q is the 3-form

$$q = \bar f_+ \bar\kappa_{0(0,1)} \wedge \kappa_1 \wedge \bar\kappa_1 - f_- \kappa_{0(0,1)} \wedge \kappa_1 \wedge \bar\kappa_1 - \sigma_{0,1} \wedge i F_a. \tag{5.32}$$

Here, the subscript $_{0,1}$ indicates that projection onto $\pi^* T^{0,1}C$ should be performed. In this regard, note that $|\kappa_{0(0,1)}| \leq \zeta \cdot |s|$ and the $\mathcal{O}(|s|)$ part of $\sigma_{0,1}$ equals $\nu \cdot s + \mu \cdot \bar{s}$. (See Section 2a and (2.3) of [28].) With these estimates in hand, and with Lemma 5.5, one finds that

$$\begin{aligned}\bar{\partial}_\theta h_p = &- p \cdot \nu \cdot h_p + \mu \cdot (m 2\pi^2 \delta^2)^{-1} p \delta^{-p} \\ &\cdot \int_C \pi_*(s^{p-1} \bar{s} i F_a) \chi_{C,z} \omega_C + \text{error},\end{aligned} \tag{5.33}$$

where the error is now bounded in norm by

$$\zeta \cdot (\delta^{1/2} + (\sqrt{r}\delta)^{-1}). \tag{5.34}$$

Here, the appearance of $\delta^{1/2}$ is due to Lemma 5.5's estimate for $|f_-|$; and the appearence of $(\sqrt{r} \cdot \delta)^{-1}$ is due to Lemma 5.5's estimate for $|f_+|$.

The assertion that $|\bar{\partial}_\theta h_p|$ is bounded follows now by bounding the explicit integral term in (5.34); and the latter is bounded by mimicking the argument (which was given above) that bounded h_p.

To complete the proof of Lemma 5.8, consider the $p = 1$ case of (5.33). Here, the explicit integral term in (5.33) is equal to $\mu \cdot \bar{h}_1$.

Step 3. The purpose of this step is to state and then prove:

Lemma 5.9. *For each $p = 1, 2, \ldots$, there exists $\zeta_p \geq 1$ such that when $r > \zeta_p$ and $\delta < \zeta_p^{-1}$, then there is a uniform bound on the $C^{0,\varepsilon}$ Hölder norm of h_p when $\varepsilon < 1$. This bound depends only ε, the local geometry of C and on $c_1(E)$. In addition, for each $k \geq 0$, the section h_p has a uniform bound on its L_1^k Sobolev norm by a constant which depends only on k, the local geometry of C and on $c_1(E)$. (This norm controls the integral over C of the k'th power of $|\nabla_\theta h_p|$.)*

Proof of Lemma 5.9. Use standard elliptic regularity results to the fact that h_p and $\bar{\partial}_\theta h_p$ are uniformly bounded in norm.

Step 4. This step proves Proposition 5.3 in the case where $m = 1$. By assumption, one starts with a sequence $(a_j, (\alpha_j, \beta_j))$ of solutions to (2.4) where the corresponding sequence $\{r_j\}$ of r values is unbounded, while the corresponding sequence $\{\delta_j\}$ of δ values is decreasing to zero in such a way that the sequence $\{\sqrt{r_j} \cdot \delta_j\}$ is also unbounded. By passing to a subsequence, one can assume that each of these three sequences moves monotonically in the appropriate direction. With this as background, construct, for each j, the section $h_1 = h_{1,j}$ of N. It follows from the previous two lemmas that there is a subsequence of $\{h_{1,j}\}$

which converges strongly in $C^{0,1/2}$ and L^2_1 to a section, t, of N which is annihilated by the operator D from (5.2).

Meanwhile, it follows from Proposition 4.2 in [27] that for all j sufficiently large, the set $\Lambda = \Lambda_j$ from Lemma 5.4 consists of a single point. With this understood, Lemmas 5.4, 5.5 and the exponential decay estimate in (1.24) of [27] imply the following: Given $\varepsilon > 0$, then for all j sufficiently large and for all $z \in C$, the distance between $\Sigma_j \cap \pi^{-1}(z)$ and $h_{1,j}(z)$ is smaller than ε. This last observation implies that the limit, t, from the preceding paragraph, is non-trivial, since each Σ_j has a point with distance 1 from the zero section. It also yields that the image of t intersects an element of the given constraint set K when each $\alpha_j^{-1}(0)$ intersects the corresponding element of the extended constraint set.

The argument is completed with the observation that the image in N of a section which is in the kernel of D is a J_∞-pseudo-holomorphic submanifold. That is, take $C = C_0$, and then the section t gives the map ψ of the proposition.

Step 5. To proceed with the argument in the general case, consider some given configuration $(a, (\alpha, \beta))$ as in Steps 1-3 where r is large, δ is small and $\sqrt{r} \cdot \delta$ is large. The section $h = (h_1, \ldots, h_m)$ of $\oplus_{1 \le q \le m} N^q$ defines a section $y = \{y_1, \ldots, y_m\}$ of the same bundle as follows: The section y is determined by the condition that the zeros, $\{\lambda_j\}_{1 \le j \le m} \in \mathrm{Sym}^m(N)$ of the polynomial section

$$p[y] = s^m + \pi^* y_1 \cdot s^{m-1} + \cdots + \pi^* y_m \tag{5.35}$$

of $\pi^* N^m \to N$ satisfy $\Sigma_j \lambda_j^p = h_p$ for each p. Thus, y is a polynomial function of h, and thus Hölder continuous with some positive exponent and Sobolev class L^2_1.

Here is the relationship between $p[y]^{-1}(0)$ and the set

$$\Sigma = \{\xi \in N : \alpha(\delta \cdot \xi) = 0\} :$$

Lemma 5.10. *Given $\varepsilon > 0$, there exists $\zeta(\varepsilon) \ge 1$ which depends only on the geometry of X near C and on $c_1(E)$ and has the following significance: Suppose $r > \zeta(\varepsilon)$ and $\delta < \zeta(\varepsilon)^{-1}$. Then for each $z \in C$, every point of $\pi^{-1}(z) \cap \Sigma$ has distance ε or less from a zero of $p[y]$ on $N|_z$, and vice versa.*

Proof of Lemma 5.10. This follows from Lemmas 5.4 and 5.5 using (1.24) in [27].

Step 6. Now, consider a sequence $\{(a_j, (\alpha_j, \beta_j))\}$ as in Proposition 5.3 corresponding to an unbounded sequence, $\{r_j\}$, of r values, and a sequence $\{\delta_j\}$ of δ values which limits to zero. It follows then from Lemma 5.9 that the corresponding sequence $\{y_{p,j}\}_{1 \leq p \leq m}$ has a subsequence (hence relabled consecutively) which converges strongly in L^2_1 and also in $C^{0,1/2}$ to $\underline{y} = \{\underline{y}_p\}_{1 \leq p \leq m}$. Let $\underline{\Sigma}$ denote the zero set of $p[\underline{y}]$ with the latter defined in (5.35). Note that as z varies in C, this $\underline{\Sigma}$ defines a closed subspace in N. And, Lemma 5.10 implies that the set $\{\Sigma_j\}$ has a subsequence which converges in the distance measure to $\underline{\Sigma}$. In particular, it follows from this that if, for each j, the set $\alpha_j^{-1}(0)$ intersects an element of an extended constraint set for C, then $\underline{\Sigma}$ must intersect the corresponding element of the constraint set.

The set $\underline{\Sigma}$ has two types of points, the *regular* points, and the *singular* points. The meaning of these terms is as follows: A point $\xi \in \underline{\Sigma}$ is regular when the following is true: There is a neighborhood $U \subset N$ of ξ with the property that when $z \in \pi(U)$, then the intersection of $\pi^{-1}(z)$ with $\underline{\Sigma} \cap U$ has exactly one point. A point in $\underline{\Sigma}$ is singular when it is not regular.

By definition, the set of regular points of $\underline{\Sigma}$ is open. This set is also dense in $\underline{\Sigma}$. (To see that $\underline{\Sigma}$ is dense, introduce the multiplicity function on $\underline{\Sigma}$ which counts the multiplicity of a point ξ as a zero of the restriction of $p(\underline{y})$ to $\pi^{-1}(\pi(\xi))$. The latter function takes values in $\{1, \ldots, m\}$ and its local minima are necessarily regular. Furthermore, being a function with a finite number of values, it takes on a local minimum on every open set.)

The nature of $\underline{\Sigma}$ near a regular point is described by

Lemma 5.11. *The set $\underline{\Sigma}$ is a J_∞-pseudo-holomorphic submanifold in a neighborhood of any regular point.*

As for the set of singular points, consider:

Lemma 5.12. *There are only finitely many singular points in $\underline{\Sigma}$.*

These two lemmas are proved in Steps 8 and 9, respectively.

Step 7. Given Lemmas 5.11 and 5.12, the proof of Proposition 5.3 is completed with the following observation: Let $\underline{\Sigma}_r \subset \underline{\Sigma}$ denote the set of regular points. It follows from Lemma 5.11 that $\underline{\Sigma}_r$ is a J_∞-pseudo-holomorphic submanifold of N, and thus inherits the structure of a complex curve C_r for which the tautological embedding, ψ, into N as $\underline{\Sigma}_r$ is J_∞-pseudo-holomorphic. Furthermore, according to Lemma 5.12, this C_r has a finite number of ends, with each being diffeomorphic to the compliment of the origin in the unit disk. Let C_0 denote the complex curve which is obtained from C_r by adding the origin to each of these disks. Standard arguments prove that the complex structure on C_r extends uniquely over C_0 to give the latter the structure of a complex curve. Then, the map ψ extends in the obvious way as a continuous map from C_0 into N which is J_∞-pseudo-holomorphic on the compliment of a finite number of points. And, standard elliptic regularity arguments prove that this map ψ is everywhere J_∞-pseudo-holomorphic. (Note that these arguments are much simpler than the removable singularities arguments in Sacks and Uhlenbeck [24] since it is known here apriori that the map ψ is continuous.) It was remarked previously that the image of ψ (i.e., $\underline{\Sigma}$) intersects any element of a contraint set which is intersects all of the Σ_j's.

Step 8. This step contains the

Proof of Lemma 5.11. It follows from the definition that the set of regular points of $\underline{\Sigma}$ defines a continuous submanifold of N which fibers locally over C. A point $z \in C$ will be called a *regular value* when $\underline{\Sigma} \cap \pi^{-1}(z) = \underline{\Sigma}_r \cap \pi^{-1}(z)$. The set of regular values of C is open and dense. (A regular value near a given z' can be found by taking a neighborhood of z' and asking for a point in said neighborhood which is a local maximum for the integer valued function which assigns to z the number of points in $\underline{\Sigma} \cap \pi^{-1}(z)$.). If z is a regular value, then the projection $\pi : \underline{\Sigma} \to C$ is a proper covering map of some neighborhood of z with some number $k \in \{1, \dots, m\}$ sheets.

At a point $z \in C$, write $p[\underline{y}]^{-1}(0)$ on $N|_z$ as

$$\{(t_1(z), n_1) \dots, (t_k(z), n_k)\},$$

where $t_i(z) \in \pi^{-1}(z)$ is a zero of $p[\underline{y}]$ and where n_i is the multiplicity of $t_i(z)$ as a zero of $p[\underline{y}]$ on $\pi^{-1}(z)$. Note that when z is a regular value, then the integers $\{n_i\}$ are constant on a neighborhood of z, and each $t_i(\cdot)$ is a continuous and L^2_1 section of N over some neighborhood of z whose image defines a sheet of $\underline{\Sigma}$ over said neighborhood. With this understood, it follows that $\underline{\Sigma}$ is J_∞-pseudo-holomorphic near the π-inverse image of a regular value, z, if each section t_i is annihilated by the operator D (in (5.2)) on some neighborhood of z.

To see that such is the case, use (5.33) plus (1.24) from [27] to conclude that

$$\Sigma_i n_i D t_i t_i^{p-1} = 0. \tag{5.36}$$

Now, p can take values from 1 to m in (5.36), and the totality of these m equations can be solved if and only if $Dt_i = 0$ for all i.

Now, suppose that ξ is a point in $\underline{\Sigma}_r$ which does not project to a regular value. None the less, a neighborhood of ξ in $\underline{\Sigma}_r$ is the image of a continuous section, t_1, over a disk in C. Furthermore, as just observed, this t_1 is annihilated by D on a dense, open set of the disk in C. Elliptic regularity for the operator D implies that t_1 is annihilated by D everywhere.

Step 9. This step contains the

Proof of Lemma 5.12. It is sufficient to prove that the set of points in any disk $B \subset C$ which are not regular values is finite. With this understood, let $C' \subset B$ denote a path component of the set of regular values. Since $\underline{\Sigma}|_{C'} \to C'$ is a covering map, there is a positive integer $k \leq m$ such that locally $p[\underline{y}]^{-1} = \{(t_i, n_i)\}_{1 \leq i \leq k}$ on C'. Here, the integers n_i are constant, and each t_i is, locally, a section of N over C' which is annihilated by D. (A given t_i is globally defined on C' up to possible confusion with t_j for which $n_j = n_i$.)

In any event, the section $q = \Pi_{i \neq j}(t_i - t_j)^{n_i n_j}$ defines a smooth section over C' of the appropriate power of the line bundle N. The significance of this q is that it is non-vanishing on C', and that it extends continously as the zero section to $B - C'$. This extension of q to the whole of B will be denoted by q'. However, note that q has a second extension to the whole of B in that q is the restriction to C' of a section of a power of N which is obtained as products from the set $\{\underline{y}_p\}_{1 \leq p \leq m}$. This means, in particular, that q has bounded L^2_1 Sobolev norm over C', and thus the extension, q', of q by zero on $B - C'$ has bounded L^2_1 norm over the whole of B. This observation will be relevant momentarily.

Since each t_i is annihilated by the operator D, the section q obeys a differential equation of the form

$$\bar{\partial}_\theta q = w \cdot q, \tag{5.37}$$

where w is uniformly bounded on the closure of C' and smooth inside. Let $o^{C'}$ denote the characteristic function of C'. It is a straightforward matter to find a continuous, Sobolev class L^2_1 function u on B which satisfies the equation

$$\bar{\partial} u = o^{C'} w. \tag{5.38}$$

With u understood, consider that $e^{-u}\cdot q'$ is a continuous, Sobolev class L^2_1 section of N over B which obeys

$$\bar{\partial}_\theta(e^{-u}q) = 0 \tag{5.39}$$

on C' and which is zero on $B - C'$. It follows that $e^{-u}\cdot q'$ is holomorphic in B, and thus $B - C'$ is a finite set of points.

6. The image of Ψ_r

The purpose of this section is to give a proof of Proposition 5.2. What follows is an outline of the seven parts to the proof.

Part 1. Proposition 5.2 follows from the following slightly weaker claim:

Claim. Given an initial sequence $\{(a_j,(\alpha_j,\beta_j))\}$ which satisfies the assumptions of Proposition 5.2, there exists a subsequence for which each $(a_j,(\alpha_j,\beta_j))$ has a point on its gauge orbit of the form

$$\left(\underline{a}_r + \frac{\sqrt{r_j}}{2\sqrt{2}}a',\ (\underline{\alpha}_r + \alpha', \underline{\beta}_r + \beta')\right), \tag{6.1}$$

where $(\underline{a}_r,(\underline{\alpha}_r,\underline{\beta}_r))$ and $(a',(\alpha',\beta'))$ are obtained from a point in $\times\mathcal{K}^{(k)}_\Lambda$ as described in Proposition 5.2 of [28].

The proof of the preceding claim is complicated by the fact that the object of the search is a point on a gauge orbit. (The gauge orbit is the orbit of data under the natural action of the group $C^\infty(X;S^1)$.) To be more explicit, the search for the gauge orbit point requires apriori estimates and the apriori estimates (which come via an elliptic differential equation) require an apriori choice of a point on the gauge orbit. (The Seiberg-Witten equations are not, by themselves, elliptic. They become so only after an appropriate choice of point on a gauge orbit.) Thus, there is a chicken versus egg problem here which must be solved.

The resolution chosen below finds the appropriate gauge choice in steps; where each step gives estimates (from the Seiberg-Witten equations) which facilitate the gauge choice for the subsequent step. Various aspects of the proof of Proposition 5.2 are established as part of some of these intermediate steps, and are then plugged in to subsequent estimates.

For example, the first step of the proof of the claim above uses a subsequence of $\{(a_j,(\alpha_j,\beta_j))\}$ (hence renumbered consecutively) to construct data $y = \{y^k\}$, where each y^k is a section of the corresponding (C_k, m_k) version of $\oplus_{1\leq q\leq m} N^q$. This step relies heavily on arguments from the preceding section. (It turns out in the end that $(a_j, (\alpha_j, \beta_j))$ has a point on its gauge orbit which is described by (6.1) where $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ and $(a', (\alpha', \beta'))$ are obtained from a point in $\times\mathcal{K}_\Lambda^{(k)}$ which is close to y.)

Now, each y^k lies in the corresponding (C_k, m_k) version of $\mathcal{Z}_0$. However, the proof of this assertion requires a number of intermediate steps. For example, the initial construction of $\{y^k\}$ does not establish that y^k is smooth; rather, only a Hölder estimate and an apriori bound on the Sobolov L_1^n norm for $n < \infty$ are initially available. Were each $\{y^k\}$ known to be C^2 apriori, then the whole argument would be considerably shorter. As it is, the fact that y^k is not known apriori to be of class C^2 accounts for at least one level of complexity in the ensuing arguments.

In any event, once the appropriate gauge choice for $(a_j, (\alpha_j, \beta_j))$ is secured, and once the necessary apriori estimates are derived, then the argument for the claim above closes by invoking the uniqueness aspects of the contraction mapping arguments which underly all of the steps in Section 5 of [28]. These uniqueness assertions insure that $(a_j, (\alpha_j, \beta_j))$ is, for large j, obtained from the constructions in [28].

All of this results in an admittedly lengthy and complicated argument.

Part 2. As remarked above, the first step of the proof uses a subsequence of the sequence $\{(a_j, (\alpha_j, \beta_j))\}$ (hence renumbered consecutively) to construct a point $y = \{y^k\} \in \times\mathcal{K}_\Lambda^{(k)}$. Near $C = C_k$, a given point on the gauge orbit of $(a_j, (\alpha_j, \beta_j))$ can be written in the form $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))+$ remainder, where $r = r_j$ and $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ are determined by y^k as described in Sections 2 and 3b of [28]. The goal will be to find a point on the gauge orbit where certain apriori estimates are available for the corresponding remainder. This step makes a preliminary choice for the gauge orbit point; a choice which comes via a two step process. The first step (Section 6b and Lemma 6.1) chooses an initial gauge for the data $(a_j, (\alpha_j, \beta_j))$ near each $C = C_k$ using results from Section 4 in [27]. This initial choice manifestly gives pointwise control of $\alpha_j - \underline{\alpha}_r$ and of the pull-back to the fibers of the normal bundle $N \to C$ of the 1-form $a_j - \underline{\alpha}_r$. The gauge only controls the L^2 norm of the horizontal part of $a_j - \underline{\alpha}_r$ (and its vertical derivative).

The gauge choice from the first step is then modified (in Section

6c and Lemma 6.3) to insure that $a_j - \underline{a}_r$ and the pull-back of $a_j - \underline{a}_r$ satisfy a zero-divergence like (Coulomb) condition along each fiber of N. In particular, only the components of the 1-form $a_j - \underline{a}_r$ along the fibers of N are involved. A useful gauge choice which involves all components of the 1-form $a_j - \underline{a}_r$ is unattainable at this point in the argument because of the paucity of apriori estimates.

Part 3. With the gauge choice from Part 2, the Seiberg-Witten equations are employed to derive a refined apriori L^2 bound on $a_j - \underline{a}_r$ (See Section 6d and Lemma 6.4.) The argument here is lengthy and convoluted, in part because the Seibeg-Witten equations with the gauge choice from Part 2 are not completely elliptic.

Part 4. This part uses the refined L^2 estimates from Part 3 to prove that each y^k lies in $\mathcal{Z}_0$. (See Section 6e and Lemma 6.5.) The argument here boils down to the following: For each j, the data $(a_j, (\alpha_j, \beta_j))$ solves the $r = r_j$ version of the Seiberg-Witten equations in (2.4). Furthermore, this data can be written as $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r)) + p_j$, and because of the estimates from Part 3, it is known that the p_j's tend to zero rapidly as j gets large. The Seiberg-Witten equations for $(a_j, (\alpha_j, \beta_j))$ and the minute size of p_j at large j constrains the data y^k; and this constraint says no more nor less than $y^k \in \mathcal{Z}_0$.

Part 5. This part (Section 6f and Lemma 6.6) finds, for each y' near y in $\times_k \mathcal{K}_\Lambda^{(k)}$ and for large j, a point on the gauge orbit of the data $(a_j, (\alpha_j, \beta_j))$ which satisfies a zero-divergence gauge condition on the whole of X with respect to $(\underline{a}_r[y'], (\underline{\alpha}_r[y'], \underline{\beta}_r[y']))$. As indicated, $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ is now defined as in Sections 2 and 3b of [28] by y' instead of y. The point y' will be treated as a parameter until its value is fixed in the last step of the proof of the claim. The construction of the zero-divergence gauge orbit point first uses the fact that each y^k lies in $\mathcal{Z}_0$ to refine the apriori estimate for $\alpha_j - \underline{\alpha}_r$ and $a_j - \underline{a}_r$ in the gauge from Lemma 6.1. These refined estimates are then exploited to find a solution to a certain differential equation which defines the zero-divergence gauge.

Part 6. This step (Section 6g and Lemma 6.7) employs $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ as defined by y' (near y) to write the zero-divergence gauge orbit point $(a_j, (\alpha_j, \beta_j))$ (from Lemma 6.6) as in (6.1) (Here, the label by the index j is implicit. This will be the policy throughout this section when the chance for confusion is small.)

With the help of Lemma 6.6's apriori estimates, the data $q' =$

$(a', (\alpha', \beta'))$ is then cut (as in (4.6) of [28]) into constituent pieces $(q_0, \{q_k\})$ which satisfy (5.3) of [28].

Part 7. This last part (Section 6h) completes the proof of Proposition 5.2 with a proof that $(q_0, \{q_k\})$ as obtained in Lemma 6.7 from $(a_j, (\alpha_j, \beta_j))$ are, in fact, given by the constructions in Section 5 of [28]. The arguments here use the previously established apriori estimates while invoking the uniqueness assertions of the contraction mapping arguments which underly all of the constructions in Section 5 of [28].

Before starting with the details, be forwarned of two conventions which are used in the proof: First, the symbol ζ will represent a constant which is independent of the index j and of any other choices of parameters. Furthermore, the precise value of ζ is allowed to change from line to line. The second convention uses r for r_j when no confusion is likely to arise.

The reader should also be aware that the notation and conventions of [28] are used heavily here, often without comment.

a) The data $\{y^k\}$

To begin the construction of $\{y^k\}$, let $\{(a_j, (\alpha_j, \beta_j))\}$ be a sequence as in the statement of Proposition 5.2. Focus attention on a component $C \subset \cup_k C_k$ and introduce the integer m which is the multiple of $[C]$ given by the value of $c_1(E)$ on a rational homology class which has intersection 1 with C and zero with the other components of $\cup_k C_k$. For j large, one can assume that the number δ as defined in (5.7) using $\alpha = \alpha_j$ is less than $R/\sqrt{r_j}$, where $R \geq 1$ is j-independent.

For each j, and for $p \in \{1, \ldots, m\}$, construct the section $h_p = h_{p,j}$ of N^p over C as in (5.26), but use everywhere $r_j^{-1/2}$ in place of δ. Then, introduce the section $\{y_1, \ldots, y_m\}$ of $\oplus_{1 \leq q \leq m} N^q$ which is defined by the condition that the zeros of $p[y] = s^m + y_1 s^{m=1} + \cdots + y_m$ give the unique point $\{\lambda_j\}_{1 \leq j \leq m}$ in $\mathrm{Sym}^m(N)$ with the property that $h_p = \Sigma_j \lambda_j^p$ for each p. It follows from Lemma 5.9 that for each such p, the sequence $\{y_{p,j}\}$ converges in the $C^{0,1/2} \cap L^2_1$ topology to a $C^{0,1/2} \cap L^2_1$ section, $\underline{y}_p$ of N^p. For each k, let y^k denote the (C_k, m_k) version of $(\underline{y}_1, \ldots, \underline{y}_m)$.

By the way, suppose that when j is large, each $\alpha_j^{-1}(0)$ intersects each member of a fixed extension of a given constraint set G_k for C_k. Then, as in the proof of Step 6 of the proof of Proposition 5.3, it follows that for each k, the zero set of $p[y^k]$ must intersect all members of G_k.

b) The first choice of gauge

Although each y^k may not be a smooth section of the corresponding (C_k, m_k) version of $\oplus_{1\le q\le m} N^q$, none-the-less, $y = \{y^k\}$ still defines, for each r, data $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ via the constructions in Sections 2 and 3b. The latter consists of a Hölder continuous and L^2_1 connection a_r and section $(\underline{\alpha}_r, \underline{\beta}_r)$ of S_+.

Note that the construction of $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ involves fixing a positive, r-independent constant δ. The latter is chosen given y, but in any event, can be assumed smaller by a factor of 10^{-6} or more than the distance between any two distinct members of $\{C_k\}$. Note that this δ is not the same as that which appears in (5.7). The latter will not be used below. Also, remember that the size of δ is further restricted to be less (by a factor of 10^{-3}) than the radius of a certain tubular neighborhood N^0 about each $C = C_k$. Such tubular neighborhoods for different elements in $\{C_k\}$ are disjoint. Here, N^0 is identified via the exponential map of Lemmas 2.1 and 2.2 of [28] with a disk bundle of the same name in the normal bundle to C. The radius of N^0 is denoted below by δ_0.

With the preceding understood, a point on the gauge orbit, o_j, of $(a_j, (\alpha_j, \beta_j))$ can be written as in (6.1). In particular, at points with distance δ or less from a given C_k, this representation of the given gauge orbit point has the form

$$\left\{\left(\theta + \rho_r^* v + \frac{\sqrt{r}}{2\sqrt{2}} b_r + \frac{\sqrt{r}}{2\sqrt{2}} a',\ (\rho_r^* \tau + \alpha',\ \lambda_r + \beta')\right)\right\}. \tag{6.2}$$

Here, $r = r_j$, while $c = (v, \tau)$ is the section of the (C_k, m_k) version of vortex bundle ((2.15) in [28]) defined by y^k, and (b_r, λ_r) are defined from y^k as in Section 3b of [28]. (See (3.12) in [28], where (6.2) is valid. Write a' as

$$a' = a_V \bar{\kappa}_1 - \bar{a}_V \kappa_1 + a_C - \bar{a}_C. \tag{6.3}$$

Here, a_C is a section of $T^{0,1}N$ which lies in $\pi^* T^* C$. (Remember that κ_1 is defined in Part 4 of Section 2a in [28].)

The purpose of this subsection is to find a point on the aforementioned gauge orbit where pointwise estimates for $(a_V, (\alpha', \beta'))$ are available. (It will become abundantly clear that estimates for a_C are hard to come by.) Such estimates are provided by Lemma 6.1, below. In the statement of the lemma, and subsequently, ∇ denotes the covariant derivative using the connection $\theta + \rho_r^* v$ when the object of the derivative is a section of a bundle which involves E. And, ∇^V denotes the covariant derivatives along the fibers of N. Also, in the statement of the lemma, $||\cdot||_2$ denotes the L^2 norm over N^0.

Lemma 6.1. *Fix $\varepsilon > 0$ and a positive integer d. For all sufficiently large j, a point on the orbit o_j can be chosen so that the data $q = (a', (\alpha', \beta'))$ has the following properties:*

1. *On the tubular neighborhood N^0 of any $C \in \{C_k\}$,*
$$\sum_{0 \le d' \le d} r^{-d'/2} |(\nabla^V)^{\otimes d'}(a_V, \alpha', \beta')| \le \varepsilon \cdot e^{-\sqrt{r} \cdot |s|/\zeta}.$$
2. *On this same tubular neighborhood,*
$$||\nabla(a_V, \alpha', \beta')||_2 + ||\nabla^V a_C||_2 + \sqrt{r} \cdot ||a_C||_2 \le \zeta.$$
3. *Where the distance to any C_k is greater than $\delta_0/2$, the norms of a', α', and β' and of their covariant derivatives to order d are bounded by* $\exp(-\sqrt{r}/\zeta)$.

Here, $r = r_j$, and the constant ζ is independent of ε and j.

(Note that the second point above gives L^2 control over all derivatives of (a_V, α', β'), but only the derivative of a_C along the fibers.)

Proof of Lemma 6.1. The proof is carried out in five steps.

Step 1. The pointwise estimate in the lemma for β' follows from Proposition 4.4 in [27]. The pointwise estimates for the covariant derivative of β' along the fibers of N follow from Proposition 4.4 in [27] given the pointwise estimate for a_V. Likewise, the bounds for the higher order covariant derivative for β' along the fibers of N follow from those for a_V given the following generalization of Proposition 4.4 in [27]:

Proposition 6.2. *Fix a complex line bundle $E \to X$ and a integer $d \ge 0$. There is a constant ζ which depends only on $c_1(E), d$ and on the Riemannian metric and has the following significance: Let $r > \zeta$ and suppose that $(a, (\alpha, \beta))$ is a solution to the r-version of the Seiberg-Witten equations for the* $\mathrm{Spin}^{\mathbb{C}}$ *structure with S_+ defined by E. Then at any point $x \in X$ and for any $d' \in \{0, 1, \dots, d\}$*
$$|(\nabla_a^{d'}(\alpha, \sqrt{r} \cdot \beta))|(x) \le \zeta r^{d'/2} \exp(-\sqrt{r}\,\mathrm{dist}(x, \alpha^{-1}(0))/\zeta).$$

Proof of Proposition 6.2. The proof is obtained by a straightforward application of the strategy which proved Proposition 4.4 of [27].

The L^2 estimate for the covariant derivative of β' follows from the pointwise estimate of β' using the L^2 estimate for a_C and Proposition 4.4 in [27].

Step 2. This step begins the task of choosing a point on the orbit of $(a_j, (\alpha_j, \beta_j))$. To start, note that Proposition 4.4 in [27] has the following consequence: Fix $\delta_1 > 0$ and when j is sufficently large, then $||\alpha_j| - 1|$ and the norms of the a_j covariant derivative of α_j and of the curvature of a_j are bounded by $\zeta \cdot e^{-\sqrt{r}/\zeta}$ at points where the distanct to any C_k is greater than δ_1.

One can conclude from the preceding that for all sufficiently large j, there is a gauge for $(a_j, (\alpha_j, \beta_j))$ for which the corresponding pair (a', α') and their covariant derivatives to order d are point-wise bounded by $\zeta \cdot e^{-\sqrt{r}/\zeta}$ where the distance to any C_k is greater than $\delta_0/4$. Indeed, the gauge should be chosen so that $\alpha' = g_j \cdot \alpha_r$ with g_j real. Then, the estimates from Proposition 4.4 in [27] imply the stated supremum bound for α'. Here, α_r is constructed from the data $\{y^k\}$ as detailed in Section 2 of [28]. The estimates from this same proposition for the a_j-covariant derivative of α' give both the C^0 estimate for a' and the C^1 bound for the covariant derivative of α'.

The bound for the covariant derivative of a' and the higher covariant derivatives of $(a', (\alpha', \beta'))$ follow by similar arguments from Proposition 6.2.

Step 3. This step extends the previous gauge choice to obtain estimates where the distance from any given $C \in \{C_k\}$ is $\mathcal{O}(1/\sqrt{r_j})$. To procede, suppose first that some $\varepsilon_1 > 0$ has been specified. Now, remark that the assumptions of Proposition 5.2 together with Proposition 4.4 in [27] find $R_0 \geq 1$ such that for large j, the section α_j has norm greater than $1/2$ where $|s| \geq R_0/\sqrt{r}$. Thus, if the gauge choice here also requires $\alpha_j = g_j \cdot \alpha_r$ with g_j real, then Proposition 4.4 in [27] and Proposition 6.2, above, imply that the conditions of Assertion 1 of Lemma 6.1 are met using ε_1 for ε as long as $|s| \geq R_1/\sqrt{r_j}$, where R_1 is determined by ε_1, but not j. (A lower bound for R_1 is given by $\zeta \cdot |\ln(\varepsilon_1)|$.) The argument here is the same as that employed by the previous step. However, in this case, note that only estimates for (a_V, α') and their derivatives along the fibers of N are available. This is because the vortex (v, τ) is smooth along the fibers of N, but of undetermined differentiability in horizontal directions.

On the otherhand, because (v, τ) is an L^2_1 section of the (C, m) version of (2.11) in [28], one can conclude that Assertion 2 of Lemma 6.1 holds as long as the integration domain is restricted to where $R_1/\sqrt{r_1} \leq |s| \leq \delta_0$.

Step 4. This step begins the task of choosing the gauge for $(a_j, (\alpha_j, \beta_j))$ where $|s| < 2 \cdot R_1/\sqrt{r_j}$. Begin with the observation from Proposition 4.2 in [27], that for large j, the $\sqrt{r_j}$ rescaled restriction of (a_j, α_j) to $N|_z$ must be close to some vortex at every $z \in C$. To be precise, fix $\varepsilon_1 > 0$, $R \geq 1$ and the non-negative integer d. Choose $z \in C$ and dilate (in Gaussian coordinates centered at z) the ball of radius $R_2/\sqrt{r_j}$ with center z to a ball of radius R_2. Use the Gaussian coordinates to identify this ball with the centered, radius R_2 ball in $\mathbb{R}^4$. Also, use the almost complex structure $J|_z$ to identify this $\mathbb{R}^4$ with $\mathbb{C}^2$.

For sufficiently large index j, the pull-back to this radius R_2 ball (by the dilation map) of some point on the gauge orbit of $(a_j, (\alpha_j, \beta_j))$ will have C^d norm within ε_1 of the pull-back to the radius R_2 ball (by a $\mathbb{C}$ linear projection from $\mathbb{C}^2$ to $\mathbb{C}$) of a vortex solution, say (v', τ'), on $\mathbb{C}$.

Next, note that the dilation map pulls back the fibers of N to give a codimension 2 foliation of the ball of radius R_2 in $\mathbb{C}^2$ by submanifolds. These submanifolds are close to being linear complex lines. Indeed, there is a complex linear vector space isomorphism $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$ and a diffeomorphism $\phi : \mathbb{C} \oplus \mathbb{C} \to \mathbb{C} \oplus \mathbb{C}$ with the following properties: Write $\phi(z, \eta) = (z', \eta')$. Then, in the radius R_2 ball about the origin:

- $\eta' = 0$ gives the pull-back by the dilation map of the zero section of N,
- the $z' =$ constant planes give the pull-back by the dilation map of the fibers of N,
- $|\phi(x) - x| \leq \zeta r^{-1/2}|x|^2$ and $|\nabla(\phi - id)| \leq \zeta r^{-1/2}$,
- $|\nabla^s \phi| \leq \zeta_s r^{-(s-1)/2}$ for $s \geq 2$.

(6.4)

With this understood, one can suppose, without loss of generality, that (v', τ') is pulled back by the composition of the map ϕ followed by a $\mathbb{C}$ linear map to $\mathbb{C}$. This is to say that the zero's of τ' define a set of parallel, complex lines in the (z', η') plane. Furthermore, since the zeros of the rescaled a_j stay uniformly close to the zero section of N, one can assume (for large j) that (v', τ') is pulled back by the map which sends (z', η') to η'.

Meanwhile, on each of the z' = constant complex lines, the zeros of the pull-back of α_j are (assuming that j is large), ε_1 close to those of the pull-back by the dilation map of the complex function τ as defined by the vortex (v, τ). (This follows from the definition of y^k.) The last fact implies that the zeros of τ' must be within $2 \cdot \varepsilon_1$ to those of τ and vice versa on each of the z = constant complex lines. Since the vortex solution (v', τ') on $\mathbb{C}$ is determined by the zeros of τ, for sufficiently large j, the pull-back to the ball of radius R_2 in $\mathbb{C}^2$ by the dilation map of a point on the gauge orbit of (a_j, α_j) must be ε_1 close in the C^0 topology to the pull-back of $(\theta + \rho_r^* v, \rho_r^* \tau)$ by this same dilation map.

Furthermore, using elliptic regularity for the vortex equations, it follows from the preceding that for sufficiently large j, the pull-back of $(\theta + \rho_r^* v, \rho_r^* \tau)$ by the dilation map must restrict to each of the z' = constant complex lines to be ε_1-close in the C^d topology to (v', τ'). This argument shows that there is a gauge for (a_j, α_j), when j is large, for which, in the ball of radius R_2 in $\mathbb{C}^2$, the following is true: The derivatives to order d along the z' = constant complex lines of the pull-back of $(a_j - (\theta + \rho_r^* v), \alpha_j - \rho_r^* \tau)$ are ε_1-close to zero.

With the preceding understood, one can (when j is sufficiently large) undo the rescaling in the ball, to obtain a point on the gauge orbit of $(a_j, (\alpha_j, \beta_j))$ with the following property: The corresponding data (a_V, a_C, α') restricts to the ball of radius $R_2/\sqrt{r}$ and center z to obey $|(\nabla^V)^{\otimes d'}(a_V, a_C, \alpha')| \leq \varepsilon_1 \cdot r^{d'/2}$ for any integer $d' \leq d$. Here, ε_1 is any fixed apriori, positive number.

Step 5. The preceding estimates are uniform in the coordinate z, which implies the following: Given $\varepsilon_1 > 0$ and $R_2 \geq 1$, there is, for all sufficiently large j, a set $\Lambda_j \subset C$ which has the properties listed below.

- Λ_j contains no more than $\zeta \cdot r_j / R_2^2$ members.
- Distinct points in Λ_j have distance greater than $R_2/(4\sqrt{r_j})$ apart.
- The balls of radius $R_2/(4 \cdot \sqrt{r_j})$ and centers at points of Λ_j cover C.
- For each point in Λ_j, there is a gauge for $(a_j, (\alpha_j, \beta_j))$ such that the resulting (a_V, a_C, α') has its derivatives along the fibers of N to each order $d' \leq d$ bounded by $\varepsilon_1 r_j^{d'/2}$ in the ball of radius $R_2/\sqrt{r_j}$ about the given point.

(6.5)

Here, ζ is independent of ε_1, R_2 and j.

Step 6. The next task is to glue these local gauges together to find a gauge on the orbit of $(a_j, (\alpha_j, \beta_j))$ over a radius $\zeta^{-1} \cdot R_2/\sqrt{r_j}$ tubular neighborhood of C for which the resulting (a_V, a_C, α') have derivatives of order $d' \leq d$ along the fibers of N bounded by $\zeta \cdot \varepsilon_1 \cdot r^{d'/2}$. This last job is a straightforward exercise with cut-off functions and is left to the reader. Here, ζ can be assumed to be independent of R, ε_1 and j. And, for the purposes of the proof of Lemma 6.1, it is sufficient to take $R_2 = \zeta'' \cdot R_1$. When gluing the local gauges together, remember that where two of the balls overlap, the corresponding two gauges for (a_j, α_j) must differ by a small gauge transformation because in either gauge, the pair (a_j, α_j) is close to the same rescaled vortex $(\theta + \rho_r^* v, \rho_r^* \tau)$.

Step 7. The final task is to modify the gauges from Steps 3 and 6 in the region where $|s| \in (R_1/\sqrt{r_j},\ 2 \cdot R_1/\sqrt{r_j})$ to make a globally defined gauge for $(a_j, (\alpha_j, \beta_j))$. This is another straightforward exercise with cut-off functions which is also left to the reader. One is aided here also by the fact that in each gauge, (a_j, α_j) is close to the same rescaled vortex.

The result of this step is a gauge for $(a_j, (\alpha_j, \beta_j))$ (when j is large) for which the estimates in Assertion 1 of Lemma 6.1 are met using $\zeta \cdot \varepsilon_1$ instead of ε. Also, the estimates in Assertion 2 are also met, except possibly for the finite bound on the L^2 norm of the horizontal derivatives of a_V and α'. The latter bounds follow, respectively, from L^2 bounds for the component f_+ of the curvature of a_j (see (5.8)), and for the a_j-covariant derivative of α_j. Both of these L^2 bounds follow directly from the estimates in (1.24) of [27]. (Note that a crucial minus sign is missing from the exponent of the last expression in (1.24) of [27]. The sign is corrected, however, in the reprint of the article which appears in the Internat. Press volume.) In this regard, note that the complex conjugate of $r^{-1/2} \cdot f_+$ involves the horizontal anti-holomorphic derivative of a_V, together with vertical derivatives of a_V and also of a_C. Thus, the L^2 norm of $r^{-1/2} \cdot f_+$ over N^0 (with the now established parts of Lemma 6.1) bounds the L^2 norm of the horizontal, anti-holomorphic derivative of a_V. Finally, integration by parts can be used to parlay the latter bound into a bound for all horizontal derivatives of a_V. Alternately, one can argue from (1.24) in [27] that $r^{-1/2} \cdot f_-$ also has a uniform L^2 norm bound. The latter controls the L^2 norm of the holomorphic, horizontal derivatives of a_V in terms of the vertical derivatives of a_V and a_C.

To summarize: Given $\varepsilon_1 > 0$, all of the Assertions of Lemma 6.1 can be met for sufficiently large j using $\zeta \cdot \varepsilon_1$ instead of ε, where ζ is independent of ε_1 and j. And, with this understood, take ε_1 here to equal $\zeta^{-1} \cdot \varepsilon$ to obtain Lemma 6.1.

c) A fiberwise "Coulomb" gauge

This subsection modifies the gauge choice from Lemma 6.1 so that the resulting pair (a', α') obeys the fiberwise coulomb gauge condition

$$\partial^V a_V - \bar{\partial}^V \bar{a}_V + \frac{\sqrt{r}}{2\sqrt{2}}(\rho_r^* \bar{\tau}\alpha' - \rho_r^* \tau \bar{\alpha}') = 0, \tag{6.6}$$

where $|s| \leq \delta_0/2$. Here is a precise statement:

Lemma 6.3. *Fix $\varepsilon > 0$ and a positive integer d. For all sufficiently large j, a point on the orbit o_j can be chosen so that the data $q = (a', (\alpha', \beta'))$ has the following properties:*

1. *On the radius δ_0 tubular neighborhood of any $C \in \{C_k\}$,*
$$\sum_{0 \leq d' \leq d} r^{-d'/2} |(\nabla^V)^{\otimes d'}(a_V, \alpha', \beta')| \leq \varepsilon e^{-\sqrt{r} \cdot |s|/\zeta}.$$
2. *On this same tubular neighborhood,*
$$||\nabla(a_V, \alpha', \beta')||_2 + ||\nabla^V a_C||_2 + \sqrt{r} \cdot ||a_C||_2 \leq \zeta.$$
3. *Where $|s| < \delta_0/2$ on this same tubular neighborhood, (6.6) holds.*
4. *Where the distance to any C_k is greater than $\delta_0/2$, the norms of a', α', and β' and of their covariant derivatives to order d are bounded by* $\exp(-\sqrt{r}/\zeta)$.

Here, $r = r_j$, and the constant ζ is independent of ε and of j.

The remainder of this subsection is occupied with the

Proof of Lemma 6.3. Let $(a'_{\text{old}}, \alpha'_{\text{old}}, \beta'_{\text{old}})$ be given by the gauge for $(a_j, (\alpha_j, \beta_j))$ as specified by Lemma 6.1, and using $d + 1$ instead of d.

Lemma 6.3's data (a', α', β') is defined in terms of a function u' on N^0 by the following rule:

$$
\begin{aligned}
a' &= a'_{\text{old}} + i \cdot r^{-1/2} du', \\
\alpha' &= e^{-i \cdot r \cdot u'} \alpha'_{\text{old}} + (e^{-i \cdot r \cdot u'} \rho_r^* \tau - \rho_r^* \tau), \\
\beta' &= e^{-i \cdot r \cdot u'} (\beta'_{\text{old}} + \underline{\beta}_r) - \underline{\beta}_r.
\end{aligned}
\tag{6.7}
$$

Meanwhile, write $u' = \chi_{\delta_0/2} \cdot u$, where the function u is required to be square integrable on N^0 and is chosen to insure that (6.6) holds where $|s| < \delta_0/2$. (Here, χ is a standard cut-off function on $[0, \infty)$ which is 1 on $[0, 1]$ and 0 on $[2, \infty)$. Then, with the choice of $t > 0$, the function χ is promoted to a function, $\chi_t \equiv \chi(|s(\cdot)|/t)$ on N.) The latter condition can be implimented by requiring u to solve a differential equation on N which has the schematic form:

$$
\begin{aligned}
&-2\partial^V \bar{\partial}^V u + 2^{-1/2} r |\rho_r^* \tau|^2 u \\
&+ \sqrt{r} \chi_{\delta_0/2} \left(\partial^V a_{V,\text{old}} - \bar{\partial}^V \bar{a}_{V,\text{old}} + \frac{\sqrt{r}}{2\sqrt{2}} (\rho_r^* \bar{\tau} \alpha'_{\text{old}} - \rho_r^* \tau \bar{\alpha}'_{\text{old}}) \right) \\
&+ r \mathcal{R}(u, \alpha'_{\text{old}}) = 0.
\end{aligned}
\tag{6.8}
$$

Here, the term $\mathcal{R}$ has the property that $|\mathcal{R}(u, \alpha'_{\text{old}})| \leq \zeta \cdot (|u|^2 + |u| \cdot |\alpha'_{\text{old}}|)$. Also, $\mathcal{R}$ has compact support where $|s| < \delta_0$. With the preceding understood, the proof of Lemma 6.3 requires an existence theorem for (6.8) and apriori estimates for the solution u. This is the next order of business.

The first observation is that (6.8) differentiates only along the fibers of N, so one can restrict attention to the fiber of N over any given point $z \in C$. Thus restricted, (6.6) becomes an equation on $\mathbb{C}$. The strategy then is to find a solution, u, which decays to zero at infinity. Such a solution will be found as the fixed point of a certain map with the help of the contraction mapping theorem. (See, e.g. Section 4a in [28].) The map in question sends a function u on $N|_z$ to $T(u)$, where the latter is given by

$$
\begin{aligned}
G_r \Big[\sqrt{r} \chi_{\delta_0/2} \Big(&\partial^V a_{V,\text{old}} - \bar{\partial}^V \bar{a}_{V,\text{old}} \\
&+ \frac{\sqrt{r}}{2\sqrt{2}} (\rho_r^* \bar{\tau} \alpha'_{\text{old}} - \rho_r^* \tau \bar{\alpha}'_{\text{old}}) \Big) + r \mathcal{R}(u, \alpha'_{\text{old}}) \Big].
\end{aligned}
\tag{6.9}
$$

Here, G_r is the Greens function kernel for the operator $(-2\partial^V \bar{\partial}^V + 2^{-1/2} r |\rho_r^* \tau|^2)$ on $N|_z$.

Concerning the Greens function G_r, remark that

$$(-2\partial^V\bar{\partial}^V + 2^{-1/2}r|\rho_r^*\tau|^2)$$

is obtained by rescaling the operator

$$(-2\partial^V\bar{\partial}^V + 2^{-1/2}|\tau|^2)$$

on $\mathbb{C}$. The latter defines a bounded map from the L^2_2 Sobolev space to L^2 with bounded inverse. This inverse, G, is given by an integral kernel; and an exercise with the maximum principle shows that

$$G(\eta,\eta') \leq \zeta|\ln(|\eta-\eta'|)|\exp(-|\eta-\eta'|/\zeta). \tag{6.10}$$

Here, ζ is independent of the pair η and η'. In any event, G_r is obtained from G by scaling by r. To be precise,

$$G_r(\eta,\eta') = r^{-1}\cdot G(\sqrt{r}\cdot\eta,\ \sqrt{r}\cdot\eta').$$

It is convenient to set up the contraction mapping argument for (6.9) as follows: Given $x \geq 1$, define the Banach space $\mathcal{B}_x$ by completing the space of functions on N with compact support using the norm

$$||u|| = \sup_N\{\exp(\sqrt{r}|s|/x)\cdot|u|\}. \tag{6.11}$$

It is an exercise with the maximum principle (use $\exp(-\sqrt{r}\cdot|s|/x)$ as a comparison function) to find a constant ζ which is independent of r (that is, j) and is such that if $x > \zeta$ and $\varepsilon < \zeta^{-1}$, then T maps $\mathcal{B}_x$ to itself and is a contraction on a ball centered about the origin of radius ζ^{-1}. (The version of ζ here is determined by the constants ζ which appear (6.10) and in the estimates for $(a_{V,\text{old}}, \alpha'_{\text{old}})$ in Lemma 6.1.) Thus, when $\varepsilon < \zeta^{-1}$ and j is sufficiently large, the contraction mapping theorem provides the function u which is a fixed point of the map T in (6.9). Apriori estimates for this function u then follow by differentiating the map T and invoking the estimates in Lemma 6.1 for $a_{V,\text{old}}$ and α'_{old}. (Remember that (v,τ) are infinitely differentiable along the fibers of N, and also that Lemma 6.1 has been invoked using $d+1$ instead of d.) In any event, one finds by straightforward arguments, that the derivatives of u along the fibers of N to order $d' \leq d+1$ of u are apriori bounded by $\zeta\cdot\varepsilon\cdot r^{d'/2}\cdot e^{-\sqrt{r}|s|/\zeta}$.

The preceding bounds plus (6.7) imply the first assertion of Lemma 6.3.

To prove the second assertion of Lemma 6.3, apriori bounds are required for the L^2 norm over N of ∇u and $\nabla^V \nabla u$. The latter are obtained as follows: Start by differentiating (6.8). This can be done on almost every fiber of N since $c = (v, \tau)$ is a Sobolev class L^2_1 section of (2.11). The result is an inhomogeneous equation for the derivatives of u of the form

$$(6.12) \qquad (-2\partial^V \bar{\partial}^V + 2^{-1/2} r |\rho_r^* \tau|^2)\nabla u + \text{Remainder} = 0.$$

The term here marked "Remainder" involves smaller terms with ∇u, terms which are linear in $\nabla_v \tau$, $\nabla_v \alpha'_{\text{old}}$ and terms which are linear in $\nabla a_{V,\text{old}}$ and $\nabla^V \nabla a_{V,\text{old}}$. With the preceding understood, take the pointwise inner product of (6.12) with ∇u, and then integrate the result over N^0. Integration by parts with the help of Lemma 6.1 and the triangle inequality yields apriori L^2 bounds for ∇u and $\nabla^V \nabla u$. The latter bounds plus (6.7) lead to Assertion 2 of Lemma 6.3.

The third assertion of Lemma 6.3 follows by construction, and the fourth assertion can be obtained directly from Lemma 6.1.

d) L_2 bounds on a_C

The purpose of this subsection is to employ the Seiberg-Witten equations to refine parts of the L^2 estimate from Assertion 2 of Lemma 6.3. The statement of the refined estimates requires the digression that follows. To start the digression, fix $z \in C$, and let η be a complex coordinate on $N|_z$ which isometrically identifies the latter with $\mathbb{C}$. By Lemma 6.3, when j is large, then

$$(6.13) \quad (v, \tau) + \chi_{\sqrt{r\delta_0}/4} \left(\frac{1}{2\sqrt{2}} (\rho^*_{r^{-1}}(a_V) d\bar{\eta} - \rho^*_{r^{-1}}(\bar{a}_V) d\eta),\ \rho^*_{r^{-1}}(\alpha') \right)$$

is close to (v, τ), but (v, τ) may not be the closest solution to the vortex equations. To find the closest solution, introduce the map Υ of Proposition 3.2 in [28]. The latter maps sections of $\oplus_{1 \le q \le m} N^q$ to sections of the vortex bundle of (2.15) in [28]. Also, introduce the operator Θ_c as defined in (2.12) for $c = (v, \tau)$. Remember from Part 2 of Section 2c of [28] that the kernel of Θ_c can be naturally identified with the tangent space at (v, τ) to the vortex moduli space.

Now note that when j is large, there exists a unique section y'^k of $\oplus_{1 \le q \le m} N^q$ which has the following properties:

- $|y'^k - y^k| \le \varepsilon$ and $||y'^k - y^k||_{1,2} \le \zeta$.

- Write
$$\Upsilon(y'^k) = (v, \tau) + (\frac{1}{2\sqrt{2}}(t_1 \cdot d\bar{\eta} - \bar{t}_1 \cdot d\eta), t_0).$$
Then
$$\chi_{\sqrt{r}\delta_0/4}\rho^*_{r^{-1}}(a_V, \alpha') - (t_1, t_0)$$
is L^2 orthogonal on each fiber of N to the kernel of Θ_c.

(6.14)

To find y'^k, first fix $z \in C$ and take any $y'^k \in \oplus_{1\le q\le m} N^q|_z$. Define a map from $\oplus_{1\le q\le m} N^q|_z$ to kernel(Θ_c) by assigning the L^2 orthogonal projection along $N|_z$ of $\chi_{\sqrt{r}\delta_0/4}\rho^*_{r^{-1}}(a_V, \alpha') - (t_1, t_0)$ to each y'^k. The fact that there is a unique, y'^k close to y^k which makes this projection zero follows using the implicit function theorem because the differential of the assignment of (t_1, t_0) to y'^k at y^k is the map Υ_1 of Proposition 3.2 in [28]. (Remember that the latter defines an isomorphism between $\oplus_{1\le q\le m} N^q|_z$ and kernel(Θ_c).) The estimates on the size of $y'^k - y^k$ in the first line of (6.14) follows from the bounds on (a_V, α') in the first two assertions of Lemma 6.3.

Given y'^k, and (t_1, t_0) as in (6.14), introduce $t = (\rho^*_r t_0, \rho^*_r t_1)$ as an element of the vector bundle $\mathcal{V}_0$ from (4.16a) in [28]. With t understood, introduce

- $\eta_V = (a'_V, \alpha'') = \chi_{\delta_0/4}(a_V, \alpha') - t.$
- $\eta_C = (a_C, \beta').$

(6.15)

Of course, η_C has no need of y'^k for its definition. For use later, note that the definition of t insures that η_V is L^2 orthogonal on each fiber of N to the kernel of the operator Θ as defined in (4.28) of [28].

The following lemma provides an apriori estimate of the size of η_V and η_C :

Lemma 6.4. *Under the assumptions of Lemma 6.3, there exists a constant ζ which is independent of ε and the index j, and is such that for all j sufficiently large,*

$$\begin{aligned} ||\nabla^V(\eta_V, \eta_C)||_2^2 + r||(\eta_V, \eta_C)||_2^2 + ||\nabla^H t||_2^2 \\ + ||\nabla^H \eta_V||_2^2 + ||\nabla^H \beta'||_2^2 \le \zeta r^{-1} \end{aligned} \tag{6.16}$$

Here, ∇^H denotes the covariant derivative in directions which are horizontal in TN. (Use the connection $\theta + \rho_r^* v$ on sections of bundles which involve E.)

Remark that (6.16) bounds $||a_C||_2$ by $\zeta \cdot r^{-1}$, which is a factor of $r^{-1/2}$ smaller than the estimate from Assertion 2 of Lemma 6.3. Likewise, (6.16) gives bounds for $\nabla^V a_C$ and $\nabla^H a_V$ which are a factor of $r^{-1/2}$ smaller than those from Lemma 6.3.

The remainder of this subsection is occupied with the

Proof of Lemma 6.4. As hinted, these estimates are obtained using the elliptic properties of the Seiberg-Witten equations. The derivation requires six steps.

Step 1. This step reintroduces the Seiberg-Witten equations to the story. For this purpose, reintroduce, from (4.28) of [28], the operator Θ and its formal L^2 adjoint $\Theta^\dagger$ on each fiber of N. Also, reintroduce on N the operators ∂^H and $\bar\partial^H$ which appear in (4.17) of [28].

Following the discussion in Section 4c of [28], rewrite the large r version of the Seiberg-Witten equations (where $|s| \leq \delta_0/8$) as equations for the pair η_V and η_C in the schematic form given below. (This equation is written out under the assumption that (6.6) holds.)

1. $\Theta \cdot \eta_V + (2^{-1}(\lambda^H a_C + \bar\lambda^H \bar a_C), \lambda^H \beta') + \mathcal{R}_V + \mathcal{R}_V^0 = 0.$

2. $\Theta^\dagger \eta_C + \bar\lambda^H (t(w) + \eta_V) + \mathcal{R}_C + \mathcal{R}_C^0 = 0.$

(6.17)

Here, $\lambda^H = a \cdot (\partial^H + b \cdot \bar\partial^H)$, where $a = 1 + \mathcal{O}(|s|^2)$ and $b = \mathcal{O}(|s|)$; and both are determined solely by the J and ω near C. (Note that the derivatives which are defined by ∂^H and $\bar\partial^H$ are covariant derivates, using the natural connections on the various summands of (4.16) of [28] as defined from θ and the Levi-Civita connection on TC.) Meanwhile, $\mathcal{R}_V^0$ and $\mathcal{R}_C^0$ are determined by the data (v, τ). In particular, their L^2 norms on N obey

$$||\mathcal{R}_{V,C}^0||_2 \leq \zeta r^{-1/2}. \tag{6.18}$$

Furthermore, $\mathcal{R}_V$ and $\mathcal{R}_C$ satisfy the pointwise bounds:

$$\begin{aligned}|\mathcal{R}_V| &\leq \zeta[\sqrt{r}(|t|\cdot|\eta_V| + |\eta_V|^2 + |\beta'|^2 + |\beta'||a_C|) + |t| \\ &\quad + |\eta_V| + |\eta_C| + |s|(|\nabla t| + |\nabla \eta_V| + |\nabla^V \eta_C|)] \\ |\mathcal{R}_C| &\leq \zeta[\sqrt{r}(|t|\cdot|\eta_C| + |\eta_V||\eta_C|) + |t| + |\eta_V| + |\eta_C| \\ &\quad + |s|(|\nabla t| + |\nabla \eta_V| + |\nabla^V \eta_C|)].\end{aligned} \tag{6.19}$$

Here is one further, crucial remark concerning $\mathcal{R}_V$ and $\mathcal{R}_V^0$: Both $\mathcal{R}_V$ and $\mathcal{R}_V^0$ are sections over N of the trivial complex line bundle. As a real bundle, the trivial complex line bundle can be written as the direct sum of two trivial real bundles, $\pi^*\Lambda^2T^*C \oplus \mathbb{R}$. The isomorphism from the latter to the former involves the volume form ω_C of C; it sends a pair $(a \cdot \omega_C, b)$ to the complex number $a + i \cdot b$. In particular, this isomorphism defines a real structure on the trivial complex line bundle. With the preceding understood, note that both $\mathcal{R}_V$ and $\mathcal{R}_V^0$ correspond to real sections of the trivial complex line bundle, since both come from sections of the $\pi^*\Lambda^2T^*C$ summand.

Step 2. The plan now is to consider (6.17) as an elliptic system for (η_V, η_C) and the section ω of V^c, and so obtain apriori L^2_1 estimates via the associated Weitzenboch formula. To start, take the L^2 norm of (6.17.2) over the subset of N where $|s| < \delta_0/16$, and so obtain, with the help of the triangle inequality, the inequality

$$\begin{aligned}&||\Theta^\dagger\eta_C||_2^2 + ||\bar{\lambda}^H(t+\eta_V)||_2^2 \\ &\quad + 2\cdot \mathrm{Re}\langle\Theta^\dagger\eta_C, \bar{\lambda}^H t\rangle_2 + 2\cdot\mathrm{Re}\langle\Theta^\dagger\eta_C, \bar{\lambda}^H\eta_V)\rangle_2 \\ &\qquad \leq \zeta(r^{-1} + (\varepsilon^2 r + 1)\cdot||\eta_C||_2^2 \\ &\qquad\quad + r^{-1}||\nabla^H(t+\eta_V)||_2^2 + r^{-1}||\nabla^V\eta_C||_2^2).\end{aligned} \tag{6.20}$$

Here, the constant ζ is independent of the index j and of the parameter ε. The derivation of (6.20) uses the apriori bounds

$$r^{-1/2}\cdot|\nabla^V(t+\eta_V)| + |(t+\eta_V)| \leq \zeta\cdot e^{-|\sqrt{r}|s|/\zeta}$$

from Lemma 6.3. The derivation also uses the apriori estimates from (1.24) in [27] which bound $|\beta'|$ by a multiple of $r^{-1/2}$.

The most troublesome terms in (6.20) are the third and fourth terms on the left side. The analysis of the third term occupies the remainder of this step, and the fourth term is treated in the subsequent step.

To begin, write $t = t' + t''$, where t' is obtained by rescaling $\Upsilon_1 \cdot (y'^k - y^k)$ using ρ_r. Here, Υ_1 is defined using y^k as in Proposition

3.2 of [28]. In particular, note that at each $z \in C$, this Υ_1 maps $\oplus_{1\leq q\leq m} N^q|_z$ isomorphically onto the kernel of the operator Θ_c as defined by $c = \Upsilon(y^k)$. Thus, $\Theta t' = 0$. Furthermore, since Υ_1 is essentially the differential of the map Υ at y^k, it follows that $|t''| \leq \zeta \cdot \varepsilon^2$. Likewise, $||\nabla^V t''||_2 \leq \zeta \cdot \varepsilon \cdot ||\nabla^V t||_2$ and also $||\nabla^H t''||_2 \leq \zeta \cdot \varepsilon \cdot ||\nabla^H t||_2$.

Given the preceding, integrate by parts in the third term in (6.20) to obtain the following inequality:

$$
\begin{aligned}
&|2 \cdot \mathrm{Re}\langle \Theta^\dagger \eta_C, \bar{\lambda}^H t\rangle_2| \\
&\quad \leq 2|\,\mathrm{Re}\langle \lambda^H \eta_C, \Theta t''\rangle_2| \\
&\qquad + \zeta(r^{-1} + R_0||\eta_C||_2^2 + \varepsilon||\nabla^V \eta_C||_2^2 + R_0^{-1}||\nabla^H t||_2^2).
\end{aligned} \tag{6.21}
$$

Here, ζ is independent of the index j and the choice of ε. Also, $R_0 \geq 1$ can be anything, in principle, although a particularly useful choice is made below. (The last terms on the right side of (6.21) arise from boundary terms where $|s| = \delta_0/16$, and from the failure of the horizontal derivatives and vertical derivatives to commute. These terms can all be handled with the help of Lemma 6.3 and with the apriori estimates in (2.5) and Lemma 2.4 of [28].)

Now, integrate by parts in reverse to rewrite the first term on the ride side of (6.21) in terms of $2 \cdot |\,\mathrm{Re}\langle \Theta^\dagger \eta_C,\, \bar{\lambda}^H t_2\rangle_2|$. The resulting inequality will have the same form as (6.21) but for a different choice of the parameter ζ. (As before, this ζ can be chosen so as to be independent of ε and the index j.) Finally, use the fact that the norms of t'' are a factor of ε smaller than those of t to bound the expression $2 \cdot |\,\mathrm{Re}\langle \Theta^\dagger \eta_C,\, \bar{\lambda}^H t''\rangle_2|$ by $\zeta \cdot \varepsilon \cdot ||\Theta^t \eta_C||_2 \cdot ||\nabla^H t||_2$, where ζ is again, independent of ε and of j. Thus, the (6.20) can be replaced by

$$
\begin{aligned}
&||\Theta^\dagger \eta_C||_2^2 + ||\bar{\lambda}^H(t + \eta_V)||_2^2 + 2\,\mathrm{Re}\langle \Theta^\dagger \eta_C,\, \bar{\lambda}^H \eta_V\rangle_2 \\
&\qquad \leq \zeta(r^{-1} + (\varepsilon^2 r + R_0)||\eta_C||_2^2 + R_0^{-1}||\nabla^H t||_2^2 \\
&\qquad\quad + r^{-1}||\nabla^H \eta_V||_2^2 + \varepsilon||\nabla^V \eta_C||_2^2).
\end{aligned} \tag{6.22}
$$

Here, and subsequently, it is assumed that $R_0^{-1} \gg \varepsilon \gg r^{-1}$. Note that the constant ζ which appears here can be taken to be independent of ε, R_0 and also the index j.

Step 3. This step considers the fourth term on the left in (6.20), which is to say the third term on the left in (6.22). To begin, integrate by parts to replace the third term on the left in (6.22) with $-2 \cdot \mathrm{Re}\langle \lambda^H \eta_C,\, \Theta \eta_V\rangle_2$ plus a remainder. This remainder term (which arises from boundary terms where $|s| = \delta_0/8$ and from the failure of derivatives to commute) can be bounded by an expression which has the same form as the right-hand side of (6.22). (Again, use Lemma 5.16 and (1.25) from [27].) The result of this replacement is the inequality

$$\begin{aligned} &\|\Theta^\dagger \eta_C\|_2^2 + \|\bar\lambda^H(t+\eta_V)\|_2^2 - 2\,\mathrm{Re}\langle \lambda^H \eta_C,\ \Theta\eta_V\rangle_2 \\ &\quad \le \zeta(r^{-1} + (\varepsilon^2 r + R_0)\|\eta_C\|_2^2 + R_0^{-1}(\|\nabla^H t\|_2^2 \\ &\qquad + \|\nabla^H \eta_V\|_2^2 + \varepsilon\|\nabla^V \eta_C\|_2^2)). \end{aligned} \tag{6.23}$$

(The R_0 which appears here is different than that which appears in (6.22). Even so, $R_0 \geq 1$ is free to choose, although a particular choice given below proves most useful. Also, the constant ζ in (6.23) is bigger than that which appears in (6.22), but it is still independent of the index j or the choice of the parameters ε and R_0.)

To make further progress, use (6.17.1) to substitute for $\Theta\eta_V$ to obtain the inequality

$$\begin{aligned} &\|\Theta^\dagger \eta_C\|_2^2 + \|\bar\lambda^H(t+\eta_V)\|_2^2 \\ &\quad + \mathrm{Re}\langle \lambda^H a_C, \lambda^H a_C + \bar\lambda^H \bar a_C \\ &\quad + (\mathcal{R}_V + \mathcal{R}_V^0)_1\rangle_2 + \|\lambda^H \beta'\|_2^2 \\ &\qquad \ge \zeta(r^{-1} + (\varepsilon^2 r + R_0)(\|\eta_C\|_2^2 + \|\eta_V\|_2^2) \\ &\qquad\quad + R_0^{-1}(\|\nabla^H t\|_2^2 + \|\nabla^H \eta_V\|_2^2) + \varepsilon\|\nabla^V \eta_C\|_2^2). \end{aligned} \tag{6.24}$$

To deal with the third term on the left side of (6.24) it is necessary to exploit the fact that $(\mathcal{R}_V + \mathcal{R}_V^0)_0$ is real. With this understood, the triangle inequality can be used to obtain

$$\begin{aligned} &\mathrm{Re}\langle \lambda^H a_C, \lambda^H a_C + \bar\lambda^H \bar a_C + (\mathcal{R}_V + \mathcal{R}_V^0)_1\rangle_2 \\ &\quad \ge 4^{-1}\|\lambda^H a_C + \bar\lambda^H \bar a_C\|_2^2 - \zeta\|\mathcal{R}_V + \mathcal{R}_V^0\|_2^2. \end{aligned} \tag{6.25}$$

Furthermore, the L^2 norm of $\mathcal{R}_V + \mathcal{R}_V^0$ is bounded by an expression which is similar to that appearing on the right-hand side of (6.24). This all means that (6.24) can be replaced by the inequality

$$\begin{aligned} &\|\Theta^\dagger \eta_C\|_2^2 + \|\bar\lambda^H(t+\eta_V)\|_2^2 + \|\lambda^H a_C + \bar\lambda^H \bar a_C\|_2^2 + \|\lambda^H \beta'\|_2^2 \\ &\quad \le \zeta(r^{-1} + (\varepsilon^2 r + R_0)\cdot(\|\eta_C\|_2^2 + \|\eta_V\|_2^2) \\ &\qquad + R_0^{-1}(\|\nabla^H t\|_2^2 + \|\nabla^H \eta_V\|_2^2) + \varepsilon\|\nabla^V \eta_C\|_2^2). \end{aligned} \tag{6.26}$$

Here, of course, the constant ζ is not the same as that which appears previously, but in any event, it can be taken to be independent of the index j and the parameters ε and R_0.

Step 4. This step uses (6.17.1) to bound the L^2 norm of $\Theta \cdot \eta_C$ by a multiple of the sum of the L^2 norm of $(2^{-1} \cdot (\lambda^H a_C + \bar{\lambda}^H \bar{a}_C), \lambda^H \beta')$ and of $\mathcal{R}_V + \mathcal{R}_V^0$. As a result, (6.26) implies that

$$\begin{aligned}
&||\Theta\eta_V||_2^2 + ||\Theta^\dagger \eta_C||_2^2 + ||\bar{\lambda}^H(t+\eta_V)||_2^2 \\
&\qquad + ||\lambda^H a_C + \bar{\lambda}^H \bar{a}_C||_2^2 + ||\lambda^H \beta'||_2^2 \\
&\qquad \le \zeta(r^{-1} + (\varepsilon^2 r + R_0)(||\eta_C||_2^2 + ||\eta_V||_2^2) \\
&\qquad\quad + R_0^{-1}(||\nabla^H t||_2^2 + ||\nabla^H \eta_V||_2^2) + \varepsilon||\nabla^V \eta_C||_2^2).
\end{aligned} \tag{6.27}$$

Here, again, the constant ζ can be assumed to be independent of ε, R_0 and the index j.

Step 5. This step invokes Lemma 4.6 in [28] and the fact that η_V is fiberwise L^2 orthogonal to any element in the kernel of Θ to conclude that

1. $||\nabla^V \eta_V||_2^2 + r||\eta_V||_2^2 \le \zeta||\Theta\eta_V||_2^2$.
2. $||\nabla^V \eta_C||_2^2 + r||\eta_C||_2^2 \le \zeta||\Theta^\dagger \eta_V||_2^2$.

(6.28)

Step 6. This step considers the relationship between $||\bar{\lambda}^H(t+\eta_V)||_2^2$ and the sum of $||\bar{\lambda}^H t||_2^2$ and $||\bar{\lambda}^H \eta_V||_2^2$. For this purpose, write $t = t' + t''$ as in Step 2. The first observation here is that

$$||\bar{\lambda}^H t''||_2^2 \le \zeta\varepsilon^2||\nabla^H t'||_2^2. \tag{6.29}$$

Thus,

$$||\bar{\lambda}^H(t+\eta_V)||_2^2 \le \zeta^{-1}(||\bar{\lambda}^H t'||_2^2 + ||\bar{\lambda}^H \eta_V||_2^2) + 2\,\mathrm{Re}\langle\bar{\lambda}^H t', \, \bar{\lambda}^H \eta_V\rangle_2. \tag{6.30}$$

To estimate the cross term in (6.30), it proves useful to introduce over an open ball $B \subset C$ an orthonormal basis $\{w_q\}_{1\le q\le m}$ for the bundle $\oplus_{1\le q\le m} N^q$. For each q, write the corresponding $\Upsilon_1 \cdot w_q$ as (b, δ) and then define $u_q = (r^{-1/2}\rho_r^* b, \, \rho_r^* \delta)$. Thus, $\{u_q\}$ defines, fiberwise over B, a basis for the kernel of Θ. Remark that because $c = (v, \tau)$ defines a Sobolev class L_1^2 section of (2.15) in [28], each u_q obeys $||\nabla^H u_q||_2 \le \zeta \cdot r^{-1/2}$.

Now, t' is a linear combination of the $\{u_q\}$. And, since each u_q is L^2 orthogonal on each fiber to η_V, and $|t'|$ and $|\eta_V|$ are both pointwise smaller than ε, it follows that

$$|\langle\bar{\lambda}^H t', \bar{\lambda}^H \eta_V\rangle_2| \le \zeta\varepsilon r^{-1/2}(||\nabla^H t||_2 + ||\nabla^H \eta_C||_2). \tag{6.31}$$

Given the preceding, one can conclude from (6.30) that

$$||\bar{\lambda}^H(t+\eta_V)||_2^2 \geq \zeta^{-1}(||\bar{\lambda}^H t'||_2^2 + ||\bar{\lambda}^H \eta_V||_2^2) - \zeta\varepsilon r^{-1}, \tag{6.32}$$

where, again, ζ is independent of ε and the index j.

Given (6.23), the plan now involves integration by parts to compare the L^2 norm of $\bar{\lambda}^H t$ with that of $\nabla^H t$, and similarly for η_C. This procedure is straightforward and yields from (6.32) the inequality

$$||\bar{\lambda}^H(t+\eta_V)||_2^2 \geq \zeta^{-1}(||\nabla^H t'||_2^2 + ||\nabla^H \eta_V||_2^2) - \zeta\varepsilon r^{-1}, \tag{6.33}$$

where ζ here is different than in (6.32), but still independent of ε and the index j.

Step 7. This step plugs (6.28) and (6.33) into (6.27). The resulting inequality gives the assertion in Lemma 6.4.

e) The appearance of $\mathcal{Z}_0$

Here is the important feature of $\{y^k\}$ from Section 6a:

Lemma 6.5. *For each k, the data y^k from Section 6a lies in the (C_k, m_k) version of the variety $\mathcal{Z}_0$. In particular, this implies that each y^k is a smooth section of the corresponding (C_k, m_k) version of $\oplus_{1\leq q\leq m} N^q$.*

Note: The assertion that y^k lies in $\mathcal{Z}_0$ is equivalent to the assertion that the (3.5) of [28] is identically zero when $c = \Upsilon(y^k)$.

The remainder of this subsection is occupied with the.

Proof of Lemma 6.5. Return to the milieu of Section 4 in [28], and reintroduce the bundle $\mathcal{V}_1 \to N^0$ in (4.16b). Lemma 4.4 in [28] describes a natural identification between $i \cdot (e \oplus \Lambda_+) \oplus S_-$ and $\mathcal{V}_1$. As such, the left hand side of (6.17) defines a canonical section of $\mathcal{V}_1$ from data $\eta_V = (a'_V, \alpha'')$ and $\eta_C = (a_C, \beta')$. That is, the left hand side of (6.17) is defined whether or not the Seiberg-Witten equations are satisfied; the Seiberg-Witten equations assert only that this section vanishes.

With the preceding understood, reintroduce from Lemma 4.5 in [28] the vector bundle $K_1 \to C$ whose fiber over z is a certain space of sections over $\pi^{-1}(z)$ of the vector bundle $\mathcal{V}_1$. This bundle K_1 depends on the data (v, τ), so it can not yet be said to be smooth, but it is $C^{0,1/2}$ and one can talk about a Sobolev class L^2_1 section. This bundle K_1 depends on r in a simple fashion; an appropriate r-dependent fiberwise rescaling canonically identifies K_1 with the bundle $V^c \otimes T^{0,1}C \to C$ as defined in Section 3 of [28] for c.

Now, multiply the left hand side of (6.17) by $\chi_{\delta_0/4}$ and take the fiberwise L^2 orthogonal projection of the result onto K_1 to define a section, $\mathfrak{s} = \mathfrak{s}[r; (\eta_V, \eta_C)]$ over C of K_1. This section is, of course, identically zero when (η_V, η_C) comes from $(a_j, (\alpha_j, \beta_j))$, a solution of the $r = r_j$ and $\mu_0 = 0$ Seiberg-Witten equations. But for the generic choice of (η_V, η_C), the section $\mathfrak{s}$ need not vanish.

Meanwhile, the fact that the bundle K_1 (as defined for a given r) is obtained by scaling the fixed bundle $V^c \otimes T^{0,1}C$ can be exploited in the following way: With r fixed, reverse this scaling to obtain from the expression in (3.5) of [28] a canonical section, ϑ_r, of K_1. Note that with the norm on K_1 defined as in (4.30) of [28], the norm of ϑ_r is independent of r.

Given the preceding, here is the strategy for the proof of Lemma 6.5: As described above, $(a_j, (\alpha_j, \beta_j))$ defines a pair (η_V, η_C) and thus the section $\mathfrak{s}[r_j, (\eta_V, \eta_C)]$ of K_1 which is, apriori, known to be zero. But, if (η_V, η_C) are sufficiently small, then this section will hardly differ from $\mathfrak{s}[r_j, (0,0)]$. However, the latter section is almost ϑ_j as defined using $r = r_j$. Indeed, the section $\mathfrak{s}[r, (0,0)]$ differs from ϑ_r by a term which is bounded pointwise by a multiple of $r^{-1/2}$. Thus, if (η_V, η_C) are, for large j, sufficiently small, then the vanishing of $\mathfrak{s}[r_j, (\eta_V, \eta_C)]$ will imply that the norm of ϑ_r tends to zero as $r = r_j$ gets large. However, as remarked above, the norm of ϑ_r is independent of r and equals the norm of (3.5) in [28]. Of course, this implies that (3.5) in [28] vanishes which is what needed proving.

This strategy is carried out in three steps.

Step 1. With the help of (6.17) and the analysis of Section 4 of [28], (especially (4.19) in [28]), one can now compute the section $\mathfrak{s}$ in terms of the data (η_V, η_C). Here is s:

$$\begin{aligned}\vartheta_r &+ \Pi \cdot \chi_{\delta_0/4}(\Theta^\dagger \eta_C + \bar{\lambda}^H(t(w) + \eta_V)) \\ &+ \Pi \cdot \chi_{\delta_0/4} \frac{\sqrt{r}}{2\sqrt{2}} (\bar{\alpha}'\beta',\, \alpha' a_C) \\ &+ \text{remainder}_1 + \text{remainder}_2,\end{aligned} \tag{6.34}$$

where, Π is the fiberwise L^2-orthogonal projection onto the span of the kernel of the operator Θ which appears in (4.28) of [28]. In this last expression, the term marked as "remainder$_1$", contains the contribution from the operators Q and Rem which appear in (4.19) of [28]. This term has the schematic form

$$\begin{aligned} &\Pi\cdot\mathcal{R}_1\cdot(\nabla^V a_V, \nabla^V\alpha', \nabla^V\beta') \\ &\qquad + \Pi\cdot\mathcal{R}_2\cdot(\nabla^H a_V, \nabla^H a_C, \nabla^H\alpha', \nabla^H\beta') \\ &\qquad + \Pi\cdot\mathcal{R}_3\cdot(a_V, \alpha', a_C, \beta'). \end{aligned} \tag{6.35}$$

Here, $\mathcal{R}_1$ is a compactly supported, $\mathbb{R}$-linear bundle endomorphism over N^0 which is bounded in norm by $\zeta\cdot|s|$ and whose covariant derivative is bounded in norm by ζ. Meanwhile, $\mathcal{R}_2$ is a compactly supported, $\mathbb{R}$-linear bundle endomorphism over N^0 which is bounded in norm by $\zeta\cdot|s|$ and has horizontal covariant derivative which are also bounded in norm by $\zeta\cdot|s|$. Finally, $\mathcal{R}_3$, is a compactly supported, $\mathbb{R}$-linear bundle endomorphism over N^0 which is bounded in norm by ζ. This constant ζ can be taken to be independent of j. This term contains the contributions to $\mathfrak{s}$ from $\mathcal{R}_C$ in (6.17).

Meanwhile, the term labeled as "remainder$_2$" has norm bounded by $\zeta\cdot r^{-1/2}$ and has, itself, two pieces. The first piece is obtained from an appropriate r-dependent rescaling of the section of $V^c\otimes T^{0,1}C$ which is obtained from the expression in (3.5) of [28] by the replacement of $\Pi^c\cdot(\cdot)$ with $\Pi^c\cdot(1-\chi_{\delta_0/4})\cdot(\cdot)$. The second piece is accounted for by the difference between the ϑ_r and $\mathfrak{s}[r,(0,0)]$. This term contains the contributions to $\mathfrak{s}$ from $\mathcal{R}_C^0$ in (6.17).

Step 2. Let $B\subset C$ be a ball, and let $\{w'_q\}_{1\le q\le m}$ be an orthonormal basis for the bundle $(\oplus_{1\le q\le m}N^q)\otimes T^{0,1}C$ over C. The homomorphism Υ_1 maps $\{w'_q\}$ to give a basis for the bundle $V^c\otimes T^{0,1}C$ over B. (Remember that V^c is defined using $c=(v,\tau)$ and thus is a bundle with Hölder continuous transition functions.) Now, fix $r=r_j$. After the appropriate r dependent rescaling, the basis $\{\Upsilon_1 w'_q\}$ of $V^c\otimes T^{0,1}C$ defines a basis $\{w_q\}$ for K_1 over B. These basis elements have r-independent norm and inner product as measured by the metric on K_1 which comes from the norm in (4.30) of [28].

Note that each w_q defines an element, $\underline{w}_q$, over $\pi^{-1}(B)$ which is, on each fiber of N, in the kernel of the operator Θ from (4.28) in [28]. Furthermore, the supremum norm of $\underline{w}_q$ is bounded by a j-independent constant ζ. Finally, the L^2 norm over $\pi^{-1}(B)$ of $\underline{w}_q$ and of $\nabla^H\underline{w}_q$ are bounded by $\zeta\cdot r^{-1/2}$. (These estimates are simple consequences of the scaling relationship between $\underline{w}_q$ and $\Upsilon_1 w'_q$.)

Step 3. Let $u=(u_1,\dots,u_q):B\to\mathbb{C}^m$ be a smooth, compactly supported map. Then $\Sigma_q u_q\cdot w_q$ defines a section of K_1 over B. The

L^2 inner product of this section with ϑ_r is independent of r as rescaling finds it equal to the L^2 inner product of $\Sigma u_q \cdot w'_q$ with the section in (3.5) of [28]. On the other hand, the L^2 inner product of $\Sigma_q u_q \cdot w_q$ with ϑ_r can be computed using the fact that the sum in (6.34) is supposed to be zero. With this understood, consider the L^2 inner product of $\Sigma_q u_q \cdot w_q$ with the various terms (other than the first) in (6.34).

The L^2 inner product of $\Sigma_q u_q \cdot w_q$ with the second term in (6.34) is bounded in norm by $e^{-\sqrt{r}/\zeta}$ for an appropriate, j-independent constant ζ because $\underline{w}_q$ is annihilated on each fiber by the operator Θ. (Up to contributions from where $|s| \geq \delta_0/8$, integration by parts identifies the inner product of this second term with the inner product of $\Sigma_q u_q \cdot \Theta \underline{w}_q$ with something. Meanwhile, the contributions from where $|s| \geq \delta_0/8$ can be estimated using Lemma 6.3.)

The L^2 inner product of $\Sigma_q u_q \cdot w_q$ with the third term in (6.34) is given by

$$r\langle |\Sigma_q u_q \underline{w}_q, \chi_{\delta_0/4} \bar{\lambda}^H (t(w) + \eta_V)\rangle_2, \tag{6.36}$$

where $\langle \cdot, \cdot \rangle_2$ is short-hand for the L^2 inner product over $N^0|_B$. After integrating by parts, Lemma 6.3 can be invoked to bound (6.36) by $\zeta \cdot ||u||_{1,2} \cdot \varepsilon$, where ζ is a j and ε independent constant, and $|| \cdot ||_{1,2}$ signifies the L^2_1 norm of a $\mathbb{C}^m$-valued function on B.

The L^2 inner product of $\Sigma_q u_q \cdot w_q$ with the fourth term in (6.34) has the form

$$r\left\langle \Sigma_q u_q \underline{w}_q, \chi_{\delta_0/4} \frac{\sqrt{r}}{2\sqrt{2}} (\bar{\alpha}'\beta',\, \alpha' a_C) \right\rangle_2. \tag{6.37}$$

Together, Lemma 6.3 can be used to bound the norm of this term by $\zeta \cdot \varepsilon \cdot ||u||_2$. (Of particular importance here is the bound of $||\eta_C||_2$ by $\zeta \cdot r^{-1}$.)

The L^2 inner product of $\Sigma_q u_q \cdot w_q$ with terms "remainder$_1$" and "remainder$_2$" in (6.34) can be bounded in norm by $\zeta \cdot (\varepsilon + r^{-1/2}) \cdot ||u||_{1,2}$ by using Lemma 6.3 and (1.24) in [27] together with the stated properties of the endomorphisms $\{\mathcal{R}_b\}_{b=1,2,3}$. (Integrate by parts to move the derivatives onto u_q and $\underline{w}_q$.)

The preceding estimates imply that the L^2 inner product on C between $\Sigma_q u_q \cdot w_q$ and ϑ_r is zero, since its norm is independent of r and thus arbitrarily small. Hence, (3.5) of [28] must vanish for $c = \Upsilon(y^k)$ as claimed.

f) Rechoosing the gauge

With each y^k now known to be a smooth section of the (C_k, m_k) version of $\oplus_{1\le q\le m} N^q$, the gauge choice for $(a_j, (\alpha_j, \beta_j))$ at large j can be refined further so as to coincide with the gauge choice which is implicit in the constructions of Section 5 in [28]. To elaborate, note that Sections 2 and 3b of [28] takes data $\{y'^k\} \in \times_k \mathcal{K}_\Lambda^{(k)}$ near $\{y^k\}$ and constructs an approximate solution, $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r)) = (\underline{a}_r[y'], (\underline{\alpha}_r[y'], \underline{\beta}_r[y']))$ of the $r = r_j$ and $\mu_0 = 0$ version of the Seiberg-Witten equations in (2.4). Then, Proposition 5.2 and Section 5 in [28] describe a perturbation of $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ which gives an honest solution to (2.4) when $\{y'^k\}$ is mapped to zero by the map ψ_r in Proposition 5.2 of [28]. The perturbation of $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ is expressed as in (6.1) where the data $(a', (\alpha', \beta'))$ are constrained to satisfy the "zero divergence" gauge condition:

$$*d*a' + i\frac{\sqrt{r}}{\sqrt{2}} im(\bar{\underline{\alpha}}_r \alpha' + \bar{\underline{\beta}}_r \beta') = 0, \tag{6.38}$$

on the whole of X. Thus, the proof of Proposition 5.2 requires the following: Given $\{y'^k\} \in \times_k \mathcal{K}_\Lambda^{(k)}$ near $\{y^k\}$ and j large, a point on the orbit of $(a_j, (\alpha_j, \beta_j))$ must be found, for which the corresponding $(a', (\alpha', \beta'))$ satisfies (6.38).

Here is the main result of this subsection:

Lemma 6.6. *There is a constant $\zeta > 1$ with the following significance: Fix $\varepsilon > 0$ but less than ζ^{-1} and a positive integer d. Take the index j large. For each k, choose a point $y'^k \in \mathcal{K}_\Lambda^{(k)}$ with L^2-distance $\zeta^{-1}\cdot\varepsilon$ or less from y^k. Use $\{y'^k\}$ to define the data $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ for $r = r_j$ as specified in Sections 2 and 3b of [28]. Then, there exists a point on the orbit of $(a_j, (\alpha_j, \beta_j))$ which is such that the data $q' = (a', (\alpha', \beta'))$ as defined by comparison with $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ as in (6.1) obeys:*

1. *At each $x \in X$,*

$$\sum_{0\le d'\le d} r^{-d'/2} |(\nabla)^{\otimes d'}(a', (\alpha', \beta'))| \le \varepsilon \cdot \exp(-\sqrt{r_j}\, \mathrm{dist}(x, \cup_k C_k)/\zeta). \tag{6.39}$$

2. *Equation (6.38) holds.*

The remainder of this subsection is occupied with the proof of this last lemma.

Proof of Lemma 6.6. The proof is accomplished via seven steps.

Step 1. Return to the gauge for $(a_j, (\alpha_j, \beta_j))$ which is given in Lemma 6.1. For the moment, define $(a', (\alpha', \beta'))$ using $\{y^k\}$ to define the reference data $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$. Given that each y^k is now known to be smooth, the proof of Lemma 6.1 can be readily modified to prove that the data $(a', (\alpha', \beta'))$ as just defined is smooth and satisfies (6.39) when j is large.

With the preceding understood, agree to label the data from Lemma 6.1's gauge using $\{y^k\}$ for reference data now as $(a'_{\text{old}}, (\alpha'_{\text{old}}, \beta'_{\text{old}}))$. Also, agree to label the reference data $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ as defined by $\{y'^k\}$ as $(\underline{a}_{r,y'}, (\underline{\alpha}_{r,y'}, \underline{\beta}_{r,y'}))$.

Step 2. The gauge from Lemma 6.6 for $(a_j, (\alpha_j, \beta_j))$ will be determined by a function u on X as follows:

$$\begin{aligned}(a', (\alpha', \beta')) = \Big(& a'_{\text{old}} - \frac{i}{\sqrt{r_j}} du + \underline{a}_{r,y} - \underline{a}_{r,y'}, (e^{i\cdot u}\alpha'_{\text{old}} \\ & + (e^{i\cdot u}\underline{\alpha}_{r,y} - \underline{\alpha}_{r,y'}), e^{i\cdot u}\beta'_{\text{old}} + (e^{i\cdot u}\underline{\beta}_{r,y} - \underline{\beta}_{r,y'}))\Big).\end{aligned} \tag{6.40}$$

The function u will be found by considering (6.38) as an equation for u on X. The latter equation is equivalent to an equation of the form:

$$- * d * du + 2^{-1/2} r_j |\underline{\alpha}_{r,y'}|^2 u + Z(u) = 0, \tag{6.41}$$

where

$$\begin{aligned} Z(u) = {} & i\sqrt{r} * d * a'_{\text{old}} \\ & + 2^{-1/2} r(|\underline{\alpha}_{r,y'}|^2 \cdot (sin(u) - u) + im(\bar{\underline{\alpha}}_{r,y'} e^{i\cdot u} \cdot \alpha'_{\text{old}},) \\ & + i\sqrt{r} * d * (\underline{a}_{r,y} - \underline{\alpha}_{r,y'}) \\ & + 2^{-1/2} r \; im(\bar{\underline{\alpha}}_{r,y'} e^{i\cdot u} \cdot (\underline{\alpha}_{r,y} - \underline{\alpha}_{r,y'})) \\ & + 2^{-1/2} r \; im(\bar{\underline{\beta}}_{r,y'} (e^{iu}\beta'_{\text{old}} + e^{i\cdot u}(\underline{\beta}_{r,y} - \underline{\beta}_{r,y'}))). \end{aligned} \tag{6.42}$$

With regard to forthcoming arguments, remark that Z obeys the following apriori, pointwise bound:

$$\begin{aligned} |Z(u)|(x) \leq {} & \zeta r(\varepsilon e^{-\sqrt{r\cdot d(x)}/\zeta} \\ & + |u|^2 + |u| \, ||y' - y|| e^{-\sqrt{r d(x)}/\zeta} \\ & + ||y' - y|| e^{-\sqrt{r\cdot d(x)}/\zeta}). \end{aligned} \tag{6.43}$$

Here, $d(x) = \text{dist}(x, \cup_k C_k)$. Also, $||y' - y|| = \Sigma_k ||y'^k - y^k||_2$. And, as always, $r = r_j$. Meanwhile, the constant ζ is independent of r, ε and the index j. Note that the derivation of (6.43) invokes (6.39). Remark also that the second to last line in (6.42) is bounded by $\zeta r ||y' - y|| \cdot e^{-\sqrt{r} \cdot d/\zeta}$. This fact follows from the definition of $\underline{\alpha}_{r,y'}$ and from the manner in which y' parameterizes $(\underline{a}_{r,y'}, (\underline{\alpha}_{r,y'}, \underline{\beta}_{r,y'}))$. See Sections 2 and 3b of [28].

Step 3. The function u in question will be found by a contraction mapping argument on a ball about the origin in a certain Banach space. The Banach space in question, $\mathcal{L}$, is obtained as the completion of the space of smooth functions on X using the norm,

$$||u||_* = ||\nabla u||_2 + \sqrt{r}||u||_2 + \sup |u|. \tag{6.44}$$

The map in question, T, sends a function u to $-G[Z(u)]$. Here, $G[\cdot]$ is the integral operator which is defined by the Greens function for the operator $- * d * d + 2^{-1/2} \cdot r \cdot |\underline{\alpha}_{r,y}|^2$. In order to affect this strategy, it is necessary to first prove that T maps a ball about the origin in $\mathcal{L}$ to itself. Subsequently, one must demonstrate that T is a contraction mapping on such a ball.

Step 4. Consider first the question of the range of the map T when restricted to the ball in $\mathcal{L}$ where $||u||_* < 1$. To estimate the L^2_1 norm of $T = T(u)$, multiply both sides of the equation

$$(- * d * d + 2^{-1/2} r |\underline{\alpha}_{r,y}|^2) T = -Z(u) \tag{6.45}$$

by T and integrate the result over X. After integrating by parts on the left side of the resulting equation and employing Hölder's inequality to bound the right side of the resulting equation, one finds that

$$\begin{aligned} ||\nabla T||_2^2 + r|| \, |\underline{\alpha}_{r,y}| T||_2^2 \leq & \zeta \sqrt{r} ||T||_2 (\varepsilon + ||y' - y|| \\ & + \sqrt{r} ||u||_2 (\sup |u|)). \end{aligned} \tag{6.46}$$

This last equation is derived with the help of (6.43).

Meanwhile, the right-hand side of (6.46) is greater than $\zeta^{-1} \cdot r \cdot ||T||_2^2$ where ζ is independent of T and r. (A heuristic argument procedes as follows: If a significant fraction of the L^2 norm were concentrated where $|\underline{\alpha}_{r,y}| > 1/2$, then the estimate follows immediately. On the otherhand, when a significant fraction of the L^2 norm of T comes from where $|\underline{\alpha}_{r,y}| < 3/4$, then this fraction must sit where the distance to

$\cup_k C_k$ is $\mathcal{O}(r^{-1/2})$. For such T, the size of $||\nabla T||_2$ is bounded from below by $\zeta^{-1} \cdot \sqrt{r} \cdot ||T||_2$. In fact, for this last estimate, one needs only the components of ∇T which lie along the fibers of the normal bundle N to each $C = C_k$. (In any event, it is straightforward to produce a completely rigorous argument along these lines using cut-off functions.)

Given these last facts, then (6.46) implies

$$||\nabla T||_2^2 + r||T||_2^2 \leq \zeta(\varepsilon^2 + ||y' - y||^2 + ||u||_*^4). \tag{6.47}$$

Step 5. Turn now to the task of estimating the sup norm of T. The tool of choice for this task is the maximum principle whose application requires the introduction of a certain comparison function. To define this function, let $d(\cdot) = \mathrm{dist}(\cdot, \cup_k C_k)$ and reintroduce a number R_1 with the property that $|\underline{\alpha}_{r,y}| > 1/2$ where $d(x) > R_1/\sqrt{r}$. Now, introduce $\zeta \geq 16$ and the function w on X which is defined as follows:

$$w(\cdot) = \exp[-\sqrt{r} d(\cdot)/(\zeta R_1)] + \zeta R_1^{-2}. \tag{6.48}$$

A straightforward calculation finds a constant $\zeta \geq 16$ which is independent of j and hence r (when j is large) and is such that

$$(- * d * d + 2^{-1/2} r |\underline{\alpha}_{r,y}|^2) w \geq \zeta^{-1} r. \tag{6.49}$$

Here, ζ depends on R_1.

Given w, consider the function

$$w' = |T| - \zeta' \cdot (\varepsilon + ||y' - y|| + ||u||_*^2) \cdot w$$

for various choices of ζ'. It follows from (6.43) that there exists $\zeta' > 1$ which is independent of ε, r and the index j, and is such that

$$(- * d * d + 2^{-1/2} \cdot r \cdot |\underline{\alpha}_{r,y}|^2) w' \leq 0$$

on X. With this understood, the maximum principle asserts that w' can not have a positive maximum. This implies that

$$|T(u)| \leq \zeta(\varepsilon + ||y' - y|| + ||u||_*^2). \tag{6.50}$$

Step 6. It follows from (6.47) and (6.50) that there is a constant $\zeta \geq 1$ which is independent of ε and j and is such that when

$$\varepsilon + ||y' - y|| < \zeta^{-2},$$

then T maps the ball of radius ζ^{-1} in $\mathcal{L}$ to itself.

It remains as yet to prove T is a contraction mapping on the radius ζ ball in $\mathcal{L}$ for a suitable choice of ζ. This task can be accomplished by straightforward modifications of the preceding argument, and is left to the reader.

Step 7. The proof of Lemma 6.6 is completed with the verification of the estimates in (6.40). For this purpose, remark that the contraction mapping construction with (6.50) bounds $\sup_X |u|$ by $\zeta \cdot \varepsilon$ when $||y|| \leq \zeta^{-1} \cdot \varepsilon$. This last fact, plus (6.43), implies that u obeys the following differential inequality where $d(x) \geq R_1/\sqrt{r}$:

$$(6.51) \qquad (- * d * d + (4\sqrt{2})^{-1} r)|u| \leq \zeta \varepsilon r e^{-\sqrt{r} d(x)/\zeta}.$$

Here, $||y' - y|| \leq \zeta^{-1} \cdot \varepsilon$ is assumed.

Now, given $\zeta' \geq 1$, introduce the comparison function

$$w = e^{-\sqrt{r} d(x)/\zeta'} + e^{-\sqrt{r} \cdot/\zeta'}.$$

For a suitable (r and j independent) choice of ζ' and ζ'', the function $w' = |u| - \zeta'' \varepsilon w$ obeys the inequality

$$(6.52) \qquad (- * d * d + (4\sqrt{2})^{-1} r \cdot) w' \leq 0,$$

where $d(x) \geq R_1/\sqrt{r}$ and w' is non-positive where $d(x) = R_1/\sqrt{r}$. Thus, the maximum principle implies that $w' \leq 0$ everywhere; and this last fact implies the pointwise bound of $|u|$ by $\zeta \varepsilon e^{-\sqrt{r} d(x)/\zeta}$ for some r, ε and j independent choice of ζ.

Given this supremum bound and the fact that $(a'_{\text{old}}, (\alpha'_{\text{old}}, \beta'_{\text{old}}))$ obeys (6.39), a bound by $\zeta_p r^{p/2} e^{-\sqrt{r} d(x)/\zeta}$ for the order p derivatives of u at a point $x \in X$ follows by standard elliptic regularity estimates. (To invoke standard techniques, use a cut-off function to localize (6.41) to the ball of radius $2^{-1} \cdot (d(x) + r^{-1/2})$ about the given x. Then refer to [17], Chapter 6.)

These estimates for u and its derivatives imply via (6.40) the asserted estimates for $(a', (\alpha', \beta'))$ in Lemma 6.6.

g) The definition of q^0 and $\{q^k\}$

Fix small $\varepsilon > 0$ and take the index j to be large. For each k, choose a point $y'^k \in \mathcal{K}_\Lambda^{(k)}$ in the ball of L^2- radius ε about y^k. Use $\{y'^k\}$ to define the data $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ choosing $r = r_j$ as specified in Sections 2 and 3b of [28]. (Here, and below, the dependence of this data on

y' will generally not be noted explicitly.) Take the point on the orbit of $(a_j, (\alpha_j, \beta_j))$ as described by Lemma 6.6 and use this point in (6.1) along with $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ in order to define $(a', (\alpha', \beta'))$ on X.

With the preceding understood, the goal for this subsection is to decompose the data $q' = (a', (\alpha', \beta'))$ as in (4.6) of [28] in terms of $(q^0, \{q^k\})$ where $(q^0, \{q^k\}$ satisfy (5.3) in [28]. That is, the goal is to write

$$q' = \Pi_k(1 - \chi_{4\delta,k})q^0 + \Sigma_k \chi_{100\delta,k} q^k. \tag{6.53}$$

Here, q^0 is a section of $i \cdot T^*X \oplus S_{+,0}$, where $S_{+,0}$ is the plus spin bundle for the canonical $\mathrm{Spin}^{\mathbb{C}}$ structure on X. (See (1.7).) And, q^k is a section over the normal bundle, N, of C_k of the C_k version of the bundle $\mathcal{V}_0$ in (4.16a) of [28]. Meanwhile, $\chi_{b,k}$ is a bump function which is defined for $0 \le b \le \delta_0/2$. It has support in the radius b tubular neighborhood of C_k and it is given in terms of a standard bump function on $[0, \infty)$ by $\chi_{b,k} = \chi(|s|/b)$. (The function χ takes values in $[0, 1]$; it is non-increasing, it vanishes on $[2, \infty)$ and it is 1 on $[0, 1]$.) In (6.53), the constant δ is chosen once and for all, with positive value less than $10^{-3} \cdot \delta_0$.

The interpretation of (6.53) requires an identification of $S_{+,0}$ with the given $\mathrm{Spin}^{\mathbb{C}}$ bundle S_+ on the support of $\Pi_k(1 - \chi_{4\delta,k})$. This identification is made via (1.9) by using the section $\underline{\alpha}_r$ of E to trivialize E where $\underline{\alpha}_r \neq 0$. The interpretation of (6.53) also requires an identification of the C_k version of $\mathcal{V}_0$ with $i \cdot T^*X \oplus S_+$ on the support of $\chi_{100\delta,k}$. The latter identification is explained in Lemma 4.4 of [28].

With the preceding understood, consider:

Lemma 6.7. *There exists a constant $\zeta \ge 1$ with the following significance: Fix $\varepsilon > 0$, but less than ζ^{-1} and a positive integer d. Take the index j large. For each k, fix a point $y'^k \in \mathcal{K}_\Lambda^{(k)}$ with L^2-distance $\zeta^{-1} \cdot \varepsilon$ or less from y^k. Use $\{y'^k\}$ to define the data $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ for $r = r_j$ as specified in Sections 2 and 3b of [28]. Define $q' = (a', (\alpha', \beta'))$ as in Lemma 6.6 from $(a_j, (\alpha_j, \beta_j))$. Then, there exists data $(q^0, \{q^k\})$ as described above such that (6.53) holds, and such that (5.3) of [28] holds as well. Furthermore,*

- $\Sigma_{0 \le d' \le d} \, |\nabla^{\otimes d'} q^0| \le e^{-\sqrt{r}/\zeta}$.
- *For each k,* $\Sigma_{0 \le d' \le d} \, r^{-d'/2} |\nabla^{\otimes d'} q^k| \le \varepsilon e^{-\sqrt{r}|s|/\zeta}$.

(6.54)

(This last line holds at each point in the normal bundle N of C_k.)

The remainder of this subsection is occupied with the

Proof of Lemma 6.7. The proof is accomplished in five steps.

Step 1. The data $(q^0, \{q^k\})$ are defined from q using some auxiliary data, $\{f^k\}$. Here, f^k is a section over the normal bundle of the $C = C_k$ and trivial E version of the bundle $\mathcal{V}_1$ from (4.16b) in [28]. The data $\{f^k\}$ is specified below. However, given this data, here is $(q^0, \{q^k\})$:

- $q^0 = \Pi_k(1 - \chi_{25\delta,k})q + \Sigma_k \chi_{100\delta,k} f^k$.
- $q^k = \chi_{25\delta,k} q - (1 - \chi_{4\delta,k}) f^k$.

(6.55)

Note that (6.53) holds, as required. When interpreting the first line in (6.55), use Lemma 4.4 in [28] to interpret $\chi_{100\delta,k} \cdot f^k$ as a section of $i \cdot T^*X \oplus S_{+,0}$, where $S_{+,0}$ is the plus spin bundle for the canonical Spin$^{\mathbb{C}}$ structure as given in (1.7). In this regard, remember that q^0 should be interpreted as a section of $i \cdot T^*X \oplus S_{+,0}$ also. The latter interpretation requires the identification of the given S_+ as in (1.9) and it requires the identification using $\underline{\alpha}_r$ of E with the trivial line bundle where $\underline{\alpha}_r \neq 0$. The interpretation of the second line in (6.55) also uses Lemma 4.4 in [28] and the identification via $\underline{\alpha}_r$ of E with the trivial bundle where $\underline{\alpha}_r \neq 0$.

Step 2. The data $\{f^k\}$ is constrained by the requirement that (5.3) of [28] holds. This constraint is implied by the requirement that f^k obey

$$L_k f^k + \sqrt{r}(1 - \chi_{2\delta,k})\chi_{200\delta,k}[\varpi(q', f^k) + \varpi(q^k, q^0)] - \wp(d\chi_{25\delta,k})q' = 0. \tag{6.56}$$

on the normal bundle N of C_k. Here, L_k is as defined in Section 4 of [28] but with the trivial vortex $(v = 0, \tau = 1)$ replacing $c = \Upsilon(y'^k)$. This is to say that L_k is defined using the $m_k = 0$ version of (2.15) in [28]. Note that when viewing (6.56) as an equation to determine f^k from q', one should look at q^0 and q^k as functionals of q' and f^k through (6.55). In this way, (6.56) reads:

$$L_k f^k - \sqrt{r}(1 - \chi_{4\delta,k})\chi_{100\delta,k}\varpi(f^k, f^k) + \sqrt{r}(1 - \chi_{25\delta,k})\chi_{25\delta,k}\varpi(q', q') - \wp(d\chi_{25\delta,k})q' = 0. \tag{6.56}$$

It is left as an exercise for the reader to verify that (6.56) (or, equivalently, (6.57)) insures that $(q^0, \{q^k\})$ solves (5.3) in [28].

Step 3. This step verifies that (6.57) has a unique, small solution for large j. In this regard, the key point is that both L_k and its formal L^2-adjoint are robustly invertible. Indeed, using the analysis in Sections 4d-e of [28], one finds that for large r,

- $||L_k b||_2 \geq \zeta^{-1}(||\nabla b||_2 + \sqrt{r} \cdot ||b||_2)$.
- $||L_k^\dagger b||_2 \geq \zeta^{-1}(||\nabla b||_2 + \sqrt{r} \cdot ||b||_2)$.

(6.58)

In both lines, $\zeta \geq 1$ is independent of b and r. (The top line holds for all smooth sections b of $\mathcal{V}_0$ with compact support, and the second line holds for all smooth, compactly supported sections of the bundle $\mathcal{V}_1$ from (4.16) of [28].) With regard to applying the analysis in Sections 4d-e of [28], note that neither K_0 nor K_1 arises in this case because the vortex which defines L_k has vortex number zero.

With the preceding understood, one can mimick the discussion in Section 4e (specificaly Lemma 4.8) of [28] to prove that L_k has a bounded inverse (called P in Lemma 4.8 of [28]) which maps $L^2(\mathcal{V}_1)$ to $L^2_1(\mathcal{V}_0)$. Furthermore, this inverse obeys the apriori bound

$$||\nabla P(h)||_2 + \sqrt{r}||P(h)||_2 \leq \zeta \cdot ||h||_2.$$

Here, ζ is independent of r and h.

Step 4. Write (6.57) as the condition for a fixed point of the map T on $L^2(\mathcal{V}_1)$ to itself which sends h to

$$L_k^{-1} \cdot (\sqrt{r}(1 - \chi_{4\delta,k})\chi_{100\delta,k}\varpi(h, h) + \Phi(q')).$$

The existence of a constant $\zeta \geq 1$ and a unique fixed point, h_0, of T with L^2 norm bounded by $\zeta^{-1}r^{-1/2}$ follows from standard dimension 4 Sobolev inequalities when Lemma 6.6 is invoked to estimate the size of the q' dependent terms in (6.57). Indeed, with the help of Lemma 6.6, one finds that $||h_0||_2 \leq \zeta e^{-\sqrt{r}/\zeta}$ when j is large. Here, ζ is independent of j.

Step 5. Pointwise estimates for $f^k = P(h_0)$ can be obtained by employing the L^2_1 estimate for f^k from the previous step with standard elliptic "bootstrapping" arguments. The result is a bound on the derivatives of f^k to order d by $\zeta e^{-\sqrt{r}/\zeta}$. (The bootstrapping arguments are of the sort given in Chapter 6 of [17].)

h) The proof of Proposition 5.2

It has now been established that q' is given by (6.53) with the data $(q^0, \{q^k\})$ as described in Lemma 6.7. In particular, (5.3) of [28] is satisfied. To complete the proof of Proposition 5.2, it remains still to verify that $\{y'^k\} \in \times_k \mathcal{K}_\Lambda^{(k)}$ can be found for which Lemma 6.7's data $(q^0, \{q^k\})$ is constructed as described in Section 5 of [28]. This verification requires six steps.

Step 1. Fix $\varepsilon > 0$, but very small, and a positive integer d. For each k, fix a point $y'^k \in \mathcal{K}_\Lambda^{(k)}$ with distance $\zeta^{-1} \cdot \varepsilon$ or less from y^k. When the index j is very large, use $\{y^k\}$ to define the data $(q^0, \{q^k\})$ as described in Lemma 6.7.

Step 2. This step verifies that when the index j is large, then the resulting q^0 is described by Lemma 5.3 in [28]. The argument here is straightforward because the first line of (5.3) is satisfied and, for large index j, the assumptions in Lemma 5.3 of [28] concerning the L^2 norm of the data $\{q^k\}$ is met. Since Lemma 6.7's q^0 is small when the index j is large, the uniqueness assertion of Lemma 5.3 in [28] implies that the q^0 from Lemma 6.7 must come via Lemma 5.3 in [28]. This is to say that Lemma 5.3 in [28] finds a solution to the first line of (5.3) in [28] as a function of extra data. The extra data in (5.3) of [28] is q^k. With this understood, the point is that q^0 from Lemma 6.7 is given by Lemma 5.3 in [28] when the extra data for the latter is $\{q^k\}$ from Lemma 6.7.

Step 3. This step begins the process of verifying that each q^k from Lemma 6.7 is also described by Section 5 of [28] when j is large. For this purpose, fix $y' \in \times_k \mathcal{K}_\Lambda^{(k)}$ near to y as before. Then, fix k. When x^k is a section of the (C_k, m_k) version of $\oplus_{1 \leq q \leq m} N^q$, write

$$\Upsilon(y'^k + x^k) = \Upsilon(y'^k) + ((2 \cdot \sqrt{2})^{-1}(b - \bar{b}, \lambda)$$

and let

$$t^k(x) = (r^{-1/2} \rho_r^* b, \rho_r^* \lambda, 0, 0)$$

as a section of $\mathcal{V}_0$. (The latter is defined in (4.16a) of [28].)

According to Lemma 5.4 of [28] and Lemma 6.7 here, each q^k has a unique decomposition as

$$q^k = P(h^k) + t^k(x^k), \tag{6.59}$$

where x^k is a section of $\oplus_{1\le q\le m} N^q$, h^k is in the $c = \Upsilon(y'^k)$ version of $L^2(\mathcal{V}_1; K_1)$, and

- $||h^k||_2 \le \zeta\varepsilon$.
- $\sup_C |x^k| + ||x^k||_2 + r^{-1/2}||\nabla x||_2 \le \zeta\varepsilon$.
- h^k obeys (5.12) in [28].
- x^k obeys (5.20) in [28] with $\mathcal{R}^k$ as in (5.25) and (5.26) of [28].

(6.60)

(The last two points hold because (2.4) is obeyed.)

The estimates above for the norms of h^k and x^k can be further refined:

Lemma 6.8. *There is a constant $\zeta \ge 1$ with the following significance: Fix $\varepsilon > 0$ but less than ζ^{-1}. Then fix $\{y'^k\} \in \times_k \mathcal{K}_\Lambda^{(k)}$ with L^2 distance ε or less from $\{y^k\}$. When j is large, use $\{y'^k\}$ to define q' from $(a_j, (\alpha_j, \beta_j))$ and introduce q^0 and $\{q^k\}$ as in Lemma 6.7. In addition, for each k, introduce h_k and x_k as in (6.59) and (6.60). Then, for each k,*

- $||h^k||_2 \le \zeta\varepsilon r^{-1/2}$,
- $||\nabla x||_2 \le \zeta\varepsilon$,
- $\sup_C |x^k| + ||x^k||_2 \le \zeta\varepsilon$.

(6.61)

Step 4. This step consists of the

Proof of Lemma 6.8. To start, rewrite (5.12) of [28] as $h^k = -\mathbb{Y}^k$ and then square both sides of this equation. Straightforward manipulations with (6.60) find that

$$||h^k||_2 \le \zeta\varepsilon r^{-1/2} + \zeta\varepsilon r^{-1/2}||\nabla x^k||_2 + \zeta\varepsilon||P(h^k)||_2. \tag{6.62}$$

With regard to (6.62), note that the terms with $\sqrt{r}\cdot\varpi(\cdot, P(h^k))$ enter into the third term on the right side of (6.62). The extra factor of ε appears here because of the bound in Lemma 6.7 of q^k by $\zeta\varepsilon e^{-\sqrt{r}|s|/\zeta}$. The second term on the right side of (6.62) comes from the term in

(5.12) of [28] that contains $(1-\Pi)\cdot T(t^k)$. (The operator T is defined in (4.45) of [28].) The point is that the leading order in x^k part of t^k contributes to $||(1-\Pi)\cdot T(t^k)||_2^2$ an amount which is no greater than $\zeta r^{-1/2}||x^k||_2 \leq \zeta r^{-1/2}\varepsilon$. The higher order in x^k parts of t^k contribute to $||(1-\Pi)\cdot T(t^k)||_2$ as in the second term on the right side of (6.62). To see that the leading order in x^k part of t^k contributes as stated, note, on the one hand, that the differential of t^k at zero maps x^k to a section of $\mathcal{V}_0$ which comes from a section of the vector bundle $K_0 \to C$. (This bundle is defined in Lemma 4.5 of [28].) On the other hand, the symbol of T maps a section of $\mathcal{V}_0$ which comes from a section of K_0 to a section of $\mathcal{V}_1$ having the form $\underline{w}'$ where w' is a section of the bundle $K_1 \to C$. This last property of the symbol of T implies that $(1-\Pi)\cdot T$ acts as a zero'th order operator on sections of $\mathcal{V}_0$ which come from sections of K_0.

In any event, Lemma 4.8 in [28] and (6.62) imply that when ε is small and j is large, then

$$||h^k||_2 \leq \zeta\varepsilon r^{-1/2} + \zeta\varepsilon r^{-1/2}||\nabla x^k||_2. \tag{6.63}$$

The next step in the proof of Lemma 6.8 uses (5.20), (5.25) and (5.26) in [28] to bound the L^2 norm of the derivative of x^k. In this regard, introduce $\Delta_{y'}$ to denote the operator

$$\bar{\partial} + \nu\aleph + \mu\mathbb{F}_{*y'^k} : C^\infty(\oplus_{1\leq q\leq m}N^q) \to C^\infty((\oplus_{1\leq q\leq m}N^q)\otimes T^{0,1}C)$$

and let $\Lambda = \Lambda_k$. Introduce the L^2-orthogonal projection Q_Λ on

$$C^\infty((\oplus_{1\leq q\leq m}N^q)\otimes T^{0,1}C)$$

onto Λ. Because $\Delta_{y'}$ is Fredholm and Q_Λ is finite rank,

$$\begin{aligned} ||\nabla x^k||_2 \leq& \zeta||(1-Q_\Lambda)\cdot\Delta_{y'}x^k||_2 + \zeta||x^k||_2 \\ \leq& \zeta||(1-Q_\Lambda)\cdot\Delta_{y'}x^k||_2 + \zeta\varepsilon. \end{aligned} \tag{6.64}$$

Meanwhile, the L^2 norm on the far right side of (6.64) is given, courtesy of (5.20) in [28], as

$$\zeta||(1-Q_\Lambda)\cdot\mathcal{R}^k||_2. \tag{6.65}$$

A bound for (6.65) can be obtained using (5.25) and (5.26) in [28]. In particular, the term $g_0 = \mathfrak{f}^k(x^k)\cdot\nabla x^k$ and $g_1,\ldots,g_4$ in (5.25) of [28] contribute, respectively, no more than the following to (6.65):

- $\zeta(\varepsilon + r^{-1/2})||\nabla x^k||_2$.
- $\zeta||x^k||_2 \leq \zeta\varepsilon$.
- $\zeta\varepsilon r||p(h^k)||_2 \leq \zeta\varepsilon\sqrt{r}||h^k||_2 \leq \zeta\varepsilon^2(1 + ||\nabla x^k||_2)$.
- $\zeta\varepsilon r||p(h^k)||_2 \leq \zeta\varepsilon^2(1 + ||\nabla x^k||_2)$.
- $\zeta\varepsilon(1 + \sqrt{r}||(1 - Q_\Lambda)L_k P(h^k)||_2) \leq \zeta\varepsilon(1 + ||h^k||_2) \leq \zeta\varepsilon$.

(6.66)

Here are some comments with respect to (6.66): First, remember that the L^2 norm over C of a section w of $(\oplus_{1\leq q\leq m} N^q) \otimes T^{0,1}C$ determines the L^2 norm over N of the corresponding section $\underline{\Upsilon_1(w)}$ of $\mathcal{V}_1$ (and vice-versa) by the rule

$$\zeta^{-1}\sqrt{r}||\underline{w}||_2 \leq ||w||_2 \leq \zeta\sqrt{r}||\underline{w}||_2. \tag{6.67}$$

With the preceding understood, the bounds in (6.66) come about as follows: The bound for g_0 follows from the bound for $|x^k|$ by $\zeta \cdot \varepsilon$ and the first point in (5.26) of [28]. The bound for g_1 from the pointwise bound of $|t^k|$ and $|s| \cdot |\nabla^V t^k|$ on N by $\zeta\varepsilon e^{-\sqrt{r}|z|/\zeta}$. The bounds for g_2 and g_3 use the preceding bound for t^k and the bound of $|P(h^k)|$ by $\zeta \cdot \varepsilon$ which comes from the $|q^k|$ bound in Lemma 6.7. (Use (6.63) also for the g_2 and g_3 bounds.) The bound for g_4 comes from the second line in (4.31) of [28] which allows a bound of $||\Pi \cdot L_k P(h^k)||_2$ by $\zeta r^{-1/2}||h^k||_2$. The contributions from the remaining terms in g_4 can be analyzed in a completely straightforward manner using Lemma 6.7.

Note that the second assertion of Lemma 6.8 follows when ε is small and j is large directly from (6.64)–(6.66). Then, the first assertion of Lemma 6.8 follows from the second with (6.63). The third assertion of Lemma 6.8 reiterates part of (6.60).

Step 5. Given Lemma 6.8, when the index j is large, then $\{h^k\}$ from Lemma 6.8 is determined by the data $\{x^k\}$ from Lemma 6.8 as described by Lemma 5.5 of [28]. Indeed,the second and third lines of (6.61) insure that $\{x^k\}$ from Lemma 6.8 is suitable data for Lemma 5.5 of [28]. And, the uniqueness assertion in this last lemma plus the first line of (6.61) insure the claim.

Step 6. Here is the situation: Take ε small (there is j-independent upper bound) and j large. In addition, take $\{y'^k\} \in \times_k \mathcal{K}_\Lambda^{(k)}$ within $\zeta^{-1} \cdot \varepsilon$

of $\{y^k\}$ in the L^2 norm. Then q^k can be written as in (6.59) where (6.60) and (6.61) hold, and $\{h^k\}$ is determined from the data $\{x^k\}$ by Lemma 5.5.

The claim now is that when j is large, then $\{y'^k\}$ can be chosen near to $\{y^k\}$ (as above) so that the corresponding $\{x^k\}$ is described by Lemma 5.6 of [28]. If such $\{y'^k\}$ can be found, then Proposition 5.2 has been proved.

To characterize such $\{y'^k\}$, note that because of the first two lines in (6.61), and the uniqueness assertion in Lemma 5.6 of [28], the condition on $\{y'^k\}$ is simply that each x^k be L^2-orthogonal to the tangent space to $\mathcal{K}_\Lambda^{(k)}$ at y'^k. Thus, the goal is to find $\{y'^k\}$ so that each of the corresponding x^k is L^2-orthogonal to the tangent space at y'^k to $\mathcal{K}_\Lambda^{(k)}$. With the preceding understood, remark that the existence of such $\{y'^k\}$ follows from the implicit function theorem and the fact that the directional derivative of $x^k(\cdot)$ in the direction of some tangent vector w^k to $\times_k \mathcal{K}_\Lambda^{(k)}$ at y'^k obeys

- $||x_*^k \cdot w^k||_2 \leq \zeta\varepsilon ||w^k||_2$ when $k \neq k'$.
- $||x_*^k \cdot w^k - w^k||_2 \leq \zeta\varepsilon ||w^k||_2$.

(6.68)

The proof of (6.68) is a straightforward, but tedious exercise which will be omitted except for the following comments: A change in $\{y'^k\}$ changes $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ and thus q' in Lemma 6.7. Unwind the definitions to see that the resulting change in x^k is the $\mathcal{O}(w^k)$ contribution to the second line in (6.68). However, changing $\{y'^k\}$ also changes both $t^k(\cdot)$ and the splitting in (6.59). These other effects of changing $\{y'^k\}$ produce effects which are $\mathcal{O}(\varepsilon \cdot ||w^k||_2)$. (The justification of these remarks refers to the definition of $(\underline{a}_r, (\underline{\alpha}_r, \underline{\beta}_r))$ in Sections 2 and 3b of [28], to arguments in the proof of Lemma 5.4 of [28], and to those in the proof of Lemma 6.6, here.)

7. Proof of Proposition 2.13

The purpose of this section is to prove Propositions 2.1 and 2.13. The proof for Proposition 2.1 is given in the first subsection. The second subsection proves, as a warm up, the $d(e) = 0$ case of Proposition 2.11 (a special case of Proposition 2.13.) The remaining subsections are devoted to the proof of Proposition 2.13. In this regard, the reader should review the discussion in Part 5 of Section 2g which outlines the proof. In particular, said discussion reduces the proof to a question of verifying the points in (2.29). Thus, the discussion below starts in Subsection 7c where (2.29) leaves off.

By the way, for both Propositions 2.11 and 2.13, the key to the proof are Lemmas 4.11 and 6.7 of [28] which translate statements about operators on X to statements about operators on pseudo-holomorphic submanifolds of X.

a) Proof of Proposition 2.1

As discussed in Section 2, there is a unique element in the $e = 0$, $\mu_0 = 0$ and large r version of $\mathcal{M}^{(r)}$ so it remains only to verify that this element should be counted with sign $+1$. As this unique element is $(a = 0, (\alpha = 1, \beta = 0))$, Lemma 4.3 in [28] can be used to analyze the operator L for this solution. In particular, L has trivial kernel and cokernel, and $||Lu||_2 \geq \zeta^{-1} r ||u||_2$ for any $u \in i \cdot \Omega^1 \oplus C^\infty(X)$. Meanwhile, the $\mathbb{C}$-linear operator in (4.2) differs from L by a zero'th order multiplication operator which has an r-independent bound on its norm. These last two facts imply that for large r, there is no $t \in [0, 1]$ where $L + t \cdot n^1$ has non-trivial kernel. Thus, the sign in question is $+1$.

b) Proof of Proposition 2.11 when $d(e) = 0$

To set the stage, fix a class $e \in H^2(X; \mathbb{Z})$ and also a triple (J, Γ, Ω) as instructed in Proposition 2.10. For each

$$h = \{(C_k, m_k)\}_{1 \leq k \leq n} \in \mathcal{H},$$

introduce the data $\{\mathcal{K}_\Lambda^{(k)}\}$ as in Proposition 2.10 and the subspace $Y_h \subset \times_k \mathcal{K}_\Lambda^{(k)}$. When r is large, Proposition 2.10 refers to a certain embedding,

$$\Psi_r = \Psi_{r,h} : \times_k \mathcal{K}_\Lambda^{(k)} \to (\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1)$$

as in (2.20) whose image contains an open subset of the $\mu_0 = 0$ version of $\mathcal{M}^{(r)}$.

The simplest case to consider has $d = d(e) = 0$ and also makes the assumption in Proposition 2.9 that all $\Lambda_k = \{0\}$ for each $h \in \mathcal{H}$.

Here is the outline for the argument in this case: Given such $h = \{(C_k, m_k)\}$, let e_k denote the Poincaré dual to C_k. Then each $d(e_k) = 0$ and $Y_h = \times_k \mathcal{K}_\Lambda^{(k)}$ in Proposition 2.10 is a finite set of points as each $\mathcal{K}_\Lambda^{(k)}$ is a finite set of points in the (C_k, m_k) version of $\mathcal{Z}$. For large r, the map $\cup_{h\in\mathcal{H}}\Psi_{h,r}$ identifies $\cup_{h\in\mathcal{H}}\Psi_h$ with the $\mu_0 = 0$ version of $\mathcal{M}^{(r)}$.

With the preceding understood, take $h \in \mathcal{H}$ and a point $y \in Y_h$. Then take a reasonable path of Fredholm operators that interpolates between $L_{\Psi_r(y)}$ and a $\mathbb{C}$-linear operator. Lemma 6.7 in [28] will construct a corresponding path for $\oplus_k \Delta_{y_k}$. The latter path will be such that its spectral flow is the same as that of the former. Since the mod(2) spectral flow for $\oplus_k \Delta_{y_k}$ is independent of the precise path and the $\mathbb{C}$-linear endpoint, the just mentioned equality of spectral flows implies the $d(e) = 0$ case of Proposition 2.11.

The details of this strategy are carried out in the five steps that follow.

Step 1. This step does not require all $\Lambda_k = \{0\}$. The following lemma summarizes the contents of this first step. The statement of the lemma introduces almost complex structures J_D on $i \cdot T^*X \oplus S_+$, and J_R on $i \cdot (\varepsilon_{\mathbb{R}} \oplus L_+) \oplus S_-$. The former acts as the given almost complex structure on the $i \cdot T^*X$ summand, and it acts as multiplication by $\sqrt{-1}$ on the complex vector bundle S_+. Meanwhile, the action of J_R preserves the $i \cdot (\varepsilon_{\mathbb{R}} \oplus L_+)$ summand and acts there as described in Step 3 of Section 1c. And, J_R also preserves the S_- summand where it acts as multiplication by $\sqrt{-1}$.

Lemma 7.1. *The conclusions of Proposition 2.10 can be augmented with the following: Fix $h = \{(C_k, m_k)\} \in \mathcal{H}$. There exists a constant $\zeta \geq 1$ which depends on $\{\mathcal{K}_\Lambda^{(k)}\}$ and has the following significance: Suppose that $r \geq \zeta$ and also that*

$$y = (y^1, \dots, y^k) \in \psi_r^{-1}(0) \subset \times_k \mathcal{K}_\Lambda^{(k)}.$$

Then, there exists a smooth, one parameter family of operators

$$\{n_t : L^2(i \cdot T^* \oplus S_+) \to L^2(i \cdot (\varepsilon_{\mathbb{R}} \oplus \Lambda_+) \oplus S_-)\}_{t\in[0,1]}$$

which obeys:

- $n_0 = 0$.
- $L_{\Psi,(y)} + n_1$ *is $\mathbb{C}$-linear in that it intertwines J_D with J_R.*
- $\sup_{p\neq 0} ||n_t(p)||_2 \cdot ||p||_2^{-1} \leq \zeta$.

Proof of Lemma 7.1. To find such a constant ζ, note that n_t can be taken to be an arbitrarily small perturbation of the operator $t \cdot 2^{-1}(L + J_R \cdot L \cdot J_D)$ in the case where $L = L_{\Psi_r(y)}$. In this case, $L^{\mathbb{C}} = 2^{-1}(L - J_R \cdot L \cdot J_D)$. Thus, the size of n_t is determined by the failure of $L_{\Psi_r(y)}$ to intertwine J_D with J_R. The r-dependence in this failure is due to the appearance of $\underline{\beta}_r + \beta'$ in the definition of Ψ_r. That is, the norm of the r- dependent term is bounded by $\zeta \cdot \sqrt{r} \cdot (|\underline{\beta}_r| + |\beta'|)$. Since both $\underline{\beta}_r$ and β' are bounded in norm by $\zeta \cdot r^{-1/2}$, this r-dependent term is uniformly bounded, with an r-independent bound.

Step 2. There is one additional constraint to impose on the family $\{n_t\}$. To state this constraint and simplify notation later on, introduce for each k the symbol $N^{(k)}$ to denote the $C = C_k$ and $m = m_k$ version of the vector bundle $\oplus_{1 \le q \le m} N^q$. Now, remark that for each k, the operator n_1 induces an operator $n_1^k : \oplus_{k'} L^2(N^{(k')}) \to L^2(N^{(k)})$ by first sending $v = (v_1, \dots, v_k)$ to $p(v) = \Sigma_k \chi_{\delta,k} \underline{\Upsilon}_1 \underline{v}^k \in C^\infty(\mathcal{V}_0)$ and then writing

$$n_1^k(v) = \Upsilon_1^{-1} \cdot x(\chi_{\delta,k}\varpi'(q', p(v)) + \chi_{\delta,k} n_1(p(v))). \tag{7.1}$$

Here, $x(g)$ is the section of $V^c \otimes T^{0,1}C_k$ for which the corresponding $\underline{x}$ equals $\Pi \cdot g$; while $c = \Upsilon(y^k)$. (In writing $n_1(p(v))$, the element $\chi_{\delta,k}\underline{\Upsilon}_1\underline{v}^k$ in the (C_k, m_k) version of $\mathcal{V}_0$ from (4.16a) of [28] has been identified, as directed in Lemma 4.4 of [28], with an element in $i \cdot \Omega^1 \oplus C^\infty(S_+)$. Likewise, $\chi_{\delta,k} n(p(v))$ has been identified as an element in $\mathcal{V}_1$ from (4.16b) of [28].)

Lemma 7.2. *Given $\varepsilon > 0$, then for all r sufficiently large, the constant ζ and the path $\{n_t\}$ in Lemma 7.1 can be chosen so that the following additional conclusion holds: For each k, there is a bounded operator n_{y^k} which maps $L^2(N^{(k)})$ to $L^2(N^{(k)} \otimes T^{0,1}C_k)$ and obeys:*

- $\Delta_{y^k} + n_{y^k}$ *is $\mathbb{C}$-linear.*
- $||(\Delta_{y^k} + n_{y^k}) \cdot v_k||_2 \ge \zeta^{-1} \cdot ||v_k||_{1,2}$ *for all v_k in $L^2_1(N^{(k)})$.*
- $\sup_{p \neq 0} ||n_{y^k}(p)||_2 / ||p||_2 \le \zeta$.
- *When $v = (v^1, \dots, v^n) \in \oplus_{k'} L^2(N^{(k')})$, then $||n_1^k(v) - n_{y^k} v^k||_2 \le \varepsilon \Sigma_{k'} ||v^{k'}||_2$.*

Step 3. This step and Step 4 contain the:

Proof of Lemma 7.2. In proving the lemma, remark that it is sufficient to establish that there exists ζ, an operator

$$n_1 : L^2(i \cdot T^* \oplus S_+) \to L^2(i \cdot (\varepsilon_{\mathbb{R}} \oplus \Lambda_+) \oplus S_-)$$

and operators $\{n_{y^k}\}$ which obey the assertions of Lemma 7.2, the second assertion of Lemma 7.1 and the $t = 1$ version of the third assertion of Lemma 7.1. (Given n_1, then $\{n_t\}$ can be obtained as a small perturbation of $t \cdot n_1$.)

To find ζ, n_1, and $\{n_{y^k}\}$ start with the observation that because each $\mathcal{K}_\Lambda^{(k)}$ has compact closure in $C^\infty(N^{(k)})$, there exists a constant $\zeta_0 > 1$ with the following significance: For each $y^k \in \mathcal{K}_\Lambda^{(k)}$, there exists

$$n_{y^k} : L^2(N^{(k)}) \to L^2(N^{(k)} \otimes T^{0,1}C_k)$$

so that the first three assertions of Lemma 7.2 hold using ζ_0 instead of ζ. (Note that it may not be possible to choose n_{y^k} to vary continuously with $y^k \in \mathcal{K}_\Lambda^{(k)}$.)

Now, let n_1 be as in Lemma 7.1. The next part of the argument uses the set $\{n_{y^k}\}$ to modify n_1 so that the result satisfies the additional fourth assertion of Lemma 7.2. The arguments here require a preliminary digression.

The digression starts by defining, for each k, a map, I_+^k, from $\Omega^1_{\mathbb{C}} \oplus C^\infty(S_+)$ to $C^\infty(N^{(k)})$ by associating to v the element $\Upsilon_1^{-1} \cdot x(\chi_{\delta,k} v)$, where

$$x : C^\infty(N; \mathcal{V}_0) \to C^\infty(V^c)$$

is defined by requiring that $\underline{x}(v) = \Pi \cdot v$. (This uses the identification near C_k, described in Lemma 4.4 of [28], between $\mathcal{V}_0$ and $i \cdot T^* \oplus S_+$.)

The map I_+^k intertwines the action of J_D with multiplication by $\sqrt{-1}$. Here is why: First, Υ_1 is a $\mathbb{C}$-linear map from $N^{(k)}$ to V^c. (Here $c = \Upsilon(y^k)$.) Second, the rescaling map from V^c to K_0 is also $\mathbb{C}$-linear. (See Lemma 4.5 of [28].) Meanwhile, as in Sections 4d and 4e of [28], a section (a_V, α') of K_0 over C_k defines $\chi_{\delta,k}(a_V \bar{\kappa}_1 - \bar{a}_V \kappa_1, (\alpha', 0))$ in $i \cdot \Omega^1 \oplus C^\infty(S_+)$. (The 1-form κ_1 is described in Part 4 of Section 2a of [28]. The identification of a section of K_0 with a pair of 1-form and spinor comes via the identification in Lemma 4.4 of [28] between $i \cdot T^{0,1}N_0 \oplus S_+$ and the bundle $\mathcal{V}_0$ from (4.16a) of [28].)

Next, define a map,

$$I_-^k : C^\infty(N^{(k)} \otimes T^{0,1}C_k) \to i \cdot (\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_-)$$

as follows: Use Υ_1 and then rescale to first map $N^{(k)}$ to K_1. Then, associate to any section (b, λ) of the bundle $K_1 \to C_k$ the element

$$\chi_{\delta,k} \cdot (b \cdot \bar{\kappa}_1 \wedge \bar{\kappa}_0 - \bar{b}\kappa_1 \wedge \kappa_0, \lambda\bar{\kappa}_0) \in i \cdot (\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_-).$$

(Here, κ_0 is described in Part 4 of Section 2a, and the identification of a section of K_1 with a pair of self-dual form and spinor comes via the identification in Lemma 4.4 of [28] between the bundles $i \cdot (\varepsilon_{\mathbb{R}} \oplus \Lambda_+) \oplus S_-$ and $\mathcal{V}_1$ from (4.16b) of [28].) Note that I_-^k intertwines multiplication by $\sqrt{-1}$ with J_D.

End the digression and consider $n_1' = n_1 + \Sigma_k I_-^k \cdot (n_{y^k} - n_1^k) \cdot I_+^k$. Here are its key properties: There exists ζ and, given $\varepsilon > 0$, then for sufficiently large r and $y \in \psi_r^{-1}(0)$,

- $\sup_{p \neq 0} ||n_1' \cdot p||_2 \, ||p||_2^{-1} \leq \zeta$,
- $||n_1'(\underline{\Upsilon_1 u^k}) - I_-^k(n_{y^k} u^k)||_2 \leq \zeta e^{-\sqrt{r}/\zeta} ||u^k||_2$,
- $||(L_{\Psi_r(y)} + n_1')J_D \cdot p - J_R \cdot (L_{\Psi_r(y)} + n_1')p||_2 \leq \varepsilon ||p||_2$ for all $p \in i \cdot \Omega^1 \oplus C^\infty(S_+)$.

(7.2)

These last assertions will be proved momentarily. To complete the proof of Lemma 7.2, remark that because of the third assertion in (7.2), there is an operator which can be added to n_1' whose L^2-operator norm is bounded by ε; and is such that the result, n_1'', when subtracted from $L_{\Psi_r(y)}$ gives a $\mathbb{C}$-linear operator. Furthermore, the latter will obey the last assertion of Lemma 7.2 because of the second assertion of (7.2).

Step 4. To prove (7.2) remark that the first line in (7.2) follows from the fact that n_1 and each n_{y^k} are uniformly bounded in the L^2-operator norm. The second line follows directly from the definition n_1'. (Here, use the last line of (5.27) in [28].)

Here is the argument for the third line of (7.2): Decompose

$$p = \Pi_k(1 - \chi_{4\delta,k}) \cdot p^0 + \Sigma_k \chi_{100\delta,k} p^k$$

by analogy with (4.6) in [28]. Since $n_1' = n_1$ on $\Pi_k(1 - \chi_{4\delta,k}) \cdot p^0$, it is enough to consider n_1' on some $p = \chi_{100\delta,k} p^k$. With this understood, write $p^k = P(h) + \underline{w}$ as in (4.45) of [28]. Then,

$$(7.3) \quad ||n_1'(\chi_{100\delta,k} P(h)) - n_1(\chi_{100\delta,k} P(h))||_2 \leq \zeta e^{-\sqrt{r}/\zeta} ||\chi_{100\delta,k} P(h)||_2,$$

because $P(h)$ is L^2 orthogonal to all sections of $\mathcal{V}_0$ which come from sections of V^c. Use the last line of (5.27) in [28] to help derive (7.3). Thus, the argument is reduced to the case where $p = \chi_{100\delta,k} \cdot \underline{w}$.

To consider the case where $p = \chi_{100\delta,k}\underline{w}$, note first that given $\varepsilon > 0$, there is an operator

$$\phi = \phi^k : L_1^2(N^{(k')}) \to L^2(N^{(k)} \otimes T^{0,1}C_k)$$

such that:

- $\oplus_k(\Delta_{y^k} + n_1^k + \phi)$ is $\mathbb{C}$-linear,
- $||\phi(v^k)||_2 \le \varepsilon ||v^k||_{1,2}$.

(7.4)

Indeed, this follows by arguing as in the proofs of Lemma 6.7 in [28] using

- the fact that both $I_\pm^k$ are $\mathbb{C}$-linear;
- the last line in (5.27) in [28];
- the fact that $L_{\Psi_r(y)} + n_1$ is $\mathbb{C}$-linear;
- the first line of (4.47) in [28].
- Given $\varepsilon > 0$, Assertion 4 of Proposition 5.2 in [28] finds ζ such that when $r > \zeta$, then each $y = \{y^k\} \in \psi_r^{-1}(0)$ has each y^k at distance ε or less from $\mathcal{Z}_0$.

(7.5)

(The fifth point in (7.5) is used to control the contribution of the term v_y in (6.29a) of [28].)

Note that ϕ in (7.4) comes from the Rem term in (4.19) of [28] (see Lemma 4.4 of [28]) and v_y in (6.29a). A closer inspection of this term (following the lines of the proof of Lemma 4.4 of [28]) shows that when r is large, then this ϕ differs from a $\mathbb{C}$-linear operator by a term, ϕ', which obeys

$$||\phi'(v)||_2 \le \varepsilon \cdot ||v||_2. \tag{7.6}$$

With regard to (7.6), remember that the symbol of the operator L is $\mathbb{C}$ linear, as are the maps $I_\pm$. Also, use the fifth point in (7.5) to control the contribution of v_y.

It follows from (7.6) and the first line of (7.4) that $n_1^k - \oplus_k n_{y^k}$ differs from a $\mathbb{C}$-linear operator by a term whose L^2-operator norm is bounded by ε when r is large. This last point implies that $n_1' - n_1$ differs on $p = \chi_{100\delta,k} \cdot \underline{w}$ by a $\mathbb{C}$-linear operator plus an operator which might not be $\mathbb{C}$-linear, but whose L^2-operator norm is bounded in any event by ε at large r. The latter fact implies the third assertion in (7.2).

Step 5. It follows from Lemma 6.7 of [28] that there is, for each k, a smooth family of operators $\{\gamma_t^k\}_{t\in[0,1]}$ mapping $\oplus_{k'} L_1^2(N^{(k')})$ to $L_1^2(N^{(k)} \otimes T^{0,1}C_k)$ with the following properties:

- There is an isomorphism $\Xi_t : \mathrm{kernel}(\oplus_k(\Delta_{y^k} + \gamma_t^k + n_t^k)) \to \mathrm{kernel}(L_{\Psi_r(y)} + n_t)$.
- $||\gamma_t^k(v)||_2 \leq \zeta(\varepsilon ||v^k||_2 + r^{-1/4}\Sigma_{k'} ||v^{k'}||_{1,2})$.

(7.7)

Here, $\{n_t^k\}$ is defined as in (7.1) but with n_t replacing n_1. Also, ε contribution comes from the v_{y^k} term in (6.29a) of [28]. In this last equation, ζ is independent of r and $\{y^k\}$.

Meanwhile, for all $v \in L_1^2(N^{(k)})$, one has $||\Delta_{y^k} v^k||_2 \geq \zeta^{-1} ||v^k||_{1,2}$ (by assumption). Thus, there exists an r and $\{y^k\}$-independent constant ζ such that when $r > \zeta$, then the family of operators $\{\oplus_k(\Delta_{y^k} + \gamma_t^k + n_t^k)\}_{t\in[0,1]}$ defines a smooth family of Fredholm operators. And, according to (7.7) and Lemma 6.7 in [28], the spectral flow for the latter family is equal to that for $\{L_{\Psi_r(y)} + n_t\}_{t\in[0,1]}$. Moreover, because of the second assertion of (7.7), the manifolds $\{\mathcal{K}_\Lambda^{(k)}\}$ and this ζ can be chosen so that no $\{\oplus_k(\Delta_{y^k} + t \cdot \gamma_0^k\}_{t\in[0,1]}$ has cokernel or kernel. Likewise, because of the second assertion in Lemma 7.2, this same data can be chosen so that no member of $\{\oplus_k(\Delta_{y^k} + t \cdot (\gamma_1^k + n_1^k - n_{y^k}) + n_1^k)\}_{t\in[0,1]}$ has cokernel or kernel.

Thus, the mod(2) spectral flow for $\{L_{\Psi_r(y)} + n_t\}_{t\in[0,1]}$ is equal to that for a path of Fredholm operators which interpolate between $\oplus_k \Delta_{y^k}$ and $\oplus_k(\Delta_{y^k} + n_{y^k})$. As any path of Fredholm operators between the latter two yields the same mod(2) spectral flow, Proposition 2.11 is proved in the case all Λ_k are trivial.

b) The proof of Proposition 2.13: The construction of Φ

As remarked in the introduction to this section, the verification of the assertions in (2.29) establishes Proposition 2.13.

To begin, make Proposition 2.10's assumptions and then introduce the space Y as in (2.28). The first point in (2.29) concerns the vector bundle $W \to Y$ and an isomorphism, Φ, of the latter with a certain trivial bundle over a neighborhood of $\psi_r^{-1}(0)$. The construction of Φ is the subject of this subsection.

Before beginning, recall that the fiber of W at a point

$$\Xi = (a, (\alpha, \beta)) \in Y$$

is the quotient of $i \cdot \Omega^{2+} \oplus C^\infty(S_+)$ by the L_Ξ image of the vector space N_Ξ. Here, N_Ξ is the L^2 orthogonal compliment of $TY|_\Xi$ in the subspace of $i \cdot \Omega^1 \oplus C^\infty(S_+)$ where the first line of (2.6) gives zero. (The latter subspace is denoted by $\mathcal{T}_\Xi$.) In particular, this means that $W|_\Xi$ can be viewed as the L^2-orthogonal compliment of $L_\Xi(N_\Xi)$ in $i \cdot \Omega^{2+} \oplus C^\infty(S_+)$. This realizes W as a subbundle of the restriction of the bundle in (4.4) to Y.

Now, let $h = \{(C_k, m_k)\} \in \mathcal{H}$. The vector bundle homomorphism Φ in (2.29) will be defined as the projection onto W of a homomorphism, Φ', from $(\times_k \mathcal{K}_\Lambda^{(k)}) \times (\times_k \Lambda_k)$ to the Ψ_r-pull-back of the bundle in (7.4).

The homomorphism Φ' is constructed as follows: Fix $y \in \times_k \mathcal{K}_\Lambda^{(k)}$. Then, $\Phi'|_y$ is the map from $\times_k \Lambda_k$ to $i\Omega^{2+} \oplus C^\infty(S_-)$ which sends $\xi = \{\xi_1, \ldots, \xi_k\} \in \times_k \Lambda_k$ to

$$\Phi'_y \cdot \xi = \Sigma_k \chi_{100\delta,k} \underline{\Upsilon_1 \cdot \xi_k}. \tag{7.8}$$

Here are some explanatory remarks: First, fix k and introduce the (C_k, m_k) version of (2.15) or (3.1) of [28]. Second, reintroduce the map Υ of Proposition 3.2 of [28] which identifies $\oplus_{1 \le q \le m} N^q$ with the (C_k, m_k) version of the space of sections of (2.15) or (3.1) of [28]. Third, set $c = \Upsilon(y^k)$ and then introduce the isomorphism $\Upsilon_1 : \oplus_{1 \le q \le m} N^q \to V^c$ as in Proposition 3.2 of [28]. Extend Υ_1 to the $\mathbb{C}$-linear isomorphism $\Upsilon_1 : (\oplus_{1 \le q \le m} N^q) \otimes T^{0,1}C \to V^c \otimes T^{0,1}C$ and so interpret $\Upsilon_1 \cdot \xi_k$ as a section of the latter. Fourth, remember that a section w of $V^c \otimes T^{0,1}C$ corresponds to a section of the bundle K_1 from Lemma 4.5 in [28] and thus a section, $\underline{w}$, of the bundle $\mathcal{V}_1$ in (4.16b) of [28]. And, remember that when u is a section of $\mathcal{V}_1$, then the identifications from Lemma 4.4 of [28] define $\chi_{100\delta,k} u$ as an element in $i \cdot (\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_-)$. Finally, note that the sections of $\mathcal{V}_1$ which come from sections of K_1 have trivial projection into the $i \cdot \Omega^0$ summand of $i \cdot (\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_-)$.

c) Proof of Proposition 2.13: Φ is an isomorphism

The purpose of this subsection is to prove the following lemma:

Lemma 7.3. *The conclusions of Proposition 2.10 and Lemmas 7.1-7.2 can be augmented with the following: For all sufficiently large r, there exists a neighborhood $Y' \subset \times_k \mathcal{K}_\Lambda^{(k)}$ of $\psi_r^{-1}(0)$ over which the map Φ defines an isomorphism between $Y' \times (\times_k \Lambda_k)$ and $\Psi_r^* W$.*

This lemma gives the first point of (2.29).

Proof of Lemma 7.3. Were the lemma false, then

$$y = (y^1, \dots, y^k) \in \times_k \mathcal{K}_\Lambda^{(k)}$$

and a non-zero

$$\xi = (\xi_1, \dots, \xi_k) \in \times_k \Lambda_k$$

would exist such that $\Phi'_y \cdot \xi = L_{\Psi_r(y)} p$ where p lies in the space

$$\mathcal{T}_{\Psi_r(y)} \in i \cdot \Omega^1 \oplus C^\infty(S_+)$$

and is L^2 orthogonal to the the image of the differential of Ψ_r. The argument below will show that no such pair (y, ξ) exists when r is large. There are five steps to this argument.

Step 1. Think of $\Phi'_y \cdot \xi$ in $i \cdot (\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_-)$ with zero component in the $i \cdot \Omega^0$ summand. Likewise, think of p as an element in $i \cdot \Omega^1 \oplus C^\infty(S_+)$. With this understood, invoke Lemma 6.7 of [28] in the case where $g' = \Phi'_y \cdot \xi$. Let $u = (u^1, \dots, u^k)$, p^0 and $\{h'^k\}$ be as described in said lemma. It then follows from (6.29b,c) of [28] that

- $||\nabla p^0||_2 + \sqrt{r}||p^0||_2 \le \zeta(r^{-1/2}\Sigma_k ||u^k||_2 + e^{-\sqrt{r}/\zeta}\Sigma_k ||\xi_k||_2)$,
- $||\nabla P(h'^k)||_2 + \sqrt{r}||h'^k||_2 \le \zeta(r^{-1/2}\Sigma_{k'} ||u^{k'}||_{1,2} + e^{-\sqrt{r}/\zeta}\Sigma_{k'} ||\xi_{k'}||_2)$.

(7.9)

The reason for this is that $(1 - \chi_{25\delta,k})\Phi'_y\xi$ and $(1 - \Pi)\chi_{25\delta,k}\Phi'_y\xi$ are both $\mathcal{O}(e^{-\sqrt{r}/\zeta})$. Indeed, the former expression obeys such a bound because

$$|\Phi' \cdot \xi| \le \zeta e^{-\sqrt{r}/\zeta} \cdot \Sigma_{k'} ||\xi_{k'}||_2$$

at points with distance δ or more from any C_k. Meanwhile, the latter expression also obeys this bound for the same reason. (For this last case, use the vanishing of $(1-\Pi)\cdot\underline{\Upsilon}_1\underline{\xi}$ to conclude that the $(1-\Pi)$ projection of $\chi_{25\delta,k}\underline{\Upsilon}_1\underline{\xi}$ is minus that of $(1-\chi_{25\delta,k})\cdot\underline{\Upsilon}_1\underline{\xi}$.)

Furthermore, write $u^k = u_0^k + u_1^k$, where u_0^k is annihilated by $(1-Q_\Lambda)\cdot\Delta_y$, and u_1^k is L^2-orthogonal to the kernel of $(1-Q_\Lambda)\cdot\Delta_y$. Here (and below), $y=y^k$ and $\Lambda=\Lambda_k$. Then u_1^k obeys

$$(1-Q_\Lambda)\cdot(\Delta_y u_1^k + \upsilon_y(u^k) + \phi_0^k(u)) = \mathfrak{f}^k, \tag{7.10}$$

where $\mathfrak{f}^k$ satisfies $||\mathfrak{f}^k||_2 \le \zeta e^{-\sqrt{r}/\zeta}\Sigma_{k'}||\xi_{k'}||_2$. (This last equation is (6.29a) in [28].)

Step 2. Since u_1^k is L^2 orthogonal to the kernel of $(1-Q_\Lambda)\cdot\Delta_y$, from (7.10), the fifth line of (7.5) and then the estimate in Lemma 6.7 of [28] for ϕ_0^k and υ_y it follows that

$$||u_1^k||_{1,2} \le \zeta(\varepsilon||u^k||_2 + r^{-1/4}\Sigma_{k'}||u^{k'}||_{1,2}) + \zeta e^{-\sqrt{r}/\zeta}\Sigma_{k'}||\xi_{k'}||_2, \tag{7.11}$$

where $\varepsilon > 0$ can be arranged as small as desired by making r large. In particular, when $\varepsilon < (2\cdot\zeta)^{-1}$, then summing (7.11) over k finds that

$$\Sigma_k||u_1^k||_{1,2} \le \zeta(\varepsilon||u_0^k||_2 + r^{-1/4}\Sigma_k||u_0^k||_{1,2}) + \zeta e^{-\sqrt{r}/\zeta}\Sigma_k||\xi_k||_2. \tag{7.12}$$

Since all L_k^p norms of an element in the kernel of $(1-Q_\Lambda)\cdot\Delta_y$ can be uniformly bounded in terms of the L^2 norm, this last equation implies the inequality

$$\Sigma_k||u_1^k||_{1,2} \le \zeta(\varepsilon + r^{-1/4})\Sigma_k||u_0^k||_2 + \zeta e^{-\sqrt{r}/\zeta}\Sigma_k||\xi_k||_2. \tag{7.13}$$

Step 3. This step constitutes a digression to consider the image of the differential of Ψ_r, where the latter is thought of as a map from $\times_k\mathcal{K}_\Lambda^{(k)}$ into $\mathrm{Conn}(E)\times C^\infty(S_+)$. For this purpose, suppose that $v=(v^1,\dots,v^k)\in T(\times_k\mathcal{K}_\Lambda^{(k)})|_y$. Let $p_v \in i\cdot\Omega^1\oplus C^\infty(S_+)$ denote the push-forward of v by the differential of Ψ_r. Then p_v appears (by construction) as

$$p_v = \Pi_k(1-\chi_{4\delta,k})\cdot p_v^0 + \Sigma_k\chi_{25\delta,k}p_v^k. \tag{7.14}$$

Furthermore, one can write, for each k, $p_v^k = P(h_v^k) + \underline{\Upsilon}_1\cdot\underline{u}_v^k$. And, one can decompose each u_v^k as $v^k + u_{v1}^k$, where the former is in the kernel of $(1 - Q_\Lambda) \cdot \Delta_y$, and the latter is L^2 orthogonal to this kernel. It then follows from Lemma 6.6 of [28] that

$$||p_v^0||_2 + ||P(h_v^k)||_2 + ||u_{v1}^k||_2 \le \zeta r^{-1/2} ||v^k||_2. \tag{7.15}$$

Step 4. If p is to be L^2 orthogonal to the image of the differential of Ψ_r, then p must also be L^2 orthogonal to p_v. (Note that p_v is not necessarily in the subspace $\mathcal{T}_{\Psi_r(y)}$. However, it differs from an element in this subspace by a tangent vector to the orbit at $\Psi_r(y)$ of $C^\infty(X; S^1)$. And, as $p \in \mathcal{T}_{\Psi_r(y)}$, the latter is L^2-orthogonal to all elements in said orbit. Thus, p must be L^2-orthogonal to p_v.)

The L^2-orthogonality of p to all p_v implies (using (7.15) and (7.9)) that $||u_0^k||_2 \le \zeta \cdot r^{-1/2} \cdot ||u_1^k||_2$. With the preceding given, take r large so that ε in (7.12) is bounded by $(2 \cdot \zeta)^{-1}$. Then, (7.12) implies that

$$\Sigma_k ||u^k||_{1,2} \le \zeta e^{-\sqrt{r}/\zeta} \Sigma_k ||\xi_k||_2. \tag{7.16}$$

And, this last estimate plus (7.9) implies that

$$||\nabla p||_2 + \sqrt{r}||p||_2 \le \zeta e^{-\sqrt{r}/\zeta} \Sigma_k ||\xi_k||_2. \tag{7.17}$$

Step 5. By squaring both sides of the equation $L_{\Psi_r(y)} p = \Phi'_y \cdot \xi$ and integrating over X, one learns (after some straightforward algebraic manipulations) that

$$||\nabla p||_2 + \sqrt{r}||p||_2 \ge \zeta ||\Phi'_y \cdot \xi||_2. \tag{7.18}$$

Since $||\Phi'_y \cdot \xi||_2 \ge \zeta^{-1} \cdot r^{-1/2} \cdot ||\xi||_2$, these inequalities in (7.17) and (7.18) are contradictory when r is larger than some constant ζ which is independent of $y \in \times_k \mathcal{K}_\Lambda^{(k)}$. Thus, when r is large, there is no (y, ξ) with the properties assumed in the introduction.

d) Proof of Proposition 2.13: Orientations and Φ

This section considers the following lemma:

Lemma 7.4. *The conclusions of Proposition 2.10 and Lemmas 7.1-7.3 can be augmented to include the following: For each k with $m_k > 1$, orient Λ_k. Use the latter to orient $\mathcal{K}_\Lambda^{(k)}$ as described in Parts 1 and 2 of Section 2g and then use Proposition 2.10's isomorphism $Y_h = \times_{k:m_k>1} \mathcal{K}_\Lambda^{(k)}$ to orient Y_h by multiplying the product orientation with $\varepsilon(\sigma) \cdot \Pi_{k:m_k=1} r(C_k, 1)$. Here, $r(C_k, 1)$ is defined as in Part 5 of Section*

1e with the choice of a cyclic ordering of $\Gamma_k = \{\gamma \in \Gamma : \gamma \cap C_k \neq \varnothing\}$. *Also,* $\varepsilon(\sigma) = \pm 1$ *is defined from these orderings of* $\{\Gamma_k\}$ *as in Part 7 of Section 1e. Next, orient* $\Psi_r(Y_h)$ *when* r *is large using the* Ψ_r*, and use the orientation on* $\Psi_r(Y_h)$ *to orient* W *as described in the second assertion of Proposition 2.14. Then, when* r *is sufficently large, the map* Φ *is orientation preserving over a neighborhood of* $\psi_r^{-1}(0) \cap Y_h$ *in* Y_h.

Note that this lemma implies the second point in (2.29).

Lemma 7.4 follows directly from two auxilliary lemmas which are given below. The first lemma requires the following preliminary digression. To begin the digression, for each k with $m_k > 1$, choose an orientation for Λ_k and use the orientation for the virtual bundle $T\mathcal{K}_\Lambda^{(k)} - \mathcal{K}_\Lambda^{(k)} \times \Lambda_k$ from Parts 1 and 2 of Section 2g to orient $\mathcal{K}_\Lambda^{(k)}$. Meanwhile, for those k with $m_k = 1$, the space $\mathcal{K}_\Lambda^{(k)}$ is an open neighborhood of 0 in the kernel of the C_k version of the operator D in (1.11). Orient this manifold as in the $m_k = 1$ discussions in Part 5 of Section 1e. Then, orient $\times_k \mathcal{K}_\Lambda^{(k)}$ using the product orientation. Next, use the differential of Ψ_r to orient $Y = \Psi_r(\times_k \mathcal{K}_\Lambda^{(k)})$ and use the orientation for the virtual bundle TY-W in Step 2 of Section 4c to orient W. End the digression.

Lemma 7.5. *The conclusions of Proposition 2.10 and Lemma 7.1-7.3 can be augmented with the following: Orient each* Λ_k *when* $m_k > 1$ *and take the product orientation for* $\times_k \Lambda_k$*. Orient* W *as in the preceding digression. When* r *is large, then the isomorphism* Φ *is orientation preserving at points near* $\psi_r^{-1}(0)$.

Lemma 7.6. *The conclusions of Proposition 2.10 and Lemma 7.1-7.3 and 7.5 can be augmented with the following: Fix an orientation for each* $\mathcal{K}_\Lambda^{(k)}$ *and orient* $\times_k \mathcal{K}_\Lambda^{(k)}$ *with the product orientation. When* r *is large, then there is a neighborhood of* $\psi_r^{-1}(0)$ *where the following two orientations for* $Y_h \subset \times_k \mathcal{K}_\Lambda^{(k)}$ *agree:*

- *Orient* Y_h *by using Proposition 2.10 to write the latter as* $\times_{k:m_k>1} \mathcal{K}_\Lambda^{(k)}$*. Then, multiply the product orientation by* $\varepsilon(\sigma) \cdot \Pi_{k:d(e_k)>0} r(C_k, 1)$*. Here,* $r(C_k, 1)$ *in the case where* $d(e_k) > 0$ *is defined as in Part 5 of Section 1e using a choice of an ordering for the set*
$$\Gamma_k = \{\gamma \in \Gamma : \gamma \cap C_k \neq \varnothing\}.$$
Also, $\varepsilon(\sigma)$ *is defined as in Part 6 of Section 1e.*

- *For $y \in Y_h$, introduce the oriented vector space V of (2.8) and then Proposition 2.10's epimorphism $G_y : T(\times_k \mathcal{K}_\Lambda^{(k)})|_y \to V$. Identify $TY_h|_y$ with the kernel of G_y and then use the given orientations on $T(\times_k \mathcal{K}_\Lambda^{(k)})|_y$ and V to induce an orientation on $TY_h|_y$. (Note that V is even dimensional, so there is no ordering issue here.)*

The proof of Lemma 7.5 is given in the next subsection. The remainder of this subsection contains the

Proof of Lemma 7.6. For each k with $d(e_k) > 0$, let

$$\Gamma_k = \{\gamma \in \Gamma : \gamma \cap C_k = \varnothing\}.$$

Order the elements in Γ_k and then introduce the corresponding vector space V_k as defined in Part 3 of Section 1e. Note that an ordering of Γ_k determines a canonical orientation of V_k. For such k, introduce the linear surjection $G_k : \mathcal{K}_\Lambda^{(k)} \to V^k$ from Part 3 of Section 1e. (Remember that $\mathcal{K}_\Lambda^{(k)}$ in this case is the kernel of the C_k version of the operator D from (1.11).) Now consider $\Psi_r(y) = (a, (\alpha, \beta))$. As described in Proposition 5.2 and (4.1) of [28], the component α is a sum, $\alpha = \underline{\alpha}_r + \alpha'$, where $|\alpha'| \leq \zeta \cdot r^{-1/2}$. Furthermore, the differential of α' in a tangent direction p to $\times_k \mathcal{K}_\Lambda^{(k)}$ obeys $|\alpha'_* p| \leq \zeta \cdot r^{-1/2} \cdot ||p||_2$. (See Lemma 6.6 of [28].) With the preceding understood, note that were $\alpha = \underline{\alpha}_r$, then the assertions of the lemma would follow directly from the definitions in Sections 2 and 3b of $\underline{\alpha}_r$. Indeed, in this case, V would be identical to $\oplus_{k:d(e_k)>0} V_k$ and the homomorphism G_y would equal $\oplus_k G_k$. (The factor of $\varepsilon(\sigma)$ in this case results by comparing the given orientation on V with that on $\oplus_{k:d(e_k)>0} V_k$.) This last fact with the small size of α' and its differential at large r then imply the lemma. (With regard to this last point, note from the definition that the behavior of $\underline{\alpha}_r$ and its differential along a tangent vector to $\times_k \mathcal{K}_\Lambda^{(k)}$ is suitably uniform in r.)

e) Proof of Lemma 7.5

The lemma compares an orientation on the virtual bundle $T(\times_k \mathcal{K}_\Lambda^{(k)}) - \Psi_r^* W$ with one on the virtual bundle $T(\times_k \mathcal{K}_\Lambda^{(k)}) - \times_k \Lambda_k$. The plan will be to compare these relative orientations at points $y \in \psi_r^{-1}(0)$. If they agree at such points, then they will agree on some neighborhood of $\psi_r^{-1}(0)$. Thus, it is enough to compare the orientations only for points $y \in \psi_r^{-1}(0)$.

The proof has two parts. The first part of the proof considers the case where some Λ_k is non trivial and translates this comparison question in this case to a question of the equality of two relative orientations on $T(\times_k \mathcal{K}_\Lambda^{(k)})|_y - \times_k \Lambda_k$. That is, the first part of the proof reduces the comparison of orientations on distinct virtual bundles to a question about orientations on the same virtual bundle. The remaining part of the proof compares the two orientations on $T(\times_k \mathcal{K}_\Lambda^{(k)})|_y - \times_k \Lambda_k$.

The latter two orientations are defined as follows: The first orientation is defined by considering a path of operators that perturb $\oplus_k \Delta_{y^k}$ to a $\mathbb{C}$-linear operator. The second orientation is defined by considering a path of operators which perturb $L_{\Psi_r(y)}$ to a $\mathbb{C}$-linear operator. The paths in question, and the comparison of the kernels along the paths are compared with the help of Lemma 6.7 in [28]. As in the case where all Λ_k are trivial, these lemmas are used to construct a path for $\oplus_k \Delta_{y^k}$ from one for $L_{\Psi_r(y)}$. Furthermore, this construction allows for a direct comparison of the kernels at corresponding points along the two paths. The resulting comparison leads to the conclusions of Lemma 7.5. The argument here is similar in most respects to that in Section 7a, above.

Part 1. As just remarked, this part of the proof translates the comparison question in the case where some $\Lambda_k \neq \{0\}$ to a question of the equality of two orientations on the virtual bundle $T(\times_k \mathcal{K}_\Lambda^{(k)})|_y - \times_k \Lambda_k$. Thus, this part assumes that $\Lambda_k \neq \{0\}$ for some k. There are three steps in this part of the proof.

Step 1. This first step contains some preliminary constructions and remarks.

To begin, note that the map Φ can be rephrased as follows: Take $y \in \times_k \mathcal{K}_\Lambda^{(k)}$ and remember that $W|_{\Psi_r(y)}$ is the L^2- orthogonal compliment inside $i \cdot \Omega^{2+} \oplus C^\infty(S_-)$ of the $L_{\Psi_r(y)}$ image of the the subspace $N_{\Psi_r(y)}$; the L^2 compliment of the image of the differential of Ψ_r in $\mathcal{T}_{\Psi_r(y)}$. (Remember that the latter is the subspace of $i \cdot \Omega^1 \oplus C^\infty(S_+)$ where the first line in (2.6) vanishes in the case where $(a, (\alpha, \beta)) = \Psi_r(y)$.) Then, let $\Pi_{\Psi_r(y)}$ denote the L^2-orthogonal projection onto $W|_{\Psi_r(y)}$. With this understood, then $\Phi_y = \Pi_{\Psi_r(y)} \cdot \Phi'_y$.

Define a homotopy, $\underline{\Phi}'$, of the homomorphism Φ' in (7.8) so that the resulting homomorphism at $(t, y) \in [0,1] \times (\times_k \mathcal{K}_\Lambda^{(k)})$ sends $\xi \in \times_k \Lambda_k$ to

$$(7.19) \qquad \underline{\Phi}'_{(t,y)} \cdot \xi = \Pi_{\Psi_r(y)} \cdot \Phi'_y \cdot \xi + t \cdot (1 - \Pi_{\Psi_r(y)}) \cdot \Phi'_y \cdot \xi.$$

Here are some facts about $\underline{\Phi}'$:

- $\underline{\Phi}'_{(t,y)}$ is injective for each pair (t, y).
- The projection from the image of $\underline{\Phi}'_{(t,y)}$ to cokernel$(L_{\Psi_r(y)})$ is surjective.

(7.20)

The injectivity of $\underline{\Phi}'_{(t,y)}$ is a consequence of the fact that $\Phi_y = \Pi_{\Psi_r(y)}\Phi'_y$. The validity of the second assertion of (7.20) can be argued as follows: Identify the cokernel of $(L_{\Psi_r(y)})$ with the kernel of $(L_{\Psi_r(y)})^\dagger$ to turn the projection in question into the L^2 orthogonal projection onto the kernel of $(L_{\Psi_r(y)})^\dagger$. However, kernel$((L_{\Psi_r(y)})^\dagger) \subset W|_{\Psi_r(y)}$ since $W|_{\Psi_r(y)}$ is the L^2 orthogonal compliment of the $(L_{\Psi_r(y)})$ image of the subspace, $N_{\Psi_r(y)}$. And, according to Lemma 7.3 and (7.19), the L^2 orthogonal projection from the image of $K_{(t,y)}$ gives the image of $\underline{\Phi}'_{(0,y)}$ which is $W|_{\Psi_r(y)}$.

Step 2. Depending on the context, it is convenient to consider the image of $\underline{\Phi}'_{(t,y)}$ as either a subspace of $i \cdot \Omega^{2+} \oplus C^\infty(S_-)$ or else a subspace in $i \cdot (\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_-)$ with zero projection onto the $i \cdot \Omega^0$ summand. When $t \in [0, 1]$, let $\Pi_{(t,y)}$ denote the L^2-orthogonal projection (on either $i \cdot \Omega^{2+} \oplus C^\infty(S_-)$ or $i \cdot (\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_-)$) onto the image of $\underline{\Phi}'_{(t,y)}$. Note that $\Pi_{(1,y)}$ is the L^2-orthogonal projection onto the image of Φ'_y, while $\Pi_{(0,y)} = \Pi_{\Psi_r(y)}$.

With $\Pi_{(t,y)}$ understood, then (7.20) implies that

- $(1 - \Pi_{(t,y)}) \cdot L_{\Psi_r(y)} : \mathcal{T}_{\Psi_r(y)} \to (1 - \Pi_{(t,y)}) \cdot (i \cdot \Omega^{2+} \oplus C^\infty(S_-))$,
- $(1 - \Pi_{(t,y)}) \cdot L_{\Psi_r(y)} : i \cdot \Omega^1 \oplus C^\infty(S_+) \to (1 - \Pi_{(t,y)}) \cdot (i \cdot (\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_-))$

(7.21)

are both surjections for each $t \in [0, 1]$ and $y \in \times_k \mathcal{K}_\Lambda^{(k)}$.

Fix $y \in \times_k \mathcal{K}_\Lambda^{(k)}$, and let V_t denote the kernel of the operators in (7.21) as a function of t. (Both operators have the same kernel.) Note that $V_0 \approx T(\times_k \mathcal{K}_\Lambda^{(k)})$ in a natural way. Indeed,

$$V_0 = \{(\Psi_r)_* v + \varsigma(v) : v \in T(\times_k \mathcal{K}_\Lambda^{(k)})|_y\}$$

where

$$\varsigma : T(\times_k \mathcal{K}_\Lambda^{(k)})|_y \to N_{\Psi_r(y)}$$

is an appropriate linear map. On the otherhand, when $y \in \psi_r^{-1}(0)$, then

$$V_1 = \{(\Psi_r)_* v : v \in T(\times_k \mathcal{K}_\Lambda^{(k)})|_y\}. \tag{7.22}$$

To prove (7.22), consider first that V_1 is a vector space whose dimension is the same as that of $T(\times_k \mathcal{K}_\Lambda^{(k)})|_y$. Thus, if the former contains the latter, then they are equal. To see that the former contains the latter, note that when $(a, (\alpha, \beta)) = \Psi_r(y)$, then (4.5) defines an element $\mathbb{H}(y)$ which lives in the image of Φ'_y. (This is by construction.) And, when $\mathbb{H}(y) = 0$, then the differential of $\mathbb{H}$ along a direction $v \in T(\times_k \mathcal{K}_\Lambda^{(k)})$ will also live in the image of Φ'_y and so is annihilated by $(1 - \Pi_{(1,y)})$. Finally, note that the differential of $\mathbb{H}$ at y computes $L_{\Psi_r(y)}((\Psi_r)_* v)$.

Step 3. When $y \in \psi_r^{-1}(0)$, then the association of V_t to $t \in [0,1]$ defines a vector bundle, $V \to [0,1]$, and so an orientation for $V_0 \approx \Psi_r^* W|_y$ induces an orientation on $V_1 \approx T(\times_k \mathcal{K}_\Lambda^{(k)})|_y$. In this step, implicitly orient the formal difference $V_1 - V_0$ using this induced orientation.

Now, consider mapping V_t to $\times_k \Lambda_k$ as follows: Associate to $v \in V_t$ the element $\underline{\Phi}'^{-1}_{(t,y)} \cdot \Pi_{\Psi_t(y)} \cdot L_{\Psi_r(y)} v$. Let T_t denote this map. If $y \in \psi_r^{-1}(0)$ and $\text{kernel}(L_{\Psi_r(y)}) = \{0\}$, then $\text{kernel}(T_t) = 0$ for all t. Thus, in this case, the relative orientation defined by T_0 on the virtual bundle $V_0 - W$ agrees with that defined by T_1 on $V_1 - \times_k \Lambda_k$. This case occurs when $d(e) = 0$ and $\Psi_r(y)$ is a smooth point of $\mathcal{M}^{(r)}$.

Now suppose that $y \in \psi_r^{-1}(0)$, but that $\text{kernel}(L_{\Psi_r(y)}) \neq \{0\}$. In this case, the kernel of T_t is the kernel of $L_{\Psi_r(y)}$ for all $t \in [0,1]$. With the preceding understood, choose a continuous family of isomorphisms $w_t : \text{kernel}(L_{\Psi_r(y)}) \to \text{cokernel}(T_t) \subset \times_k \Lambda_k$. Then, consider the path of operators $L_{\Psi_r(y)} + \underline{\Phi}'_{(t,y)} \cdot w_t$ where w_t is to be thought of as a map from $i \cdot \Omega^1 \oplus C^\infty(S_+)$ which is zero on the L^2-orthogonal complement of $\text{kernel}(L_{\Psi_r(y)})$. It then follows that for every t, the map from V_t to $\times_k \Lambda_k$ which assigns to v the element $\underline{\Phi}'^{-1}_{(t,y)} \cdot \Pi_{(t,y)} \cdot (L_{\Psi_r(y)} + \underline{\Phi}'_{(t,y)} \cdot w_t) \cdot v$ is an isomorphism.

Using the preceding family of isomorphism, an orientation for $T(\times_k \mathcal{K}_\Lambda^{(k)})|_y - \times_k \Lambda_k$ is, again, induced by one on $T(\times_k \mathcal{K}_\Lambda^{(k)})|_y - W|_{\Psi_r(y)}$.

Part 2. Here again, suppose that $y \in \psi_r^{-1}(0)$. It follows from the preceding remark that the question of whether

$$\text{Identity - } \Phi : T(\times_k \mathcal{K}_\Lambda^{(k)})|_y - \times_k \Lambda_k \to T(\times_k \mathcal{K}_\Lambda^{(k)})|_y - W|_{\Psi_r(y)} \tag{7.23}$$

preserves orienation can be decided by whether two orientations on $T(\times_k \mathcal{K}_\Lambda^{(k)})|_y - \times_k \Lambda_k$ agree. This part of the proof of Lemma 7.5 compares the two relative orientations. There are eight steps to this part of the proof.

Step 1. The first orientation on $T(\times_k \mathcal{K}_\Lambda^{(k)})|_y - \times_k \Lambda_k$ comes from Part 1 of Section 2g where an orientation is defined on each of the virtual vector spaces $T(\mathcal{K}_\Lambda^{(k)})|_{y^k} - \Lambda_k$. The resulting orientation on $T(\times_k \mathcal{K}_\Lambda^{(k)})|_y - \times_k \Lambda_k$ is insensitive to a permutation of the labels of the set $\{(C_k, m_k)\}$. This is because $\dim(T(\mathcal{K}_\Lambda^{(k)})|_{y^k}) = \dim(\Lambda_k) \bmod(2)$ for each k.

Briefly, Section 2g's orientation of $T(\mathcal{K}_\Lambda^{(k)})|_{y^k} - \Lambda_k$ is obtained as follows: In the case where $d(e_k) = 0$ and $\Lambda_k = \{0\}$, the orientation is the mod(2) count of the number of points along the path in (2.24) where the operator $(\Delta_{y^k} + n_t)$ has non-trivial cokernel. In the case where $d(e_k) > 0$ or when $\Lambda = \Lambda_k \neq \{0\}$, the orientation is defined in three steps. First, define a vector bundle over $[0,1]$ by assigning to each $t \in [0,1]$ the kernel of the operator $(1 - Q_\Lambda) \cdot (\Delta_{y^k} + n_t)$. Second, trivialize this vector bundle. Third, in the case where $m_k = 1$, orient kernel$(\Delta_{y^k} + n_1)$ with its complex orientation. In the case where $m_k > 1$ and $\Lambda = \Lambda_k \neq \{0\}$, orient the virtual vector space kernel$((1 - Q_\Lambda) \cdot (\Delta_{y^k} + n_1)) - \Lambda$ by mapping the former to the latter using $Q_\Lambda \cdot (\Delta_{y^k} + n_1)$. In all of these cases, remember that n_1 should be chosen so that the latter map is an isomorphism. In fact, in the arguments that follow, take $n_1 = n_{y^k}$, where the n_{y^k} is given in Lemma 7.2.

With regard to the preceding definition, be aware that the same orientation on the virtual bundle $T(\times_k \mathcal{K}_\Lambda^{(k)})|_y - \times_k \Lambda_k$ is obtained by taking any continous path, $\{T^{(t)}\}_{t \in [0,1]}$ of Fredholm operators from $\oplus_k L_1^2(N^{(k)})$ to $\oplus_k L^2(N^{(k)} \otimes T^{0,1} C_k)$ with the following properties:

- $T^{(0)} = \oplus_k \Delta_{y^k}$,
- $T^{(1)} = \oplus_k (\Delta_{y^k} + n_{y^k})$,
- The map sending $t \in [0,1]$ to $(\oplus_k (1 - Q_\Lambda)) \cdot T^{(t)}$ is suitably generic. In particular, its cokernel in $\oplus_k (1 - Q_\Lambda) \cdot L^2(N^{(k)} \otimes T^{0,1} C_k)$ should be trivial when $\dim(\times_k \mathcal{K}_\Lambda^{(k)}) > 0$.

(7.24)

Step 2. The second orientation for $T(\times_k \mathcal{K}_\Lambda^{(k)})|_y - \times_k \Lambda_k$ is defined as follows: Consider a smooth family of operators

$$\{n_t : L^2(i \cdot T^* \oplus S_+) \to L^2(i \cdot (\varepsilon_{\mathbb{R}} \oplus \Lambda_+) \oplus S_-)\}_{t \in [0,1]}$$

and the corresponding 1-parameter family

$$L^{(t)} = L_{\Psi_r(y)} + \chi(3t) \cdot \Phi'_y \cdot w_1 + n_t \tag{7.25}$$

of differential operators. (Remember that $\underline{\Phi}'_{(1,y)} = \Phi'$.) Choose $\{n_t\}$ as in Lemmas 7.1 and 7.2. Also, choose $\{n_t\}$ so that for each t,

$$\begin{aligned} (1 - \Pi_{(1,y)}) \cdot L^{(t)} &: i \cdot \Omega^1 \oplus C^\infty(S_+) \\ &\to (1 - \Pi_{(1,y)}) \cdot (i \cdot (\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_-)) \end{aligned} \tag{7.26}$$

is surjective. Finally, choose n_t so that the assignment of

$$\Phi'^{-1} \cdot \Pi_{(1,y)} \cdot L^{(1)} v \in \times_k \Lambda_k$$

to $v \in \text{kernel}((1 - \Pi_{(1,y)}) \cdot L^{(1)})$ defines surjective linear map,

$$O : \text{kernel}(1 - \Pi_{(1,y)}) \cdot L^{(1)} \to \times_k \Lambda_k. \tag{7.27}$$

It is an exercise with analytic perturbation theory to see that such a smooth family $\{n_t\}$ can be chosen.

With the preceding understood, let $U_t \subset i \cdot \Omega^1 \oplus C^\infty(S_+)$ denote the kernel of the operator in (7.26). Since this operator changes smoothly with $t \in [0, 1]$, the collection of vector spaces $\{U_t\}_{t \in [0,1]}$ fit together to define a vector bundle $U \to [0, 1]$.

The operator O induces a canonical orientation on $U_1 - \times_k \Lambda_k$ as follows: First, the kernel of O is the kernel of the operator $L_{\Psi_r(y)} + n_1$. As the latter intertwines the almost complex structures J_D and J_R, the kernel of O has a natural complex vector space structure (induced by J_D) and so has a natural orientation. On the otherhand, O maps U_1 onto $\times_k \Lambda_k$ and this means that $U_1/\,\text{kernel}(O)$ inherits an orientation from $\times_k \Lambda_k$ using O to identify these two spaces. Since $\times_k \Lambda_k$ is even dimensional, there are no ordering issues and so an orientation for both kernel(O) and $U_1/\,\text{kernel}(O) - \times_k \Lambda_k$ orients $U_1 - \times_k \Lambda_k$.

The orientation on $U_1 - \times_k \Lambda_k$ induces one for the formal difference $U_0 - \times_k \Lambda_k$ since vector bundles over the interval are trivial. In this way, a relative orientation is defined for $T(\times_k \mathcal{K}_\Lambda^{(k)}) - \times_k \Lambda_k$ because

$$U_0 = \text{kernel}((1 - \Pi_{(1,y)}) \cdot L^{(0)}) = T(\times_k \mathcal{K}_\Lambda^{(k)})|_y.$$

Step 3. Consider now U_t, the kernel of the operator in (7.26). An element $p \in U_t$ is characterized as a solution to the equation

$$L_{\Psi_r(y)}p = n_t(p) + \Phi'_y \cdot \xi, \tag{7.28}$$

where $\xi = \Phi'^{-1}_y \cdot \Pi_{(1,y)} \cdot (L_{\Psi_r(y)}p - n_t(p))$.

Equation (7.28) has the form $L_{\Psi_r(y)}p = g'$, so Lemma 6.7 in [28] can be invoked in the analysis. Here $g' = n_t(p) + \Phi'_y \cdot \xi$.

Lemma 7.7. *The conclusions of Proposition 2.10 and Lemmas 7.1-7.3 and 7.6 can be augmented with the following assertion: There is a constant $\zeta \geq 1$ and, given $\varepsilon > 0$, there is a constant $\zeta' \geq 1$ such that when $r \geq \zeta'$, $y \in \psi_r^{-1}(0)$, and $\{n_t\}$ is as above and satisfies the constraints of Lemmas 7.1 and 7.2; then for each k, there is a family of linear maps*

$$\{\tau_t^k : \oplus_{k'} L_1^2(N^{(k')}) \to L^2(N^{(k)} \otimes T^{0,1}C_k)\}_{r\in[0,1]}$$

which obeys:

- $||\tau_t^k \cdot u||_2 \leq \varepsilon \cdot \Sigma_{k'}||u^{k'}||_{1,2}$.
- *Let $H_t \subset \oplus_k L_1^2(N^{(k)})$ denote the vector space of $u = (u^1, \dots, u^n)$ for which*

$$(1 - Q_\Lambda) \cdot (\Delta_{y^k} u^k + n_t^k + \tau_t^k) = 0 \tag{7.29 a}$$

  holds for each k. Here, n_t^k is defined as in (7.1) but with n_t replacing n_1. Then, there is a linear isomorphism $\mathbb{Q}_t$, from H_t to the vector space U_t of solutions to (7.28).

- *The map $\mathbb{Q}_t$ sends u to $p = \Pi_k(1 - \chi_{4\delta,k}) \cdot p^0 + \Sigma_k \chi_{100\delta,k} p^k$, where p^0 is a linear functional of $u \in H_t$ which obeys*

$$||\nabla p^0||_2 + \sqrt{r} \cdot ||p^0||_2 \leq \zeta \cdot r^{-1/2} \cdot \Sigma_k ||u^k||_{1,2}. \tag{7.29 b}$$

- *Also, $p^k = P(h_u^k) + \underline{\Upsilon}_1 \cdot \underline{u}^k$, where u^k obeys (7.29a) and $h_u^k \in L^2(\mathcal{V}_1; K_1)$ obeys*

$$||h_u^k||_2 \leq \zeta \cdot (r^{-1/2} \cdot ||u^k||_{1,2} + r^{-1} \cdot \Sigma_{k'}||u^{k'}||_{1,2}). \tag{7.29 c}$$

- *The assignment of $t \in [0,1]$ to H_t defines a smooth vector bundle $H \to [0,1]$, and then the assignment of $t \in [0,1]$ to $\mathbb{Q}_t$ defines a smooth section, $\mathbb{Q}$, of* $\mathrm{Hom}(H, U)$.

Proof of Lemma 7.7. This lemma follows from Lemma 4.11 in [28] with the following two additional comments: First, (7.28) can be thought of as an equation for a pair, (p, ξ) with $p \in i \cdot \Omega^1 \oplus C^\infty(S_+)$ and $\xi \in \times_k \Lambda_k$. And, as in the proof of Lemma 7.3, one has the pointwise bound $(1 - \chi_{25\delta,k})|\Phi'_y \cdot \xi| \leq \zeta e^{-\sqrt{r}/\zeta} \Sigma_k ||\xi_k||_2$. Also, the L^2 norm of $(1 - \Pi)\chi_{25\delta,k}\Phi'_y \cdot \xi$ is similarly bounded by $\zeta e^{-\sqrt{r}/\zeta}||\xi_k||_2$. (As remarked earlier, the latter follows from the fact that $(1-\Pi) \cdot \underline{\Upsilon}_1 \cdot \underline{\xi}_k = 0$.) Finally, the fifth point in (7.5) should be used to control the size of the term with v_y in (6.29a) of [28].

Step 5. Lemma 7.7 describes a linear isomorphism, $\mathbb{Q}$, between the vector bundles H and U over [0, 1]. Given an orientation of the latter, orient the former by declaring that $\mathbb{Q}$ preserve orientation. This $\mathbb{Q}$ translates the orientation for $U_0 - \times_k \Lambda_k$ in Step 3 into one for $H_0 - \times_k \Lambda_k$. Since bundles over [0, 1] are always trivial, the latter is induced from an orientation for $H_1 - \times_k \Lambda_k$.

Step 6. Now, $\{H_t\}_{t \in [0,1]}$ are the kernels of the family of operators

$$\{S^{(t)} \equiv \oplus_k (1 - Q_\Lambda) \cdot (\Delta_{y^k} + n^k_t + \tau^k_t)\}_{t \in [0,1]}. \tag{7.30}$$

By connecting $S^{(0)}$ to $\oplus_k(1 - Q_\Lambda) \cdot \Delta_{y^k}$ by a path of surjective Fredholm operators, and likewise $S^{(1)}$ to $\oplus_k(1 - Q_\Lambda) \cdot (\Delta_{y^k} + n_{y^k})$, a path of operators $\{T^{(t)}\}$ will have been constructed as in (7.24) from which one can compute the orientation described in Step 2.

To connect $S^{(0)}$ to $\oplus_k(1-Q_\Lambda) \cdot \Delta_{y^k}$, remark that standard perturbation theory arguments find ζ (independent of y) such that when $r \geq \zeta$, then each of the operators

$$\{\oplus_k (1 - Q_\Lambda) \cdot (\Delta_{y^k} + (t+1) \cdot \tau^k_0)\}_{t \in [-1,0]} \tag{7.31}$$

has trivial cokernel. What is more, for each t in question, the L^2 orthogonal projection from the kernel to the kernel of $\oplus_k(1 - Q_\Lambda) \cdot \Delta_{y^k}$ is an isomorphism. (These assertions use Lemma 7.7's estimate for τ^k_0.)

To connect $S^{(1)}$ to $\oplus_k(1 - Q_\Lambda) \cdot \Delta_{y^k} + n_{y^k}$, remark that standard perturbation theory arguments find ζ (independent of y) such that when

$r \geq \zeta$, then each of the operators

$$(7.32)\quad \{\oplus_k (1 - Q_\Lambda) \cdot (\Delta_{y^k} + (t-1) \cdot n_{y^k} + (2-t) \cdot (-n_1^k + \tau_1^k))\}_{t\in[1,2]}$$

has trivial cokernel. And, those same arguments prove that for each t in question, L^2 orthogonal projection maps the kernel isomorphically to that of $\oplus_k(1 - Q_\Lambda) \cdot (\Delta_{y^k} + n_{y^k})$.

Together, the family $S^{(\cdot)}$ with (7.31) and (7.32) defines (after changing the scale for t) a family $\{T^{(t)}\}$ as in (7.24).

Step 7. The orientation on $H_1 - \times_k \Lambda_k$ that comes via U_1 is defined by declaring that $O \cdot \mathbb{Q}$ preserve orientation as a map from $H_1/\operatorname{kernel}(O \cdot \mathbb{Q})$; and by orienting the kernel of $O \cdot \mathbb{Q}$ by identifying the latter using $\mathbb{Q}$ with the complex vector space kernel(O). A second orientation on $H_1 - \times_k \Lambda_k$ is obtained from an orientation on the virtual vector bundle kernel$(\oplus_k(1 - Q_\Lambda) \cdot (\Delta_{y^k} + n_{y^k})) - \times_k \Lambda_k$ by using the family of kernels from the operators in (7.32) to define a vector bundle over the interval [1, 2] whose fiber over 1 is H_1 and whose fiber over 2 is kernel$(\oplus_k(1 - Q_\Lambda) \cdot (\Delta_{y^k} + n_{y^k}))$. Here, the orientation on the virtual vector space kernel$(\oplus_k(1-Q_\Lambda)\cdot(\Delta_{y^k} + n_{y^k})) - \times_k \Lambda_k$ is defined as follows: First, the kernel of the linear map $\oplus_k Q_\Lambda(\Delta_{y^k} + n_{y^k})$ on kernel$(\oplus_k(1 - Q_\Lambda) \cdot (\Delta_{y^k} + n_{y^k}))$ is oriented by observing that the kernel of said linear map is the kernel of $\oplus_k(\Delta_{y^k} + n_{y^k})$ which is a complex vector space. Second, use $\oplus_k Q_\Lambda(\Delta_{y^k} + n_{y^k})$ to identify $\times_k \Lambda_k$ with the quotient vector space kernel$(\oplus_k(1 - Q_\Lambda) \cdot (\Delta_{y^k} + n_{y^k}))/\operatorname{kernel}(\oplus_k Q_\Lambda(\Delta_{y^k} + n_{y^k}))$.

The claim here is that there exists $\zeta \geq 1$ independent of y such that when $r \geq \zeta$ these two orientations on $H_1 - \times_k \Lambda_k$ agree. Here is why: First,

$$(7.33)\quad |O\cdot\mathbb{Q}(u) - \oplus_k Q_\Lambda \cdot (\Delta_{y^k} u^k + n_1^k(u) + \tau_1^k(u))| \leq \zeta e^{-\sqrt{r}/\zeta} \cdot \Sigma_k ||u^k||_{1,2};$$

which is a consequence of the bounds for $(1 - \chi_{25\delta,k}) \cdot |\Phi'_y \cdot \xi|$ and $||(1-\Pi)\cdot\chi_{25\delta,k}\Phi'_y\cdot\xi||_2$ by $\zeta \cdot e^{-\sqrt{r}} \cdot \Sigma_{k'} ||u^{k'}||_2$. Second, $\oplus_k(\Delta_{y^k} + n_{y^k})$ differs from $\oplus_k(\Delta_{y^k} + n_1^k + \tau_1^k)$ by a term whose k'th coordinate has norm (as an operator from the space $\oplus_k L_1^2(N^{(k)})$ to $L^2(N^{(k)} \otimes T^{0,1}C_k)$) which is bounded by $\zeta \cdot r^{-1/4}$. This means, as noted above, that L^2 orthogonal projection from the kernel of the t version of any of the operators in (7.32) to kernel$(\oplus_k(1-Q_\Lambda)\cdot(\Delta_{y^k} + n_{y^k}))$ is an isomorphism which differs from the identity by a term with norm bounded by $\zeta \cdot r^{-1/4}$. Thus, the inverse map is also an isomorphism which differs from the identity by a term whose norm is bounded by $\zeta \cdot r^{-1/4}$. The composition of this inverse map with the map $\oplus_k Q_\Lambda \cdot (\Delta_{y^k} + (t-1)\cdot n_{y^k} + (2-t)\cdot(-n_1^k + t_1^k))$ thus differs from the map $\oplus_k Q_\Lambda \cdot (\Delta_{y^k} + n_{y^k})$ on $\text{kernel}(\oplus_k (1-Q_\Lambda)\cdot(\Delta_{y^k} + n_{y^k}))$ by a term which has norm bounded by $\zeta \cdot r^{-1/4}$. However, the latter map is surjective with an inverse having norm less than ζ, this curtesy of the second assertion of Lemma 7.2. Thus, when r is large, the maps which define the orientation for H_1 extend across the vector bundle defined by the kernels of (7.32). And, on the fiber over 2, said extension gives the second orientation for H_1.

Step 8. The identification of H_0 via $\mathbb{Q}$ with U_0 defines an identification of the former space with $T(\times_k \mathcal{K}_\Lambda^{(k)})|_y$. Given the results of the preceding step, then Lemma 7.5 follows with a demonstration that the orientation which is induced on H_0 by the latter identification is identical to that which is induced by considering U_0 as the fiber over the point $0 \in [-1, 0]$ and $T(\times_k \mathcal{K}_\Lambda^{(k)})|_y = \text{kernel}(\times_k \Delta_{y^k})$ as the fiber over -1 for the vector bundle whose fiber over t is the kernel of the t version of the operator in (7.31). The claim here is that there exists $\zeta \geq 1$ independent of y, such that when $r \geq \zeta$ these two orientations agree. Here is why: The first orientation is that which is obtained by L^2 orthogonal projection from H_0 to $\text{kernel}(\oplus_k (1 - Q_\Lambda) \cdot \Delta_{y^k})$. And, as remarked, this same L^2-orthogonal projection defines an isomorphism from the t version of any of the operators in (7.31).

f) Comparing the sections w and ψ_r

With Lemmas 7.3 and 7.4 understood, it follows that the section w of W from Proposition 2.14 can be pulled back by Ψ_r on a neighborhood $\mathcal{O}_h \subset Y_h$ of $\psi_r^{-1}(0)$ to define the map $\Phi^{-1}(\Psi_r^* w) : \mathcal{O}_h \to \times_k \Lambda_k$. This map can then be compared with ψ_r :

Lemma 7.8. *The conclusions of Proposition 2.10 and Lemmas 7.1-7.7 can be augmented as follows: When r is large, there is a smooth, $\dim(Y_h) + 1$ dimensional, oriented manifold with boundary Z, and a smooth map $p : Z \to \times_k \Lambda_k$ with the following properties:*

- $\partial Z = \mathcal{O}_h \cup -\mathcal{O}_h$.
- $p|_{Y'} = \Phi^{-1}(\Psi_r^* w)$.
- $p|_{(-Y')} = \psi_r$.
- $p^{-1}(0)$ *is compact.*

Note that this last lemma implies the third point of (2.28) using standard, finite dimensional arguments about counting the zeros of sections of vector bundles.

Proof of Lemma 7.8

When $\Xi = (a, (\alpha, \beta)) \in \mathrm{Conn}(E) \times C^\infty(S_+)$, use $\mathcal{T}_\Xi$ to denote the subspace in $i \cdot \Omega^1 \oplus C^\infty(S_+)$ where the first line in (2.6) holds. And, when $y \in \times_k \mathcal{K}_\Lambda^{(k)}$, use $N_{\Psi_r(y)} \subset \mathcal{T}_{\Psi_r(y)}$ to denote the L^2 orthogonal compliment of the image of the differential of Ψ_r. Note that the assignment of $N_{\Psi_r(y)}$ to $y \in \times_k \mathcal{K}_\Lambda^{(k)}$ gives a smooth vector bundle $\underline{N} \to \times_k \mathcal{K}_\Lambda^{(k)}$. Also, the map from $\underline{N}$ to $(\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1)$ which assigns to (y, x) the orbit of $\Psi_r(y) + x$ restricts to an L^2_2 neighborhood $\underline{O} \subset \underline{N}$ of the zero section as a diffeomorphism onto an open neighborhood of $\Psi_r(\times_k \mathcal{K}_\Lambda^{(k)})$ in the space $(\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1)$.

When $t \in [0,1]$, and $\mathcal{O}_h \subset Y_h$ is an open neighborhood of $\psi_r^{-1}(0)$ use Z_t to denote the set of solutions $\{(y, x) \in \underline{O} : y \in \mathcal{O}_h\}$ of

$$(1 - \Pi_{(t,y)}) \cdot \mathbb{H}(\Psi_r(y) + x) = 0. \tag{7.34}$$

Here, As before, when

$$\Xi = (a, (\alpha, \beta)) \in \mathrm{Conn}(E) \times C^\infty(S_+),$$

use (4.5) to define $\mathbb{H}(\Xi)$ as an element in $i \cdot \Omega^{2+} \oplus C^\infty(S_-)$ or in $i \cdot (\Omega^0 \oplus \Omega^{2+}) \oplus C^\infty(S_-)$ with zero projection onto the $i \cdot \Omega^0$ summand. Note that standard elliptic regularity techniques (as in Chapter 6 of [17]) can be used to prove that an L^2_1 solution x to (7.34) is smooth. This is because the range of $\Pi_{(t,y)}$ consists of smooth elements.

By shrinking $\mathcal{O}_h$ and $\underline{O}$ if necessary (replacing $\mathcal{O}_h$ by its intersection with a smaller open neighborhood of $\Psi_r(Y_h)$), one can arrange the following:

- $\Psi_r^{-1}(\mathcal{M}^{(r)}_{\Gamma,\Omega}) \subset Z_t$.
- The tautological map from Z_t into $(\mathrm{Conn}(E) \times C^\infty(S_+))/C^\infty(X; S^1)$ intersects $\mathcal{M}^{(r)}$ only in $\mathcal{M}^{(r)}_{\Gamma,\Omega}$.
- Each Z_t is a smooth, submanifold of $\underline{O}$ near $\Psi_r^{-1}(\mathcal{M}^{(r)}_{\Gamma,\Omega})$ of dimension $\dim(Y_h)$.
- The set of triples $Z = \{(t, (y, x)) : (y, x) \in Z_t\}$ is a smooth manifold near $\Psi_r^{-1}(\mathcal{M}^{(r)}_{\Gamma,\Omega})$ of dimension $1 + \dim(Y_h)$.

- $Z_1 = \mathcal{O}_h$.
- Z_0 is the image of a section of $\underline{O}$ over $\mathcal{O}_h$. Thus, the bundle projection gives a canonical diffeomorphism between Z_0 and $\mathcal{O}_h$.
- Z can be oriented so that the induced orientations on Z_1 and Z_0 are opposite.

(7.35)

The preceding assertions can be established as follows: The first assertion follows directly from (7.34). The second assertion is a consequence of the fact that each $x \in N_{\Psi_r(y)}$ is normal to $\mathcal{M}^{(r)}$. To establish the third and fourth assertions, note first that if $y \in \Psi_r^{-1}(\mathcal{M}^{(r)}_{\Gamma,\Omega})$, then the linearization of (7.34) at $(y, 0)$ is

$$(1 - \Pi_{(t,y)}) \cdot L_{\Psi_r(y)} : \mathcal{T}_{\Psi_r(y)} \to (1 - \Pi_{(t,y)}) i \cdot \Omega^{2+} \oplus C^\infty(S_+);$$

and the latter is surjective. Now appeal to the implicit function theorem. Indeed, it follows from the implicit function theorem that by shrinking each $\mathcal{O}_h$ if necessary, and also shrinking $\underline{O}$ in each fiber over $\times_k \mathcal{K}^{(k)}_\Lambda$, one can arrange, without loss of generality, that each Z_t is a smooth submanifold of $\underline{O}$ of dimension Y_h and that Z is a smooth submanifold of the product $[0,1] \times \underline{O}$ having dimension $1 + Y_h$.

Next, consider the assertion about Z_1. This follows from the fact that $\mathbb{H}(\Psi_r(y)) \in \Phi'(\times_k \Lambda_k)$ by the very definition of Ψ_r in Section 5. Thus, $\mathcal{O}_h$ is a component of Z_1, and it follows that by shrinking $\mathcal{O}_h$ and $\underline{O}$ again if necessary, one can assume, without loss of generality, that $Z_1 = \mathcal{O}_h$.

Next, consider the assertion about Z_0. In this regard, remember that $(1 - \Pi_{(0,\Psi_r(y))}) = (1 - \Pi_{\Psi_r(y)})$ was *defined* as the L^2 orthogonal projection onto $L_{\Psi_r(y)}(N_{\Psi_r(y)})$. Thus, the linearization of the $t = 0$ version of (7.34) at $y \in \Psi_r^{-1}(\mathcal{M}^{(r)}_{\Gamma,\Omega})$ is already surjective on $N_{\Psi_r(y)}$. With this understood, an appeal to the implicit function theorem finds a neighborhood of $\Psi_r^{-1}(\mathcal{M}^{(r)}_{\Gamma,\Omega})$ in Z_0 which has the form $(y, x(y))$, where $x(y) \in N_{\Psi_r(y)}$ satisfies the equation

$$(1 - \Pi_{\Psi_r(y)}) \cdot \mathbb{H}(\Psi_r(y) + x(y)) = 0.$$

Then, by shrinking $\mathcal{O}_h$ and $\underline{O}$ if necessary, the assertion about Z_0 can be arranged.

Finally, consider the orientation question. In this regard, remark that Z contains the product $[0,1] \times \Psi_r^{-1}(\mathcal{M}_{\Gamma,\Omega}^{(r)})$ as a subspace. Then, the tangent space to Z at $(t,(y,0))$ in the latter subspace is naturally isomorphic to $\mathbb{R} \times TZ_t|_y$. Now, with y fixed, the assignment to t of $TZ_t|_y$ defines a vector bundle over $[0,1]$ whose fiber at $t=1$ or $t=0$ is naturally isomorphic to $TY_h|_y$. Thus, TZ is oriented along $[0,1] \times \Psi_r^{-1}(\mathcal{M}_{\Gamma,\Omega}^{(r)})$, and the implicit function theorem implies that by shrinking $\mathcal{O}_h$ and $\underline{O}$ if necessary, one can obtain a consistent orientation for TZ with the asserted properties.

With Z understood as above, define the map

$$H : Z \to \times_k \Lambda_k$$

by assigning to each triple $(t,(y,x))$ the element $\underline{\Phi}'^{-1}_{(t,y)} \Pi_{(t,\Psi_r(y))} \mathbb{H}(\Psi_r(y)+x)$.

The identification of Z_0 as the set of $(y, x(y))$ where $(1-\Pi_{\Psi_r(y)}) \cdot \mathbb{H}(\Psi_r(y)+x(y)) = 0$ identifies the map H on Z_0 with $\Phi^{-1} \cdot \Psi_r^* w$. Meanwhile, the identification of Z_1 with the zero section in $\underline{O}$ identifies H on Z_1 with ψ_r.

Finally, note that if H sends $(y,x) \in Z_t$ to zero, then $\mathbb{H}(\Psi_r(y)+x) = 0$ and by definition, $\Psi_r(y)+x \in \mathcal{M}^{(r)}$. However, since $\mathcal{M}^{(r)} \subset \Psi_r(\times_k \mathcal{K}_\Lambda^{(k)})$ and x is a section of the normal bundle to the submanifold $\Psi_r(\times_k \mathcal{K}_\Lambda^{(k)})$, one can assume (again, by shrinking both $\mathcal{O}_h$ and $\underline{O}$ if necessary) that $H((t,(y,x)) = 0$ if and only if $x = 0$ and $y \in \Psi_r^{-1}(\mathcal{M}_{\Gamma,\Omega}^{(r)})$.

g) The Euler number for ψ_r and the weight for h

The validity of the final point in (2.29) when r is large follows from the fourth assertion of Proposition 5.2 of [28] using Proposition 2.7 to insure compactness.

References

[1] N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations of the second order*, J. Math. Pure Appl. **36** (1957) 235–249.

[2] M. F. Atiyah, *K-Theory*, Benjamin, New York, 1967.

[3] R. Fintushel & R. Stern, *Immersed spheres in 4-manifolds and the immersed Thom conjecture*, Turkish J. Math. **19** (1995) 145–157.

[4] M. Freed & K. K. Uhlenbeck, *Instantons and four-manifolds*, Springer, New York, 1984.

[5] M. Gromov, *Pseudo-holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985) 307–347.

[6] A. Jaffe & C. H. Taubes, *Vortices and monopoles*, Birkhäuser, Boston 1980.

[7] T. Kato, *Perturbation theory for linear operators*, Springer, Berlin, 1976.

[8] D. Kotshick, P. B. Kronheimer & T. S. Mrowka, in preparation.

[9] P. B. Kronheimer & T. S. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Letters **1** (1994) 797–808.

[10] D. Kotschick, *The Seiberg-Witten invariants of symplectic 4-manifolds (after C.H. Taubes)*, Seminaire Bourbaki, 48eme annee (1995-96) No. 812.

[11] T. Li & A. Liu, *General wall crossing formula*, Math. Res. Letters **2** (1995) 797–810.

[12] D. McDuff, *Lectures on Gromov invariants for symplectic 4-manifolds*, to appear in the Proc. NATO Summer School, Montreal, 1995.

[13] ———, *Singularities and positivity of intersections of J-holomorpic curves, with Appendix by Gang Liu*, Proc. CIMPA Summer School on Symplectic Topology, Nice 1992, Birkhäuser, Basel, 1994.

[14] D. McDuff & D. Salamon, *J-Holomorphic curves and quantum cohomology*, Amer. Math. Soc., Providence, 1996.

[15] J. W. Morgan, *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, Math. Notes **44** Princeton Univ. Press, 1996.

[16] J. Morgan, Z. Szabo & C. H. Taubes, *A product formula for the Seiberg-Witten invariants and the generalized Thom conjecture*, J. Differential Geom. **44** (1996) 706–788.

[17] C. B. Morrey, *Multiple integrals in the calculus of variations*, Springer, Berlin 1966.

[18] H. Ohta & K. Ono, *Note on symplectic 4-manifolds with* $b^{2+} = 1$. *II*, Preprint, 1995.

[19] T. H. Parker & J. G. Wolfson, *Pseudo-holomorphic maps and bubble trees*, J. Geom. Anal. **3** (1993) 63–98.

[20] Y. Ruan, *Symplectic topology and complex surfaces*, Geometry and Topology on Complex Manifolds, (T. Mabuchi, J. Noguchi and T. Ochial, eds.), World Scientific Publ., Singapore, 1994.

[21] N. Seiberg & E. Witten, *Electro-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, Nucl. Phys. **B426** (1994) 19–52.

[22] ———, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric Yang-Mills theory*, Nucl. Phys. **B426** (1994) 581–640.

[23] S. Smale, *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1968) 861–866.

[24] J. Sacks & K. K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. of Math. **113** (1981) 1–24.

[25] C. H. Taubes, *The Seiberg-Witten and the Gromov invariants*, Math. Res. Letters **2** (1995) 221–238.

[26] ———, *Counting pseudo-holomorphic submanifolds in dimension 4*, J. Differential Geom. **44** (1996) 818–893.

[27] ———, *SW => Gr : From the Seiberg-Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. **9** (1996) 845–918.

[28] ———, *Gr => SW. From pseudo-holomorphic curves to Seiberg-Witten solutions*, J. Differential Geom. **51** (1999) 203–334 and reprinted in Proc. First IP Lecture Series, Vol 2, Internat. Press, to appear

[29] ———, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Letters **1** (1994) 809–822.

[30] ———, *Arbitrary N-vortex solutions to the first order Ginzburg-Landau equations*, Commun. Math. Phys. **72** (1980) 277–292.

[31] E. Witten, *Monopoles and 4-manifolds*, Math. Res. Letters **1** (1994) 769–796.

[32] ———, *Two dimensional gravity and intersection theory on moduli space*, Surveys in Differential Geom. **1** (1991) 243–310.

[33] R. Ye, *Gromov's compactness theorem for pseudo-holomorphic curves*, Trans. Amer. Math. Soc. **342** (1994) 671–694.

Harvard University