ASSIGNMENTS AND ABSTRACT MOMENT MAPS

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Abstract

Abstract moment maps arise as a generalization of genuine moment maps on symplectic manifolds when the symplectic structure is discarded, but the relation between the mapping and the action is kept. Particular examples of abstract moment maps had been used in Hamiltonian mechanics for some time, but the abstract notion originated in the study of cobordisms of Hamiltonian group actions.

In this paper we answer the question of existence of a (proper) abstract moment map for a torus action and give a necessary and sufficient condition for an abstract moment map to be associated with a pre-symplectic form. This is done by using the notion of an assignment, which is a combinatorial counterpart of an abstract moment map.

Finally, we show that the space of assignments fits as the zeroth cohomology in a series of certain cohomology spaces associated with a torus action on a manifold. We study the resulting "assignment cohomology" theory.

1. Introduction

Abstract moment maps arise as a generalization of genuine moment maps on symplectic manifolds. The essence of their definition is that the symplectic structure is discarded, but the relation between the mapping and the action is kept intact. To be precise, an abstract moment map on a G-manifold M is an equivariant mapping $\Psi\colon M \to \mathfrak{g}^*$ satisfying the following constancy condition: for every Lie algebra element $\xi \in \mathfrak{g}$, the component $\langle \Psi, \xi \rangle$ is locally constant on the set $\{\xi_M = 0\}$ where the action generating vector field ξ_M vanishes.

Moment maps on symplectic manifolds are among examples of abstract moment maps. In general, however, an abstract moment map is not associated with a symplectic form or even a closed two-form. An abstract moment map is an additional structure on a G-manifold, and a given G-manifold admits many abstract moment maps. Thus G-manifolds equipped with abstract moment maps occupy an intermediate place between pure G-manifolds and symplectic manifolds with Hamiltonian G-actions.

The goal of this paper is to find a relationship between G-manifolds, G-manifolds equipped with abstract moment maps, and Hamiltonian G-manifolds. This is done by using a new notion, the notion of an *assignment*, comprising certain combinatorial data extracted from an abstract moment map. An assignment should be thought of as a combinatorial counterpart of an abstract moment map. For a torus action, an assignment is a function from the set of orbit type strata to the dual spaces to the Lie algebras of the stabilizers. (Thus a manifold with a finite orbit type stratification has only a finite–dimensional space of assignments.)

First we address the existence and uniqueness question for abstract moment maps with a fixed assignment. We prove that, for a torus action, every assignment is associated with an abstract moment map. Furthermore, two abstract moment maps give rise to the same assignment if and only if they differ, roughly speaking, by a Hamiltonian moment map arising from an exact two-form.

For some problems concerning non-compact manifolds, it is important to consider abstract moment maps which are proper. (Noncompact G-manifolds with proper abstract moment maps share some of the appealing properties of compact G-manifolds. See [23], [14], [13].) We show that, for a given assignment, an abstract moment map can be chosen proper if the assignment is proper or, to be more precise, "polarized"; see Section 3.3.

Then we use the notion of an assignment to answer the question whether a given abstract map is associated with a two-form.

Moment maps associated with true symplectic forms must additionally satisfy some non-degeneracy requirements. These requirements are analyzed in [13]. On the other hand, moment maps on Poisson manifolds (see, e.g., [8]) are not in general abstract moment maps because they need not be locally constant on the fixed point set.

This paper is organized as follows. In Section 2 we recall the definition of abstract moment maps and illustrate it by a number of examples.

In Section 3 we give a necessary and sufficient condition (in terms of assignments) for a G-manifold to admit a (polarized) abstract moment map. In Section 4 we show that two abstract moment maps with the same assignment differ by one which is associated with an exact two-form (or, to be more precise, with a one-form). We call such abstract moment maps *exact*. Section 5 is devoted to the question of which abstract moment maps are Hamiltonian, i.e., associated with closed two-forms. We show that every abstract moment map is locally Hamiltonian. Globally, there is an obstruction, which is stated again in terms of assignments. In Section 6 we prove a technical theorem on which the results of Sections 4 and 5 heavily rely: an abstract moment map on a linear representation is exact if and only if it vanishes at the origin. Finally, we show that the space of assignments and some of its generalizations fit as the zeroth cohomology in a series of certain cohomology spaces associated with a G-manifold. This cohomology is introduced and studied in Section 7.

Abstract moment maps were introduced in [23] to study geometric equivariant G-cobordisms and to state and prove the cobordism linearization theorem. This theorem, whose earliest version was given in [12] (see also [19] for important related work), asserts that under certain hypotheses a manifold with a torus action is equivariantly cobordant to the disjoint union of the linear isotropy representations at the fixed points.[1]

One of the main conceptual difficulties in the formulation of this theorem is to find a notion of non-compact cobordism which would not render every compact manifold cobordant to the empty set. (This is the central problem arising when non-compact manifolds are introduced in a cobordism theory: all compact manifolds may become cobordant to each other and so to zero.) The notion of an abstract moment map provides a solution to this problem: two G-manifolds equipped with proper abstract moment maps are cobordant if there exists a G-cobordism between them and a proper abstract moment map on it extending those on the boundary. This definition leads to a non-trivial cobordism theory in which the theory of compact (geometric) G-cobordisms is embedded.[2] The non-compact theory appears to be in some sense simpler

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[1] A number of applications of the linearization theorem are outlined in [12] and [23]. See [32] and [33] for some new applications, and see [13] and [14] for further developments.

[2] Strictly speaking, this is true for geometric stable complex G-cobordisms. See [14] and references therein.

than the compact one. The reason is that the non-compact theory has a well-understood set of generators and probably fewer relations than the compact one. (See [23] and [13].)

We feel, however, that, as some examples in Sections 2 and 7 indicate, abstract moment maps and assignments may have uses beyond those connected with geometric equivariant G-cobordisms.

Notation and conventions.   Throughout this paper, M is a G-manifold (C^∞-smooth), with G being a torus, except in some rare cases where G is allowed to be a more general compact Lie group. As usual, $\mathfrak{g}$ denotes the Lie algebra of G, and ξ_M is the vector field induced by the action of $\xi \in \mathfrak{g}$ on M. The stabilizer of $x \in M$ is denoted G_x, and the fixed point set of G on M by M^G. All ordinary and equivariant cohomology groups are assumed to have real coefficients unless specified otherwise.

We consider only abstract moment maps for torus actions. Many (but not all) of our results should extend to proper actions of other Lie groups.

Acknowledgment

2. Abstract moment maps

Let us first recall some standard facts about smooth group actions. Let a Lie group G act smoothly on a manifold M. Assume that G is compact or, more generally, that the action is proper (that is, that the map $(a, m) \mapsto (a \cdot m, m)$ from $G \times M$ to $M \times M$ is proper). Each Lie algebra element $\xi \in \mathfrak{g}$ gives rise to a vector field ξ_M on M. The *stabilizer* of a point $m \in M$ is the group $\{a \in G \mid a \cdot m = m\}$. The Lie algebra of this group is equal to $\{\xi \in \mathfrak{g} \mid \xi_M|_m = 0\}$ and is called the *infinitesimal stabilizer* of m. For each subgroup $H \subseteq G$, the connected components of the set of points whose stabilizer is conjugate to H (hence equal to H, if G is Abelian) are smooth sub-manifolds of M. These connected components are the *orbit type strata* of M. They are partially ordered: $X \preceq Y$ if and only if the stratum X is contained in the closure of the stratum Y. Similarly, the connected components of the sets of points

whose infinite stabilizers are conjugate to given sub-algebras $\mathfrak{h} \subseteq \mathfrak{g}$ (hence equal to $\mathfrak{h}$, if G is Abelian) form the *infinitesimal orbit type strata* of M. These are, too, partially ordered by inclusions of closures of strata. The infinitesimal orbit type stratification is more coarse than the orbit type stratification, because two points x and y can have different stabilizers but the same infinitesimal stabilizer. In what follows we will mainly work with the infinitesimal orbit type stratification because this stratification is more suitable for the goals of this paper than the orbit type stratification.

Next, let us recall the definition of abstract moment maps, as given in [23]. For a map $\Psi \colon M \to \mathfrak{g}^*$, we denote by $\Psi^{\mathfrak{h}}$ or Ψ^H the composition of Ψ with the natural projection $\mathfrak{g}^* \to \mathfrak{h}^*$, where $\mathfrak{h}$ is the Lie algebra of H. Similarly, for any Lie algebra element $\xi \in \mathfrak{g}$, we denote by $\Psi^\xi \colon M \to \mathbb{R}$ the ξth component of Ψ, i.e., $\Psi^\xi = \langle \Psi, \xi \rangle$.

Definition 2.1. An *abstract moment map* on M is a smooth map $\Psi \colon M \to \mathfrak{g}^*$ with the following properties:

1. Ψ is G-equivariant, and

2. for any subgroup H of G, the map $\Psi^H \colon M \to \mathfrak{h}^*$ is locally constant on the submanifold M^H of points fixed by H.

Remark 2.2. For the second requirement to hold, it is enough to assume that for any Lie algebra element $\xi \in \mathfrak{g}$, the function Ψ^ξ is locally constant on the set of zeros of the corresponding vector field ξ_M. If G is compact, it is enough to demand the requirement for circle subgroups of G.

In this paper we mainly consider the case where G is a torus and we often focus on abstract moment maps that are proper.

Example 2.3. The constant function zero is an abstract moment map. It is proper if and only if M is compact.

Example 2.4. If the fixed point set M^G has a non-compact component, M does not admit a proper abstract moment map.

Example 2.5. Let G be the circle group, and let us identify $\mathfrak{g}^*$ with $\mathbb{R}$. Then an abstract moment map is a real valued invariant function that is constant on each connected component of the fixed point set. In particular, if the set of fixed points is discrete, any invariant function is an abstract moment map.

Example 2.6. Recall that a Hamiltonian G-manifold is a triple (M, ω, Ψ), where M is a G-manifold, ω is a closed invariant two-form (which in some contexts – not here – is required to be symplectic), and Ψ is a moment map, i.e., a G-equivariant function $\Psi \colon M \to \mathfrak{g}^*$, such that Hamilton's equation,

$$(1) \qquad \iota(\xi_M)\omega = -d\Psi^\xi \quad \text{for all } \xi \in \mathfrak{g},$$

holds. Then Ψ is an abstract moment map.

If equation (1) holds, we say that ω is compatible with Ψ, or that Ψ is associated with ω. An abstract moment map associated with some two-form will be called a *Hamiltonian* moment map.

Example 2.7. Let a Lie group G act on a manifold M and let μ be any invariant one-form. Then the function $\Psi \colon M \to \mathfrak{g}^*$ defined by

$$(2) \qquad \Psi^\xi = \mu(\xi_M)$$

is an abstract moment map. Moreover, for each $H \subset G$, the function Ψ^H vanishes on M^H.

An abstract moment map that arises from Equation (2) is called *exact*. A compatible two-form is then given by $\omega = d\mu$. Many "classical" moment maps, e.g., the canonical moment map on a cotangent bundle, are exact. Also, in the pre-quantization of a Hamiltonian action, the pullback of the moment map to the pre-quantum circle bundle is an exact abstract moment map.

The advantage of working with exact moment maps over Hamiltonian ones is that if Ψ_0 and Ψ_1 are exact moment maps, then so is $(1 - \rho)\Psi_0 + \rho\Psi_1$ for any smooth function ρ. Indeed, if Ψ_0 and Ψ_1 arise from one-forms μ_0 and μ_1, then $(1-\rho)\Psi_0 + \rho\Psi_1$ arises from the one-form $(1 - \rho)\mu_0 + \rho\mu_1$.

Remark 2.8. Recall that for any Lie group G an equivariant differential two-form on a G-manifold M is a formal sum $\omega + \Psi$, where ω is an invariant two-form on M, and Ψ is a smooth equivariant function from M to $\mathfrak{g}^*$. This equivariant form is said to be equivariantly closed if and only if it satisfies (1); it is said to be equivariantly exact if and only if there exists an invariant one-form μ such that $\omega = d\mu$ and $\Psi^\xi = \mu(\xi_M)$ for all $\xi \in \mathfrak{g}$. The second equivariant cohomology, denoted $H^2_G(M)$, is the quotient of the space of equivariantly closed equivariant two-forms by the subspace of those that are equivariantly exact.

Example 2.9. The pull-back of an abstract moment map is an abstract moment map. More precisely, let $f\colon N \to M$ be an equivariant map of G-manifolds and let Ψ be an abstract moment map on M. Then $f^*\Psi = \Psi \circ f$ is an abstract moment map on N; the map $f^*\Psi$ is proper, provided that f and Ψ are proper.

For instance, following [40] and [29], consider a G-manifold Q and denote by $J\colon T^*Q \to \mathfrak{g}^*$ the canonical moment map: $J^\xi(\mu) = \mu(\xi_Q(x))$, where $\mu \in T_x^*Q$. The action map $F\colon Q \times \mathfrak{g} \to TQ$ is defined as $F(x,\xi) = \xi_Q(x)$. Consider a Lagrangian on Q with Legendre transformation $\mathcal{L}\colon TQ \to T^*Q$. Then

$$\mathbb{I} = (\mathcal{L}F)^* J\colon Q \times \mathfrak{g} \xrightarrow{F} TQ \xrightarrow{\mathcal{L}} T^*Q \xrightarrow{J} \mathfrak{g}^*$$

is an exact abstract moment map. For example, assume that $\mathcal{L}$ arises from a Riemannian metric $\langle\ ,\ \rangle$ on Q so that $\mathcal{L}(v) = \langle v, \cdot \rangle$ for a tangent vector v. Then $\mathbb{I}^\xi(x,\xi) = \langle \xi_Q(x), \zeta_Q(x) \rangle$. The map $\mathbb{I}$, called the locked inertia tensor, is used in the analysis of relative equilibria. (See [40] and [29].) Note that in general $Q \times \mathfrak{g}$ is not a symplectic manifold and G, in this example, does not have to be commutative.

Remark 2.10. In view of Example 2.6, it is worth pointing out that a moment map on a Poisson manifold (see, e.g., [8]) may not be an abstract moment map even when the Poisson structure is preserved by the action. The reason is that, since a moment map is defined only up to addition of Casimir functions, and since on a Poisson manifold Casimir functions often exist in abundance, a moment map on a Poisson manifold may not be constant on the fixed point set.

3. Existence of abstract moment maps

Every manifold with a G-action admits an abstract moment map: the zero map. This map is never proper unless the manifold is compact. In this section we answer the question of when a G-manifold admits a proper (in fact, polarized, see Definition 3.19) abstract moment map.

A necessary condition for an action to admit a proper abstract moment map Ψ is that each component of the fixed point set be compact. (Recall that Ψ is constant on each such component.) Is this condition sufficient? Moreover, does a (proper) abstract moment map exist with prescribed values at the fixed points?

3.1. Existence of abstract moment maps for circle actions.

Answers to the above questions take a particularly simple and attractive form when G is a circle, when abstract moment maps are simply G-invariant functions that are constant on the connected components of the fixed point set.

Theorem 3.1. *Let G be a circle acting on M, and let $\psi\colon M^G \to \mathbb{R}$ be a locally constant function.*

1. *There exists an abstract moment map $\Psi\colon M \to \mathbb{R}$ with $\Psi|_{M^G} = \psi$.*

2. *Assume that ψ is proper and bounded from below. Then Ψ can be chosen to be proper and bounded from below.*

Remark 3.2. In other words, if G is the circle group, we can prescribe the values of an abstract moment map on the connected components of M^G completely arbitrarily. If M^G is compact, the condition of the second assertion is satisfied automatically, and every locally constant function on M^G extends to a proper abstract moment map.

Proof. The theorem follows from the following two facts, applied to $X = M^G$ and $f = \psi$.

1. Let $X \subset M$ be a closed sub-manifold and $f\colon X \to \mathbb{R}$ a smooth function. Then there exists a smooth function $F\colon M \to \mathbb{R}$ such that $F|_X = f$. Moreover, if f is bounded from below, F can be chosen to be bounded from below too, and if f is proper and bounded from below, F can be chosen to be proper and bounded from below.

2. Let $F\colon M \to \mathbb{R}$ be proper and bounded from below. Then the average $\overline{F}$ of F by a compact group action is also proper and bounded from below.

Let us prove the first fact. Fix a tubular neighborhood U of X in M, and let $\pi\colon U \to X$ be a smooth projection which extends to a proper map from the closure $\overline{U}$ to X. Let ρ, $1-\rho$ be a smooth partition of unity subordinate to the covering of M by the two open sets U and $M \smallsetminus X$. Pick a smooth function $\varphi\colon M \to \mathbb{R}$ which is proper and bounded from below (see, e..g., [16], Chapter 1, Section 8). Then $F = \rho\pi^*f + (1-\rho)\varphi$ has the desired properties.

To prove the second fact, notice that $\overline{F}^{-1}([-a,a])$ is contained in $G \cdot F^{-1}([-a,a])$, which is the image of the compact set $G \times F^{-1}([-a,a])$ under the continuous action mapping $G \times M \to M$.   q.e.d.

Remark 3.3. In Theorem 3.1 it is not true that if ψ is just proper (but not bounded), then Ψ can be chosen to be proper. In general, a proper map on a closed submanifold X of M might not extend to a proper map on M. For instance, the function $f(0, y) = y$ on the y-axis does not extend to a continuous proper function F from $\mathbb{R}^2$ to $\mathbb{R}$. A similar counterexample involving abstract moment maps is given below.

Example 3.4. Let M be obtained by the following plumbing construction:

$$M = \mathbb{Z} \times S^2 \times D^2 / \sim,$$

where

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$
$$D^2 = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < \epsilon^2\},$$

and

$$(n, x, y, \sqrt{1 - x^2 - y^2}, u, v) \sim (n+1, u, v, -\sqrt{1 - u^2 - v^2}, x, y)$$

for all n. Take the diagonal circle action:

$$e^{i\theta} \cdot [n, x, y, z, u, v] = [n, x', y', z, u', v'],$$

where

$$\begin{bmatrix} x' & u' \\ y' & v' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x & u \\ y & v \end{bmatrix}.$$

The function

$$(3) \qquad\qquad \psi([n, 0, 0, 1, 0, 0]) = n$$

is a locally constant function on the fixed point set and is proper, but it does not extend to a proper function $\Psi \colon M \to \mathbb{R}$. (The function ψ extends to a (non-proper) Hamiltonian moment map, for a closed two-form whose pullback to each $\{n\} \times S^2 \times \{0\}$ is non-negative and has total area one.)

3.2. Assignments. Let us now investigate more closely the question of existence of an abstract moment map for an action of a torus whose dimension is greater than one. Theorem 3.1 is no longer true in this case:

Example 3.5. Let $\Psi = (\Psi_1, \Psi_2)$ be an abstract moment map on $M = S^2 \times S^2$ with $G = S^1 \times S^1$ acting by rotating each of the two factors. Then Ψ must send the four fixed points to the corners of a rectangle in $\mathbb{R}^2$ whose sides are parallel to the axes. Thus the values $\Psi(M^G)$ cannot be assigned arbitrarily.

Moreover, the abstract moment maps might not even separate the components of the fixed point set:

Example 3.6. Let S^4 be the unit sphere in $\mathbb{C} \times \mathbb{C} \times \mathbb{R}$, and let $G = S^1 \times S^1$ act on it by rotating each of the first two factors. There are two fixed points: the North Pole and the South Pole. The set of points fixed by the first S^1 is connected and contains both poles. The same is true for the set of points fixed by the second S^1. Consequently, any abstract moment map on S^4 must have the same value at the poles.

These examples stress the role of the orbit type strata other than the components of the fixed point set.

For a stratum X we denote by $\mathfrak{g}_X$ the infinitesimal stabilizer of any of its points.

Suppose that $\Psi \colon M \to \mathfrak{g}^*$ is an abstract moment map. Then for each infinitesimal orbit type stratum X in M, the map Ψ followed by the projection $\mathfrak{g}^* \to \mathfrak{g}_X^*$ gives a single element $A(X)$ of $\mathfrak{g}_X^*$. The existence question for an abstract moment map is equivalent to the existence question for such an assignment, $X \mapsto A(X)$. We make this precise in Theorems 3.18 and 3.21, which rely on the following definition and example.

Definition 3.7. An *assignment* is a function A that associates to each infinitesimal orbit type stratum X in M an element $A(X)$ of $\mathfrak{g}_X^*$ and that satisfies the following compatibility condition: if X is contained in the closure of Y, then $A(Y)$ is the image of $A(X)$ under the restriction map $\mathfrak{g}_X^* \to \mathfrak{g}_Y^*$. The linear space of all assignments on M is denoted by $\mathcal{A}(M)$.

In Section 7 we discuss assignments in a broader, more abstract, context.

Example 3.8. Let $\Psi \colon M \to \mathfrak{g}^*$ be an abstract moment map. Then $A(X) = \Psi^{\mathfrak{g}_X}(X)$ is an assignment. The assignment A and the moment map Ψ are said to be associated with each other. If the abstract moment map is exact, i.e., Ψ arises from a one-form μ so that $\Psi^\xi = \mu(\xi_M)$, the corresponding assignment is zero.

Example 3.9. When G is the circle group, an assignment simply associates a real number to each component of the fixed point set. Thus, in this case, $\mathcal{A}(M) = (\mathfrak{g}^*)^{\pi_0(M^G)}$.

Example 3.10. Consider the action of the two-dimensional torus $G = S^1 \times S^1$ on $M = \mathbb{CP}^2$ given by $(t_1, t_2)[z_0 : z_1 : z_2] = [z_0 : t_1 z_1 : t_2 z_2]$. This action has three fixed points, and every assignment A is uniquely determined by its value at the fixed points. There are, however, three relations between the values $A|_{M^G} \in (\mathfrak{g}^*)^3$, coming from the strata with one-dimensional stabilizers. As a result, $\mathcal{A}(M)$ is three-dimensional. Geometrically, the assignment values at the fixed points are the vertices of a triangle in $\mathbb{R}^2$ with two equal sides that are parallel to the coordinate axes.

Example 3.11. Let M_1 be a G_1-manifold and M_2 be a G_2-manifold. Then

$$\mathcal{A}(M_1 \times M_2) = \mathcal{A}(M_1) \oplus \mathcal{A}(M_2)$$

for the $G_1 \times G_2$-action on $M_1 \times M_2$.

Remark 3.12. Replacing in Definition 3.7 the infinitesimal orbit type stratification by the orbit type stratification notion leads to the same class of assignments $\mathcal{A}(M)$. Namely, a function that associates to each orbit type stratum X an element of $\mathfrak{g}_X^*$ and that satisfies the compatibility condition of Definition 3.7 is in fact constant on each infinitesimal orbit type stratum, and, hence, is an assignment.

Example 3.13. Let M be an n-dimensional complex manifold and let G be an n-dimensional torus that acts on M. Suppose that each point of M with stabilizer $H \subseteq G$ has a neighborhood which is biholomorphic to a neighborhood of the origin in $\mathbb{C}^n$ with an action of H of the following form. The H-action is obtained as the composition of an isomorphism $H \to (S^1)^{\dim H}$ with the $(S^1)^{\dim H}$-action on $\mathbb{C}^{\dim H} \times \mathbb{C}^{n-\dim H}$ which is standard on the first factor and trivial on the second. (For instance, this is the case if M is a toric manifold; see [11] or [4].) It follows that for any stratum X, the natural map

$$\mathfrak{g}_X^* \xrightarrow{\oplus_Y \pi_Y^X} \bigoplus_{\{Y \;|\; X \preceq Y, \; \dim \mathfrak{g}_Y = 1\}} \mathfrak{g}_Y^*$$

is a linear isomorphism. Therefore, a moment assignment is determined by its values on the strata Y with $\dim \mathfrak{g}_Y = 1$, and these values can be

prescribed arbitrarily. So for such M,

$$\mathcal{A}(M) = \bigoplus_Y \mathfrak{g}_Y^* \cong \mathbb{R}^{\#\{Y \ | \ \dim \mathfrak{g}_Y = 1\}}.$$

Example 3.14. Let M be a Kähler toric manifold (also see Example 3.13) with moment map $\Psi \colon M \to \mathfrak{g}^*$. The image $\Psi(M)$ is a convex polytope [1], [18]. A convex polytope is stratified, with the strata being its open faces of various dimensions. The orbit type strata in M are exactly the preimages in M of the open faces in $\Psi(M)$, [9]. As a consequence, the poset of (infinitesimal) orbit type strata of M is isomorphic to the poset of faces of $\Psi(M)$. Moreover, the stabilizers can be read from the faces and vice versa: the affine plane spanned by the face $\Psi(X)$ is a shift of the annihilator in $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}_X$. The shifts are exactly given by the assignment $A(X) = \Psi^{\mathfrak{g}_X}(X)$:

$$\text{affine span}(\Psi(X)) = \text{ preimage of } A(X) \text{ under } \mathfrak{g}^* \to \mathfrak{g}_X^* \ .$$

The polytope $\Psi(M)$, and hence the moment assignment A, determine the manifold, the G-action, and the symplectic form up to an equivariant symplectomorphism, [9], and the equivariant Kähler structure on the strata, [17].

Remark 3.15. In Example 3.14 we saw that the moment assignment of a symplectic toric manifold M determines its moment polytope $\Psi(M)$. Similarly, for a toric manifold with a closed invariant two-form which may have degeneracies, the moment assignment determines its *twisted polytope* in the sense of [24].

An obvious, but important, fact is

Lemma 3.16. *Let Ψ_0 and Ψ_1 be abstract moment maps which have the same assignment, A. Then $(1 - \rho)\Psi_0 + \rho\Psi_1$ is also an abstract moment map with assignment A, for any invariant smooth function ρ.*

Proof. For any $H \subseteq G$, on every component X of M^H, we have

$$(1 - \rho)\Psi_0^H + \rho\Psi_1^H = (1 - \rho)A(X) + \rho A(X) \equiv A(X).$$

q.e.d.

Remark 3.17. The definition of an assignment can be extended to non-commutative groups G. An assignment can then be defined as a function $x \mapsto A(x) \in \mathfrak{g}_x^*$ on M such that the following conditions hold:

- $A(g \cdot x) = Ad_g^* A(x)$ for all $x \in M$ and $g \in G$.

- $A^{\mathfrak{h}}$ is locally constant on the set $M^{\mathfrak{h}}$ of points x with $\mathfrak{h} \subseteq \mathfrak{g}_x$.

In the non-commutative case, as in Example 3.8, an abstract moment map gives rise to an assignment.

3.3 Existence of abstract moment maps for torus actions.
The relation between abstract moment maps and assignments is expressed in the following:

Theorem 3.18. *Let M be a manifold with an action of a torus G. Let $A\colon X \mapsto A(X)$ be an assignment. Then there exists an abstract moment map $\Psi\colon M \to \mathfrak{g}^*$ which is associated with A, i.e., such that $\Psi^{\mathfrak{g}_X}(X) = A(X)$ in $\mathfrak{g}_X^*$ for every orbit type stratum X.*

Proof. Let m be a point in M, let $\mathfrak{h}$ be the infinitesimal stabilizer of m, and let $A(m) \in \mathfrak{h}^*$ be the element assigned to the orbit type stratum containing m. Let $\Psi_m \in \mathfrak{g}^*$ be any element whose projection to $\mathfrak{h}^*$ is $A(m)$. Pick an open neighborhood U_m of the orbit $G \cdot m$ which equivariantly retracts to the orbit. The constant function Ψ_m is an abstract moment map on U_m whose assignment is $A|_{U_m}$. Choose an invariant partition of unity $\{\rho_j\}$ subordinate to the covering of M by the open subsets U_m, with the support of ρ_j contained in the open set U_{m_j}. The convex combination $\Psi = \sum \rho_j \Psi_{m_j}$ is an abstract moment map; this follows from Lemma 3.16, applied to open subsets of the manifold.   q.e.d.

On a non-compact manifold, it is sometimes required that an abstract moment map be proper. (See [23], [13], [14].) In fact, we often need a component of Ψ to be proper and bounded from below.

Definition 3.19. Let $\eta \in \mathfrak{g}$ be a Lie algebra element. A function $\Psi\colon M \to \mathfrak{g}^*$ is said to be η-*polarized* if its ηth component, $\Psi^\eta\colon M \to \mathbb{R}$, is proper and bounded from below.

Note that an η-polarized function is necessarily proper, because its η-component is proper. However, not every proper map is η-polarized for some η. If M is compact, Ψ is automatically η-polarized for all η.

Polarized abstract moment maps posses the following two properties, which may in generally fail for proper abstract moment maps:

1. A linear combination of η-polarized abstract moment maps on the same manifold is again an η-polarized (hence proper) abstract moment map.

2. Let $\Psi_j \colon M_j \to \mathfrak{g}^*$, $j = 1, 2$, be η-polarized abstract moment maps. Consider the product $M_1 \times M_2$ with the diagonal G-action; let π_1, π_2 be the projection maps to M_1 and M_2. Then $\Psi_1 \circ \pi_1 + \Psi_2 \circ \pi_2$ is an η-polarized (hence proper) abstract moment map.

Fix a G-manifold M and a vector $\eta \in \mathfrak{g}$. Denote the set of zeros of η_M by M^η. This is exactly the set of points whose infinitesimal stabilizer contains η. Therefore, the η-coordinate of any assignment is well defined on this set.

Definition 3.20. An assignment A is η-*polarized* if its η-component on M^η,

$$A^\eta \colon M^\eta \to \mathbb{R},$$

is proper and bounded from below.

Theorem 3.21. *Let M be a manifold with a G action. For every η-polarized assignment $X \mapsto A(X)$ on M there exists an η-polarized abstract moment map $\Psi \colon M \to \mathfrak{g}^*$ whose assignment is A.*

Corollary 3.22. *Assume that M^G is compact. Then every assignment extends to a proper abstract moment map. (Indeed, for a generic η, $M^\eta = M^G$ is compact. Hence any assignment is η-polarized.)*

As a consequence, if M^G is compact, there always exists a proper abstract moment map (e.g., one which extends the zero assignment).

Proof of Theorem 3.21. The function $A^\eta \colon M^\eta \to \mathbb{R}$ is well defined, proper, and bounded from below. Since M^η is closed, A^η extends to a function $\varphi \colon M \to \mathbb{R}$ that is proper and bounded from below. (See item 1 in the proof of Theorem 3.1.)

For each $m \in M$, let $\Psi_m \in \mathfrak{g}^*$ be an element whose projection to $\mathfrak{g}_m^*$ is $A(m)$. We choose $\Psi_m \in \mathfrak{g}^*$ to meet the following additional requirement: $\Psi_m^\eta = \langle \Psi_m, \eta \rangle = \varphi(m)$. If $m \in M^\eta$, this condition is automatically satisfied, and if $m \notin M^\eta$, this choice is possible because $\eta \notin \mathfrak{g}_m$.

Let U_m be a tubular neighborhood of the orbit through m which equivariantly retracts to the orbit and on which the function φ differs from the value $\varphi(m)$ by less than 1. Then the constant function Ψ_m is an abstract moment map on U_m with assignment A and whose η-component is bounded from below by $\varphi - 1$.

Choose an invariant partition of unity $\{\rho_j\}$ subordinate to the covering of M by the open subsets U_m, with the support of ρ_j contained in the open set U_{m_j}. Then the convex combination $\Psi = \sum \rho_j \Psi_{m_j}$ is an

abstract moment map; this follows from Lemma 3.16, applied to open subsets of the manifold. Moreover, since the η-component of each Ψ_m is bounded from below by $\varphi - 1$, the same holds for Ψ. Since $\Psi^\eta \geq \varphi - 1$, and $\varphi - 1$ is proper and bounded from below, Ψ^η is proper and bounded from below.   q.e.d.

In general, a G-manifold M may admit no proper abstract moment maps, even when every connected component of the fixed point set M^G is compact (and so a proper locally constant map $\psi \colon M^G \to \mathfrak{g}^*$ does exist). The obstruction lies in the compatibility condition; the manifold M might not admit a proper assignment. We will now construct an example of such a G-manifold.

Example 3.23. Let $G = S^1 \times S^1$ act on the four–dimensional sphere S^4 as in Example 3.6. Recall that the fixed points are the North and South Poles and that any abstract moment map on S^4 must take the same value at these points. Fix some small $\epsilon > 0$, and let D^4 be the ϵ-ball in $\mathbb{C} \times \mathbb{C}$ with the G-action that rotates each of the two factors. Take the trivial disk bundle over S^4,

$$
\begin{aligned}
N &= S^4 \times D^4 \\
&= \{(z, z', x, w, w') \mid |z|^2 + |z'|^2 + x^2 = 1 \text{ and } |w|^2 + |w'|^2 < \epsilon^2\} \\
&\subset \mathbb{C}^2 \times \mathbb{R} \times \mathbb{C}^2
\end{aligned}
$$

with the diagonal action of G. Since the neighborhood of each of the two fixed points in N is equivariantly diffeomorphic to $D^4 \times D^4$, we can plumb an infinite sequence of such N's. More explicitly, take $M = N \times \mathbb{Z}/\sim$ where the equivalence relation $\sim$ is

$$
\begin{aligned}
(4) \qquad & (z, z', \sqrt{1 - |z|^2 - |z'|^2}, w, w', n) \\
& \sim (w, w', -\sqrt{1 - |w|^2 - |w'|^2}, z, z', n + 1)
\end{aligned}
$$

for all $n \in \mathbb{Z}$, whenever both $|z|^2 + |z'|^2$ and $|w|^2 + |w'|^2$ are less than ϵ. Then M is a G-manifold. The gluing map (4) reverses the orientation; however, we can get an orientation on M by flipping the orientation of every other copy of N. An abstract moment map on M must take a constant value on the infinite sequence of fixed points; such a map cannot be proper.

3.4. Minimal stratum assignments. Theorem 3.18 can be understood as that assignments are combinatorial counterparts of abstract

moment maps. The amount of information needed to determine an assignment can be further reduced by taking a full advantage of the compatibility condition, as follows. Recall that the (infinitesimal) orbit type strata in M are partially ordered; $X \preceq Y$ if and only if X is contained in the closure of Y. The strata that are minimal under this ordering are exactly those that are closed subsets of M. The closure of any orbit type stratum in M is a smooth sub-manifold which contains a minimal stratum.

Every component of the fixed point set is a minimal stratum. However, there can exist minimal strata outside the fixed point set M^G. Whether or not the existence of such strata is related to an algebraic property called *formality*. Recall that a compact manifold M is formal if one of the following equivalent conditions (see, e.g., [7], [20], or [26]) is satisfied:

1. $H_G^*(M) = H^*(M) \otimes H^*(BG)$ as an $H^*(BG)$-module;

2. $H_G^*(M)$ has no $H^*(BG)$-torsion;

3. the restriction $j^*\colon H_G^*(M) \to H_G^*(M^G) = H^*(M^G) \otimes H^*(BG)$ is a mono-morphism.

The following is an interesting geometric consequence of formality.

Proposition 3.24. *On a compact formal manifold M, every minimal stratum is a connected component of M^G.*

Proof. Let X be a minimal stratum and let H be the connected component of identity of G_x for $x \in X$. Assume $H \neq G$. Then the equivariant Thom class τ of the normal bundle to X is a non-zero torsion element in $H^*(BG)$. In fact, τ is annihilated by the image of $H^*(B(G/H)) \to H^*(BG)$. Alternatively, $j^*\tau = 0$, because $X \cap M^G = \emptyset$. q.e.d.

For example, when M is compact symplectic with G acting Hamiltonianly, M is equivariantly perfect, and hence formal, (see [26]), and the above analysis applies. In this case, however, to show that a minimal stratum X consists of fixed points, it suffices to observe that X is a compact symplectic manifold and G acts Hamiltonianly on X, so G must have fixed points on X.

Definition 3.25. A *minimal stratum assignment* is an assignment of an element $A(X) \in \mathfrak{g}_X^*$ to each minimal stratum X, where $\mathfrak{g}_X$ is the infinitesimal stabilizer of $x \in X$, such that the following compatibility

condition is satisfied: *if two minimal strata X_1 and X_2 are such that $X_1 \preceq Y$ and $X_2 \preceq Y$ for some stratum Y, then the restrictions to $\mathfrak{g}_Y$ of $A(X_1)$ and $A(X_2)$ are the same: $A(X_1)^{\mathfrak{g}_Y} = A(X_2)^{\mathfrak{g}_Y}$.*

Notice that this condition holds automatically for the zero assignment.

The following theorem is an immediate consequence of the definitions and Theorem 3.18.

Theorem 3.26. *The restriction of any assignment to the minimal strata is a minimal stratum assignment. Conversely, any minimal stratum assignment extends to a unique assignment. Hence, every minimal stratum assignment is associated with an abstract moment map.*

Remark 3.27. It appears that in Theorem 3.26 the minimal stratum assignment cannot be replaced by a function defined only on the fixed point set. Namely, we expect there to exist a G-manifold M with isolated fixed points and a function $\psi \colon M^G \to \mathfrak{g}^*$ which does not extend to an assignment (hence does not extend to an abstract moment map), but which satisfies the following compatibility condition: if $x, y \in M^G$ belong to the same connected component of M^H, then $\psi^H(x) = \psi^H(y)$.

Question 3.28. In Remark 3.17 we proposed a definition of assignments for an action of a non-abelian Lie group. It appears to be an interesting and feasible problem to check whether or not the results of this section generalize to such actions.

4. Exact moment maps

We have already shown that the natural forgetful homomorphism from the space of abstract moment maps on a G-manifold M to the space $\mathcal{A}(M)$ of assignments on M is onto (Theorem 3.18). In this section we study the kernel of this epimorphism.

Recall that an abstract moment map Ψ is said to be exact if there exists a G-invariant one-form μ with $\Psi^\xi = \mu(\xi_M)$ for all $\xi \in \mathfrak{g}$ (see Example 2.7). The assignment associated with such a map is zero. The following result, which is proved later in this section, shows that the converse is also true.

Theorem 4.1. *An abstract moment map whose assignment is identically zero is exact. More explicitly, suppose that $\Psi \colon M \to \mathfrak{g}^*$ is an abstract moment map such that for each subgroup $H \subset G$, the function*

$\Psi^H \colon M \to \mathfrak{h}^*$ *vanishes on the H-fixed point set M^H. Then there exists an invariant one-form μ such that $\Psi^\xi = \mu(\xi_M)$ for all $\xi \in \mathfrak{g}$.*

Combining Theorems 3.18 and 4.1 we obtain

Corollary 4.2. *The sequence*

$$0 \to \left\{ \begin{array}{c} exact \\ moment \\ maps \end{array} \right\} \to \left\{ \begin{array}{c} abstract \\ moment \\ maps \end{array} \right\} \to \mathcal{A}(M) \to 0$$

is exact.

The proof of Theorem 4.1 relies on the following key result, which we will prove in Section 6:

Theorem 4.3. *Let G be a torus acting linearly on $\mathbb{R}^m$, and let Ψ be an abstract moment map on a neighborhood of the origin, vanishing at the origin. Then there exists a G-invariant one-form μ on a neighborhood of the origin such that $\mu(\xi_M) = \Psi^\xi$ for all $\xi \in \mathfrak{g}$.*

We will also need a parametric version of this theorem:

Corollary 4.4. *Let G be a torus acting linearly on the fibers of a vector bundle $\mathcal{V} \to Y$, and let Ψ be an abstract moment map on a neighborhood of the zero section, vanishing on the zero section. Then there exists a smooth family μ of G-invariant one-forms on the fibers of $\mathcal{V}$, such that $\mu(\xi_M) = \Psi^\xi$ near the zero section.*

Proof of Corollary 4.4. By using a partition of unity on Y, the corollary can be reduced to the case where Y is a linear space and $\mathcal{V} = \mathbb{R}^m \times Y$. This case follows immediately from Theorem 4.3 when $\mathbb{R}^m$ is replaced by $\mathbb{R}^m \times Y$ with the trivial G-action on the second factor.

q.e.d.

Assuming Theorem 4.3, let us prove a preliminary result, which is a local version of Theorem 4.1 that will be used in the next section, and deduce Theorem 4.1 from it.

Proposition 4.5. *Let G be a torus acting on a manifold M and let $\Psi \colon M \to \mathfrak{g}^*$ be an abstract moment map. Let p be a point in M and $H = G_p$ its stabilizer. Suppose that $\Psi^H(p) = 0$. Then there exist an open G-invariant neighborhood V of p in M and a G-invariant one-form μ on V such that*

$$(5) \qquad\qquad \mu(\xi_M) = \Psi^\xi$$

on V for all $\xi \in \mathfrak{g}$.

Proof. Let us first examine the case where the action is locally free near p. Fix a basis $\xi_1, \ldots, \xi_n$ in $\mathfrak{g}$. Then the vector fields $(\xi_i)_M$ form a basis in the tangent space to the orbit at every point of a G-invariant neighborhood V of the orbit through p. By setting

$$(6) \qquad\qquad \mu((\xi_i)_M) = \Psi^{\xi_i},$$

we thus obtain a form defined along the orbits in V. We extend it to a differential form μ on V by taking its composition with an orthogonal projection to the orbit with respect to a G-invariant metric. It is easy to see that μ satisfies the condition $\mu(\xi_M) = \Psi^{\xi}$.

Let us now prove the proposition in the general case. Pick a closed subgroup $K \subset G$ whose Lie algebra $\mathfrak{k}$ is complementary to $\mathfrak{h}$ in $\mathfrak{g}$. A small G-invariant neighborhood V of the orbit Y through p can be identified, by the slice theorem, with a neighborhood of the zero section in the normal bundle $\pi \colon \mathcal{V} \to Y$ to Y in M, with the action induced by that on M. We can apply Corollary 4.4 to the linear H-action on the fibers of $\mathcal{V}$, equipped with the abstract moment map Ψ^H induced from M. As a result, we get a smooth family μ of one-forms on the fibers of $\mathcal{V}$, such that $\mu(\xi_M) = \Psi^{\xi}$ for all $\xi \in \mathfrak{h}$. The K-orbits form a foliation which is transverse to the fibration π. We extend μ to a one-form on a whole neighborhood of Y by making μ vanish on the vectors tangent to the K orbits. The resulting form is a G-invariant form μ_H on V so that $\mu_H(\xi_M) = \Psi^{\xi}$ for all $\xi \in \mathfrak{h}$, and $\mu_H(\xi_M) = 0$ for all $\xi \in \mathfrak{k}$.

The K-action on V is locally free. Let μ_K be the form defined as above by (6) and extended to V so that it vanishes on the vectors tangent to the fibers of π. Then $\mu_K(\xi_M) = \Psi^{\xi}$ for all $\xi \in \mathfrak{k}$. Since the vector fields ξ_M for $\xi \in \mathfrak{h}$ are tangent to the fibers of π, we also have $\mu_K(\xi_M) = 0$ for all $\xi \in \mathfrak{h}$. The form $\mu = \mu_H + \mu_K$ has the desired property, that $\mu(\xi_M) = \Psi^{\xi}$ for all $\xi \in \mathfrak{g}$.   q.e.d.

Proof of Theorem 4.1. By Proposition 4.5, there exists an open covering of M by invariant sets U_α, and on each U_α there exists an invariant one-form μ_α such that $\Psi^{\xi} = \mu_\alpha(\xi_M)$ for all $\xi \in \mathfrak{g}$. Let ρ_j be a partition of unity subordinate to this covering, with ρ_j supported in U_{α_j} for each j. Define $\mu = \sum \rho_j \mu_{\alpha_j}$. Then $\Psi^{\xi} = \mu(\xi_M)$ on M for all $\xi \in \mathfrak{g}$.   q.e.d.

5. Hamiltonian moment maps

We have already seen (Example 2.6) that every moment map which is associated with a closed invariant two-form is an abstract moment map. We will examine now the question of which abstract moment maps arise in this way. Recall that such abstract moment maps are called Hamiltonian. Thus we fix an abstract moment map Ψ, and we look for a closed two-form ω with $\iota(\xi_M)\omega = -d\Psi^\xi$. Note that such an ω would necessarily be G-invariant.

Our first observation is an immediate consequence of the fact that every exact moment map, $\Psi^\xi = \mu(\xi_M)$, is automatically Hamiltonian with $\omega = d\mu$. Thus Theorem 4.1 implies

Corollary 5.1. *Let* $\Psi\colon M \to \mathfrak{g}^*$ *be an abstract moment map with zero assignment. Then* Ψ *is associated with an exact two-form. In particular,* Ψ *is Hamiltonian.*

5.1. Local existence of two-forms.
Our next result shows that there are no local obstructions to the existence of ω, if G is abelian. Quite surprisingly, a similar local existence result fails to hold for non-abelian compact groups, [5].

Corollary 5.2 (Local existence of two-forms). *Let* G *be a torus acting on a manifold* M, *and let* $\Psi\colon M \to \mathfrak{g}^*$ *be an abstract moment map. For every* $p \in M$, Ψ *is associated with an exact two-form* ω *on some open* G-*invariant neighborhood* V *of* p. *In particular,* Ψ *is Hamiltonian on a neighborhood of* p.

Proof. Consider the new abstract moment map $\Psi - \Psi(p)$. By Proposition 4.5, there exist an invariant neighborhood V of p in M and a G-invariant one-form μ on V such that $\mu(\xi_M) = \Psi^\xi - \Psi^\xi(p)$ on V. Let $\omega = d\mu$; then $\iota(\xi_M)\omega = -d\Psi^\xi$ on V.   q.e.d.

The following semi-local result is also of interest.

Corollary 5.3. *On a manifold with a unique minimal stratum,* X, *every abstract moment map is Hamiltonian.*

Proof. Pick an element $\gamma \in \mathfrak{g}^*$ whose restriction to $\mathfrak{g}_X$ is equal to $\Psi^{\mathfrak{g}_X}(X)$, and apply Theorem 4.1 to the abstract moment map $\Psi - \gamma$.
$$\text{q.e.d.}$$

Remark 5.4. It is well known that moment maps Ψ associated with symplectic forms satisfy a certain non-degeneracy condition. For

example, for circle actions the Hessian $d^2\Psi$ must be non-degenerate on the normal bundle to the fixed point set. In [13], we state explicitly a necessary and sufficient condition for Ψ to be locally associated with a symplectic form. Furthermore, we prove that abstract moment maps satisfying this non-degeneracy condition globally have many properties of moment maps on symplectic manifolds. These include the convexity theorem ([1] and [18]) and formality ([26], see also Section 3.4 above).

5.2. Global existence of two-forms.   Let us now turn to the problem of global existence for ω. The following example shows that not every abstract moment map is Hamiltonian.

Example 5.5. Let S^1 act on $\mathbb{CP}^2$ by

$$\lambda \cdot [z_0 : z_1 : z_2] = [z_0 : \lambda z_1 : \lambda^2 z_2].$$

There are three fixed points: $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$. Denote by a, b, c their respective images by an abstract moment map. If the abstract moment map is associated with a closed two form, ω, then it is an easy consequence of Stokes's theorem that the differences, $b - a$ and $c - b$ are, respectively, equal (up to a factor) to the integrals of ω on the 2-spheres $[* : * : 0]$ and $[0 : * : *]$ in $\mathbb{CP}^2$. Since these lie in the same cohomology class, the values a, b, c must then be equidistant: $a - b = b - c$. However, an abstract moment map can take arbitrary values a, b, c at the three fixed points, by Theorem 3.1.

Recall that the equivariant cohomology classes in $H^2_G(M)$ are represented by the sums $\omega + \Psi$ where Ψ is a Hamiltonian moment map and ω is a compatible two-form; see Example 2.6. The forgetful mapping which sends $\omega + \Psi$ to the assignment corresponding to Ψ gives rise to a homomorphism

$$(7) \qquad\qquad \rho \colon H^2_G(M) \to \mathcal{A}(M).$$

Theorem 5.6. *An abstract moment map is Hamiltonian if and only if its assignment is in the image of* (7).

Proof. Let Ψ be an abstract moment map and A its assignment. It is clear by definition that $A \in \operatorname{im} \rho$ if Ψ is Hamiltonian.

Conversely, assume that $A \in \operatorname{im} \rho$. Then there exists a G-equivariant equivariantly closed two-form $\omega + \Phi$ such that the assignment of Φ is also A. The difference $F = \Psi - \Phi$ is an abstract moment map with the zero assignment. By Theorem 4.1, F is exact and therefore Hamiltonian

(Corollary 5.1). Thus Ψ, as the sum of the two Hamiltonian moment maps F and Φ, is Hamiltonian.   q.e.d.

The space of Hamiltonian assignments, i.e., assignments associated with Hamiltonian abstract moment maps, is the quotient of the space of all Hamiltonian abstract moment maps by the space of exact abstract moment maps. This follows from Corollary 4.2. These three spaces fit together to form a part of a commutative square of exact sequences which summarizes some of our results.

Proposition 5.7. *The following diagram is commutative and all of its rows and columns are exact:*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \left\{\begin{array}{c} exact \\ basic \\ 2\text{-}forms \end{array}\right\} & \to & \left\{\begin{array}{c} equivariantly \\ exact \\ 2\text{-}forms \end{array}\right\} & \to & \left\{\begin{array}{c} exact \\ moment \\ maps \end{array}\right\} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \left\{\begin{array}{c} closed \\ basic \\ 2\text{-}forms \end{array}\right\} & \to & \left\{\begin{array}{c} equivariantly \\ closed \\ 2\text{-}forms \end{array}\right\} & \to & \left\{\begin{array}{c} Hamiltonian \\ moment \\ maps \end{array}\right\} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^2(M/G) & \to & H^2_G(M) & \to & \left\{\begin{array}{c} Hamiltonian \\ assignments \end{array}\right\} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Proof. The exactness of the left column is a particular case of a more general fact, that the cohomology of the basic De Rham complex of M is equal to $H^*(M/G)$, if G is compact or, more generally, if the G-action is proper, even when the action is not free. This result, due to Koszul [27], is similar to the De Rham theorem and can be proved in the same way. An easy proof is as follows. Recall that the sequence of sheaves of singular cochains on M/G (with real coefficients) is a fine resolution of the constant sheaf $\mathbb{R}$ on M/G. Furthermore, basic forms on G-invariant open subsets of M form a sheaf on M/G. This sheaf is a resolution of the locally constant sheaf because it is locally acyclic. Indeed, by using the fact that G is compact (or that the action is proper) and adapting the proof of the Poincaré lemma, one can show that the basic cohomology of a neighborhood of an orbit is the same as of the orbit itself, i.e., zero in positive degrees. It is easy to see that this sheaf is also fine because it admits partitions of unity. Thus basic forms on M provide another fine resolution of the constant sheaf on M/G. Since the cohomology of both resolutions are equal to the Čech cohomology of the constant sheaf on M/G, they are equal to each other.

The middle column is exact by the definition of equivariant cohomology via the equivariant De Rham complex. (See, e.g., [2] and [10].)

Exactness of the right column follows from Corollary 4.2.

The fact that the top two rows are exact follows directly from the definitions of the spaces involved.

The commutativity of the diagram is clear. Finally, commutativity with the exactness of the columns and the top two rows implies that the bottom row is exact by simple diagram chasing.   q.e.d.

Remark 5.8. Our notion of assignments has an interesting connection with a recent theorem of Goretsky-Kottwitz-MacPherson, [15]. Assume that a compact oriented G-manifold M is formal and satisfies in addition the so-called GKM condition, which we will recall below. Then the Goretsky-Kottwitz-MacPherson theorem implies that every assignment is Hamiltonian, and hence every abstract moment map is associated with a two-form.

The "GKM condition" is that the fixed points are isolated, and, additionally, every orbit type stratum with stabilizer of codimension one is two-dimensional.

Let us now show how the above assertion follows from the theorem. Let A be an assignment. Its restriction to the fixed point set is a locally constant function, $A^G \colon M^G \to \mathfrak{g}^*$. Such a function can be identified with an element of $H_G^2(M^G)$. For each subgroup $H \subset G$ of codimension one, the set of H-fixed points is a disjoint union of two-spheres in M, on each of which G/H acts with exactly two fixed points; this is a consequence of the "GKM condition" and formality. The assignment compatibility condition implies that in each such a two-sphere, the images in $\mathfrak{h}^*$ of A^G are the same at the two fixed points. The Goretsky-Kottwitz-MacPherson theorem asserts that this condition on A^G implies that there exists an equivariantly closed equivariant two-form, $\omega + \Psi$, on M, whose restriction to M^G is A^G. By formality, an assignment on M is uniquely determined by its restriction to M^G (see 3.4). Thus, A is the assignment associated with Ψ; hence, it is Hamiltonian.

6. Abstract moment maps on linear spaces: the proof of Theorem 4.3

A crucial step in the proof of Theorem 4.1 and Proposition 4.5, on which many of our subsequent results rely, is Theorem 4.3. In this section we recall this theorem and prove it. A different proof can be found in [13].

Theorem 4.3. *Let G be a torus acting linearly on $\mathbb{R}^m$, and let Ψ be an abstract moment map on a neighborhood of the origin, vanishing at the origin. Then there exists a G-invariant one-form μ on a neighborhood of the origin such that $\mu(\xi_M) = \Psi^\xi$ for all $\xi \in \mathfrak{g}$.*

Proof of Theorem 4.3. First note that by adding, if necessary, an additional copy of $\mathbb{R}$ (with the trivial G-action) to $\mathbb{R}^m$, we can always make m even.

There exists a G-invariant complex structure on the vector space $\mathbb{R}^m$; fix one. We obtain a representation ρ of G on $\mathbb{C}^d$ with weights $\alpha_1, \dots, \alpha_d$. The infinitesimal action of the Lie algebra $\mathfrak{g}$ of G on $\mathbb{C}^d$ is then given by the vector fields

$$(8) \qquad \xi_M = \sqrt{-1} \sum_{i=1}^d \alpha_i(\xi) \left(z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} \right).$$

From now on we will forget about the reality conditions and assume that μ is a complex-valued one-form. When such a form is found, it will suffice to replace it by the real form $(\mu + \bar{\mu})/2$, which still has the desired properties because ξ_M are real vector fields.

We will also forget about the equivariance conditions. If a form μ such that $\mu(\xi_M) = \Psi^\xi$ for all $\xi \in \mathfrak{g}$ is constructed, and if Ψ is invariant, we can replace μ by its average.

Any one-form on $\mathbb{C}^d$ can be written in the form

$$(9) \qquad \mu = -\sqrt{-1} \sum_{i=1}^d f_i dz_i - g_i d\bar{z}_i$$

for some smooth functions f_j and g_j, $j = 1, \dots, d$. For such a one-form, the function $\Psi \colon \mathbb{C}^d \to \mathfrak{g}^*$ defined by $\Psi^\xi = \mu(\xi_M)$, with the vector fields ξ_M given by (8), is

$$(10) \qquad \Psi = \sum_{j=1}^d (z_j f_j + \bar{z}_j g_j) \alpha_j.$$

Conversely, for any function Ψ which has the form (10) for some smooth functions f_j, g_j, there exists a one-form μ such that $\Psi^\xi = \mu(\xi_M)$ for all $\xi \in \mathfrak{g}$. Namely, just take μ to be given by (9).

To prove Theorem 4.3, it is thus enough to prove

Proposition 6.1. *Let Ψ be a $\mathfrak{g}^*$-valued function on a neighborhood of the origin in $\mathbb{C}^d$, vanishing at the origin and satisfying the second condition of an abstract moment map:*

for any subgroup $H \subset G$, the function $\Psi^H \colon \mathbb{C}^d \to \mathfrak{h}^*$ is locally constant on the set of H-fixed points.

Then there exist smooth functions f_j and g_j such that Ψ is given by (10) on a neighborhood of the origin.

Remark 6.2. The converse is easy: any Ψ of the form (10) satisfies the second condition of an abstract moment map.

Let us start with polynomial functions and one-forms:

Proposition 6.3. *Let $\Psi \colon \mathbb{C}^d \to \mathfrak{g}^*$ be a polynomial function which vanishes at the origin and satisfies the second condition of an abstract moment map. Then there exist polynomials f_j and g_j on $\mathbb{C}^d$ such that Ψ is given by (10).*

Proof of Proposition 6.3. Since Ψ is polynomial, we can write it uniquely as a sum of monomials,

$$\Psi = \sum_{k,l} \beta_{k,l} z^k \bar{z}^l,$$

summing over $k = (k_1, \ldots, k_d)$ and $l = (l_1, \ldots, l_d)$ in $\mathbb{N}^d$, where the coefficients $\beta_{k,l}$ are in $\mathfrak{g}^*$.

For every subset $I \subset \{1, \ldots, d\}$, denote by $(\mathbb{C}^\times)^I$ the subset of $\mathbb{C}^d$ consisting of all vectors $(z_1, \ldots, z_d)$ for which $z_i \neq 0$ if and only if $i \in I$. All the points z in $(\mathbb{C}^\times)^I$ have the same stabilizer, G_I, whose Lie algebra is

$$(11) \qquad\qquad \mathfrak{g}_I = \bigcap_{i \in I} \ker \alpha_i.$$

Since Ψ satisfies the second condition of an abstract moment map, $\Psi^{\mathfrak{g}_I}$ is constant on $(\mathbb{C}^\times)^I$. Since, additionally, Ψ is continuous and vanishes at the origin, $\Psi^{\mathfrak{g}_I}$ vanishes on $(\mathbb{C}^\times)^I$.

Let us analyze what this condition tells us about the coefficients $\beta_{k,l}$. The polynomial $\Psi^\xi = \sum_{k,l} \beta_{k,l}(\xi) z^k \bar{z}^l$ vanishes on $(\mathbb{C}^\times)^I$ for all $\xi \in \mathfrak{g}_I$ if and only if for each k, l, the summand $\beta_{k,l}(\xi) z^k \bar{z}^l$ vanishes on $(\mathbb{C}^\times)^I$ for all $\xi \in \mathfrak{g}_I$. Fix k and l, and restrict attention to

$$I = I_{k,l} = \{i \mid k_i \neq 0 \text{ or } l_i \neq 0\}.$$

Since the monomial $z^k \bar{z}^l$ does not vanish on $(\mathbb{C}^\times)^I$, its coefficient, $\beta_{k,l}(\xi)$, must vanish for all $\xi \in \mathfrak{g}_I$. By (11), a linear functional that vanishes on $\mathfrak{g}_I$ is a linear combination of α_i, $i \in I$. Therefore, $\beta_{k,l} = \sum_{i \in I_{k,l}} \lambda_{i,k,l} \alpha_i$, and

$$\Psi = \sum_i \alpha_i \sum_{k,l \text{ such that } i \in I_{k,l}} \lambda_{i,k,l} z^k \bar{z}^l.$$

Since for each $i \in I_{k,l}$, either z_i or $\bar{z}_i$ factors out of the monomial $z^k \bar{z}^l$, Ψ is of the form (10).   q.e.d.

We will now show that the theorem we need to prove in the smooth category follows from its polynomial version, which has already been proved. In other words, we will deduce Proposition 6.1 from Proposition 6.3. To this end, let us reformulate these propositions as assertions that certain sequences of homomorphisms are exact.

Denote by $\mathcal{P}$ the ring of complex-valued polynomials in z_j and $\bar{z}_j$, $j = 1, \dots, d$. Define the modules $\mathcal{M}_i$, $i = 1, 2, 3$, over $\mathcal{P}$ as follows:

- $\mathcal{M}_1$ is the space of one-forms $\sum f_i \, dz_i + g_i \, d\bar{z}_i$ with f_i and g_i in $\mathcal{P}$.

- $\mathcal{M}_2$ is the tensor product $\mathcal{P} \otimes \mathfrak{g}^*$ over $\mathbb{C}$.

- For each subset $I \subseteq \{1, \dots, n\}$, denote by $\mathcal{P}_I$ the ring of polynomial functions on $(\mathbb{C}^\times)^I$. The restriction homomorphism $\mathcal{P} \to \mathcal{P}_I$ makes $\mathcal{P}_I$ into a $\mathcal{P}$-module. Set

$$\mathcal{M}_3 = \bigoplus_I \mathcal{P}_I \otimes \mathfrak{g}_I^*,$$

  where $\mathfrak{g}_I$ is the Lie algebra of the stabilizer of $(\mathbb{C}^\times)^I$, given by (11).

Define the sequence of homomorphisms

$$(12) \qquad\qquad\qquad \mathcal{M}_1 \xrightarrow{\alpha} \mathcal{M}_2 \xrightarrow{\beta} \mathcal{M}_3$$

by setting $\alpha\colon \mu \mapsto \Psi$ with $\Psi^\xi = \mu(\xi_M)$ and β to be the homomorphism

$$\mathcal{P} \otimes \mathfrak{g}^* \to \bigoplus_I \mathcal{P}_I \otimes \mathfrak{g}_I^*$$

associated with the restrictions $\mathcal{P} \to \mathcal{P}_I$ and $\mathfrak{g}^* \to \mathfrak{g}_I^*$. In other words, the Ith component of $\beta(\Psi)$ is the $\mathfrak{g}_I^*$-component of Ψ restricted to $(\mathbb{C}^\times)^I$. Hence, Ψ satisfies the second condition of an abstract moment map if and only if $\beta(\Psi) = 0$, and Ψ is associated with a one-form if and only if it is in the image of α. Proposition 6.3 is equivalent to the sequence (12) being exact.

Denote by $\mathcal{O}$ and $\mathcal{E}$, respectively, the algebras of germs of analytic, resp. smooth, functions on $\mathbb{C}^d$ at the origin. Let $\mathcal{M}_i^{smooth}$, where $i = 1,\ 2,\ 3$, be the modules defined similar to $\mathcal{M}_i$ but in the category of smooth germs at the origin. Note that $\mathcal{M}_i^{smooth}$ are modules over $\mathcal{E}$.

As before, we have a sequence of homomorphisms

(13)
$$\mathcal{M}_1^{smooth} \xrightarrow{\alpha} \mathcal{M}_2^{smooth} \xrightarrow{\beta} \mathcal{M}_3^{smooth}.$$

To prove the theorem in the smooth category, it suffices to show that this sequence is exact.

Note that the inclusions $\mathcal{P} \to \mathcal{O}$ and $\mathcal{P} \to \mathcal{E}$ make $\mathcal{O}$ and $\mathcal{E}$ into $\mathcal{P}$-modules. The following lemma is standard:

Lemma 6.4. $\mathcal{M}_i^{smooth} = \mathcal{M}_i \otimes_\mathcal{P} \mathcal{E}$.

To finish the proof of Theorem 4.3, we need to recall some facts from commutative algebra. Let B be a commutative ring and A a sub-ring of B. The ring B is said to be *flat* over A if for every exact sequence of A-modules

$$\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3$$

the sequence

$$\mathcal{M}_1 \otimes_A B \to \mathcal{M}_2 \otimes_A B \to \mathcal{M}_3 \otimes_A B$$

is also exact.

It is known that $\mathcal{O}$ is flat over $\mathcal{P}$ (see [30], page 45, Example 4.11) and $\mathcal{E}$ is flat over $\mathcal{O}$ (see [30], page 88, Corollary 1.2). This in turn implies that $\mathcal{E}$ is flat over $\mathcal{P}$. Therefore, the exactness of (12) implies the exactness of (13).   q.e.d.

7. Assignment cohomology

In this section we show that the space of assignments $\mathcal{A}(M)$ on a G-manifold fits as the zeroth space in a sequence of vector spaces $\mathrm{HA}^*(M)$, called the assignment cohomology.

7.1. Construction of assignment cohomology.

Let M be a manifold with an action of a torus G. Denote by P_M the set of its infinitesimal orbit type strata (see Section 2). For each stratum X, denote by $\mathfrak{g}_X$ the infinitesimal stabilizer of the points of X, and let

$$V(X) = \mathfrak{g}_X^*$$

be the dual space. Recall that P_M is a partially ordered set, a *poset* for brevity, with $X \preceq Y$ if X is contained in the closure of Y. Denote by $\pi_Y^X \colon V(X) \to V(Y)$ the natural projection dual to the inclusion map $\mathfrak{g}_Y \subseteq \mathfrak{g}_X$ when $X \preceq Y$. Recall that the space of assignments is

$$\mathcal{A}(M) = \{v \in \prod_{X \in P_M} V(X) \mid \pi_Y^X v_X = v_Y \text{ for all } X \preceq Y\}.$$

Every abstract moment map induces an assignment. We will call elements of $\mathcal{A}(M)$ *moment assignments*, to distinguish them from assignments with other coefficients $V(X)$, which we introduce later.

Define the *assignment cohomology* $\mathrm{HA}^*(M)$ to be the cohomology of the following cochain complex which we will denote by $C^*(M; V)$. A k-cochain is a function φ that associates to each ordered $(k+1)$-tuple $X_0 \preceq \ldots \preceq X_k$ of elements of P_M an element in $V(X_k)$. The differential d is defined by the formula

$$
\begin{aligned}
d\varphi(X_0, \ldots, X_{k+1}) &= \sum_{\ell=0}^{k}(-1)^\ell \varphi(X_0, \ldots, \widehat{X_\ell}, \ldots, X_{k+1}) \\
&\quad +(-1)^{k+1}\pi_{X_{k+1}}^{X_k}\varphi(X_0, \ldots, X_k),
\end{aligned}
$$

(14)

where, as usual, the hat over X_ℓ means that X_ℓ is omitted.

Example 7.1. The zeroth assignment cohomology is simply the space of assignments:

$$\mathrm{HA}^0(M) = \mathcal{A}(M).$$

Remark 7.2 (Functoriality). Assignment cohomology is functorial with respect to equivariant maps of manifolds:

Let M and N be G-manifolds and let $f \colon M \to N$ be a G-equivariant map. Such a map might not send a stratum in M to a stratum in N. (For example, the function $f(z,w) = z$ from $\mathbb{C}^2$ with the diagonal circle action to $\mathbb{C}$ with the standard circle action sends the open dense stratum $\mathbb{C}^2 \smallsetminus 0$ to the union of strata $\{0\} \sqcup \mathbb{C}^\times = \mathbb{C}$.) However, it does induce a monotone mapping of posets, $\tilde{f} \colon P_M \to P_N$, in the following way. For each stratum X in M there exists a unique stratum Y in N with $\mathfrak{g}_X \subseteq \mathfrak{g}_Y$ whose closure contains $f(X)$. (To see this, consider the infinitesimal $\mathfrak{g}_X$-action in N. Since X is connected, $f(X)$ is contained in a unique component of the $\mathfrak{g}_X$-fixed point set of N. This component is a G-invariant submanifold of N. The stratum Y is the open dense stratum in this component.) We set $\tilde{f}(X) = Y$. Note that $\mathfrak{g}_X \subseteq \mathfrak{g}_{\tilde{f}(X)}$ for all X. By definition, the pullback map on the cochain complexes sends a cochain $\varphi \in C^k(N; V)$ to the cochain $f^*\varphi \in C^k(M; V)$ given by

$$(f^*\varphi)(X_0, \dots , X_k) = \pi_{X_k}^{\tilde{f}(X_k)} \varphi(\tilde{f}(X_0), \dots , \tilde{f}(X_k)).$$

The map f^* commutes with d and thus induces a pullback map in cohomology.

Theorem 7.3. $\mathrm{HA}^k(M) = 0$ when $k \geq \dim M$ or $k \geq \dim G$.

The proof of Theorem 7.3 will be an easy application of the following alternative construction of assignment cohomology. Let $C_0^k(M; V)$ be the space of functions φ that associate to each ordered $(k+1)$-tuple $X_0 \prec X_1 \prec \dots \prec X_k$ of *distinct* elements of P_M an element of $V(X_k)$. This can be identified with the subspace of $C^k(M; V)$ consisting of those cochains that vanish on $(X_0, \dots , X_k)$ whenever $X_i = X_{i+1}$ for some i.

Theorem 7.4. $C_0^*(M; V)$ *is a subcomplex of* $C^*(M; V)$, *and the inclusion map of complexes induces an isomorphism in cohomology.*

This result is standard. However, for the sake of completeness, we prove it below.

The complex $C_0^*(M; V)$ is much smaller than $C^*(M; V)$ and is more convenient to use for explicit calculations. Its disadvantage is that this complex is not functorial with respect to mappings of posets: a map $f \colon M \to N$ that send strata to strata sends a tuple $X_0 \prec \dots \prec X_k$ to a tuple $f(X_0) \preceq \dots \preceq f(X_k)$, but the $f(X_j)$ might not be distinct even if the X_j are.

Proof of Theorem 7.3. By Theorem 7.4 it suffices to prove Theorem 7.3 for the cohomology of the complex $C_0^*(M; V)$.

The first part of the theorem follows from the fact that if $X \prec Y$ then $\dim X < \dim Y$. Therefore, the longest possible tuple $X_0 \prec \ldots \prec X_k$ of distinct strata has $k = \dim M$. Thus $C_0^k(M; V) = 0$ for $k > \dim M$. When $k = \dim M$, the maximal stratum X_k is the open dense stratum in M, on which $V(X_k) = 0$. As a result, $C_0^{\dim M}(M; V) = 0$.

The second part of the theorem follows from the fact that if $X \prec Y$ then $\dim \mathfrak{g}_X > \dim \mathfrak{g}_Y$. The same argument as before shows that $C_0^k(M; V) = 0$ whenever $k \geq \dim G$.   q.e.d.

Notice that the proof of the second part of the theorem breaks down if the infinitesimal orbit type stratification is replaced by the orbit type stratification.

Proof of Theorem 7.4. Denote by $(X_0^{k_0}, \ldots, X_l^{k_l})$ the tuple

$$(X_0, \ldots, X_0, \ldots, X_l, \ldots, X_l)$$

in which each X_j occurs k_j times and the strata X_j are ordered and distinct: $X_0 \prec X_1 \prec \ldots \prec X_l$. Denote by $n(X_0^{k_0}, \ldots, X_l^{k_l})$ the number of j's such that $k_j > 1$; call this number the *fatness* of the tuple. As is easy to check, the fatness of a $(k+1)$-tuple is no greater than k.

For each integer $n \geq 0$ let $C_n^k(M; V)$ be the space of $(k+1)$-cochains that are supported on tuples of fatness n. (This is consistent with the previous definition of $C_0^k(M; V)$.) Then

$$C^k(M; V) = C_0^k(M; V) \oplus \ldots \oplus C_k^k(M; V)$$

as vector spaces. Set

$$C_{>0}^k(M; V) = \bigoplus_{n=1}^{k} C_n^k(M; V).$$

For every nonzero cochain φ in this space, there exists a unique integer n between 1 and k and a unique decomposition

$$(15) \qquad \varphi = \varphi_n + \varphi_{n+1} + \ldots + \varphi_k$$

such that $\varphi_j \in C_j^*(M; V)$ for all j and such that $\varphi_n \neq 0$.

An easy computation shows that

$$d(C_n^k(M; V)) \subseteq C_n^{k+1}(M; V) + C_{n+1}^{k+1}(M; V)$$

for all $n \geq 1$ and that

$$d(C_0^k(M; V)) \subseteq C_0^{k+1}(M; V).$$

In particular, $C_0^*(M; V)$ and $C_{>0}^*(M; V)$ are subcomplexes, and the assignment cohomology splits:

$$\mathrm{HA}^*(M; V) = \mathrm{HA}_0^*(M; V) \oplus \mathrm{HA}_{>0}^*(M; V).$$

It remains to show that $\mathrm{HA}_{>0}^*(M; V)$ vanishes.

Define a linear map $L \colon C^k(M; V) \to C^{k-1}(M; V)$ by

$$(L\varphi)(X_0^{k_0}, \dots, X_l^{k_l}) = \sum_{j=0}^{l} (-1)^{k_0 + \dots + k_{j-1}} \varphi(X_0^{k_0}, \dots, X_j^{k_j+1}, \dots, X_l^{k_l}).$$

An explicit computation shows that

(16) $$dL\varphi + Ld\varphi = \pi\varphi,$$

where

$$(\pi\varphi)(X_0^{k_0}, \dots, X_l^{k_l}) = n(k_0, \dots, k_l)\varphi(X_0^{k_0}, \dots, X_l^{k_l}).$$

Therefore,
$$\pi \colon C^*(M; V) \to C^*(M; V)$$

is chain homotopic to zero and, as a consequence, induces the zero map π_* in homology. Denote by

$$j \colon C_{>0}^*(M; V) \to C^*(M; V)$$

the natural inclusion. It is clear that

$$\pi j \colon C_{>0}^*(M; V) \to C_{>0}^*(M; V)$$

is an isomorphism. Thus

$$\pi_* j_* \colon \mathrm{HA}_{>0}^*(M; V) \to \mathrm{HA}_{>0}^*(M; V)$$

is an isomorphism. Since $\pi_* = 0$, this is possible only when $\mathrm{HA}_{>0}^*(M; V) = 0$.   q.e.d.

7.2. Assignments with other coefficients.   The definitions of assignments and assignment cohomology extend word–for–word to other systems of coefficients. A system of coefficients V on the poset P_M is a function that associates a vector space $V(X)$ to each stratum X and a linear map

$$\pi_Y^X \colon V(X) \to V(Y)$$

to each pair $X \preceq Y$, so that $\pi_Z^Y \pi_Y^X = \pi_Z^X$ whenever $X \preceq Y \preceq Z$ and $\pi_X^X = \mathrm{id}$. We define a differential complex $(C^*(M;V),d)$ as before and denote its cohomology by $\mathrm{HA}^*(M;V)$.

A morphism $V_1 \to V_2$ of two systems of coefficients consists of a linear map $V_1(X) \to V_2(X)$ for each $X \in P_M$, such that the squares

$$
\begin{array}{ccc}
V_1(X) & \longrightarrow & V_2(X) \\
\downarrow & & \downarrow \\
V_1(Y) & \longrightarrow & V_2(Y)
\end{array}
$$

commute for all $X \preceq Y$. The systems of coefficients V on P_M form a category. The assignment cohomology groups, $\mathrm{HA}^n(M;V)$, and in particular the space of assignments $\mathcal{A}(M;V)$, are functorial in V.

Remark 7.5. One can think of the poset P_M as a category in which there exists a single arrow $Y \to X$ whenever $X \preceq Y$. A system of coefficients is a contravariant functor from P_M to the category of vector spaces, and a morphism of systems of coefficients is a natural transformation. The space of assignments is the inverse limit:

$$
(17) \qquad\qquad \mathcal{A}(M;V) = \varprojlim_{X \in P_M} V(X),
$$

which is a functor in V.[3]

A system of coefficients V can be viewed as a *presheaf* on the category P_M, in the sense of [39, Expose I, Definition 1.2]. (Also see [36, Ch. I §2 and §4].) The assignments are the global sections:

$$
(18) \qquad\qquad \mathcal{A}(M;V) = \Gamma(P_M;V),
$$

and the assignment cohomology is equal to the cohomology of this presheaf. More explicitly, the cohomology groups of the presheaf are defined to be the derived functors of the global section functor (18) (as a functor in V); equivalently, the cohomology groups are the derived functors of the inverse limit functor (17). The fact that these derived functors are the same as the cohomology of the complex $C^*(M;V)$ is Proposition 6.1 in [36, ch. II].

---

[3] Note that the convention on the direction of morphisms commonly used to turn a poset into a category is opposite of the one employed in this paper. Alternatively, the poset of strata is sometimes given the order inverse of the one used above. However, the only essential point in the choice of directions of morphisms is that a poset should be made into a category so that V becomes a *contravariant* functor.

For some applications it is beneficial to work with systems of coefficients with values in categories other than the category of vector spaces. Let us illustrate this by some examples.

Example 7.6. Assignments with values in the functor

$$(19) \qquad V(X) = \{ \text{ equivalence classes of representations of } \mathfrak{g}_X \}$$

are called *isotropy assignments*. A G-action gives rise to a canonical isotropy assignment defined as follows: to each X associate the isotropy representation of $\mathfrak{g}_X$ on T_pM, $p \in X$. In a similar manner, any G-equivariant vector bundle over M gives rise to an isotropy assignment.

Example 7.7. For a symplectic manifold with a Hamiltonian torus action, its X-ray is, roughly speaking, the direct sum of the isotropy assignment and the moment map assignment. The notion of X-rays, introduced in [41], is central to the study of Hamiltonian torus actions. See [41], [22], [25], [34], [35].

A broad class of systems of coefficients can be obtained by the following construction. Consider the category of sub-algebras of $\mathfrak{g}$ with morphisms given by inclusion maps. Let V' be any functor from this category to the category of vector spaces (or modules, abelian groups, etc.). Then we get a system of coefficients with $V(X) = V'(\mathfrak{g}_X)$. For instance, we get moment assignments from $V'(\mathfrak{h}) = \mathfrak{h}^*$, and we get isotropy assignments from $V'(\mathfrak{h}) = \{ \text{ equivalence classes of representations of } \mathfrak{h} \}$.

Example 7.8. Assume that M has a unique minimal stratum, X_0. This is the case, for instance, when M is a vector space on which G acts linearly. Then $\mathrm{HA}^0(M;V) = V(X_0)$ and $\mathrm{HA}^k(M,V) = 0$ for all $k > 0$. Indeed, the operator

$$(Q\varphi)(Y_1, \dots, Y_k) = \varphi(X_0, Y_1, \dots, Y_k)$$

satisfies $dQ\varphi + Qd\varphi = \varphi$, hence it is a homotopy operator for the complex $C^*(M;V)$. (Also see Remark 7.16.)

Theorem 7.4 holds for assignment cohomology with many other systems of coefficients. For example, the theorem clearly holds whenever V takes values in the category of vector spaces. When Theorem 7.4 applies, we also have the following variant of Theorem 7.3:

Proposition 7.9.

1. *Let V be such that $V(X) = 0$ for the open stratum X. Then $\mathrm{HA}^k(M;V) = 0$ when $k \geq \dim M$.*

2. *Let V be obtained as the pull-back of a functor on the sub-algebras of $\mathfrak{g}$ which vanishes on the zero sub-algebra. Then $\mathrm{HA}^k(M; V) = 0$ when $k \geq \dim G$.*

The proof of this fact is entirely similar to the proof of Theorem 7.3.

7.3. Assignment cohomology for pairs. Let N be a subset of M which is a union of strata. Define the *relative* assignment cohomology $\mathrm{HA}^*(M, N; V)$ to be the cohomology of the sub-complex $C^*(M, N; V)$ of $C^*(M; V)$ formed by those cochains which vanish on all $(k+1)$-tuples $X_0 \preceq \ldots \preceq X_k$ in which all of the strata X_j are in N.

Theorem 7.10. *There is a long exact sequence*

$$
(20) \qquad
\begin{aligned}
&\ldots \to \mathrm{HA}^*(M, N; V) \to \mathrm{HA}^*(M; V) \to \mathrm{HA}^*(N) \\
&\to \mathrm{HA}^{*+1}(M, N; V) \to \ldots,
\end{aligned}
$$

where the connecting homomorphism is given by the standard formula.

Proof. The theorem is an immediate consequence of the fact that the sequence of complexes

$$
(21) \qquad 0 \to C^*(M, N; V) \to C^*(M; V) \to C^*(N; V) \to 0
$$

is exact. To prove the exactness of (21), note that the space $C^*(M, N; V)$ is the kernel of the restriction map $C^*(M; V) \to C^*(N; V)$ by its definition. To see that the restriction map is onto, note that every cochain φ in $C^*(N; V)$ can be extended to a cochain $\tilde{\varphi}$ in $C^*(M; V)$ by declaring $\tilde{\varphi}(X_0, \ldots, X_k)$ to be zero whenever not all of X_j's are in N.   q.e.d.

Remark 7.11. It is not clear if there is a way to define relative assignment cohomology so that the sequence (20) is exact in the case where $N \subseteq M$ is G-invariant but not a union of strata. For instance, consider a G-manifold B and let G act on $M = B \times [0, 1]$ by acting on the first factor. Let $N = B \times \{0, 1\}$. Since every stratum in M meets N, it seems reasonable to set $\mathrm{HA}^*(M, N) = 0$. Then an exact sequence (20) would give an isomorphism between $\mathcal{A}(M)$ and $\mathcal{A}(N)$. However, these spaces are not isomorphic. In fact, $\mathcal{A}(N)$ is isomorphic to $\mathcal{A}(M) \oplus \mathcal{A}(M)$.

As is with other "cohomology theories", an exact sequence of coefficients gives rise to a long exact sequence in assignment cohomology:

Theorem 7.12. *A short exact sequence of systems of coefficients,*

$$(22) \qquad\qquad 0 \to V_1 \to V_2 \to V_3 \to 0,$$

induces a long exact sequence in cohomology,

$$(23) \qquad \begin{aligned} &\to \mathrm{HA}^k(M, N; V_1) \to \mathrm{HA}^k(M, N; V_2) \to \mathrm{HA}^k(M, N; V_3) \\ &\xrightarrow{\delta} \mathrm{HA}^{k+1}(M, N; V_1) \to \end{aligned}$$

Proof. The short exact sequence (22) naturally induces a short exact sequence of complexes,

$$0 \to C^*(M, N; V_1) \to C^*(M, N; V_2) \to C^*(M, N; V_3) \to 0,$$

and hence the long exact sequence (23) in cohomology.   q.e.d.

Remark 7.13. Suppose that $N \subseteq M$ is an *open* subset which is a union of strata. Note that being open is equivalent to the following condition:

$$(24) \qquad \begin{aligned} &\text{for every pair } X \prec Y \text{ of strata in } M, \\ &\quad \text{if } X \subseteq N \text{ then also } Y \subseteq N. \end{aligned}$$

Then the relative assignment cohomology is equal to the ordinary assignment cohomology with a different system of coefficients:

$$\mathrm{HA}^n(M, N; V) = \mathrm{HA}^n(M; V_{M/N}),$$

where $V_{M/N}$ is the system of coefficients given by

$$V_{M/N}(X) = \begin{cases} V(X) & \text{if } X \not\subseteq N, \\ 0 & \text{otherwise} \end{cases}$$

for any stratum X in M, with the projection maps

$$(\pi_{M/N})^X_Y = \begin{cases} \pi^X_Y & \text{if both } X \text{ and } Y \text{ are not in } N, \\ 0 & \text{otherwise.} \end{cases}$$

The compatibility condition,

$$(25) \qquad (\pi_{M/N})^Y_Z \circ (\pi_{M/N})^X_Y = (\pi_{M/N})^X_Z \text{ whenever } X \preceq Y \preceq Z,$$

which is required for $V_{M/N}$ to form a system of coefficients, is satisfied if and only if N meets the requirement (24).

Remark 7.14. If $N \subseteq M$ is a union of strata and is open, the long exact sequence for the pair (M, N) in Theorem 7.10 follows from the long exact sequence for coefficients in Theorem 7.12. To see this, recall that $\mathrm{HA}^n(M, N; V) = \mathrm{HA}^n(M; V_{M/N})$ as explained in Remark 7.13. Furthermore, we set $V_N(X) = V(X)/V_{M/N}(X)$ with $(\pi_N)_Y^X = \pi_Y^X$ if X and Y are both in N and $(\pi_N)_Y^X = 0$ otherwise. Then we have $\mathrm{HA}^*(M; V_N) = \mathrm{HA}^*(N; i^*V)$, where $i: N \to M$ is the inclusion map. The sequence of systems of coefficients $0 \to V_{M/N} \to V \to V_N \to 0$ is exact. By Theorem 7.12, this sequence gives rise to the long exact sequence which coincides with the sequence (20) of Theorem 7.10.

Remark 7.15. Relative assignment cohomology is a sequence of functors

$$V \mapsto \mathrm{HA}^n(M, N; V)$$

from the category $\mathcal{C}_M$ of systems of coefficients on P_M to the category of vector spaces. This sequence, together with the maps δ of (23), forms a δ-*functor*. (Essentially, this means that short exact sequences in $\mathcal{C}_M$ induce long exact sequences in cohomology, as in Theorem 7.12. See [28].)

In the non-relative case $N = \emptyset$ this δ-functor is *universal*, i.e, the functors $V \mapsto \mathrm{HA}^n(M; V)$ are the derived functors of the assignment functor $V \mapsto \mathcal{A}(M; V)$; see Remark 7.5. This remains true in the relative case if $N \subseteq M$ is a union of strata and is open: the relative assignment cohomology functors $V \mapsto \mathrm{HA}^n(M, N; V)$ are then the derived functors of the relative assignment functor $V \mapsto \mathcal{A}(M, N; V)$ which associates to each V the space of assignments that vanish on N. However, for a general N, it is not clear if these functors are universal or, equivalently, whether or not $V \mapsto \mathrm{HA}^*(M, N; V)$ are the derived functors for $V \mapsto \mathcal{A}(M, N; V)$.

Remark 7.16. The poset of strata P_M does not in general satisfy the following condition which is routinely required in some sources (e.g., [21], [31], and [37]):

$$\text{(26)} \qquad \begin{array}{l} \text{for any } X \in P_M \text{ and } Y \in P_M \text{ there exists } Z \in P_M \\ \qquad \text{such that } Z \preceq X \text{ and } Z \preceq Y. \end{array}$$

(The reader should keep in mind that our order convention is opposite of the standard one; see footnote 3.) This condition is met, for example, when P_M has a minimal element, i.e., an $X_0 \in P_M$ such that $X_0 \preceq X$ for all $X \in P_M$. (Equivalently X_0 is a stratum which is contained in

the closure of every stratum X.) With the condition (26), the poset P is called a *directed set*. Under this condition, an inverse system V of finite–dimensional vector spaces is automatically flabby (see, e.g., [21] and [37]) and $\varprojlim^{(k)} V = 0$ for all $k > 0$. This generalizes Example 7.8.

7.4. Examples of calculations of assignment cohomology.
The following simple example shows that relative assignment cohomology can be non-trivial in degrees greater than zero.

Example 7.17. Let $M = \mathbb{CP}^2$, and G be the torus $\mathbb{T}^2$ acting on M as in Example 3.10, and $N = M^G$. Then $\mathcal{A}(M, N) = 0$, $\mathcal{A}(N) = (\mathfrak{g}^*)^3$ is six-dimensional, and $\dim \mathcal{A}(M) = 3$. Furthermore,

$$\mathrm{HA}^{*>0}(M) = \mathrm{HA}^{*>0}(N) = 0.$$

Thus (20) turns into the exact sequence

$$0 \to \mathcal{A}(M) \to \mathcal{A}(N) \to \mathrm{HA}^1(M, N) \to 0,$$

where $\dim \mathrm{HA}^1(M, N) = 3$, as can also be checked by a direct calculation.

Example 7.18 (Assignment cohomology for toric varieties). Let M be a compact smooth Kähler toric manifold of complex dimension n with moment map $\Psi\colon M \to \mathfrak{g}^*$. (See Examples 3.13 and 3.14.) Recall that the poset P_M of orbit type strata is isomorphic to the poset of faces $\Psi(X)$ of a simple polytope $\Psi(M)$, and for each stratum X,

$$\dim_{\mathbb{C}} X = \dim \Psi(X) = n - \dim \mathfrak{g}_X^*.$$

We will work with the system of coefficients $V(X) = \mathfrak{g}_X^*$. The zeroth assignment cohomology is the space of assignment which was computed in Example 3.13. Namely,

$$\mathrm{HA}^0(M; V) = \bigoplus_Y \mathfrak{g}_Y^*,$$

where the summation is over all strata Y with $\dim \mathfrak{g}_Y = 1$. These strata correspond to the $(n-1)$-dimensional faces of $\Psi(M)$. In particular,

$$\dim \mathrm{HA}^0(M; V) = \text{the number of facets of } \Psi(X).$$

Let us prove that the higher cohomology groups vanish:

$$(27) \qquad\qquad \mathrm{HA}^k(M; V) = 0 \quad \text{for all } k \geq 1.$$

By Theorem 7.4 it is enough to work with the complex $C_0^*(M;V)$. For a closed cochain $\varphi \in C_0^k(M;V)$, $k \geq 1$, we will find a primitive $(k-1)$-cochain $\psi \in C_0^{k-1}(M;V)$, i.e., a cochain ψ such that $d\psi = \varphi$.

Let $X_0 \prec \ldots \prec X_{k-1}$ be any ordered k-tuple of distinct strata. Recall that the natural map

$$(28) \qquad V(X_{k-1}) \overset{\oplus \pi_{X_k}^{X_{k-1}}}{\longrightarrow} \bigoplus_{X_k} V(X_k),$$

where X_k is such that $\operatorname{codim} X_k = 1$, and $X_{k-1} \preceq X_k$, is a linear isomorphism. Therefore, to define the value $\psi(X_0, \ldots, X_{k-1})$, which is an element of $V(X_{k-1})$, it is enough to specify the projections of these elements to all of the spaces $V(X_k)$ with X_k as above. We require these projections to be

$$(29) \qquad \pi_{X_k}^{X_{k-1}} \psi(X_0, \ldots, X_{k-1}) = (-1)^k \varphi(X_0, \ldots, X_k).$$

Let us show that $d\psi = \varphi$. Set $\varphi' = \varphi - d\psi$. Note that $\psi(X_0, \ldots, X_{k-1}) = 0$ when $\operatorname{codim} X_{k-1} = 1$. Then it follows from the definition (29) of ψ and the definition (14) of the differential that φ' vanishes on all tuples $X_0 \prec \ldots \prec X_k$ in which $\operatorname{codim} X_k = 1$. Again, by (14),

$$d\varphi'(X_0, \ldots, X_{k+1}) = \pi_{X_{k+1}}^{X_k} \varphi'(X_0, \ldots, X_k)$$

for all tuples

$$X_0 \prec \ldots \prec X_k \prec X_{k+1}$$

in which $\operatorname{codim} X_{k+1} = 1$. Since $d\varphi' = 0$, and (28) is a linear isomorphism, this implies that $\varphi' \equiv 0$. Hence, $\varphi = d\psi$ is exact.

We now give an example of a manifold which has a non-trivial (absolute, not relative) first assignment cohomology.

Example 7.19. Let $M = S^2 \times S^2 \times S^2$, and let $G = S^1 \times S^1$ act by

$$(a, b) \cdot (u, v, w) = (a \cdot u, b \cdot v, ab^{-1} \cdot w),$$

where on the right the dot denotes the standard S^1 action on S^2 by rotations. The moment assignments can be drawn as pictures showing the moment map images of the orbit type strata (the "x-ray"). Such a picture is shown in Figure 1. Notice that this picture is two–, not three–, dimensional. This arrangement can be moved around as long as the edges are shifted but not rotated. An assignment is therefore

determined by the location of the bottom left vertex and the lengths of the three edges coming out of it. Therefore,

$$(30) \qquad \dim \mathrm{HA}^0(M;V) = \dim \mathcal{A}(M;V) = 5.$$

We will find the dimension of the first assignment cohomology space by using the Euler characteristic of the complex $C_0^*(M;V)$ of Theorem 7.4. We have

$$(31) \qquad \begin{aligned} \dim \mathrm{HA}^0(M;V) &- \dim \mathrm{HA}^1(M;V) \\ &= \sum_k (-1)^k \dim C_0^k(M;V) \\ &= \dim C_0^0(M;V) - \dim C_0^1(M;V), \end{aligned}$$

because $C_0^k(M;V) = 0$ for all $k \geq 2$ (see also Theorem 7.3). A 0-cochain associates to each vertex an element of a two–dimensional space and to each edge an element of a one–dimensional space. Therefore,

$$(32) \qquad \begin{aligned} \dim C_0^0(M;V) &= 2(\text{number of vertices}) + (\text{number of edges}) \\ &= 2 \cdot 8 + 12 \\ &= 28. \end{aligned}$$

A 1-cochain associates an element of a one–dimensional space to each pair consisting of a vertex and an edge coming out of it. Therefore

$$(33) \qquad \dim C_0^1(M;V) = 24.$$

Substituting (32), (33), and (30) in (31), we get

$$5 - \dim \mathrm{HA}^1(M;V) = 28 - 24,$$

hence

$$\dim \mathrm{HA}^1(M;V) = 1.$$

Generalizations of assignment cohomology. The results and definitions of this section can be generalized or altered in many natural ways. For instance, the assignment cohomology can be defined for an arbitrary system of coefficients with values in an abelian category on an arbitrary poset. In particular, in a more geometrical realm, the infinitesimal orbit type stratification can be replaced by the orbit type stratification, and V can then be the pull-back of a contravariant functor on subgroups of G. Furthermore, instead of working with functors with values in finite–dimensional vector spaces, one may consider functors with values in abelian groups or graded vector spaces or rings. In
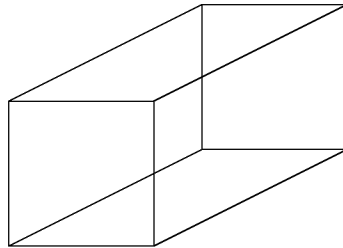
FIGURE 1.   A manifold with nonzero first assignment cohomology.

fact, such functors do arise in the study of symplectic manifolds, and $\mathcal{A}(M;V)$ can be viewed as a repository containing many of the invariants of the action. One important example is the isotropy assignments of Example 7.6 above.

Moreover, most of the results of this section can be extended with obvious modifications to actions of finite or compact *non-abelian* groups. For example, we can take V to be the pull-back of a functor on the subalgebras of $\mathfrak{g}$ which is invariant under conjugations. However, the space of moment map assignments for actions of non-abelian groups (Remark 3.17) does not arise as the zeroth cohomology groups of this type. In the non-abelian case, the space of moment map assignments does not seem to be associated with a functor on the poset of strata. Instead, to obtain this space one should work with the singular foliation of M given by the decomposition of M according to actual stabilizers, but not just their conjugacy classes. This renders a correct generalization of assignment cohomology to actions of non-abelian groups much less straightforward.

Remark 7.20. The assignment cohomology appears to be related to Bredon's equivariant cohomology, [6], and, perhaps, to Borel's equivariant cohomology with twisted coefficients. The nature and explicit form of these relations are, however, unclear to the authors.

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