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ON THE CONVERGENCE AND COLLAPSING OF KÄHLER METRICS

WEI-DONG RUAN

Abstract

In this paper we consider the convergence and collapsing of Kähler manifolds. While the convergence and collapsing of Riemannian manifolds have been discussed by many people and applied to many fields, how to generalize it to Kähler case is not apriorily clear. Our paper is an attempt in this direction. We discussed the corresponding concepts of convergence and collapsing for Kähler manifolds. We proved that when a sequence of Kähler manifolds with the fixed background complex compact manifold is not collapsing, it will converge to a complete Kähler manifold which is biholomorphic to a Zariski open set of the original background complex manifold with some possible "bubbling" on the complement of that Zariski open set. We also discussed the structure of collapsing. Especially we show the resulting Monge-Ampère foliation is holomorphic, produce some holomorphic vector fields with respect to the foliation, and also give some applications of our results. The main methods we are using are estimates from the theories of harmonic maps and partial differential equations, some results from several complex variables, and ideas from Riemannian geometry.

1. Introduction and background

Given a compact complex manifold M with a fixed complex structure, for a given sequence of Kähler metrics $\{g_i\}$ on M, consider $\{(M, g_i)\}$ as a sequence of Kähler manifolds. We want to study how do they converge.

In the most generality, one has the following Gromov-Hausdorff convergence. On the space of compact metric spaces, define

$$d(A,B) = inf\left\{\epsilon \;\middle|\; \begin{array}{c} A \hookrightarrow X \hookleftarrow B, \text{isometric embedding}, \\ A \subset U(B,\epsilon), B \subset U(A,\epsilon) \end{array}\right\}$$

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where A, B are compact metric spaces, the infimum is taken over all compact metric space X and the corresponding ϵ satisfies given conditions. This is called the Gromov-Hausdorff distance. The space of compact metric spaces is complete under the Gromov-Hausdorff distance. Therefore we can take limit for a Cauchy sequence of compact metric spaces under this distance.

Compact Riemannian manifolds with their metrics form compact metric spaces. To have geometrically interesting convergence property, one would hope the limit of a family of manifolds to be like a manifold with similar property, and convergence stronger than Gromov-Hausdorff convergence. This is similar to the situation in PDE, where we first get the existence of a weak solution of the equation by some functional analysis methods, and then improve its regularity according to given regularity conditions. An important result in convergence of Riemannian manifolds is the following theorem:

Theorem 1.1 ([8], [11],[13]). *For $\{(M_i, g_i)\}$ compact Riemannian manifolds (possibly with boundary), assume sectional curvatures $|K_{g_i}| < C$, $diam_{g_i}(M_i) < C$ and injectivity radius $i_{g_i} > c$. Then there exists a subsequence (denote by the same index for simplicity) $\{(M_i, g_i)\}$ converging to (M_∞, g_∞) in the following sense:*

$\forall \epsilon$, let $M_i(\epsilon) = \{x \in M_i, dist_{g_i}(x, \partial M_i) > \epsilon\}$,

For all compact $K \subset M_\infty$, $\exists \epsilon$, such that for i large, there exist diffeomorphisms $\phi_i : M_i(\epsilon) \longrightarrow M_\infty$ that satisfy the following:

(i) $K \subset \phi_i(M_i(\epsilon))$ for i large.

(ii) $(\phi_i^{-1})^* g_i$ converge to g_∞ in $C^{1,\alpha}(K)$ norm, $\forall \alpha < 1$.

This theorem was first formulated by Gromov et al [8] and was given a sketched proof, later rigorously proved by Greene-Wu [11], Peters [13] for the precise convergence based on some early result by Jost-Karcher [10].

Remark. A precursor of Theorem 1.1 (essentially an important special case) is Cheeger's finiteness theorem [4]. He showed that there are only finitely many diffeomorphism types of n-manifolds satisfying the geometric condition in Theorem 1.1. Many of the key ideas of the later development were already there. Theorem 1.1 can be viewed as a reformulation and a more precise refinement in the convergence language. For this reason, the above theorem is usually called Cheeger-Gromov convergence theorem.

Since then many works have been done in this field and there are also many applications of these convergence results in solving geometric problems.

If one weakens the assumption, the conclusion will be weaker. An important case is to assume $|K_{g_i}| < C$, $\max_{x \in M_i} i_{g_i}(x) \to 0$, where K_g and i_g denote the sectional curvature and injective radius of a Riemannian metric g respectively. This is called collapsing and has been studied by Cheeger-Gromov [5] and Fukaya [7]. They found out that under the above geometric condition there are " flat structures of positive rank" on the manifold, namely some sort of local torus action, also some sort of fiber structure which roughly indicate the direction of collapsing. Usually the sequence will converge(collapse) under the Hausdorff distance to a lower dimension length space.

In general, if we only assume $|K_{g_i}| < C$, drop the boundedness of diameter, the limit will consists of "thick part" (non-collapsing part) and "thin part" (collapsing part). In some sense the sequence will converge to a union of complete Riemannian manifolds, denoted by $(M_\infty, g_\infty) = \bigcup_i (M_{\infty,i}, g_{\infty,i})$. This is illustrated in the following picture.

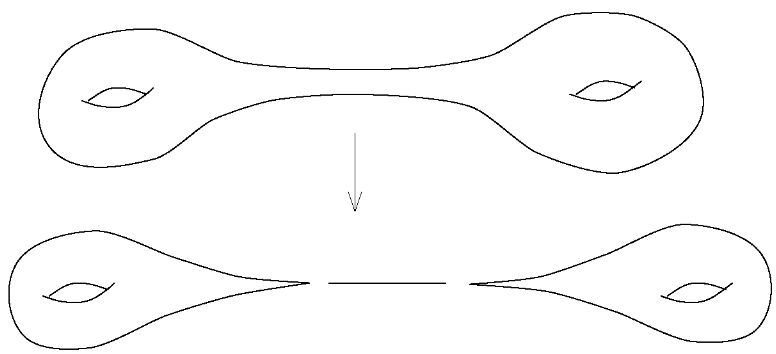

FIGURE 1. Riemannian convergence.

Remark. Convergence concept we use is based on Riemannian pointed convergence. In our language, (M_n, g_n) converges to $(M_\infty, g_\infty) = \bigcup_i (M_{\infty,i}, g_{\infty,i})$ if for any i there exist a sequence of points $p_{n,i} \in M_n$ such that $(M_n, p_{n,i}, g_n)$ converge to $(M_{\infty,i}, p_{\infty,i}, g_{\infty,i})$ in the sense of pointed convergence.

These kind of results provide systematic tools for attacking problems in Riemannian geometry and other related fields.

Main Results. In the above subsection we gave a brief introduction to the convergence and collapsing of Riemannian manifolds. Our problem is: if the manifolds are Kähler, what phenomenon can occur?

The results of convergence of Riemannian manifolds have been used in complex geometry. One example is Gang Tian's proof of Calabi conjecture for complex surface with $c_1 > 0$. He used the convergence property in one of the estimates. But in his geometric situation the conditions $i_{g_i} > c$, $diam_{g_i} < C$ are satisfied, nothing special about Kähler manifolds was used.

Our work tries to explore the implication of Kähler condition on the convergence property. Being Kähler is a very strong restriction. Kähler manifolds are more rigid than Riemannian manifolds. One would expect stronger and more global results. But what to expect is not a priori clear. Our work is an attempt toward this direction. We have the following theorem.

Theorem 1.2. *Fix a compact complex manifold M. Let $\{g_i\}$ be a sequence of Kähler metrics on M with bounded Kähler classes. Consider the sequence of Kähler manifolds $\{(M, g_i)\}$. Assume that $|K_{g_i}| < C$ uniformly. Then there exist a subsequence, still use the same subscripts for simplicity, $(M, g_i) \longrightarrow (M_\infty, g_\infty)$, and a Zariski open subset $U \subset M$, on which one of the following will happen:*

(i) (Collapsing) $\omega_{g_i}|_U \longrightarrow \omega_{g_\infty}|_U$ uniformly and $\omega_{g_\infty}|_U$ degenerate everywhere, i.e., $\omega^n_{g_\infty}|_U = 0$, here ω_g denotes the Kähler form of g.

(ii) (Non-collapsing) There exists one of the components $(M_{\infty,0}, g_{\infty,0})$ of the limit, which is a complete Kähler manifold, and a biholomorphic map

$$i : M_{\infty,0} \longrightarrow U.$$

For other components $(M_{\infty,i}, g_{\infty,i})$, there are maps

$$\pi_i : M_{\infty,i} \longrightarrow E = M - U.$$

Remark. There are two kinds of convergence involved in the theorem (and also in the rest of our paper). One kind (as in (i)) is the classical convergence of $(1,1)$-forms on a complex manifold. The other is convergence of Riemannian manifolds as described in the previous remark. In a sense, we are exploring the relation of the two kinds of convergence to relate the limiting manifolds to the original fixed manifold we start with.

Roughly speaking, the theorem is saying that there exists a subsequence that converges to a complete Kähler manifold, such that one of the connected components of the limit is biholomorphic to a Zariski open subset of M. For other components, there is a holomorphic fibration to some subvariety of M in the compliment of the Zariski open set.

These kind of results cannot be expected in Riemannian manifolds. Kähler manifolds are more rigid. Therefore we can get more detailed description of the convergence and structure of the limit.

For example, we have the following picture in case of Riemann surfaces. The situation in Figure 1 will not occur.

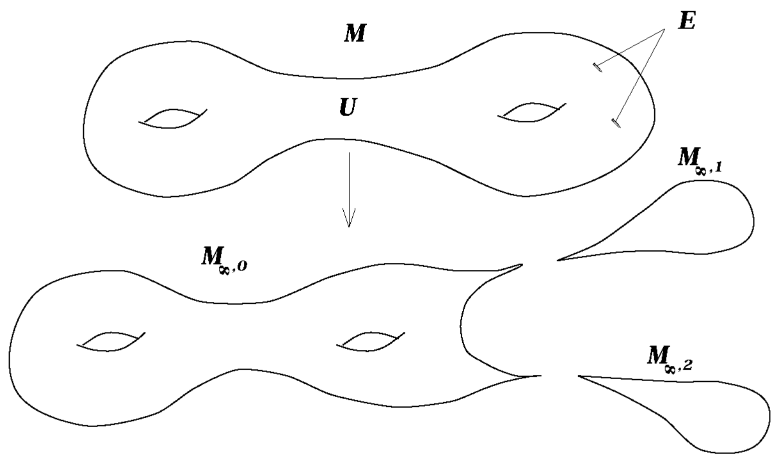

FIGURE 2. Kähler Convergence.

Using the above theorem we can get the following corollaries:

Corollary 1.1. *In the non-collapsing case, there exists a Zariski open set $U \subset M$, $g_i|_U \longrightarrow g_\infty|_U$, and $g_\infty|_U$ is a complete Kähler metric on U.*

Corollary 1.2. *Assume $c_1(M) < 0$, and $\{g_i\}$ be a family of Kähler metrics with bounded curvature. Then $\{(M, g_i)\}$ is non-collapsing.*

More on Collapsing. From the theorem in the last subsection, we see that in the collapsing case there exists a Zariski open subset $U \subset M$, such that $g_\infty|_U$ is a degenerate metric. Locally in a ball,

$$g_\infty = \partial\bar{\partial}u, \qquad (\partial\bar{\partial}u)^n = 0.$$

This is called the homogeneous Monge-Ampère equation.

Homogeneous Monge-Ampère equations have been studied in the literature. One result is that the homogeneous Monge-Ampère equation gives rise to Monge-Ampère foliation. $Ker(g_\infty)$ is an integrable distribution. The leaves of this foliation locally are complex submanifolds. But in general, the foliation could have different ranks in different open sets, it is not even necessary a holomorphic foliation, i.e., the leaves do not necessarily vary holomorphically.

In our case we can show the following:

Theorem 1.3. *When the homogeneous Monge-Ampère equation comes from a collapsing, the foliation is holomorphic, and of constant rank on a smaller Zariski open subset $U' \subset U$.*

We also have a way to produce holomorphic vector fields which are along the fibers and globally results in meromorphic vector fields.

Our method is partly based on the corresponding results in Riemannian geometry, some techniques from complex analysis and properties of harmonic maps.

Our paper is organized in the following way. In Section 2 we give some preliminary and basic estimates on Kähler manifolds and Kähler potentials. In Section 3 we discuss various weak and strong convergences of Kähler metrics, using estimates from Section 2. In Section 4 we consider the problem from the point of view of convergence of Kähler manifolds instead of metrics. We will first discuss the problem using the results from Riemannian Geometry, and then try to explore a more natural approach more suitable for the Kähler situation. In Section 5 we construct the holomorphic map which connect the original manifold to the limit and prove its surjectivity. In Section 6 we will discuss some basic facts of homogeneous Monge-Ampère equations which will be relevant to the collapsing problem. Collapsing Situation will be discussed in Section 7. Some explicit general meromorphic vector fields along the collapsing directions will be constructed. In Section 8 we will explore some finer structure of collapsing, especially we will construct some natural global holomorphic vector fields along the collapsing directions which are more closely related to the collapsing structure. We will also discuss its applications.

2. Preliminary on the relation between two Kähler metrics

The starting point of the discussion of convergence and collapsing of Kähler metrics on a fixed compact complex manifold is the following very simple observation: Holomorphic maps between Kähler manifolds are harmonic, and the relation between two Kähler metrics on a fixed complex manifold can be analyzed by results of harmonic maps. For example Schoen-Uhlenbeck's estimate for the energy density of a harmonic map is very important in our work. In this section we will explain the relevants of this result and various other facts that will be needed later.

Let us consider a compact complex manifold M with a background Kähler metric g_0 and another Kähler metric g. It is a well-known fact that the map:

$$id : (M, g_0) \longrightarrow (M, g)$$

is a harmonic map. Here id is the identity map of the manifold M, and $e(g) = tr_{g_0} g$ is the energy density function of the map id; it satisfies:

$$\frac{1}{2}\Delta_0 e(g) = \|\nabla' du\|^2 - \sum_{\alpha,\beta} \langle R^g(e_\alpha, e_\beta)e_\alpha, e_\beta\rangle + \sum_\alpha Ric_{g_0}(e_\alpha, e_\alpha).$$

If we assume $Ric_{g_0} \geq -C_1$, $R^g \leq C_2$, then,

$$-\frac{1}{2}\Delta_0 e(g) \leq C_1 e(g) + C_2 e(g)^2. \tag{1}$$

Notice that $\int_M e(g) dV_{g_0}$ is a topological invariant, and if we assume that the Kähler class of g is bounded, then this integral will be bounded.

This equation is a quasi-linear elliptic equation. Standard technique from PDE for this kind of equation is the Moser iteration method. But the estimates one gets from the Moser iteration are not quite enough for the requirements from the geometric problems. Schoen-Uhlenbeck's brilliant estimate aimed at the geometric problem of harmonic maps exactly suits the geometric situation; one can find its proof in [14]. For reader's convenience we will state the theorem and provide a proof here.

Proposition 2.1. *There exists a constant C' such that if*

$$r^{2-2n} \int_{B_r} e(g) dV_{g_0} < C',$$

then,

$$\sup_{\sigma \in [0,r]} \sigma^2 \sup_{B_{r-\sigma}} e(g) < \frac{1}{C'} r^{2-2n} \int_{B_r} e(g) dV_{g_0} < 1.$$

Especially,

$$\sup_{B_{r/2}} e(g) < \frac{4}{C'} \int_{B_r} e(g).$$

Proof. Assume

$$D = \sup_{\sigma \in [0,r]} \sigma^2 \sup_{B_{r-\sigma}} e(g)$$

is achieved at $\sigma_0 \in (0, r)$, $x_0 \in B_{r-\sigma_0}$, i.e.,

$$D = \sigma_0^2 e(g)(x_0) = \sup_{\sigma \in [0,r]} \sigma^2 \sup_{B_{r-\sigma}} e(g).$$

Notice $B_{\sigma_0/2}(x_0) \subset B_{r-\sigma_0/2}$, which implies that for $x \in B_{\sigma_0/2}(x_0)$,

$$e(g)(x) \leq 4e(g)(x_0).$$

Using this condition, on $B_{\sigma_0/2}(x_0)$ we can reduce (1) to

$$-\Delta_0 e(g) \leq C(1 + e(g)(x_0))e(g).$$

Here we need the following estimate lemma of linear elliptic equation. The proof can be found in Gilbarg-Trudinger's book [9].

Lemma 2.1 (elliptic equation estimate). *For the second order linear elliptic equation*

$$-\Delta u \leq Ku \quad \text{in } B_r(x_0),$$

if $u > 0$ is a solution on $B_r(x_0)$, then we have

$$sup_{x \in B_{r/2}} u(x) \leq C(1 + (Kr^2)^n) \frac{1}{Vol(B_r)} \int_{B_r} u dx.$$

Now we continue the proof. Apply Lemma 2.1 to the last equation before the lemma, we get

$$e(g)(x_0) \leq C(1 + (e(g)(x_0)\sigma^2)^2) \frac{1}{Vol(B_{\sigma/2}(x_0)))} \int_{B_{\sigma/2}(x_0)} e(g) dx$$

for any $\sigma \in (0, \sigma_0)$.

By the monotonicity formula for harmonic map

$$\sigma^2 \frac{1}{Vol(B_{\sigma/2}(x_0)))} \int_{B_{\sigma/2}(x_0)} e(g) dx \leq C\sigma_0^2 \frac{1}{Vol(B_{\sigma_0}(x_0)))} \int_{B_{\sigma_0}(x_0)} e(g) dx$$

we have

$$1 \leq C(e(g)(x_0)\sigma^2 + \frac{1}{e(g)(x_0)\sigma^2})\sigma_0^2 \frac{1}{Vol(B_{\sigma_0}(x_0)))} \int_{B_{\sigma_0}(x_0)} e(g)dx$$

for any $\sigma \in (0, \sigma_0)$.

Let $\tilde{C} = 1/2C$, and assume

$$\sigma_0^2 \frac{1}{Vol(B_{\sigma_0}(x_0)))} \int_{B_{\sigma_0}(x_0)} e(g)dx < \tilde{C}.$$

Either we will get:

$e(g)(x_0)\sigma_0^2 > 1$ (then let $\sigma^2 = \frac{1}{e(g)(x_0)} < \sigma_0^2$ we have

$$1 \leq 2C\sigma_0^2 \frac{1}{Vol(B_{\sigma_0}(x_0)))} \int_{B_{\sigma_0}(x_0)} e(g)dx < 1,$$

which is a contradiction).

Or we will get: $e(g)(x_0)\sigma_0^2 < 1$, which gives that

$$1 \leq C(1 + \frac{1}{e(g)(x_0)\sigma^2})\sigma_0^2 \frac{1}{Vol(B_{\sigma_0}(x_0)))} \int_{B_{\sigma_0}(x_0)} e(g)dx.$$

Solving this inequality yields

$$e(g)(x_0)\sigma^2 \leq 2C\sigma_0^2 \frac{1}{Vol(B_{\sigma_0}(x_0)))} \int_{B_{\sigma_0}(x_0)} e(g)dx$$

for any $\sigma \in (0, \sigma_0)$.

Use the monotonicity formula again, we get

$$\sup_{\sigma \in [0,r]} \sigma^2 \sup_{B_{r-\sigma}} e(g) \leq \frac{1}{\tilde{C}} r^{2-2n} \int_{B_{\frac{3}{2}r}} e(g) dV_{g_0}$$

If we adjust some of the constant right from the beginning, we have

$$\sup_{B_{\frac{r}{2}}} e(g) < \frac{4}{\tilde{C}} \frac{1}{B_r} \int_{B_r} e(g).$$

q.e.d.

Now consider $g = g_1 + \partial\bar{\partial}u$, and u is normalized so that $\int_M u dV_0 = 0$; we want to estimate the Kähler potential u.

For $y \in M$ satisfying

$$r^{2-2n} \int_{B_r(y)} e(g) dV_{g_0} < C', \qquad x \in B_{r/2}(y),$$

we have

$$tr_{g_0} g_1(x) + \Delta_0 u(x) = e(g)(x) \leq \frac{C}{r^{2n}} \int_{B_{r(y)}} e(g) dV_0.$$

The following well-known lemma can be found in [16]. For completeness we will give the proof.

Lemma 2.2. *That g is positive definite implies $u \leq C$; C depends on upper bound of g_1 with respect to g_0.*

Proof. First notice

$$u(x) = \frac{1}{Vol M} \int_{y \in M} u(y) dy - \int_{y \in M} G(x,y)(\Delta u(y)) dy,$$

where $G(x,y)$ is the Green function of the Laplacian which is normalized as

$$\int_{y \in M} G(x,y) dy = 0 \quad \text{for any } x \in M.$$

Recall that when g is positive, u satisfies

$$-\Delta_0 u(x) \leq tr_{g_0} g_1(x) \leq C, \qquad \int_M u dV_0 = 0.$$

Substituting the above in the previous formula and taking the supremum, we get

$$\sup_{x \in M} u(x) \leq C Vol(M) \sup_{x,y \in M} (-G(x,y)).$$

Since the Green function is bounded from below, $u \leq C$. q.e.d.

By this lemma we can normalize u so that $u < 0, \int_M u dV_0 = C$. Thus,

$$-C_1 \leq \Delta_0(-u) \leq \frac{C_2}{Vol(B_r(x))} \int_{B_r(x)} e(g) dV_0 - C_3.$$

The following PDE lemma is standard:

Lemma 2.3. *$-\Delta v \leq K, v > 0$ in $B_r(x)$ imply that*

$$\sup_{B_{r/2}(x)} v(x) \leq C(\frac{1}{Vol(B_r(x))} \int_{B_r} v dx + r^2 K).$$

In our case, use $-u$ in place of v; when

$$r^{2-2n} \int_{B_r(x)} e(g) dV_{g_0} < C'$$

we get

$$\begin{aligned} \sup_{B_{r/4}(x)} (-u) \leq & C(\frac{1}{Vol(B_r(x))} \int_{B_{r/2}(x)} (-u) dx + r^{2-2n} \int_{B_r(x)} e(g) dV) \\ \leq & r^{-2n} C + C'. \end{aligned}$$

Remark. We are not requiring g to be in the interior of Kähler cone, so $g \geq 0$ and $g_1 \geq 0$ are enough. Every estimate depends only on the geometry of the background metric g_0.

3. Convergence of Kähler metrics with bounded curvature

Consider a sequence of Kähler metrics $\{g_m, m = 1, 2, ...\}$ on M. In this section we would like to use the estimates from the last section to discuss the convergence of the sequence. Suppose $[g_m] \longrightarrow [\bar{g}]$ as Kähler classes, where $[g]$ represents the Kähler class of g. We require that $\bar{g} \geq 0$, g_0 is the background metric, and $\{g_m\}$ have sectional curvature uniformly bounded from above.

We will first use a weak convergence argument to get a weak limit as positive $(1,1)$ current of the sequence of Kähler metrics. This is a very standard argument in complex analysis. Then we will use the estimates discussed from last section combined with a theorem of Siu to show that the limiting current is smooth away from a closed subvariety(a sort of partial regularity), also we will show the strong convergence of the sequence away from that subvariety, and estimate the limiting Kähler potential around the subvariety.

Weak convergence argument. Recall that a Kähler metric can be viewed as a closed positive $(1,1)$ current. Currents simply mean differential forms with distribution as coefficients, and differentials are understood in the distribution sense.

Viewing $\{g_m\}$ as closed positive $(1,1)$ currents, since $[g_m] \longrightarrow [\bar{g}]$ as Kähler classes, we have $\int g_m \wedge g_0^{n-1} < C$ uniformly. Then by the general theory of closed positive current there exists a subsequence (for simplicity of notation, we will always use the same letter for the subscripts of subsequences) g_m converge to g_∞ weakly as Borel measure. Since closedness and positivity are preserved under weak limit, g_∞ is a closed positive (1,1)-current. Before further discussion, we want to introduce the definition and basic property of Lelong number $\nu(g_\infty, x)$ of closed positive (1,1)-current as follows.

Definition 3.1. Let ω be a closed positive (1,1)-current defined on a neighborhood of zero in $\mathbf{C^n}$, and

$$\omega_0 = dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2 + \cdot + dz^n \wedge d\bar{z}^n.$$

The **Lelong number** of ω at point 0 is defined as:

$$\nu(\omega, 0) = \lim_{r\to 0} cr^{2-2n} \int_{B_r(x)} \omega \wedge \omega_0^{n-1}.$$

Lelong number of closed positive (1,1)-current is a generalization of the concept of multiplicity of complex codimension 1 subvariety. The constant c in the definition can be chosen to make the Lelong number coincide with multiplicity when taking the (1,1)-current that represents a complex codimension 1 subvariety. In our paper for simplicity we will omit this constant c. Especially for g_∞

$$\nu(g_\infty, x) = \lim_{r\to 0} r^{2-2n} \int_{B_r(x)} e(g_\infty).$$

For a plurisubharmonic function w on M, $\partial\bar{\partial}w$ is a closed positive (1,1)-current. One can define the Lelong number of w to be the Lelong number of $\partial\bar{\partial}w$ at corresponding point. A useful formula for $\nu(w, x)$ is

$$\nu(w, x) = \liminf_{z\to x} \frac{w(z)}{log|z-x|}.$$

One result from complex analysis which makes the concept of Lelong number very useful is the following theorem of Siu.

Theorem 3.1 ([15]). *If u is a d-closed positive (k,k)-current on an open set Ω of $\mathbf{C^n}$, then for $c > 0$ the set E_c of the points of Ω where the Lelong number of u is $\geq c$ is a subvariety of codimension $\geq k$ in Ω.*

Define $E_\infty = \{x \in M | \nu(g_\infty, x) \geq C'/2\}$. Then by the above Theorem of Siu, E_∞ is a complex subvariety of complex dimension $\leq n-1$.

For any $x \in M \setminus E_\infty$, $\nu(g_\infty, x) < C'/2$. One can choose small r such that

$$r^{2-2n} \int_{B_r(x)} e(g_\infty) < 3C'/4.$$

Since g_m converges to g_∞ weakly, we have

$$r^{2-2n} \int_{B_r(x)} e(g_\infty) = \lim_{m\to\infty} r^{2-2n} \int_{B_r(x)} e(g_m) < 3C'/4.$$

For m large we will have

$$r^{2-2n} \int_{B_r(x)} e(g_m) < C'.$$

By Proposition 1.1,

$$\sup_{B_{r/2}} e(g) < \frac{4}{C' r^{2n}} \int_{B_r} e(g) \leq \frac{4}{r^2}.$$

This implies that away from E_∞, g_m converges to g_∞ weakly in L^∞ norm and strongly in L^p norm for $p < \infty$.

Now we have finished the proof of the following:

Proposition 3.1. *Given Kähler manifold (M, g_0) and a family of Kähler metrics $\{g_m\}$ such that $\int g_m \wedge g_0^{n-1} < C$ uniformly, then (by possibly taking subsequence), $g_m \longrightarrow g_\infty$ weakly in the sense of distribution. Here g_∞ is a closed positive (1,1)-current. and there exists a $c > 0$, away from subvariety*

$$E_\infty = \{x | \nu(g_\infty, x) \geq c\}, g_m \longrightarrow g_\infty$$

weakly in L^∞_{loc}, strongly in L^p_{loc}, for any $p < \infty$.

To get the limit potential we need the following Lemma:

Lemma 3.1. *Let $g_m = g_m^0 + \partial\bar\partial u_m$. Then there exists a subsequence still denoted by $\{u_m\}$ such that $u_m \longrightarrow u_\infty$ in L^1 norm and $g_\infty = g_\infty^0 + \partial\bar\partial u_\infty$.*

Proof. By the above proposition, $\Delta u_m = e(g_m) - e(g_m^0)$ is weakly convergent to $e(g_\infty) - e(g_\infty^0)$. Since the Green operator is a compact operator from Borel measure to $L^1(M)$, $u_m \longrightarrow u_\infty$ strongly in L^1 norm.

$\partial\bar{\partial}$ is weakly continuous, so $g_m^0 + \partial\bar{\partial}u_m \longrightarrow g_\infty + \partial\bar{\partial}u_\infty$ weakly. Clearly g_∞ is a closed positive (1-1)-current. q.e.d.

We will need the following asymptotic behavior of Kähler potential later.

Lemma 3.2. *$\forall x \in E_\infty$, $\exists C(x)$, r, such that $\forall z \in B_r(x)$,*

$$u_\infty(z) \leq C(x) + \frac{C'}{2} log|z-x|^2.$$

Proof. Consider $v = \frac{C'}{2} log|z-x|^2$, $(\partial\bar{\partial}v)^n = \frac{C'}{2}\delta_x$. Then, $\nu(v,x) = C'/2$, and $\nu(v,z) = 0, \forall z \neq x$. By comparing with g_∞ and use the formula of Lelong number for plurisubharmonic functions which we mentioned earlier when we define Lelong number, we immediately get the lemma. q.e.d.

Remark. For all the arguments we do not need the limit $[\bar{g}]$ in the interior of Kähler cone.

4. A different perspective: convergence of Kähler manifolds

So far we have been discussing the convergence of Kähler metrics on a fixed compact complex manifold. In this section, we will analyze from a different point of view by considering $\{(M, g_m)\}$ as a sequence of Kähler(Riemannian) manifolds, and its convergence in the sense of Cheeger-Gromov.

From the discussion in the introduction, we can discuss their convergence under quite general condition. But we need some non-collapsing condition to ensure that we do not end up getting nothing, and maintain the relation with the original complex manifold. It turns out that the following very weak non-collapsing assumption is good enough in our situation. This is very special for Kähler manifolds compared to Riemannian manifolds. Under this very weak assumption we will show that the sequence do not collapse, and we can get a $C^{1,\alpha}$ limiting complete "Kähler manifold". For Riemannian Manifolds one would stop here, but in Kähler case we have an integrable almost complex structure. Using Newlander-Nirenberg theorem one can actually see that the limiting "Kähler manifold" is actually complex analytic.

The above discussion uses the Riemannian convergence result almost directly, which is a kind of convenience. On the other hand, to go back to complex analytic category in the end, one has to use non-trivial theorem of Newlander-Nirenberg. At last in this section we will look at the Riemannian convergence argument much closely and explore an argument which will stay within the holomorphic category. Therefore will avoid using Newlander-Nirenberg theorem.

First, let us start with the following assumption.

Assumption (*). Assume $\exists r > 0, C > 0$, such that $\forall m, \exists x_m$ satisfying

$$dist_{g_0}(x_m, E_\infty) \geq r, \qquad \frac{det(g_m)}{det(g_0)}|_{x_m} \geq C.$$

Roughly speaking, we assume the volume of $\{g_m\}$ is not collapsing everywhere away from E_∞. This is a very weak assumption, because the opposite of it is exactly that the sequence of metrics are collapsing everywhere away from E_∞; the case we will consider in Section 7. The following lemma will show that under the bounded curvature condition the above very weak non-collapsing assumption will imply non-collapsing away from blow-up locus.

Lemma 4.1. *Under Assumption(*), one can find $r_1 > 0$, $C_1, C_2 > 0$ such that in $B_{r_1}(x_m)$, $C_1 g_0 \leq g_m \leq C_2 g_0$.*

Proof. Since $dist_{g_0}(x_m, E_\infty) \geq r$, one can find r_1, such that

$$(2r_1)^{2-2n} \int_{B_{4r_1}(x_m)} e(g_m) < C'.$$

Then in $B_{2r_1}(x_m)$ by Proposition 2.1 we have

$$e(g_m) \leq \frac{C}{B_{4r_1}(x_m)} \int_{B_{4r_1}(x_m)} e(g_m).$$

Hence, there exists C_2, such that

$$g_m \leq C_2 g_0 \quad \text{in } B_{2r_1}(x_m).$$

Let $V_m(x) = \frac{det(g_m)}{det(g_0)}|_x$. Then

$$-\partial\bar{\partial} log V_m(x) = Ric_{g_m} - Ric_{g_0},$$

$$-\Delta_{g_0} log V_m(x) = tr_{g_0} Ric_{g_m} - r_0,$$

where r_0 is the scalar curvature of g_0.

Since Ric_{g_m} is bounded with respect to g_m, we get

$$|tr_{g_0} Ric_{g_m}| \leq Ce(g_m) < C$$

in $B_{2r_1}(x_m)$ (possibly taking r_1 smaller.)

Notice the assumption (*) yields

$$|\Delta_{g_0} log V_m(x)| = |tr_{g_0} Ric_{g_m} - r_0| \leq C,$$

and

$$V_m(x_m) \geq c, \quad V_m \leq \left(\frac{tr_{g_0} g_m}{n}\right)^{\frac{n}{2}} \leq C.$$

Use the standard Harnack estimate for Laplacian, we obtain

$$\frac{det(g_m)}{det(g_0)} = V_m \geq C \qquad \text{in } B_{r_1}(x_m),$$

which together with

$$g_m \leq C_2 g_0 \quad \text{in } B_{2r_1}(x_m).$$

gives the conclusion of the lemma. q.e.d.

Let us consider the convergence property of the sequence of Kähler manifolds $\{(M, x_m, g_m), m = 1, 2, ...\}$ as introduced in Section 2. By the above lemma, assumption(*) is enough to ensure that $\{(M, x_m, g_m)\}$ do not collapse. We first state a version of Gromov compactness theorem here.

Theorem 4.1. *Let $\{(M_i, x_i, g_i), x_i \in M_i\}$ be a sequence of n-dimensional pointed Kähler manifolds, and Ω_i a sequence of subdomain in M_i with boundary $\partial\Omega_i$. Suppose the following is true for all i:*

(i) $\|R_{g_i}\|_{g_i}$ are uniformly bounded for x in Ω_i;

(ii) $i_{g_i}(x) \geq C$ for $x \in \Omega_i$;

(iii) $0 \leq C' \leq Vol_{g_i}(\Omega_i) \leq C''$ for some uniform constants C' and C''.

Then, given any $\epsilon > 0$, there exist a subsequence also denoted by $\{(\Omega_i(\epsilon), X_i, g_i), i = 1, 2, ...\}$, where,

$$\Omega_i(\epsilon) = \{x \in \Omega_i | dist_{g_i}(x, \partial\Omega_i) > \epsilon\}$$

and a Kähler manifold $(\Omega_\infty(\epsilon), x_\infty, g_\infty)$ such that for compact $K \subset \Omega_\infty(\epsilon)$, $\exists\epsilon' > \epsilon$, such that for i large, there exist diffeomorphisms $\phi_i : \Omega_i(\epsilon') \longrightarrow \Omega_i(\epsilon)$ satisfying the following:

(1) $K \subset \phi_i(\Omega_i(\epsilon'))$ *for any* $i \geq 0$.

(2) $(\phi_i^{-1})^* g_i$ *converge to* g_∞ *in* $C^{1,\alpha}(K), \forall \alpha < 1$.

(3) $(\phi_k)_* J_k \circ (\phi_k^{-1})_*$ *converge uniformly to* J_∞ *in* $C^{1,\alpha}(K)$*, where* J_k, J_∞ *are the almost complex structures of* Ω_i,$\Omega_\infty(\epsilon)$ *respectively.*

This version of compactness theorem can be found in [16]. For the readers' convenience we give a sketch of the idea of the proof.

Sketch of the Proof. The standard argument in Riemannian geometry of Gromov compactness will show the theorem except for (3).

For (3), since $\tilde{g}_i = (\phi_i^{-1})^* g_i \longrightarrow g_\infty$ in $C^{1,\alpha}(K)$, $\tilde{J}_i = (\phi_i)_* \circ J_i \circ (\phi_i^{-1})_*$ is uniformly bounded, and satisfy a first order differential equation: $\nabla_{\tilde{g}_i} \tilde{J}_i = 0$, which only involve the Christoffel symbols. These imply that $\{\tilde{J}_i\}$ is uniformly bounded in $C^{1,\alpha}(K)$, $\forall \alpha < 1$.

With possibly taking subsequence, we may assume that $\tilde{J}_k \longrightarrow J_\infty$ in $C^{1,\alpha}(K)$, $\forall \alpha < 1$. It is easy to see J_∞ is integrable, since $\nabla_{g_\infty} J_\infty = 0$, and $(\Omega_\infty(\epsilon), g_\infty)$ is a Kähler manifold.

Clearly we may require $\phi_k(x_i) \longrightarrow x_\infty \in \Omega_\infty(\epsilon)$.

Remark. Condition on curvature may be relaxed. For example $\sup_{x\in K} |Ric_{g_i}|_{g_i}$ bounded and $\int |R|^p$ bounded for some $p > n$. Reducing to $\int |R|^n$ may induce orbifold singularity.

Now go back to our situation. Let

$$\Omega_{m,\delta} = \{x \in M | i_{g_m}(x) > \delta\}_0,$$

where the subscript 0 indicates taking the connected component which contains x_m. $\{(\Omega_{m.\delta}, x_m, g_m)\}$ clearly satisfies the conditions of the above theorem.

By possibly taking subsequence we have $\{(\Omega_{n,\delta}(\delta), x_n, g_n)\}$ converging to $(\Omega_\infty(\delta), x_\infty^\delta, g_\infty^\delta)$ in $C^{1,\alpha}$ norm.

Let $\delta \longrightarrow 0$, by possibly taking diagonal subsequence,

$$\{(\Omega_{n,1/n}(1/n), x_n, g_n)\}$$

converges to $(M_\infty, x_\infty, g_\infty)$ in $C^{1,\alpha}_{loc}$ norm. Notice here $(M_\infty, x_\infty, g_\infty)$ only represents one of the complete connected component (say $M_{\infty,0}, g_{\infty,0}$) of $(M_\infty, g_\infty) = \bigcup_i (M_{\infty,i}, g_{\infty,i})$ as discussed in the introduction. For simplicity of notation, we omit the subscript 0 here.

To show that (M_∞, J_∞) is a complex manifold we need Newlander-Nirenberg Theorem. Since in our case, we only have $J_\infty \in C^{1,\alpha}$,. We need a improved version as follows, whose proof can be find in [12].

Theorem 4.2 (Newlander-Nirenberg). *Let (M, J) be an almost complex manifold with an integrable almost complex structure $J \in C^{1,\alpha}$. Then (M, J) is a complex manifold.*

The above argument uses the non-trivial theorem of Newlander-Nierenberg. As mentioned at the beginning of this section, it clearly would be of interest to see if one can do all the arguments in complex analytic category to avoid the use of Newlander-Nirenberg theorem. This is indeed possible. First let us analyze the convergence argument in the Riemannian case. It is well known that at the core of this kind of convergent result is the proof of the existence of harmonic coordinates, i.e., charts for which the coordinate functions are harmonic functions, on balls of uniform size(depending on the geometric conditions), and uniform $C^{1,\alpha}$ estimates of the metric tensor g_{ij} in these coordinates. For our situation, if we use holomorphic coordinate chart coordinate functions will be automatically harmonic. The above argument seems to be extremely well adapted to the Kähler case. For reader's convenience we will give the detail which will also indicate the proof of Riemannian convergence results.

Definition 4.1. A compact Kähler manifold (M, g) is said to have an adapted holomorphic coordinate atlas(more precisely an $(r, N, C^{1,\alpha})$ adapted atlas), if there is a covering $\{B_{x_k}(r)\}_1^n$ of M by geodesic r-balls, for which the balls $B_{x_k}(r/2)$ also cover M and the balls $B_{x_k}(r/4)$ are disjoint, such that each $B_{x_k}(10.r)$ has a holomorphic coordinate chart $U_k = U = \{z^j\}_1^n$, such that the metric tensor in these coordinates is $C^{1,\alpha}$ bounded. i.e., if $g_{i\bar{j}} = g(\partial/\partial z^i, \partial/\partial z^j)$ on $B_{x_k}(10.r)$, then

$$C^{-1}.\delta_{ij} \leq g_{i\bar{j}} \leq C.\delta_{ij} \text{ (as bilinear forms)}$$

and

$$r^{1+\alpha}\|g_{i\bar{j}}\|_{C^{1,\alpha}} \leq C,$$

for some constant $C > 1$, where the norms are taken with respect to the coordinates z^j on $B_{x_k}(10.r)$.

Finding good holomorphic coordinates with respect to geometry is an interesting question. The following lemma will show that once a good holomorphic coordinate chart is constructed the convergence result will be immediate for the proof one may see the analogue Riemannian case in [1]

Lemma 4.2. *Let (M_i, g_i) be a sequence of compact Kähler manifolds(with or without boundary), which have an $(r, N, C^{1,\alpha})$ adapted* *holomorphic coordinate atlas, for some fixed r, N and $C^{1,\alpha}$ Hölder constant C. Then a subsequence of $\{(M_i, g_i)\}$ converges, in the $C^{1,\alpha'}$ topology, for $\alpha' < \alpha$, to a $C^{1,\alpha}$ Kähler manifold (M, g).*

With this lemma in mind, it is thus natural to seek geometric conditions which imply that a manifold has an adapted harmonic coordinate chart. It turns out that by replacing harmonic with holomorphic, the Riemannian argument will naturally go through in the Kähler case. For completeness we will state and sketch a proof of the Kähler version of this result. One may look into [1] for the detail of the Riemannian case.

Lemma 4.3. *Let (M, g) be a compact Kähler manifold (with or without boundary), such that*

$$|Ric_M| \leq \lambda,\ inj_M B_x(\frac{1}{2}.dist(x, \partial M)) \geq i_0(x) > 0.$$

Then given any $C > 1$ and $\alpha \in (0,1)$, there is an $\epsilon_0 = \epsilon_0(\lambda, C, n, \alpha)$ with the following property: given any $x \in M$, there is a holomorphic coordinate system $U = {z^i}_1^n$ defined on $B_x(\epsilon(x)) \subset M$ such that if $g_{i\bar{j}} = g(\partial/\partial z^i, \partial/\partial z^j)$, then $g_{i\bar{j}}(x) = \delta_{ij}$ and

$$C^{-1}.\delta_{ij} \leq g_{i\bar{j}} \leq C.\delta_{ij} \text{ (as bilinear forms)} \\ \epsilon(x)^{1+\alpha}\|g_{i\bar{j}}(y)\|_{C^{1,\alpha}} \leq C, \tag{4.1}$$

for all $y \in B_x(\epsilon(x))$, where

$$\frac{\epsilon(x)}{i_0(x)} \geq \epsilon_0 \frac{dist(x, \partial M)}{diam M} > 0.$$

Remark. It is easier to understand the statement of the lemma when M is compact without boundary. In that case the lemma claims that if

$$|Ric_M| \leq \lambda,\ inj_M \geq i_0 > 0,$$

then there is an adapted holomorphic coordinate atlas on (M, g).

Proof. The lemma is clearly true if ϵ_0 is allowed to depend on x. We need to remove this dependence.

We argue by contradiction. If the lemma does not hold, then given any $C > 1$ and $\alpha \in (0,1)$, together with the bound λ, there is a sequence of Kähler manifolds (M_i, g_i), points $x_i \in M_i$ and balls $B_{x_i}(\epsilon(x_i)) \subset M_i$ such that

$$\frac{diam M_i}{dist(x_i, \partial M_i)} \frac{\epsilon(x_i)}{i_0(x_i)} = \epsilon_i \to 0, \quad \text{as } i \to \infty, \tag{4.2}$$

with (4.1) valid for $\{g_i\}$. We may assume (and will) that the points x_i realize the minimum value of the ratio (4.2), as a function of x, where $\epsilon(x)$ is the maximum radius of the geodesic ball at x on which (4.1) holds. Rescale the metrics so that $h_i = \epsilon(x_i)^{-2} g_i$ which have the effect of expanding the ball $B_{x_i}(\epsilon(x_i)) \subset M_i$ to a ball of radius 1 in the h_i-metric. We now consider, the sequence of pointed Kähler manifolds (M_i, x_i, h_i). These manifolds have the property that,

$$\begin{aligned} &(i) \quad \|Ric_{h_i}\|_{C^0} \to 0, \\ (4.3) \qquad &(ii) \quad inj_{h_i} \to \infty, \\ &(iii) \quad dist_{h_i}(x_i, \partial M_i) \to \infty \end{aligned}$$

uniformly on compact subsets. On any uniform sized balls with center x_i, we have adapted holomorphic coordinate atlases of uniform size $C > 1$ independent of i. Also by the definition of $\epsilon(x_i)$, for any $r > 1$, $B_{x_i}(r) \subset M_i$ are not adapted holomorphic coordinate balls with constant C.

Now by using Lemma 4.3, the sequence $\{(M_i, x_i, h_i)\}$ converges in $C^{1,\alpha'}$ sense to a $C^{1,\alpha}$ Kähler manifold (N, x, h), for $\alpha' < \alpha$, with $x = \lim x_i$.

Conditions (4.3) imply that (N, h) is Ricci flat, complete and that injective radius is infinity. Then by an argument of Anderson using Cheeger-Gromoll splitting theorem one can show that $(N, h) \cong (\mathbf{C^n}, \delta)$. For detail we refer to [1].

This will lead to that for i large enough, there will exist adapted holomorphic coordinate balls centered at x_i with constant C and arbitrary large radius, a contradiction. q.e.d.

5. Construction and surjectivity of the map

In the last two sections, we discussed convergence from two different perspectives. In the case of convergence, we constructed two objects: a Zariski open set $U = M - E_\infty$ of original complex manifold M, and a connected component M_∞ of the complete limiting Kähler manifold with possibly many components. Since Kähler manifolds are quite rigid, the two constructions are closely related. In this section we will construct a holomorphic map $\psi_\infty : M - E_\infty \longrightarrow M_\infty$ and prove its surjectivity to M_∞.

Recall our results in Section 2. $\{g_m\}$ converges weakly to g_∞ and strongly away from E_∞. E_∞ is a subvariety of M of codimension ≥ 1.

First we want to construct the map. Let

$$M_\epsilon = \{x \in M | dist_{g_0}(x, E_\infty) \geq \epsilon\}.$$

Clearly (M_ϵ, g_n) have bounded diameter. When δ is small relative to ϵ, the inclusion map

$$i_\epsilon : (M_\epsilon, g_n) \hookrightarrow (\Omega_{n,\delta}, g_n)$$

is well defined.

Consider maps:

$$\psi_k^\epsilon = \phi_k \circ i_\epsilon : (M_\epsilon, g_0) \longrightarrow (M_\infty, g_\infty) \quad \text{(not holomorphic!)}.$$

From the harmonic map estimate in Section 2, it is clear that $\|d\psi_k^\epsilon\| \leq C_\epsilon$ uniformly with respect to k. Noticing that $g_k \longrightarrow g_\infty$ and $J_k \longrightarrow J_\infty$ in $C^{1,\alpha}_{loc}$, also $\bar\partial_{g_k}\psi_k^\epsilon = 0$, one can derive the uniform $C^{2,\alpha'}_{loc}$ convergence for ψ_k^ϵ.

By possibly taking subsequence, we may assume that ψ_k^ϵ converges to ψ_∞^ϵ;

$$\psi_\infty^\epsilon : (M_\epsilon, g_0) \longrightarrow (M_\infty, g_\infty)$$

is holomorphic. Taking a diagonal subsequence, let $\epsilon_k = 1/k \longrightarrow 0$, $\psi_k^{1/k}$ converges to ψ_∞ locally;

$$\psi_\infty : M - E_\infty \longrightarrow M_\infty$$

is holomorphic. By assumption (*) and Lemma 4.1, ψ_∞ is an open map.

Now we have constructed the map ψ_∞, and next we will show that ψ_∞ is surjective; we need the following theorem of Bishop.

Theorem 5.1 (Bishop). *Let W be a subvariety of a domain U. If V is a purely k-dim subvariety of $U - W$ whose $2k$-volume is finite, then $\bar V$ is an analytic subvariety of U, and $\bar V - V$ is a variety of dim $\leq k-1$.*

In our case, consider

$$Graph(\psi_\infty) \subset (M - E_\infty) \times M_\infty \subset M \times M_\infty.$$

Using Theorem 5.1, we can conclude that

$$\psi_\infty(M - E_\infty) \subset M_\infty$$

is Zariski open, and

$$M_\infty - \psi_\infty(M - E_\infty) = D_\infty$$

is a subvariety of codim$\leq n-1$.

We want to show D_∞ is empty (i.e., ψ_∞ is surjective).

Lemma 5.1. *ψ_∞ is surjective.*

Proof. Notice $g_\infty = \psi_{\infty *} g_{\infty,0} + \partial\bar{\partial} u_\infty$. Recalling in Section 2, we showed that when

$$x_i \longrightarrow x_\infty \in E_\infty,$$

$$u_\infty(x_i) \longrightarrow -\infty.$$

Now $\forall x \in D_\infty$, $\exists$ a regular disc

$$\alpha : B \longrightarrow M_\infty,$$

$\alpha(0) = x$, $\alpha(B) \cap D_\infty = x$. $\hat{g}_\infty = \alpha^* g_\infty$,$\hat{g}_0 = \alpha^* \psi_{\infty *} g_{\infty,0}$, $\hat{u}_\infty = \alpha^* u_\infty$. Then $\hat{g}_\infty = \hat{g}_0 + \partial\bar{\partial}\hat{u}_\infty$.

Consider the pullback of $\alpha(B)$ to M via ϕ_n. When N is large, $\phi_n^{-1}\alpha(B)$ is almost holomorphic in $(\Omega_{n,1/1}(1/n), g_n)$. We may find $\alpha'(B)$close to $\phi_n^{-1}\alpha(B)$ in $C^{2,\alpha}$ norm, such that $\alpha'(B)$ is a holomorphic disc in M.

$$\hat{g}_n = \hat{g}_0 + \partial\bar{\partial}\hat{u}_n, \qquad 1 - \Delta_{\hat{g}_n}\hat{u}_n = tr_{\hat{g}_n}\hat{g}_0 > 0.$$

Around $\alpha'(\partial B)$, $\hat{u}_n$ is bounded uniformly for n. $\Delta_{\hat{g}_n}\hat{u}_n \leq 1$ implies

$$\sup_{\alpha'(B)} -\hat{u}_n \leq \sup_{\alpha'(\partial B)} -\hat{u}_n + C.$$

Namely, $\hat{u}_n(0)$ bounded from below. But $\hat{u}_\infty(0) = -\infty$, contradiction. So D_∞ is empty and $\psi : M - E_\infty \longrightarrow M_\infty$ is surjective. q.e.d.

Now summarizing the results from the last several sections we have already proved the following theorem except for the injectivity of the map which will be a simple corollary of Lemma 8.1 in Section 8.

Theorem 5.2. *Fix a compact complex manifold M. Let $\{g_i\}$ be a sequence of Kähler metrics on M with bounded Kähler classes. Consider the sequence of Kähler manifolds $\{(M, g_i)\}$. Assume that $|K_{g_i}| < C$ uniformly. Then there exist a subsequence(still use the same subscript for simplicity) $\{(M, g_i)\}$ converging to (M_∞, g_∞) (in the sense as discussed in introduction), and a Zariski open subset $U \subset M$, on which one of the following will happen:*

(i) (Collapsing) $\omega_{g_i}|_U \longrightarrow \omega_{g_\infty}|_U$ *uniformly and* $\omega_{g_\infty}|_U$ *degenerate everywhere, i.e.,* $\omega^n_{g_\infty}|_U = 0$.

(ii) (Non-collapsing) *There exists one of the components*

$$(M_{\infty,0}, g_{\infty,0}) \subset (M_\infty, g_\infty),$$

which is a complete Kähler manifold, and a biholomorphic map

$$\psi_\infty : U \longrightarrow M_{\infty,0}.$$

So far, we have finished the discussion of the convergence part of the argument, and starting from next section we will concentrate on collapsing related problem.

Remark. The techniques developed from previous several sections can also be used to discuss the blow-up components of the limiting manifold, such as $(M_{\infty,i}, g_{\infty,i}) \subset (M_\infty, g_\infty)$ for $i > 0$. The idea is to reverse the role of M_∞ and M, consider the corresponding equations, and also use the Schoen-Uhlenbeck estimate to construct holomorphic maps from $(M_{\infty,i}, g_{\infty,i})$ to $E_\infty \in M$. This will result in some fiber structure, or be considered as foliation. Later on we will use the technique developed for collapsing to analyze the detail of the fiber structure resulting from "blow-up".

6. Homogeneous Monge-Ampère equation

As mentioned in the introduction, the collapsing metric locally can be written as a homogeneous complex Monge-Ampère equation. In this section we would do some computations which generalize a result of D. Burns[3]. Then we will discuss its relevance to our collapsing problems and construct local holomorphic vector fields along collapsing direction.

Consider locally in $\mathbf{C^n}$ the following homogeneous complex Monge-Ampère equation.

$$(\partial\bar{\partial}u)^n = 0. \tag{6.1}$$

Denote $\omega = \partial\bar{\partial}u$, and assume $\omega^{n-r+1} = 0, \omega^{n-r} \neq 0, (1 \leq r \leq n)$, i.e., ω has rank $n-r$.

It is clear that solution to (6.1) is not unique; different solutions differ by pluriharmonic functions. Let $u_a, (a = 1, 2, ...r)$ satisfy:

$$\omega = \partial\bar{\partial}u_a, \qquad (\partial\bar{\partial}u_a)^n = 0. \qquad (a = 1, 2, ..., r).$$

Denote $\tau_a = exp(u_a), \tau = \Sigma_{a=1}^r \tau_a$. By the condition $\omega^{n-r} \neq 0$ we have that if u_a's are chosen general enough, then $\Omega = \partial\bar{\partial}\tau$ locally will be a Kähler metric. In local coordinates, $ds^2 = \tau_{i\bar{j}} dz^i d\bar{z}^j$ (summation convention $i, j = 1, ..., n$) where $\tau_i = \partial\tau/\partial z^i$, etc. Define $\tau^{i\bar{j}}$ by the relation $\tau^{i\bar{j}}\tau_{k\bar{j}} = \delta^i_k$. Following D. Burns [3], we consider the vector fields of type (1,0):

$$X_a = \tau^{i\bar{j}}\tau_{a\bar{j}}\frac{\partial}{\partial z^i}.$$

Invariantly, X is determined uniquely by

$$\eta_a = \mathrm{Re}(\mathrm{X_a}) = \frac{1}{4}\mathrm{grad}(\tau_\mathrm{a}),$$

where "grad" means the gradient of τ in the τ-metric.

Notice the following facts:

$$\partial\bar{\partial}\tau_a = \tau_a(\partial\bar{\partial}u_a + \partial u_a \bar{\partial} u_a),$$

$$\Omega = \partial\bar{\partial}\tau = \tau\omega + \sum_a \tau_a \partial u_a \bar{\partial} u_a,$$

$$\begin{aligned}
\Omega^{n-1} =& \frac{(n-1)!}{(n-r)!}\tau^{n-r}\omega^{n-r}(\sum_a(\prod_{b\neq a}(\tau_b\partial u_b\bar{\partial}u_b)) \\
&+ \frac{(n-1)!}{(n-r-1)!}\tau^{n-r-1}\omega^{n-r-1}\prod_a(\tau_a\partial u_a\bar{\partial}u_a), \\
\Omega^{n-1}\wedge\bar{\partial}\tau_b =& \frac{(n-1)!}{(n-r)!}\tau^{n-r}\omega^{n-r}(\prod_a(\tau_a)\prod_{a\neq b}\partial u_a\bar{\partial}u_a)\bar{\partial}u_b, \\
\Omega^n =& \frac{n!}{(n-r)!}\tau^{n-r}\omega^{n-r}(\prod_a(\tau_a)\prod_a\partial u_a\bar{\partial}u_a),
\end{aligned}$$

$$X_a = n\frac{\Omega^{n-1}\wedge\bar{\partial}\tau_a}{\Omega^n}.$$

Proposition 6.1.

$$i(X_a)\omega = 0.$$

Proof. Need to show $< i(X_a)\omega, \bar{v} >= 0, \forall\bar{v}$, i.e.,

$$\Omega^{n-1}\wedge\bar{\partial}\tau_a\wedge(i(\bar{v})\omega) = 0.$$

By the "facts" we have,

$$i(X_a)\omega = (r-1)!\tau^{n-r}\omega^{n-r}\prod_{b\neq a}(\tau_b\partial u_b\bar{\partial}u_b)\tau_a i(\bar{v})\omega.$$

Since

$$\omega^{n-r} i(\bar{v})\omega = \frac{1}{n-r+1} i(\bar{v})\omega^{n-r+1} = 0,$$

we get the desired conclusion. q.e.d.

Notice that

$$\tau_a \Omega^n = n\Omega^{n-1} \wedge \partial\tau_b \wedge \bar{\partial}\tau_b,$$

or, equivalently,

$$X_a(\tau_a) = \tau_a, \tag{6.2}$$

also

$$X_a(\tau_b) = 0, \qquad a \neq b. \tag{6.3}$$

Let L denote a leaf of F (the Monge-Ampère foliation). Then $u_a|_L$ is pluriharmonic. We may locally introduce a complex coordinate $z^a = r_a e^{i\theta_a}$, where $log r_a^2 = u_a$, and θ is a pluriharmonic conjugate(on L)of $u_a/2$. Hence

$$\tau_a|_L = |z^a|^2, \qquad\qquad u_a|_L = log|z^a|2,$$

$$\partial\bar{\partial}\tau|_L = \Sigma_a \tau_a \partial u_a \bar{\partial} u_a = \Sigma_a dz^a \wedge d\bar{z}^a.$$

This imply that $\Omega|_L$ is a flat metric. Extend z^a locally to coordinate holomorphic functions, and complete the system of coordinates such that L is given by $z^{r+1} = ... = z^n = 0$. Proposition 6.1, (6.2) and (6.3) imply that,

$$X_a = z^a \frac{\partial}{\partial z^a}, \qquad \text{along L.}$$

Proposition 6.2. *The leaves of F are totally geodesic in the τ-metric.*

Proof. Need to show $\nabla_{\frac{\partial}{\partial z^a}} \frac{\partial}{\partial z^b} = 0, \;\; 1 \leq a, b \leq r$. We have

$$\begin{aligned}\Gamma_{ab}^k =& \tau^{k\bar{l}} \frac{\partial \tau_{a\bar{l}}}{\partial z^b}\\ =& \tau^{k\bar{l}} \frac{\partial}{\partial z^b}\Big(\sum_c (\tau_c u_{c,a\bar{l}} + \tau_c u_{c,a} u_{c,\bar{l}})\Big).\end{aligned}$$

Notice that

$$u_{c,a\bar{l}}|_L = 0, \qquad \text{and} u_{c,a\bar{l}b} = 0,$$

So,

$$\Gamma^k_{ab} = \tau^{k\bar{l}} \sum_c u_{c,\bar{l}}(u_{c,ab} + u_{c,a}u_{c,b})\tau_c.$$

As

$$u_c = log|z^c|^2, \qquad u_{c,ab} + u_{c,a}u_{c,b} = 0,$$

we get,

$$\Gamma^k_{ab} = 0.$$

i.e.,

$$\nabla_{\frac{\partial}{\partial z^a}} \frac{\partial}{\partial z^b} = 0.$$

q.e.d.

We have seen from the above that the vector fields $X_a, a = 1, 2, ..., r$, tangent to F, are holomorphic along the (holomorphic) leaves of F. To understand when the leaves of F vary holomorphically, following Burns, we introduce the "twist" tensor $\mathcal{L}$ of F, which measures how non-holomorphically the leaves of F vary. The following proposition shows the precise relations.

Proposition 6.3. *The following are equivalent:*
(1) The foliation F is holomorphic.
(2) $\mathcal{L} = 0$.
(3) $X_a, (a = 1, 2, ..., r)$ are holomorphic on M.

Before getting into the proof, let us first explain $\mathcal{L}$.

$\mathcal{L} : \mathcal{T} \otimes \bar{\mathcal{N}} \longrightarrow \mathcal{N}$ is a bundle map, where $\mathcal{T}$ is the tangent bundle of F, and $\mathcal{N}$ is the normal bundle of F, $(\mathcal{T}M/\mathcal{T}.)$

$\forall Z \in \mathcal{T}_p, \bar{W} \in \bar{\mathcal{N}}_p$,

$$\mathcal{L}_p(Z, \bar{W}) = [\tilde{Z}, \tilde{\bar{W}}] \qquad mod(\mathcal{T} \oplus \bar{\mathcal{T}}(M)),$$

where $\tilde{Z}$ is a local section of $\mathcal{T}$ with $\tilde{Z}(p) = Z$, and $\tilde{\bar{W}}$ is a local section of $\bar{\mathcal{T}}(M)$ with $\tilde{\bar{W}}(p) = \bar{W}$.

Take $Z_a = \frac{\partial}{\partial z^a}, a = 1, 2, ..., r$, and $\bar{W}_i = \frac{\partial}{\partial \bar{z}^a}, a = r+1, ..., n$. Let $\tilde{Z}_a = \frac{\partial}{\partial z^a} - u^{j\bar{k}} u_{a\bar{k}} \frac{\partial}{\partial z^j}$. (From now on, $a, b \in \{1, 2, ..., r\}, i, j, k \in \{r+1, ..., n\}$.)

Since $u^{j\bar{k}} u_{j\bar{l}} = \delta^k_l$ locally, we need to check $\tilde{Z}_a \in \mathcal{T}$, which amounts to show that:

$$u_{a\bar{i}} - u^{j\bar{k}} u_{a\bar{k}} u_{j\bar{i}} = 0,$$

and,

$$u_{a\bar{b}} - u^{j\bar{k}} u_{a\bar{k}} u_{j\bar{b}} = 0.$$

The first one is trivial. For the second one we use

$$det \begin{pmatrix} u_{a\bar{b}} & u_{a\overline{(r+1)}} & u_{a\overline{(r+2)}} & \cdots & u_{a\bar{n}} \\ u_{(r+1)\bar{b}} & u_{(r+1)\overline{(r+1)}} & \cdots\cdots & \cdots & u_{(r+1)\bar{n}} \\ \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ u_{n\bar{b}} & u_{n\overline{(r+1)}} & \cdots\cdots & \cdots & u_{n\bar{n}} \end{pmatrix} = 0.$$

Put all these together we get,

$$\mathcal{L}(\frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar{z}^l}) = u^{j\bar{k}} u_{a\bar{k}\bar{l}} \frac{\partial}{\partial z^j} \qquad \mod (\mathcal{T}).$$

So, $\mathcal{L} = 0$ is equivalent to $u_{a\bar{k}\bar{l}} = 0$.

Now we get to the proof of Proposition 6.3.

Proof. The equivalence of (1) and (2) is a fairly routine exercise. (3) implies (2) obviously. We only need to prove (2) implies (3):

Since $X_b = \tau^{i\bar{j}} \tau_{b,\bar{j}} \frac{\partial}{\partial z^a}$, we have:

$$(\tau^{i\bar{j}} \tau_{b,\bar{j}})_{\bar{l}} = -\tau^{i\bar{j}} (\tau_{k\bar{j}\bar{l}} \tau^{k\bar{m}} \tau_{b,\bar{m}} - \tau_{b,\bar{j}\bar{l}}) \qquad \text{along } \mathcal{L},$$

$$\tau^{k\bar{m}} \tau_{b,\bar{m}} = \begin{cases} z^b, & \text{for } k = b; \\ 0, & \text{otherwise.} \end{cases}$$

So,

$$(\tau^{i\bar{j}} \tau_{b\bar{j}})_{\bar{l}} = -\tau^{i\bar{j}} (\tau_{b\bar{j}\bar{l}} z^b - \tau_{b,\bar{j}\bar{l}}).$$

We compute the right-hand side,

$$\begin{aligned}
&\tau_{b,\bar{j}\bar{l}} - \tau_{b\bar{j}\bar{l}} z^b \\
&= \tau_b (u_{b,\bar{j}\bar{l}} + u_{b,\bar{j}} u_{b,\bar{l}}) - z^b (\sum_c \tau_c (u_{c,b\bar{j}} + u_{c,b} u_{c,\bar{j}}))_{\bar{l}} \\
&= \tau_b (u_{b,\bar{j}\bar{l}} + u_{b,\bar{j}} u_{b,\bar{l}}) - z^b \sum_c \tau_c u_{c,\bar{l}} (u_{c,b\bar{j}} + u_{c,b} u_{c,\bar{j}}) \\
&\quad - z^b \sum_c \tau_c (u_{c,b\bar{j}\bar{l}} + u_{c,b} u_{c,\bar{j}\bar{l}} + u_{c,b\bar{l}} u_{c,\bar{j}}) \\
&= \tau_b (u_{b,\bar{j}\bar{l}} + u_{b,\bar{j}} u_{b,\bar{l}}) - z^b \sum_c \tau_c u_{c,b\bar{j}\bar{l}} \\
&\quad - z^b \sum_c \tau_c (u_{c,\bar{l}} u_{c,\bar{j}} + u_{c,\bar{j}\bar{l}}) u_{c,b} \\
&= z^b \tau u_{b\bar{j}\bar{l}} = 0. \qquad \text{q.e.d.}
\end{aligned}$$

On normal bundle $\mathcal{N}$ we can consider matrix H of the metric,

$$H_{j\bar{k}} =< \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k} >= u_{j\bar{k}}, \quad r+1 \leq j,k \leq n.$$

The curvature 2-form of this metric on $\mathcal{N}$ over L is:

$$\eta = 2iB_{a\bar{b}} dz_a \wedge dz_{\bar{b}},$$

where

$$B_{a\bar{b}} = -H^{-1}H_{a\bar{b}} + H^{-1}H_a H^{-1}H_{\bar{b}}.$$

Let

$$A_{a\bar{b}} = \begin{pmatrix} u_{a\bar{b}} & u_{a\overline{(r+1)}} & u_{a\overline{(r+2)}} & \cdots & u_{a\bar{n}} \\ u_{(r+1)\bar{b}} & u_{(r+1)\overline{(r+1)}} & \cdots\cdots & \cdots & u_{(r+1)\bar{n}} \\ \cdots\cdots & \cdots\cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ u_{n\bar{b}} & u_{n\overline{(r+1)}} & \cdots\cdots & \cdots & u_{n\bar{n}} \end{pmatrix}.$$

Clearly $detA_{a\bar{b}} = 0$ by $(\partial\bar{\partial}u)^{n-r+1} = 0$.
Expand $(detA_{a\bar{b}})_{k\bar{l}} = 0$, $r+1 \leq k,l \leq n$:

$$0 = (detA_{a\bar{b}})_{k\bar{l}} = det \begin{pmatrix} u_{a\bar{b}k\bar{l}} & u_{a\overline{(r+1)}k} & \cdot\cdot & u_{a\bar{n}k} \\ u_{(r+1)\bar{b}\bar{l}} & & & \\ \cdots\cdots & & H & \\ u_{n\bar{b}\bar{l}} & & & \end{pmatrix}$$

$$+ \sum_{i,j \geq r+1} det \begin{pmatrix} 0 & 0 & \cdots & u_{a\bar{j}\bar{l}} & \cdots & 0 \\ 0 & u_{(r+1)\overline{(r+1)}} & \cdots & u_{(r+1)\bar{j}\bar{l}} & \cdots & u_{(r+1)\bar{n}} \\ \cdots & \cdots\cdots\cdots & \cdots & \cdots\cdots & \cdots & \cdots\cdots \\ u_{i\bar{b}k} & \cdots\cdots\cdots & \cdots & \cdots\cdots & \cdots & \\ \cdots & \cdots\cdots\cdots & \cdots & \cdots\cdots & \cdots & \cdots\cdots \\ 0 & u_{n\overline{(r+1)}} & \cdots & \cdots\cdots & \cdots & u_{n\bar{n}} \end{pmatrix}$$

$$= u_{a\bar{b}k\bar{l}} detH + det \begin{pmatrix} 0 & u_{a\overline{(r+1)}k} & \cdots & u_{a\bar{n}k} \\ u_{(r+1)\bar{b}\bar{l}} & & & \\ \cdots\cdots & & \mathbf{H} & \\ u_{n\bar{b}\bar{l}} & & & \end{pmatrix}$$

$$+ \sum_{i,j \geq r+1} (-1)^{i+j+1} u_{a\bar{j}\bar{l}} u_{i\bar{b}k} det(H_{\hat{i},\hat{j}}).$$

Moreover,

$$0 = u_{a\bar{b}k\bar{l}} detH - \sum_{i,j \geq r+1} (-1)^{i+j} u_{a\bar{j}k} u_{i\bar{b}\bar{l}} det(H_{\hat{i},\hat{j}})$$

$$+ \sum_{i,j \geq r+1} (-1)^{i+j+1} u_{a\bar{j}\bar{l}} u_{i\bar{b}k} det(H_{\hat{i},\hat{j}}).$$

Thus,

$$-H_{a\bar{b}} det(H) + H_a C(H) H_{\bar{b}} = -\overline{C_a^t C(H)} C_b.$$

Multiplying the left by $(detH)^{-1} H^{-1}$, we get

$$B_{a\bar{b}} = -H^{-1} \overline{C}_a H^{-1} C_{\bar{b}},$$

or

$$u_{a\bar{b}k\bar{l}} - u_{a\bar{j}k} u_{i\bar{b}\bar{l}} u^{i\bar{j}} = u_{a\bar{j}\bar{l}} u_{i\bar{b}k} u^{i\bar{j}}.$$

Notice that $\mathcal{L} \in \mathcal{N} \otimes \overline{\mathcal{N}}^* \otimes \mathcal{T}^*$,

$$\eta = \overline{\mathcal{L}} \wedge \mathcal{L}.$$

From the above computations, one can see that the Monge-Ampère foliation is holomorphic if only if the "twist tensor" vanish, i.e., if only if the connection on normal bundle is flat.

Recall that being holomorphic foliation means a locally one having a holomorphic map $\pi : U \longrightarrow V$ such that leaves of the foliation coincide with the fiber of π; here U is an open neighborhood in M such that foliation has constant rank, V is an open set in $\mathbf{C}^m$.

Since our foliation comes from the homogeneous Monge-Ampère equation(or equivalently, degenerate Kähler metric), one would expect that the foliation respect the "metric", i.e., the degenerate Kähler form be the pull back of a Kähler metric on V. Clearly, the pull back of any Kähler metric on V to U gives rise to a homogeneous holomorphic Monge-Ampère foliation. The above computation and discussion indicate the converse is also true. We state it as follows:

Proposition 6.4. *If Monge Ampère foliation is holomorphic, then it is given by the pull back of a Kähler metric by the degenerating map.*

Now we would like to discuss the implication of this general construction to our situation. Let ω be the Kähler form of the limiting degenerate "Kähler metric" g_∞. By the result from the next section, ω has generically constant rank. Assume $\omega^{n-r+1} = 0, \omega^{n-r} \neq 0, (1 \leq r \leq n)$, namely ω has rank $n-r$. For simplicity we would like to assume that there is a line bundle K such that $C_1(K)=\omega$(general situation can be done by writing ω as linear combination of integer ample classes). Let $\| \|_K$ be the Hermitian metric on K corresponding to ω, which means that for any holomorphic section $s\in H^0(K)$ we have $\omega=\partial\bar{\partial}log(\|s\|_L^{-2})$.

For large N, $NK=K^{\otimes N}$ have many section. Choosing generic sections $s_a\in H^0(NK), a=1,2,...,r$, define

$$\tau_a=\|s_a\|_L^{-2/N},\ u_a=log\tau_a.\ \tau=\sum_a\tau_a.$$

Since $\{u_a\}$ are generic, $\Omega=\partial\bar{\partial}\tau$ is a nondegenerate Kähler metric generically on M.

By the early construction of this section, $\{u_a\}$ will give rise to vector fields: $X_a,\ a=1,2,...,r$. Result of next section will show they are global meromorphic tangent fields generically spanning the leaves of the foliation.

7. Collapsing

The meaning of collapsing in this paper is quite different from the collapsing in Riemannian case as briefly indicated in the introduction. We would like to clarify its meaning before discuss it. In Riemannian geometry, collapsing means that the injective radius of the sequence of Riemannian manifolds goes to zero uniformly everywhere. In our case, we use the term "collapsing" to indicate that the sequence does not satisfy the assumption (*) in Section 4. More precisely, in any compact subset of the Zariski open set $M-E_\infty$ the sequence collapses in the usual sense. We can illustrate this by the following examples.

Example. Consider $\mathbf{CP^n}$ with the standard Fubini-Study metric $(\mathbf{CP^n},\mathbf{g_{FS}})$, fix a natural embedding:

$$\mathbf{C^n}\longrightarrow\mathbf{CP^n},$$

multiplication by $1/m$

$$m:\mathbf{C^n}\longrightarrow\mathbf{C^n},$$

and extend to a map from $\mathbf{CP^n}$ to $\mathbf{CP^n}$, still denote it by m. Then the sequence m^*g_{FS} collapses in our sense, but "blows-up" along infinite divisor.

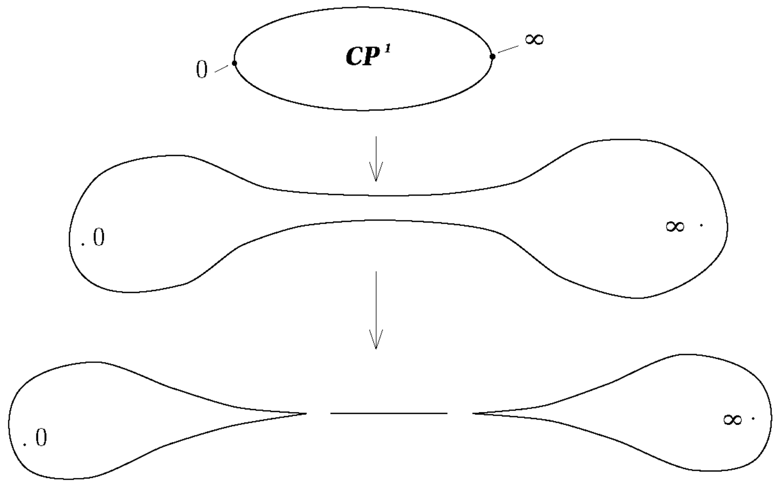

FIGURE 3.

Another example can be indicated by the following picture. $\mathbf{CP^1}$ "blows-up" 0 and ∞ and collapses otherwise.

In the collapsing case, metrics converge to a degenerate Kähler metric and we get a Monge-Ampère foliation. In general Monge-Ampère foliations are not necessary holomorphic. In this section we will show that the foliation we get from degeneration of bounded curvature metrics are holomorphic.

For any $x \in M \setminus E_\infty$, take r small such that $B_{g_0}(x,r) \subset M \setminus E_\infty$. By the harmonic map estimate under the bounded curvature condition,

$$g_m \leq C^2 g_0 \qquad\qquad \text{in } B_{g_0}(x,r)$$

which imply:

$$B_{g_0}(x,r) \hookrightarrow B_{g_m}(x,Cr).$$

Take $\widetilde{B_{g_m}}(x,Cr)$ as the ball of radius Cr in the universal cover space of $B_{g_m}(x,Cr)$. Since $B_{g_0}(x,r)$ is simply connected, one can lift the inclusion to

$$B_{g_0}(x,r) \hookrightarrow \widetilde{B_{g_m}}(x,Cr).$$

$\{\widetilde{B_{g_m}}(x,Cr), g_m\}$ as sequence of pointed Kähler manifolds have bounded curvature and injective radius bounded from below. Use the general convergence theorem as stated in the Introduction, we may assume that they converge to Kähler manifold $\{B_\infty(x,Cr), g_\infty\}$. By a similar argument as in Section 5 the sequence of maps as above gives a limiting

holomorphic map
$$B_{g_0}(x,r) \hookrightarrow B_\infty(x,Cr),$$
and fibers of the map are exactly the foliation fiber. Therefore the foliation is a holomorphic foliation.

Now we have proved the following theorem as stated in the introduction.

Theorem 7.1. *When the homogeneous Monge-Ampère equation comes from a collapsing, the foliation is holomorphic, and of constant rank on a smaller Zariski open subset $U' \subset U$.*

Remark. This result together with propositions in the last section implies that $X_a, a = 1, 2, ..., r$ are locally holomorphic vector fields and globally meromorphic vector fields, and the normal bundles along the leaves are metrically flat.

In general, different collapsing sequences which result in different degenerate "Kähler metrics" may give rise to the same foliation. From the discussion of these two sections, the difference between these degenerate "Kähler metrics" are exactly reflected by that they are pullbacks of different Kähler metrics by the same holomorphic collapsing map.

8. More on collapsing and applications

In Section 6, we were able to construct holomorphic vector fields along the collapsing directions using a general construction from Monge-Ampère foliations. Locally around the points where the foliation has constant rank the construction gives holomorphic vector fields which globally can be extended to meromorphic vector fields. But the vector fields are usually not natural with respect to the foliation, in the sense that poles and zeros of the vector fields constructed are usually not too much related to the collapsing structure and blow-up subvariety E. One naively would hope to use the collapsing structure to produce natural holomorphic vector fields. For example construct holomorphic vector fields pointing to the "most collapsing direction". In this section we will try to explore this idea in detail, and analyze the foliation more closely and create vector fields which are natural for the foliation.

Recall from the last section we have two objects: one is a neighborhood of the original complex manifold, another is the "limiting object", an open neighborhood as limit of neighborhoods in local universal covers, and a nilpotent local group action as limit of local fundamental groups of the the corresponding local neighborhoods; we will follow Fukaya to discuss this group action in more detail later in this section. We also have a holomorphic map from the first object to the second object. Limiting group action will naturally produce vector fields on the limiting object. The problem is how to pull back. Recall also from the last section that the above mentioned holomorphic map is the limit of a sequence of biholomorphic map. One can pull back the vector fields by the biholomorphic maps. Renormalize them to make sure they are bounded, and then take limit. The problem is that even if one normalizes the vector fields to be norm 1, the limit can still be zero in general (think of the example z^n for n going to infinity for $|z| < 1$). Apparently one needs some kind of Harnack estimate to ensure the limit to be non-vanishing. To this point, it is easier to handle when collapsing fibers have dimension 1.

First we will prove a Harnack type lemma for volume form and clarify some result from Section 5, then we will give a proof of an application (Corollary 1.2) which we mentioned in the instruction.

Lemma 8.1. *Let (M, g_0) be a compact complex Kähler manifold, and g be another Kähler metric on M. Assume the Ricci curvature of g is bounded from above and below. Then there exist C_1 and C_2 such that*

$$|log\frac{detg}{detg_0} - \frac{1}{Vol(M)}\int_m log\frac{detg}{detg_0}dV_{g_0}| < C_1 - C_2 u,$$

where $g = g_0 + \partial\bar{\partial}u$ and $\int_M u = 0$.

Proof. Let

$$f = log\frac{detg}{detg_0} - \frac{1}{Vol(M)}\int_m log\frac{detg}{detg_0}dV_{g_0}.$$

Then

$$Ric_g = -\partial\bar{\partial}f + Ric_{g_0}.$$

Since the Ricci curvature of g is bounded, we can find $C_1, C_2 > 0$ such that

$$-C_1 g_0 - C_2(g_0 + \partial\bar{\partial}u) \leq -\partial\bar{\partial}f \leq C_1 g_0 + C_2(g_0 + \partial\bar{\partial}u),$$

i.e.,

$$(C_1 + C_2)g_0 + \partial\bar{\partial}(C_2 u + f) \geq 0,$$

and

$$(C_1 + C_2)g_0 + \partial\bar{\partial}(C_2 u - f) \geq 0.$$

By Lemma 2.2, we get

$$C_2 u + f \leq C$$

and

$$C_2 u - f \leq C,$$

i.e., exist $C_1, C_2 > 0$ (use the same notation for simplicity) such that

$$|f| \leq C_1 - C_2 u.$$

q.e.d.

Remark 1. In the lemma, we assumed $g = g_0 + \partial\bar{\partial}u$. One can of course replace g_0 in the above formula by any Kähler metric bounded by g_0.

Remark 2. This simple lemma implies that in any compact subset $K \subset (M - E_\infty)$, the ratio between Vol_g and Vol_{g_0} can only vary by a bounded amount. Recall that in Section 5, we constructed a surjective holomorphic map ψ from the original $M - E_\infty$ to the limit M_∞. This lemma in particular yields that this surjective map ψ is actually injective, and therefore biholomorphic. We will sketch below a rough argument.

From Section 3 we already have

$$g \leq C g_0 \quad \text{in } K,$$

and by noticing the fact that in our case maps are non-collapsing from the above lemma we also have

$$C^{-1} det g_0 \leq det g \leq C det g_0 \ \text{ in } K.$$

The above two formulas together imply that

$$C^{-1} g_0 \leq g \leq C g_0 \quad \text{in } K.$$

This shows that the map ψ is a biholomorphism.

Now we would give a proof of the following theorem mentioned in the introduction as a corollary. Recall, from Theorem 1.2, roughly a sequence of bounded curvature Kähler manifolds will either be collapsing or non-Collapsing. Therefore it will be of interest to know for what kind of manifold one can have a collapsing sequence, and for what kind of manifold there will be none. The following theorem conforms in some way the general belief one gets from the examples that "positive curved manifolds" tend to have collapsing sequences and "negative curved manifolds" tend to have none. Its proof is based on Yau's Schwarz lemma. For reader's convenience we will give the detail of the proof.

Theorem 8.1. *Assume $C_1(M) < 0$, and g_i is a sequence of Kähler metrics with bounded curvature. Then (M, g_i) is non-collapsing.*

Proof. By Aubin-Yau's solution of Calabi Conjecture for $C_1(M) < 0$, there exists a Kähler-Einstein metric g_0 on M such that

$$Ric_{g_0} = -g_0.$$

Notice the formula

$$Ric_g = -\partial\bar{\partial} log \frac{detg}{detg_0} + Ric_{g_0}.$$

Using the fact that $Ric_{g_0} = -g_0$ and taking the trace with respect to g, we get

$$-\Delta_g(log \frac{detg}{detg_0}) = r_g + tr_g g_0,$$

where r_g denotes the scalar curvature of the Kähler metric g.

Now we want to apply the maximal principle to this equation. Assume that $detg/detg_0$ achieve its minimal at $x \in M$. Then

$$r_g(x) + tr_g g_0(x) = -\Delta_g(log \frac{detg}{detg_0})(x) \leq 0,$$

$$tr_g g_0(x) \leq -r_g(x) \leq C,$$

which together yield

$$max(\frac{detg_0}{detg}) = \frac{detg_0}{detg}(x) \leq (\frac{tr_g g_0(x)}{n})^n < C,$$

or

$$detg \geq C detg_0,$$

i.e., volume is not collapsing. In our case we already have $g \leq Cg_0$ away from the blow-up subvariety; together we get $g \geq Cg_0$. Therefore the sequence is not collapsing and some subsequence of it converge on a Zariski open set to a complete Kähler metric. q.e.d.

From Remark 2 after Lemma 8.1, and the proof of Theorem 8.1, we have already seen the usefulness of Lemma 8.1. Now we will try to explore further the meaning of Lemma 8.1 in order to construct holomorphic vector fields along the collapsing directions as mentioned in the beginning of this section. From now on we will assume that the resulting Monge-Ampère foliation has fiber dimension one, unless otherwise stated.

Remark. According to Remark 2 following Lemma 8.1, we can rescale g_m such that,

$$C_m \cdot \omega_{g_m}^n \longrightarrow \Omega_n,$$

where Ω_n is a (n,n) volume form on $M \backslash E$. On the other hand,

$$\omega_{g_m}^{n-1} \longrightarrow \omega_{g_\infty}^{n-1} = \Omega_{n-1}.$$

Ω_n and Ω_{n-1} together will naturally determine a metric along the foliation leaves. Apriori there is no reason why this metric should be flat, but the vector fields which we will construct will have constant norm under this metric. Therefore the metric should be flat. Thus we have proved

Theorem 8.2. *When the Monge-Ampère foliation with leaves of dimention one comes from a collapsing, the metric along the leaves of the foliation as constructed above is flat.*

Proof. Recalling our collapsing construction, we have a sequence:

$$B_{g_m}(p,r) \hookrightarrow (M, g_m) \text{ here } p \in M,$$

the r balls in the universal covers:

$$\widetilde{B_{g_m}}(p,r) \text{ converge to } B_r \text{ (in } C^{1,\alpha}).$$

Consider $B_{g_0}(p,r_0) \subset (M,g)$, since $g_m \leq Cg_0$, one can assume

$$B_{g_0}(p,r_0) \subset B_{g_m}(p,r).$$

Then we can consider maps

$$I_m : \ B_{g_0}(p,r_0) \hookrightarrow B_{g_m}(p,r),$$

which lift to:

$$\widetilde{I_m} : \ B_{g_0}(p,r_0) \hookrightarrow \widetilde{B_{g_m}}(p,r).$$

The right-hand side will converge to B_r. Notice that $|D\widetilde{I_m}| \leq C$ uniformly, when it goes to limit, we will get a map:

$$I_0: \; B_{g_0}(p, r_0) \hookrightarrow B_r.$$

$Im(I_0)$ will be a subvariety of B_r, and I_0 will give the desired Monge-Ampère foliation on $B_{g_0}(p, r_0)$. Since we are considering foliation with fiber dimension one, $Im(I_0)$ will have dimention $n-1$.

Since each $\widetilde{I_m}$ is a biholomorphism, if we choose a holomorphic vector field v on B_r, which is not along $Im(I_0)$, then we can pull it back, and properly rescale it to get

$$v_m = C_m(\widetilde{I_m}^{-1})_* v.$$

To guarantee their convergence, we need to look more closely at the maps $(\widetilde{I_m}^{-1})_*$. Locally at a point, I_{0*} is like a linear projection operator with kernel of complex dimension one, i.e., looks like

$$I_{0*} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

$\widetilde{I_{m*}}$ are perturbations of I_{0*}, look like

$$\widetilde{I_{m*}} = \begin{pmatrix} 1+a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & 1+a_{2,2} & \cdots & a_{2,n-1} & a_{2,n} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots\cdots\cdots & \cdots\cdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & 1+a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & a_{n,n} \end{pmatrix},$$

$a_{i,j}$ are small,

$$(\widetilde{I_m}^{-1})_* = (\widetilde{I_{m*}})^{-1} = \frac{1}{det\widetilde{I_{m*}}} \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n,1} & A_{n,2} & \cdots & A_{n,n} \end{pmatrix},$$

$A_{i,j}$ are the minors of $\widetilde{I_{m*}}$.

Clearly, all $A_{i,j}$ are of order $a_{i,j}$, except that $A_{n,n}$ are of order 1. Let $C_i = det(\widetilde{I_{m*}})$. As long as $v^n \neq 0$ (the n-th component of v), or in other words, v is not tangent to the image of I_{0*}, $\{v_m\}$ will converge to a vector field v_0 on $B_{g_0}(p, r_0) \subset M$. v_0 vanishes if and only if the Monge-Ampère foliation is singular or v is tangent to the image of I_0.

On the other hand, recall that by comparing Ω_n and Ω_{n-1} the collapsing sequence induces a singular metric (bounded from below) on the leaves of the foliation. It is easy to check that the norm of v_0 under this metric is constant; this means that the induced fiber metric blows-up exactly when the foliation is singular. The vector field v_0 naturally gives a canonical flat structure on any leaf with respect to the collapsing sequence. This is not trivial as remarked earlier, since apriorily the fiber metric constructed from Ω_n and Ω_{n-1} has no reason to be flat. q.e.d.

In higher fiber dimension, the above argument can be generalized to construct a canonical holomorphic "multi-vector" (section of anti-canonical bundle). which is of constant norm under the natural metric on relative anti-canonical bundle of fibers. In other words, we are getting a holomorphic complex volume form along the fiber which is non-vanishing and blow-up at singular locus of the foliation. So instead of flat structure we get a sort of Ricci flat structure. Of course with the courvature bound, we actually expect this Ricci flat structure to be flat.

Remark. The above construction of the pull back vector field is quite delicate. For example, one can restrict the metrics g_m to the fibers of the foliation, and it seems quite reasonable to expect that after suitable rescaling the sequence will converge to our canonical fiber metric. But this is actually false. Even if the limit exist, it will not in general coincide with our canonical fiber metric.

From the above construction, we see the vector fields we constructed are actually holomorphic away from the blow-up locus E. A natural question is then whether the vector fields are holomorphic everywhere. This is very important if one wants to deduce any global implication of the construction. We believe this is the case, and we will discuss this problem and more result on collapsing of Kähler manifolds in a forthcomming paper.

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Columbia University