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THE NIETO QUINTIC IS JANUS-LIKE

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Introduction

The Nieto quintic, which we denote by $\mathcal{N}_5$ in this paper, is a very special quintic hypersurface in $\mathbb{P}^4 = \{\sum_{i=0}^5 x_i = 0\} \subset \mathbb{P}^5$ given by the equations

$$\mathcal{N}_5 := \{\sum_{i=0}^{5} x_i = \sum_{i \neq j \neq k \neq l \neq m} x_i x_j x_k x_l x_m = 0\} \subset \mathbb{P}^5.$$

It is the most singular member of the pencil of quintics which are invariant under the natural action of the symmetric group Σ_6 acting by permutation of coordinates. This quintic hypersurface was discovered by Nieto in his thesis [10], as a closely related variety to the variety which parametrizes those lines in $\mathbb{P}^3$ which lie on some smooth Heisenberg-invariant quartic surface. The Nieto quintic is the Hessian variety of the Segre cubic (cf. [8, 3.2]) $\mathcal{S}_3$, which in its turn is the unique cubic invariant under Σ_6. The singular locus of $\mathcal{N}_5$ consists of the following: 10 isolated nodes p_i; 20 singular lines ℓ_k which meet in 15 singular points q_j. Each of the points q_j has multiplicity 3.

In [8, 3.4.4] I made a conjecture concerning a certain birational image of $\mathcal{N}_5$, to the effect that the image is (precisely) the Satake compactification of a ball quotient, and it is that conjecture which will be verified in this paper. We identify modular subvarieties and compactification locus, and that is what leads to the above: the compactification divisors of the ball quotient will consist of the proper transforms of the ten points p_i as well as the proper transforms of the 20 singular lines ℓ_k.

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The points of multiplicity three, on the other hand, will not belong to the boundary but are instead (upon resolution) modular subvarieties. There are also two sets of 15 planes on $\mathcal{N}_5$, and all 30 of these turn out to be modular subvarieties. These two-dimensional ball quotients are all arithmetic, as we can locate them in the list of [3] as arithmetic. Using the techniques developed in [9] and [8], we can then show that our group is arithmetic. This will be done in another paper.

In the first section we examine two surfaces which are (new) ball quotients, at least in the form they appear here, which will be an important step in showing that the birational model of $\mathcal{N}_5$ is a Satake compactification of a ball quotient. Indeed, we use a Theorem, proved in [8], which reduces the checking of Yau's inequality to the checking of certain proportionalities for *divisors* on a variety. These divisors are the ball quotient surfaces we discuss. Then we do the necessary Chern class computations, using the usual methods of projective geometry. The combinatorics of the situation is decisive.

1. Two surface ball quotients

There is a unique cubic surface in $\mathbb{P}^3$ with four ordinary double points; it is called the Cayley cubic and can be given by the equation

$$\mathfrak{C} := \left\{ \frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 0 \right\} \subset \mathbb{P}^3,$$

by which is meant of course $x_1x_2x_3 + x_0x_2x_3 + x_0x_1x_3 + x_0x_1x_2 = 0$. Then $x_0 = 0, \dots, x_3 = 0$ define the faces of the coordinate tetrahedron, the vertices of which are the nodes of $\mathfrak{C}$. There are 9 lines on $\mathfrak{C}$, the six exceptional $\mathbb{P}^1$'s, viewing $\mathfrak{C}$ as the blow-up of $\mathbb{P}^2$ at the six points depicted in Figure 1, as well as the three diagonals which are denoted in Figure 1 by N_{0124}, N_{0125} and N_{0123} (see [8, 4.1.3] for details about and pictures of this and other cubic sufaces).

As described in [8, p. 75], there are 15 hyperplane sections of the Segre cubic $\mathcal{S}_3$ which are copies of $\mathfrak{C}$; in the coordinates used in *loc. cit.*, $\mathcal{S}_3 = \{\sum x_i = \sum x_i^3 = 0\} \subset \mathbb{P}^5$, these hyperplanes are $\mathcal{T}_{ij} = \{x_i - x_j = 0\}$, $j < i = 0, \dots, 5$. We know that $\mathcal{S}_3$ is a three-dimensional ball quotient (*loc. cit.*, Thm. 3.2.5), such that the 15 Segre planes on $\mathcal{S}_3$ are two-dimensional subball quotients (*loc. cit.*, Prop. 3.2.7). Note that $\mathcal{T}_{ij}$ is invariant under the permutation $x_i \leftrightarrow x_j$.

Theorem 1.1. *The Cayley cubic $\mathfrak{C}$ is the Satake compactification* *of an arithmetic quotient* $\Gamma\backslash\mathbb{B}_2$, *where* Γ *is an arithmetic subgroup, contained in* $SU(3,1;\mathcal{O}_K)$, *and commensurable with* $SU(2,1;\mathcal{O}_K)$. *Here* K *is the field of Eisenstein numbers,* $K=\mathbb{Q}(\sqrt{-3})$.

Proof. We use the moduli interpretation of $\mathcal{S}_3$ and $\mathfrak{C}$ (*loc. cit.*, 3.2.6). $\mathfrak{C}$ is some two-dimensional variety parametrizing abelian fourfolds with the relevant PEL-structure. From the fact that, as just mentioned, there is a $\mathbb{Z}/2$ *centralizing* $\mathcal{T}_{ij}$, this same $\mathbb{Z}/2$ centralizes $\mathfrak{C}$. Since we know that the symmetry group is $\Sigma_6=\Gamma(1)/\Gamma(\sqrt{-3})$, using the usual notation for the subgroups of $SU(3,1;\mathcal{O}_K)$, it follows that every automorphism in Σ_6 can be lifted to an automorphism of $\mathbb{B}_3$. Being an involution, it fixes a subball, so the surface $\mathfrak{C}$ is the quotient of a totally geodesic subball $(\mathbb{B}_2\cong)B\subset\mathbb{B}_3$. Hence $\mathfrak{C}$ is the Satake compactification of a ball quotient, the first statement. The commensurability now follows from the fact that $\mathcal{S}_3$ is the Picard variety for the $\sqrt{-3}$-level variety of the three-dimensional Picard modular group for the field $K=\mathbb{Q}(\sqrt{-3})$. The commensurability follows just as easily from *loc. cit.*, Lemma 3.2.5.1 as follows. Of the 15 Segre planes, there are six passing through each of the nodes, and one sees easily that on the exceptional $\mathbb{P}^1$ (of one of the nodes of $\mathfrak{C}$) there are three lines (just look at Figure 1, where the six lines are the points denoted Q_{ij}, while the four nodes are the lines denoted L_{ijk}). The action of the group of cube roots of unity on each of the six lines is easily seen, and this means that the compactifying divisor on a torsion-free quotient covering $\mathfrak{C}$ is the elliptic curve $\mathbf{E}_\varrho$ (here $\varrho=e^{\frac{2\pi i}{3}}$ and $\mathbf{E}_\varrho$ denotes the elliptic curve with modulus ϱ); 3.2.5.1 now implies the commensurability statement. q.e.d.

Remark. A picture of this cubic surface can be found in [8, p. 117] and on the web at the *Gallery of Algebraic Surfaces* at the URL `http://www.mathematik.uni-kl.de/~wwwagag/Galerie.html`.

Let $Y^*_{\mathcal{S}}\longrightarrow\mathcal{S}_3$ be a torsion-free ball quotient such that it induces $Y^*_{\mathfrak{C}}\longrightarrow\mathfrak{C}$, a torsion-free ball quotient covering $\mathfrak{C}$. The branching locus on $\mathfrak{C}$ consists of the 9 lines on $\mathfrak{C}$. Next we note that, from the fact that $\mathfrak{C}$ is birational to $\mathbb{P}^2$, such that the branching locus of $Y^*_{\mathfrak{C}}\longrightarrow\mathfrak{C}$ maps to the lines in Figure 1, there must be a birationally equivalent cover $Y^*\longrightarrow\mathbb{P}^2$, which is branched along the line arrangement of Figure 1. This arrangement is called Ceva(2,1), and consists of a subset of the inverse image of the famous arrangement $\mathrm{A}_1(6)$, the complete quadrilateral (see (1)), under the squaring map $(z_0:z_1:z_2)\mapsto(z_0^2:z_1^2:z_2^2)$ of $\mathbb{P}^2$ to itself. More precisely, numbering the the lines in $\mathrm{A}_1(6)$ by 12, 13, 14, 23, 24, 34 and the four threefold points of the arrangement by

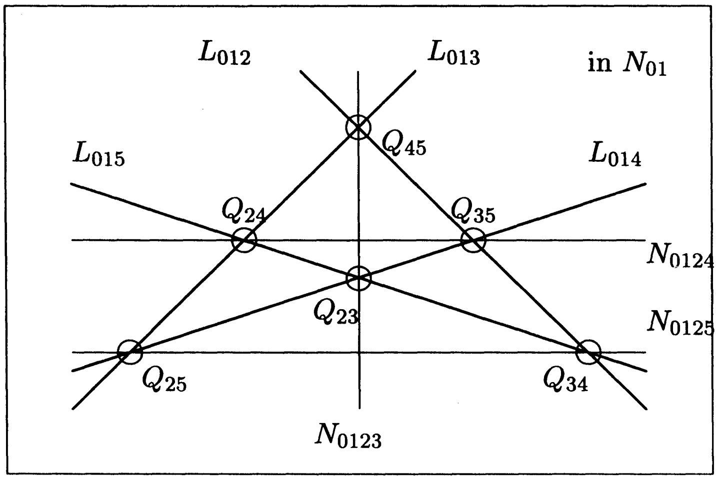
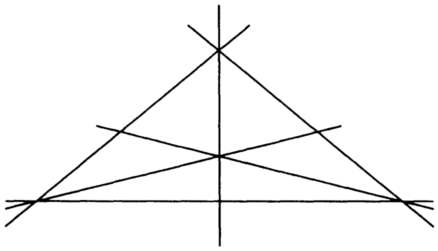


FIGURE 1. The plane N_{01}

(1)



01, 02, 03 and 04, then Ceva(2,1) consists of the inverse image of 12, 13, 23, 14 and the point $0i$, $i = 1, \ldots, 4$, while the inverse images of the lines 24 and 34 do not belong to the arrangement. Then the lines denoted L_{ijk} in Figure 1 are now labeled 12, 12 and 13, 13 (these pairs consisting of two perpendicular lines). The six points denoted Q_{ij} in Figure 1 are the inverse image of 04 (consisting of the four points, the center and the three outer vertices), and the inverse image of 02 and 03. Because of the fact that the lines 14, 24 and 34 belong to the branching locus of the squaring map, it follows that the branching degrees are:

- the same for Ceva(2,1) and $\mathrm{A}_1(6)$, for the lines 12, 23 and 13, as well as the point 04;
- multiplied by two for $\mathrm{A}_1(6)$, for the other lines and points.

Since, on $\mathcal{S}_3$, the branching degrees of $Y_{\mathcal{S}}^* \longrightarrow \mathcal{S}_3$ are 3 along all Segre planes (and at the cusps ∞ in the formalism developed by Höfer, [5, 5.2.3], also explained in [2, 4.3]), the degrees on Ceva(2,1) are:

- 3 at the six points Q_{ij},
- ∞ along the four lines L_{ijk},
- 3 along the three diagonals N_{ijkl}.

This gives the following branch degrees for the complete quadrilateral (the negative weight below corresponds to an exceptional curve, which after being blown up and the cover being constructed, is exceptional of the first kind and can be blown down; see [5, Section 3.3] and [2, 4.2]):

01	02	03	04	12	13	14	23	24	34
-6	6	6	3	∞	∞	6	3	2	2

where the 2's come from the fact that those lines do not belong to the arrangement Ceva(2,1), hence are unbranched, then multiplied by 2. Under the permutation $0 \leftrightarrow 1$, this maps to the case #8 in the list of Höfer [5, 5.1.5], which is #24 in the list (12) in [2, p. 201] and #7 in the Deligne-Mostow list [3].

The other surface we are interested in does not occur on $\mathcal{S}_3$, so the first argument above will not work. However, this one does derive from $\mathrm{A}_1(6)$, and is described as follows. Here the arrangement is Ceva(2,3), which is the full inverse image of $\mathrm{A}_1(6)$ under the squaring map above; it consists of 9 lines, and is pictured in Figure 2. For this line arrangement we want the following branching degrees: at the four points denoted P_{ij} in Figure 2, the branching degree is ∞, at all other lines and points it is 3. By the prescription above, this implies the following degrees for the complete quadrilateral:

01	02	03	04	12	13	14	23	24	34
6	6	6	∞	3	3	6	3	6	6

This is, after the permutation $1 \leftrightarrow 4$, #7 in Höfer's list, #22 in the list (12) in [2, p. 201] and #5 in the list of [3].

Corollary 1.2. *There is a torsion-free ball quotient $Y \longrightarrow \mathbb{P}^2$, which is branched over the arrangement Ceva(2,3) to degree 3 along all lines, i.e., this is the Fermat cover of the arrangement. Furthermore, the group Γ is commensurable with $SU(2,1;\mathcal{O}_K)$.*

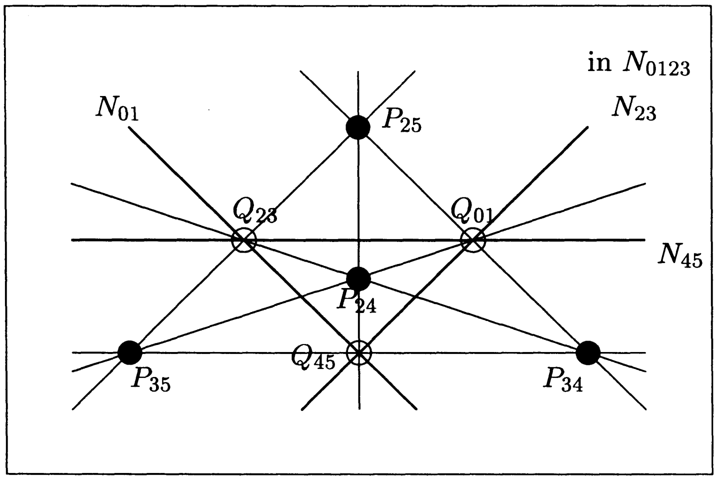


FIGURE 2. The plane N_{0123}

Proof. The first statement follows from the uniformization provided by the hypergeometric differential equation. The furthermore statement follows precisely as above; here the compactification curves are again clearly $\mathbf{E}_\varrho$. q.e.d.

2. Geometry on the Nieto Quintic

For details on this beautiful variety, we refer the reader to [1] and [8, 3.4]. Here we just recall the loci that we require in the sequel.

Proposition 2.1 *([1, 3.1]). $\mathcal{N}_5$ has the following singular locus:*

(i) 20 lines $L_{ijk} = \{x_i = x_j = x_k = 0 = \sum x_i\}$;

(ii) ten isolated points, the Σ_6-orbit of $(1,1,1,-1,-1,-1)$, which are the points $P_{ij} = (1, \pm 1, \dots \pm 1)$, with $+1$ in the i-th and j-th positions.

The 20 lines L_{ijk} of 2.1 meet at the following 15 points:

$$
\begin{aligned}
Q_{ij} =&(0, \dots, 1, \dots, -1, \dots) \\
=&\{\Sigma_6 - \text{orbit of } Q_{56} = (0:0:0:0:1:-1)\}.
\end{aligned} \tag{2}
$$

Lemma 2.2 *([8, 3.4.2]). The 20 lines L_{ijk} of Proposition 2.1 meet four at a time at the 15 points Q_{ij}; each line L_{ijk} contains three of the points, namely we have $Q_{ij} \in L_{klm} \iff \{i,j\} \cap \{k,l,m\} = \emptyset$.*

Lemma 2.3 *$\mathcal{N}_5$ contains the following 30 $\mathbb{P}^2$'s:*

(i) 15 Segre planes $N_{ijkl} = \{x_i + x_j = x_k + x_l = x_m + x_n = 0\}$;

(ii) 15 other planes $N_{ij} = \{x_i = x_j = 0 = \sum_{k \neq i,j} x_k\}$.

If we define the hyperplanes $\mathcal{H}_{ij} = \{x_i + x_j = 0\}$, then these cut out on $\mathcal{N}_5$ the Segre planes, and the hyperplanes $\mathcal{J}_i = \{x_i = 0\}$ cut out the other planes.

Lemma 2.4. *The fifteen hyperplanes $\mathcal{H}_{ij} = \{x_i + x_j = 0\}$ meet $\mathcal{N}_5$ each in the union of the three planes N_{ijkl}, N_{ijkm} and N_{ijkn}, and the plane N_{ij}, counted twice; the six hyperplanes $\mathcal{J}_i = \{x_i = 0\}$ meet $\mathcal{N}_5$ each in the union of five planes N_{ij}, N_{ik}, N_{il}, N_{im} and N_{in}.*

In this notation, the geometry of each of the 30 planes is precisely as depicted in Figures 1 and 2.

3. The conjecture

First we define a birational model of $\mathcal{N}_5$, which we will denote by $\widehat{\mathcal{N}}_5$.

(3)

a) Blow up the 15 points Q_{ij} of (2); let $p_1 : \tilde{\mathcal{N}}_5 \longrightarrow \mathcal{N}_5$ denote this blow up.

b) As each of the lines L_{ijk} contains three of the Q_{ij} (see Lemma 2.2), the proper transform of each L_{ijk} on $\tilde{\mathcal{N}}$ can be blown down to an isolated singular point (the normal bundle is $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$). Let $p_2 : \tilde{\mathcal{N}}_5 \longrightarrow \widehat{\mathcal{N}}_5$ denote this blow down.

The following is easy to see (see Figures 2 and 1).

Lemma 3.1. *The singular locus of $\widehat{\mathcal{N}}_5$ consists of the 20 isolated points from (3) b), and the ten ordinary double points, the images of the singular points P_{ij} of Proposition 2.1, (ii). The proper transforms of the N_{ijkl} of Lemma 2.3, (i) on $\mathcal{N}_5$ are $\mathbb{P}^2$'s blown up in three points, a del Pezzo surface; the proper transforms of the N_{ij} of Lemma 2.3, (ii) are $\mathbb{P}^2$'s blown up in six points, then the L_{ijk} are blown down to four nodes, so this is the Cayley cubic. Likewise, the proper transforms of the exceptional divisors over the points Q_{ij} of (3), a) are all copies of the Cayley cubic.*

The conjecture is that $\widehat{\mathcal{N}}_5$ is the Satake compactification of a ball quotient. First note that this is equivalent to the following. Let $\overline{\mathcal{N}}_5$ denote the blow-up of $\widehat{\mathcal{N}}_5$ at the 30 singular points; $\overline{\mathcal{N}}_5$ is smooth, and we let $\mathcal{E}$ denote the exceptional locus, the disjoint union of 30 copies of $\mathbb{P}^1 \times \mathbb{P}^1$. Then the conjecture above is equivalent to: $\overline{\mathcal{N}}_5 - \mathcal{E}$ is an open ball quotient, and $\overline{\mathcal{N}}_5$ is a smooth toroidal compactification of $\overline{\mathcal{N}}_5 - \mathcal{E}$.

4. Relative proportionality

We now sum up the technology involved in proving the conjecture. The idea is simple: it will suffice to check certain conditions among Chern numbers of certain *surfaces* on the threefold. This is explained in great detail in [8, Sections 5.5-5.6].

One defines:

Definition 4.1. *Let $\overline{X}$ be projective and smooth of dimension N, and $\Delta \in \overline{X}$ a normal crossings divisor. The* Yau proportionality factor *is:*

$$\begin{aligned} \mathcal{Y}(\overline{X},\Delta) &= N\bar{c}_1^N(\overline{X},\Delta) - 2(N+1)\bar{c}_1^{N-2}(\overline{X},\Delta)\bar{c}_2(\overline{X},\Delta) \\ &= Nc_1^N(\overline{X}) - 2(N+1)c_1^{N-2}(\overline{X})c_2(\overline{X}) + (-1)^N N(\Delta)^N. \end{aligned}$$

Let $\overline{D}_\alpha \subset \overline{X}$ be the compactification of a codimension one subvariety with compactification divisor $\Delta_\alpha = \overline{D}_\alpha - D_\alpha = \overline{D}_\alpha \cap \Delta$. The relative proportionality factors *are*

$$\begin{aligned} \mathcal{R}_1(\overline{D}_\alpha,\Delta_\alpha) &= \bar{c}_1^{N-1}(\overline{X},\Delta)_{|\overline{D}_\alpha} - \left(\frac{N+1}{N}\right)^{N-1} \bar{c}_1^{N-1}(\overline{D}_\alpha,\Delta_\alpha), \\ \mathcal{R}_2(\overline{D}_\alpha,\Delta_\alpha) &= \bar{c}_1^{N-3}(\overline{X},\Delta)\bar{c}_2(\overline{X},\Delta)_{|\overline{D}_\alpha} \\ &\quad -\frac{N+1}{N-1}\left(\frac{N+1}{N}\right)^{N-3} \bar{c}_1^{N-3}(\overline{D}_\alpha,\Delta_\alpha)\bar{c}_2(\overline{D}_\alpha,\Delta_\alpha). \end{aligned}$$

Note that, if $\overline{D}_\alpha$ is a surface, then $\mathcal{R}_2(\overline{D}_\alpha,\Delta_\alpha) = 0$ is a vacant condition.

Theorem 4.2 ([8, 5.5.6]). *Let $(\overline{X},\Delta)$ be above, with Δ a disjoint union of abelian varieties. Suppose $N \geq 3$ and there exist divisors $\overline{D}_\alpha \subset \overline{X}$ and rational numbers λ_α such that, in $H^2(\overline{X},\mathbb{Q})$, $\bar{c}_1(\overline{X},\Delta) =$*

$\sum \lambda_\alpha \overline{D}_\alpha$ *holds. If the* $\overline{D}_\alpha$ *fulfill:*

$$\mathcal{R}_1(\overline{D}_\alpha, \Delta_\alpha) = \mathcal{R}_2(\overline{D}_\alpha, \Delta_\alpha) = \mathcal{Y}(\overline{D}_\alpha, \Delta_\alpha) = 0,$$

then

$$\mathcal{Y}(\overline{X}, \Delta) = 0.$$

Consequently, if $K_{\overline{X}}+\Delta$ *is ample on* $X := \overline{X}-\Delta$*, then by Yau's theorem* $\overline{X}$ *is a compactification of an open ball quotient,* $X = \Gamma_X \backslash \mathbb{B}_N$*.*

5. The cover

We now check this condition for the case at hand. We first construct a cover which is branched to degree 3 along the 15 Segre planes as well as along the 15 other planes. We utilize the construction of Fermat covers, as explained in great detail in [6] as well as [8, 3.1.5]. As we noted above the union of the 15 hyperplanes $\mathcal{H}_{ij}$ cuts out all 30 planes on $\mathcal{N}_5$, so we use the Fermat cover associated to the arrangement of the 15 planes $\mathcal{H}_{ij}$. This arrangement has a *singular locus* (cf. [8, Definition 3.1.1]). This is because there are three of the hyperplanes meeting at each of the Segre planes. The Fermat cover is then constructed by resolving the singularities, and the result is a cover of $\mathbb{P}^4$ branched the proper transforms of the 15 hyperplanes, as well as along the exceptional loci, everywhere of branching degree 3. The induced branched cover of $\mathcal{N}_5 \subset \mathbb{P}^4$ is branched to degree 3 along all 30 planes. This cover lifts naturally to a cover of the desingularization $\overline{\mathcal{N}}_5$ discussed above; in other words we have a commutative diagram

$$\begin{array}{ccc} \overline{Y} & \longrightarrow & \overline{\mathcal{N}}_5 \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathcal{N}_5, \end{array}$$

and we apply the methods introduced above to the cover $\overline{Y} \longrightarrow \overline{\mathcal{N}}_5$. We use Theorem 4.2. First, the divisors $\overline{D}_\alpha$ and Δ are defined as follows. The irreducible components of Δ are the irreducible components of the inverse image of the divisor Π below under the projection $\overline{Y} \longrightarrow \overline{\mathcal{N}}_5$, while the components $\overline{D}_\alpha$ are the irreducible components of the inverse image of the $\overline{E}_\beta$ below under the projection $\overline{Y} \longrightarrow \overline{\mathcal{N}}_5$.

(4)

1. Π is the union of the 30 exceptional $\mathbb{P}^1 \times \mathbb{P}^1$'s on $\mathcal{N}_5$.
2. The divisors $\overline{E}_\beta$ are the proper transforms on $\mathcal{N}_5$ of the following:
   (a) The 15 Segre planes.
   (b) The 15 other planes, whose proper transforms on $\widehat{\mathcal{N}}_5$ are copies of the Cayley cubic.
   (c) The 15 *exceptional* divisors, resulting from the first step of (3); each of these is isomorphic on $\widehat{\mathcal{N}}_5$ to a copy of the Cayley cubic.

By the results of Section 1, if $\Delta_\alpha := \Delta \cap \overline{D}_\alpha$, $D_\alpha := \overline{D}_\alpha - \Delta_\alpha$, then each of the divisors D_α covering one of the E_β ($E_\beta := \overline{E}_\beta - \Pi_\beta$) of 2. is a ball quotient, in other words the Yau proportionality factor vanishes, $\mathcal{Y}(\overline{D}_\alpha, \Delta_\alpha) = 0$. By the remark preceeding the Theorem, the relative proportionality factor $\mathcal{R}_2$ is an empty condition for surfaces; hence it will suffice to show $\mathcal{R}_1 = 0$ for each of the ball quotients. We follow the procedure detailed in [8, Section 5.6] for the case of the Burkhardt quartic. First, we require that the first Chern class of $\overline{Y}$ can be written as a rational linear combination of the divisors $\overline{D}_\alpha$. This requires some notation. Let Q_β, $\beta = 1, \dots, 15$ denote the exceptional divisors on the blow-up X_1 of $\mathbb{P}^4$ at the 15 points denoted Q_{ij} above; let P_m, $m = 1, \dots, 10$ denote the exceptional divisors of the blow-up X_2 of X_1 at the 10 points denoted P_{ij} above; finally let L_n, $n = 1, \dots, 20$ denote the 20 exceptional divisors resulting from blowing up X_2 at the proper transforms of the 20 lines denoted L_{ijk} above; let this last blow-up be denoted X_3. Then we have the formula for the first Chern class of X_3,

$$c_1(X_3) = 5H - 3\sum_1^{15} Q_\beta - 3\sum_1^{10} P_m - 2\sum_1^{20} L_n, \tag{5}$$

and also that of the class of $\overline{\mathcal{N}}_5$ in X_3:

$$[\overline{\mathcal{N}}_5] = 5H - 3\sum_1^{15} Q_\beta - 2\sum_1^{10} P_m - 2\sum_1^{20} L_n, \tag{6}$$

where the coefficient 3 is due to the fact that the singularities in the points Q_{ij} have multiplicity 3. Now applying adjunction, we get

Lemma 5.1. *In $H^2(\overline{\mathcal{N}}_5,\mathbb{Q})$ we have $c_1(\overline{\mathcal{N}}_5) = -(\sum_1^{10} P_m)_{|\overline{\mathcal{N}}_5}$.*

Proof. By adjunction, $c_1(\overline{\mathcal{N}}_5) = (c_1(X_3) - \overline{\mathcal{N}}_5)_{|\overline{\mathcal{N}}_5}$, so the statement follows from (5) and (6). This was also shown in [1, §9], and amounts to the fact that, blowing down only the 10 P_m, the result is a nodal Calabi-Yau. q.e.d.

Consider one of the branching planes $\mathcal{H}_{ij}$, and let $\tilde{\mathcal{H}}_{ij}$ denote the proper transform. Since the multiplicity of $\mathcal{N}_5$ in the points P_{ij} and lines L_{ijk} is 2 while it is 3 at the points Q_{ij}, for the proper transform of $\mathcal{H}_{ij}$ we have:

$$\tilde{\mathcal{H}}_{ij} = \rho^*\mathcal{H}_{ij} - \sum_m \sigma_{ij,m}P_m - \sum_n \sigma_{ij,n}L_n - \sum_\beta \sigma_{ij,\beta}Q_\beta;$$

$$(\tilde{\mathcal{H}}_{ij})_{|\mathcal{N}_5} = \left(\rho^*\mathcal{H}_{ij} - \frac{1}{2}\sum_m \sigma_{ij,m}P_m - \frac{1}{2}\sum_n \sigma_{ij,n}L_n - \frac{1}{3}\sum_\beta \sigma_{ij,\beta}Q_\beta\right)_{|\mathcal{N}_5};$$

here the symbols $\sigma_{ij,m}$, $\sigma_{ij,n}$ and $\sigma_{ij,\beta}$ are 1 or 0, depending on whether the indicated objects are incident or not. From the combinatorics of the situation we have

- 6 other planes containing each P_{ij}.
- 6 Segre planes containing each Q_{ij}.
- 3 other planes containing each Q_{ij}.
- 3 Segre planes containing each L_{ijk}.

Therefore, the total transform of the union of the 30 planes is

$$\begin{aligned}\sum_1^{30}[H_i] &:= \sum_1^{15}[N_{ij}] + \sum_1^{15}[N_{ijkl}] \\ &\sim (\frac{30}{5}H - \frac{6}{2}\sum_1^{10} P_m - \frac{9}{3}\sum_1^{15} Q_\beta - \frac{3}{2}\sum_1^{20} L_n)_{|\overline{\mathcal{N}}_5},\end{aligned}$$

so the ramification divisor of the cover $\overline{\pi} : \overline{Y} \longrightarrow \overline{\mathcal{N}}_5$ is

$$\begin{aligned}\mathcal{R} &= \sum_1^{30}[H_i] + \left(\sum_1^{10} P_m + \sum_1^{15} Q_\beta + \sum_1^{20} L_n\right)_{|\overline{\mathcal{N}}_5} \\ &\sim (6H - 2\sum_1^{10} P_m - 2\sum_1^{15} Q_\beta - \frac{1}{2}\sum_1^{20} L_n)_{|\overline{\mathcal{N}}_5}.\end{aligned}$$

For the cover $\overline{Y}$ we have the standard formula

$$\begin{aligned} c_1(\overline{Y}) =&\overline{\pi}^* \left(c_1(\overline{\mathcal{N}}_5) - \frac{2}{3}\mathcal{R}\right) \\ =&\overline{\pi}^* \left(-\sum_1^{10} P_m - \frac{2}{3}(6H - 2\sum_1^{10} P_m - 2\sum_1^{15} Q_\beta - \frac{1}{2}\sum_1^{20} L_n)_{|\overline{\mathcal{N}}_5}\right) \\ =&\overline{\pi}^* \left(-4H + \frac{4}{3}\sum_1^{15} Q_\beta + \frac{1}{3}(\sum_1^{10} P_m + \sum_1^{20} L_n)\right). \end{aligned}$$

Consequently,

$$\begin{aligned} c_1(\overline{Y}, \Delta) &= \overline{\pi}^* \left(c_1(\overline{\mathcal{N}}_5) - \frac{2}{3}\mathcal{R} - \frac{1}{3}\Pi\right) \qquad (\Pi = \sum P_m + \sum L_n) \\ &= \overline{\pi}^* \left(-4H + \frac{4}{3}\sum_1^{15} Q_\beta\right). \qquad (7) \end{aligned}$$

Observe the factor 4/3 occuring, which is required by relative proportionality. We have the following immediate corollary.

Lemma 5.2. *There are $\lambda_\alpha \in \mathbb{Q}$ such that $\overline{c}_1(\overline{Y}, \Delta)$ can be written*

$$\overline{c}_1(\overline{Y}, \Delta) = \sum \lambda_\alpha \overline{D}_\alpha,$$

where the $\overline{D}_\alpha$ denote the inverse images of the $\overline{E}_\beta$ on $\overline{Y}$.

Proof. The right-hand side of (7) can clearly be so written. q.e.d.

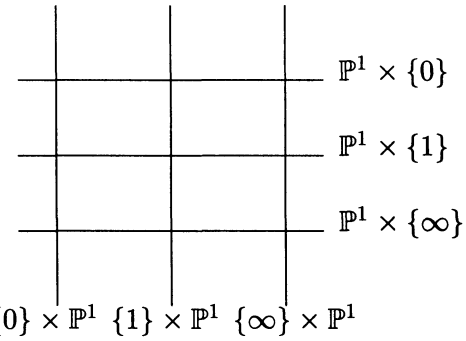
Corollary 5.3. *$K_{\overline{Y}} + \Delta$ is ample away from Δ.*

Proof. Using the Nakai-Moisezon criterion, it is sufficient to check that $(K_{\overline{Y}} + \Delta) \cdot C > 0$ for all effective curves C not contained in Δ. We leave this easy verification to the reader. q.e.d.

In order to apply Theorem 4.2, we require the following.

Lemma 5.4. *The irreducible components $\Delta_i \subset \Delta \subset \overline{Y}$ are isomorphic to the product $\mathbf{E}_\varrho \times \mathbf{E}_\varrho$, where $\varrho = e^{2\pi i/3}$, and $\mathbf{E}_\varrho$ denotes the elliptic curve with modulus ϱ. Furthermore, the Δ_i are disjoint.*

Proof. Both statements follow from the properties of the branched cover $\overline{Y} \longrightarrow \overline{\mathcal{N}}_5$. First note that on $\overline{\mathcal{N}}_5$, all components of Π (the 30 exceptional $\mathbb{P}^1 \times \mathbb{P}^1$'s) are disjoint. Moreover, on each $\mathbb{P}^1 \times \mathbb{P}^1$, the branch locus is the following divisor:



Let Δ_i be an irreducible component of Δ, mapping to $\Pi_\lambda \subset \Pi$. Since the branch locus is a product, it follows that the cover $\Delta_i \longrightarrow \Pi_\lambda$ is also a product, each factor of which is the Galois branched cover of $\mathbb{P}^1$, branched over $\{0, 1, \infty\}$ of degree 9 (a Fermat cover). It is well known that this cover is the elliptic curve $\mathbf{E}_\varrho$, hence $\Delta_i \cong \mathbf{E}_\varrho \times \mathbf{E}_\varrho$. The inverse image of the six branch curves on Π_λ are the intersections of Δ_i with branch divisors which cover the 30 planes on $\overline{\mathcal{N}}_5$. Since on $\overline{\mathcal{N}}_5$ all components of Π_λ are disjoint, the furthermore statement now follows.

q.e.d.

6. The Nieto quintic is Janus-like

Next we verify $\mathcal{R}_1 = 0$ for the components $\overline{D}_\alpha$ covering the three types of surfaces occuring in equation (4), 2. To do this we will work downstairs. Since we know $c_1(\overline{Y}, \Delta) = \overline{\pi}^*(c_{\overline{Y}})$, and $c_1(\overline{D}_\alpha, \Delta_\alpha) = \overline{\pi}^*(c_{\overline{D}_\alpha})$, the relation

$$c_1(\overline{Y}, \Delta)_{|\overline{D}_\alpha} = \frac{4}{3} c_1(\overline{D}_\alpha, \Delta_\alpha)$$

will follow from the relation

$$c_{\overline{Y}} \cdot \overline{E}_\beta = \frac{4}{3} c_{\overline{D}_\alpha}, \tag{8}$$

for any $\overline{E}_\beta$ in the image of $\overline{D}_\alpha$ under the projection $\overline{Y} \longrightarrow \overline{\mathcal{N}}_5$, which is what we will in fact show.

- The Segre planes —

Let $\pi : D^* \longrightarrow E^*$ be a (singular) branched cover, branched along the arragement Λ of type $A_1(9)$=Ceva(2,3) to degree 3, such that $\overline{\pi} : \overline{D} \longrightarrow \overline{E}$, the resolution of this singular cover, is the smooth compactification of the ball quotient. We calculate $c_1(\overline{D}, \Delta_{\overline{D}})$. Let $\mathcal{R}_D \subset E$ be the ramification divisor of $\overline{\pi}$, and let $\omega \in H^2(\overline{E}, \mathbb{Z})$ denote the hyperplane class. Let, for each $m = 1, \dots, 10$, P_m *denote* the intersection $P_m \cap \overline{E}$ (so that there are four of these for each $\overline{E}$ which are non-vanishing), and let Q_β *denote* $Q_\beta \cap \overline{E}$ (so that there are three of these which are non-vanishing). Then $E = \overline{E} - \sum P_m$, i.e., $\Pi_E = \sum P_m$ is the union of four rational curves:

$$\begin{aligned}
c_1(\overline{D}, \Delta_{\overline{D}}) =& \overline{\pi}^* \left(c_1(\overline{E}) - \frac{2}{3}\mathcal{R}_D - \frac{1}{3}\Pi_E \right) \\
=& \overline{\pi}^* \left(3\omega - \sum_1^4 P_m - \sum_1^3 Q_\beta \right. \\
& \left. -\frac{2}{3}(9\omega - 2\sum P_m - 3\sum Q_\beta) - \frac{1}{3}\sum P_m \right) \\
=& \overline{\pi}^* \left(-3\omega + \frac{1}{3}\sum P_m + \sum Q_\beta - \frac{1}{3}\sum P_m \right) \\
=& \overline{\pi}^*(-3\omega + \sum_\beta Q_\beta).
\end{aligned}$$

A comparison with (7) shows the proportionality (8) for the Segre planes.

- The other planes —

  Notation as above, we have

$$\begin{aligned}
c_1(\overline{D}, \Delta_{\overline{D}}) =& \overline{\pi}^* \left(3\omega - \sum_1^6 Q_\beta - \frac{2}{3}(7\omega - 2\sum Q_\beta) - \frac{1}{3}\Pi_E \right) \\
=& \overline{\pi}^* \left(-\frac{5}{3}\omega + \frac{1}{3}\sum Q_\beta - \frac{1}{3}\Pi_E \right), \\
\Pi_E =& \sum_1^4 [L_{ijk}] = 4\omega - 2\sum Q_\beta, \\
c_1(\overline{D}, \Delta_{\overline{D}}) =& \overline{\pi}^* \left(-3\omega + \sum Q_\beta \right),
\end{aligned}$$

and again a comparison with (7) verifies the required proportionality.

- The exceptional cubics —

  It suffices to remark that for both the previous and this case $\overline{E}$ is $\mathbb{P}^2$ blown up in 6 points, and the configuration of branch loci is identical on all 30 such divisors. Note that in this case the 6 curves on $\overline{E}$ occuring in the sum above is the intersection of $\overline{E}$ with the proper transforms of the six Segre planes meeting at that Q_{ij}, while the class ω on that exceptional $\mathbb{P}^2$ is in reality $(Q_\beta)_{|Q_\beta}$.

This also completes the proof of

Theorem 6.1. *$\overline{Y}$ is a smooth compactification of the torsion-free ball quotient $\overline{Y} - \Delta$.*

Proof. We apply Theorem 4.2. Lemmas 5.2 and 5.4 and Corollary 5.3 show the assumptions on the pair $(\overline{X}, \Delta)$ in the theorem are satisfied. For all types of $\overline{D}_\alpha$, Theorem 1.1 and Corollary 1.2 leads to $\mathcal{Y}(\overline{D}_\alpha, \Delta_\alpha) = 0$. $\mathcal{R}_2 = 0$ is an empty condition for surfaces, and we have shown above that $\mathcal{R}_1(\overline{D}_\alpha, \Delta_\alpha) = 0$. Theorem 4.2 yields the vanishing of $\mathcal{Y}(\overline{Y}, \Delta)$, and Yau's theorem implies the statement. q.e.d.

Corollary 6.2. *The open set $\overline{\mathcal{N}}_5 - \Pi$ is an open ball quotient, $\overline{\mathcal{N}}_5 - \Pi = \Gamma_{\mathcal{N}_5} \backslash \mathbb{B}_3$, where $\Gamma_{\mathcal{N}_5}$ has torsion.*

Proof. We use the following principle. If the branch locus of the cover, which is left fixed by elements of the Galois group, are subballs (upstairs), then these automorphisms can be extended to automorphisms of the ball itself. It thus follows that the Galois group can be lifted, and hence that the quotient under the Galois group is again a ball quotient. We have already verified that the branch divisors are either subball quotients or compactification divisors, and the result follows. q.e.d.

Now we combine the results of [1] with the previous theorem. The following is essentially proved in [1, 8.1 and 8.4].

Theorem 6.3. *There is a Zariski-open subset of $\overline{\mathcal{N}}_5$ which is the quotient of the Siegel space of degree 2. Consequently $\overline{\mathcal{N}}_5$ is a smooth compactification of a quotient of the Siegel space.*

Proof. In [1, 8.1] it is shown that a Zariski open subset parametrizes $H_{2,2}$-invariant quartic surfaces with 16 skew lines. These are K3 surfaces, and the fact that they contain 16 lines means that they generically have isomorphic Picard groups. By the general theory (see [8, Propositions 2.3.1 and 2.3.2]), it follows that a Zariski open set is a quotient of a domain of type $\mathbf{IV}_3$ by an arithmetic group. Since by one of the exceptional isomorphisms this domain is isomorphic to the Siegel domain of genus 2, the result follows. q.e.d.

Remark. We note that in [1] it is proved that a double cover of $\mathcal{N}_5$ is a Siegel quotient by a subgroup which is commensurable to $Sp(4,\mathbb{Z})$, namely, it is a moduli space of certain abelian surfaces which admit a $(1,3)$ polarization. Since the passage from this double cover to the variety $\mathcal{N}_5$ is by taking the Kummer variety of the corresponding abelian surfaces, it follows that the Galois automorphism of this double cover lifts to an automorphism of Siegel space $\mathbb{S}_2$. This in turn tells us that the group occuring in the previous Theorem is commensurable with the same group $Sp(4,\mathbb{Z})$.

The notion of *Janus-like* varieties was introduced in [9]. This is the following two-way likeness for a smooth projective variety X: there exist two different domains $\mathcal{D}_1$, $\mathcal{D}_2$ and discrete groups $\Gamma_i \subset \mathrm{Aut}(\mathcal{D}_i)$ and there exist smooth compactifications $\overline{X}_{\Gamma_i}$ of the quotients $X_{\Gamma_i} := \Gamma_i \backslash \mathcal{D}_i$, such that we have isomorphisms $\overline{X}_{\Gamma_1} \cong \overline{X}_{\Gamma_2} \cong X$. For a general variety V we say it is Janus-like if it has a $\mathrm{Aut}(V)$−equivariantly birational model X which is Janus-like as just defined. It now follows immediately that:

Corollary 6.4. *The Nieto quintic is Janus-like.*

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