J. DIFFERENTIAL GEOMETRY
53 (1999) 489-545

SYMPLECTIC LEFSCHETZ FIBRATIONS WITH ARBITRARY FUNDAMENTAL GROUPS

J. AMORÓS, F. BOGOMOLOV, L. KATZARKOV & T. PANTEV

Abstract

In this paper we give an explicit construction of a symplectic Lefschetz fibration whose total space is a smooth compact four dimensional manifold with a prescribed fundamental group. We also study the numerical properties of the sections in symplectic Lefschetz fibrations and their relation to the structure of the monodromy group.

1. Introduction

In this paper we give an explicit construction of a symplectic Lefschetz fibration whose total space is a smooth compact four dimensional manifold with a prescribed fundamental group. The existence of such a fibration is also a consequence of the remarkable recent work of Donaldson [5] (see also [1]) who proved the existence of a Lefschetz pencil structure on any symplectic 4-manifold and the results of Gompf [9], who proved that any finitely presentable group can be realized as the fundamental group of a symplectic 4-manifold.

Since Donaldson's proof is non-constructive, as an alternative we present a direct purely topological construction of symplectic Lefschetz fibrations which is effective and allows an explicit control on the number of singular fibers. The construction is based on an algebraic geometric method for creating positive relations among right-handed Dehn twists.

Received April 16, 1999, and, in revised form, May 5, 2000. The first author was supported by DGYCYT grant PB96-0234, the second author was partially supported by NSF Grant DMS-9801591, the third author was partially supported by NSF Grant DMS-9700605 and A.P. Sloan research fellowship, and the fourth author was partially supported by NSF Grant DMS-9800790 (with an appendix by Ivan Smith).

The ubiquity of such relations combined with a simple group theoretic characterization of symplectic Lefschetz fibrations due to Gompf (see Proposition 2.3) turns out to be sufficient for the construction.

Before we can state our main theorem we need to introduce some notation. For any integer $n \geq 0$ denote by π_n the fundamental group of a compact Riemann surface of genus n. As usual a group G is called *finitely presentable* if it can be written as a quotient of a free group on finitely many generators by a subgroup generated by the conjugacy classes of finitely many elements. By a *finite presentation* of a group G we mean a surjective homomorphism $A \twoheadrightarrow G$ from some finitely presentable group A onto G so that $\ker[A \to G]$ is generated as a normal subgroup by finitely many elements in A.

Theorem A. *Let Γ be a finitely presentable group with a given finite presentation $a : \pi_g \twoheadrightarrow \Gamma$. Then there exists a surjective homomorphism $b : \pi_h \to \pi_g$ for some $h \geq g$ and a symplectic Lefschetz fibration $f : X \to S^2$ such that*

(i) *the regular fiber of f is of genus h,*

(ii) $\pi_1(X) \cong \Gamma$,

(iii) *the natural surjection of the fundamental group of the fiber of f onto the fundamental group of X coincides with $a \circ b$.*

Note that any finitely presentable group Γ admits a finite presentation of the form $\pi_g \twoheadrightarrow \Gamma$ since π_g surjects onto a free group on g generators.

The map b in the above theorem is not arbitrary. It factors as

$$\begin{tikzcd} \pi_h \arrow[rr, "b"] \arrow[dr] & & \pi_g \\ & \pi_e \arrow[ur] & \end{tikzcd}$$

where the surface of genus e is obtained form the surface of genus g by adding handles and the surface of genus h is obtained from the surface of genus e as a ramified finite covering.

Our second theorem concerns symplectic fibrations of Lefschetz type over curves of higher genus.

Theorem B. *Let Γ be a finitely presentable group with a given presentation $a : \pi_g \twoheadrightarrow \Gamma$. Then there exist a surjective homomorphism*

$\pi_e \to \pi_g$ and a symplectic Lefschetz fibration $f : X \to C_k$ over some smooth surface C_k of genus k so that

(i) *the regular fiber of f is of genus e,*

(ii) *f has a unique singular fiber,*

(iii) *the fundamental group of X fits in a short exact sequence*

$$1 \to \Gamma \to \pi_1(X) \to \pi_k \to 1,$$

(iv) *the natural surjective map from the fundamental group of the fiber of f to Γ coincides with the composition $\pi_e \to \pi_g \xrightarrow{a} \Gamma$.*

This work is an elaboration on a discussion at the end of [3]. We describe in details an enhancement of the general technique for constructing examples of symplectic fibrations used in [3]. Our proof is based on exploiting the correspondence between subgroups of the mapping class group and graphs of vanishing cycles.

In general it is expected that every smooth four-dimensional manifold is diffeomorphic to an achiral Lefschetz fibration possibly after some stabilization. The purpose of this paper is to study what other conditions besides chirality determine the SLF (Symplectic Lefschetz Fibration) among all fibrations.

The paper is organized as follows: In section two we give some preliminaries on subgroups of mapping class groups generated by Dehn twists and recall an important result of Gompf which characterizes symplectic Lefschetz fibrations via their monodromy representations. In section three we explain the general construction and prove Theorems A and B. In section four we give an explicit example of a symplectic Lefschetz fibration whose total space has a fundamental group isomorphic to $\mathbb{Z}$. All this demonstrates the flexibility of the construction for obtaining interesting examples.

The construction is similar in spirit to a computation done by Donaldson in which he represented Thurston's example as a symplectic Lefschetz fibration of genus three Riemann surfaces. A whole series of examples of the same flavor was constructed independently in [25] and [28], [29].

In the last section we apply the group theoretic part of the construction to the study of the numerical properties of sections in symplectic Lefschetz fibrations. Finally Appendix A, written by Ivan Smith, presents a short proof of the non-existence of SLF with monodromy contained in the Torelli group.

Acknowledgments. The authors would like to thank UC Irvine and MPI-Bonn for their hospitality. Parts of this work were done during the authors, participation in the UCI Hodge theory activities in June 1998 and during the first three authors, participation in the MPI activities dedicated to the work of B. Moishezon organized by F.Catanese, F.Hirzebruch and M.Teicher. We would like to thank the organizers of both events for the nice working atmosphere they have created.

Special thanks are due to D.Auroux, S.Donaldson, T.Fuller, R.Hain, B.Ozbagci, I.Smith, R.Stern and D.Toledo for their nice suggestions and constant attention to this work. We also thank D.Zagier and V.Platonov for useful group theoretic comments and D. Auroux and R. Hain for pointing out some mistakes in a preliminary version of this paper.

Finally we would like to thank the anonymous referee for pointing out serious gaps in a previous version of Lemma 3.2 and in one of the examples.

Notation and terminology

C_g a smooth compact oriented surface of genus g.

Δ an analytic disk.

Γ a finitely presentable group.

$\mathrm{Map}_g(-)$ the subsemigroup in Map_g generated by the right Dehn twists.

Map_g^n the mapping class group of a smooth genus g surface with n punctures.

$\mathrm{Map}_{g,r}^n$ the mapping class group of a smooth genus g surface with n punctures and r boundary components.

Map_R the subgroup of the mapping class group Map_g generated by the Dehn twists $t_r, r \in R$.

$\mathrm{Map}_R(-)$ the subsemigroup in Map_R generated by all conjugates of $t_r, r \in R$ within Map_R.

Mon the geometric monodromy group of a TLF (Topological Lefschetz Fibration), i.e., the image $\mathrm{Mon} := \mathrm{mon}(\pi_1(S^2 \setminus \{q_1, \ldots, q_\mu\}, o))$ of the geometric monodromy representation.

mon the geometric monodromy representation associated with a TLF of genus g, i.e.,

$$\mathrm{mon} : \pi_1(S^2 \setminus \{q_1, \dots, q_\mu\}, o) \to \mathrm{Map}_g \,.$$

μ the number of singular fibers in a TLF or a SLF.

o a base point in $S^2 \setminus \{q_1, \dots, q_\mu\}$.

$\mathcal{O}_V$ the structure sheaf of an algebraic variety V.

$\mathcal{O}_V(1)$ a very ample line bundle on an algebraic variety V.

$f : X \to S^2$ a topological or symplectic Lefschetz fibration (TLF or SLF).

$f : X \to C_k$ a symplectic fibration of Lefschetz type over a higher genus surface.

π_g^n the fundamental group of a smooth genus g surface with n punctures.

π_g the fundamental group of a smooth complete surface of genus g.

$\pi_g \twoheadrightarrow \Gamma$ a finite presentation of Γ, i.e., a surjective homomorphism whose kernel is finitely generated as a normal subgroup.

Q_i a critical point of a topological or symplectic Lefschetz fibration.

q_i a critical value of a topological or symplectic Lefschetz fibration.

$R \subset C_g$ a graph connected collection of circles on the surface C_g.

$s \subset C_g$ a circle in C_g or in other words a smooth connected one dimensional submanifold in C_g.

Σ a smooth projective algebraic curve.

T_s a right-handed Dehn twist diffeomorphism associated with a circle $s \in C_g$. In other words for any $s \subset C_g$ one chooses an orientation preserving identification of a tubular neighborhood of s with the oriented cylinder $[0,1] \times S^1 \subset \mathbb{R} \times \mathbb{C}$ and then defines $T_s \in \mathrm{Diff}^+(C_g)$ as the diffeomorphism that acts a $(t, z) \mapsto (t, e^{2\pi i t} z)$ on the cylinder and as identity everywhere else.

t_s the mapping class of a right-handed Dehn twist T_s. The element $t_s \in \mathrm{Map}_g$ depends only on the isotopy class of T_s.

U_i a small neighborhood of a critical value of a TLF or SLF.

U_{Q_i} a small neighborhood of a critical point in a TLF or SLF.

X_i the singular fiber of a TLF or a SLF corresponding to the critical value q_i.

X_o a regular fiber of a TLF or a SLF.

2. Symplectic Lefschetz fibrations

First we recall some basic definitions and results. For more details the reader may wish to consult [27, Section 3.2.7] and [14].

Definition 2.1. Let X be a smooth compact 4-manifold equipped with a smooth surjective map $f : X \to S^2$. We shall call it a *topological Lefschetz fibration (TLF)* if the following conditions hold:

(i) The differential df is surjective outside a finite subset of points $\{Q_1, \dots, Q_\mu\} \subset X$.

(ii) Whenever $p \in S^2 \setminus \{f(Q_1), \dots, f(Q_\mu)\}$ the fiber $f^{-1}(p)$ is a smooth orientable Riemann surface of a given genus g.

(iii) The images $q_i := f(Q_i)$ are different for different Q_i.

(iv) Let X_i denote the fiber of f containing Q_i. Then for any i there are small disks $q_i \in U_i$ and $Q_i \subset U_{Q_i} \subset X$ with $f : U_{Q_i} \to U_i$ being a complex Morse function in some complex coordinates (x, y) on U_{Q_i} and z on U_i, i.e., $z = f(x, y) = x^2 + y^2$.

A topological Lefschetz fibration $f : X \to S^2$ will be called *orientable* (or *chiral*) if there exists an orientation on X so that the complex coordinates in (iv) above can be chosen in a way compatible with the orientations on X and S^2. Note that the definition of a topological Lefschetz fibration is designed in such a way that the function $|f|^2$ is a Morse function near the singular fibers X_i. The Morse flow gives a handle body decomposition of X and in particular one gets standard retractions $cr_i : f^{-1}(\overline{U}_i) \to X_i$. Fix a base point $o \in S^2$ which is a regular value of f and choose arcs $a_1, \dots, a_\mu$ in $S^2 \setminus (\cup_i U_i)$ which connect o with some point on ∂U_i, $i = 1, \dots, \mu$ as in Figure 2.1

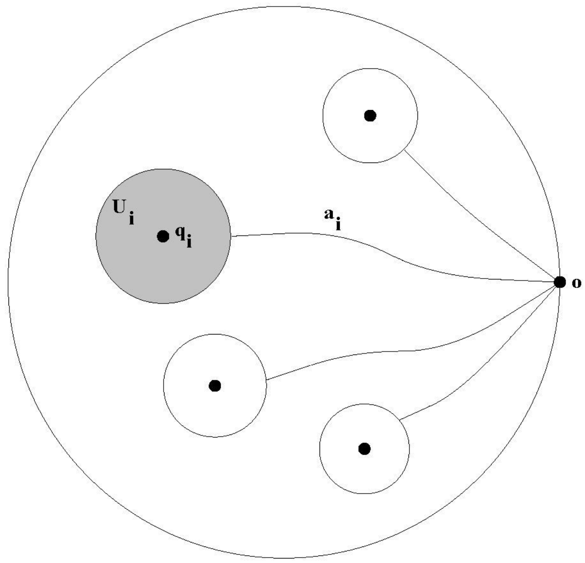

Figure 2.1: An arc system for $f : X \to S^2$.

Such a collection of arcs and discs is called an *arc system* for the TLF. A choice of an arc system gives a presentation of the fundamental group of $S^2 \setminus \{q_1, \dots, q_\mu\}$:

$$\pi_1(S^2 \setminus \{q_1, \dots, q_\mu\}, o) = \langle c_1, \dots, c_\mu | c_1 \cdot \dots \cdot c_\mu = 1 \rangle,$$

where geometrically c_i is represented by the o-based loop in $S^2 \setminus \{q_1, \dots, q_\mu\}$ obtained by tracing a_i followed by tracing ∂U_i counterclockwise and then tracing back a_i in the opposite direction.

Since the family $f : X \to S^2$ is locally trivial when restricted to each a_i we get well defined retractions of X_o onto each of the singular fibers X_i. By abuse of notation these retractions will be denoted by cr_i as well. Each $cr_i : X_o \to X_i$ contracts a smooth circle $s_i \subset X_o$ - *the geometric vanishing cycle*. The boundary of $f^{-1}(U_i)$ is diffeomorphic to a smooth fiber bundle over the circle ∂U_i with X_o as a fiber. This fiber bundle is determined by a gluing diffeomorphism $T_{s_i} : X_o \to X_o$ which is the usual Dehn twist along the circle s_i. More precisely T_{s_i} is the right-handed Dehn twist along s_i with respect to the orientation on X_o compatible with the orientation on U_{Q_i} given by the complex coordinates around Q_i. The circles s_i and the Dehn twists T_{s_i} are uniquely determined up to a smooth isotopy and thus give well defined elements

$t_i \in \mathrm{Diff}^+(X_o)/\,\mathrm{Diff}_0^+(X_o) =: \mathrm{Map}_g \subset \mathrm{Out}(\pi_1(X_o))$ - the group of mapping classes of an oriented surface of genus g. The homomorphism $\mathrm{mon} : \pi_1(S^2 \setminus \{q_1, \dots, q_\mu\}, o) \to \mathrm{Map}_g$, $\mathrm{mon}(c_i) = t_i$ is called the *geometric monodromy representation* of $f : X \to S^2$ and its image is called the *geometric monodromy group* of the TLF. It is known [14, Theorem 2.4] that if $f : X \to S^2$ is an orientable TLF of genus $g \geq 2$, then the geometric monodromy representation of f uniquely determines the diffeomorphism type of f.

Note that if C_g is an oriented surface and $s \subset C_g$ is a smoothly embedded, homotopically non-trivial circle one can perform both the right-handed Dehn twist $T_s : C_g \to C_g$ and the left-handed Dehn twist $T_s^{-1} : C_g \to C_g$. If however $f : X \to S^2$ is an orientable TLF and $C_g = X_o$ is given the induced orientation from X, then all of the geometric monodromy transformations $\{T_{s_1}, \dots, T_{s_\mu}\}$ are right-handed Dehn twists. This property actually characterizes the orientable TLF completely.

We also consider topological Lefschetz fibrations which are compatible with an additional closed non-degenerate 2-form w on X.

Definition 2.2. Assume that $f : X \to S^2$ is a TLF and that (X, w) is a symplectic manifold. We say that $(f : X \to S^2; w)$ is a *symplectic Lefschetz fibration (SLF)* if for any $p \in S^2$ the form w is non-degenerate on the fiber X_p at p in the sense that the smooth locus of X_p is a symplectic submanifold in X and for every i the symplectic form w_{Q_i} is non-degenerate on each of the two planes contained in the tangent cone of X_i at Q_i.

Gompf [8] had shown that under some mild restrictions the SLF can also be characterized in purely topological terms. For the convenience of the reader we recall the proof of this very useful fact. Different proofs can be found in [8, Theorem 10.2.18] or [30].

Proposition 2.3 (R.Gompf). *A topological Lefschetz fibration $f : X \to S^2$ of curves of genus $g \geq 2$ admits a symplectic structure if and only if it is orientable.*

Proof. We will only prove the "if" part of the Proposition. The "only if" part is not used anywhere in the paper and is left as an exercise.

Let $f : X \to S^2$ be an orientable TLF with singular values $q_1, \dots, q_\mu$. Take a topologically simple cover of S^2 by open disks $\{D_\alpha\}$, such that it includes a disk U_i centered at every singular value q_i, and $q_i \notin \bar{U}_\alpha$ if $\alpha \neq i$. We will put symplectic structures on the families over the disks

U_α first, and then glue them by adapting an argument of Thurston (see [18, Theorem 6.3]) for symplectic fibrations.

For every disk D_i containing a singular value of f, take the trivial family $C_g \times U_i \to U_i$, endowed with a symplectic form $w_{\text{fiber}} \oplus w_{\text{base}}$, the summands being symplectic forms on the factors. By identifying C_g with a regular fiber of $f_{|f^{-1}D_i}$ and choosing an arc from the basepoint regular value to the singular value of the fibration we get a vanishing loop $s_i \subset C_g$ that determine the diffeomorphism type of the pencil f over D_i. We will perform symplectic surgery on the trivial family $C_g \times U_i$ to make it diffeomorphic to f.

Let $q : \mathbb{C}^2 \to \mathbb{C}$ be the standard quadratic map $q(x,y) = x^2 + y^2$. Take a small ball B centered at $(0,0) \in \mathbb{C}^2$, a disk $D \subset q(B)$. The restricted family $q_{|B} : B \to D$ has a single quadratic singular fiber B_0 over $0 \in D \subset \mathbb{C}$.

Let $s \subset C_g$ be one of the vanishing simple loops of the fibration X. Consider the normalization $\widetilde{B}_0 \to B_0$ and let o_1 and o_2 be the preimages of the singular point $(0,0) \in B_0 \subset B$ in $\widetilde{B}_0$. Choose small analytic discs in $\widetilde{B}_0$ centered at o_1 and o_2 respectively and let Δ_1, Δ_2 denote their images in B_0.

Select two open annuli $A_1 \subset \Delta_1 \backslash \{(0,0)\}$ and $A_2 \subset \Delta_2 \backslash \{(0,0)\}$ such that the point $(0,0)$ does not lie in the closure $\bar{A}_1 \cup \bar{A}_2$ (see Figure 2.2). Using Moser's characterization of symplectic type of surfaces by volume, we may choose also two open cylinders C_1, C_2 on the opposite sides of a bicollar neighborhood of c, such that both C_1, C_2 are retracts of the bicollar neighborhood, their adherence $\bar{C}_1 \cup \bar{C}_2$ does not intersect c, and they are symplectomorphic to A_1, A_2 respectively.

The annuli A_1, A_2, C_1, C_2 are embedded in their respective total spaces with trivial normal bundle, so by Weinstein's symplectic neighborhood theorem [9, Lemma 2.1], theorem exists an $\varepsilon > 0$ and open neighborhoods W_i of A_i in B, V_j of C_j in $C_g \times S^2$ for $i, j = 1, 2$ such that the W_i, V_j are symplectomorphic to $A_i \times D_\varepsilon, C_j \times D_\varepsilon$ respectively. Shrinking D if necessary, we may now perform a surgery by inserting the Dehn twist of $q_{|B} : B \to D$ by the identifications $W_i \cong V_i$. The symplectic structures on B and $C_g \times D_i$ define a symplectic structure on f over D_i in this way.

We repeat this surgery over all the critical values of f, rescaling the obtained symplectic structures $\omega_1, \ldots, \omega_\mu$ so that they induce the same symplectic structure $[\sigma]$ on the regular fibers of the pencil. Through diffeomorphisms with trivial families we may also endow the restrictions

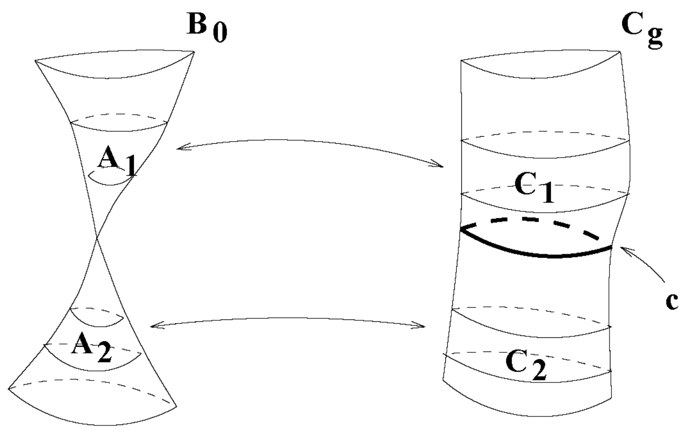

Figure 2.2: The cylinders A_i and C_i, $i = 1, 2$.

of f over the regular disks U_α with symplectic structures ω_α inducing the same symplectic structure $[\sigma]$ on the regular fiber of the pencil as the twist families over the singular values.

Let now $\tau_0 \in \Omega^2(X)$ be a closed 2–form such that its restriction to the fibers represents the cohomology class $[\sigma]$. Such a form exists due to the assumption $g \geq 2$. Indeed, consider the differential map $df : TX \to f^*TS^2$. Over the open set $X \setminus \{Q_1, \ldots, Q_\mu\}$ the map df is a surjective morphism of vector bundles and so its kernel is a well defined rank two real vector bundle on $X \setminus \{Q_1, \ldots, Q_\mu\}$. On the other hand by the definition of a TLF we can find local charts $q_i \in U_i \subset S^2$, $Q_i \in U_{Q_i} \subset X$ and complex coordinates z_i on U_i and (x_i, y_i) on U_{Q_i} so that $f((x_i, y_i)) = x_i y_i = z_i$. Since $f_{|U_{Q_i}}$ is an algebraic map we can interptret the restriction $df_{|U_{Q_i}}$ as a morphism between the holomorphic tangent bundles of U_{Q_i} and U_i respectively. Explicitly

$$df_{|U_{Q_i}}\left(a(x_i, y_i)\frac{\partial}{\partial x_i} + b(x_i, y_i)\frac{\partial}{\partial y_i}\right) = (y_i a + x_i b) \cdot \frac{\partial}{\partial z_i}.$$

In particular the kernel of the algebraic map $df_{|U_{Q_i}}$ is the coherent sheaf on U_{Q_i} corresponding to the $\mathbb{C}[x_i, y_i]$-module

$$\mathbb{C}[x_i, y_i] \cdot \left(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}\right) \subset \mathbb{C}[x_i, y_i] \cdot \frac{\partial}{\partial x_i} \oplus \mathbb{C}[x_i, y_i] \cdot \frac{\partial}{\partial y_i}$$

and is therefore an invertible sheaf on U_{Q_i}. In other words the real rank two vector bundle $\ker(df_{X\setminus\{q_1,\ldots,q_\mu\}})$ glues with the real rank two vector bundles underlying the invertible sheaves $\ker(df_{|U_{Q_i}})$ to a global rank two vector bundle T_f on X. Moreover the orientability assumption on the TLF implies that T_f is an orientable real rank two vector bundle and so T_f admits an almost complex structure. Finally by construction T_f restricts to the tangent bundle of the general fiber of f and so $c_1(T_f)$ restricts to the first Chern class of the tangent bundle of the general fiber. But the fibers of f are assumed to have genus $g \geq 2$ and so the first Chern class of the general fiber of f is $c[\sigma]$ for some non-zero constant c. Hence $c^{-1}c_1(T_f) \in H^2(X, \mathbb{R})$ is a cohomology class that restricts to $[\sigma]$ on the general fiber of f. If all the vanishing cycles of f are non-separating, the only closed surfaces in X contained in fibers of f are the fibers themselves. Therefore $c^{-1}c_1(T_f)$ will restrict to $[\sigma]$ on every fiber of f and so we may take τ_0 to be a closed 2–form representing the cohomology class $c^{-1}c_1(T_f)$. If however the fibration f has separating vanishing cycles one needs to modify the class $c^{-1}c_1(T_f)$ further so that it restricts to a non-trivial cohomology class on every closed surface contained in a fiber of f. This can be easily achieved by adding to $c^{-1}c_1(T_f)$ suitable (rational) multiples of Poincare duals to the closed surfaces contained in the singular fibers of f (see also [8, Solution to Exercise 10.2.19]).

Let τ_0 be a closed 2–form representing the modification of the cohomology class $c^{-1}c_1(T_f)$. Over every disk D_α we have that τ_0 and the symplectic form ω_α are cohomologous, thus we may select 1–forms $\lambda_\alpha \in \Omega^1(f^{-1}(D_\alpha))$ so that

$$\omega_\alpha - \tau_0 = d\lambda_\alpha.$$

Choose now a partition of unity $\{\rho_\alpha : D_\alpha \to [0,1]\}$ subordinate to the cover $\{U_\alpha\}$ and such that for every critical value q_i the function ρ_i has constant value 1 on its neighborhood. Define $\tau \in \Omega^2(X)$ by

$$\tau = \tau_0 + \sum_\alpha d((\rho_\alpha \circ f)\lambda_\alpha).$$

This is a closed form, cohomologous to τ_0, restricting to the class $[\sigma]$ in every fiber and equal to the previously found symplectic ω_i in neighborhoods of the singular fibers. As in the case of smooth symplectic fibrations, this form τ is nondegenerate on the tangent spaces to the fibers, and as it is defined on the compact total space, if we select a symplectic form $\beta \in \Omega^2(S^2)$ the forms

$$\omega_K = \tau + Kf^*\beta$$

are nondegenerate for sufficiently large K. q.e.d.

3. The main construction

3.1 Positive relations among right Dehn twists

Let g be a nonnegative integer. Fix a compact oriented reference surface C_g of genus g and an infinite sequence of distinct points $x_0, x_1, \dots, x_n, \dots \in C_g$. Put $\pi_g^n := \pi_1(C_g \setminus \{x_1, \dots, x_n\}, x_0)$ and let $\mathrm{Diff}^+(C_g)^n_r$ denote the group of all orientation preserving diffeomorphisms of C_g that fix the points $x_1, x_2, \dots, x_{n+r}$ and induce the identity on the tangent spaces $T_{x_i}C_g$ for $i = n+1, \dots, n+r$. For any triple of non-negative integers (g, n, r) such that $2g - 2 + n + 2r > 0$ define the *mapping class group* $\mathrm{Map}^n_{g,r}$ as the group of connected components of $\mathrm{Diff}^+(C_g)^n_r$, i.e.,

$$\mathrm{Map}^n_{g,r} := \pi_0(\mathrm{Diff}^+(C_g)^n_r).$$

As usual we will skip the labels n and r if they happen to be equal to zero.

By definition the mapping class group Map^n_g acts by outer automorphisms on π^n_g. In fact this action identifies [10] Map_g with the index two subgroup of $\mathrm{Out}(\pi_g)$ consisting of outer automorphisms acting trivially on $H^2(\pi_g, \mathbb{Z}) \cong H^2(C_g, \mathbb{Z})$. Similarly one can interpret the group Map^1_g as the group of all automorphisms of π_g acting trivially on $H^2(\pi_g, \mathbb{Z})$. The group $\mathrm{Map}^n_{g,r}$ is generated by the right-handed Dehn twists along all (unoriented) non separating loops $c \subset C_g \setminus \{x_1, \dots, x_{n+r}\}$.

Remark 3.1. For any g, n the natural forgetful map $\mathrm{Map}_{g,n} \to \mathrm{Map}^n_g$ is surjective and has a central kernel which can be identified with the free abelian group generated by the Dehn twists along simple loops around the punctures.

For future reference define $\mathrm{Map}^n_{g,r}(-) \subset \mathrm{Map}^n_{g,r}$ to be the subsemigroup of $\mathrm{Map}^n_{g,r}$ generated by (the images of) all right-handed Dehn twists.

The first step in the construction is the following simple observation.

Lemma 3.2. *Let $s_1, \dots, s_m \subset C_g$ be free simple closed loops (not necessarily distinct) and let $t_1, \dots, t_m$ be the corresponding right-handed Dehn twists. Suppose that there exist integers $\{n_i\}_{i=1}^m$ so that the t_i's* *satisfy the relation* $\prod_i t_i^{n_i} = 1$ *in the mapping class group* Map_g^1. *Then there exists a TLF* $f : X \to S^2$ *with the following properties:*

(a) *The regular fiber of* f *can be identified with* C_g *so that the vanishing cycles of* f *are precisely the* s_i*'s.*

(b) *The geometric monodromy group* $\mathrm{Mon}(f)$ *of* f *is just the subgroup of the mapping class group generated by the* t_i*'s.*

(c) *The fundamental group of* X *is isomorphic to the quotient of* π_g *by the normal subgroup generated by all the* s_i*'s.*

(d) *If all the* n_i*'s are positive, then* f *is a SLF.*

Proof. To construct the fibration $f : X \to S^2$ start with the direct product $D_0 \times C_g$ where $D_0 \subset \mathbb{C}$ is a small disk around zero. Next attach $|n_i|$-copies of small discs D_i, $i = 1, \dots, n$, along the boundary of D_0. Over each D_i choose a standard holomorphic Lefschetz fibration $U_{X_i} \to D_i$ with a unique singular fiber X_i at the center so that the vanishing cycle is exactly s_i. The union of $D_0 \times C_g$ and the fibrations $U(X_i)$ (each appearing $|n_i|$-times respectively) has a structure of a topological Lefschetz fibration $u : U \to D$ over a larger disc D. By construction the Lefschetz fibration u has a regular fiber isomorphic to C_g and vanishing cycles $s_1, \dots, s_m$. Moreover the geometric monodromy transformation $\mathrm{mon}(\partial D) \in \mathrm{Map}_g$ for u is precisely the product $\prod_i t_i^{n_i}$. Since the latter is equal to identity in Map_g^1 and a posteriori in Map_g we get that $U_{|\partial D}$ is homotopy equivalent (and hence diffeomorphic) to a product $C_g \times S^1$. In particular we can extend u to a TLF over S^2.

The conditions (a) and (b) are satisfied by construction. Let $f^\sharp : X^\sharp \to S^\sharp$ denote the fibration obtained from f by removing the singular fibers. Now $f^\sharp$ is a fiber bundle by construction. Also the condition that $\prod_i t_i^{n_i} = 1$ is equivalent to requiring that $f^\sharp$ admits a topological section. Consequently we can identify the fundamental group $\pi_1(X^\sharp)$ with the semidirect product $\pi_1(S^\sharp) \ltimes_{\mathrm{mon}} \pi_g$ where $\mathrm{mon} : \pi_1(S^\sharp) \to \mathrm{Aut}(\pi_g)$ is the monodromy representation. Now the Seifert-van Kampen theorem implies that $\pi_1(X)$ is isomorphic to the quotient of π_g by the normal subgroup generated by the orbits of the vanishing loops $\{s_1, \dots, s_m\}$ under the monodromy group Mon of f. On the other hand any Dehn twist t_s is uniquely characterized by the property that the maximal quotient of π_g on which t_s acts as the identity is the quotient of π_g by the normal subgroup generated by s. In particular for any $\gamma \in \pi_g$

the element $\gamma^{-1}t_s(\gamma)$ is a product of conjugates of s and so the normal subgroup of π_g generated by the Mon-orbits of $s_1, \dots, s_m$ coincides with the normal subgroup generated by $s_1, \dots, s_m$ only which proves part (c) of the Lemma.

Finally the fact that condition (d) holds for f is a consequence of Proposition 2.3. The Lemma is proven. q.e.d.

Remark 3.3. **(i)** If the condition $\prod_i t_i^{n_i} = 1 \in \mathrm{Map}_g^1$ in the hypothesis of Lemma 3.2 is replaced by the milder condition $\prod_i t_i^{n_i} = 1 \in \mathrm{Map}_g$, the assertions of the lemma are still true provided that the center of the quotient of π_g by the normal subgroup generated by the s_i's is a torsion group.

Indeed the construction of the TLF $f : X \to S^2$ described in the proof of Lemma 3.2 requires only that $\prod_i t_i^{n_i} = 1 \in \mathrm{Map}_g$. In particular exactly as in Lemma 3.2 we have a topological Lefschetz fibration $u : U \to D$ over a disk D so that ∂U is diffeomorphic to the product $C_g \times \partial D$. The full TLF $f : X \to S^2$ is again the gluing of U and the product $C_g \times D_\infty$ (where D_∞ is a small disk around $\infty \in S^2$) along the boundary $\partial U \cong C_g \times S^1 \cong \partial(C_g \times D_\infty)$. Next by the Seifert-van Kampen theorem we can identify the fundamental group $\pi_1(U)$ of the manifold with boundary U exactly with the quotient of π_g by the normal subgroup generated by the s_i's. Furthermore the quotient map $h : \pi_g \twoheadrightarrow \pi_1(U)$ is induced from the inclusion $C_g \hookrightarrow U$ as one of the smooth fibers of u. On the other hand the homomorphism h extends naturally to a homomorphism $h^\partial : \pi_g \times \mathbb{Z} \to \pi_1(U)$ induced from the inclusion $C_g \times \partial D = \partial U \hookrightarrow U$ of the boundary. Let $\mathfrak{c} \in \pi_g \times \mathbb{Z}$ be one of the two standard generators of $\{1_{\pi_g}\} \times \mathbb{Z}$. In other words $\mathfrak{c}$ is represented by a loop which projects homeomorphically to ∂D. Since $g \geq 2$ we see that $\mathfrak{c}$ generates the center of $\pi_g \times \mathbb{Z}$ and so $h^\partial(\mathfrak{c}) \in \pi_1(U)$ is a central element. The gluing of $C_g \times D_\infty$ to U adds exactly one more relation to $\pi_1(U)$, namely the relation $h^\partial(\mathfrak{c}) = 1$. Therefore $\pi_1(X) = \pi_1(U)/\mathfrak{C}$ where $\mathfrak{C} \subset \pi_1(U)$ is the cyclic central group generated by $h^\partial(\mathfrak{c})$. In particular if the center of $\pi_1(U)$ is trivial we get $\pi_1(X) = \pi_1(U)$ as desired.

If $Z(\pi_1(U))$ is only torsion we have to modify the construction. In this case $\mathfrak{C} = \mathbb{Z}/m$ for some integer m and we can remedy the situation by gluing m copies of X along smooth fibers. Clearly the resulting TLF will satisfy assertions (a), (b) and (d) of Lemma 3.2. To see that it will also satisfy (c) one can argue as follows. Observe first that the image of the edge homomorphism $\pi_2(U, \partial U) \to \pi_1(\partial U)$ is precisely $\ker(h^\partial)$. In particular, the natural projection $(u, u_{|\partial U}) : (U, \partial U) \to (D, \partial D)$ induces an isomorphism $\mathrm{im}[\pi_2(U, \partial U) \to \pi_1(\partial U)]/\ker(h) \to \pi_1(\partial D) = \mathbb{Z}$ and hence every element in $\pi_2(U, \partial U)$ that maps with degree one on the disk D goes to an element of the coset $\mathfrak{c} + \ker(h) \subset \ker(h^\partial)$. It is now clear that if we glue two copies of U along a smooth fiber of u we similarly obtain elements in the coset $2\mathfrak{c} + \ker(h)$. Thus after takling the connected sum of m copies of U we will get a new TLF $f : Y \to S^2$ for which $\pi_1(Y) = \pi_1(U)/\langle h^\partial(\mathfrak{c}^m)\rangle = \pi_1(U)$.

(ii) It is possible to find relations $\prod_i t_i^{n_i} = 1 \in \mathrm{Map}_g$ for which $h^\partial(\mathfrak{c}) \neq 1$. The following example was suggested to us by the referee.

Fix an integer $m > 1$ and a generic TLF of degree m in $\mathbb{CP}^2$. Delete a 4-ball around each of the m^2 points in the base locus. Now double the resulting manifold with boundary (i.e., cross with $[0, 1]$ and take the resulting boundary), to obtain a TLF structure on $X = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \# (m^2 - 1)(S^1 \times S^3)$. The general fiber F of X has genus $(m-1)(m-2) + m^2 - 1$ and is m-divisible in the homology of X. In particular $h^\partial(\mathfrak{c})$ corresponding to the TLF $X \setminus F$ is of order m.

The previous lemma shows that the construction of symplectic Lefschetz fibrations reduces to the problem of finding relations in the semigroup $\mathrm{Map}_g^1(-) \subset \mathrm{Map}_g^1$. In order to find such relations and to be able to modify them we will need to study certain configurations of embedded circles in C_g.

Recall [13] that given two isotopy classes ρ and σ of smooth circles in C_g one defines the *geometric intersection number* $i(\rho, \sigma)$ as the minimum number of points of $r \cap s$ over all representatives r of ρ and s of σ. A finite set R of smooth circles in C_g is said to be in *minimal position* if every two elements $r, s \in R$ intersect transversally in exactly $i([r], [s])$ points and the inersection of every three distinct elements in R is empty. It is well known that any finite collection of circles on a surface can be isotoped to minimal position.

Let G_R be the unique graph which is dual to the one dimensional cell complex $\cup\{r | r \in R\} \subset C_g$, and has no multiple edges. In other words the graph G_R has vertices labeled by the elements of R and two vertices $s, r \in R$ are connected by an edge if and only if the two loops s and r in C_g intersect at a unique point.

Definition 3.4.

(a) Two loops $s, r \in R$ will be called *adjacent* if they intersect transversally at a single point, i.e., if the vertices s and r in G_R are connected by a single edge.

(b) Two circles $s, r \in R$ will be called *graph connected* if the corresponding vertices are connected by a path of edges in G_R. In other words $s, r \in R$ are graph connected if there exist a sequence of circles $s_1, \dots, s_m \in R$ so that $s = s_1$, $r = s_m$ and s_i is adjacent to s_{i+1} for all $i = 1, \dots, m-1$.

(c) Let R and S be two finite sets of circles in C_g so that $R \cup S$ is in minimal position. We will say that R *is graph connected to* S if for any loop $r \in R$ there exists a loop $s \in S$ so that r and s are graph connected in $R \cup S$.

Remark 3.5. The adjacency condition in part (b) of Definition 3.4 is imposed only on two consecutive loops in the sequence $s_1, \dots, s_m$. In particular, non-consecutive loops may intersect in an arbitrary transverse fashion.

For any pair of smooth circles $a, b \subset C_g$ intersecting at exactly one point one can find a neighborhood $C_{a,b}$ of $a \cup b$ in C_g diffeomorphic to a torus with one hole as depicted on Figure 3.1.

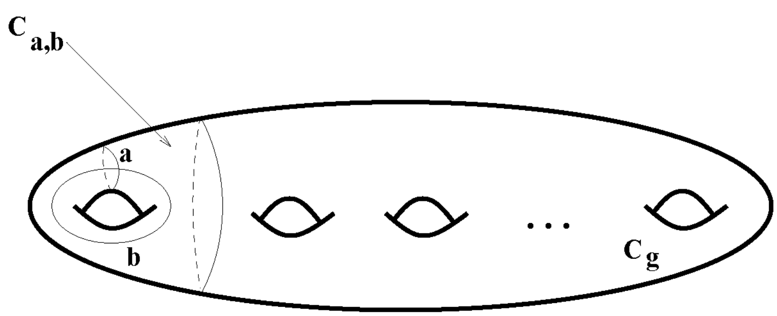

Figure 3.1: A handle determined by two adjacent cycles.

Thus we obtain a homomorphism $h_{a,b} : \mathrm{Map}_{1,1} \to \mathrm{Map}_g^1$ defined by the handle $C_{a,b} \subset C_g$.

For a set R of circles in minimal position denote by Map_R^1 the subgroup of Map_g^1 generated by the right Dehn twists $\{t_r\}_{r \in R}$. Let

$\mathrm{Map}^1_R(-)$ denote the subsemigroup in Map^1_R generated by the right twists $\{t_r\}_{r\in R}$ and all of their conjugates in Map^1_R. Note that $\mathrm{Map}^1_R(-)$ is contained in $\mathrm{Map}^1_g(-)$ as a subsemigroup since for every $\phi \in \mathrm{Map}^1_g$ and every isotopy class ρ of circles on C_g one has $\phi \circ t_\rho \circ \phi^{-1} = t_{\phi(\rho)}$. Moreover by definition the semigroup $\mathrm{Map}^1_R(-) \subset \mathrm{Map}^1_R$ is invariant under conjugation in Map^1_R.

The next two lemmas give a topological characterization of the existence of relations in $\mathrm{Map}^1_g(-)$.

Lemma 3.6. *Let L be a finite set of circles in minimal position in C_g. Then $\mathrm{Map}^1_L(-) = \mathrm{Map}^1_L$ if and only if there exists a finite relation $\alpha = 1$ where α is a product of only positive powers of $t_{\phi(l)}$'s with $l \in L$, $\phi \in \mathrm{Map}^1_L$ and each $t_l, l \in L$ occurs at least once in α.*

Proof. For the proof of the "if" part consider a relation $\alpha = 1$ as in the statement of the lemma. By hypothesis there exist a positive integer μ and a map $l : \{1, 2, \ldots, \mu\} \to \{\phi(x) | x \in L, \phi \in \mathrm{Map}^1_L\}$ so that $\mathrm{im}(l) \supset L$ and

$$1 = \alpha = \prod_{i=1}^{\mu} t_{l(i)}$$

in Map^1_L. If we multiply both sides of the above relation by $t^{-1}_{l(1)}$ on the left we get that

$$t^{-1}_{l(1)} = \prod_{i=2}^{\mu} t_{l(i)} \in \mathrm{Map}^1_L(-).$$

Since the relation is cyclic this implies that $t^{-1}_{l(i)} \in \mathrm{Map}^1_L(-)$ for all $i = 1, \ldots, k$. Combining this with the surjectivity of l yields the inclusion $\{t_l^{-1}\}_{l\in L} \subset \mathrm{Map}^1_L(-)$ and hence $\mathrm{Map}^1_L(-) = \mathrm{Map}^1_L$.

To prove the "only if" part of the lemma note that the assumption $\mathrm{Map}^1_L(-) = \mathrm{Map}^1_L$ implies that $t_l^{-1} \in \mathrm{Map}^1_L(-)$ for all $l \in L$. In other words each t_l^{-1} can be written as a product of finitely many of the right Dehn twists $\{t_{\phi(s)}\}_{s\in L, \phi\in \mathrm{Map}^1_L}$. Let now m be the cardinality of L and let $l_1, \ldots, l_m$ be an ordering of L. Next take the obvious relation

$$1 = t_{l_1} t^{-1}_{l_1} t_{l_2} t^{-1}_{l_2} \ldots t_{l_m} t^{-1}_{l_m}$$

and then replace each of the left Dehn twists $t^{-1}_{l_i} \in \mathrm{Map}^1_L = \mathrm{Map}^1(-)$ with the corresponding product of right Dehn twists. The resulting right-hand side will be a word α with the desired property. q.e.d.

The next lemma gives a convenient criterion guaranteeing the existence of positive relations among right Dehn twists.

Lemma 3.7. *Let R and S be two sets of circles in minimal position. Assume that R is graph connected to S and that* $\mathrm{Map}^1_S(-) = \mathrm{Map}^1_S$. *Then* $\mathrm{Map}^1_{R\cup S}(-) = \mathrm{Map}^1_{R\cup S}$.

Proof. By definition $\mathrm{Map}^1_{R\cup S}(-)$ is the subsemigroup in Map^1_g generated by $t_{\phi(c)}$, $\phi \in \mathrm{Map}^1_{R\cup S}$, $c \in R\cup S$. Therefore we need to show that for any $\phi \in \mathrm{Map}^1_{R\cup S}$ and $c \in R\cup S$ the left twist $t^{-1}_{\phi(c)}$ also belongs to $\mathrm{Map}^1_{R\cup S}(-)$. Since $t_{\phi(c)} = \phi \circ t_c \circ \phi^{-1}$ and $\mathrm{Map}^1_{R\cup S}(-)$ is conjugation invariant in $\mathrm{Map}^1_{R\cup S}$, it suffices to check that $t_c^{-1} \in \mathrm{Map}^1_{R\cup S}(-)$ whenever $c \in R\cup S$.

If $c \in S$ then by assumption $t_c^{-1} \in \mathrm{Map}^1_S(-) \subset \mathrm{Map}^1_{R\cup S}(-)$. If $c \in R$, then by the hypothesis of the lemma c is graph connected in $G_{R\cup S}$ with some element $d \in S$. On the other hand according to Lemma 5.1 (b) for any pair a, b of adjacent cycles on C_g the right twists t_a, t_b are conjugate in $h_{a,b}(\mathrm{Map}_{1,1}) \subset \mathrm{Map}^1_{R\cup S}$. In particular t_c^{-1} and t_d^{-1} belong to the same conjugacy class in $\mathrm{Map}_{R\cup S}$. Since by assumption $t_d^{-1} \in \mathrm{Map}^1_S(-) \subset \mathrm{Map}^1_{R\cup S}(-)$ and due to the conjugation invariance of $\mathrm{Map}^1_{R\cup S}(-)$ in $\mathrm{Map}^1_{R\cup S}$ we conclude that $t_c^{-1} \in \mathrm{Map}^1_{R\cup S}(-)$ as well, which concludes the proof of the lemma. q.e.d.

Let now Γ be a finitely presentable group and let $a : \pi_g \twoheadrightarrow \Gamma$ be a given presentation. As we can see from the previous lemmas the construction of a SLF whose total space has fundamental group Γ reduces to finding a graph connected system of circles R so that R generates $\ker(a)$ as a normal subgroup and the right Dehn twists about some subsystem of S satisfy a positive relation in Map^1_g.

To achieve this we will have to modify the presentation a and in particular enlarge the genus g. This will be done in two steps which are explained in the next two sections.

3.2 Geometric presentations

The first step is purely topological. Starting with a presentation $a : \pi_g \twoheadrightarrow \Gamma$ we show how to add handles to C_g to obtain a new presentation $\psi : \pi_e \twoheadrightarrow \Gamma$ for which the cycles generating $\ker(\psi)$ are nicely situated on C_e.

We begin with the following definition:

Definition 3.8. Let Γ be a finitely presentable group. A *geometric presentation* of Γ is a surjective homomorphism $\psi : \pi_e \to \Gamma$, where π_e is the fundamental group of a compact Riemann surface C_e of genus e such that:

(a) The group of relations $\ker(\psi)$ is generated as a normal subgroup by finitely many simple closed loops $r_1, \dots, r_m \subset C_e$.

(b) Every two of the loops $r_1, \dots, r_m$ intersect transversally at most at one point and the subspace $\cup_i r_i \subset C_e$ is connected.

As we will see in the next section the geometrically presented groups are well suited for algebraic geometric manipulations. Thus it is important to find a procedure for constructing geometric presentations of finitely presentable groups. We have the following

Lemma 3.9. *Let $a : \pi_g \twoheadrightarrow \Gamma$ be a given presentation. Then there exists a surjective homomorphism $\pi_e \twoheadrightarrow \pi_g$ so that the composition $\pi_e \to \pi_g \to \Gamma$ is a geometric presentation.*

Proof. Let $R \subset \ker(a)$ be a finite subset of elements which generates $\ker(a)$ as a normal subgroup (such a set exists since by assumption a is a finite presentation of Γ). Represent each element $r \in R$ by a free immersed loop $c_r \subset C_g$ so that the one dimensional simplicial subcomplex $M := \cup_{r\in R} c_r \subset C_g$ is connected and has only ordinary double points as singularities. Such a collection of immersed loops c_r can be easily found as a small perturbation of a standard representation of the elements in R as immersed loops based at some fixed point $x_0 \in C_g$.

For each singular point $p \in M$ choose a small disk $p \in D_p \subset C_g$ which doesn't contain any other singularity of M. Now for each $p \in \operatorname{Sing}(M)$ delete from D_p a smaller disk centered at p and glue a handle C_p to C_g along the inner rim of the resulting annulus. The two branches of $M \cap D_p$ meeting at p can be then completed to two smooth disjoint curves on C_p as shown on Figure 3.2.

In this way we obtain a new smooth surface

$$C_e := (C_g \setminus \operatorname{Sing}(M)) \cup (\cup_{p\in \operatorname{Sing}(M)} C_p)$$

of genus $e = g + \#(\operatorname{Sing}(M))$ and a system of smooth disjoint circles $\tilde{c}_r \subset C_e$, $r \in R$. Next for every $p \in \operatorname{Sing}(M)$ we choose standard generators $a_p, b_p \subset C_p \subset C_e$ of the fundamental group of C_p as on

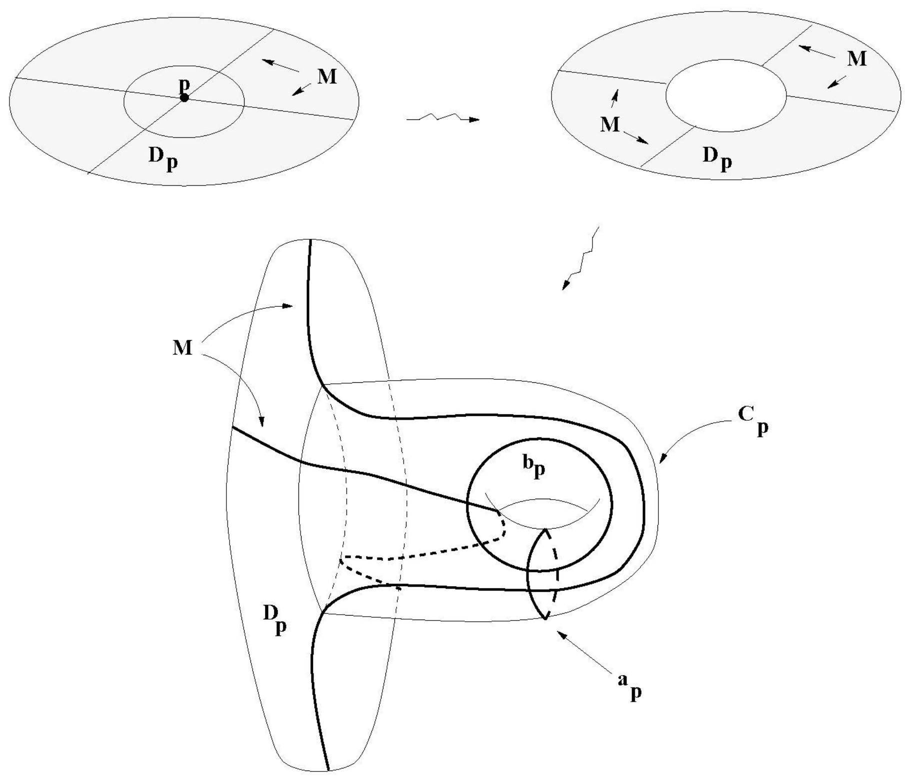

Figure 3.2: Gluing the handle C_p to $D_p \setminus \{p\}$.

Figure 3.2. By construction Γ is isomorphic to the quotient of π_e by the normal subgroup $N \lhd \pi_e$ generated by the finite set of elements

$$\{\tilde{c}_r\}_{r \in R} \cup (\cup_{p \in \mathrm{Sing}(M)} \{a_p, b_p\}) \subset \pi_e.$$

But we have glued the handles in such a way that the $\tilde{c}_r$'s are disjoint from each other, $(a_p \cup b_p) \cap (a_q \cup b_q) \neq \varnothing$ only if $p = q$ and each a_p or b_p intersects exactly one of the $\tilde{c}_r$'s transversally at a single point. Finally since every a_p intersects b_p at one point we conclude that the union of all these cycles is connected in C_e and hence $\pi_e \to \pi_e/N \cong \Gamma$ is a geometric presentation. q.e.d.

3.3 The proof of Theorem A

Assume now that Γ is a group with a fixed geometric presentation $\psi : \pi_e \to \Gamma$. Let $R = \{r_1, \ldots, r_m\}$ be a set of circles in minimal position in

C_e so that R generates $\ker(\psi)$ as a normal subgroup as in Definition 3.8. Note that without a loss of generality we may assume that R contains a non-separating circle $s \in R$. Indeed, if $\ker(\psi)$ happens to be contained in $[\pi_e, \pi_e]$ we can always glue an extra handle to C_e and add to R the three standard generators of the first homology of the handle (plus some extra loops if required).

Let Σ be a smooth projective algebraic curve of genus e which we have identified as a C^∞ manifold with C_e. Fix a point $p \in s \subset \Sigma$ which does not lie on any of the circles in $R \setminus \{s\}$ and let $p \in \Delta \subset \Sigma$ be a small analytic disc which is disjoint from all the circles in $R \setminus \{s\}$.

Let $V = \Sigma \times \mathbb{P}^1$ and let $D \subset V$ be a very ample divisor such that $D = \sum_i D_i$ with each $D_i \subset V$ being a section for the projection $p_\Sigma : V \to \Sigma$. By replacing $\mathcal{O}_V(D)$ by its third power if necessary we may further assume that the degree of D on $\mathbb{P}^1$ (= the number of sections D_i) is divisible by three.

Let p_i, Δ_i and s_i denote the preimages in D_i of p, Δ and s respectively. Similarly let R_i be the set of circles in D_i consisting of the preimages of the circles in R via the projection $p_{\Sigma|D_i} : D_i \to \Sigma$. When $\mathcal{O}_V(D)$ is chosen to be sufficiently ample and the divisor D is chosen to be a generic deformation of a set of sections of p_Σ which pass through a given point lying over $p \in \Sigma$ we may easily arrange that all intersections of the D_i's are transverse and that for every pair of indices $i \neq j$ we have $\Delta_i \cap \Delta_j \neq \varnothing$ (see Figure 3.3 below).

For any $i \neq j$ pick a point $p_{i,j} \in \Delta_i \cap \Delta_j$. Choose arcs $a^i_{i,j} \subset \Delta_i$ connecting p_i with $p_{i,j}$ which meet only at p_i and do not intersect s_i at any other point (see Figure 3.4).

Remark 3.10. For each i and j only *one* point is chosen in $\Delta_i \cap \Delta_j$. Therefore the set of points $\{p_{i,j}\}$ will be only a subset in $\cup_{i<j}(D_i \cap D_j)$ in general.

Consider a generic pencil in the linear system $|D|$ which contains D as a member and has a smooth general member. After blowing up the base points of this pencil on V we obtain a smooth surface $\widehat{V}$ and a projective morphism $v : \widehat{V} \to \mathbb{CP}^1$ having D as a fiber over some point $d \in \mathbb{CP}^1$ and a smooth connected general fiber. Let V_o be a smooth fiber of v over some reference point $o \in \mathbb{CP}^1$ which is close to d. Then we have (once we choose an arc system for v) a well defined deformation retraction from a tubular neighborhood of D in $\widehat{V}$ onto D. Let $cr : V_o \to D$ denote the restriction of this retraction to the curve V_o. The continuous map cr collapses certain smooth circles on V_o to the

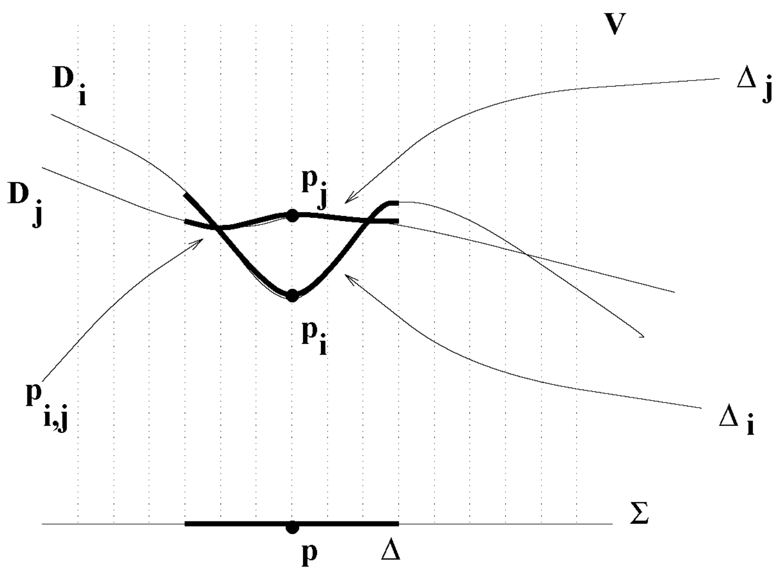

Figure 3.3: The divisors D_i.

points in the finite set $\cup_{i<j}(D_i \cap D_j)$. Note that via the map cr we may view the $\Delta_i \setminus \{p_{i,j}\}_{j \neq i}$ as punctured discs on the smooth curve V_o, the set $\cup_i R_i$ as a set of circles in minimal position on V_o and the arcs $a^i_{i,j}$ as arcs on V_o.

In particular we see that the points $p_{i,j} \in D$ correspond to vanishing cycles $c_{i,j} \subset V_o$ for the pencil v which are disjoint from $s_i \subset V_o$. Moreover by a slight perturbation within each Δ_i we can arrange that for each pair of indices $i \neq j$ the arcs $a^i_{i,j}, a^j_{i,j} \in V_o$ join smoothly at a point on the circle $c_{i,j}$ (see Figure 3.5).

For each triple of distinct indices i, j, k denote by $l_{i,j,k}$ the smooth unoriented circle in V_o obtained by tracing the segments $a^x_{x,y}$, $\{x, y\} \subset \{i, j, k\}$ in the following order (see Figure 3.5)

$$l_{i,j,k} = a^k_{k,i} \circ \bar{a}^k_{k,j} \circ a^j_{j,k} \circ \bar{a}^j_{j,i} \circ a^i_{i,j} \circ \bar{a}^i_{i,k},$$

where we have adopted the convention, that a^x_{xy} is oriented from p_x to $p_{x,y}$ and $\bar{a}^x_{xy}$ is oriented from $p_{x,y}$ to p_x.

By construction, the l_{ijk}'s are free loops representing elements in the kernel of the natural surjection $(p_{\Sigma|V_o})_* : \pi_1(V_o) \twoheadrightarrow \pi_1(\Sigma)$.

Before we state the main result of this section we need to introduce some notation. Let h be the genus of V_o and let $S \subset \pi_1(V_o)$ be the set

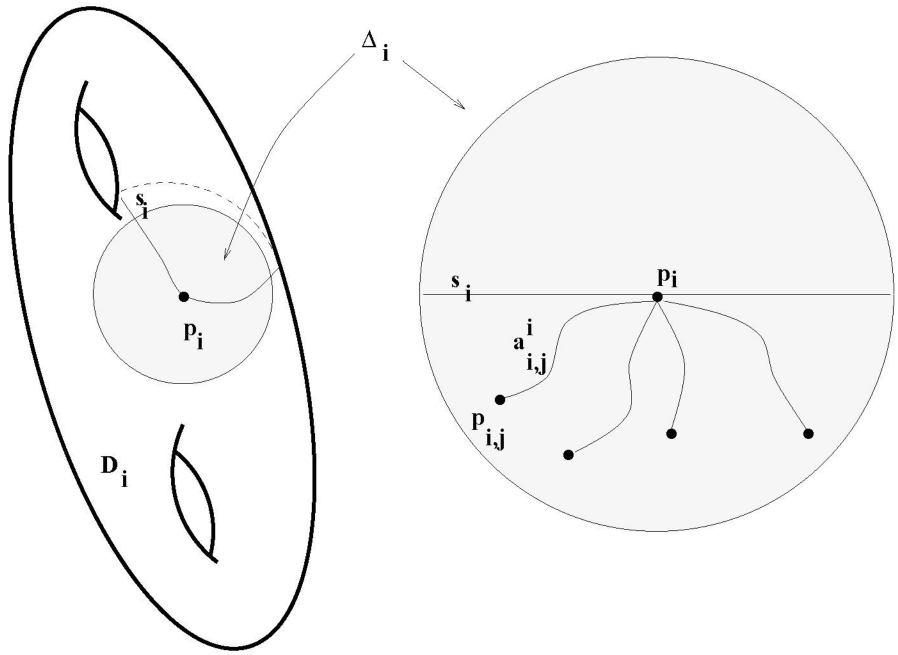

Figure 3.4: The system of arcs in Δ_i.

of geometric vanishing cycles for the algebraic pencil v. We have the following:

Proposition 3.11. *Let L be the set of circles in minimal position on C_h defined as*

$$L := (\cup_i R_i) \cup S \cup \{l_{123}, l_{456}, \ldots\}.$$

Then there exists a SLF $f : X \to S^2$ of genus h such that:

- *The set of vanishing cycles of f is exactly L.*
- *The fundamental group of X is canonically isomorphic to Γ.*
- *The identifications $\pi_1(X) \cong \Gamma$ and $V_o \cong C_h$ can be chosen so that the natural epimorphism $\pi_h \to \pi_1(X)$ (induced from the inclusion of C_h in X as a regular fiber of f) becomes the composition*

$$\pi_h = \pi_1(V_o) \xrightarrow{(p_{\Sigma|V_o})_*} \pi_1(\Sigma) = \pi_e \xrightarrow{\psi} \Gamma.$$

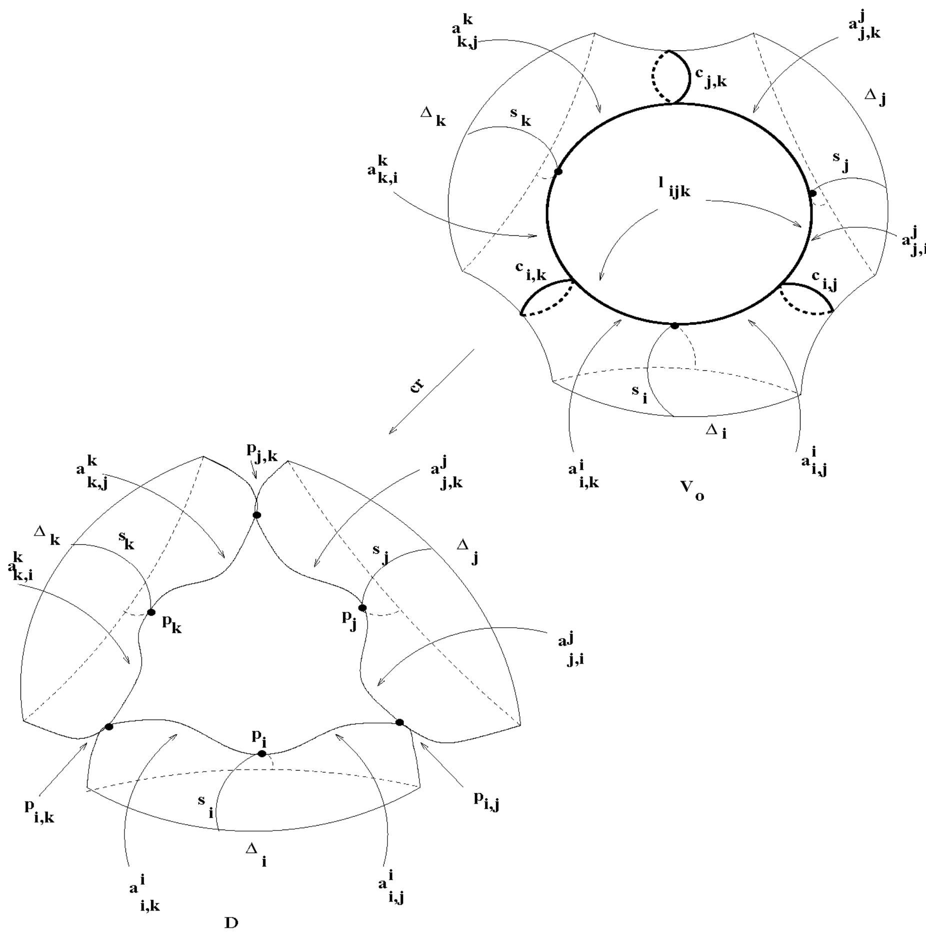

Figure 3.5: Retraction of V_o onto D.

Proof. Let $\iota : V_o \hookrightarrow V$ be the natural inclusion of the divisor V_o in the surface V. Consider the induced map $\iota_* : \pi_1(V_o) \to \pi_1(V)$. Since V_o is the regular fiber of the algebraic Lefschetz fibration $v : \widehat{V} \to \mathbb{P}^1$ we conclude by Lemma 3.2 (c) that $\ker(\iota_*)$ is the normal subgroup generated by the circles in S. On the other hand since $p_{\Sigma *}$ identifies $\pi_1(V)$ with $\pi_1(\Sigma)$ we have that $\ker(\iota_*) = \ker((p_{\Sigma|V_o})_*)$ and so $l_{ijk} \in \ker(\iota_*)$ for all triples of indices i, j, k. Due to this observation the normal subgroup in $\pi_1(V_o) = \pi_h$ generated by the elements in L actually coincides with the normal subgroup generated by the elements in $(\cup_i R_i) \cup S$. Moreover by the handle body decomposition for the

Lefschetz pencil v we have a commutative diagram

$$\begin{tikzcd} \pi_1(V_o) \arrow[rr, "(p_{\Sigma|V_o})_*"] \arrow[dr, "cr_*"'] & & \pi_1(\Sigma) \\ & \pi_1(D) \arrow[ur, "(p_{\Sigma|D})_*"'] & \end{tikzcd}$$

and hence each set $R_i \subset \pi_1(V_o) = \pi_h$ is mapped bijectively to the set $R \subset \pi_e$ by the map ι_*. This shows that $L \subset \ker(\psi \circ \iota_*)$ and that moreover L generates $\ker(\psi \circ \iota_*)$ as a normal subgroup in π_h. Therefore By Lemma 3.2 the proposition will be proven if we can find a positive relation in Map_g^1 among the right Dehn twists corresponding to elements in L.

The set of circles S is the set of vanishing cycles in an algebraic Lefschetz pencil corresponding to an ample line bundle. In particular this algebraic pencil has a base point and so we have a positive relation in Map_g^1 among the Dehn twists corresponding to elements in S as in Lemma 3.2. Thus $\mathrm{Map}_S^1(-) = \mathrm{Map}_S^1$ by the "if" part of Lemma 3.6.

Put $R := \cup_i R_i \cup \{l_{123}, l_{456}, \ldots\}$. By the geometric presentation assumption we know that each R_i is graph connected. Also by construction l_{ijk} is graph connected to the circles $s_i \in R_i$, $s_j \in R_j$ and $s_k \in R_k$ and to the vanishing cycles $c_{i,j}, c_{j,k}, c_{i,k} \in S$. Therefore R is graph connected to S and so by Lemma 3.7 we have $\mathrm{Map}^1_{R\cup S}(-) = \mathrm{Map}^1_{R\cup S}$. Now we can apply the "only if" part of Lemma 3.6 to the set $L = R \cup S$ which concludes the proof of the proposition. q.e.d.

Granted the previous proposition Theorem A now becomes a tautology:

Proof of Theorem A. Given a finite presentation $a : \pi_g \to \Gamma$ of Γ first use Lemma 3.9 to obtain a geometric presentation $\psi : \pi_e \to \Gamma$ and then apply Proposition 3.11 to obtain a SLF with fundamental group Γ. q.e.d.

3.4 Symplectic fibrations over bases of higher genus

The construction explained in the two previous sections can be easily modified to produce examples of symplectic Lefschetz fibrations over surfaces of genus bigger than zero (see Remark 3.12 for an explanation of the terminology). Moreover in that case the algebraic geometric part of the construction becomes superfluous and the resulting fibrations have monodromy groups generated essentially by the Dehn twists with respect to the circles giving the relations in some geometric presentation. This phenomenon is illustrated in Theorem B which we prove below.

Remark 3.12. Before we prove the theorem a terminological remark is in order. Note that the notion of a symplectic Lefschetz fibration introduced in Definition 2.2 makes sense without the assumptions that the base of the fibration is a sphere and that the total space is four dimensional. In fact the whole setup of Definition 2.2 easily generalizes to fibrations of even dimensional manifolds over a base of even dimension.

In view of this we will adopt a slightly different terminology for the remainder of this section. Namely a symplectic Lefschetz fibration will mean a map $f : X \to C$ from a smooth 4-fold X to a smooth compact Riemann surface C which satisfies all the conditions (i)-(iv) from Definition 2.1 but with S^2 replaced by C.

This abuse of terminology should not cause any confusion since symplectic Lefschetz fibrations like that will be considered only in this section.

We can now state the main result of this section.

Proposition 3.13. *Let Γ be a finitely presentable group with a given presentation $a : \pi_g \twoheadrightarrow \Gamma$. Then there exist a geometric presentation $\pi_e \to \pi_g \stackrel{a}{\to} \Gamma$ and a symplectic Lefschetz fibration $f : X \to C_k$ for some $k >> 0$ so that*

- *the regular fiber of f is of genus e,*
- *the fundamental group of X fits in a short exact sequence*
$$1 \to \Gamma \to \pi_1(X) \to \pi_k \to 1,$$
- *the natural surjective map from the fundamental group of the fiber of f to Γ coincides with the composition $\pi_e \to \pi_g \stackrel{a}{\to} \Gamma$.*

Proof. Let $\psi : \pi_d \to \pi_g \to \Gamma$ be a geometric presentation as in Lemma 3.9. Let R be a generating set of circles for $\ker(\psi)$ as in Definition 3.8. Even though the circles in R need not satisfy a positive relation in Map_d^1 the subsemigroup $\mathrm{Map}_R^1(-)$ is very close to being the whole group Map_R^1 as the following arguments show.

Lemma 3.14. *The abelianization of Map_R^1 is a cyclic group.*

Proof. Fix a reference element $c \in R$. The group Map^1_R is generated as a normal subgroup in Map^1_g by the Dehn twists $\{t_s\}_{s\in R}$. Since by definition any two circles in R are graph connected we conclude as in the proof of Lemma 3.7 that all generators of Map^1_R are conjugate to each other. Therefore any character $\mathrm{Map}^1_R \to S^1$ is defined by its value on t_c and so the quotient $\mathrm{Map}^1_R/[\mathrm{Map}^1_R, \mathrm{Map}^1_R]$ is generated by $t_c[\mathrm{Map}^1_R, \mathrm{Map}^1_R]$. The lemma is proven. q.e.d.

The previous lemma shows that if we can find an integer n such that $t_s^n \in [\mathrm{Map}^1_R, \mathrm{Map}^1_R]$ for any $s \in R$, then the abelianization of Map_R is of order at most n. In fact the same statement holds for any subgroup G of the mapping class group $G \subset \mathrm{Map}^1_d$ which is contained in the normal closure of Map^1_R in Map^1_d.

For the next step in the proof we will need to assume that the set R is big enough. More precisely let $r, s \in R$ be two adjacent circles on C_d and let $C_{r,s} \subset C_d$ be the corresponding handle. Let p be the intersection point of r and s. By deleting a small disk $D_p \subset C_d$ centered at p and gluing a handle C_p in its place as in the proof of Lemma 3.9 we obtain a new surface C_e of genus $e = d+1$. The two open curves $r, s \subset C_d \setminus D_p$ can be completed to smooth circles by adding arcs in $C_{r,s}$ as shown on Figure 3.6. Furthermore, we can enlarge the set of relations R to a new set of circles $S = \{q\} \cup R \cup \{a, b\}$ where $q \subset C_{r,s}$ is a circle isotopic to r such that $q \cap D_p = \varnothing$ and a, b are the standard generators of the fundamental group of C_p (see Figure 3.6).

By construction we have a geometric presentation $\pi_e \to \Gamma$ whose kernel is generated as a normal subgroup by the elements of S. By Lemma 3.14 the abelianization of the group Map^1_S is cyclic. In fact since we have ensured that S is big enough we have the following:

Lemma 3.15. *The abelianization of* Map^1_S *is a finite cyclic group of order dividing* 10.

Proof. The union $(C_{r,s} \setminus D_p) \cup C_p$ is a genus two handle on C_e and therefore the inclusion $(C_{r,s} \setminus D_p) \cup C_p \subset C_e$ induces a natural homomorphism $h : \mathrm{Map}_{2,1} \to \mathrm{Map}^1_S$. Moreover by construction the subgroup $h(\mathrm{Map}_{2,1}) \subset \mathrm{Map}^1_S$ contains a non-trivial Dehn twist - e.g. the twist along the circle $a \subset C_e$. Therefore we conclude as in the proof of Lemma 3.14 that the group Map^1_S is generated by conjugates of the element $t_a \in h(\mathrm{Map}_{2,1}) \subset \mathrm{Map}^1_S$.

On the other hand it is known (see e.g. [32]) that the group $\mathrm{Map}_{2,1}$ has a presentation with several relations of degree zero and one relation

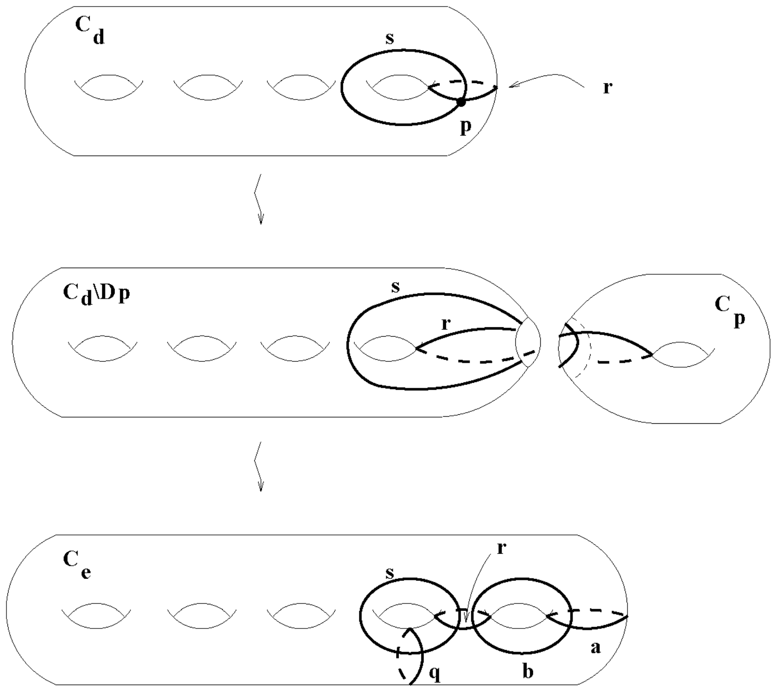

Figure 3.6: Enlarging a geometric presentation.

of degree ten. More precisely one has

$$\mathrm{Map}_{2,1} = \left\langle t_1, \dots, t_5 \;\middle|\; \begin{array}{l} t_i t_j t_i = t_j t_i t_j \text{ if } |i-j| = 1,\ t_i t_j = \\ t_j t_i \text{ if } |i-j| \neq 1 \text{ and } (t_1 t_2 t_3)^4 = \\ t_5 (t_4 t_3 t_2 t_1^2 t_2 t_3 t_4)^{-1} t_5 t_4 t_3 t_2 t_1^2 t_2 t_3 t_4 \end{array} \right\rangle.$$

The braid relations $t_i t_j t_i = t_j t_i t_j$ for $|i-j| = 1$ in this presentation show that any homomorphism from $\mathrm{Map}_{2,1}$ to an abelian group G will have to map all t_i's to the same element $g \in G$. Moreover the degree ten relation in the presentation implies that $g^{10} = 1 \in G$ and so the abelianization of $\mathrm{Map}_{2,1}$ is isomorphic to $\mathbb{Z}/10$.

Due to this we see that the tenth power of any generator of Map^1_S will be conjugate to t_a^{10} and so will belong to the commutator subgroup $[\mathrm{Map}^1_S, \mathrm{Map}^1_S]$. Thus by the remark after the proof of Lemma 3.14 the abelianization of Map^1_S will be a cyclic group of order dividing ten. The lemma is proven q.e.d.

Now we can finish the proof of the proposition. Let m be the cardinality of S and let $s_1, \dots, s_m$ be an ordering of the elements in S. By

Lemma 3.15 we can find elements $\xi_i, \eta_i \in \mathrm{Map}^1_S$, $i = 1, \dots, k$ such that the relation

$$\prod_{j=1}^{m} t_{s_j}^{10} = \prod_{i=1}^{k} [\xi_i, \eta_i] \tag{3.1}$$

holds in Map^1_S.

For every $i = 1, \dots, k$ choose diffeomorphisms $\Xi_i, E_i \in \mathrm{Diff}^+(C_e)$ representing the mapping classes ξ_i, η_i. Let K be a Riemann surface of genus k with one boundary component. The fundamental group $\pi_1(K)$ is generated by $2k+1$ simple closed curves $\alpha_1, \beta_1, \dots, \alpha_k, \beta_k, c$ satisfying the only relation

$$\prod_{i=1}^{k} [\alpha_i, \beta_i] = c,$$

and so we can construct a representation $\pi_1(K) \to \mathrm{Diff}^+(C_e)$ given by $\alpha_i \mapsto \Xi_i$, $\beta_i \mapsto E_i$, $c \mapsto \prod[\Xi_i, E_i]$. Moreover since we have the freedom of changing the diffeomorphisms Ξ_i and E_i up to isotopy we may assume without a loss of generality that all the Ξ_i's and E_i's preserve a given symplectic form on C_e. With this choice the group $\pi_1(K)$ acts symplectically and freely on the product $\widetilde{K} \times C_e$ of the universal cover of K and C_e, and by passing to the quotient of this action we obtain a smooth symplectic fibration $Y \to K$ with fiber C_e. The restriction of this fibration over the boundary circle c of K is a smooth C_e fibration over S^1 corresponding to the element $\prod_i [\xi_i, \eta_i] \in \mathrm{Map}_e$. On the other hand exactly as in the proof of Lemma 3.2 we can construct a symplectic Lefschetz fibration $u : U \to D$ whose restriction $U_{|\partial D}$ to the boundary circle is a principal C_e fibration over S^1 corresponding to the element $\prod_j t_j^{10}$. Due to the relation (3.1) this implies that the restrictions of Y and U to $c \subset K$ and $\partial D \subset D$ respectively are isomorphic C_e bundles over a circle and so we can glue Y and U along their boundaries to obtain a smooth orientable compact fourfold X which fibers over the genus k curve $C_k = K \cup_{c=\partial D} D$ with a regular fiber of genus e. Since Proposition 2.3 is true for abitrary base surfaces (see [8, Theorem 10.2.18] for a proof) we conclude that the projection map $f : X \to C_k$ is a symplectic Lefschetz fibration with $10m$ singular fibers and vanishing cycles $s_1, \dots, s_m$. In particular $\ker[\pi_1(X) \to \pi_1(C_k)]$ is isomorphic to the quotient of π_e by the normal subgroup generated by the s_j's and so the proposition is proven. q.e.d.

Remark 3.16. It is clear from the proof of Proposition 3.13 that with a little extra care one can control effectively the genus of the fibers of f and the number of singular fibers of f but not the genus of the base C_k.

The trick of enlarging the geometric presentation by creating a genus two handle seems to be really necessary since a priori one can guarantee only the existence of a non-trivial homomorphism from $\mathrm{Map}_{1,1}$ to Map_d^1 which is not enough since the abelianization of $\mathrm{Map}_{1,1}$ is $\mathbb{Z}$ as explained in the proof of Corollary 5.2. On the other hand it is known that the group $\mathrm{Map}_{g,r}$ is perfect for $g \geq 3$, $r \geq 0$ [26]. Hence same argument as in the proof of Proposition 3.13 shows that we can glue in an extra handle to C_e to obtain a non-trivial homomorphism $\mathrm{Map}_{3,1} \to \mathrm{Map}_{d+2,1}$ and a geometric presentation $\pi_{d+2} \to \Gamma$ whose kernel is generated as a normal subgroup by a connected graph of circles P with cardinality $\#P = \#R + 5 = m + 2$. In this case already the Dehn twists t_s, $s \in P$ themselves will be products of commutators in Map_P^1 and so we will get a SLF with $m + 2$ singular fibers.

In fact by further enlarging the genus of the base one can modify the fibration described in Remark 3.16 to obtain a proof of Theorem B. Actually we have the following precise version of Theorem B:

Corollary 3.17. *Let Γ be a finitely presentable group with a given geometric presentation $a : \pi_d \to \Gamma$. Then there exist a geometric presentation $\pi_{d+2} \to \pi_d \xrightarrow{a} \Gamma$ and a symplectic Lefschetz fibration $f : Y \to C$ over a smooth compact Riemann surface C so that*

- *the regular fiber of f is diffeomorphic to C_{d+2},*
- *f has a unique singular fiber,*
- *the fundamental group of X fits in a short exact sequence*

$$1 \to \Gamma \to \pi_1(X) \to \pi_1(C) \to 1,$$

- *the natural surjective map from the fundamental group of the fiber of f to Γ coincides with the composition $\pi_{d+2} \to \pi_d \xrightarrow{a} \Gamma$.*

Proof. We will use the notation of Remark 3.16. As a first approximation to $f : X \to C$ we will use the genus $d + 2$ SLF constructed in Remark 3.16. By construction the homomorphism $\pi_{d+2} \to \Gamma$ is a geometric presentation whose kernel is generated as a normal subgroup by a connected graph of circles P. Observe that since each t_s, $s \in P$ is a product of commutators in Map^1_P similarly to the proof of Theorem B we can replace a tubular neighborhood of the singular fiber at which the circle s vanishes with a smooth C_{d+2} fibration over a high genus Riemann surface corresponding to a representation of t_s as a product of commutators. If we do this for all elements in S but one we will get a symplectic Lefschetz fibration with one singular fiber and the same monodromy group as that of the original fibration. Let $f : X \to C$ be this fibration and let $q \in C$ be the only critical value of f. Let $r \in P$ be the cycle vanishing at q. The map f induces a surjection on fundamental groups $f_* : \pi_1(X) \to \pi_1(C)$ and the kernel $\ker(f_*)$ is naturally a quotient of the fundamental group π_{d+2} of the regular fiber of f. To calculate $\ker(f_*)$ note that we have a natural surjection $\pi_1(X \setminus f^{-1}(q)) \to \pi_1(X)$ which restricts to the quotient map $\pi_{d+2} \twoheadrightarrow \ker(f_*)$. Using this map one can identify $\ker[\pi_{d+2} \twoheadrightarrow \ker(f_*)]$ with the subgroup in $\pi_1(X \setminus f^{-1}(q))$ which is generated as normal subgroup (in $\pi_1(X \setminus f^{-1}(q))$) by the cycle $r \in \pi_{d+2} \subset \pi_1(X \setminus f^{-1}(q))$. Indeed, let $\tilde{f} : \widetilde{X} \to \Delta$ be a SLF over a disk Δ of genus $d+2$. If $\tilde{f} : \widetilde{X} \to \Delta$ has a section we can use Seifert-van Kampen theorem in the same way as in the proof of Lemma 3.2 to argue that $\pi_1(\widetilde{X})$ is the quotient of π_{d+2} by the normal subgroup generated by all vanishing cycles. Now if we take $\tilde{f} : \widetilde{X} \to \Delta$ to be the pullback of $f : X \to C$ to the universal cover Δ of C it follows that $\ker(f_*) = \pi_1(\widetilde{X})$ and so $\ker[\pi_{d+2} \twoheadrightarrow \ker(f_*)]$ is generated as a normal subgroup by the vanishing cycles of $\tilde{f}$. But by construction the set of vanishing cycles of $\tilde{f}$ is simply the set $\{hrh^{-1}\}_{h\in\pi_1(C)}$, i.e., $\ker[\pi_{d+2} \twoheadrightarrow \ker(f_*)]$ is generated by r as a normal subgroup.

On the other hand the fact that $X \setminus f^{-1}(q) \to C \setminus \{q\}$ is a smooth fibration identifies $\pi_1(X \setminus f^{-1}(q))$ with the semidirect product $\pi_1(C \setminus \{q\}) \ltimes_{\mathrm{mon}} \pi_{d+2}$ where $\mathrm{mon} : \pi_1(C \setminus \{q\}) \to \mathrm{Aut}(\pi_{d+2})$ is the monodromy representation. By the definition of a semidirect product the subgroup $\pi_1(C \setminus \{q\}) \subset \pi_1(X \setminus f^{-1}(q))$ normalizes $\pi_{d+2} \subset \pi_1(X \setminus f^{-1}(q))$ and the inner action of $\pi_1(C \setminus \{q\})$ on π_{d+2} coincides with the representation mon. Consequently $\ker[\pi_{d+2} \twoheadrightarrow \ker(f_*)]$ is the subgroup of π_{d+2} which is generated as a normal subgroup (in π_{d+2}) by the orbit of the vanishing cycle $r \in \pi_{d+2}$ under the monodromy group of f.

The next step is to recall that by construction the monodromy group of f is exactly Map^1_P. Moreover since P is a set of relations in a geometric presentation, we know that the Dehn twists about any two circles in P are conjugate and hence for any $s \in P$ there exists an element $\phi \in \mathrm{Map}^1_P$ for which $t_s = \phi \circ t_r \circ \phi^{-1} = t_{\phi(r)}$. However we have argued above that $\phi(r) \in \ker(f_*)$ for any $\phi \in \mathrm{Map}^1_P$ and hence $\ker f_*$ contains a circle which induces the same Dehn twist as s. To finish the proof we only need to observe that for any surface C_g and any circle $c \subset C_g$ the Dehn twist t_c is uniquely characterized by the property that the maximal quotient of π_g on which it induces the identity is the quotient of π_g by the normal subgroup generated by c. Thus if we have two circles inducing the same Dehn twist the normal subgroups in π_g generated by those circles coincide. In combination with the above discussion this implies that $P \subset \ker(f_*)$ and so $\ker(f_*)$ coincides with the quotient of π_{d+2} by the normal subgroup generated by the elements in P. The corollary is proven. q.e.d.

Remark 3.18. Observe that even though there are no obvious restrictions for the Lefschetz fibrations constructed in Proposition 3.13 to be Kähler it will be very hard to construct projective examples like that. In general the algebraic Lefschetz fibrations have much bigger monodromies and as a result the fundamental group of an algebraic family $f : X \to C$ as above is an extension of $\pi_1(C)$ with a rather small (in general trivial) group. The reason for this phenomenon is that as it follows from the above argument the only condition for the existence of a symplectic Lefschetz pencil with a given fundamental group is some equation in the mapping class group. On the other hand Hodge theory imposes many other restrictions in the algebraic situation - for example it is shown in [15] that if the first Betti number of X is trivial, then the monodromy group of an algebraic Lefschetz pencil of big enough degree cannot fix any loop on the fiber. An interesting question to investigate is if this restriction exists for SLP.

4. An example of a SLF with $\pi_1(X) = \mathbb{Z}$

In this section we will use a slight modification of the proof of Proposition 3.11 to construct an example of a symplectic Lefschetz fibration with fundamental group equal to $\mathbb{Z}$.

Let E be an elliptic curve. Consider the complex projective surface $V := E \times \mathbb{P}^1$ and let $p_E : V \to E$ be the natural projection. Let $D_\nu \subset V$ be the section of p_E given by the graph of the natural degree two covering $\nu : E \to \mathbb{P}^1$. Fix a point $p \in \mathbb{P}^1$ which is not a branch point for the covering $\nu : E \to \mathbb{P}^1$ and let $D_p = E \times \{p\}$ be the corresponding section of p_E.

Consider a divisor $D = D_p + D_\nu$. By construction D is a strict normal crossing curve with exactly two singular points $\{p_1, p_2\} = D_p \cap D_\nu$. Suppose that the linear system $|D|$ contains a smooth divisor Y. We will assume that Y is close enough to D in $|D|$ so that there is a well defined map $cr : Y \to D$ coming from a deformation retraction of a tubular neighborhood of D onto D. The curve Y is of genus 3 and the natural map $p_{E|Y} : Y \to E$ is a two sheeted covering branched at four points which get glued in pairs to the points $p_E(p_1)$ and $p_E(p_2)$ respectively when we deform Y to D. Let seg_{12} be a contractible segment in E connecting $p_E(p_1)$ and $p_E(p_2)$ and let $s \subset E$ be a nonseparating circle intersecting seg_{12} at a single point (see Figure 4.1). Denote by S the set of geometric vanishing cycles for an algebraic pencil $\mathbb{P}^1 \subset |D|$ containing both Y and D. As in the proof of Proposition 3.11 consider the system of circles in minimal position

$$L := \{l_{12}, s_p, s_\nu\} \cup S \subset Y$$

where $l_{12} := p_{E|Y}^{-1}(\mathrm{seg}_{12})$ is the double cover of seg_{12} and s_p and s_ν are the preimages of s in E_p and E_ν respectively, viewed as cycles on Y via cr (see Figure 4.1). By definition the set S contains two cycles v_1 and v_2 corresponding to p_1 and p_2 respectively (i.e., the cycles contracted by cr) and so L is graph connected to S.

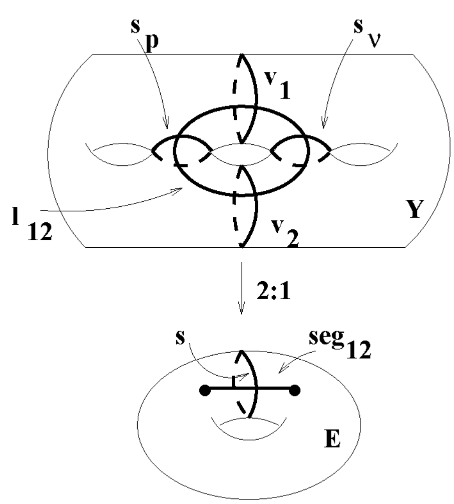

Figure 4.1: The covering $p_{E|Y} : Y \to E$.

The same argument as in the proof of Proposition 3.11 shows that the quotient Γ of $\pi_1(Y)$ by the normal subgroup generated by the circles in L is isomorphic to the quotient of $\pi_1(E)$ by the normal subgroup generated by s, i.e., is isomorphic to $\mathbb{Z}$. Furthermore since L is graph connected to S the combination of Lemma 3.2 and the "only if" part of Lemma 3.6 a yields a SLF with monodromy group Map^1_L, regular fiber isomorphic to Y and fundamental group Γ.

Therefore in order to finish the construction we only need to show that the linear system $|D|$ contains a smooth divisor. Let $\mathbb{F}_0 := \mathbb{P}^1 \times \mathbb{P}^1$ and let $a := \nu \times \mathrm{id} : V \to \mathbb{F}_0$ be the double cover induced from ν. As usually $\mathcal{O}_{\mathbb{F}_0}(m,n)$ will denote the line bundle $p_1^*\mathcal{O}_{\mathbb{P}^1}(m) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(n)$ where $p_i : \mathbb{F}_0 \to \mathbb{P}^1$, $i = 1, 2$ are the projections on the first and second factors respectively. By definition we have $D_p \in |a^*\mathcal{O}_{\mathbb{F}_0}(0,1)|$ and $D_\nu \in |a^*\mathcal{O}_{\mathbb{F}_0}(1,1)|$. Thus $D \in |a^*\mathcal{O}_{\mathbb{F}_0}(1,2)|$ and so we need to find a smooth divisor in the linear system $|a^*\mathcal{O}_{\mathbb{F}_0}(1,2)|$. But the double cover $a : V \to \mathbb{F}_0$ is a root cover branched along a section of $\mathbb{O}_{\mathbb{F}_0}(4,0)$ and hence $a_*\mathcal{O}_V = \mathcal{O} \oplus \mathcal{O}(-2,0)$. This yields

$$\begin{aligned} H^0(V, a^*\mathcal{O}_{\mathbb{F}_0}(1,2)) &= H^0(\mathbb{F}_0, a_*a^*\mathcal{O}_{\mathbb{F}_0}(1,2)) \\ &= H^0(\mathbb{F}_0, \mathcal{O}_{\mathbb{F}_0}(1,2) \otimes (\mathcal{O}_{\mathbb{F}_0} \oplus \mathcal{O}_{\mathbb{F}_0}(-2,0))) \\ &= H^0(\mathbb{F}_0, \mathcal{O}_{\mathbb{F}_0}(1,2)), \end{aligned}$$

and so every member of the linear system $|D|$ is a pullback of a divisor in $|\mathcal{O}_{\mathbb{F}_0}(1,2)|$. Finally since $\mathcal{O}_{\mathbb{F}_0}(1,2)$ is obviously base point free, the general point in the five dimensional projective space $|\mathcal{O}_{\mathbb{F}_0}(1,2)|$ will represent a smooth rational curve on $\mathbb{F}_0$, which is a double cover of $\mathbb{P}^1$ via the projection p_1, and intersects the branch divisor of a in eight distinct points. Therefore the preimage of such a curve in V is a smooth curve Y of genus three which finishes the construction.

Remark 4.1. The family $f : X \to S^2$ just constructed is an example of a symplectic Lefschetz fibration whose monodromy group does not act semi simply on the first cohomology of the fiber. This can be seen directly from the explicit description of the Dehn twists t_{w_i}, t_u and t_v given above. In fact it is proven in [15] that as long as the total space of a SLF has an odd first Betti number the monodromy action on the first cohomology of the fiber has a non-trivial unipotent radical.

Remark 4.2. Explicit examples of symplectic Lefschetz fibrations with prescribed first homology were also constructed by Ivan Smith [30]. In particular he shows that any abelian group which is a quotient of a free group on g generators can be realized as the first homology of a SLF with fiber genus $2g$.

Remark 4.3. As an alternative to the previous construction one can use our main theorem in the following way. Start with a finitely presentable group Γ and let $a_1 : \pi_g \twoheadrightarrow \Gamma$ and $a_2 : \pi_g \twoheadrightarrow \Gamma$ be two presentations. Use Theorem A to construct two SLF $f_1 : X_1 \to S^2$ and $f_2 : X_2 \to S^2$ of the same genus $h > g$ so that $\pi_1(X_i) = \Gamma$ and the natural maps from the fundamental groups of the general fibers of f_i onto Γ factor as $\pi_h \to \pi_g \xrightarrow{a_i} \Gamma$ for $i = 1, 2$. If we now glue the two pencils X_1 and X_2 along a regular fiber we obtain a new SLF $f : X \to S^2$ of genus h which by van Kampen's theorem has fundamental group $\pi_g/(\ker(a_1) * \ker(a_2))$. In particular by gluing two genus two pencils with fundamental groups $\mathbb{Z}^2$ one can get a genus two Lefschetz pencil with fundamental group $\mathbb{Z}$. In fact this procedure is the baby version of a sophisticated technique employed by Ivan Smith [28] who showed that every quotient of $\mathbb{Z}^{\oplus 2}$ can be realized as the fundamental group of a symplectic Lefschetz fibration.

Similar series of very elegant examples of genus two SLF with fundamental groups $\mathbb{Z} \oplus \mathbb{Z}/n$ was constructed in [25].

Remark 4.4. It was pointed out to us by Ron Stern that similarly to [25] and [28] the examples with a fundamental group $\mathbb{Z}$ and $b_1 = 1$ we construct above are all not even homotopic to complex surfaces. Indeed from the classification of complex surfaces it follows that if the above surfaces are complex they are either secondary Kodaira surfaces, class VII surfaces or elliptic surfaces. The fact that our examples are symplectic excludes the first two possibilities. The third possibility is ruled out by the observation that if this is an elliptic fibration then it should be over S^2. But then $b_1 \neq 1$ as it follows from [6, Section 2.2].

5. Sections in symplectic Lefschetz fibrations

In this section we study the relation between numerical properties of the sections in a symplectic Lefschetz fibration $f : X \to S^2$ and the geometric monodromy of f.

5.1 Mapping class groups in genus one

First we recall some well known facts on the structure of the mapping class groups in genus one (see e.g. [2, 7]).

Since any translation on a torus is homotopic to the identity, the natural map $\mathrm{Map}_1^1 \to \mathrm{Map}_1$ is an isomorphism. Furthermore $\mathrm{Map}_1^1 = SL(2,\mathbb{Z})$ and coincides with the group of linear automorphisms of the two-dimensional torus. The group $\mathrm{Map}_{1,1}$ is naturally identified with the mapping class group of a two-dimensional torus with one hole. As in Remark 3.1 the natural forgetful map $\mathrm{Map}_{1,1} \to \mathrm{Map}_1^1$ realizes $\mathrm{Map}_{1,1}$ as a central extension

$$0 \to \mathbb{Z} \to \mathrm{Map}_{1,1} \to SL(2,\mathbb{Z}) \to 1, \tag{5.2}$$

where the kernel $\mathbb{Z}$ is generated by the right twist t_c along the boundary circle c of the hole.

On the other hand the group $\mathrm{Map}_{1,1}$ admits a presentation

$$\mathrm{Map}_{1,1} = \langle t_a, t_b | t_a t_b t_a = t_b t_a t_b \rangle, \tag{5.3}$$

with generators t_a and t_b corresponding to the Dehn twist along a standard symplectic basis of cycles $H_1(C_1,\mathbb{Z}) = \mathbb{Z}a \oplus \mathbb{Z}b)$ on the non-punctured torus C_1. In particular under the natural map $\mathrm{Map}_{1,1} \to SL(2,\mathbb{Z})$ we have

$$t_a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, t_b \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

From this it is clear that the element $(t_a t_b)^3$ is central and maps to $-1 \in SL(2,\mathbb{Z})$. Thus $(t_a t_b)^6$ generates $\mathbb{Z} = \ker[\mathrm{Map}_{1,1} \to SL(2,\mathbb{Z})]$, i.e., $(t_a t_b)^6 = t_c$ (see also [13, Theorem 4.3]).

The central extension (5.2) corresponds to an element $\tau \in H^2(SL(2,\mathbb{Z}),\mathbb{Z})$. Since $SL(2,\mathbb{Z})$ can be identified with the fundamental group of the moduli stack $\mathcal{M}_1^1$ of elliptic curves, the element τ can be interpreted as the first Chern class of a line bundle on $\mathcal{M}_1^1$. In fact Mumford had shown [23, Main Theorem] that τ generates $\mathrm{Pic}(\mathcal{M}_1^1) \cong \mathbb{Z}/12$ and that τ corresponds to the line bundle on $\mathcal{M}_1^1$

which to every flat family $p : E \to S$ of elliptic curves associates the element $\bar{\omega}_f \in \mathrm{Pic}(S)$ where $\bar{\omega}_f$ is the pullback of the relative dualizing sheaf of f via the zero section.

More algebraically the central extension (5.2) can be described as follows. Consider the universal covering $\widetilde{SL}(2,\mathbb{R}) \to SL(2,\mathbb{R})$. Since $\pi_1(SL(2,\mathbb{R})) \cong \mathbb{Z}$ is a central subgroup in $\widetilde{SL}(2,\mathbb{R})$, we can take the preimage $\widetilde{SL}(2,\mathbb{Z})$ of $SL(2,\mathbb{Z})$ in $\widetilde{SL}(2,\mathbb{R})$. By construction there is a natural central extension

$$0 \to \mathbb{Z} \to \widetilde{SL}(2,\mathbb{Z}) \to SL(2,\mathbb{Z}) \to 1. \tag{5.4}$$

One has the following simple but somewhat tedious lemma:

Lemma 5.1.

(a) *The group* $\mathrm{Map}_{1,1}$ *is isomorphic to* $\widetilde{SL}(2,\mathbb{Z})$.

(b) *The elements* $t_a, t_b \in \mathrm{Map}_{1,1}$ *are conjugate in* $\mathrm{Map}_{1,1}$.

As an immediate corollary we get

Corollary 5.2. *The subsemigroup of* $\mathrm{Map}_{1,1}$ *generated by the conjugates of the element* t_a *does not contain the identity element.*

The proofs of these statements will be given in Sections 5.2 and 5.3 respectively. The proof of Lemma 5.1 is based on the standard description [16, Section 1.8-1.9] of the universal cover of $SL(2,\mathbb{R})$ which we recall next.

5.2 Universal covers of symplectic groups

Let V be a two dimensional real vector space with basis $p, q \in V$ and coordinate functions $x, y \in V^\vee$. The group $SL(2,\mathbb{R})$ is naturally identified with the group of linear automorphisms of V preserving the standard symplectic form $w := x \wedge y \in \wedge^2 V^\vee$. Let Λ denote the Grassmanian of all Lagrangian subspaces in V. Since $\dim_{\mathbb{R}}(V) = 2$ we have $\Lambda = Gr(1,V) \cong \mathbb{RP}^1 \cong S^1$. In fact any line $\ell \subset V$ is uniquely determined by the angle ($\theta \mod \pi$) where θ is the angle ℓ forms with the x-axis. If we identify $V \cong \mathbb{C}$ via $(x,y) \mapsto x + iy$, then the identification $u : \Lambda \widetilde{\to} S^1$ is given explicitly as $u(\ell) = e^{2i\theta}$, where $\ell = \mathbb{R}e^{i\theta}p$. The symplectic group $SL(2,\mathbb{R})$ acts on Λ and this action lifts to a well defined continuous action of $\widetilde{SL}(2,\mathbb{R})$ on the universal cover $\widetilde{\Lambda} \cong \mathbb{R}$ of Λ which essentially determines $\widetilde{SL}(2,\mathbb{R})$

Explicitly the group $\widetilde{SL}(2,\mathbb{R})$ can be described in terms of the $SL(2,\mathbb{R})$ action on Λ by means of the Maslov index [4, 16]. Recall [16, Section 1.5] that for a real symplectic vector space (V, w) with a Lagrangian Grassmanian Λ, the corresponding Maslov index is the function $\tau : \Lambda^3 \to \mathbb{Z}$ defined as follows. For each triple (ℓ_1, ℓ_2, ℓ_3) of Lagrangian subspaces of V consider the quadratic form $Q_{\ell_1\ell_2\ell_3}(x)$ on the vector space $\ell_1 \oplus \ell_2 \oplus \ell_3$ given by

$$Q_{\ell_1\ell_2\ell_3}(x_1 \oplus x_2 \oplus x_3) = w(x_1, x_2) + w(x_2, x_3) + w(x_3, x_1).$$

Next put $\tau(\ell_1, \ell_2, \ell_3) :=$ the signature of $Q_{\ell_1\ell_2\ell_3}$. By construction the function τ is antisymmetric in the three arguments and is invariant under the diagonal action of $Sp(V, w)$ on Λ^3. Moreover if $\ell \in \Lambda$ is a fixed Lagrangian subspace of V, then the $\mathbb{Z}$-valued function

$$\begin{array}{llcl} \tau_\ell : & Sp(V,w) \times Sp(V,w) & \longrightarrow & \mathbb{Z} \\ & (g,h) & \longrightarrow & \tau(\ell, g\ell, gh\ell), \end{array}$$

satisfies (see [16, Lemma 1.6.13])

$$\tau_\ell(g_1g_2, g_3) + \tau_\ell(g_1, g_2) = \tau_\ell(g_1, g_2g_3) + \tau_\ell(g_2, g_3).$$

Thus τ_ℓ is a cocycle for the group $Sp(V, w)$ with coefficients in the trivial module $\mathbb{Z}$ and so determines a central extension

$$0 \to \mathbb{Z} \to G_\ell \to Sp(V, w) \to 1. \tag{5.5}$$

As a set $G_\ell = Sp(V, w) \times \mathbb{Z}$ with the group structure being given by

$$(g_1, n_1) \cdot (g_2, n_2) = (g_1g_2, n_1 + n_2 + \tau_\ell(g_1, g_2)). \tag{5.6}$$

The group G_ℓ acts naturally on the universal cover $\widetilde{\Lambda}$ of Λ. It turns out [16, Section 1.9] that there is a unique topology on G_ℓ for which this action becomes continuous, and so G_ℓ has a natural structure of a Lie group. Furthermore the universal covering group $\widetilde{SP}(V, w)$ of $Sp(V, w)$ is naturally identified with the identity component of G_ℓ. In fact there is a canonical character $s : G_\ell \to S^1$ of order four so that $\widetilde{SP}(V, w) = \ker(s)$.

This description of $\widetilde{SP}(V, w)$ can be made completely explicit for the case of a two dimensional symplectic vector space (V, w) considered above. Indeed the Maslov index map τ admits a very concrete description in this case. The natural (counterclockwise) orientation of $S^1 \subset \mathbb{R}^2$

gives a cyclic ordering for any triple of points in S^1. Let $\ell_1, \ell_2, \ell_3 \in \Lambda$ be three lines in V. Then the Masloiv index $\tau(\ell_1, \ell_2, \ell_3)$ is given by

$$\tau(\ell_1, \ell_2, \ell_3) = \begin{cases} 0 & \text{if } \ell_1, \ell_2, \ell_3 \text{ are not all distinct} \\ 1 & \text{if } u(\ell_2) \text{ is between } u(\ell_1) \text{ and } u(\ell_3) \\ -1 & \text{if } u(\ell_2) \text{ is between } u(\ell_3) \text{ and } u(\ell_1) \end{cases}$$

Geometrically this just means that $\tau(\ell_1, \ell_2, \ell_3) = 1$ if ℓ_2 is inside the angle (mod π) formed by ℓ_1 and ℓ_3 and $\tau(\ell_1, \ell_2, \ell_3) = -1$ if ℓ_2 is outside that angle.

Choose $\ell := \mathbb{R}p$ and consider the central extension (5.5). The function $s : G_\ell \to S^1$ defined by

$$s\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, n\right) := \begin{cases} \operatorname{sgn}(c) i^{n+1} & \text{if } c \neq 0, \\ \operatorname{sgn}(a) i^n & \text{if } c = 0 \end{cases}$$

is a character of G_ℓ and $\widetilde{SL}(2, \mathbb{R}) = \ker(s)$. In other words we have (5.7)

$$\widetilde{SL}(2,\mathbb{R}) = \left\{ (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, n) \in SL(2,\mathbb{R}) \times \mathbb{Z} \;\middle|\; \begin{array}{l} \text{If } c = 0, \text{ then } n \text{ is even} \\ \text{and } \operatorname{sgn}(a) = (-1)^{n/2}, \\ \text{and if } c \neq 0, \text{ then } n \text{ is odd} \\ \text{and } \operatorname{sgn}(c) = (-1)^{(n+1)/2} \end{array} \right\}$$

with a group law given by the formula (5.6).

Using this explicit description we can prove Lemma 5.1.

Proof of Lemma 5.1. Put

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad B := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since $B = JAJ^{-1}$ it suffices to check that there exist lifts $\widetilde{A}, \widetilde{B}, \widetilde{J} \in \widetilde{SL}(2, \mathbb{R})$ of A, B and J respectively so that:

- $\widetilde{B} = \widetilde{J}\widetilde{A}\widetilde{J}^{-1}$,
- $\widetilde{A}\widetilde{B}\widetilde{A} = \widetilde{B}\widetilde{A}\widetilde{B}$,
- $(\widetilde{A}\widetilde{B})^6$ generates $\mathbb{Z} \cong \ker[\widetilde{SL}(2, \mathbb{Z}) \to SL(2, \mathbb{Z})]$.

Let $k \in \mathbb{Z}$. In terms of the description (5.7) choose

$$\widetilde{A}_k = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 4k\right), \quad \widetilde{J} = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1\right),$$

$$\widetilde{B}_k = \widetilde{J}\widetilde{A}\widetilde{J}^{-1} = \left(\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, 4k + 1\right).$$

Using the group law (5.6) one calculates

$$\widetilde{A}_k\widetilde{B}_k = \left(\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, 8k+1+\tau\left(\mathbb{R}p, \mathbb{R}p, \mathbb{R}q\right)\right) = \left(\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, 8k+1\right).$$

Similarly we have

$$\begin{aligned} \widetilde{A}_k\widetilde{B}_k\widetilde{A}_k &= \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 12k+1+\tau\left(\mathbb{R}p, \mathbb{R}q, \mathbb{R}q\right)\right) \\ &= \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 12k+1\right) \end{aligned}$$

and

$$\begin{aligned} \widetilde{B}_k\widetilde{A}_k\widetilde{B}_k &= \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 12k+2+\tau\left(\mathbb{R}p, \mathbb{R}(p-q), \mathbb{R}q\right)\right) \\ &= \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 12k+1\right). \end{aligned}$$

Thus $\widetilde{A}_k\widetilde{B}_k\widetilde{A}_k = \widetilde{B}_k\widetilde{A}_k\widetilde{B}_k$ as required. Furthermore we have

$$\begin{aligned} (\widetilde{A}_k\widetilde{B}_k\widetilde{A}_k)(\widetilde{B}_k\widetilde{A}_k\widetilde{B}_k) &= \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 24k+2+\tau\left(\mathbb{R}p, \mathbb{R}q, \mathbb{R}p\right)\right) \\ &= \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 24k+2\right), \end{aligned}$$

and hence

$$(\widetilde{A}_k\widetilde{B}_k)^6 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 48k+4\right).$$

In addition from the description (5.7) we see that

$$\ker[\widetilde{SL}(2,\mathbb{Z}) \to SL(2,\mathbb{Z})] = \left\{\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 4k\right) | k \in \mathbb{Z}\right\},$$

and so $(\widetilde{A}_k\widetilde{B}_k)^6$ will generate $\ker[\widetilde{SL}(2,\mathbb{Z}) \to SL(2,\mathbb{Z})]$ if $k=0$. The Lemma is proven. q.e.d.

Remark 5.3. From the description (5.7) it is clear that the element $\widetilde{A}_k$ is the most general lift of A. Since any two lifts of J differ by a central element in $\widetilde{SL}(2,\mathbb{Z})$, it follows that the pair $(\widetilde{A}_k, \widetilde{B}_k)$ used in the proof of Lemma 5.1 represents the most general way to lift A and B to elements in $\widetilde{SL}(2,\mathbb{Z})$ so that that they still remain conjugate. The choice of $k = 0$ guaranteeing the identification of $\mathrm{Map}_{1,1}$ with $\widetilde{SL}(2,\mathbb{Z})$ has the following simple geometric interpretation.

The natural action of $SL(2,\mathbb{R})$ on the Lagrangian Grassmanian $\Lambda \cong S^1$ lifts to a non-trivial action of the universal covering group $\widetilde{SL}(2,\mathbb{R})$ on the universal cover $\widetilde{\Lambda} \cong \mathbb{R}$ of Λ. The group $\mathbb{Z}$ $= \ker[\widetilde{SL}(2,\mathbb{R}) \to SL(2,\mathbb{R})]$ acts discretely on $\widetilde{\Lambda}$ with a fundamental domain isomorphic to an interval of length 2π, and the action of $k \in \mathbb{Z}$ on $\widetilde{L}$ is by translation by $2k\pi$. In particular for a point $\tilde{\ell} \in \widetilde{\Lambda} = \mathbb{R}$ and an element $\tilde{g} \in \widetilde{SL}(2,\mathbb{R})$ we can define the *displacement angle* of $\tilde{g}$ at $\tilde{\ell}$ as the real number $(\tilde{g}\tilde{\ell} - \tilde{\ell}) \mod 2\pi\mathbb{Z}$. For example if we identify the universal cover $\widetilde{U}(1) \cong \mathbb{R}$ with a subgroup of $\widetilde{SL}(2,\mathbb{R})$, then the elements in $\widetilde{U}(1) \cong \mathbb{R}$ have displacement angles that are independent of the choice of the point $\tilde{\ell}$ and thus act as translations on $\widetilde{\Lambda} \cong \mathbb{R}$.

If now $1 \neq g \in SL(2,\mathbb{R})$ is a nilpotent element, then g has a fixed point on Λ, and hence among all lifts of g in $\widetilde{SL}(2,\mathbb{R})$ there is a unique one having a periodic sequence of fixed points in $\widetilde{\Lambda}$ with period π.

The lifts $\widetilde{A}_0$ and $\widetilde{B}_0$ are characterized uniquely (see the explanation below) as the lifts of A and B having periodic sequences of fixed points in $\widetilde{\Lambda}$. In fact, granted this characterization, one can easily prove Lemma 5.1. Indeed, the matrix $AB \in SL(2,\mathbb{R})$ is conjugate to a rotation by $-\pi/3$, and so the displacement angle of any element in $\widetilde{SL}(2,\mathbb{R})$ lifting AB at any point in $\widetilde{\Lambda}$ will be exactly $-\pi/3$. On the other hand since $\widetilde{A}_0$ and $\widetilde{B}_0$ each have a sequence of fixed points which is periodic with period π we see that the displacement angles of $\widetilde{A}_0$ and $\widetilde{B}_0$ can not be smaller than $-\pi$. Thus both $\widetilde{A}_0\widetilde{B}_0\widetilde{A}_0$ and $\widetilde{B}_0\widetilde{A}_0\widetilde{B}_0$ have displacement angles which are strictly bigger than -2π, and since they lift the same element $ABA = BAB \in SL(2,\mathbb{R})$ we must have $\widetilde{A}_0\widetilde{B}_0\widetilde{A}_0 = \widetilde{B}_0\widetilde{A}_0\widetilde{B}_0$.

To justify the characterization of $\widetilde{A}_0$ and $\widetilde{B}_0$ as lifts of A and B having fixed points one proceeds as follows. Note that similarly to (5.7) the space $\widetilde{\Lambda}$ can be identified [16, Section 1.9] as a set with

$$\widetilde{\Lambda} = \{(\ell, k) \in \Lambda \times \mathbb{Z} | k \equiv 1 + \dim_{\mathbb{R}}(\ell \cap \ell_0) \mod 2\}.$$

Under this identification the action of $(g, n) \in \widetilde{SL}(2,\mathbb{R})$ on $\widetilde{\Lambda}$ is given by $(g, n) \cdot (\ell, k) = (g\ell, n + k + \tau(\ell_0, g\ell_0, g\ell))$, and we immediately see that $\widetilde{A}$ fixes each of the points $\{(\mathbb{R}p, 2k)\}_{k\in\mathbb{Z}}$ and that $\widetilde{B}$ fixes each of the points $\{(\mathbb{R}q, 2k+1)\}_{k\in\mathbb{Z}}$.

Remark 5.4. Lemma 5.1 (a) easily implies Mumford's theorem asserting that $\mathrm{Pic}(\mathcal{M}_1^1)$ is a cyclic group of order 12. Indeed due to Lemma 5.1 (a) we only need to check that the group of central extensions $H^2(SL(2,\mathbb{Z}),\mathbb{Z})$ is isomorphic to $\mathbb{Z}/12$ and is generated by the class of (5.4). Recall that $SL(2,\mathbb{Z}) = (\mathbb{Z}/6) *_{(\mathbb{Z}/2)} (\mathbb{Z}/4)$ where the cyclic subgroups $\mathbb{Z}/6$ and $\mathbb{Z}/4$ are generated by the matrices AB and ABA respectively. The commutator subgroup of $SL(2,\mathbb{Z})$ is a free subgroup Φ on two generators, namely Φ is the subgroup generated by $[B,A]]$ and $[B,A^{-1}]$. Hence $SL(2,\mathbb{Z})$ fits in a short exact sequence $0 \to \Phi \to SL(2,\mathbb{Z}) \to \mathbb{Z}/12 \to 0$, and we have the Hochschild-Serre spectral sequence

$$E_2^{pq} := H^p(\mathbb{Z}/12, H^q(\Phi,\mathbb{Z})) \Rightarrow H^{p+q}(SL(2,\mathbb{Z}),\mathbb{Z}) \tag{5.8}$$

abutting to the cohomology of $SL(2,\mathbb{Z})$. This is a first quadrant spectral sequence, and since Φ is of cohomological dimension one only the first two rows of (5.8) are non-trivial. In particular (5.8) degenerates in the E_3-term and so $E_\infty^{02} = E_2^{02} = 0$, $E_\infty^{11} = \ker d_2^{11}$, $E_\infty^{20} = E_2^{20}$. Also since $\mathbb{Z}$ is the trivial $SL(2,\mathbb{Z})$ module we have

$$\begin{aligned} E_2^{20} &= H^2(\mathbb{Z}/12, H^0(\Phi,\mathbb{Z})) = H^2(\mathbb{Z}/12, (\mathbb{Z})^\Phi) = H^2(\mathbb{Z}/12,\mathbb{Z}), \\ E_2^{11} &= H^1(\mathbb{Z}/12, H^1(\Phi,\mathbb{Z})) = H^1(\mathbb{Z}/12, \mathrm{Hom}_{\mathbb{Z}}(\Phi,\mathbb{Z})) \\ &= H^1(\mathbb{Z}/12,\mathbb{Z}^2) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/12,\mathbb{Z}^2) = 0. \end{aligned}$$

In particular we have an isomorphism

$$H^2(SL(2,\mathbb{Z}),\mathbb{Z}) = H^2(SL(2,\mathbb{Z})/\Phi,\mathbb{Z}) = H^2(\mathbb{Z}/12,\mathbb{Z}).$$

On the other hand the pullback of the canonical central extension

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

via any homomorphism $\mathbb{Z}/12 \to \mathbb{Q}/\mathbb{Z}$ gives an identification $H^2(\mathbb{Z}/12,\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/12,\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/12$.

Finally since both AB and ABA are conjugate to rotations of finite order we can find maximal compact subgroups $U', U'' \subset SL(2,\mathbb{R})$ so that $AB \in U'$ and $ABA \in U''$. Furthermore since the inclusions $U' \subset SL(2,\mathbb{R})$ and $U'' \subset SL(2,\mathbb{R})$ are homotopy equivalences and since U' and U'' are both isomorphic to $\mathbb{R}/\mathbb{Z}$ we see that the pull backs of the extension (5.4) by the inclusions $\mathbb{Z}/6 \subset SL(2,\mathbb{Z})$ and $\mathbb{Z}/4 \subset SL(2,\mathbb{Z})$ are just the standard extensions

$$0 \to \mathbb{Z} \xrightarrow{6} \mathbb{Z} \to \mathbb{Z}/6 \to 0 \quad \text{and} \quad 0 \to \mathbb{Z} \xrightarrow{4} \mathbb{Z} \to \mathbb{Z}/4 \to 0$$

respectively. In particular the extension (5.4) is the pullback of

$$0 \to \mathbb{Z} \xrightarrow{12} \mathbb{Z} \to \mathbb{Z}/12 \to 0$$

via the homomorphism $SL(2,\mathbb{Z}) \cong (\mathbb{Z}/6) *_{(\mathbb{Z}/2)} (\mathbb{Z}/4) \twoheadrightarrow \mathbb{Z}/12$ and hence corresponds to the generator of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/12, \mathbb{Q}/\mathbb{Z})$.

5.3 Sections in elliptic symplectic fibrations

Now we apply the above considerations to the study of symplectic Lefschetz fibrations of genus one.

Consider a relatively minimal symplectic Lefschetz fibration $f : X \to S^2$ of genus one which corresponds to a relation

$$t_1^{n_1} t_2^{n_2} \ldots t_m^{n_m} = 1, \text{ with } n_i > 0 \text{ for all } i = 1, \ldots, m, \tag{5.9}$$

in $\mathrm{Map}_1^1 = SL(2,\mathbb{Z})$ as in Lemma 3.2. To avoid pathologies we will assume that f has a continuous section and that at least one of the Dehn twists t_i is a twist along a non-separating cycle. In that case it is known [19, Theorem 9] [14, Theorem 2.4] that the relation determines $f : X \to S^2$ up to a diffeomorphism. Let σ be a smooth section of f. We would like to find the relationship between σ^2 and the numbers n_i (see Corollary 5.5).

The Dehn twists t_i correspond to a sequence of smooth circles $s_1, \ldots, s_m$ in the generic fiber $X_o \cong C_1$ of f. Since the twisting diffeomorphisms T_{s_i} are nontrivial only in small neighborhoods of s_i in C_1, we may assume without loss of generality that there is point $p \in X_o$ and a small disc $p \in \Delta \subset X_o$ so that all T_{s_i} act identically on Δ. Recall (Lemma 3.2) that the fibration f was reconstructed from the relation (5.9) in two steps. We produce first a SLF $u : U \to D$ over a disc D with a boundary monodromy transformation $\prod_i T_{s_i}^{n_i} : C_1 \to C_1$. Next, since by assumption $\prod_i T_{s_i}^{n_i}$ is homotopic to the identity we can complete the resulting fibration into an elliptic fibration over S^2 by gluing the trivial genus one family along the boundary of D.

Now we can construct one section where the computation of the square can be made explicitly. Indeed the fibration $u : U \to D$ has a section $D \times \{p\}$ by construction. Also the fiber bundle $u_{|u^{-1}(\partial D)} : u^{-1}(\partial D) \to \partial D$ is diffeomorphic by a fiber-preserving diffeomorphism to $\partial D \times X_o$. Since $\mathrm{Map}_1^1 = \mathrm{Map}_1$ we may choose this homotopy equivalence so that it stabilizes the point $p \in X_o$ and thus glue the section $D \times \{p\}$ with the constant section through p in the trivial family we are gluing to D in order to complete u to f. Let σ denote the resulting section.

As we have just seen the right twists t_i can be viewed as elements of the group $\mathrm{Map}_{1,1}$ which is the group of classes of maps stabilizing $\Delta \subset X_o = C_1$. Then from the handle body decomposition described in Lemma 3.2 we conclude that there is a non-negative integer n so that

$$\prod t_i^{n_i} = t_c^n,$$

where c is the boundary circle of $C_1 \setminus \Delta$. In particular the degree of the normal bundle to σ in X is exactly $-n$.

As a corollary from this observation one can obtain a symplectic version of a well known result of L. Szpiro [31] who proved that for any jacobian elliptic fibration (elliptic fibration with a holomorphic section) over $\mathbb{P}^1$ with only multiplicative fibers we have an inequality between the number D of singular fibers and the number N of irreducible components of the singular fibers. More precisely he showed that $N \leq 6D$.

Since in the monodromy group the multiplicative fiber with m components corresponds to the element t^m for some right Dehn twist t we see that the following Corollary is a straightforward generalization of Szpiro's result to the symplectic category.

Corollary 5.5. *Assume that in* $\mathrm{Map}_{1,1} = \widetilde{SL}(2,\mathbb{Z})$ *we have a relation* $\prod_{i=1}^m t_i^{n_i} = t_c^n$ *with* $n_i > 0$. *Then* $\sum n_i = 12n$ *and* $m \geq 2n$.

Proof. From the standard presentation (5.3) of $\mathrm{Map}_{1,1}$ we see that there exists a unique homomorphism $\chi : \mathrm{Map}_{1,1} \to \mathbb{Z}$ characterized by the property that $\chi(t_a) = 1$ and $\chi(t_b) = 1$. Alternatively by Remark 5.4 the pushout of the extension (5.2) via the multiplication map $\mathrm{mult}_{12} : \mathbb{Z} \to \mathbb{Z}$ is a split extension and so we can choose for χ the composition of mult_{12} with a splitting of this extension. Since $t_c = (t_a t_b)^6$ we have that $\chi(t_c) = 12$ and that

$$\sum n_i = \chi(\prod_{i=1}^m t_i^{n_i}) = \chi(t_c^n) = 12n.$$

Furthermore as explained in Remark 5.3 the transformation t_c^n translates every point in $\mathbb{R} = \widetilde{\Lambda}$ by $-2n\pi$. On the other hand $t_i^{n_i}$ projects to a nilpotent transformation in $SL(2, \mathbb{Z})$ and so has a displacement angle at least $-\pi$ at any point in $\widetilde{\Lambda}$. Hence $m \geq 2n$. q.e.d.

We are now in a position to prove Corollary 5.2.

Proof of Corollary 5.2. If the subsemigroup in $\widetilde{SL}(2, \mathbb{Z})$ generated by the conjugates of t_a contains the identity element, then we can find a relation of the form (5.9) in $\mathrm{Map}_{1,1}$. Hence there exist a SLF $f : X \to S^2$ where the section corresponding to the puncture p has a trivial tubular neighborhood and normal bundle. By the above calculation this yields $0 = \sigma^2 = -\sum n_i$, i.e., $n_i = 0$ for all i. This proves the corollary. q.e.d.

Remark 5.6. (i) It is very tempting to try to extend this purely group theoretic proof of Szpiro's result to the elliptic curves over number fields. It is expected that the analogue of Szpiro's inequality with any constant instead of 6 will lead to a solution of the ABC-conjecture.

(ii) S. Zhang pointed out to us that the same reasoning as in the proof of Corollary 5.5 proves the inequality $N \leq 6D + 6k$ for an elliptic Lefschetz fibration $f : X \to C_k$ over a smooth surface of genus k.

Indeed the fibration f in this case is determined by right-handed Dehn twists $\{t_i\}_{i=1}^m$, $\{a_j, b_j\}_{j=1}^k$ satisfying the monodromy relation

$$\prod_{i=1}^{m} t_i^{n_i} \prod_{j=1}^{k} [a_j, b_j] = 1$$

in $SL(2, \mathbb{Z})$. Again a choice of a continuous section of f determines a natural lifting of all t_i's, a_j's and b_j's to elements in $\mathrm{Map}_{1,1} = \widetilde{SL}(2, \mathbb{Z})$ satisfying the relation $\prod_{i=1}^m t_i^{n_i} \prod_{j=1}^k [a_j, b_j] = t_c^n$. But a natural lifting of a commutator is also a commutator and hence is contained in the kernel of the character $\chi : \widetilde{SL}(2, \mathbb{Z}) \to \mathbb{Z}$. In particular each of the commutators $[a_j, b_j] \in \widetilde{SL}(2, \mathbb{Z})$ has invariant points when acting on $\widetilde{\Lambda}$ and so is a hyperbolic transformation. Therefore the displacement angle for every commutator $[a_j, b_j]$ will be at least $-\pi$. This yields $m + k \geq 2n$ or equivalently $6m + 6k \geq \sum n_i$.

5.4 Symplectic fibrations with fibers of higher genus

Some of the features of $\mathrm{Map}_{1,1}$ that allowed us to restrict the numerical properties of the sections in a genus one SLF carry over to the case of

SLF of higher genus. Here we outline an approach to the study of the numerical behavior of the sections in a higher genus SLF.

The main ingredient in the discussion in the previous section was the existence of a natural action of $\mathrm{Map}_{1,1}$ on the universal cover $\widetilde{\Lambda} = \mathbb{R}$ of the Lagrangian Grassmanian Λ and the notion of a displacement angle.

There is a similar action of $\mathrm{Map}_{g,1}$ which we proceed to describe. For more details the reader may wish to consult the excellent exposition of S. Morita [22].

As explained in Remark 3.1 the group $\mathrm{Map}_{g,1}$ is a central extension

$$0 \to \mathbb{Z} \to \mathrm{Map}_{g,1} \to \mathrm{Map}_g^1 \to 1 \tag{5.10}$$

of Map_g^1 by an infinite cyclic group generated by the Dehn twist around the puncture.

This central extension has many remarkable properties but we want to emphasize the ones which are reflected in the structure of the semi-group $\mathrm{Map}_{g,1}(-) \subset \mathrm{Map}_{g,1}$ generated by all right-handed Dehn twists.

Let $e \in H^2(\mathrm{Map}_g^1), \mathbb{Z})$ denote the class of the central extension (5.10). The class e is dubbed the *Euler class* by Morita and admits the following simple description [22]. As in the beginning of Section 3.1 we will view the elements in Map_g^1 as isotopy classes of orientation preserving diffeomorphisms of C_g that preserve the point $x_1 \in C_g$.

Fix an isomorphism of the universal covering of C_g with the unit disk $D \subset \mathbb{C}$, e.g. by fixing a point j in the Teichmüller space $\mathcal{T}_g$. Let $\nu : D \to C_g$ be the corresponding covering map. Any orientation preserving diffeomorphism $\Phi : C_g \to C_g$ with $\Phi(x_1) = x_1$ defines a quasi-conformal map $h^{\Phi} : D \to D$ which preserves the preimage $\nu^{-1}(x_1)$ of x_1 in D. This map extends to an orientation preserving homeomorphism $h^{\Phi}_{\partial} : S^1 \to S^1$ of the boundary $\partial D \cong S^1$ of D. Modulo isometries of D the homeomorphism h^{Φ}_{∂} is uniquely defined by the isotopy class of Φ relative to p. If in addition we require that the isomorphism $j : \widetilde{C}_g \widetilde{\to} D$ sends a marked point $\tilde{x}_1 \in \nu^{-1}(x_1)$ to $0 \in D$ and the differential dj induces a fixed isomorphism of the tangent spaces $T_{\tilde{x}_1}\widetilde{C}_g$ and $T_0 D$, then the element $h^{\Phi}_{\partial} \in \mathrm{Homeo}^+(S^1)$ is uniquely determined by the mapping class $[\Phi] \in \mathrm{Map}_g^1$ of Φ.

Therefore we obtain a natural action $\rho : \mathrm{Map}_g^1 \to \mathrm{Homeo}^+(S^1)$ given by $\rho([\Phi]) = h^{\Phi}_{\partial}$. The group $\mathrm{Homeo}^+(S^1)$ has a natural central extension by an infinite cyclic group which is the group $\mathrm{Homeo}^{\mathrm{per}}(\mathbb{R})$ of 2π-periodic orientation preserving homeomorphisms of the line $\mathbb{R}$ (here as before one should view the line $\mathbb{R}$ as the universal cover of the circle

S^1). One can check [20] that the central extension (5.10) is just the pullback of the extension $0 \to \mathbb{Z} \to \mathrm{Homeo}^{\mathrm{per}}(\mathbb{R}) \to \mathrm{Homeo}^{+}(S^1) \to 1$ by the homomorphism ρ. In particular we have a natural homomorphism $\rho^{\mathrm{per}} : \mathrm{Map}_{g,1} \to \mathrm{Homeo}^{\mathrm{per}}(\mathbb{R})$ and so in the same way as in Remark 5.3 we can define the *displacement angle* associated with any pair (ϕ, ℓ), $\phi \in$, $\ell \in \mathbb{R}$ as the real number $(\rho^{per}(\phi)(\ell) - \ell) \mod 2\pi\mathbb{Z}$.

We propose the following conjectural characterization of the semigroup $\mathrm{Map}_{g,1}(-)$ which generalizes Remark 5.3:

Conjecture 5.7. *The subsemigroup* $\mathrm{Map}_{g,1}(-) \subset \mathrm{Map}_{g,1}$ *consists of mapping classes whose displacement angle is non-negative at every point in* $\mathbb{R}$.

Before we give some evidence for the validity of this conjecture we need to recall Morita's analysis of the Euler class e.

Similarly to the genus one case we can view the Euler class e as an element of Picard group of the moduli space $\mathcal{M}_g^1$ of smooth curves of genus g with one marked point. As such e can be written explicitly [12], [21] as a linear combination with $\mathbb{Q}$-coefficients of two cocycles of geometric origin: e_1 and c. The cocycle e_1 is just the first Mumford-Miller-Morita class and is proportional to the generator of the Picard group of the moduli space $\mathcal{M}_g$ of smooth curves of genus g. Also as in the genus one case the cocycle e_1 is induced from a cocycle for $Sp(H_1(C_g, \mathbb{Z}))$ via the natural homomorphism $\mathrm{Map}_g^1 \to \mathrm{Map}_g \to Sp(H_1(C_g, \mathbb{Z}))$. The cocycle c is somewhat more mysterious. Geometrically it can be defined as follows. Let Θ denote the pullback of the first Chern class of the relative theta line bundle on the universal degree $g-1$ Jacobian $\mathcal{J}^{g-1} \to \mathcal{M}_g$ via the canonical Abel-Jacobi map from the universal curve $\mathcal{M}_g^1 \to \mathcal{M}_g$ to $\mathcal{J}^{g-1} \to \mathcal{M}_g$ defined by the point $x_1 : \mathcal{M}_g \to \mathcal{M}_g^1$. Then it can be shown that $c = 8\Theta - e_1/3$ [21, (1.7)] and [11, Proposition 2].

Observe that these geometric definitions of the classes e_1 and c make sense in the genus one case as well. However it follows from the proof of Mumford's result [23, Main Theorem] discussed in Section 5.1 that in the genus one case the two classes e_1 and c coincide and so we are not getting anything new for the group $\mathrm{Map}_1 = SL(2, \mathbb{Z})$. In contrast for $g > 2$ the classes e_1 and c are linearly independent in $H^2(\mathrm{Map}_g^1, \mathbb{Q})$ and are related to the class e by the following explicit formula (see e.g. [21])

$$e = \frac{1}{4g(1-g)}(e_1 + c).$$

Algebraically the classes e_1 and c can be interpreted as follows. Consider the lattice $H := H_1(C_g, \mathbb{Z})$ together with its natural symplectic form θ given by the intersection pairing. Morita has shown (see e.g. [20]) that there exists a two-step nilpotent group $\mathcal{U}$ equipped with a $Sp(H,\theta)$-action and a homomorphism $\kappa : \mathrm{Map}_g^1 \to Sp(H,\theta) \ltimes (\mathcal{U} \otimes \mathbb{Q})$ so that the classes e_1 and c are pullbacks of natural cohomology classes on $Sp(H,\theta) \ltimes (\mathcal{U} \otimes \mathbb{Q})$. More precisely $\mathcal{U}$ is a $Sp(H,\theta)$-invariant central extension of $\wedge^3 H$ by a certain $Sp(H,\theta)$-module of finite rank and e_1 is a pullback of a cohomology class of $Sp(H,\theta)$ and c is a pullback of a cohomology class of $Sp(H,\theta) \ltimes \frac{1}{2} \wedge^3 H$ via the natural maps

$$\mathrm{Map}_g^1 \xrightarrow{\kappa} Sp(H,\theta) \ltimes (\mathcal{U} \otimes \mathbb{Q}) \to Sp(H,\theta), \tag{5.11}$$

$$\mathrm{Map}_g^1 \xrightarrow{\kappa} Sp(H,\theta) \ltimes (\mathcal{U} \otimes \mathbb{Q}) \to Sp(H,\theta) \ltimes (\wedge^3 H \otimes \mathbb{Q}) \tag{5.12}$$

respectively.

To identify the element in $H^2(Sp(H,\theta), \mathbb{Z})$ that pulls back to e_1 consider the $2g$-dimensional vector space $H_{\mathbb{R}} := H \otimes \mathbb{R}$. The symplectic group $Sp(H_{\mathbb{R}}, \theta)$ is homotopy equivalent to its maximal compact subgroup which in turn is isomorphic to the unitary group $U(g)$. In particular $\pi_1(Sp(H_{\mathbb{R}}, \theta)) \cong \pi_1(U(g)) \cong \mathbb{Z}$ and so the universal cover $\widetilde{Sp}(H_{\mathbb{R}}, \theta)$ of $Sp(H_{\mathbb{R}}, \theta)$ is naturally a central extension of $Sp(H_{\mathbb{R}}, \theta)$ by an infinite cyclic group. By pulling back this extension by the natural inclusion $Sp(H,\theta) \subset Sp(H_{\mathbb{R}}, \theta)$ we get a central extension

$$0 \to \mathbb{Z} \to \widetilde{Sp}(H,\theta) \to Sp(H,\theta) \to 1. \tag{5.13}$$

From the geometric description of e_1 given above it is clear that e_1 is proportional to the pullback of the extension class (5.13)[1].

To identify the element in $H^2(Sp(H,\theta) \ltimes \frac{1}{2} \wedge^3 H, \mathbb{Z})$ that pulls back to c notice that the contraction with θ gives a well defined homomorphism of $Sp(H,\theta)$-modules $C : \wedge^3 H \to H$. Consider the Heisenberg central extension $0 \to \mathbb{Z} \to \mathcal{H} \to H \to 0$ corresponding to the class $\theta \in H^2(H, \mathbb{Z})$. Since by definition θ is $Sp(H,\theta)$-invariant the pullback of the Heisenberg extension via the map C will also be $Sp(H,\theta)$-invariant and will so determine an element $H^2(Sp(H,\theta) \ltimes \frac{1}{2} \wedge^3 H, \mathbb{Z})$. In [20, Theorem 3.1] Morita shows that this element pulls back to c via the homomorphism (5.12).

[1] In fact it was pointed out to us by R. Hain that the extension class (5.13) determines the universal central extension of $Sp(H,\theta)$ and so with the correct choice of the orientation of the fiber must be equal to the pullback of the determinant of the pushforward of the relative cotangent bundle on the universal abelian variety or in other words to $e_1/12$.

It is not hard to see that the above interpretation of c gives rise to a representation of $\mathrm{Map}_{g,1}$ into a covering of a different symplectic group. Indeed, let us fix a symplectic basis of H and identify $Sp(H,\theta)$ with the group $Sp(2g,\mathbb{Z})$. Denote by $ASp(2g,\mathbb{Z})$ the subgroup of the group $Sp(2g+2,\mathbb{Z})$ which stabilizes the orthogonal complement of the element b_{g+1} of the standard symplectic basis $a_1, b_1, \ldots, a_{g+1}, b_{g+1}$ of $\mathbb{Z}^{2g+2}$. The natural map $ASp(2g,\mathbb{Z}) \to GL(2g+1,\mathbb{Z})$, $X \mapsto X_{|b_{g+1}^{\perp_\theta}}$ maps the group $ASp(2g,\mathbb{Z})$ onto $Sp(2g,\mathbb{Z}) \ltimes \mathbb{Z}^{2g}$ and so $ASp(2g,\mathbb{Z})$ fits into a central extension

$$0 \to \mathbb{Z} \to ASp(2g,\mathbb{Z}) \to Sp(2g,\mathbb{Z}) \ltimes \mathbb{Z}^{2g} \to 1 \tag{5.14}$$

which is clearly isomorphic to the central extension of $Sp(H,\theta) \ltimes H$ considered above.

The fact that e is proportional to $e_1 + c$ combined with the above discussion implies that the group $\mathrm{Map}_{g,1}$ has a well defined homomorphism $\delta : \mathrm{Map}_{g,1} \to \widetilde{ASp}(2g,\mathbb{Z})/\mathbb{Z}$ where $\widetilde{ASp}(2g,\mathbb{Z})$ is the the universal cover of $ASp(2g,\mathbb{Z})$ and $\mathbb{Z} \subset \widetilde{ASp}(2g,\mathbb{Z})$ is generated by an element proportional to the difference of the extension classes (5.13) and (5.14).

On the other hand it is clear from the construction that $\widetilde{ASp}(2g,\mathbb{Z})$ is also isomorphic to the preimage of $ASp(2g,\mathbb{Z})$ into the universal cover of $Sp(2g+2,\mathbb{R})$. The choice of an inclusion $U(g+2) \subset Sp(2g+2,\mathbb{R})$ induces an isomorphism of the Lagrangian Grassmanian Λ of $\mathbb{R}^{2g+2}$ with the quotient $U(g+1)/O(2g+2)$. Since $O(2g+2) \subset SU(g+1)$ we have a well defined map $\det : \Lambda \to U(g+1)/SU(g+1) \cong S^1$ and in particular a well defined map of the universal covers $\widetilde{\det} : \widetilde{\Lambda} \to \mathbb{R}$. Since the group $\widetilde{Sp}(2g+2,\mathbb{R})$ acts naturally on $\widetilde{\Lambda}$ we can use the map $\widetilde{\det}$ to define a displacement angle for any element $A \in \widetilde{Sp}(2g+2,\mathbb{R})$ and any $\ell \in \mathbb{R}$.

Since the homomorphism $\delta : \mathrm{Map}_{g,1} \to \widetilde{ASp}(2g,\mathbb{Z})/\mathbb{Z}$ is essentially given by the class e it is reasonable to expect that for any ϕ and any $\ell \in \mathbb{R}$ the displacement angles of $(\rho^{\mathrm{per}}(\phi), \ell)$ and $(\delta(\phi), \ell)$ will have the same sign. If this is the case, then the validity of Conjecture 5.7 will easily follow since similarly to the genus one case we can split the all unipotent elements in $ASp(2g,\mathbb{Z})$ into two classes according to their displacement angle. That is - unipotent elements whose lifts in $\widetilde{ASp}(2g,\mathbb{Z})$ have a periodic sequence of points with displacement angle zero and unipotent elements elements whose lifts in $\widetilde{ASp}(2g,\mathbb{Z})$ do not have points with displacement angle zero.

Since the Dehn twists obviously belong to the first class and since the elements in the first class will generate the subsemigroup in $\widetilde{ASp}(2g,\mathbb{Z})$

consisting of elements with non-positive displacement angle this argument will prove Conjecture 5.7.

Similar arguments should also lead to a proof of the following theorem, which is obviously correct in the case of projective Lefschetz pencils.

Theorem 5.8 (I.Smith). *There are no non-trivial SLF whose monodromy group is contained in the Torelli group.*

A special case of the above theorem was originally proven by B.Ozbagci [24, Corollary 7] who showed that hyperelliptic SLF (and in particular all Lefschetz fibrations of fiber genus two) cannot have monodromy contained in the Torelli group. The general statement of Theorem 5.8 appeared as a conjecture in a preliminary version of the present paper. The conjecture was settled affirmatively by Ivan Smith who graciously provided us with the proof appearing in Appendix 5.4 below.

In this direction we would like to ask a couple of questions:

Let $\mathcal{L} \to \mathcal{M}_g$ be the Hodge line bundle and let $c_1(\mathcal{L}) \in CH^1(\overline{\mathcal{M}}_g, \mathbb{Q})$ denote the natural extension of the first Chern class of $\mathcal{L}$ to the Deligne-Mumford compactification of $\mathcal{M}_g$.

Question 5.9. *Let $f : X \to C$ be a symplectic Lefschetz fibration of fiber genus g over an arbitrary Riemann surface C. Is it true that*

$$\langle c_1(\mathcal{L}), [C] \rangle \geq 0?$$

This question has an affermative answer (see Corollary A.6) whenever C is of genus zero.

Question 5.10. *Let $f : X \to C$ be a symplectic Lefschetz fibration of fiber genus g over an arbitrary Riemann surface C. Is it true that*

$$(8g + 4)\langle c_1(\mathcal{L}), [C] \rangle - g \cdot \mu \geq 0?$$

Here as usual μ denotes the number of the singular fibers in the fibration $f : X \to C$.

The last question is a symplectic version of the Moriwaki inequality and was suggested by R. Hain. If the answers of the above questions are positive we get additional restrictions on the words in the mapping class group defining SLF. Some analogues of Corollary 5.5 for high genus SLF can be expected as well.

Remark 5.11. It is sometimes profitable to consider other central extensions of Map_g^1. For example we can pullback the central extension (5.13) via the natural map $\mathrm{Map}_g^1 \to Sp(H, \theta)$. The resulting group Π is an extension of Map_g^1 corresponding to the extension class $e_1/12$ and is the exact analogue of the group $\widetilde{SL}(2, \mathbb{Z})$ from the genus one case. Even though the group Π is not directly related to $\mathrm{Map}_{g,1}$ and does not have any geometric interpretation in general, one might be able to use Π in concrete geometric situations. For example since the classes e_1 and e become proportional when restricted to the Torelli group one might hope to use the group Π for studying SLF over a base of high genus whose monodromy group is contained in the Torelli group.

The advantantage of working with Π is that it has a simpler structure than $\mathrm{Map}_{g,1}$. In particular by definition Π comes equipped with a surjective homomorphism to $Sp(2g, \mathbb{Z})$ and hence with an action on the universal cover of the Lagrangian Grassmanian $\Lambda := U(g)/O(g)$. As before we can use the map $\widetilde{\det} : \widetilde{\Lambda} \to \mathbb{R}$ to define a displacement angle for any element in $Sp(2g, \mathbb{R})$ and by composition with $\Pi \to Sp(2g, \mathbb{Z})$ for all elements in Π.

Note that the unipotent matrices $u \in Sp(2g, \mathbb{R})$ with $\mathrm{rank}(u - \mathrm{id}) = 1$ split into two conjugacy classes in $Sp(2g, \mathbb{R})$. One of these classes corresponds to the images of the right-handed Dehn twists in Map_g^1 and the other to the images of the left Dehn twists. In particular the lifts of all $u \in Sp(2g, \mathbb{R})$ with $\mathrm{rank}(u - \mathrm{id}) = 1$ to elements in $\widetilde{Sp}(2g, \mathbb{R})$ generate a subgroup of $\widetilde{Sp}(2g, \mathbb{R})$ which contains the image of Π.

Consider now the action of $u \in Sp(2g, \mathbb{R})$ with $\mathrm{rank}(u - \mathrm{id}) = 1$ on $\Lambda = U(g)/O(g)$. It stabilizes a subvariety of Lagrangian subspaces which have a trivial projection on the first coordinate line. If in addition $u(\lambda) \neq \lambda$ for some $\lambda \in \Lambda$, then the closure of the orbit of $\langle u \rangle \cdot \lambda$ in Λ is a circle which maps isomorphically onto S^1 under the natural map $\det : \Lambda \to S^1$.

Indeed if $W \cong \mathbb{R}^{2g-1}$ is the kernel of $u - \mathrm{id}$, then any subspace of W is invariant under u, and hence any Lagrangian subspace $L \subset \mathbb{R}^{2g}$ is either invariant under u or contains a u-invariant subspace $L' \subset L$ of codimension 1. Let L be the subspace corresponding to the point $\lambda \in \Lambda$ and let $l \in L$ be a vector which is transversal to L'. Then the linear span of l and $u(l)$ is a two-dimensional subspace $S \cong \mathbb{R}^2$ in $\mathbb{R}^{2g}$ and all Lagrangian subspaces $\xi \in \langle u \rangle \cdot L \subset \Lambda$ are contained in $L' \oplus S$. The rank of symplectic form on $L' \oplus S$ is 2 and in fact the form on $L' \oplus S$ is pulled back from S. Consequently the orbit $\langle u \rangle \cdot \lambda$ is the same as for the standard action of $SL(2,\mathbb{R})$ and so the restriction of det to $\overline{\langle u\rangle \cdot \lambda}$ is an isomorphism.

With all of this said we see that the elements of Π that project to right-handed Dehn twists in Map_g^1 are all contained in the subsemigroup $\Pi(-) \subset \Pi$ consisting of all elements in Π with a non-negative displacement angle. In fact the matrices $u \in Sp(2g,\mathbb{R})$ with $\mathrm{rank}(u - \mathrm{id}) = 1$ and such that u belongs to the right-handed conjugacy class lift to elements in $\widetilde{Sp}(2g,\mathbb{R})$ which generate a subsemigroup $\widetilde{Sp}(2g,\mathbb{R})(-) \subset \widetilde{Sp}(2g,\mathbb{R})$. By construction the semigroup $\widetilde{Sp}(2g,\mathbb{R})(-)$ consists of group elements $a \in \Pi$ satisfying $\widetilde{\det}(a \cdot x) \geq \widetilde{\det}(x)$ for all $x \in \widetilde{\Lambda}$. This semigroup contains a discrete subsemigroup corresponding to the elements $\widetilde{Sp}(2g,\mathbb{Z})$ as well as the image of $\Pi(-)$. The geometry of the semigroup $\widetilde{Sp}(2g,\mathbb{R})(-)$ played a crucial role in our analysis of the genus one case and we expect it to play an extremely important role in general.

Appendix A (by Ivan Smith) Torelli fibrations

The purpose of this Appendix is to present a short proof of Theorem 5.8.

Suppose $f : X \to S^2$ is a SLF whose monodromy is contained in the Torelli group. The SLF f induces a sphere $S \cong S^2$ in the compactified moduli space of genus g curves. Indeed let us choose an almost complex structure on X compatible with the symplectic form. The restriction of the almost complex structure on each smooth fiber is integrable and so one obtains a map u of a punctured sphere into $\mathcal{M}_g$. By assumption the map f has a local complex model near the singular points, and it is easy to see that this gives an integrable almost complex structure in an entire neighborhood of the singular fibers. Thus we can extend u smoothly to a map of the closed sphere into the compactified moduli space. The isotopy class of this map is independent of the choices of almost complex structures on X and in the neighborhoods of the singular fibers.

Let $\mathcal{L} \to \mathcal{M}_g$ be the Hodge line bundle and let $c_1(\mathcal{L}) \in CH^1(\overline{\mathcal{M}}_g, \mathbb{Q})$ denote the natural extension of the first Chern class of $\mathcal{L}$ to the Deligne-Mumford compactification of $\mathcal{M}_g$. We will need the following two preliminary lemmas:

Lemma A.1. *For any symplectic Lefschetz fibration,*

$$\mathrm{sign}(X) = 4\langle c_1(\mathcal{L}), [S]\rangle - \mu,$$

where μ is the number of the singular fibers of f and $\operatorname{sign}(X)$ *is the signature of X.*

Proof. See [30]. q.e.d.

Lemma A.2. *For a fibration with only separating vanishing cycles,* $\operatorname{sign}(X) = -\mu$.

Proof. See [24]. q.e.d.

As a consequence of the above two lemmas we get that for a symplectic Lefschetz fibration with monodromy group contained in the Torelli group:

$$\langle c_1(\mathcal{L}), [S] \rangle = 0.$$

Therefore Theorem 5.8 will be proven if we know the following:

Lemma A.3. *Let X be a symplectic Lefschetz fibration with monodromy group contained in the Torelli group. Then:*

$$\langle c_1(\mathcal{L}), [S] \rangle > 0.$$

Remark A.4. We need base S^2 for this entire argument. There are fibrations over T^2 with monodromy group in the Torelli group and no singular fibres.

Proof. Let n denote the number of exceptional (-1)-spheres in the space X and let $\varepsilon : X \to X_{\min}$ denote the contraction of all the (-1) spheres. Clearly since each new exceptional sphere contributes a homology class, we know that n is bounded above by the second Betti number $b_2(X)$. In fact without a loss of generality we may assume that $n < b_2(X)$ since any symplectic manifold satisfies $b_2(X)^+ > 0$.

Furthermore we may assume that $X_{\min}$ is not symplectomorphic to an irrational ruled surface. Indeed if $\pi : X_{\min} \to C$ is a sphere bundle over a surface C of genus ≥ 1, then $H_1(C, \mathbb{Z}) = H_1(X_{\min}, \mathbb{Z}) = H_1(X, \mathbb{Z})$. On the other hand since the geometric monodromy of f is contained in the Torelli group it follows that $H_1(X_s, \mathbb{Z}) = H_1(X, \mathbb{Z})$ for all smooth fibers $X_s \subset X$ of f. In particular $g(C) = g(X_s) = g$ and so $\pi_{|X_s} : X_s \to C$ must be a homotopy equivalence for all s, i.e., the fibration f must have a trivial monodromy, which is a case we exclude.

Since X is a symplectic Lefschetz fibration of fiber genus g one has $c_2(X) = 4 - 4g + \mu$. Also observe that $c_2(X) = \operatorname{euler}(X) = 2 - 2b_1(X) + b_2(X)$ and $b_1(X) = 2g$ since all vanishing cycles are by assumption null-homologous. So we get the estimate:

$$n < b_2(X) = 2 + \mu,$$

Now $c_1^2(X)+c_2(X) = c_1^2(X_{\min})-n+c_2(X)$ and using the above estimate on n, we see that

$$c_1^2(X) + c_2(X) > c_1^2(X_{\min}) + 2 - 4g \geq 2 - 4g.$$

The last inequality follows from a powerful theorem of A.K.Liu [17] asserting that for any *minimal* symplectic four-manifold Y which is not irrational ruled one has $c_1^2(Y) \geq 0$.

On the other hand the signature formula in Lemma A.1 gives

$$\begin{aligned}\frac{1}{12}[c_1^2(X) + c_2(X)] &= \frac{1}{4}[\operatorname{sign}(X) + c_2(X)] \\ &= \frac{1}{4}[4\langle c_1(\mathcal{L}), [S]\rangle - \mu + 4(1-g) + \mu] \\ &= \langle c_1(\mathcal{L}), [S]\rangle + 1 - g\end{aligned}$$

and hence

$$\langle c_1(\mathcal{L}), [S]\rangle > \frac{4g-5}{6},$$

i.e., $\langle c_1(\mathcal{L}), [S]\rangle > 0$ as long as $g > 1$. This proves the lemma since the case $g = 1$ is clear. q.e.d.

As a consequence of Theorem 5.8 we get

Lemma A.5. *Let X be an arbitrary symplectic Lefshetz fibration. Then*

$$\operatorname{sign}(X) + \mu > 0.$$

Proof. As it follows from Theorem 5.8 there always exists a vanishing cycle non-homologous to zero. By applying the local signature formula of [24] we see that the contribution of this singular fiber to $\operatorname{sign}(X)$ is either zero or one. The latter implies the lemma. q.e.d.

As corollary we get:

Corollary A.6. *Let X be an arbitrary symplectic Lefshetz fibration. Then:*

$$\langle c_1(\mathcal{L}), [S]\rangle > 0.$$

Remark A.7. The above lemmas puts many restrictions on the possible monodromies for hyperelliptic families (see [24]).

Remark A.8. The proof of Theorem 5.8 is related to a question of Gompf who asked whether $c_2(X)$ is positive for symplectic four-manifolds which are not irrational ruled. In particular it is an interesting question if the minimal number of the singular fibers in a SLF is at least $4(g-1)$ if X is not irrational ruled symplectic four-manifold.

IVAN SMITH, NEW COLLEGE, OXFORD UNIVERSITY

References

[1] D. Auroux, *Géométrie approximativement holomorphe des variétés symplectiques compactes,* École Polytechnique Thesis, 1998.

[2] J. Birman, *Mapping class groups of surfaces*, Braids (Santa Cruz, CA, 1986), Contemp. Math., Amer. Math. Soc., Providence, RI, Vol. 78, 1988, 13–43.

[3] F. Bogomolov & L. Katzarkov, *Symplectic fourmanifolds and projective surfaces,* Topology Appl. **88** (1998) 79–110.

[4] S. Cappell, R. Lee & E. Miller, *On the Maslov index*, Comm. Pure Appl. Math. **47** (1994) 121–186.

[5] S. Donaldson, *Lefschetz fibrations in symplectic geometry*, Doc. Math. J. DMV. Extra Volume ICMII (1998) 309–314.

[6] R. Friedman & J. W. Morgan, *Smooth four-manifolds and complex surfaces*, Springer, Berlin, 1994.

[7] S. Gervais, *Presentation and central extensions of mapping class groups*, Trans. Amer. Math. Soc. **348** (1996) 3097–3132.

[8] R. Gompf & A. Stipsicz, *4-manifolds and Kirby calculus*, Amer. Math. Soc., 1999.

[9] R. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. **142** (1995) 527–595.

[10] R. Hain & E. Looijenga, *Mapping class groups and moduli spaces of curves*, Algebraic geom.—Santa Cruz 1995, Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, Vol. 62, 1997, 97–142.

[11] R. Hain & D Reed, *Geometric proofs of some results of Morita*, e-print: math.AG/9810054.

[12] J. Harris, *Families of smooth curves,* Duke Math. J. **51** (1984) 409–419.

[13] N. Ivanov & J. D. McCarthy, *On injective homomorphisms between Teichmüler modular groups*, Preprint, available `http://www.math.msu.edu/ivanov/i.ps`, 1995.

[14] A. Kas, *On the handlebody decomposition associated to a Lefschetz fibration*, Pacific J. Math. **89** (1980) 89–104.

[15] L. Katzarkov, T. Pantev & S. Simpson, *Non-abelian open orbit theorems*, Preprint, 1998.

[16] G. Lion & M. Vergne, *The Weil representation, Maslov index and theta series*, Progr. Math., Birkhäuser, Boston, MA, 6 (1980).

[17] A. K. Liu, *Some new applications of general wall crossing formula, Gompf's conjecture and its applications*, Math. Res. Lett. **3** (1996) 569–585.

[18] D. McDuff & D. Salamon, *Introduction to symplectic topology*, 2nd ed., Oxford University Press, 1998.

[19] B. Moishezon, *Complex surfaces and connected sums of complex projective planes*, Lecture Notes in Math., Springer, Vol. 603, 1977.

[20] S. Morita, *Characteristic classes of surface bundles and bounded cohomology,* A fête of topology, Academic Press, Boston, MA, 1988, 233–257.

[21] ———, *Families of Jacobian manifolds and characteristic classes of surface bundles. II,* Math. Proc. Cambridge Philos. Soc. **105** (1989) 79–101.

[22] ———, *The structure of the mapping class group and characteristic classes of surface bundles*, Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), Contemp. Math., Amer. Math. Soc., Providence, RI, Vol. 150, 1993, 303–315.

[23] D. Mumford, *Picard groups of moduli problems*, Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963), pp. 33–81. Harper & Row, New York, 1965.

[24] B. Ozbagci, *Signatures of Lefschetz fibrations*, e-print: math.GT/9809178.

[25] B. Ozbagci & A. Stipsicz, *Noncomplex smooth 4-manifolds with genus-2 Lefschetz fibrations*, e-print: math.GT/9810007.

[26] J. Powell, *Two theorems on the mapping class group of a surface*, Proc. Amer. Math. Soc. **68** (1978) 347–350.

[27] *Groupes de monodromie en géométrie algébrique. II,* 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Dirigé par P. Deligne et N. Katz, Lecture Notes in Math., Vol. 340.

[28] I. Smith, *Lefschetz fibrations of genus two and sphere bundles over spheres*, in preparation.

[29] ———, *Lefschetz fibrations of odd genus and torus bundles over tori*, in preparation.

[30] ———, PhD thesis, Oxford University, 1998.

[31] L. Szpiro, *Discriminant et conducteur des courbes elliptiques*, Astérisque **183** (1990) 7–18. Séminaire sur les Pinceaux de Courbes Elliptiques (Paris, 1988).

[32] B. Wajnryb, *A simple presentation for the mapping class group of an orientable surface*, Israel J. Math. **45** (1983) 157–174.

IUPC, Barcelona, Spain
Courant Institute, New York University
University of California, Irvine
University of Pennsylvania, Philadelphia