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CONSTANT MEAN CURVATURE SURFACES IN SUB-RIEMANNIAN GEOMETRY

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Abstract

We investigate the minimal and isoperimetric surface problems in a large class of sub-Riemannian manifolds, the so-called Vertically Rigid spaces. We construct an adapted connection for such spaces and, using the variational tools of Bryant, Griffiths and Grossman, derive succinct forms of the Euler-Lagrange equations for critical points for the associated variational problems. Using the Euler-Lagrange equations, we show that minimal and isoperimetric surfaces satisfy a constant horizontal mean curvature conditions away from characteristic points. Moreover, we use the formalism to construct a horizontal second fundamental form, II_0, for vertically rigid spaces and, as a first application, use II_0 to show that minimal surfaces cannot have points of horizontal positive curvature and that minimal surfaces in Carnot groups cannot be locally strictly horizontally geometrically convex. We note that the convexity condition is distinct from others currently in the literature.

1. Introduction

Motivated by the classical problems of finding surfaces of least area among those that share a fixed boundary (the minimal surface problem) and surfaces of least area enclosing a fixed volume (the isoperimetric problem), several authors have recently formulated and investigated similar problems in the setting of sub-Riemannian or Carnot-Carathédory spaces. In particular, N. Garofalo and D.M. Nheiu in [**15**] laid the foundations of the theory of minimal surfaces in Carnot-Carathéodory spaces and provided many of the variational tools necessary to make sense of such a problem. Building on this foundation, Danielli, Garofalo and Nhieu, [**9**, **10**], investigated aspects of minimal and constant mean curvature surfaces in Carnot groups. Among many other results, these authors showed the existence of isoperimetric sets, and that, when considering the isoperimetric problem in the Heisenberg groups, if one restricts to the set of surfaces which are the union of

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two graphs over a ball, then the minimizers satisfy an analogue of the constant mean curvature equation. In this setting, the authors identify the absolute minimizer bounding a fixed volume and show that it is precisely the surface that Pansu conjectured to be the solution to the isoperimetric problem [**22**]. Further in this direction of the isoperimetric problem, Leonardi and Rigot [**20**] independently showed the existence of isoperimetric sets in any Carnot group and investigated some of their properties. Leonardi and Masnou, [**19**], investigated the geometry of the isoperimetric minimizers in the Heisenberg groups and also showed a more restricted version of the result in [**10**] showing that among sets with a cylindrical symmetry, Pansu's set is the isoperimetric minimizer.

In addition to this more general work in Carnot groups and Carnot-Carathéodory spaces, a great deal of work has been done on the minimal surface problem in more specialized settings. For example, the second author, in [**23**], showed a connection between Riemannian minimal graphs in the Heisenberg group and those in the Carnot Heisenberg group and used this connection to prove $W^{1,p}$ estimates for solutions to the minimal surface equation. In addition, he found a number of initial examples of minimal surfaces in the Heisenberg group and used them to demonstrate non-uniqueness of the solution to the Dirichlet problem in the Heisenberg group. Recently, both Garofalo and the second author, [**16**], and Cheng, Huang, Malchiodi and Yang, [**7**], independently investigated minimal surfaces in some limited settings. In [**7**], the authors investigate C^2 minimal surfaces in three dimensional pseudohermitian geometries (including, of course, the Heisenberg group) and, using the techniques of CR geometry, investigate the structure of minimal surfaces in this setting and, among many results, prove, under suitable conditions, a uniqueness result for the Dirichlet problem for minimal surfaces in the Heisenberg group and classify the complete minimal graphs over the xy-plane. Garofalo and the second author restricted their view to the Heisenberg group and provided, among other results, a representation theorem for smooth minimal surfaces, a horizontal regularity theorem and proved an analogue of the Bernstein theorem, showing that a minimal surface in the Heisenberg group that is a graph over some plane satisfies a type of constant curvature condition. This theorem of Bernstein type also leads to the classification of the complete minimal graphs over the xy-plane. Cheng and Hwang, [**6**], classified all properly embedded minimal surfaces in $\mathbb{H}$ of helicoid-type. In [**24**], the second author extends the representation theorem of [**16**] to minimal surfaces in $\mathbb{H}^1$ with lower regularity, provides examples of continuous (but not smooth) minimal surfaces, and shows a geometric obstruction to the existence of smooth minimal solutions to the Plateau Problem in the Heisenberg group. In [**8**], Cole examines smooth minimal surfaces in spaces of Martinet-type. While this collection of spaces includes the

Heisenberg group and many of those considered in [**7**], Cole's thesis also treats three dimensional spaces that do not have equiregular horizontal subbundles. In [**8**], Cole derives the minimal surface equation and examines the geometry and existence of smooth solutions to the Plateau Problem.

While there has been great progress in the understanding of minimal and constant mean curvature surfaces in the setting of Carnot-Carathéodory spaces, there are still many fundamental open questions left to address. Most notably, much of the focus has been on the minimal surface equation and the majority of the work has focused on more limited settings such as the Heisenberg group, groups of Heisenberg type or three dimensional pseudohermitian manifolds. In this paper, we will address more general problems using a new tool to help discriminate between the various types of constant mean curvature surfaces that abound in different Carnot-Carathéodory spaces.

Question: *In a Carnot-Carathéodory manifold M, do the surfaces of least perimeter or the surfaces of least perimeter enclosing a fixed volume satisfy any partial differential equations? Can the solutions be characterized geometrically?*

In Euclidean space, there is a beautiful connection between the geometry of surfaces and the solutions to these variational problems: minimal surfaces are characterized as zero mean curvature surfaces while isoperimetric surfaces have constant mean curvature.

In this paper, we restrict ourselves to a large class of sub-Riemannian manifolds which we call vertically rigid sub-Riemannian (VR) spaces. These spaces are defined in Section 2 and include basically all examples already studied (including Carnot groups, Martinet-type spaces and pseudohermitian manifolds) but are a much larger class. On such spaces, we define a new connection, motivated by the Webster-Tanaka connection of strictly pseudoconvex pseudohermitian manifolds, that is adapted to the sub-Riemannian structure. Using this connection and the variational framework of Bryant, Griffiths and Grossman [**3**], we investigate minimal and isoperimetric surface problems. The framework of [**3**] provides a particularly nice form of the Euler-Lagrange equations for these problems and leads us to define a *horizontal second fundamental form*, II_0, and the *horizontal mean curvature*, $\text{Trace}(II_0)$, associated to a hypersurface in a Carnot group. Given a noncharacteristic submanifold Σ of a VR space M, let $\{e_1, \ldots, e_k\}$ be an orthonormal basis for the horizontal portion of the tangent space to Σ and let e_0 be the unit horizontal normal to Σ (see the next section for precise definitions).

Then, we define the horizontal second fundamental form as

$$II_0 = \begin{pmatrix} \langle \nabla_{e_1} e_0, e_1 \rangle & \dots & \langle \nabla_{e_1} e_0, e_k \rangle \\ \vdots & \vdots & \vdots \\ \langle \nabla_{e_k} e_0, e_1 \rangle & \dots & \langle \nabla_{e_k} e_0, e_k \rangle \end{pmatrix}$$

and define the horizontal mean curvature as the trace of II_0. We note that the notion of horizontal mean curvature has appeared in several contexts (see [**2**, **7**, **9**, **10**, **11**, **16**, **23**]), and that this notion coincides with the others, possibly up to a constant multiple. However, we emphasize that the version of the mean curvature above applies to all VR spaces (before this work, only [**9**, **10**] deals with mean curvature in any generality, but again is limited to Carnot groups) and has the advantage of being written in an invariant way with respect to the fixed surface. With this notion in place, we have a characterization of C^2 solutions to the two variational problems discussed above:

Theorem 1.1. *Let M be a vertically rigid sub-Riemannian manifold and Σ a noncharacteristic C^2 hypersurface. Σ is a critical point of the first variation of perimeter if and only if the horizontal mean curvature of S vanishes.*

Similarly,

Theorem 1.2. *Let M be a vertically rigid sub-Riemannian space and Σ a C^2 hypersurface. If Σ is a solution to the isoperimetric problem, then the horizontal mean curvature of Σ is locally constant.*

Thus, we recover an analogue of the classical situation: the solutions to these two problems are found among the critical points of the associated variational problems. Moreover, these critical points are characterized by having the trace of the second fundamental form be constant.

We note that the characterization of minimal surfaces in terms of both a PDE and in terms of mean curvature was achieved first by Danielli, Garofalo and Nhieu, [**9**], in Carnot groups and by the second author, [**23**], for minimal graphs in the Heisenberg group. The technique described above provides a broad extension of this characterization and describes mean curvature in a geometrically motivated manner. From this point of view, this is most similar to the treatment of mean curvature by Cheng, Huang, Malchiodi and Yang, [**7**], who use the Webster-Tanaka connection to investigate the minimal/CMC surface problem. In contrast, some of the earlier definitions of mean curvature relied on the minimal surfaces equation for the definition (as in [**23**]) or via a different geometric analogue such as a symmetrized horizontal Hessian (as in [**11**]).

On the other hand, for isoperimetric surfaces, the only known links between isoperimetric sets and constant mean curvature are under certain restrictions on the class of sets in the first Heisenberg group ([**9**,

19]), for C^2 surfaces in 3-dimensional pseudohermitian manifolds [**7**] and for C^2 surfaces in the first Heisenberg group using mean curvature flow methods due to Bonk and Capogna [**2**] and, recently, for C^2 surfaces in all Heisenberg groups due to Ritoré and Rosales [**25**]. Thus, our treatment of these problems unifies these results and extends them to a much larger class of sub-Riemannian manifolds. Moreover, we provide a number of new techniques and tools for investigating these problems in a very general setting.

As mentioned above, the geometric structure of minimal surfaces has only been studied in cases such as the Heisenberg group, pseudohermitian manifolds, and Martinet-type spaces. In general, even in the higher Heisenberg groups, nothing is known about the structure and geometry of minimal surfaces. As an illustration of the power of this framework, we use the horizontal second fundamental form to provide some geometric information about minimal surfaces in any VR space. To better describe minimal surfaces, we introduce some new notions of curvature in VR spaces:

Definition 1.3. Let II_0 be the horizontal second fundamental form for a C^2 noncharacteristic surface, Σ, in a vertically rigid sub-Riemannian manifold M. Let $\{\mu_1, \dots, \mu_k\}$ be the eigenvalues (perhaps complex and with multiplicity) of II_0. Then, the **horizontal principle curvatures** are given by

$$\kappa_i = \mathrm{Re}\,(\mu_i)$$

for $1 \leq i \leq k$.

Moreover, given $x \in \Sigma$, we say that Σ is **horizontally positively (non-negatively) curved** at x if II_0 is either positive (semi-)definite or negative (semi-) definite at x, and is **horizontally negatively curved** at x if there is at least one positive and one negative κ_i. Σ is **horizontally flat** at x if $\kappa_i = 0$ for $1 \leq i \leq k$.

Let Σ be a C^2 hypersurface in M, a vertically rigid sub-Riemannian manifold. Then, the **horizontal exponential surface** at $x \in \Sigma$, $\Sigma_0(x)$, is defined to be the union of all the horizontal curves in Σ passing through x. The notion of horizontal principle curvatures described above gives rise to a new definition of convexity:

Definition 1.4. A subset U of a Carnot group M with C^2 boundary Σ is **(strictly) horizontally geometrically convex**, or (strictly) hg-convex, if, at each noncharacteristic point $x \in \Sigma$, $\Sigma_0(x)$ lies (strictly) to one side of $T^h_x\Sigma$, the horizontal tangent plane to Σ at x. We say that Σ is locally (strictly) hg-convex at x if there exists an $\epsilon > 0$ so that $\Sigma_0(x) \cap B(x, \epsilon)$ lies (strictly) to one side of $T^h_x\Sigma$.

We note that this notion of convexity is distinct from those described in [**11**] or [**21**]. In Section 6, we give explicit examples showing the nonequivalence of the various notions.

With these definitions in place, we prove an analogue to the classical statement that a minimal surface in $\mathbb{R}^3$ must be nonpositively curved.

Theorem 1.5. *Let Σ be a C^2 noncharacteristic minimal hypersurface in a vertically rigid sub-Riemannian space M. Then, Σ cannot contain a point of horizontal positive curvature. If we further assume that M is a Carnot group, then Σ cannot be locally strictly horizontally geometrically convex.*

We emphasize that this is the first description of the geometry of minimal surfaces in a relatively general class of spaces.

The rest of the paper is divided into five sections. In Section 2, we define vertically rigid sub-Riemannian spaces, the adapted connection we mentioned above, and an adapted frame bundle for such objects. In Section 3, we briefly review the relevant machinery from [**3**]. In Section 4, we address the minimal surface problem using the machinery of Bryant, Griffiths and Grossman. Section 5 addresses the isoperimetric problem in this setting, and finally, in Section 6, we define the horizontal second fundamental form and prove the geometric properties of minimal surfaces described above.

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2. Vertically rigid sub-Riemannian manifolds

We begin with our basic definitions:

Definition 2.1. A **sub-Riemannian (or Carnot-Carathéodory) manifold** is a triple $(M, V_0, \langle\cdot,\cdot\rangle)$ consisting of a smooth manifold M^{n+1}, a smooth $k+1$-dimensional distribution $V_0 \subset TM$ and a smooth inner product on V_0. This structure is endowed with a metric structure given by

$$d_{cc}(x, y) = \inf \left\{ \int \langle \dot\gamma, \dot\gamma\rangle^{\frac{1}{2}} | \gamma(0) = x, \gamma(1) = y, \gamma \in \mathscr{A} \right\}$$

where $\mathscr{A}$ is the space of all absolutely continuous paths whose derivatives, when they are defined, lie in V_0.

Definition 2.2. A sub-Riemannian manifold has a **vertically rigid complement** if there exist

- a smooth complement V to V_0 in TM,
- a smooth frame $T_1, \dots T_{n-k}$ for V,
- a Riemannian metric g such that V and V_0 are orthogonal, g agrees with $\langle\cdot,\cdot\rangle$ on V_0 and $T_1, \dots T_{n-k}$ are orthonormal,
- a partition of $\{1, \dots, n-k\}$ into equivalence classes such that for all sections $X \in \Gamma(V_0)$, $g([X, T_j], T_i) = 0$ if $j \sim i$.

A sub-Riemannian space with a vertically rigid complement is called a **vertically rigid (VR) space**.

Remark 2.3. The final assumption is the only one to impose any real constraints on the sub-Riemannian structure. Its value is in that it causes an important piece of torsion to vanish, specifically the second part of Lemma 2.16, which simplifies computation. As we shall soon see, it is present in most examples currently seen in the literature.

For a VR space, we shall denote the number of equivalence classes of the partition by v and the size of the partitions by $l_1, \dots l_v$. In particular, we then have $l_1 + \cdots + l_v = n - k$. After choosing an order for the partitions, for $j > 0$ we set

$$V_j = \operatorname{span}\{T_i : \ i \text{ is in the } j\text{th partition}\}.$$

Then

$$TM = \bigoplus_{j=0}^{v} V_j.$$

After reordering we can always assume that the vector fields $T_1, \dots, T_{l_1}$ span V_1, the next l_2 span V_2 etc.

There are 3 motivating examples for this definition:

Example 2.4. Let (M, θ, J) be a strictly pseudoconvex pseudohermitian structure (see [**26**]). Then $V_0 = \ker\theta$ has codimension 1 and a vertically rigid structure can be defined by letting T_1 be the characteristic (Reeb) vector field of θ and defining g to be the Levi metric

$$g(X, Y) = d\theta(X, JY) + \theta(X)\theta(Y).$$

Since T_1 is dual to θ and $T_1 \lrcorner d\theta = 0$ the required commutation property clearly holds.

Example 2.5. Let $(M, \mathfrak{v}_0)$ be a graded Carnot group with step size r. Then the Lie algebra of left-invariant vector fields of M decomposes as

$$\mathfrak{m} = \bigoplus_{j=0}^{r} \mathfrak{v}_j$$

where $\mathfrak{v}_{j+1} = [\mathfrak{v}_0, \mathfrak{v}_j]$ for $j < r$ and $[\mathfrak{v}_0, \mathfrak{v}_r] = 0$. We then set $V_j = \operatorname{span}(\mathfrak{v}_j)$ and construct a (global) frame of left invariant vector fields for each $\mathfrak{v}_j$, $j > 0$. If T is a left invariant section of V_j and X any horizontal vector field, then at every point

$$[T, X] \in V_0 \oplus V_{j+1}$$

by the defining properties of graded Carnot groups (with $V_{r+1} = 0$). Thus, the final rigidity assumption is satisfied.

Example 2.6. We now give two explicit examples of Carnot groups to more concretely illustrate the previous example. In addition, in later sections, we will construct surfaces in each of these Carnot groups as examples of different phenomena.

One of the simplest and most basic Carnot groups is the first Heisenberg group. We can identify this group, denoted $\mathbb{H}$, with $\mathbb{R}^3$, using coordinates (x,y,t). Letting $X_1 = \partial_x - \frac{y}{2}\partial_t, X_2 = \partial_y + \frac{x}{2}\partial_t, T = \partial_t$, we have a presentation of the sub-Riemannian Heisenberg group with $V_0 = \operatorname{span}\{X_1, X_2\}$. Noting that the only nontrivial bracket among these vector fields is $[X_1, X_2] = T$ and letting $V_1 = \operatorname{span}\{T\}$, we have a vertically rigid structure on $\mathbb{H}$.

Another simple example is the Carnot group $\mathbb{H}\times\mathbb{R}$. Using coordinates (x,y,t,s) we can write a presentation for the Lie algebra of this group as spanned by $\{X_1, X_2, X_3, T\}$ where

$$\begin{aligned} X_1 &= \partial_x - \frac{y}{2}\partial_t \\ X_2 &= \partial_y + \frac{x}{2}\partial_t \\ X_3 &= \partial_s \\ X_4 &= \partial_t. \end{aligned}$$

Taking $V_0 = \operatorname{span}\{X_1, X_2, X_3\}$ and $V_1 = \operatorname{span}\{X_4\}$ yields a vertically rigid structure.

In the previous examples, the vertical structure was chosen to carefully mimic the bracket-generating properties of the sub-Riemannian distribution. We include another example, where the bracket-generation step size need not be constant to illustrate the flexibility of this definition.

Example 2.7. Let $M = \mathbb{R}^3$. We define a Martinet-type sub-Riemannian structure on M by defining V_0 to be the span of

$$X = \partial_x + f(x,y)\partial_z, \qquad Y = \partial_y + g(x,y)\partial_z$$

where f and g are smooth functions. The metric is defined by declaring X, Y an orthonormal frame for V_0. (In particular, if we take $f = 0$ and $g = x^2$, we see that the step size is 1 on $x \neq 0$ and 2 at $x = 0$.) Now define $T_1 = \partial_z$ and extend the metric so that X, Y, T_1 are orthonormal. Again the commutation condition clearly holds.

To illustrate the generality of the definition, we give one last example.

Example 2.8. Let $\{X_1, \ldots, X_k\}$ be a collection of smooth vector fields on $\mathbb{R}^n$ that satisfy Hörmander's condition (see [**17**]). We will construct the $\{T_i\}$ as follows. As the X_i bracket generate, let $\{T_i\}$ be a basis for the complement of the span of the $\{X_i\}$ formed by differences of the brackets of the X_i and linear combinations of the X_i themselves. These T_i are naturally graded by counting the number of brackets of $X_i's$ it takes to include T_i in the span. Define a Riemannian inner product that makes the $\{X_1, \ldots, X_k, T_1, \ldots, T_{n-k}\}$ an orthonormal basis. This structure satisfies all the conditions for a vertically rigid structure except possibly the last.

We note that the majority of the examples in the literature, either from subelliptic PDE, control theory and/or robotic path planning, satisfy the last condition.

The advantage of vertically rigid structures is that they admit connections which are adapted to analysis in the purely horizontal directions.

Definition 2.9. A connection ∇ on TM is adapted to a vertically rigid structure if

- ∇ is compatible with g, i.e., $\nabla g = 0$,
- $\nabla T_j = 0$ for all j,
- $\mathrm{Tor}_p(X, Y) \in V_p$ for all sections X, Y of V_0 and $p \in M$.

The motivating example for this definition is the Webster-Tanaka connection for a strictly pseudoconvex pseudohermitian manifold [**26**].

Lemma 2.10. *Every vertically rigid structure admits an adapted connection.*

Proof. Let $\overline{\nabla}$ denote the Levi-Cevita connection for g. Define ∇ as follows: set $\nabla T_j = 0$ for all j. Then for a section X of V_0 and any vector field Z define

$$\nabla_Z X = (\overline{\nabla}_Z X)_0$$

where $(\cdot)_0$ denotes the orthogonal projection onto V_0. This essentially defines all the Christoffel symbols for the connection. It is easy to see that it satisfies all the required conditions. For example, to show $\nabla g = 0$, take vector fields X, Y, Z and write

$$X = X_0 + \sum x_i T_i$$
$$Y = Y_0 + \sum y_i T_i$$

where $X_0, Y_0 \in V_0$. Using the fact that $\nabla T_i = 0$, we have

$$\begin{aligned}
&\nabla g(X, Y, Z) \\
&= Zg(X, Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) \\
&= Zg(X_0, Y_0) + \sum Z(x_i y_i) - g((\overline{\nabla}_Z X_0)_0, Y_0) \\
&\quad - g(\sum Z x_i, Y) - g(X_0, (\overline{\nabla}_Z Y_0)_0) - g(\sum Z y_i, X) \\
&= g(\overline{\nabla}_Z X_0, Y_0) - g((\overline{\nabla}_Z X_0)_0, Y_0) + g(X_0, \overline{\nabla}_Z Y_0) \\
&\quad - g(X_0, (\overline{\nabla}_Z Y_0)_0) + \sum Z(x_i y_i) - \sum Z(x_i) y_i - \sum Z(y_i) x_i \\
&= 0.
\end{aligned}$$

The last equality follows since X_0 and Y_0 are horizontal vector fields and using the product rule. The statement about torsion follows directly from the definition. q.e.d.

Lemma 2.11. *If X, Y, Z are horizontal vector fields then*

$$\begin{aligned}\langle \nabla_X Y, Z\rangle = \frac{1}{2}\big(&X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle\\ &+ \langle [X, Y]_0, Z\rangle + \langle [Z, X]_0, Y\rangle + \langle [Z, Y]_0, X\rangle\big).\end{aligned}$$

Proof. We note that since $\nabla g = 0$ and V_0 is parallel,

$$\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle - \langle Y, \nabla_X Z\rangle.$$

Since the torsion of two horizontal vector fields is purely vertical, we also obtain

$$\begin{aligned}\langle \nabla_X Y, Z\rangle &= \langle \nabla_Y X, Z\rangle + g([X, Y], Z)\\ &= \langle \nabla_Y X, Z\rangle + \langle [X, Y]_0, Z\rangle.\end{aligned}$$

The remainder of the proof is identical to the standard treatment of the Levi-Cevita connection on a Riemannian manifold given in [**5**]. q.e.d.

As a consequence, the following corollary follows immediately:

Corollary 2.12. *For distinct VR structures on a subRiemannian manifold that share a vertical complement V and any connections ∇^1, ∇^2 adapted to these VR structures,*

$$\langle \nabla^1_X Y, Z\rangle = \langle \nabla^2_X Y, Z\rangle$$

for any horizontal vector fields X, Y and Z.

In other words, up to a choice of vertical complement, any adapted connection agrees with any other adapted connection when restricted to horizontal vector fields.

Remark 2.13. We note that the definition of adapted connection leaves some flexibility in its definition. In particular, we have some freedom in defining Christoffel symbols related to $\nabla_{T_i} X$ when X is a section of V_0. While we could make choices that would fix a unique adapted connection, we will not do so in order to preserve the maximum flexibility for applications. For instance, when working with strictly pseudoconvex pseudohermitian manifolds, it is natural to specify these Christoffel symbols to obtain the Webster-Tanaka connection. When working in general VR spaces, there are no preexisting adapted connections in the literature, so it is natural to choose the Christoffel symbols to eliminate horizontal components of torsion, i.e., take $\nabla_{T_i} X = [T_i, X]_0$ so that $\mathrm{Tor}\,(T_i, X)_0 = 0$. However, we shall not include this as part of our definition as it does not (in general) agree with the Webster-Tanaka connection on strictly pseudoconvex pseudohermitian manifolds.

To study these connections and sub-Riemannian geometry it is useful to introduce the idea of the graded frame bundle.

Definition 2.14. An orthonormal frame $(e,t) = e_0,\dots,e_k,t_1,\dots,t_{n-k}$ is graded if $e_0,\dots,e_k$ span V_0, $t_1,\dots,t_{n-k}$ span V and each t_j is in the span of $\{T_i : i \sim j\}$.

The bundle of graded orthonormal frames $\mathcal{GF}(M) \xrightarrow{\ \pi\ } M$ is then a $O(k+1)\times \Pi_{j=1}^{v} O(l_j)$-principle bundle.

On $\mathcal{GF}(M)$ we can introduce the canonical 1-forms ω^j, η^j defined at a point $f = (p,e,t)$ by

$$\omega^j(X)_f = g_p(\pi_* X, e_j), \quad \eta^j(X)_f = g_p(\pi_* X, t_j).$$

An adapted connection can be viewed as a affine connection on $\mathcal{GF}(M)$. The structure equations are then determined by the following lemma.

Lemma 2.15. *On $\mathcal{GF}(M)$ there exist connection 1-forms ω^i_j $0 \le i,j \le k$ and η^i_j, $1 \le i,j \le n-k$ together with torsion 2-forms τ^i $0 \le i \le k$ and $\tilde{\tau}^i$ $1 \le i \le n-k$ such that*

$$\begin{aligned} d\omega^i &= \sum_{0\le j\le k} \omega^j \wedge \omega^i_j + \tau^i, \\ d\eta^i &= \sum_{i\sim j} \eta^j \wedge \eta^i_j + \tilde{\tau}^i. \end{aligned} \tag{1}$$

Proof. The content of the lemma is in the terms that do not show up from the standard structure equations of an affine connection. However, since V_0 is parallel, we can immediately deduce that there exist forms ω^j_k such that $\nabla e_k = \omega^j_k \otimes e_j$. Furthermore, since each t_j is in the span of $\{T_i : i \sim j\}$ and all the T_i are also parallel, we must have $\nabla t_j = \sum_{i\sim j} \eta^i_j \otimes t_i$ for some collections of forms η^i_j. q.e.d.

Lemma 2.16. *The torsion forms for an adapted connection have the following properties:*

- $\tau^j(e_a, e_b) = 0$,
- $\tilde{\tau}^j(t_i, e_b) = 0$ *if* $j \sim i$

for any lifts of the vector fields.

Proof. The first of these is a direct rewrite of the defining torsion condition for an adapted connection. For the second, we observe that

$$\mathrm{Tor}(T_i, e_b) = \nabla_{T_i} e_b - \nabla_{e_b} T_i - [T_i, e_b]$$

is orthogonal to T_j if $i \sim j$ by the bracket conditions of a vertically rigid structure. The result then follows from noting that torsion is tensorial. q.e.d.

3. Exterior differential systems and variational problems

In this section, we briefly review the basic elements of the formalism of Bryant, Griffiths and Grossman which can be found in more detail in chapter one of [**3**]. Their formalism requires the following data:

(a) A contact manifold (M, θ) of dimension $2n+1$.
(b) An n-form, called the Lagrangian, Λ and the associated area functional
$$\mathscr{F}_\Lambda(N) = \int_N \Lambda,$$
where N is a smooth compact Legendre submanifold of M, possibly with boundary. In this setting, a Legendre manifold is a manifold $i : N \to M$ so that $i^*\theta = 0$.

From this data, they compute the Poincaré-Cartan form Π from the form $d\Lambda$. They show that $d\Lambda$ can be locally expressed as
$$d\Lambda = \theta \wedge (\alpha + d\beta) + d(\theta \wedge \beta)$$
for appropriate forms α, β. Then,
$$\Pi = \theta \wedge (\alpha + d\beta)$$
and they denote $\alpha + d\beta$ by Ψ. With this setup, Bryant, Griffiths and Grossman prove the following characterization of Euler-Lagrange systems ([**3**], Section 1.2):

Theorem 3.1. *Let N be a Legendre surface with boundary ∂N in M given by $i : N \to M$ as above. Then N is a stationary point under all fixed boundary variations, measured with respect to $\mathscr{F}_\Lambda$, if and only if $i^*\Psi = 0$.*

In the next two sections, we will use this formalism and the previous theorem to investigate the minimal and isoperimetric surface problems.

4. Minimal Surfaces

For a C^2 hypersurface Σ of a vertically rigid sub-Riemannian manifold we define the horizontal perimeter of Σ to be

$$P(\Sigma) = \int_\Sigma |(\nu_g)_0| \, \nu_g \lrcorner dV_g \tag{2}$$

where ν_g is the unit normal to Σ with respect to the Riemannian metric g. At noncharacteristic points of Σ, i.e., where $T\Sigma \not\subseteq V_0$, this can be re-written as

$$P(\Sigma) = \int_\Sigma \nu \lrcorner dV_g, \tag{3}$$

where ν is the horizontal unit normal vector, i.e., the projection of the Riemannian normal to V_0. We note that, when restricting to the class of C^2 submanifolds, this definition is equivalent to the perimeter measure introduced independently by Capogna, Danielli and Garofalo [**4**], Franchi, Gallot and Wheeden [**13**] and Biroli and Mosco [**1**]. These definitions are sub-Riemannian generalizations of the perimeter measure of De Giorgi introduced in [**12**]. Our primary goal for this section is to answer the following question.

Question 1. In a vertically rigid sub-Riemannian manifold, given a fixed boundary, can the hypersurfaces spanning the boundary with least perimeter measure be geometrically characterized?

In analogue with classical results in Euclidean space and Riemannian manifolds we introduce a second fundamental form adapted to the subRiemannian structure.

Definition 4.1. Consider a noncharacteristic point of a hypersurface Σ of a VR space M and fix a horizontal orthonormal frame $e_0, \dots e_k$ as before with $\nu = e_0$. Then the **horizontal second fundamental form** is the $k \times k$ matrix,

(4)
$$II_0 = \begin{pmatrix} \langle \nabla_{e_1}\nu, e_1\rangle & \dots & \langle \nabla_{e_1}\nu, e_k\rangle \\ \vdots & \ddots & \vdots \\ \langle \nabla_{e_k}\nu, e_1\rangle & \dots & \langle \nabla_{e_k}\nu, e_k\rangle \end{pmatrix} = \begin{pmatrix} \omega_0^1(e_1) & \dots & \omega_0^k(e_1) \\ \vdots & \ddots & \vdots \\ \omega_0^1(e_k) & \dots & \omega_0^k(e_k) \end{pmatrix}.$$

Further, we define the **horizontal mean curvature**, H, to be the trace of II_0.

A standard formula in Riemannian geometry (see for example [**18**]) states that for any connection for which the volume form is parallel, the divergence of a vector field can be computed by

$$\mathrm{div}_g X = \mathrm{Trace}(\nabla X + \mathrm{Tor}(X, \cdot)).$$

The adapted connection is symmetric for g, and so we can apply this result while noting that by the defining conditions $\mathrm{Trace}\,(\mathrm{Tor}(\nu, \cdot)) = 0$. Thus,

$$\mathrm{div}_g \nu = \sum \langle \nabla_{e_j}\nu, e_j\rangle + \sum g(\nabla_{t_j}\nu, t_j) = \sum \langle \nabla_{e_j}\nu, e_j\rangle. \tag{5}$$

We note that several authors have proposed other candidates for certain types of analogues of the second fundamental form and horizontal mean curvature. In particular, Danielli, Garofalo and Nhieu use a similar second fundamental form to analyze minimal and CMC surfaces in Carnot groups (in [**9**, **10**]) and the symmetrized horizontal Hessian to analyze convex sets (in [**11**]). We emphasize that, in constrast to the Riemannian setting, the definition above (as well as the one in [**9**]) is explicitly *a priori* non-symmetric.

This definition of mean curvature coincides (up to a constant multiple) with the various definitions of mean curvature in the Carnot group setting (see, for example, [**9**, **10**, **16**, **7**, **25**]). Moreover, our version of the minimal surface equation as $Trace\ II_0 = 0$ matches with others in the literature. For example, suppose Σ is described as $\varphi = 0$ for a C^2

defining function φ. Then for any orthonormal frame $\{X_j\}$ for V_0 we can write

$$\nu = \sum \frac{X_i\varphi}{\sqrt{\sum (X_i\varphi)^2}} X_i,$$

and so

$$H = \mathrm{div}_g \nu = \sum X_i \left(\frac{X_i\varphi}{\sqrt{\sum (X_i\varphi)^2}} \right).$$

To answer Question 1, we shall employ the techniques of Bryant, Griffiths and Grossman [**3**] by exhibiting the minimizing hypersurfaces as integrable Legendre submanifolds of a contact covering manifold $\widetilde{M}$. More specifically, we define $\widetilde{M}$ to be the bundle of horizontally normalized contact elements,

$$\widetilde{M} = \{\tilde{p} = (p, \nu, T) \in M \times (V_0)_p \times V_p : \|\nu\| = 1\}.$$

Thus $\widetilde{M} \xrightarrow{\pi_1} M$ has the structure of an $\mathbb{S}^{k+1} \times \mathbb{R}^{n-k}$-bundle over M. Next we define a contact form θ on $\widetilde{M}$ by

$$\theta_{\tilde{p}}(X) = g_p((\pi_1)_* X, \nu + T).$$

To compute with θ it is useful to work on the graded frame bundle. However, as there is no normalization on the T component of $\widetilde{M}$, we shall need to augment $\mathcal{GF}(M)$ to the fiber bundle $\mathcal{GF}^0(M)$ defined as follows: over each point the fibre is $O(k+1) \times \Pi_{j=1}^{v} O(l_j) \times \mathbb{R}^{l_1+\cdots+l_v}$. The left group action is extended as follows. If $h = (h_1, h_2) \in O(k+1) \times \Pi_{j=1}^{v} O(l_j)$,

$$h \cdot (p, e, t, a) = (p, (h_1 \cdot e, h_2 \cdot t), a h_2^{-1}).$$

The natural projection π from $\mathcal{GF}^0(M)$ to M now filters through $\widetilde{M}$ as

$$\mathcal{GF}^0(M) \xrightarrow{\pi_2} \widetilde{M} \xrightarrow{\pi_1} M$$

where under π_2, $(p, e, t, a) \mapsto (p, e_0, \sum a_j t_j)$. In particular, this means $\pi_2 \circ (id, h_2) = \pi_2$. This formulation now allows us to pull θ back to $\mathcal{GF}^0(M)$ by

$$\pi_2^* \theta = \omega^0 + a_j \eta^j.$$

We shall denote this pullback by θ also.

Remark 4.2. The augmented frame bundle $\mathcal{GF}^0(M)$ is not a principle bundle, and so we cannot impose an affine connection on it in the usual sense. However, since it has the smooth structure of $\mathcal{GF}(M) \times \mathbb{R}^{l_1+\cdots+l_v}$ we can naturally include the canonical forms and the connection structure equations of $\mathcal{GF}(M)$ into the augmented bundle. Thus, the results of Lemma 2.15 and Lemma 2.16 hold on $\mathcal{GF}^0(M)$ also.

Lemma 4.3. *The contact manifold $(\widetilde{M}, \theta)$ is maximally non-degenerate, i.e., $\theta \wedge d\theta^n \neq 0$.*

Proof. We shall work on the augmented graded frame bundle where

$$d\theta = \omega^j \wedge \omega_j^0 + \tau^0 + da_j \wedge \eta^j + a_j(\eta^i \wedge \eta_i^j + \tilde{\tau}^j).$$

We pick out one particular term of the expansion of $\theta \wedge d\theta^n$, namely

$$\mu = \omega^0 \wedge \omega^1 \wedge \cdots \wedge \omega^k \wedge \omega_0^1 \wedge \ldots \omega_0^k \wedge \eta^1 \wedge \ldots \eta^{n-k} \wedge da_1 \wedge \ldots da_{n-k}.$$

The connection forms are vertical (in the principle bundle sense) and the canonical forms are horizontal (again in the bundle sense). Thus μ is the wedge of $n-k$ da terms, $n+k+1$ horizontal forms and k vertical forms. Since each torsion form is purely (bundle) horizontal, μ is clearly the only term of this form in $\theta \wedge d\theta^n$. All the forms are independent so μ does not vanish. Thus we deduce that $\theta \wedge d\theta^n \neq 0$ on $\mathcal{GF}^0(M)$ and so cannot vanish on $\widetilde{M}$. q.e.d.

The transverse Legendre submanifolds of $(\widetilde{M}, \theta)$ are the immersion $\iota \colon \Sigma \hookrightarrow \widetilde{M}$ such that $\iota^*\theta = 0$ and $\pi_2 \circ \iota$ is also an immersion. These are noncharacteristic oriented hypersurface patches in M with normal directions defined by the contact element in $\widetilde{M}$.

Define

$$\Lambda = \omega^1 \wedge \cdots \wedge \omega^k \wedge \eta^1 \wedge \cdots \wedge \eta^{n-k} \tag{6}$$

on $\mathcal{GF}^0(M)$. Then $\Lambda = \pi_2^*(\nu \lrcorner \pi_1^* dV)$ and so Λ is basic over $\widetilde{M}$. Furthermore, due to (3) we see that

$$P(\Sigma) = \int_\Sigma \iota^*\Lambda. \tag{7}$$

Now on $\mathcal{GF}^0(M)$, Lemma 2.16 implies that τ^j has no component of the form $\omega^0 \wedge \omega^j$ and $\tilde{\tau}^j$ none of form $\omega^0 \wedge \eta^j$. Thus, we see from (1) that

$$\begin{aligned} d\Lambda = &\sum_j (-1)^{j-1} \omega^1 \wedge \cdots \wedge (\omega^0 \wedge \omega_0^j) \wedge \cdots \wedge \omega^k \wedge \eta^1 \wedge \cdots \wedge \eta^{n-k} \\ &+ \sum_j (-1)^{j-1} \omega^1 \wedge \cdots \wedge (\omega^j \wedge \omega_j^j) \wedge \cdots \wedge \omega^k \wedge \eta^1 \wedge \cdots \wedge \eta^{n-k} \\ &+ \sum_j (-1)^{k+j-1} \omega^1 \wedge \ldots \omega^k \wedge \eta^1 \wedge \cdots \wedge (\eta^j \wedge \eta_j^j) \wedge \cdots \wedge \eta^{n-k}. \end{aligned}$$

The connection is metric compatible so $\omega_j^j = 0$ and $\eta_j^j = 0$. Thus the second and third sums vanish identically. This implies $d\Lambda = \theta \wedge \Psi$ where

$$\Psi = \sum_j \omega^1 \wedge \cdots \wedge (\omega_0^j) \wedge \cdots \wedge \omega^k \wedge \eta^1 \wedge \cdots \wedge \eta^{n-k}. \tag{8}$$

If $\Sigma \subset \widetilde{M}$ is a transverse Legendre submanifold, then we can construct a graded frame adapted to Σ, i.e., with $e_0 = \nu$. Choosing any section immersing Σ into $\mathcal{GF}^0(M)$ we can then pull Ψ back to Σ. Switching the

ω's and η's to represent the coframe and connection form for this fixed frame, we get

$$\Psi_{|\Sigma} = \left(\sum_{j=1}^{k} \omega_0^j(e_j) \right) \omega^1 \wedge \cdots \wedge \omega^k \wedge \eta^1 \wedge \cdots \wedge \eta^{n-k}. \tag{9}$$

Theorem 4.4. *Suppose Σ is a C^2 hypersurface in the vertically rigid manifold M. Then Σ is a critical point for perimeter measure in a noncharacteristic neighborhood $U \subset \Sigma$ if and only if the unit horizontal normal ν satisfies the minimal surface equation*

$$H = 0 \tag{10}$$

in U. Equivalently, the horizontal normal must satisfy

$$div_g \nu = 0 \tag{11}$$

everywhere on U, where the divergence is taken with respect to the Riemannian metric g.

Proof. From the Bryant-Griffiths-Grossman method [**3**] we see that $\iota \colon \Sigma \hookrightarrow \widetilde{M}$ is a stationary Legendre submanifold for Λ in a small neighborhood if and only if $\iota^*\Psi = 0$. This condition is just $\sum_{j=1}^{k} \omega_0^j(\nu) = 0$ for any local orthonormal frame $(\nu, e_1, \dots e_k)$ for V_0. This can be rewritten as

$$H = \sum \langle \nabla_{e_j} \nu, e_j \rangle = 0. \tag{12}$$

q.e.d.

Corollary 4.5. *For a given a sub-Riemannian manifold, there may be many choices of vertically rigid complement and adapted connection. The minimal surface equation* (11) *may depend on the choice of orthogonal complement V, but not on the remainder of the vertically rigid structure or choice of adapted connection.*

Proof. After we write $\mathrm{div}_g \nu = \sum \langle \nabla_{e_j} \nu, e_j \rangle$, the result follows immediately from Corollary 2.12. q.e.d.

Corollary 4.6. *Any minimal C^2 noncharacteristic patch of a vertically rigid sub-Riemannian manifold $(M, V_0, \langle \cdot, \cdot \rangle)$ with*

$$dim\ V_0 = 2$$

is ruled by horizontal ∇-geodesics.

Proof. Extend ν off Σ to any unit horizontal vector field. Define $\nu^\perp$ to be any horizontal unit vector field that is orthogonal to ν. By the torsion properties of the connection and the arguments of Theorem 4.4, the minimal surface equation (11) can be written

$$0 = \langle \nabla_{\nu^\perp} \nu, \nu^\perp \rangle = -\langle \nu, \nabla_{\nu^\perp} \nu^\perp \rangle.$$

Since $\nu^\perp$ has no covariant derivatives in vertical directions, this implies that

$$\nabla_{\nu^\perp}\nu^\perp = 0.$$

In other words the integral curves of $\nu^\perp$ are ∇-geodesics. Since we are assuming C^2 regularity of the surface and no characteristic points, the vector field $\nu^\perp$ is C^1 and non-vanishing. Therefore, every point in the surface is contained in a unique integral curve of $\nu^\perp$. As $\nu^\perp$ is everywhere tangent to the surface, these integral curves must foliate the surface. q.e.d.

Remark 4.7. We note that the last corollary is a generalization of the results of Garofalo and the second author [**16**] in the Heisenberg group, those of Cheng, Hwang, Malchiodi and Yang [**7**] in three dimensional pseudohermitian manifolds, and those of Cole [**8**] in Martinet-type spaces. In those cases, the authors proved the minimal surfaces in those settings were ruled by appropriate families of horizontal curves.

5. CMC surfaces and the isoperimetric problem

We now investigate the following question

Question 2. Given a fixed volume, what are the closed surfaces bounding this volume of minimal perimeter?

Using the results of the previous section, we can now define a hypersurface of locally constant mean curvature (CMC) by requiring that $H =$ constant on each connected component of $\Sigma' = \Sigma - \text{char}(\Sigma)$. If we wish to specify the constant, we will call Σ a CMC(ρ) surface. By comparing to the Riemannian case, these are our prime candidates for solutions to Question 2. Throughout this section, we shall make the standing assumption that the volume form dV_g is globally exact, i.e., there exists a form μ such that $d\mu = dV_g$ on M. Since this is always locally true, the results of this section will hold for sufficiently small domains.

For a closed codimension 2 surface γ in M we define

(13)

$$\text{Span}(\gamma, a) = \left\{ C^2 \text{ noncharacteristic surface } \Sigma : \partial\Sigma = \gamma, \int_\Sigma \mu = a \right\}.$$

Lemma 5.1. *If Span(γ, a) is non-empty, then any element of minimal perimeter Σ_0 must have constant mean curvature.*

Proof. As the boundary of the surfaces are fixed, we can again employ the formalism of [**3**]. We permit variations that alter the integral $\int_\Sigma \mu$ and apply a Lagrange multiplier method to establish a condition for critical points of perimeter. Indeed, the minimizer Σ_0 must be a critical point of the functional

$$\Sigma \mapsto \int_\Sigma \Lambda - c \int_\Sigma \mu$$

for some constant c. Now pulled-back to the contact manifold,

$$\begin{aligned} d(\Lambda - c\mu) &= \theta \wedge \Psi - c\pi_1^* dV_g \\ &= \theta \wedge \Big((H-c)\omega^1 \wedge \cdots \wedge \omega^k \wedge \eta^1 \wedge \cdots \wedge \eta^{n-k} \Big) \\ &:= \theta \wedge \widetilde{\Psi}. \end{aligned}$$

The same methods as Theorem 4.4 then imply that $\widetilde{\Psi}_{|\Sigma_0} = 0$, and hence Σ_0 has constant mean curvature. q.e.d.

Theorem 5.2. *If a C^2 domain Ω minimizes surface perimeter over all domains with the same volume, then $\Sigma = \partial\Omega$ is locally CMC.*

Proof. Let $p \in \partial\Sigma$ be a noncharacteristic point. As Σ is C^2 there exists a noncharacteristic neighbourhood U of p in Σ with at least C^2 boundary. Now by Stokes' theorem, $\mathrm{Vol}(\Omega) = \int_\Sigma \mu$ and so U must minimize perimeter over all noncharacteristic surfaces in $\mathrm{Span}(\partial U, \int_U \mu)$. Therefore U has constant mean curvature by Lemma 5.1. q.e.d.

Under certain geometric conditions we can provide a more intuitive description of μ.

Definition 5.3. A dilating flow for a vertically rigid structure is a global flow $F \colon M \times \mathbb{R} \longrightarrow M$

- $(F_\lambda)_* E_j = e^\lambda E_j$ for some fixed horizontal orthonormal frame
$$\{E_1, \ldots, E_{k+1}\}$$
- $(F_\lambda)_* T_j = e^{\gamma_j \lambda} T_j$ for some constant γ_j.

Associated to a dilating flow are the dilation operators defined by

$$\delta_\lambda = F_{\log\lambda}$$

and the generating vector field X defined by

$$X_p = \frac{d}{d\lambda}_{|\lambda=0} F(\lambda, p).$$

The homogeneous dimension of M is given by

$$Q = k + 1 + \sum_{j=1}^{n-k} \gamma_j.$$

The dilating flow is said to have an origin O if for all p, $\delta_\lambda(p) \to O$ as $\lambda \to 0$.

In the sequel, a vertically rigid sub-Riemannian manifold that admits a dilating flow with origin will be referred to as a VRD-manifold.

Example 5.4. All Carnot groups admit a dilation with origin. On the Lie algebra level, the dilation is defined merely by defining a linear map with eigenspaces the various levels of the grading, i.e., $\delta_\lambda X = \lambda^{j+1} X$ for $X \in V_j$. The group dilations are then constructed by exponentiating.

Example 5.5. The jointly homogeneous Martinet spaces, i.e., those of Example 2.7 with the functions f and g bihomogeneous of degree m. Then the dilations are defined by

$$\delta_\lambda(x, y, z) = (\lambda x, \lambda y, \lambda^{m+1} z).$$

Then clearly

$$(\delta_\lambda)_*(\partial_x + f(x, y)\partial_z) = \lambda\partial_x + \lambda^{m+1} f(x, y)\partial_z = \lambda(\partial_x + f(\lambda x, \lambda y)\partial_z)$$

and $(\delta_\lambda)_*\partial_z = \lambda^{m+1}\partial_z$.

Lemma 5.6. *In a VRD-manifold, the form* $\mu = Q^{-1} X \lrcorner dV_g$ *satisfies*

$$d\mu = dV_g.$$

Proof. Let ω^j, η^i be the dual basis to E_j, T_i. Then

$$dV_g = \omega^1 \wedge \cdots \wedge \omega^{k+1} \wedge \eta^1 \wedge \cdots \wedge \eta^{n-k}.$$

Now $F_\lambda^* \omega^j(Y) = \omega^j((F_\lambda)_* Y)$ so $F_\lambda^* \omega^j = \lambda \omega^j$. Thus

$$\mathcal{L}_X \omega^j = \frac{d}{d\lambda}_{|\lambda=0} F_\lambda^* \omega^j = \lambda \omega^j.$$

By a virtually identical argument we see that $\mathcal{L}_X \eta^j = \gamma_j \eta^j$. Therefore

$$d\mu = Q^{-1} d(X \lrcorner dV_g) = Q^{-1} \mathcal{L}_X dV_g = dV_g.$$

q.e.d.

Every point p in a VRD-manifold can be connected to the origin by a curve of type $t \mapsto \delta_t(p)$. For any surface Σ we can then construct the dilation cone over Σ as

$$\operatorname{cone}(\Sigma) = \{\delta_t(p) : 0 \leq t \leq 1, p \in \Sigma.\} \tag{14}$$

Lemma 5.7. *Suppose* Σ *is* C^2 *surface patch in a VRD-manifold such that any dilation curve intersects* Σ *at most once. If* Σ *is oriented so that the normal points away from the origin, then*

$$\operatorname{Vol}(\operatorname{cone}(\Sigma)) = \int_\Sigma \mu.$$

Proof. This is just Stokes' theorem for a manifold with corners, for

$$\int_{\operatorname{cone}(\Sigma)} dV_g = \int_{\partial\operatorname{cone}(\Sigma)} \mu = \int_\Sigma \mu$$

as μ vanishes when restricted to any surface foliated by dilation curves.

q.e.d.

We can now interpret Lemma 5.1 as minimizing surface perimeter under the constraint of fixed dilation cone volume. Since the volume of a domain is equal to the signed volume of its boundary dilation cone, this yields some geometric intuition for the arguments of Theorem 5.2.

Remark 5.8. In [**19**], the authors characterize cylindrically symmetric minimizers in the Heisenberg group as constant mean curvature surfaces in that setting, while in [**10**], the authors reach the same characterization within the class of all surfaces which may be written as the union of two graphs over a disk in the xy-plane. Our treatment allows for such a characterization in all VR spaces without such restrictions on the shape of the surfaces. We note, however, that this method requires some regularity (the surfaces must be at least C^2 to ensure the computations work) while the work in [**10**] and [**19**] is more general in this respect, allowing for piecewise C^1 defining functions. Moreover, we emphasize again that in both these papers, the authors go further by identifying the isoperimetric minimizer in their respective settings.

Remark 5.9. We point out that Theorem 5.2 provides an approach to understanding the isoperimetric problem in VR or VRD spaces via a better understanding of their constant mean curvature surfaces. For a specific example, the reader is referred to Section 6 below.

6. The horizontal second fundamental form

We now present a more classical interpretation of these results by returning to an analysis of the second fundamental form.

In addition to the horizontal mean curvature, which we defined as the trace of the horizontal second fundamental form, we would also like to define other aspects of horizontal curvature.

Definition 6.1. Let II_0 be the horizontal second fundamental form for a C^2 noncharacteristic surface, Σ, in a vertically rigid sub-Riemannian manifold M. Let $\{\mu_0, \ldots, \mu_k\}$ be the eigenvalues (perhaps complex and with multiplicity) of II_0. Then, the **horizontal principle curvatures** are given by

$$\kappa_i = \mathrm{Re}\,(\mu_i)$$

for $1 \leq i \leq k$.

Moreover, given $x \in \Sigma$, we say that Σ is **horizontally positively (non-negatively) curved** at x if II_0 is either positive (semi-)definite or negative (semi-) definite at x and is **horizontally negatively curved** at x if there is at least one positive and one negative κ_i. Σ is **horizontally flat** at x if $\kappa_i = 0$ for $1 \leq i \leq k$.

This definition coupled with the observation that $Trace\ II_0 = \kappa_1 + \cdots + \kappa_k$ yields the following immediate corollary of Theorem 4.4:

Corollary 6.2. *If Σ is a C^2 minimal surface in a vertically rigid sub-Riemannian manifold M, then Σ has no noncharacteristic points of horizontal positive curvature.*

This is reflective of the Euclidean and Riemannian cases where minimal surfaces cannot have points of positive curvature.

Remark 6.3. In the definition of the horizontal principle curvatures, we consider the real parts of the eigenvalues of II_0 rather than, for example, the (real) eigenvalues of the symmetrized horizontal second fundamental form. We choose this definition for two reasons. First, symmetrization would break the direct link between the horizontal principle curvatures and the curvature of the horizontal curves discussed in the next three results. Second, the imaginary parts of the complex eigenvalues of II_0 (if they exist) carry information that would be lost under symmetrization. Specifically, the imaginary parts indicate the nature of the Lie bracket structure on the surface. If $\lambda \pm i\beta$ is a pair of complex eigenvalues of II_0 associated to an eigenvector pair, $u \pm iv$, then

$$II_0(u+iv) = (\lambda + i\beta)(u+iv) = (\lambda u - \beta v) + i(\lambda v + \beta u),$$

and so

$$\begin{aligned}\langle \nabla_v u, e_0\rangle &= -\langle u, II_0 v\rangle \\ &= -\langle u, \beta u + \lambda v\rangle \\ &= -\beta\langle u,u\rangle - \lambda\langle u,v\rangle\end{aligned}$$

and

$$\begin{aligned}\langle \nabla_u v, e_0\rangle &= -\langle v, II_0 u\rangle \\ &= -\langle v, \lambda u - \beta v \\ &= -\lambda\langle u,v\rangle + \beta\langle v,v\rangle.\end{aligned}$$

Recalling that the torsion of $\nabla$is purely vertical, we have that $\langle \nabla_u v - \nabla_v u - [u,v], e_0\rangle = 0$, and we conclude that $\langle [u,v], e_0\rangle = \beta(\langle v,v\rangle + \langle u,u\rangle)$.

Remark 6.4. We note that having, for example, only positive (or only negative) horizontal principle curvatures at a point is necessary but not sufficient to conclude that the surface is horizontally positively curved. Recalling that a matrix A is positive definite if and only if A^* is positive definite and that there are many examples of two by two matrices with complex eigenvalues of positive real part that, when symmetrized, have one positive and one negative eigenvalue (for example, $\begin{pmatrix} 1 & 1 \\ -5 & 1 \end{pmatrix}$), we see that for general matrices this observation is true. In this example, we show that this behavior can also appear in the horizontal second fundamental form. To see this we will consider two surfaces in $\mathbb{H} \times \mathbb{R}$ (see Example 2.6). The first is $\phi_1 = s - t^2 = 0$ and the second is $\phi_2 = s - x^2 + y^2 + t^2 = 0$. For ϕ_1, we can take

$$e_1 = \frac{-xt}{(t^2(x^2+y^2))^{\frac{1}{2}}}\, X_1 - \frac{yt}{(t^2(x^2+y^2))^{\frac{1}{2}}}\, X_2$$

and

$$\begin{aligned} e_2 = &\frac{yt}{\sqrt{t^2(x^2+y^2)+t^4(x^2+y^2)^2}}\, X_1 \frac{-xt}{\sqrt{t^2(x^2+y^2)+t^4(x^2+y^2)^2}}\, X_2 \\ &- \frac{t^2(x^2+y^2)}{\sqrt{t^2(x^2+y^2)+t^4(x^2+y^2)^2}}\, T. \end{aligned}$$

Direct computation shows that

$$II_0 = \begin{pmatrix} 0 & -\frac{t}{t^2(x^2+y^2)+1} \\ \frac{t}{t^2(x^2+y^2)+1} & -\frac{x^2+y^2}{2(t^2(x^2+y^2)+1)^{\frac{3}{2}}} \end{pmatrix}.$$

At the point $(x, y, t, s) = (0, 1, 2, 0)$, we have that the eigenvalues of II_0 are $\{-\frac{\sqrt{5}}{100} \pm i\frac{\sqrt{1595}}{100}\}$ but the eigenvalues of II_0^* are $\{0, -1/5\sqrt{5}\}$. Thus, even though both of the real parts of the complex eigenvalues of II_0 are negative, II_0^* (and hence II_0) is merely negative semi-definite. For the second surface $\phi_2 = 0$, we choose a basis for V_0 by $\{e_0, e_1, e_2\}$ where if we let $p = X_1\phi_2, q = X_2\phi_2, r = X_3\phi_2 = 1$, we have

$$e_0 = \nu = \frac{p}{\sqrt{p^2+q^2+1}}\, X_1 + \frac{q}{\sqrt{p^2+q^2+1}}\, X_2 + \frac{1}{\sqrt{p^2+q^2+1}}\, X_3.$$

And, away from points where $p = q = 0$, we may take

$$e_1 = \frac{q}{\sqrt{p^2+q^2}}\, X_1 - \frac{p}{\sqrt{p^2+q^2}}\, X_2$$

and

$$\begin{aligned} e_2 = &\frac{p}{\sqrt{p^2+q^2+(p^2+q^2)^2}}\, X_1 + \frac{q}{\sqrt{p^2+q^2+(p^2+q^2)^2}}\, X_2 \\ &- \frac{p^2+q^2}{\sqrt{p^2+q^2+(p^2+q^2)^2}}\, X_3. \end{aligned}$$

A direct, but much more involved computation, shows that at the point $(x, y, t, s) = (-3, -3, -3, -3)$,

$$II_0 = \begin{pmatrix} 0 & -\frac{1}{19} \\ \frac{5}{19} & \frac{9}{361}\sqrt{19} \end{pmatrix},$$

having eigenvalues $\frac{9\sqrt{19}}{722} \pm i\frac{\sqrt{19}\sqrt{299}}{722}$. However, II_0^* has eigenvalues $\frac{9}{722}\sqrt{19} \pm \frac{\sqrt{19}\sqrt{295}}{722}$, the first of which is positive and the second of which is negative. The details of these computations are left to the reader.

Definition 6.5. Let c be a horizontal curve on Σ a C^2 hypersurface in a vertically rigid sub-Riemannian manifold. Then at a noncharacteristic point of Σ, the **horizontal normal curvature** of c is given by

$$k_c = -\langle \nabla_{\dot{c}} \dot{c}, e_0 \rangle.$$

We note that, analogous to the Euclidean and Riemannian cases, there is a connection between the horizontal normal curvature of curves passing through a point on a hypersurface and the horizontal second fundamental form at that point:

Lemma 6.6. *Let c be a horizontal curve on Σ a C^2 hypersurface in a vertically rigid sub-Riemannian manifold. Then at noncharacteristic points,*

$$k_c = \langle II_0(\dot{c}), \dot{c} \rangle.$$

Proof. Since c is horizontal, we have that $\dot{c} = c_1\, e_1 + \cdots + c_k\, e_k$ for appropriate functions c_i. Differentiating $\langle \dot{c}, e_0 \rangle = 0$, we have:

$$\begin{aligned} -\langle \nabla_{\dot{c}} \dot{c}, e_0 \rangle &= \langle \dot{c}, \nabla_{\dot{c}} e_0 \rangle \\ k_c &= \langle II_0(\dot{c}), \dot{c} \rangle. \end{aligned}$$

q.e.d.

This gives, as an immediate corollary, an analogue of Meusnier's theorem:

Corollary 6.7. *All horizontal curves lying on a surface Σ in M, a VR space, which, at a noncharacteristic point $x \in \Sigma$, have the same tangent vector also have the same horizontal normal curvature at x.*

As in the classical case, Corollary 6.7 allows us to speak of the horizontal normal curvature associated with a direction rather than with a curve, showing that II_0 contains all of the horizontal curvature information at a point.

Lemma 6.8. *Given Σ and x as above, let l be the number of distinct principle curvatures at x. Then, there exist curves $\{c_1, \ldots, c_l\} \subset \Sigma$ so that*

$$k_{c_i} = \kappa_i.$$

Proof. Let $\{\lambda_1, \ldots \lambda_j, \lambda_{j+1} \pm i\beta_{j+1}, , \ldots, \lambda_l \pm i\beta_l\}$ be the eigenvalues of II_0 at x associated with the distinct principle curvatures. Further, let $\{u_1, \ldots, u_j, u_{j+1} \pm iv_{j+1}, \ldots, u_k \pm iv_k\}$ be the associated eigenvectors. Without loss of generality, we have ordered the eigenvalues so that the real eigenvalues appear first and the complex eigenvalues are last. We note that, for each complex conjugate pair of eigenvalues, $\lambda_j \pm \beta_j$, the associated principle curvatures, κ_j and κ_{j+1}, are equal. Using the eigenvectors, we may replace $\{e_1, \ldots, e_k\}$ by a new basis given by

$$\{u_1, \ldots, u_j, u_{j+1}, v_{j+1}, \ldots, u_l, v_l, \tilde{e}_{2l-j}, \tilde{e}_k\}$$

where $\{\tilde{e}_i\}$ form a basis for the orthogonal complement of the eigenvectors. Rewriting II_0 with respect to this new basis, there is a submatrix of II_0 which is a block matrix where the first block is of the form

$$A = \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_j \end{pmatrix}$$

and there are $k - j$ remaining blocks of the form

$$B_i = \begin{pmatrix} \lambda_i & \beta_i \\ -\beta_i & \lambda_i \end{pmatrix}.$$

Now let c_i be the integral curve of the i^{th} new basis vector for $1 < i < 2l - j$. Then,

$$k_{c_i} = \langle II_0(\dot{c}_i), \dot{c}_i \rangle = \langle II_0(e_i), e_i \rangle = \kappa_i.$$

q.e.d.

Remark 6.9. We note that since II_0 is often nonsymmetric, there is often not a full basis of eigenvectors. For example, in the Carnot group $\mathbb{H} \times \mathbb{R}$ of Example 2.6, the surface defined by $\frac{xy}{2} - t - s = 0$ has unit horizontal normal given by:

$$\nu = \frac{y}{\sqrt{1+y^2}}\, X_1 - \frac{1}{\sqrt{1+y^2}}\, X_3$$

and

$$II_0 = \begin{pmatrix} 0 & \frac{y^2}{(y^2+1)^{\frac{3}{2}}} \\ 0 & 0 \end{pmatrix}.$$

Thus, this is a minimal surface and II_0 has a double eigenvalue of 0 and a single eigenvector $(1, 0)$ in this basis. The presents an entirely different phenomena than the analogous Riemannian or Euclidean situation.

We note that this phenomena and the existence of complex eigenvalues both indicate the existence of a nontrivial bracket structure among the elements of the tangent space to Σ. Indeed, both of these indicate that there are vector fields $e_i, e_j \in \{e_1, \dots, e_k\}$ with the property that $[e_i, e_j]$ has a component in the e_0 direction. In particular, this is an indication that the distribution $\{e_1, \dots, e_k\}$ is not integrable, and hence, Σ cannot be realized as a surface ruled by codimension one horizontal submanifolds.

We pause to note that we can now state an analogue of Corollary 4.6:

Corollary 6.10. *Any $CMC(\rho)$ noncharacteristic patch of a vertically rigid sub-Riemannian manifold $(M, H, \langle \cdot, \cdot \rangle)$ with*

$$\dim V_0 = 2$$

is ruled by horizontal curves with horizontal constant curvature ρ.

Proof. We follow precisely the same proof as that of Corollary 4.6 to get that

$$\langle \nabla_{\nu^\perp} \nu^\perp, \nu \rangle = \rho.$$

Thus, the integral curves of $\nu^\perp$ have constant horizontal curvature ρ.

q.e.d.

Remark 6.11. We note that in specific cases, these rulings can be computed exactly. Basically this amounts to explicitly computing the torsion terms and solving ODEs. For example, it is straightforward to verify that in the Heisenberg group, such curves are geodesics with respect to the Carnot-Carathéodory metric and are horizontal lifts of planar circles of curvature ρ. This particular observation is also contained in [**7**].

Definition 6.12. Let Σ be a C^2 hypersurface in a vertically rigid sub-Riemannian manifold M. Then, the **horizontal exponential surface** at $x \in \Sigma$ is defined to be the union of all the horizontal curves in Σ passing through x. We denote this subset of Σ by $\Sigma_0(x)$.

Definition 6.13. Let Σ be a C^2 hypersurface in a vertically rigid sub-Riemannian manifold M. Then, the **horizontal tangent plane** at a noncharacteristic point $x \in \Sigma$, is defined as

$$T_x^h\Sigma = \{\exp_x(v) | g(v, e_0(x)) = 0, \ v \in T_xM\}$$

where exp is the Riemannian exponential map.

We says that a set $S \subset M$, containing x, **lies to one side** of $T_x^h\Sigma$ if $S \cap (M \setminus T_x^h\Sigma)$ lies entirely in a single connected component of $M \setminus T_x^h\Sigma$. We say that S **lies strictly to one side** of $T_x^h\Sigma$ if, in addition to the previous condition, $S \cap T_x^h\Sigma = \{x\}$. We say that either of these two conditions holds locally if there exists $\epsilon > 0$ so that the appropriate condition holds for $S \cap B(x, \epsilon)$.

This horizontal tangent plane in a Carnot group can also be defined by blowing up the metric at a given point (see [**14**]).

This gives us a geometric interpretation of these curvature conditions analogous to the Riemannian setting:

Theorem 6.14. *Let Σ be a C^2 hypersurface in M, a Carnot group, and let $\{\kappa_i\}$ be the set of horizontal principle curvatures of Σ at a non-characteristic point x. Then, $\Sigma_0(x)$ locally lies strictly to one side of $T_x^h\Sigma$ if and only if the surface is horizontally positively curved at x. Similarly, if Σ is horizontally negatively curved at x, then any neighborhood of x in $\Sigma_0(x)$ contains points on both sides of $T_h^h\Sigma$.*

Proof. First assume that $\Sigma_0(x)$ locally lies strictly to one side of the horizontal tangent plane at x. Then, as any curves in $\Sigma_0(x)$ must also lie strictly to one side of the tangent plane, we have that $\langle \nabla_{\dot{c}_1} \dot{c}_1, e_0 \rangle$ and $\langle \nabla_{\dot{c}_2} \dot{c}_2, e_0 \rangle$ are either both positive or both negative at x for any $c_1, c_2 \in$

$\Sigma_0(x)$. Thus, Σ is horizontally positively curved at x. Conversely, if Σ is horizontally positive definite at x and $c \in \Sigma_0(x)$, then $\langle \nabla_{\dot{c}}\dot{c}, e_0\rangle = -\langle \dot{c}, II_0\dot{c}\rangle$ is either strictly positive or strictly negative. Without loss of generality, we will assume it to be positive. But, geometrically, this says that with respect to the connection ∇, each curve c locally changes in the direction of e_0 and cannot move towards $-e_0$. To see this, consider a curve $c \in \Sigma_0(x)$ in a small neighborhood of x. If we let v_i be the left invariant unit vector field on M so that $v_i(x) = e_i(x)$, we have that $T^h_x\Sigma$ is the integral submanifold of the distribution perpendicular to v_0 (with respect to the Riemannian metric). With respect to these vector fields, we write

$$\dot{c}(t) = \dot{c}_0(t)\ v_0 + \cdots + \dot{c}_N(t)\ v_N$$

where $\dot{c}_0(0) = 0$. Then, computing with respect to the Riemannian metric, we have

$$\begin{aligned}\langle \nabla_{\dot{c}}\dot{c}, v_0\rangle &= \ddot{c}_0(t) + \sum_{i,j} \dot{c}_i(t)\dot{c}_j(t)\langle \nabla_{v_i}v_j, v_0\rangle \\ &= \ddot{c}_0(t).\end{aligned}$$

The last equation follows from the Riemannian Kozul formula because, as a Carnot group is graded, for left invariant horizontal vector fields v_1, v_2, v_3, $\langle [v_1, v_2], v_3\rangle = 0$. Note that at $t = 0$, we have

$$\langle \nabla_{\dot{c}}\dot{c}(0), e_0\rangle = \langle \nabla_{\dot{c}}\dot{c}(0), v_0(c(0))\rangle = \ddot{c}_0(0)$$

and hence our hypothesis that II_0 is positive definite shows that $\ddot{c}_0(0) > 0$ and the result follows. A similar argument shows the last statement. q.e.d.

This leads us to define a notion of convexity in sub-Riemannian manifolds.

Definition 6.15. A subset U of a Carnot group M with C^2 boundary Σ is **(strictly) horizontally geometrically convex**, or (strictly) hg-convex, if, at each noncharacteristic point $x \in \Sigma$, $\Sigma_0(x)$ lies (strictly) to one side of $T^h_x\Sigma$. We say that Σ is locally (strictly) hg-convex at a noncharacteristic point x if there exists an $\epsilon > 0$ so that $\Sigma_0(x) \cap B(x, \epsilon)$ lies (strictly) to one side of $T^h_x\Sigma$.

With this definition, we have yet another analogue of Euclidean minimal surface theory:

Corollary 6.16. *If Σ is a C^2 minimal hypersurface in a Carnot group then Σ cannot bound a locally strictly hg-convex set.*

Proof. As the distribution V_0 is non-integrable, every C^2 surface must have at least one noncharacteristic point. The result then follows from Theorem 6.14 and Corollary 6.2. q.e.d.

Theorem 1.5 in the introduction is the combination of this corollary and Corollary 6.2.

We note that this notion of convexity is distinct from some other notions in the literature. In [**11**], the authors introduce a variety of notions of convexity, two which we will discuss in relation to hg-convexity. First, they define a function, ϕ, to be weakly H-convex if its symmetrized horizontal Hessian is positive semi-definite (the reader should also see the work of Lu, Manfredi and Stroffolini [**21**], which independently presented a notion of convex functions at the same time). Second, they define a subset D of a Carnot group to be weakly H-convex if at every point $x \in D$, the intersection of $\exp_x(V_0)$ with D is starlike (in the Euclidean sense).

We present two illustrative examples. Consider the first Heisenberg group, $\mathbb{H}$, of Example 2.6. We first point out that the plane $t = 0$ in the Heisenberg group is hg-convex and, moreover, it bounds a weakly H-convex region $\{(x, y, t) | t \leq 0\}$ and its defining function is weakly H-convex. Hence, for this surface, all three notions coincide.

However, letting $\phi = x^2 - y^2 - t$ and $S = \{(x, y, t) | \phi(x, y, t) = 0\}$, if we let $p = X_1\phi, q = X_2\phi$ and $\overline{p} = p/\sqrt{p^2 + q^2}$, $\overline{q} = q/\sqrt{p^2 + q^2}$, we then have that $e_0 = \overline{p}\ X_1 + \overline{q}\ X_2$ and we may take

$$\begin{aligned} e_1 &= \overline{q}\ X_1 - \overline{p}\ X_2 \\ &= \frac{1}{\sqrt{17x^2 + 16xy + 17y^2}} \left((4x + y)\ X_1 + (x + 4y)\ X_2\right). \end{aligned}$$

Then, II_0 is a one by one matrix:

$$II_0 = (\langle \nabla_{e_1} e_0, e_1 \rangle) = \left(\frac{60(y^2 - x^2)}{(17x^2 + 16xy + 17y^2)^{\frac{3}{2}}} \right).$$

In particular, note that in a sufficiently small neighborhood of the point $(0, 1, -1)$, II_0 is positive and the surface is locally strictly hg-convex. However, the symmetrized horizontal Hessian of ϕ is

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

As this matrix is not positive definite, the function ϕ is not weakly H-convex. Moreover, the region $D = \{(x, y, t) | \phi(x, y, t) \geq 0\}$ is not weakly H-convex either. To see this, we show explicitly that the intersection of the horizontal plane at $(0, 1, -1)$, $P = exp_{(0,1,-1)}V_0 = \{(x, y, t) | t = -x/2 - 1\}$ with D is not starlike. We consider three points $p_1 = (0, 1, -1), p_2 = (-1, 1, -1/2), p_3 = (-1/4, 1, -7/8)$ on P. Note that $\phi(p_1) = 0, \phi(p_2) = 1/2, \phi(p_3) = -1/16$, and so $p_1, p_2 \in P \cap D$ and $p_3 \notin D$. However, all three points lie on a horizontal straight line connecting p_1 to p_2 in P. Hence, $P \cap D$ is not star-shaped. Thus, we conclude that the notion of hg-convexity is not equivalent to either weak H-convexity of the defining function or of the set.

Remark 6.17. We remark that the definitions and proofs in this section are direct generalizations or adaptations of the Euclidean and/or Riemannian machinery. While the proofs are quite straightforward, we point out that this is due to a correct choice of geometric structure, in this case the adapted connection, that allows for the ease of the proofs. Without such machinery, the statements about the relation between horizontal curvature and minimality/isoperimetry above were known only in more restricted settings.

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