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PROOF OF THE ANGULAR MOMENTUM-MASS INEQUALITY FOR AXISYMMETRIC BLACK HOLES

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Abstract

We prove that an extreme Kerr initial data set is a unique absolute minimum of the total mass in a (physically relevant) class of vacuum, maximal, asymptotically flat, axisymmetric data for Einstein equations with fixed angular momentum. These data represent non-stationary, axially symmetric black holes.

As a consequence, we obtain that any data in this class satisfy the inequality $\sqrt{J} \leq m$, where m and J are the total mass and angular momentum of spacetime.

1. Introduction

An *initial data set* for the Einstein vacuum equations is given by a triple (S, h_{ij}, K_{ij}) where S is a connected 3-dimensional manifold, h_{ij} a (positive definite) Riemannian metric, and K_{ij} a symmetric tensor field on S, such that the vacuum constraint equations

$$D_j K^{ij} - D^i K = 0, \tag{1}$$

$$R - K_{ij}K^{ij} + K^2 = 0, \tag{2}$$

are satisfied on S. D and R are the Levi-Civita connection and the Ricci scalar associated with h_{ij}, and $K = K_{ij}h^{ij}$. In these equations the indices are moved with the metric h_{ij} and its inverse h^{ij}.

The manifold S is called *Euclidean at infinity* if there exists a compact subset $\mathcal{K}$ of S such that $S \setminus \mathcal{K}$ is the disjoint union of a finite number of open sets U_k, and each U_k is isometric to the exterior of a ball in $\mathbb{R}^3$. Each open set U_k is called an *end* of S. Consider one end U and the canonical coordinates x^i in $\mathbb{R}^3$, which contains the exterior of the ball to which U is diffeomorphic. Set $r = \left(\sum (x^i)^2\right)^{1/2}$. An initial data set is called *asymptotically flat* if S is Euclidean at infinity, the metric h_{ij} tends to the euclidean metric, and K_{ij} tends to zero as $r \to \infty$ in an appropriate way. These fall off conditions (see [**2**], [**13**] for the optimal

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fall off rates) imply the existence of the total mass m (or ADM mass [**1**]) defined at each end U by

$$m = \frac{1}{16\pi} \lim_{r\to\infty} \oint_{\partial B_r} (\partial_j h_{ij} - \partial_i h_{jj})\, n^i \, ds, \tag{3}$$

where ∂ denotes partial derivatives with respect to x^i, B_r is the euclidean sphere $r = constant$ in U, n^i is its exterior unit normal and ds is the surface element with respect to the euclidean metric.

A central result concerning this physical quantity is the positive mass theorem [**37**], [**45**]:

$$m \geq 0, \tag{4}$$

for asymptotically flat, complete, vacuum, data; with equality only for flat data (i.e., the data for Minkowski spacetime).

We will further assume that the data are *axially symmetric*, which means that there exists a Killing vector field η^i, i.e.,

$$\pounds_\eta h_{ij} = 0, \tag{5}$$

where $\pounds$ denotes the Lie derivative, which has complete periodic orbits and is such that

$$\pounds_\eta K_{ij} = 0. \tag{6}$$

For axially symmetric data there exists another well defined physical quantity, namely the angular momentum J associated with an arbitrary closed 2-surface Σ in S (the Komar integral of the Killing vector [**28**], see also [**38**]). We define the angular momentum of Σ by the following surface integral

$$J(\Sigma) = \oint_\Sigma \pi_{ij} \eta^i n^j \, ds_h, \tag{7}$$

where $\pi_{ij} = K_{ij} - K h_{ij}$ and n^i, ds_h are, respectively, the unit normal vector and the surface element with respect to h_{ij}. As a consequence of equation (1) and the Killing equation (5), the vector $\pi_{ij}\eta^j$ is divergence free. Then, by the Gauss theorem, $J(\Sigma) = J(\Sigma')$ if $\Sigma \cup \Sigma'$ is the boundary of a region contained in S (i.e., J depends only on the homology class of S). If $S = \mathbb{R}^3$, it follows that $J(\Sigma) = 0$ for all Σ. In order to have non zero J the manifold S must have a non trivial topology; for example, S can have more than one end.

Let Σ_∞ be any closed surface in a given end U such that it encloses the corresponding ball in $\mathbb{R}^3$. The total angular momentum of the end U is defined by $J \equiv J(\Sigma_\infty)$.

Physical arguments suggest the following inequality at any end

$$m \geq \sqrt{|J|}, \tag{8}$$

for any complete, asymptotically flat, axially symmetric and vacuum initial data set (see [**17**] and reference therein). Moreover, the equality in (8) should imply that the data set is a slice of the extreme Kerr spacetime.

This inequality was proved for an initial data set close to an extreme Kerr data set in [**18**], [**17**].

The main result of this article is the following:

Theorem 1.1. *Let* (h_{ij}, K_{ij}, S) *be a Brill data set* (*see Definition* 2.1) *such that they satisfy condition* 2.5. *Then inequality* (8) *holds. Moreover, the equality in* (8) *holds if and only if the data are a slice of the extreme Kerr spacetime.*

Another way of stating this theorem is to say: *extreme Kerr initial data is the unique absolute minimum among all Brill data set* (*which satisfies Condition* 2.5) *with fixed angular momentum.*

Let us discuss the hypotheses of this theorem. The first assumption is that the data belong to the Brill class. This class of data is defined in Section 2; it involves certain technical restrictions on both the topology of the manifold and the behavior of the fields. As it was mentioned above, Theorem 1.1 is expected to be true for general asymptotically flat, axisymmetric, vacuum, complete data. Nevertheless, we emphasize that the Brill class is physically relevant in the following sense: it contains the Kerr black hole data, it also contains non stationary data (in particular small deviations from Kerr), and gravitational radiation is not constrained to be small in any sense. In Section 2 we review a well known procedure for constructing a rich class of examples of this class of data set.

The second assumption, Condition 2.5, implies that the data have non trivial angular momentum only at one end. The theorem is expected to be valid without this restriction; however, this generalization appears to be quite difficult.

Theorem 1.1 generalizes the results presented in [**18**], [**17**] in two ways. First, it does not involve any smallness assumptions on the norm of the fields. In particular, the data is not required to be close to extreme Kerr data. Second, the Killing vector η is not required to be hypersurface orthogonal.

Theorem 1.1 will be a consequence of the following result in the calculus of variations.

Let ρ denote the cylindrical radius in $\mathbb{R}^3$ and Γ the axis $\rho = 0$. Define

$$g = 2 \log \rho. \tag{9}$$

It is important to note that g is an harmonic function in $\mathbb{R}^3 \setminus \Gamma$. Let $x, Y : \mathbb{R}^3 \to \mathbb{R}$ be two arbitrary functions. Consider the following functional

$$\mathcal{M}(x, Y) = \frac{1}{32\pi} \int_{\mathbb{R}^3} \left(|\partial x|^2 + e^{-2x-2g} |\partial Y|^2 \right) \, d\mu, \tag{10}$$

where $d\mu$ is the volume element in $\mathbb{R}^3$ and the contractions are with respect to the euclidean metric. The relation between this functional and the mass of a Brill data set is discussed in Section 2; see also [**21**].

The extreme Kerr initial data (x_0, Y_0) are given by (see, for example, [**18**])

$$x_0 = \log X_0 - g, \quad Y_0 = \bar{Y}_0 - \frac{2J^2 \cos\theta \sin^4\theta}{\Sigma}, \tag{11}$$

where

$$X_0 = \left(\tilde{r}^2 + |J| + \frac{2|J|^{3/2}\tilde{r}\sin^2\theta}{\Sigma}\right)\sin^2\theta, \quad \bar{Y}_0 = 2J(\cos^3\theta - 3\cos\theta), \tag{12}$$

and

$$\tilde{r} = r + \sqrt{|J|}, \quad \Sigma = \tilde{r}^2 + |J|\cos^2\theta. \tag{13}$$

In these equations, (r, θ) are spherical coordinates in $\mathbb{R}^3$ (with $\rho = r\sin\theta$) and J is an arbitrary constant.

Let $H_0^1(\mathbb{R}^3 \setminus \{0\})$ be the completion of $C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ under the norm

$$\|\alpha\|_1 = \left(\int_{\mathbb{R}^3} |\partial\alpha|^2 \, d\mu\right)^{1/2}, \tag{14}$$

and $H^1_{0,X_0}(\mathbb{R}^3 \setminus \Gamma)$ the completion of $C_0^\infty(\mathbb{R}^3 \setminus \Gamma)$ under the norm

$$\|y\|_{1,X_0} = \left(\int_{\mathbb{R}^3} X_0^{-2}|\partial y|^2 \, d\mu\right)^{1/2}. \tag{15}$$

We define the positive and negative part of a function α by $\alpha^+ = \max\{\alpha, 0\}$ and $\alpha^- = \min\{\alpha, 0\}$.

Theorem 1.2. *Consider the functional $\mathcal{M}$ defined by* (10). *Let $\alpha \in H_0^1(\mathbb{R}^3 \setminus \{0\})$, $y \in H^1_{0,X_0}(\mathbb{R}^3 \setminus \Gamma)$. Assume in addition that $\alpha^-, yX_0^{-1} \in L^\infty(\mathbb{R}^3)$ and $\alpha, X_0^{-1}y \to 0$ as $r \to \infty$. Then, the following inequality holds:*

$$\mathcal{M}(x_0 + \alpha, Y_0 + y) \geq \mathcal{M}(x_0, Y_0), \tag{16}$$

where (x_0, Y_0) are the extreme Kerr data. Moreover, the equality in (16) *holds if and only if $\alpha = y = 0$.*

This theorem is a generalization of the results presented in [**18**] where a local version has been proved.

Remarkably, α and y are not assumed to be axially symmetric in this theorem (i.e., they can depend on the φ coordinate). However, we emphasize that Theorem 1.1 is only valid for axially symmetric data (see the remark after Theorem 2.2).

It is important to note that for the extreme Kerr data the difference $Y_0 - \bar{Y}_0 = y_0$ satisfies the hypothesis of Theorem 1.2 (see the appendix). Then inequality (16) can be written in an equivalent form

$$\mathcal{M}(x_0 + \alpha, \bar{Y}_0 + y) \geq \mathcal{M}(x_0, Y_0). \tag{17}$$

The function $\bar{Y}_0$ fixes the angular momentum of the data and it also fixes the origin of coordinates.

In Theorem 1.2, we require the boundedness of the functions α^- and yX_0^{-1}. It is possible to prove the same result without the assumption on α^- and with a stronger assumption on y, namely $ye^{-g} \in L^\infty(\mathbb{R}^3)$ (see a previous version of this article in [**20**]). The disadvantage of this choice is that the function y_0 defined above does not satisfy this assumption: $y_0 e^{-g}$ is not bounded at the origin. And hence, important examples as non-extreme Kerr and the Bowen-York data (see Section 2) are not included. Also, without the assumption $\alpha^- \in L^\infty(\mathbb{R}^3)$ the proofs are more involved. Nevertheless, I believe that for future generalization of Theorem 1.2 these arguments which do not make use of the condition $\alpha^- \in L^\infty(\mathbb{R}^3)$ can be relevant.

In Section 3, we give an equivalent norm for the Sobolev spaces $H^1_0(\mathbb{R}^3 \setminus \{0\})$ and $H^1_{0,X_0}(\mathbb{R}^3 \setminus \Gamma)$. In particular, this shows the equivalence between $H^1_0(\mathbb{R}^3 \setminus \{0\})$ and the weighted Sobolev spaces studied in [**2**].

2. Brill data

The purpose of this section is to define a class of axially symmetric initial data sets. We will call it the Brill class because it is inspired in Brill's positive mass proof for axially symmetric data [**5**]. The point in this definition is that in this class the total mass satisfies the lower bound given by Theorem 2.2.

Axial symmetry implies certain local conditions on the fields h_{ij} and K_{ij}. Let us consider first the metric h_{ij}. For any axially symmetric metric, there exists a coordinate system (ρ, z, φ) such the metric has locally the following form

$$h = e^{(x-2q)}(d\rho^2 + dz^2) + \rho^2 e^x (d\varphi + A_\rho d\rho + A_z dz)^2, \tag{18}$$

where the functions x, q, A_ρ, A_z do not depend on φ. In these coordinates, the axial Killing vector is given by $\eta = \partial/\partial\varphi$ and its norm is given by

$$X = e^{x+g}, \tag{19}$$

where g is given by (9).

Let K_{ij} be a solution of equation (1) such that it satisfies (6). Define the vector S^i by

$$S_i = K_{ij}\eta^j - X^{-1}\eta_i K_{jk}\eta^j\eta^k, \tag{20}$$

where $\eta_i = h_{ij}\eta^j$. Then, define K_i by

$$K_i = \epsilon_{ijk} S^j \eta^k, \tag{21}$$

where ϵ_{ijk} is the volume element of h_{ij}. Using equations (1), (6) and the Killing equation (5) we obtain

$$D_{[j}K_{i]} = 0. \tag{22}$$

Hence, there exists a scalar function Y such that

$$K_i = \frac{1}{2} D_i Y. \tag{23}$$

Summarizing, axial symmetry implies that locally the metric has the form (18) and there exists a potential Y for the second fundamental form.

Definition 2.1. We say that an initial data set (h_{ij}, K_{ij}, S) for the Einstein vacuum equations is a Brill data set if it satisfies the following conditions.

i) $S = \mathbb{R}^3 \setminus \sum_{k=1}^{N} i_k$ where i_k are points in $\mathbb{R}^3$ located at the axis $\rho = 0$ of $\mathbb{R}^3$.

ii) The coordinates (ρ, z, φ) form a global coordinate system on S and the metric h_{ij} is given by (18). The functions x, q, A_ρ, A_z are assumed to be smooth in S. The functions x and q satisfy

$$x = o(r^{-1/2}), \quad \partial x = o(r^{-3/2}), \tag{24}$$

$$q = o(r^{-1}), \quad \partial q = o(r^{-2}) \tag{25}$$

as $r \to \infty$, and

$$x = o(r_{(k)}^{-1/2}), \quad \partial x = o(r_{(k)}^{-3/2}), \tag{26}$$

$$q = o(r_{(k)}^{-1}), \quad \partial q = o(r_{(k)}^{-2}) \tag{27}$$

as $r_{(k)} \to 0$. $r_{(k)}$ is the euclidean distance to the end point i_k.

Let Γ' be defined as $\Gamma' = \Gamma \setminus \sum_{k=1}^{N} i_k$. We assume that

$$q|_{\Gamma'} = 0. \tag{28}$$

iii) The second fundamental form satisfies

$$\pounds_\eta K_{ij} = 0, \quad K = 0. \tag{29}$$

The corresponding potential Y is a smooth function on S such that

$$\int_{\mathbb{R}^3} |\partial Y|^2 e^{-2x-2g} \, d\mu < \infty. \tag{30}$$

Let us analyze the definition of Brill data. Condition (i) implies that S is Euclidean at infinity with $N+1$ ends. In effect, for each i_k, take a small ball B_k of radius $r_{(k)}$, centered at i_k, where $r_{(k)}$ is small enough such that B_k does not contain any other $i_{k'}$ with $k' \neq k$. Take B_R, with large R, such that B_R contains all points i_k. The compact set $\mathcal{K}$ is given by $\mathcal{K} = B_R \setminus \sum_{k=1}^{N} B_k$ and the open sets U_k are given by $B_k \setminus i_k$, for $1 \leq k \leq N$, and U_0 is given by $\mathbb{R}^3 \setminus B_R$. Our choice of coordinates makes an artificial distinction between the end U_0 (which represent $r \to \infty$) and the other ones. This is convenient for our purpose because we want to work always at one fixed end.

The fall off conditions (24)–(25) imply that the metric is asymptotically flat at the end U_0 (i.e., it satisfies the conditions given in [**2**], [**13**]). At the other ends, the fall off conditions (26)–(27) are more general; they include the standard asymptotically flat fall off and they also include the fall off of the extreme Kerr initial data.

In a Brill data set there are two geometrical scalar functions, the norm of the Killing vector X and the potential Y which is related to the twist of the Killing vector (also called the Ernst potential [**23**]). These scalars are well defined in the four dimensional spacetime which results as the evolution of the data. In contrast, the function x depends on a choice of coordinates on the data.

The total mass is essentially contained in the $1/r$ part of the conformal factor x, due to our assumption on q.

The angular momentum is determined by the potential Y in the following way. Define the intervals I_k, $0 < k < N$, to be the open sets in the axis between i_k and i_{k-1}; we also define I_0 and I_N as $z < i_0$ and $z > i_N$ respectively. That is, $\Gamma' = \cup_{k=0}^{N} I_k$. Since g is singular at the axis, the assumption (30) implies that the gradient ∂Y must vanish at each I_k and hence Y is constant at I_k. If Y is a smooth function on $\mathbb{R}^3$, this of course implies that Y is constant at the whole axis. However, as we will see, in order to have a non zero angular momentum, Y cannot be continuous at the end points i_k.

Let Σ_k be a closed surface that encloses only the point i_k. From equation (7) we deduce

$$J_k \equiv J(\Sigma_k) = \frac{1}{8}\left(Y|_{I_k} - Y|_{I_{k-1}}\right), \tag{31}$$

where J_k is the total angular momentum of the end i_k. The total angular momentum of the end $r \to \infty$ is given by

$$J = \frac{1}{8}\left(Y|_{I_0} - Y|_{I_N}\right), \tag{32}$$

which is equivalent to

$$J = \sum_{k=1}^{N} J_k. \tag{33}$$

Finally, let us discuss the restrictions involved in Definition 2.1 with respect to general asymptotically flat, axisymmetric, complete and vacuum data. Locally, there is no restriction on the metric and the only restriction on the second fundamental form is the maximal condition $K = 0$. Globally, we have assumed a particular topology on the compact core $\mathcal{K}$ of the asymptotically flat manifold S. Also, we have assumed that the metric has globally the form (18). The fall off conditions (24) for x are a consequence of the standard definition of asymptotically flatness; however, the fall off conditions (25) for q are an extra assumption. Condition (28) for q on the axis is a consequence of the regularity of the metric at the axis, and hence it is not a restriction.

The fundamental property of Brill data is the following:

Theorem 2.2. *The total mass m of a Brill data satisfies the following inequality*

$$m \geq \mathcal{M}(x, Y), \tag{34}$$

where $\mathcal{M}(x, Y)$ is given by (10).

This theorem extends Brill original proof [**5**] in two ways. First, it allows for non zero A in the metric (18). This generalization was recently given in [**25**], and we use this result in the following proof. The second extension is that the topology of the data is non trivial; this was introduced in [**21**]. In particular, this includes the topology of the Kerr initial data. It is important to recall that we are not introducing any inner boundary. The mass is obtained as an integral over S, that is, an integral over all the asymptotic regions (see the discussion in [**21**]).

Proof. Under our decay assumptions on q, we have that the total mass of a Brill data is given by

$$m = -\frac{1}{8\pi} \lim_{R\to\infty} \oint_{\partial B_R} \partial_r x \, ds. \tag{35}$$

The Ricci scalar R of the metric h_{ij} is given by (see [**25**])

$$-\frac{1}{8} R e^{(x-2q)} = \frac{1}{4}\Delta x + \frac{1}{16}|\partial x|^2 - \frac{1}{4}\Delta_2 q + \frac{1}{16}\rho^2 e^{2q}(A_{\rho,z} - A_{z,\rho})^2, \tag{36}$$

where Δ is the Laplacian in $\mathbb{R}^3$ and Δ_2 is the 2-dimensional Laplacian

$$\Delta_2 q = q_{,\rho\rho} + q_{,zz}. \tag{37}$$

We want to integrate (36) over $\mathbb{R}^3$. Let us analyze each term individually. Consider the first term in the right hand side of (36). To perform the integral, we take the compact domain $\mathcal{K}$ defined above, and we have

$$\int_{\mathcal{K}} \Delta x \, d\mu = \int_{\partial\mathcal{K}} \frac{\partial x}{\partial n} \, ds, \tag{38}$$

where $\partial/\partial n$ denotes a normal derivative. The boundary $\partial\mathcal{K}$ is formed by the boundaries ∂B_k and ∂B_R. Using the decay condition (26), we get that the contribution of ∂B_k vanishes in the limit $r_{(k)} \to 0$. Using (35), we get that the contribution of ∂B_R in the limit $R \to \infty$ is the mass.

Take the Ricci scalar in the left hand side of (36). We use the hypothesis that the data have $K = 0$ and the constraint equation (2) to get

$$R = K_{ij}K^{ij}. \tag{39}$$

We will get a lower bound to the left hand side of (39). The metric (18) can be written in the following form:

$$h_{ij} = q_{ij} + X^{-1}\eta_i\eta_j, \tag{40}$$

where q_{ij} is a positive definite metric in the orbit space. Using this decomposition we get

$$K_{ij}K^{ij} = K^{ij}K^{kf}q_{ik}q_{jf} + X^{-2}(K^{ij}\eta_i\eta_j)^2 + 2X^{-1}K^{ij}K^{kf}\eta_i\eta_k q_{jf}. \tag{41}$$

The first two terms in the right hand side of this equation are positive defined. Using the definitions (20) and (21), the last term can be written as follows:

$$K^{ij}K^{kf}\eta_i\eta_k q_{jf} = S^iS_i \tag{42}$$

$$= \frac{1}{X}K^iK_i \tag{43}$$

$$= \frac{1}{4X}D^iYD_iY \tag{44}$$

$$= \frac{1}{4X}|\partial Y|^2 e^{-x+2q}. \tag{45}$$

Then we get

$$Re^{(x-2q)} \geq \frac{1}{2X^2}|\partial Y|^2. \tag{46}$$

Take the term $\Delta_2 q$ in (36). Let K_δ be the cylinder $\rho \leq \delta$ and consider the following domain $A_\delta = \mathcal{K} \setminus K_\delta$. We integrate over A_δ and then take the limit $\delta \to 0$. The integral over A_δ can be written in the following form

$$\int_{A_\delta} \Delta_2 q \, d\mu = 4\pi \int_{A_\delta} d\rho \, dz \, (q_{,\rho\rho} + q_{,zz})\rho, \tag{47}$$

$$= 4\pi \int_{A_\delta} d\rho \, dz \, \left((\rho q_{,\rho} - q)_{,\rho} + (\rho q_{,z})_{,z}\right). \tag{48}$$

We use the divergence theorem in two dimensions to transform this volume integral in a boundary integral, that is

$$\int_{A_\delta} d\rho\, dz\, \left((\rho q_{,\rho} - q)_{,\rho} + (\rho q_{,z})_{,z}\right) = \oint_{\partial A_\delta} \bar{V} \cdot \bar{n}\, d\bar{s}, \tag{49}$$

where $\bar{n}$ is the 2-dimensional unit normal, $d\bar{s}$ is the line element of the 1-dimensional boundary, and $\bar{V}$ is the 2-dimensional vector given in coordinates (ρ, z) by

$$\bar{V} = ((\rho q_{,\rho} - q), (\rho q_{,z})). \tag{50}$$

By (28) and the assumption that q is smooth on S (and hence the derivatives $q_{,\rho}$ and $q_{,z}$ are bounded at Γ') we have that the vector V vanishes at Γ'. Then, using (47) and (49) we get

$$\lim_{\delta \to 0} \int_{A_\delta} \Delta_2 q\, d\mu = \oint_{\partial \mathcal{K}} \bar{V} \cdot \bar{n}\, d\bar{s}. \tag{51}$$

We now take the limit $R \to \infty$ and $r_{(k)} \to 0$. We use the decay conditions (25) and (27) to obtain

$$\int_{\mathbb{R}^3} \Delta_2 q\, d\mu = 0. \tag{52}$$

Since the last term in (36) is positive, collecting these calculations we get (34). q.e.d.

Since the data should satisfy the constraint equations (1)–(2), it is not obvious that we can construct non trivial examples of Brill data. One can easily check that Schwarzschild data in isotropic coordinates is in the Brill class. Other explicit examples are Brill-Lindquist data and the Kerr black hole data (i.e., Kerr data with parameters such that inequality (8) is satisfied), see [**21**].

Let us discuss a general procedure to construct a rich family of Brill data. For simplicity, we will assume that $A = 0$ in equation (18). Consider the metric

$$\tilde{h}_{ij} = e^{-2q}(d\rho^2 + dz^2) + \rho^2 d\varphi^2. \tag{53}$$

This metric will be used as a conformal background for the physical metric h_{ij}, that is, $h_{ij} = e^x \tilde{h}_{ij}$. We will take q in (53) and the potential Y as given functions.

We first discuss how to construct solutions of the momentum constraint (1) from an arbitrary potential Y, and how to prescribe the angular momentum of the solution. Consider the following tensor

$$\tilde{K}^{ij} = \frac{2}{\rho^2} \tilde{S}^{(i} \eta^{j)}, \tag{54}$$

where

$$\tilde{S}^i = \frac{1}{2\rho^2} \tilde{\epsilon}^{ijk} \eta_j \tilde{D}_k Y, \tag{55}$$

$\tilde{\epsilon}_{ijk}$ denotes the volume element with respect to $\tilde{h}_{ij}$ and $\tilde{D}$ is the connexion with respect to $\tilde{h}_{ij}$. The indices of the tilde quantities are moved with $\tilde{h}_{ij}$ and its inverse $\tilde{h}^{ij}$. The tensor $\tilde{K}^{ij}$ is symmetric, trace free, and satisfies (see, for example, the appendix in [**19**])

$$\tilde{D}_i \tilde{K}^{ij} = 0. \tag{56}$$

Hence, for an arbitrary function Y we get a solution of equation (56) given by (54). This, essentially, provides a solution of the momentum constraint (1).

To control the angular momentum of the data, we will prescribe the behavior of Y near the axis in the following way. Take spherical coordinates $(r_{(k)}, \theta_{(k)})$ centered at the end point i_k and consider the following function

$$\bar{Y}_k = 2J_k(\cos^3\theta_{(k)} - 3\cos\theta_{(k)}), \tag{57}$$

where J_k are arbitrary constants. The normalization factor is chosen to be consistent with equation (31). Define

$$\bar{Y} = \sum_{k=0}^{N} \bar{Y}_k. \tag{58}$$

Let $Y = \bar{Y} + y$, where y vanishes at the axis. Then, the angular momentum of Y at the ends i_k is given by the free constants J_k in $\bar{Y}$.

We discuss now the conditions on the function q. Define the Yamabe number of $\tilde{h}_{ij}$ to be

$$\lambda = \inf_{0 \neq \varphi \in C_0^\infty(S)} \frac{\int_{\mathbb{R}^3} \left(8\tilde{D}^i\varphi\tilde{D}_i\varphi + \tilde{R}\varphi^2\right) \, d\mu_{\tilde{h}}}{\int_{\mathbb{R}^3} \varphi^6 \, d\mu_{\tilde{h}}}. \tag{59}$$

In order to construct a Brill data, the metric $\tilde{h}_{ij}$ should satisfy the condition $\lambda > 0$, as we will see in the following theorem:

Theorem 2.3. *Let $q \in C_0^\infty(S)$ such that $\lambda > 0$ and let $Y = \bar{Y} + y$, where $\bar{Y}$ is given by* (58) *and $y \in C_0^\infty(\mathbb{R}^3 \setminus \Gamma)$. Then, there exists a function x such that*

$$h_{ij} = e^x \tilde{h}_{ij}, \quad K_{ij} = e^{-x/2}\tilde{K}_{ij} \tag{60}$$

define a Brill data set, where $\tilde{h}_{ij}$ is given by (53) *and $\tilde{K}_{ij}$ is given by* (54)*.*

This theorem was proved in [**6**] and [**7**] (see also the correction in [**30**] of this article). There exists a more general version of the theorem [**12**], [**31**]. We have assumed that the functions involved have compact support in order to simplify the assumptions, but decay conditions are also possible.

Sketch of proof. What follows is the rewriting of our setting in terms of the one used in these references. To simplify the discussion, let us follow the existence theorem in section VIII of [**12**].

Define the function ψ_0 by

$$\psi_0 = \sum_{k=1}^{N} 1 + \frac{1}{r_{(k)}}. \tag{61}$$

Consider the metric defined by the following conformal rescaling

$$\hat{h}_{ij} = \psi_0^4 \tilde{h}_{ij}. \tag{62}$$

One can easily check that this metric is asymptotically flat with $N+1$ ends. Moreover, the Yamabe number of the metric $\hat{h}_{ij}$ is the same as the the one for h_{ij} because, by construction, it is a conformally invariant quantity. Then, $\hat{h}_{ij}$ is in the positive Yamabe class. Hence, we can apply the above mentioned theorem to conclude that there exists a solution of the Lichnerowicz equation

$$\hat{D}^i \hat{D}_i \psi - \frac{\hat{R}}{8} = \hat{K}^{ij} \hat{K}_{ij} \psi^{-7}, \tag{63}$$

such that $\psi \to 1$ at the end point i_k. Where $\hat{K}_{ij}$ is given by $\hat{K}_{ij} = \psi_0^{-2} \tilde{K}_{ij}$ with $\tilde{K}_{ij}$ given by (54), hat quantities are defined with respect to the metric $\hat{h}_{ij}$ and the indices are moved with this metric and its inverse.

Define x to be $e^x = (\psi\psi_0)^4$. Then it follows, by the standard conformal transformation formulas, that (60) define a solution of the constraint equations (1)–(2).

The singular part of x is given by ψ_0, and at the end point i_k we have

$$x = O(-4 \log r_{(k)}), \quad \partial x = O(r_{(k)}^{-1}), \tag{64}$$

which is consistent with (24). q.e.d.

It remains to show how to achieve the condition $\lambda > 0$. This is given by Theorem 4.2 in [**7**]. Applying this theorem to the present case we get (see also [**32**]):

Theorem 2.4. *Let $q^0 \in C_0^\infty(S)$ and set $q = Cq^0$, where C is a constant. Then, for C small enough, we have $\lambda > 0$.*

A simple but non trivial choice for q which satisfies $\lambda > 0$ is $q = 0$. This gives conformally flat solutions for the constraint equations. These kinds of solutions are extensively used in numerical simulations for black hole collisions (see the review article [**16**]). Two examples are the Bowen-York spinning data [**4**] and the data discussed in [**22**].

The definition of Brill data is tailored to the hypothesis of Theorem 2.2. However, in order to prove Theorem 1.2 we need to impose more conditions. More precisely, we assume the following. Define $y = Y - \bar{Y}_0$ and $\alpha = x - x_0$ where $\bar{Y}_0$ and x_0 are given by (11).

Condition 2.5. We assume $y \in H^1_{0,X_0}(\mathbb{R}^3 \setminus \Gamma)$ and $\alpha^-, X_0^{-1}y \in L^\infty(\mathbb{R}^3)$ and $X_0^{-1}y \to 0$ as $r \to \infty$.

The conditions on y imply that y vanishes at the axis Γ and hence there exists only one end with non trivial angular momentum. The location of this end is fixed by the function $\bar{Y}_0$. However, let us emphasize that the data can have extra ends as long as they have zero angular momentum.

We have also assumed that $\alpha^- \in L^\infty(\mathbb{R}^3)$. This implies an extra restriction on the behavior of x near the ends. In Definition 2.1 we have assumed the fall off behavior (26) of x near the ends, on the other hand for extreme Kerr we have $x_0 = -2\ln r + O(1)$ near $r \to 0$. A relevant class of fall conditions that satisfies both (26) and $\alpha^- \in L^\infty(\mathbb{R}^3)$ is given $x = -\beta \ln r + O(1)$ near $r \to 0$, for $\beta \geq 2$. In particular, this includes the asymptotically flat ends $\beta = 4$ described in Theorem 2.3 (see equation (64)).

Let us discuss important examples of Brill data that satisfies Condition 2.5. First, extreme Kerr data. In this case, we have $\alpha = 0$ and $y = y_0 = Y_0 - \bar{Y}_0$. In the appendix we prove that the function y_0 satisfies the assumptions in 2.5. Second, non-extreme Kerr black hole data (for the explicit form of the functions X and Y for these data see the appendix of [**21**]). These data are asymptotically flat at the end $r \to 0$ and hence, by the discussion above, we have $\alpha^- \in L^\infty(\mathbb{R}^3)$. Using a computation similar to the one presented for extreme Kerr in the appendix, we conclude that the function y also satisfies 2.5. Finally, two other examples of Brill data that satisfy Condition 2.5 are the Bowen-York data for only one spinning black hole (i.e., $Y = \bar{Y}_0$ and $q = 0$) and the data constructed in [**22**] in which $Y = Y_0$ and $q = 0$.

3. Global Minimum

The crucial property of the mass functional defined in (10) is its relation to the energy of harmonic maps from $\mathbb{R}^3$ to the hyperbolic plane $\mathbb{H}^2$: they differ by a boundary term. Let g be an arbitrary harmonic function on a domain Ω in $\mathbb{R}^3$. Define the mass functional over Ω as

$$\mathcal{M}_\Omega = \frac{1}{32\pi}\int_\Omega \left(|\partial x|^2 + e^{-2x-2g}|\partial Y|^2\right)\, d\mu. \tag{65}$$

Then, using that g is harmonic, we find the following identity

$$\mathcal{M}_\Omega = \mathcal{M}'_\Omega - \oint_{\partial\Omega} \frac{\partial g}{\partial n}(g + 2x)\, ds, \tag{66}$$

where $\mathcal{M}'_\Omega$ is given by

$$\mathcal{M}'_\Omega = \frac{1}{32\pi}\int_\Omega \left(\frac{|\partial X|^2 + |\partial Y|^2}{X^2}\right) d\mu, \tag{67}$$

and we have defined the function X by

$$X = e^{g+x}. \tag{68}$$

The functional $\mathcal{M}'_\Omega$ defines an energy for maps $(X, Y) : \mathbb{R}^3 \to \mathbb{H}^2$ where $\mathbb{H}^2$ denotes the hyperbolic plane $\{(X, Y) : X > 0\}$, equipped with the negative constant curvature metric

$$ds^2 = \frac{dX^2 + dY^2}{X^2}. \tag{69}$$

The Euler-Lagrange equations for the energy $\mathcal{M}'_\Omega$ are given by

$$\Delta \log X = -\frac{|\partial Y|^2}{X^2}, \tag{70}$$

$$\Delta Y = 2\frac{\partial Y \partial X}{X}. \tag{71}$$

The solutions of (70)–(71), i.e., the critical points of $\mathcal{M}'_\Omega$, are called harmonic maps from $\mathbb{R}^3 \to \mathbb{H}^2$. Since $\mathcal{M}_\Omega$ and $\mathcal{M}'_\Omega$ differ only by a boundary term, they have the same Euler-Lagrange equations.

Harmonic maps have been intensively studied; in particular, the Dirichlet problem for target manifolds with negative curvature has been solved [**27**], [**35**], [**34**]. However, these results do not directly apply in our case because the equations are singular at the axis. In effect, the function X represents the norm of the Killing vector (see equation (19)) which vanishes at Γ', and this function appears in the denominator of equations (70)–(71). This singular behavior implies that the energy of the harmonic map is infinite as it can be seen from equation (67).

Solutions of equations (70)–(71), with this type of singular behavior at the axis, represent vacuum, stationary, axially symmetric solutions of Einstein equations. This equivalence was discovered by Carter [**11**] based in the work of Ernst [**23**]. The relation between the stationary, axially symmetric equations and harmonic maps was discovered much later by Bunting (the original work by Bunting is unpublished, see [**9**]). In General Relativity, equations (70)–(71) are important because they play a central role in the black hole equilibrium problem (see [**9**] and the review articles [**14**], [**10**]). Motivated by this problem, G. Weinstein in a series of articles, [**39**], [**40**], [**41**], [**42**], [**44**], [**43**] (see also [**29**]), studied the Dirichlet problem for harmonic maps with prescribed singularities of this type. Weinstein's work will be particularly relevant here; let us briefly describe it.

Weinstein constructs solutions of (70)–(71) which represent stationary, axially symmetric, black holes with disconnected horizons. To prove the existence of such solutions, he defines the energy $\mathcal{M}_\Omega$, with an appropriate harmonic function g. This energy plays a role of an auxiliary functional in order to "regularize" the singular energy $\mathcal{M}'_\Omega$ of the harmonic map. The solution is a minimum of $\mathcal{M}_\Omega$ and the existence is proved with a direct variational method.

Our problem is related: we have a solution of (70)–(71) (i.e., the extreme Kerr solution given by (11)–(12)) and we want to prove that it is a unique minimum of $\mathcal{M}$. There exist, however, two important differences from Weinstein's work.

The first one, which is a simplification, is that we do not want to prove existence of a solution. We already have an explicit solution; we just want to prove that it is a minimum.

The second difference, which introduces a difficulty, is that we deal with the *extreme* Kerr solution. Extreme means that $m = \sqrt{|J|}$, where m is the mass and J the angular momentum of the black hole; this definition can be also extended for multiple black holes (see [**41**]). This is a degenerate limit for black hole solutions, and it is excluded in the hypotheses of Weinstein existence theorems. Hence, these results do not directly apply to our case.

The extreme limit presents important peculiarities with respect to the non extreme cases. Remarkably enough, in this case (and only in this case) the functional $\mathcal{M}$ is the mass of the black hole (see [**21**]). In the non extreme cases, the functional defined by Weinstein is not the same as our definition because the choice of the harmonic function g is different. In particular, if we take the extreme limit of the Weinstein functional for one Kerr black hole, we get zero and not the total mass. Perhaps, Weinstein's functional describes the interaction energy of multiple black holes and this is related to the non zero force between them. The existence of this force in the general case is an open question. This question is relevant for the black hole uniqueness problem with disconnected horizons.

Another peculiarity of the extreme case is that the relevant manifold is complete without boundary; in the non extreme case the manifold has an inner boundary: the horizon of the black hole (there is no horizon in the extreme Kerr black hole).

Let us give the main ideas of the proof of Theorem 1.2. Theorems 3.1 and 3.2 establish that extreme Kerr is the unique minimum in an annulus centered at the origin, with appropriate boundary conditions. The choice of the domain is important to avoid the singularity of the extreme Kerr solution at the origin (this is the main technical difference with the non extreme case). These two theorems are analogous to Proposition 1 and Proposition 3 of [**40**] and use similar techniques. The main idea in the proof of Theorem 3.1 is the a priori bounds found by Weinstein. In Theorem 3.3 we prove a uniqueness result for extreme

Kerr in the whole domain $\mathbb{R}^3$ under appropriate decay conditions. This theorem is interesting by itself. Finally, to prove Theorem 1.2, we cover $\mathbb{R}^3$ with annulus and use a density argument together with the previous theorems. This argument will work because we know a priori the solution in $\mathbb{R}^3$. This is an important point: in this theorem we are not proving the existence of the extreme Kerr solution. Note that in [**40**], Theorem 1, where the existence of solution for the non extreme cases was proved, this proof requires the a priori bounds given by Proposition 2, which are not valid in the extreme case.

Let B_R be a ball of radius R in $\mathbb{R}^3$ centered at the origin. We define the annulus $A = B_R \setminus B_\epsilon$, where $R > \epsilon > 0$ are two arbitrary constants. Let $H_0^1(A)$ be the standard Sobolev space on A, that is, the closure of $C_0^\infty(A)$ under the norm

$$\|\alpha\|_{1;A} = \left(\int_A |\partial\alpha|^2 \, d\mu \right)^{1/2}. \tag{72}$$

And define the weighted Sobolev space $H_{0,h}^1(A)$ to be the closure of $C_0^\infty(A \setminus \Gamma)$ under the norm

$$\|y\|_{1,h;A} = \left(\int_A e^{-2g} |\partial y|^2 \, d\mu \right)^{1/2}. \tag{73}$$

Since the function x_0 is smooth on A, the norm (73) is equivalent to the norm (15) restricted to A.

Theorem 3.1. *Consider the functional defined by* (65) *on the annulus A, with $g = 2\log\rho$. Let x_0 and Y_0 be the extreme Kerr solution given by* (11)*. Then, there exist*

$$\alpha_0 \in H_0^1(A), \quad y_0 \in H_{0,h}^1(A), \tag{74}$$

such that

$$\mathcal{M}_A(x_0 + \alpha, Y_0 + y) \geq \mathcal{M}_A(x_0 + \alpha_0, Y_0 + y_0), \tag{75}$$

for all $\alpha \in H_0^1(A)$ and $y \in H_{0,h}^1(A)$. Moreover, the minimum (α_0, y_0) satisfies

$$\alpha_0 \in L^\infty(A), \quad e^{-g} y_0 \in L^\infty(A), \tag{76}$$

and the functions

$$X = e^{g + x_0 + \alpha_0}, \quad Y = Y_0 + y_0, \tag{77}$$

define a harmonic map from $(X, Y) : A \setminus \Gamma \to \mathbb{H}^2$; that is, they satisfy equations (70)–(71) *on $A \setminus \Gamma$.*

Remark. The choice of the domain is important because the function x_0 is not bounded at the origin. The proof fails if the domain includes the origin.

Proof. Define

$$m_0 = \inf_{\alpha\in H^1_0(A),\, y\in H^1_{0,h}(A)} \mathcal{M}_A(\alpha, y). \tag{78}$$

Since $\mathcal{M}$ is bounded below, m_0 is finite. Note that the functional $\mathcal{M}_A$ is not bounded for arbitrary functions in $H^1_0(A)\times H^1_{0,h}(A)$.

Let (α_n, y_n) be a minimizing sequence, that is

$$\mathcal{M}_A(\alpha_n, y_n) \to m_0 \text{ as } n\to\infty. \tag{79}$$

To prove the existence of a minimum we will prove that there exists some subsequence of (α_n, y_n) which converges to an actual minimizer (α_0, y_0). To prove this, we will show that for every minimizing sequence it is possible to construct another minimizing sequence such that α_n is uniformly bounded. Then, the existence of a convergent subsequence follows from standard arguments (see [**39**]).

We define x_n, Y_n by

$$x_n = x_0 + \alpha_n, \quad Y_n = Y_0 + y_n. \tag{80}$$

We first obtain a lower bound for x_n. Let

$$C_1 = \min_{\partial A} x_0, \tag{81}$$

the constant C_1 depends on R and ϵ, in particular $C_1\to\infty$ as $\epsilon\to 0$ because x_0 is singular at the origin. This is the reason why the proof fails if the domain includes the origin. Given (x_n, y_n), define a new sequence (x'_n, y_n) as $x'_n = \max\{x_n, C_1\}$. Then one can check that $\mathcal{M}(\alpha'_n, y_n)\le \mathcal{M}(\alpha_n, y_n)$. Moreover, $\alpha'_n\in H^1_0(A)$. This gives lower bounds for α'_n on A:

$$\alpha'_n \ge C_1 - x_0 \ge C_1 - \max_A x_0 = C'_1. \tag{82}$$

Using this lower bound, we want to prove that the minimizing sequence can be chosen such that $\alpha_n\in C^\infty_0(A)$ and $y_n\in C^\infty_0(A\setminus\Gamma)$. This is an important step in the proof, it will be used in the following to calculate boundary integrals that are not defined for generic functions in H^1. Also, it plays an essential role in the proof of Theorem 1.2.

Define the set $\mathcal{H}$ as the subset of $H^1_0(A)$ such that the lower bound (82) is satisfied. The functional $\mathcal{M}_A$ is bounded for all functions $y\in H^1_{0,h}(A)$ and $\alpha\in\mathcal{H}$. By definition, for every $\alpha\in H^1_0(A)$ and $y\in H^1_{0,h}(A)$ there exists a sequence $\alpha_n\in C^\infty_0(A)$ and $y_n\in C^\infty_0(A\setminus\Gamma)$ such that $\alpha_n\to\alpha$ and $y_n\to y$ as $n\to\infty$ in the norms (72) and (73) respectively. If $\alpha\in\mathcal{H}$, then by Lemma 5.1, we can take α_n such that $\alpha_n\in\mathcal{H}$ for all n. For such a sequence, we claim that

$$\lim_{n\to\infty}\mathcal{M}_A(\alpha_n, y_n) = \mathcal{M}_A(\alpha, y). \tag{83}$$

To prove this we compute

$$|\mathcal{M}_A(\alpha_n, y_n) - \mathcal{M}_A(\alpha, y)| \le I_1 + I_2, \tag{84}$$

where

$$I_1 = \frac{1}{32\pi}\int_A \left||\partial x_n|^2 - |\partial x|^2\right| \, d\mu, \tag{85}$$

$$I_2 = \frac{1}{32\pi}\int_A e^{-2g}\left|e^{-2x_n}|\partial Y_n|^2 - e^{-2x}|\partial Y|^2\right| \, d\mu. \tag{86}$$

For I_1 we have

$$I_1 = \frac{1}{32\pi}\int_A |\partial(x_n+x)\cdot\partial(x_n-x)| \, d\mu, \tag{87}$$

$$\leq \frac{1}{\sqrt{32\pi}}\left(\mathcal{M}_A^{1/2}(\alpha_n, y_n) + \mathcal{M}_A^{1/2}(\alpha, y)\right)\|\alpha-\alpha_n\|_{1;A}, \tag{88}$$

where in the last line we have used Hölder inequality. The first factor in the right hand side of (88) is bounded for all n and $\alpha_n \to \alpha$ in $H^1_0(A)$, and we obtain that $I_1 \to 0$ as $n \to \infty$.

A similar computation for I_2 leads to

$$I_2 = \frac{1}{32\pi}\int_A e^{-2g}\left|\left(e^{-x_n}\partial Y_n + e^{-x}\partial Y\right)\cdot\left(e^{-x_n}\partial Y_n - e^{-x}\partial Y\right)\right| \, d\mu \tag{89}$$

$$\leq \frac{1}{\sqrt{32\pi}}\left(\mathcal{M}_A^{1/2}(\alpha_n, y_n) + \mathcal{M}_A^{1/2}(\alpha, y)\right)\left(I_{2,1}+I_{2,2}\right), \tag{90}$$

where

$$I_{2,1} = \left(\int_A e^{-2g-2x_0}|\partial Y|^2\left|e^{-\alpha_n}-e^{-\alpha}\right|^2 \, d\mu\right)^{1/2}, \tag{91}$$

$$I_{2,2} = \left(\int_A e^{-2g-2x_0-2\alpha_n}|\partial(y-y_n)|^2 \, d\mu\right)^{1/2}. \tag{92}$$

The function x_0 is positive on A, so it can be trivially bounded by $e^{-2x_0} \leq 1$ and hence suppressed from the definitions of $I_{2,1}$ and $I_{2,2}$. However, for later use in the proof of Theorem 1.2, we keep it in equations (91)–(92).

We have $\alpha_n \in \mathcal{H}$, thus the integrand in $I_{2,1}$ is bounded by a summable function for all n. Since $\alpha_n \to \alpha$ a.e. we can apply the dominated convergence theorem to conclude that $I_{2,1} \to 0$ as $n \to \infty$. For $I_{2,2}$ we use again that $\alpha_n \in \mathcal{H}$ to bound the exponential factor $e^{-\alpha_n}$ for all n, and then the assumption $y_n \to y$ in $H^1_{0,h}(A)$ to conclude that $I_{2,2} \to 0$ as $n \to \infty$. Hence, we have proved (83).

Let $\alpha_k \in H^1_0(A)$, $y_k \in H^1_{0,h}(A)$ be a minimizing sequence. Let $\alpha_{k,n} \in C^\infty_0(A)$ and $y_{k,n} \in C^\infty_0(A \setminus \Gamma)$ such that $\alpha_{k,n} \to \alpha_k$ and $y_{k,n} \to y_k$ as $n \to \infty$. Then we have

$$|\mathcal{M}_A(\alpha_{k,n}, y_{k,n}) - m_0| \leq |\mathcal{M}_A(\alpha_{k,n}, y_{k,n}) - \mathcal{M}_A(\alpha_k, y_k)| + |\mathcal{M}_A(\alpha_k, y_k) - m_0|. \tag{93}$$

For an arbitrary ϵ, by (78), there exists k such that

$$|\mathcal{M}_A(\alpha_k, y_k) - m_0| \le \epsilon/2. \tag{94}$$

For this k, by (83), there exists n such that

$$|\mathcal{M}_A(\alpha_{k,n}, y_{k,n}) - \mathcal{M}_A(\alpha_k, y_k)| \le \epsilon/2. \tag{95}$$

Hence, we conclude that

$$m_0 = \inf_{k,n\in\mathbb{N}} \mathcal{M}_A(\alpha_{k,n}, y_{k,n}). \tag{96}$$

In order to obtain upper bounds, we exploit the symmetries of the hyperbolic plane. Define the following inversions

$$\bar{X} = \frac{X}{X^2 + Y^2}, \tag{97}$$

$$\bar{Y} = \frac{Y}{X^2 + Y^2}. \tag{98}$$

We have (see [**39**])

$$\frac{|\partial X|^2 + |\partial Y|^2}{X^2} = \frac{|\partial \bar{X}|^2 + |\partial \bar{Y}|^2}{\bar{X}^2}. \tag{99}$$

Let $\bar{g}$ be an arbitrary harmonic function, and define $\bar{x}$ by

$$\bar{X} = e^{\bar{g} + \bar{x}}. \tag{100}$$

Using equations (66) and (99), we obtain the following identity

$$\mathcal{M}_A = \bar{\mathcal{M}}_A + \oint_{\partial A} \left(\frac{\partial \bar{g}}{\partial n}(\bar{g} + 2\bar{x}) - \frac{\partial g}{\partial n}(g + 2x) \right) ds, \tag{101}$$

where $\bar{\mathcal{M}}_A = \mathcal{M}_A(\bar{x}, \bar{Y})$.

Take $g = \bar{g}$. Denote by K_δ the cylinder $\rho \le \delta$. Since g is singular on the axis, in order to perform the integrals we will consider the domain $A_\delta = A \setminus K_\delta$ for some small $\delta > 0$ and then take the limit $\delta \to 0$. The boundary integral in (101) reduces to

$$C_A = \lim_{\delta \to 0} \oint_{\partial A_\delta} 2 \frac{\partial g}{\partial n}(\bar{x} - x)\, ds. \tag{102}$$

From (97) and (100) we deduce

$$\bar{x} - x = -\log(e^{2g+2x} + Y^2). \tag{103}$$

Then we have

$$\lim_{\rho \to 0} (\bar{x} - x) = -2 \log |J|, \tag{104}$$

where we have used that $y \in C_0^\infty(A \setminus \Gamma)$ and $Y^2 = Y_0^2 = 4J^2$ at Γ. We assume $J \neq 0$, the case $J = 0$ is trivial. Hence we obtain

$$\mathcal{M}_A = \bar{\mathcal{M}}_A + C_A, \tag{105}$$

where

$$C_A = -16\pi(R-\epsilon)\log(4J^2) - \oint_{\partial A} 2\frac{\partial g}{\partial n}\log(e^{2g+2x_0}+Y_0^2)\,ds. \tag{106}$$

The important point is that C_A is finite.

We can use the same argument as above to obtain lower bound for the function $\bar{x}$ in A. Take

$$C_2 = \min_{\partial A} \bar{x} = \min_{\partial A}\{x_0 - \log(e^{2g+2x_0}+Y_0^2)\}. \tag{107}$$

As in the case of C_1, here we also have that $C_2 \to \infty$ as $\epsilon \to 0$. Note that C_2 and C_1 are independent of α and y.

As before, we can define a new function $\bar{x}' = \max\{\bar{x}, C_2\}$, and the energy of $\bar{x}'$ is less or equal the energy of $\bar{x}$. Then $\bar{x}' \geq C_2$. In the following we redefine $\bar{x}'$ by $\bar{x}$. From (97) we have

$$\bar{X} \leq \frac{1}{X}, \tag{108}$$

and then

$$e^x \leq e^{-2g-\bar{x}} \leq e^{-2g-C_2}, \tag{109}$$

in A. Also, from (97) we have

$$\bar{X} \leq \frac{X}{Y^2}, \tag{110}$$

and then we deduce

$$Y^2 \leq e^{-2g-2C_2}. \tag{111}$$

We have obtained the bounds (109) and (111) which are singular at the axis. To get bounds in a neighborhood of the axis we will split this neighborhood in two disconnected domains: the upper part and the lower one. More precisely, fix $\delta > 0$ (we emphasize that in this case we will not take the limit $\delta \to 0$ as before), define $K_+ = A \cap K_\delta \cap \{z \geq \epsilon\}$ and $K_- = A \cap K_\delta \cap \{z \leq \epsilon\}$; see Figure 1. We will obtain estimates for K_+ and K_- independently.

On K_+ we define the following modified inversions

$$\bar{X} = \frac{X}{X^2+(Y+2J)^2}, \tag{112}$$

$$\bar{Y} = \frac{Y}{X^2+(Y+2J)^2}. \tag{113}$$

Take $\bar{g} = -g$ and integrate (101) over K_+. The boundary term is given by

$$C_{K_+} = -2\oint_{\partial K_+} \frac{\partial g}{\partial n}(\bar{x}+x)\,ds, \tag{114}$$

where

$$\bar{x} = -\log\left(e^x + e^{-2g-x}(Y+2J)^2\right). \tag{115}$$

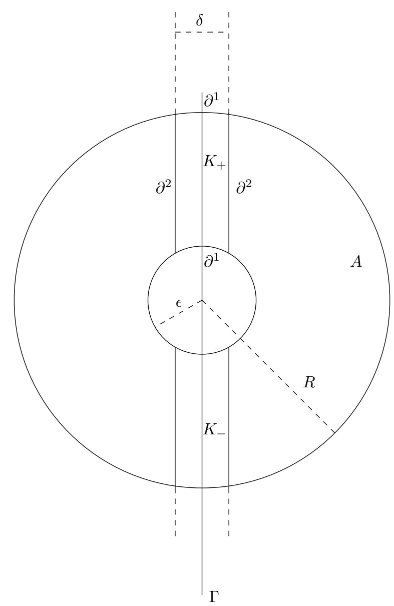

Figure 1. Domains.

We want to prove that C_{K_+} is finite and the difficulty is of course that g is singular at Γ. We decompose the boundary ∂K_+ into two pieces. The first one intersects the axis and is given by $\partial^1 = \partial K_+ \cap \partial A$, and the second does not intersect the axis and is given by $\partial^2 = \partial K_+ \cap \partial K_\delta$; see Figure 1. On ∂^2 the function g is regular and hence the integral is finite. On ∂^1 we have $y = \alpha = 0$. Using that y vanishes near the axis and the following limit

$$\lim_{\rho \to 0} e^{-2g}(Y_0 + 2J)^2 = 0, \tag{116}$$

we conclude that the integral is also finite in this piece of the boundary. Equation (116) is in fact the reason why in equations (112)–(113) we have modified the inversions (97)–(98) with the extra term $2J$.

We can use now the same idea as before to obtain upper bounds. Set

$$C_3 = \min_{\partial K_+} \bar{x} = \min_{\partial K_+} \{-\log(e^x + e^{-2g-x}(Y_0 + 2J)^2\}. \tag{117}$$

By (116) we have that this constant is finite. Then, we get that $\bar{x} \geq C_3$ in K_+ and we can use the inversion to get upper bounds for x in K_+. However, here C_3 does depend on α and y because these functions do not vanish on ∂^2. The key point is that nevertheless we can get lower bounds to C_3 which does not depend on α and y. In order to do this we will use the previously defined constants C_2 and C_1. The estimates are done in ∂^1 and ∂^2 independently. We decompose $C_3 = C_3^1 + C_3^2$ where

$$C_3^1 = \min_{\partial^1}\{-\log(e^{x_0} + e^{-2g-x_0}(Y_0 + 2J)^2\}, \tag{118}$$

$$C_3^2 = \min_{\partial^2}\{-\log(e^{x} + e^{-2g-x}(Y + 2J)^2\}. \tag{119}$$

The constant C_3^1 does not depend on α and y. For C_3^2 we use the previous estimate (109)

$$C_3^2 \geq \hat{C}_3^2, \tag{120}$$

where

$$\hat{C}_3^2 = -\log\left[\delta^{-4}\left((e^{-C_2} + e^{-C_1}(2\delta^{-4}e^{-2C_2} + 8J^2)\right)\right] \tag{121}$$

does not depend on α and y. Then, we conclude that $C_3 \geq C_3^1 + \hat{C}_3^2$. Hence, on K_+ we have

$$e^x \leq e^{-\bar{x}} \leq e^{-C_3} \leq e^{-(C_3^1+\hat{C}_3^2)}, \tag{122}$$

and

$$(Y + 2J)^2 \leq e^{-2C_3}e^{2g} \leq e^{-2(C_3^1+\hat{C}_3^2)}e^{2g}. \tag{123}$$

From (123), using $|a| - |b| \leq |a + b|$, we obtain that $e^{-g}y$ is bounded. A similar procedure can be used for K_-, replacing J by $-J$ in the inversions (112)–(113). q.e.d.

We now turn to uniqueness. Let (X_1, Y_1) and (X_0, Y_0) be two points in $\mathbb{H}^2$. The distance d between these points in $\mathbb{H}^2$ is given by (see, for example, [**3**])

$$\cosh d = 1 + \delta, \tag{124}$$

where

$$\delta = \frac{1}{2}\frac{(X_1 - X_0)^2 + (Y_1 - Y_0)^2}{X_1X_0}. \tag{125}$$

In our case, (X, Y) defines a map $(X, Y) : \mathbb{R}^3 \to \mathbb{H}^2$, hence d defines a function $d : \mathbb{R}^3 \to \mathbb{R}$. Assume that (X_1, Y_1) and (X_0, Y_0) are harmonic maps; we then have the following two fundamental inequalities proved in [**36**]

$$\Delta d^2 \geq 0, \tag{126}$$

and

$$\Delta\sigma \geq 0, \tag{127}$$

where $\sigma = \sqrt{1 + d^2}$. These inequalities constitute the basic ingredient in the uniqueness proof.

Following [**39**], we deduce from (126)

$$\Delta\delta \geq 0, \tag{128}$$

because δ is a convex function of d^2. Note that δ has a simpler expression in terms of X, Y than d.

Uniqueness proofs for the harmonic map equations (70)–(71) constitute a fundamental step in the black hole uniqueness theorems in General Relativity. The first result in this subject was proved by Carter [**8**] at the linearized level. Robinson [**33**] obtained an identity for equations (70)–(71) which lead to the first uniqueness proof. The content of the Robinson identity is essentially given by (128). However, Robinson discovered this identity independently of (126). We emphasize that (126) implies (128) but the converse is not true.

In the context of black hole theory, (126) is called the Bunting identity (see equation (6.48) in [**9**]). This identity is not only more general than the Robinson one but allows to extend the uniqueness proof to the charged case.

The following uniqueness theorem in based on (128).

Theorem 3.2. *The solution found in Theorem* 3.1 *is unique and is given by* $(0, 0)$*.*

Proof. Let (X_0, Y_0) be the extreme Kerr solution and let (X_1, Y_1) be another solution of the harmonic map equations (70)–(71) on $A \setminus \Gamma$, which satisfies (77), (74) and (76).

As usual, let x_0 and x_1 be given by

$$X_0 = e^{g+x_0}, \quad X_1 = e^{g+x_1}, \tag{129}$$

and define

$$y = Y_1 - Y_0, \quad \alpha = x_1 - x_0. \tag{130}$$

Let δ be given by (125) and set

$$\delta = \delta_x + \delta_y, \tag{131}$$

where

$$\delta_x = \cosh\alpha - 1, \quad \delta_y = \frac{1}{2} y^2 e^{-2g-2x_0-\alpha}. \tag{132}$$

Note that by hypothesis $\delta = 0$ on ∂A.

Below, we will prove that $\delta \in H^1(A)$. Let us assume that this is true. Since δ satisfies (128) in $A \setminus \Gamma$ we can apply Lemma 5.3 to conclude that (128) is satisfied in A. Hence, we can use the weak maximum principle for weak solutions (see [**26**]) in A. The function δ is non negative in A and vanishes at the boundary, so the weak maximum principle implies that $\delta = 0$ in A and hence the conclusion follows.

It remains to prove that $\delta \in H^1(A)$. In fact we will prove a stronger result: $\delta \in H^1(A) \cap L^\infty(A)$. Recall that x_0 and α are bounded on A. Then, it follows that $\delta_x \in L^\infty(A)$. From (132) we get

$$\partial\delta_x = \sinh\alpha\,\partial\alpha; \tag{133}$$

since $\alpha \in H^1(A)$ it follows that $\delta_x \in H^1(A)$.

Consider δ_y. Since x_1 and $e^{-g}y$ are bounded in A, we conclude that $\delta_y \in L^\infty(A)$. Its derivative is given by

$$\partial\delta_y = y\partial y e^{-2g-2x_0-\alpha} - y^2(\partial g + \partial x_0 + \frac{1}{2}\partial\alpha)e^{-2g-2x_0-\alpha}. \tag{134}$$

Then, we have

$$|\partial\delta_y|^2 \leq C\left(|\partial y|^2 e^{-2g} + (|\partial x_0|^2 + \frac{1}{2}|\partial\alpha|^2) - y^4 e^{-4g}|\partial h|^2\right), \tag{135}$$

where the constant C depends only the L^∞ norm of α, x_0 and ye^{-g}. When we perform the integral, the first three terms are bounded since $y \in H^1_{0,X_0}(\mathbb{R}^3 \setminus \Gamma)$ and α, x_0 are in $H^1(A)$. For the last term we use a Poincaré type inequality (see Lemma 1 of [**40**] and Lemma 2.2 in [**18**]). We conclude that δ_y, and hence δ, is in $H^1(A) \cap L^\infty(A)$. q.e.d.

Remark. The proof of Theorem 3.2 fails if we extend to the domain to $\mathbb{R}^3$ because the function δ_x is not in $H^1(B_\epsilon)$.

In order to extend this theorem to $\mathbb{R}^3$ (or, in other words, in order to generalize the uniqueness proofs to the extreme cases) we will use inequality (127) instead of (126) and (128).

It is convenient to have an equivalent expression for d in terms of δ. A straightforward computation gives

$$d = 2\log(\sqrt{\delta} + \sqrt{\delta + 2}) - \log 2, \tag{136}$$

and hence the following expression for the derivative

$$\partial d = \frac{\partial\delta}{\sqrt{\delta(\delta+2)}} = \frac{\partial\delta}{\sinh d}. \tag{137}$$

From (136) we deduce the following important inequalities

$$d \geq |\alpha|, \tag{138}$$

where α is given by (130) and

$$d \leq |\alpha| + C, \tag{139}$$

where the constant C depends only on the L^∞ norm of δ_y in $\mathbb{R}^3$.

Let us analyze the derivatives of d^2. Using (138) and (137) we obtain

$$|\partial d^2|^2 \leq 8d^2|\partial\alpha|^2 + 8d^2|\partial\delta_y|^2. \tag{140}$$

From this expression we get

$$|\partial\sigma|^2 \leq 2\left(|\partial\alpha|^2 + |\partial\delta_y|^2\right). \tag{141}$$

Before proving Theorem 3.3, we give an equivalent norm for the relevant Sobolev spaces.

Using a Poincaré type inequality (see Theorem 1.3 in [**2**]), it follows that the norm (14) on functions in $C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ is equivalent to the following weighted norm

$$\|\alpha\|_1 = \left(\int_{\mathbb{R}^3} |\partial\alpha|^2 \, d\mu\right)^{1/2} + \left(\int_{\mathbb{R}^3} \frac{\alpha^2}{r^2} \, d\mu\right)^{1/2}. \tag{142}$$

Then, the Sobolev space $H_0^1(\mathbb{R}^3 \setminus \{0\})$ is equivalent to the weighted Sobolev space $W'^{1,2}_{-1/2}$ studied in [**2**]. In particular, from (142) we deduce that if $\alpha \in H_0^1(\mathbb{R}^3 \setminus \{0\})$, then $\alpha \in H^1_{loc}(\mathbb{R}^3)$. We also mention that the Sobolev inequality

$$\left(\int_{\mathbb{R}^3} \alpha^6 \, d\mu\right)^{1/6} \leq C \left(\int_{\mathbb{R}^3} |\partial\alpha|^2 \, d\mu\right)^{1/2} \tag{143}$$

is satisfied for all functions $\alpha \in H_0^1(\mathbb{R}^3 \setminus \{0\})$.

Analogously, we can use another type of Poincaré inequality (see Lemma 5.4) to obtain an equivalent norm to (15) for functions in $C_0^\infty(\mathbb{R}^3 \setminus \Gamma)$

$$\|y\|_{1,X_0} = \left(\int_{\mathbb{R}^3} X_0^{-2} |\partial y|^2 \, d\mu\right)^{1/2} + \left(\int_{\mathbb{R}^3} \frac{|\partial X_0|^2}{X_0^4} y^2 \, d\mu\right)^{1/2}. \tag{144}$$

Theorem 3.3 (Uniqueness of extreme Kerr)**.** *Let (X, Y) be a solution of the harmonic map equations (70)–(71) in $\mathbb{R}^3 \setminus \Gamma$. Define (α, y) by $X = e^{g+x}$, $Y = Y_0 + y$, $x = x_0 + \alpha$. Assume that $\alpha \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$, $y \in H^1_{0,X_0}(\mathbb{R}^3 \setminus \Gamma)$, $yX_0^{-1}, \alpha^- \in L^\infty(\mathbb{R}^3)$ and that $\alpha, yX_0^{-1} \to 0$ as $r \to \infty$. Then, $\alpha = 0$ and $y = 0$.*

Proof. Let us analyze the function δ_y given by (132). The computations are similar as in Theorem 3.2; the difference is that here we have to take care of the singular behavior of the functions at the origin. In terms of X_0, the function δ_y is given by

$$\delta_y = \frac{y^2 e^{-\alpha}}{2X_0^2} \leq \frac{y^2 e^{-\alpha^-}}{2X_0^2}. \tag{145}$$

Using the hypothesis $yX_0^{-1}, \alpha^- \in L^\infty(\mathbb{R}^3)$ we obtain $\delta_y \in L^\infty(\mathbb{R}^3)$.

Take a ball B_R in $\mathbb{R}^3$ and consider the the derivative of δ_y in B_R

$$\partial\delta_y = e^{-\alpha}\left(\frac{y\partial y}{X_0^2} - \frac{y^2\partial\alpha}{2X_0^2} - \frac{y^2\partial X_0}{X_0^3}\right). \tag{146}$$

Using our assumptions, we conclude that the first two terms on the right hand side of equation (146) are in $L^2(B_R)$. For the third term we use the assumption $yX_0^{-1} \in L^\infty(\mathbb{R}^3)$ and the Poincaré inequality given by Lemma 5.4. Then, we conclude that δ_y is in $H^1(B_R)$.

Using inequality (139) (which holds because we have proved that δ_y is bounded) it follows that $\sigma \in L^2(B_R)$; then, using (141), we obtain

$\sigma \in H^1(B_R)$. Applying the maximum principle to the inequality (127), we get

$$\sup_{\partial B_R} \sigma \geq \sup_{B_R} \sigma \geq 1. \tag{147}$$

Using the decay conditions we get that $\sup_{\partial B_R} \sigma \to 1$ as $R \to \infty$. Then it follows that $d = 0$, and hence $\alpha = y = 0$. q.e.d.

Proof of Theorem 1.2. We first prove the inequality (16) using theorems 3.1 and 3.2. The crucial step is to prove that the minimizing sequence can be chosen among functions with compact supports in annulus centered at the origin.

Let $\alpha \in H^1_0(\mathbb{R}^3 \setminus \{0\})$ and $y \in H^1_{0,X_0}(\mathbb{R}^3 \setminus \Gamma)$. By definition, there exists a sequence $y_n \in C^\infty_0(\mathbb{R}^3 \setminus \Gamma)$ such that $y_n \to y$ in $H^1_{0,X_0}(\mathbb{R}^3 \setminus \Gamma)$ as $n \to \infty$. Let R be the radius of a ball that contains the support of y_n. The radius R depends on n and we have that $R \to \infty$ as $n \to \infty$. For $\epsilon = 1/R$, let $\chi_{\epsilon,R}$ be the cut off function defined in equation (179) of the appendix. Set $\alpha_n = \alpha\chi_{\epsilon,R}$. This function has compact support contained in the annulus $A_n = B_R \setminus B_\epsilon$ and $\alpha_n \in H^1_0(A_n)$. By Lemma 5.2 we have that $\alpha_n \to \alpha$ in $H^1_0(\mathbb{R}^3 \setminus \{0\})$ as $n \to \infty$. We claim that

$$\lim_{n\to\infty} \mathcal{M}(\alpha_n, y_n) = \mathcal{M}(\alpha, y). \tag{148}$$

This is similar to equation (83) in the proof of Theorem 3.1. Replacing the domain A by $\mathbb{R}^3$, we define the same integrals as in equations (85)–(86). Using (87)–(88) we conclude that $I_1 \to 0$ as $n \to \infty$.

For the integrals $I_{2,1}$ and $I_{2,2}$ we use the hypothesis $\alpha^- \in L^\infty(\mathbb{R}^3)$ (which plays the same role as the lower bound (82) in the proof of Theorem 3.1) and

$$e^{-\alpha_n} = e^{-\alpha^+\chi_{\epsilon,R} - \alpha^-\chi_{\epsilon,R}} \leq e^{-\alpha^-\chi_{\epsilon,R}} \leq e^{-\alpha^-}, \tag{149}$$

to bound the terms with $e^{-\alpha_n}$ by constants independent of n. Using the assumption $y \in H^1_{0,X_0}(\mathbb{R}^3 \setminus \Gamma)$ we conclude that these two integrals tend to zero as $n \to \infty$, and hence we have proved (148).

Using a similar argument as in the proof of Theorem 3.1, from equation (148) we conclude that the minimizing sequence (α_n, y_n) can be taken among functions with compact support in annulus A_n.

We apply Theorem 3.1 and Theorem 3.2 on A_n. We get

$$\mathcal{M}_{A_n}(x_0 + \alpha_n, Y_0 + y_n) \geq \mathcal{M}_{A_n}(x_0, Y_0). \tag{150}$$

Using this inequality we obtain

$$\mathcal{M}(x_0+\alpha_n, Y_0+y_n) = \mathcal{M}_{\mathbb{R}^3\setminus A_n}(x_0, Y_0) + \mathcal{M}_{A_n}(x_0+\alpha_n, Y_0+y_n) \tag{151}$$

$$\geq \mathcal{M}_{\mathbb{R}^3\setminus A_n}(x_0, Y_0) + \mathcal{M}_{A_n}(x_0, Y_0) \tag{152}$$

$$= \mathcal{M}(x_0, Y_0) \tag{153}$$

$$= \sqrt{|J|}. \tag{154}$$

And then we get (16).

We now prove the rigidity part. Assume that there exist $\alpha \in H_0^1(\mathbb{R}^3 \setminus \{0\})$ and $y \in H_{0,X_0}^1(\mathbb{R}^3 \setminus \Gamma)$ such that

$$\mathcal{M}(x_0+\alpha, Y_0+y) = \mathcal{M}(x_0, Y_0) = \sqrt{|J|}. \tag{155}$$

From inequality (16) it follows that (α, y) is a minimum of $\mathcal{M}$; hence it satisfies the harmonic maps equations. We use Theorem 3.3 to conclude that $\alpha = y = 0$. q.e.d.

Finally, let us mention that Theorem 1.1 follows directly from Theorem 1.2 and Theorem 2.2. Note that in the existence proofs of Section 2 the free data are the functions q and Y; on the other hand, in Theorem 1.1 the free functions are x and Y. Also, we emphasize that x and Y are not necessarily axially symmetric in 1.2; however, the bound given by Theorem 2.2 require this condition.

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5. Appendix

Lemma 5.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^1 boundary $\partial\Omega$. Suppose that $u \in H^1_0(\Omega)$ and*

$$u \geq K, \tag{156}$$

almost everywhere in Ω, where $K \leq 0$ is a constant. Then, there exists a sequence $u_n \in C^\infty_0(\Omega)$ such that

$$u_n \geq K, \tag{157}$$

for all n and $u_n \to u$ in the $H^1_0(\Omega)$ norm.

Proof. The proof follows similar arguments as the proof of the trace zero theorem for functions in $H^1_0(\Omega)$; see, for example, Theorem 2 in Chapter 5 of [**24**]. We will follow this reference. We will first prove the statement for functions in the half plane which vanishes at the boundary, and then we will extend this to the domain Ω.

Let (x', x_n) be coordinates in $\mathbb{R}^n$ and denote by $\mathbb{R}^n_+$ the subset $x_n > 0$. Let us assume that $u \in H^1(\mathbb{R}^n)$, it has compact support in $\bar{\mathbb{R}}^n_+$ and vanishes on $\partial\mathbb{R}^n_+$. Then, we can approximate u by smooth functions with compact support in $\bar{\mathbb{R}}^n_+$ which vanishes at the boundary $\partial\mathbb{R}^n_+$. Integrating these functions and taking the limit to u, we obtain the following estimate (see eq. (9), Chapter 5, [**24**])

$$\int_{\mathbb{R}^{n-1}} |u(x', x_n)|^2 \, dx' \leq C x_n \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |\partial u|^2 \, dx' dt, \tag{158}$$

for a.e. $x_n > 0$.

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a cut off function such that $\chi \in C^\infty(\mathbb{R})$, $0 \leq \chi \leq 1$, $\chi(t) = 1$ for $0 \leq t \leq 1$, $\chi(t) = 0$ for $2 \leq t$ and $|d\chi/dt| \leq 1$, and write $\chi_\epsilon(x) = \chi(x_n/\epsilon)$, $u_\epsilon = (1 - \chi_\epsilon)u$. We want to prove that $u_\epsilon \to u$ in $H^1(\Omega)$ as $\epsilon \to 0$. We have

$$\|u_\epsilon - u\|^2_{L^2(\Omega)} = \int_\Omega u^2 \chi^2_\epsilon \, d\mu, \tag{159}$$

since $u^2\chi^2_\epsilon \leq u^2$ (where, by hypothesis, u^2 is measurable) and $u^2\chi^2_\epsilon \to 0$ a.e. as $\epsilon \to 0$ by the dominated convergence theorem we conclude that the integral converges to zero as $\epsilon \to 0$. Consider the derivative

$$\|\partial u_\epsilon - \partial u\|_{L^2(\Omega)} \leq \|\chi_\epsilon \partial u\|_{L^2(\Omega)} + \|u \partial \chi_\epsilon\|_{L^2(\Omega)}. \tag{160}$$

Using the same argument as above, we have that the first term in the right hand side of this inequality goes to 0 as $\epsilon \to 0$. The delicate term is the second one. Note that the derivative of χ_ϵ has support in $\epsilon \leq x_n \leq 2\epsilon$ and that $|\partial\chi| \leq \epsilon^{-1}$, so we have

$$||u \partial \chi_\epsilon||^2_{L^2(\Omega)} \leq \epsilon^{-2} \int_\epsilon^{2\epsilon} \int_{\mathbb{R}^{n-1}} u^2 \, dx' dt. \tag{161}$$

Using the estimate (158) we obtain

$$\epsilon^{-2}\int_{\epsilon}^{2\epsilon}\int_{\mathbb{R}^{n-1}} u^2\,dx'dt \le C\epsilon^{-2}\int_0^{2\epsilon} t\,dt\int_{\epsilon}^{2\epsilon}\int_{\mathbb{R}^{n-1}} |\partial u|^2\,dx'dx_n \tag{162}$$

$$\le C\int_{\epsilon}^{2\epsilon}\int_{\mathbb{R}^{n-1}} |\partial u|^2\,dx'dx_n, \tag{163}$$

and this integral tends to zero as $\epsilon \to 0$. Then we conclude

$$u_\epsilon \to u \text{ in } H^1(\mathbb{R}^n_+). \tag{164}$$

Let η_δ be a mollifier. Since the functions u_ϵ have compact support in $\mathbb{R}^n_+$, we can mollify them to construct smooth functions $u_{\epsilon,\delta}$ in $\mathbb{R}^n_+$. Moreover, if u satisfies the lower bound (156), then $u_{\epsilon,\delta}$ satisfies it also. Indeed,

$$u_{\epsilon,\delta}(x) = \int_{\mathbb{R}^n} \eta_\delta(x-y)u_\epsilon(y)\,dy \ge K\int_{\mathbb{R}^n} \eta_\delta(x-y)(1-\chi_\epsilon)(y)\,dy \tag{165}$$

$$\ge K, \tag{166}$$

where in the last line we have used that $K \le 0$ and

$$\int_{\mathbb{R}^n} \eta_\delta\,dx = 1. \tag{167}$$

To show that the functions $u_{\epsilon,\delta}$ converges to u as $\epsilon, \delta \to 0$, we write

$$||u - u_{\epsilon,\delta}||_{H^1} \le ||u - u_\epsilon||_{H^1} + ||u_\epsilon - u_{\epsilon,\delta}||_{H^1}, \tag{168}$$

and then use that $u_{\epsilon,\delta} \to u_\epsilon$ as $\delta \to 0$ (this is the standard interior approximation in H^1 by smooth functions, see for example, Theorem 1, Chapter 5, of [**24**]) and that $u_\epsilon \to u$ as $\epsilon \to 0$.

We now extend this result to the domain Ω using a partition of unity and flattering out the boundary. Since $\partial\Omega$ is compact, we can find finitely many points $x_i^0 \in \partial\Omega$ and radii $r_i > 0$, such that $\partial\Omega \subset \cup_{i=1}^N B(x_i, r_i)$. Define $V_i = \Omega \cap B(x_i, r_i)$ and let $V_0 \subset\subset \Omega$, such that $\Omega \subset \cup_{i=0}^N V_i$.

Let $\{\zeta\}_{i=0}^N$ be a smooth partition of unity of $\bar{\Omega}$ subordinate to V_i. Define $u_i = u\zeta_i$, we have

$$u = \sum_{i=0}^N u_i. \tag{169}$$

Consider u_i for $i \ge 1$, since the boundary is C^1, it possible to make a coordinate transformation such that it straightens out $\partial\Omega$ near x_i. Then, we can assume that each u_i has compact support in $\bar{\mathbb{R}}^n_+$ and vanishes on $\partial\mathbb{R}^n_+$. We use the result proved above to approximate each u_i by smooth functions with compact support which satisfy the lower bound (156). Using (169) we obtain the desired conclusion. q.e.d.

The following function will be essential in the proofs of lemmas 5.2 and 5.3. It was taken from [**29**], Lemma 3.1. Define

$$t_\epsilon(\rho) = \frac{\log(-\log\rho)}{\log(-\log\epsilon)} \tag{170}$$

and

$$\chi_\epsilon(\rho) = \chi(t_\epsilon(\rho)), \tag{171}$$

where χ is the cut off function defined above. The function t_ϵ is defined for $0 < \epsilon < 1$ and $0 < \rho < 1$. We have that $t_\epsilon \geq 2$ for $\rho \leq e^{(\log\epsilon)^2}$ and $0 \leq t_\epsilon \leq 1$ for $\epsilon < \rho < e^{-1}$ (we assume ϵ small enough). It follows that the function χ_ϵ defines a smooth function in for $0 \leq \rho < \infty$ (we trivially extend the function to be zero when $\rho \geq 1$). Moreover, $\chi_\epsilon(\rho) = 0$ for $\rho \leq e^{-(\log\epsilon)^2}$ and $\chi_\epsilon(\rho) = 1$ for $r \geq \epsilon$.

The derivative of χ_ϵ has support in $e^{-(\log\epsilon)^2} \leq \rho \leq \epsilon$ and is given by

$$\partial_\rho\chi_\epsilon = -\frac{d\chi_\epsilon}{dt}\frac{1}{\log(-\log\epsilon)\rho\log\rho}. \tag{172}$$

Assume $\epsilon \leq 1/2$, then we have

$$\int_0^\infty |\partial_\rho\chi_\epsilon|^2 \rho d\rho \leq \frac{1}{(\log(-\log\epsilon))^2}\int_0^{1/2}\frac{d\rho}{\rho(\log\rho)^2}. \tag{173}$$

The integral on the right hand side is bounded since

$$\int \frac{d\rho}{\rho(\log\rho)^2} = -\frac{1}{\log\rho}. \tag{174}$$

Then we obtain

$$\lim_{\epsilon\to 0}\int_0^\infty |\partial_\rho\chi_\epsilon|^2\rho d\rho = 0. \tag{175}$$

Take cylindrical coordinates (ρ, z, ϕ) in $\mathbb{R}^3$, the integral (175) is equivalent to

$$\lim_{\epsilon\to 0}\int_0^\infty |\partial\chi_\epsilon|^2\, d\mu = 0. \tag{176}$$

This equation will be the crucial property of χ_ϵ used in the proof of Lemma 5.3.

Consider now the spherical radius r, define $\chi_\epsilon(r)$ using the function $t_\epsilon(r)$ given by (170). For $R > 1$, we also define

$$t_R(r) = \frac{\log(\log r)}{\log(\log R)}, \tag{177}$$

and

$$\chi_R(r) = \chi(t_R(r)). \tag{178}$$

Then the following function has support in an annulus of radii $e^{(\log R)^2}$ and $e^{-(\log \epsilon)^2}$

$$\chi_{\epsilon,R}(r) = \chi_R(r) + \chi_\epsilon(r) - 1. \tag{179}$$

A similar computation as above leads to

$$\lim_{\substack{\epsilon\to 0\\ R\to\infty}} \int_{\mathbb{R}^3} |\partial \chi_{\epsilon,R}|^3 \, d\mu = 0. \tag{180}$$

Lemma 5.2. *Let $u \in H^1_0(\mathbb{R}^3 \setminus \{0\})$. Then the functions $u_{\epsilon,R} = u\chi_{\epsilon,R}$ where $\chi_{\epsilon,R}$ is the cut off function defined in* (179) *converges to u in the $H^1_0(\mathbb{R}^3 \setminus \{0\})$ norm, as $R \to \infty$, $\epsilon \to 0$.*

Proof. We have

$$||\partial u_{\epsilon,R} - \partial u||_{L^2(\mathbb{R}^3)} \leq ||(1-\chi_{\epsilon,R})\partial u||_{L^2(\Omega)} + ||u\partial \chi_{\epsilon,R}||_{L^2(\Omega)}. \tag{181}$$

The first term in the right hand side of this inequality goes to 0 as $\epsilon \to 0$, $R \to \infty$. For the second term, we have

$$||u\partial\chi_{\epsilon,R}||^2_{L^2(\mathbb{R}^3)} \leq ||u^2||_{L^p(\mathbb{R}^3)} |||\partial\chi_{\epsilon,R}|^2||_{L^q(\mathbb{R}^3)} \tag{182}$$

$$\leq ||\partial u||_{L^2(\mathbb{R}^3)} |||\partial\chi_{\epsilon,R}|^2||_{L^{3/2}(\mathbb{R}^3)}, \tag{183}$$

where in the first line, we have used Hölder inequality with $1/p + 1/q = 1$ and in the second line, we chose $p = 3$ and $q = 3/2$, and use the Sobolev inequality (143). Then we use (180) to obtain the desired conclusion. q.e.d.

Lemma 5.3. *Let $u \in H^1(\Omega)$ be a weak subsolution of the Laplace equation in $\Omega \setminus \Gamma$. Then u is also a weak subsolution of the Laplace equation in Ω.*

Proof. By definition of weak subsolution in $\Omega \setminus \Gamma$, we have

$$\int_\Omega \partial u \partial v \, d\mu \geq 0, \tag{184}$$

for all $v \in C^\infty_0(\Omega \setminus \Gamma)$. We want to prove that this inequality holds also for all $v \in C^\infty_0(\Omega)$.

Take cylindrical coordinates in $\mathbb{R}^3$ where ρ is the distance to the axis Γ. Consider the cut off function $\chi_\epsilon(\rho)$ defined in (171). Let $v \in C^\infty_0(\Omega)$ and set $v = v(1-\chi_\epsilon) + v\chi_\epsilon$. Then we have

$$\begin{aligned}\int_\Omega \partial u \partial v \, d\mu &= \int_\Omega \partial u \partial (v(1-\chi_\epsilon)) \, d\mu + \int_\Omega \partial u \partial (v\chi_\epsilon) \, d\mu \\ &\geq \int_\Omega \partial u \partial (v(1-\chi_\epsilon)) \, d\mu,\end{aligned} \tag{185}$$

where we have used (184) since $v\chi_\epsilon \in C^\infty_0(\Omega \setminus \Gamma)$. We have

$$\int_\Omega \partial u \partial (v(1-\chi_\epsilon)) \, d\mu \leq C \|u\|_{H^1(\Omega)} \|\partial \chi_\epsilon\|_{L^2(\mathbb{R}^3)}. \tag{186}$$

We take the limit $\epsilon \to 0$ and use equation (176) to conclude that the integral goes to zero. Hence we conclude that

$$\int_\Omega \partial u \partial v \, d\mu \geq 0, \tag{187}$$

for all $v \in C_0^\infty(\Omega)$. q.e.d.

The following lemma gives a Poincaré type inequality for functions in $H^1_{0,X_0}(\mathbb{R}^3 \setminus \Gamma)$.

Lemma 5.4. *Let $y \in C_0^\infty(\mathbb{R}^3 \setminus \Gamma)$ and Y_0, X_0 be given by* (12)*. Then the following inequality holds*

$$\int_{\mathbb{R}^3} X_0^{-2} |\partial y|^2 \, d\mu \geq \int_{\mathbb{R}^3} \frac{(|\partial Y_0|^2 + |\partial X_0|^2)}{X_0^4} y^2 \, d\mu \tag{188}$$

$$\geq \int_{\mathbb{R}^3} \frac{|\partial X_0|^2}{X_0^4} y^2 \, d\mu. \tag{189}$$

Proof. We use the following general identity proved in Proposition C.2 of [**15**]

$$\int_{\mathbb{R}^3} e^{2v} |\partial y|^2 \, d\mu \geq \int_{\mathbb{R}^3} e^{2v} (\Delta v + |\partial v|^2) |y|^2 \, d\mu, \tag{190}$$

for $v = x_0 + g$. Using equation (70), the conclusion follows. q.e.d.

Finally, let us prove that the function

$$y_0 = Y_0 - \bar{Y}_0 = -\frac{2J^2 \cos\theta \sin^4\theta}{\Sigma}, \tag{191}$$

defined in the introduction satisfies the hypothesis of Theorem 1.2. Note that $y_0 \in C^\infty(\mathbb{R}^3 \setminus \{0\})$. Using equation (12) we obtain the lower bound

$$X_0 \geq |J| \sin^2\theta. \tag{192}$$

Then we get

$$\frac{|y_0|}{X_0} \leq \frac{2|J|}{(r + \sqrt{|J|})^2}, \tag{193}$$

which implies $|y_0|/X_0^{-1} \leq 2$ and $|y_0|/X_0^{-1} \to 0$, as $r \to \infty$. This bound also implies that $y_0/X_0^{-1} \in L^p(\mathbb{R}^3)$ for $3/2 < p$.

Remains to show that $y_0 \in H^1_{0,X_0}(\mathbb{R}^3 \setminus \Gamma)$. From (191), we can explicitly compute the norm (15) to prove that it is finite. Take the sequence $y_{\epsilon,R} = y_0 \chi_\epsilon(\rho) \chi_R(r)$ where $\chi_\epsilon(\rho)$ and $\chi_R(r)$ are given by (171) and (178). We have that $y_{\epsilon,R} \in C_0^\infty(\mathbb{R}^3 \setminus \Gamma)$. To prove that $y_{\epsilon,R} \to y$ in $H^1_{0,X_0}(\mathbb{R}^3 \setminus \Gamma)$, as $R \to \infty$, $\epsilon \to 0$, we use the same argument, as in the proof of Lemma 5.2 and the fact that $y_0/X_0^{-1} \in L^6(\mathbb{R}^3)$.

References

[1] R. Arnowitt, S. Deser, & C.W. Misner, *The dynamics of general relativity*, Gravitation: An Introduction to Current Research (L. Witten, ed.), Wiley, New York, 1962, 227–265, MR 0143629.

[2] R. Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure App. Math. **39**(**5**) (1986) 661–693, MR 849427.

[3] A.F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, **91**, Springer-Verlag, New York, 1983, MR 698777.

[4] J.M. Bowen & J.W. York, Jr., *Time-asymmetric initial data for black holes and black-hole collisions*, Phys. Rev. D **21**(**8**) (1980) 2047–2055.

[5] D. Brill, *On the positive definite mass of the Bondi-Weber-Wheeler time-symmetric gravitational waves*, Ann. Phys. **7** (1959) 466–483.

[6] M. Cantor, *A necessary and sufficient condition for York data to specify an asymptotically flat spacetime*, J. Math. Phys. **20**(**8**) (1979) 1741–1744, MR 543911, Zbl 0427.35072.

[7] M. Cantor & D. Brill, *The Laplacian on asymptotically flat manifolds and the specification of scalar curvature*, Compositio Mathematica **43**(**3**) (1981) 317–330, MR 632432.

[8] B. Carter, *Axisymmetric black hole has only two degrees of freedom*, Phys. Rev. Lett. **26**(**6**) (1971) 331–333.

[9] B. Carter, *Bunting identity and Mazur identity for nonlinear elliptic systems including the black hole equilibrium problem*, Commun. Math. Phys. **99**(**4**) (1985) 563–591, MR 796013.

[10] ———, *Has the black hole equilibrium problem been solved?*, The Eighth Marcel Grossmann Meeting, Part A, B (Jerusalem, 1997), World Sci. Publishing, River Edge, NJ, 1999, 136–155, MR 1891866, Zbl 0970.83030.

[11] B. Carter, *Black hole equilibrium states*, Black holes/Les astres occlus (École d'Été Phys. Théor., Les Houches, 1972), Gordon and Breach, New York, 1973, 57–214, MR 0465047.

[12] Y. Choquet-Bruhat, J. Isenberg, & J.W. York, Jr., *Einstein constraint on asymptotically euclidean manifolds*, Phys. Rev. D **61** (1999) 084034, MR 1791413.

[13] P. Chruściel, *Boundary conditions at spatial infinity from a Hamiltonian point of view*, Topological properties and global structure of space-time (Erice, 1985), NATO Adv. Sci. Inst. Ser. B Phys., **138**, Plenum, New York, 1986, 49–59, MR 1102938.

[14] ———, *Uniqueness of stationary, electro-vacuum black holes revisited*, Helv. Phys. Acta **69**(**4**) (1996) 529–552, Journées Relativistes, **96**, Part II (Ascona, 1996), MR 1435995.

[15] P.T. Chruściel & E. Delay, *On mapping properties of the general relativistic constraints operator in weighted function spaces, with applications*, Mem. Soc. Math. Fr. (N.S.) **94** (2003) 1–103, MR 2031583, Zbl 1058.83007.

[16] G.B. Cook, *Initial data for numerical Relativity*, Living Rev. Relativity **3**(**5**) (2001) 2000–5, 53 pp. (electronic), http://www.livingreviews.org/Articles/Volume3/2000-5cook/, MR 1799071, Zbl 1024.83001.

[17] S. Dain, *Angular momemtum-mass inequality for axisymmetric black holes*, Phys. Rev. Lett. **96** (2006) 101101, MR 2214721.

[18] ———, *Proof of the (local) angular momemtum–mass inequality for axisymmetric black holes*, Class. Quantum. Grav. **23** (2006) 6845–6855, MR 2273524,Zbl 1107.83036.

[19] ———, *Initial data for a head on collision of two Kerr-like black holes with close limit*, Phys. Rev. D **64**(**15**) (2001) 124002, MR 1878172.

[20] ———, *Proof of the angular momentum-mass inequality for axisymmetric black holes*, gr-qc/0606105, 2006.

[21] ———, *A variational principle for stationary, axisymmetric solutions of einstein's equations*, Class. Quantum. Grav. **23** (2006) 6857–6871, MR 2273525, Zbl 1107.83007.

[22] S. Dain, C.O. Lousto, & R. Takahashi, *New conformally flat initial data for spinning black holes*, Phys. Rev. D **65**(**10**) (2002) 104038, MR 1919023.

[23] F.J. Ernst, *New formulation of the axially symmetric gravitational field problem*, Phys. Rev. **167** (1968) 1175–1179.

[24] L.C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, **19**, American Mathematical Society, Providence, RI, 1998, MR 1625845, Zbl 0902.35002.

[25] G.W. Gibbons & G. Holzegel, *The positive mass and isoperimetric inequalities for axisymmetric black holes in four and five dimensions*, Class. Quant. Grav. **23** (2006) 6459–6478, MR 2272015, Zbl 1111.83029.

[26] D. Gilbarg & N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition, MR 1814364, Zbl 1042.35002.

[27] R.S. Hamilton, *Harmonic maps of manifolds with boundary*, Springer-Verlag, Berlin, 1975, Lecture Notes in Mathematics, **471**, MR 0482822, Zbl 0308.35003.

[28] A. Komar, *Covariant conservation laws in General Relativity*, Phys. Rev. **119**(**3**) (1959) 934–936, MR 0102403.

[29] Y.Y. Li & G. Tian, *Regularity of harmonic maps with prescribed singularities*, Commun. Math. Phys. **149**(**1**) (1992) 1–30, MR 1182409,Zbl 0785.53047.

[30] D. Maxwell, *Solutions of the Einstein constraint equations with apparent horizon boundaries*, Commun. Math. Phys. **253**(**3**) (2005) 561–583, MR 2116728.

[31] ———, *Rough solutions of the Einstein constraint equations*, J. Reine Angew. Math. **590** (2006) 1–29, MR 2208126, Zbl 1088.83004.

[32] N.O. Murchadha, *Brill waves*, Directions in General Relativity (B.L. Hu and T.A. Jacobson, eds.), **2**, Cambridge University Press, 1993, 210–223.

[33] D.C. Robinson, *Uniqueness of the Kerr black hole*, Phys. Rev. Lett. **34**(**14**) (1975) 905–906.

[34] R. Schoen & K. Uhlenbeck, *A regularity theory for harmonic maps*, J. Differential Geom. **17**(**2**) (1982) 307–335, MR 664498, Zbl 0521.58021.

[35] ———, *Boundary regularity and the Dirichlet problem for harmonic maps*, J. Differential Geom. **18**(**2**) (1983) 253–268, MR 710054, Zbl 0547.58020.

[36] R. Schoen & S.-T. Yau, *Compact group actions and the topology of manifolds with nonpositive curvature*, Topology **18**(**4**) (1979) 361–380, MR 551017, Zbl 0424.58012.

[37] ———, *Proof of the positive mass theorem*, II, Comm. Math. Phys. **79**(**2**) (1981) 231–260, MR 612249, Zbl 0494.53028.

[38] R.M. Wald, *General relativity*, The University of Chicago Press, Chicago, 1984, Zbl 0549.53001.

[39] G. Weinstein, *On rotating black holes in equilibrium in general relativity*, Comm. Pure App. Math. **43**(**7**) (1990) 903–948, MR 1072397, Zbl 0709.53063.

[40] ———, *The stationary axisymmetric two-body problem in general relativity*, Comm. Pure App. Math. **45**(**9**) (1992) 1183–1203, MR 1177481.

[41] ———, *On the force between rotating co-axial black holes*, Trans. Amer. Math. Soc. **343**(**2**) (1994) 899–906, MR 1214787, Zbl 0807.53080.

[42] ———, *On the Dirichlet problem for harmonic maps with prescribed singularities*, Duke Math. J. **77**(**1**) (1995) 135–165, MR 1317630, Zbl 0829.53041.

[43] ———, *Harmonic maps with prescribed singularities on unbounded domains*, Amer. J. Math. **118**(**3**) (1996) 689–700, MR 1393266, Zbl 0858.53018.

[44] ———, *N-black hole stationary and axially symmetric solutions of the Einstein/Maxwell equations*, Comm. Partial Differential Equations **21**(**9**–**10**) (1996) 1389–1430, MR 1410835, Zbl 0863.53061.

[45] E. Witten, *A new proof of the positive energy theorem*, Commun. Math. Phys. **80**(**3**) (1981) 381–402, MR 626707, Zbl 1051.83532.

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