

POLAR ACTIONS ON SYMMETRIC SPACES

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Abstract

We study isometric Lie group actions on symmetric spaces admitting a section, i.e., a submanifold that meets all orbits orthogonally at every intersection point. We classify such actions on the compact symmetric spaces with simple isometry group and rank greater than one. In particular, we show that these actions are hyperpolar, i.e., the sections are flat.

1. Introduction and main results

An isometric action of a compact Lie group on a Riemannian manifold is called *polar* if there exists a connected immersed submanifold Σ that intersects the orbits orthogonally and meets every orbit. Such a submanifold Σ is then called a *section* of the group action. If the section is flat in the induced metric, the action is called *hyperpolar*. Our main result is a classification of polar actions on compact symmetric spaces with simple isometry group and rank greater than one. This classification shows that these actions are in fact all hyperpolar.

One may think of the elements in a section as being *canonical forms*, representing the orbits of the group action uniquely up to the action of a finite group, the *Weyl group*. This point of view may be illustrated by the example of the orthogonal group $O(n)$ acting on the space of real symmetric $n \times n$ -matrices by conjugation, where the subspace of diagonal matrices is a section. Another motivation comes from submanifold geometry, in particular from the theory of isoparametric submanifolds and their generalizations [44], [45]. The orbits of polar actions have many remarkable geometric properties; for instance, the principal orbits of polar representations are isoparametric submanifolds of Euclidean space.

However, the history of the subject probably starts with an application in topology. Bott [5] and Bott and Samelson [6] considered the adjoint action of a compact Lie group on itself and on its Lie algebra [5], and more generally, the isotropy action of a compact symmetric space [6]. The motivation of Bott and Samelson to study these actions was that they are “variationally complete”, which made it possible to

apply Morse theory to the space of loops in the symmetric space. Conlon [11] proved that hyperpolar actions on Riemannian manifolds are variationally complete, referring to the sections as *K-transversal domains*. Hermann [27] found another class of examples; namely, if H and K are both symmetric subgroups of a simple compact Lie group G , then the action of H on the symmetric space G/K is hyperpolar. It was shown much later [21] that actions on compact symmetric spaces are variationally complete if and only if they are hyperpolar.

Conlon [12] observed that s-representations are hyperpolar, and later on Dadok [13] obtained a classification of irreducible polar representations. The classification shows that the connected components of the orbits of a polar representation agree with the orbits of an s-representation after a suitable identification of the representation spaces. Reducible polar representations were classified by Bergmann [1].

Cohomogeneity one actions, i.e., actions whose principal orbits are hypersurfaces, are a special case of independent interest. Cohomogeneity one actions on spheres were classified by Hsiang and Lawson [29]. Later Takagi [43], D'Atri [14], and Iwata [31] classified cohomogeneity one actions on $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{O}P^2$, respectively.

Szenthe [42], Palais and Terng [38] investigated fundamental properties of polar actions on Riemannian manifolds. Heintze, Palais, Terng and Thorbergsson [24], [25] obtained structural results for hyperpolar actions on compact symmetric spaces, and studied relations to polar actions on infinite dimensional Hilbert space and involutions of affine Kac-Moody algebras. They showed in particular [25] that compact Riemannian homogeneous spaces admitting a hyperpolar action with a fixed point are symmetric.

In [33], the author gave a classification of hyperpolar actions on the irreducible compact symmetric spaces, the main result being that these actions are orbit equivalent to the examples found by Hermann if the cohomogeneity is ≥ 2 .

Podestà and Thorbergsson [39] classified polar actions on the compact symmetric spaces of rank one. The first result on polar actions on irreducible symmetric spaces of higher rank, without assuming flatness of the sections, was obtained by Brück [7], who showed that on these spaces, polar actions with a fixed point are hyperpolar. Podestà and Thorbergsson [40] proved that polar actions on compact irreducible homogeneous Kähler manifolds are coisotropic and classified coisotropic and polar actions on the real Grassmannians $\mathbb{G}_2(\mathbb{R}^n)$ of rank two. It turned out that all polar actions on these spaces are hyperpolar.

This approach was further pursued by Biliotti and Gori [3], who classified coisotropic and polar actions on the complex Grassmannians $\mathbb{G}_k(\mathbb{C}^n)$. The classification of coisotropic actions on the compact

irreducible Hermitian symmetric spaces was recently completed by Biliotti [2], showing in particular that polar actions on these spaces are hyperpolar, which led Biliotti to conjecture that this holds for all compact irreducible symmetric spaces.

The present work extends the classification of polar actions to all irreducible symmetric spaces of type I, i.e., to the compact symmetric spaces with simple isometry group, confirming the conjecture of Biliotti for these spaces. We show that polar actions on the symmetric spaces of type I and higher rank are hyperpolar. That is, they are of cohomogeneity one or orbit equivalent to the examples found by Hermann. Our main result can be stated as follows.

Theorem 1. *Let M be a compact symmetric space of rank greater than one whose isometry group G is simple. Let $H \subset G$ be a closed connected non-trivial subgroup acting polarly on M . Then the action of H on M is hyperpolar, that is, the sections are flat in the induced metric. Moreover, the sections are embedded submanifolds.*

Type	$\pi^{-1}(H)$	\tilde{M}
A III-II	$SU(2n-2k-1) \times SU(2k+1)$	$SU(2n)/Sp(n)$
A III-II	$S(U(2n-2) \times U(1) \times U(1))$	$SU(2n)/Sp(n)$
A III-II	$S(U(2n-2) \times U(1))$	$SU(2n)/Sp(n)$
A III-III	$SU(k) \times SU(n-k)$	$SU(n)/S(U(\ell) \times U(n-\ell)), (k, \ell) \neq (\frac{n}{2}, \frac{n}{2})$
BD I-I	$G_2 \times SO(n-7)$	$SO(n)/SO(2) \times SO(n-2), n \geq 7$
BD I-I	$Spin(7) \times SO(n-8)$	$SO(n)/SO(2) \times SO(n-2), n \geq 8$
BD I-I	$Spin(7) \times SO(n-8)$	$SO(n)/SO(3) \times SO(n-3), n \geq 8$
BD I-I	$G_2 \times G_2$	$SO(14)/SO(2) \times SO(12)$
BD I-I	$G_2 \times Spin(7)$	$SO(15)/SO(2) \times SO(13)$
BD I-I	$Spin(7) \times Spin(7)$	$SO(16)/SO(2) \times SO(14)$
BD I-I	$Spin(7) \times Spin(7)$	$SO(16)/SO(3) \times SO(13)$
C I-II	$SU(n)$	$Sp(n)/Sp(k) \times Sp(n-k)$
C II-II	$Sp(a+b)$	$Sp(a+b+1)/Sp(a) \times Sp(b+1)$
C II-II	$Sp(a+b) \times U(1)$	$Sp(a+b+1)/Sp(a) \times Sp(b+1)$
D ₄ I-I'	G_2	$SO(8)/U(4)$
DI-III	$SO(2n-2)$	$SO(2n)/U(n)$
D III-I	$SU(n)$	$SO(2n)/SO(k) \times SO(2n-k), k < n$
D III-III'	$SU(n)$	$SO(2n)/\alpha(U(n))$
E II-IV	$SU(6)$	E_6/F_4
E II-IV	$SU(6) \cdot U(1)$	E_6/F_4
E III-II	$Spin(10)$	$E_6/SU(6) \cdot Sp(1)$
E VII-VI	E_6	$E_7/SO'(12) \cdot Sp(1)$

Table 1. Hyperpolar subactions of Hermann actions.

In [33], the hyperpolar actions on irreducible compact symmetric spaces were only determined up to orbit equivalence. In the present work, we obtain the complete classification of connected Lie groups acting polarly without fixed points on the symmetric spaces of higher rank with simple isometry group. For actions with fixed points the complete classification follows immediately from Corollary 6.2 and Lemma 2.6.

Theorem 2. *Let $M = G/K$ be a connected compact symmetric space of rank greater than one, whose isometry group is simple. Let $H \subset G$ be a closed connected proper subgroup such that the H -action on G/K is polar, non-trivial, non-transitive, and without a fixed point. Then*

- (i) *either $H \subset G$ is maximal connected (and as described in Theorem A of [33])*
- (ii) *or the universal cover of the symmetric space \tilde{M} and the conjugacy class of the subgroup $H \subset G$ are as given by Table 1, where $\pi: \text{Isom}(\tilde{M}) \rightarrow G$ is the covering map, and there exists a connected subgroup $H_0 \subset G$ whose Lie algebra $\mathfrak{h}_0 \subset \mathfrak{g}$ is the fixed point set of an involution of \mathfrak{g} and such that the H_0 -action on G/K has the same orbits as the H -action.*

The first column of Table 1 indicates a connected subgroup H_0 of G containing H ; see Table 3 and the remarks there. By α we denote a non-trivial outer automorphism of $\text{SO}(2n)$ of order two, given by conjugation with an element from $\text{O}(2n) \setminus \text{SO}(2n)$. The proofs of Theorems 1 and 2 are completed and summarized on pp. 469–479.

Combining Theorem 1 with the results of [39] and Corollary D of [23], we obtain the following result on sections and Weyl groups of polar actions. The Weyl group $W_\Sigma = N_H(\Sigma)/Z_H(\Sigma)$ is a quotient group of the group \hat{W}_Σ as defined in Lemma 5.1.

Corollary 1. *Let H be a connected compact Lie group acting polarly on a compact symmetric space M with simple isometry group. Then a section Σ of the H -action on M is isometric to a flat torus, a sphere or a real projective space. The group \hat{W}_Σ acting on the universal cover of Σ is an irreducible affine Coxeter group in case Σ is flat, or a finite Coxeter group of Euclidean space restricted to a sphere in case Σ is non-flat.*

In particular, the Weyl groups of such polar actions can be described by connected Dynkin diagrams of affine type (in the hyperpolar case) or Dynkin diagrams of the finite type (in the polar, non-hyperpolar case).

This article is organized as follows. We start by setting up terminology and notation. We then review examples and known results on polar actions. In Section 3 we recall some facts about symmetric spaces and their totally geodesic submanifolds; in particular, we give a characterization of maximal totally geodesic submanifolds and obtain an

upper bound on the dimension of totally geodesic submanifolds locally isometric to a product of spheres.

In Section 4, we recall a criterion that reduces the problem of deciding whether an action on a symmetric space is polar or not to a problem on the Lie algebra level. In Section 5, we prove Splitting Theorem 5.2, which says that if a section Σ of a polar action admits a local splitting $\tilde{\Sigma} = \tilde{\Sigma}_1 \times \tilde{\Sigma}_2$ such that the Weyl group acts trivially on one factor $\tilde{\Sigma}_2$, then the symmetric space is locally a Riemannian product $M \times \tilde{\Sigma}_2$. As a consequence, we show that the section of a polar action on a compact irreducible symmetric space is locally isometric to a product of spaces of constant curvature. This observation is crucial for our classification since it implies an upper bound on the cohomogeneity, reducing the classification problem to a finite number of cases.

In Section 6, we introduce another main tool by collecting various sufficient conditions for actions to be *polarity minimal*, which means that the restriction to a closed connected subgroup with orbits of lower dimension is either non-polar or trivial. This is of essential importance since it enables us to restrict our attention at first to maximal subgroups of the isometry group. In many cases we are able to show that the action of a maximal connected subgroup is non-polar and polarity minimal, thereby excluding all of its subgroups.

In the remaining part of the paper, the classification is carried out. We start with the maximal connected subgroups in the isometry group of a symmetric space. In Section 7, we consider Hermann actions, i.e., actions of symmetric subgroups of the isometry group. We show that actions of cohomogeneity ≥ 2 are polarity minimal and determine orbit equivalent subactions. We then consider maximal connected subgroups in the isometry group of classical symmetric spaces, which are given by irreducible representations of non-simple groups. It turns out that they are either non-polar and polarity minimal or of cohomogeneity one. In Section 9 we study actions of simple irreducible subgroups in the classical groups. In Section 10, we consider actions on the exceptional symmetric spaces. It turns out that the actions of non-symmetric maximal subgroups are non-polar and polarity minimal.

It then remains to study subactions of cohomogeneity one and transitive actions. Since we do not have an *a priori* proof that these actions are polarity minimal, it is necessary to descend from maximal connected subgroups $H_1 \subset G$ acting with cohomogeneity ≤ 1 to further subgroups

$$G \supset H_1 \supset H_2 \supset \dots,$$

where $H_{n+1} \subset H_n$ is maximal connected, until we arrive at an action which is polarity minimal.

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2. Preliminaries and examples

An isometric action of a compact Lie group G on a Riemannian manifold is called *polar* if there exists a connected immersed submanifold Σ such that Σ meets all G -orbits and the intersection of Σ with any G -orbit is orthogonal at all intersection points. Such a submanifold Σ is called a *section* for the G -action on M . In particular, the actions of finite groups and transitive actions are special cases of polar actions, the section being the whole space or a point, respectively.

Note that we do not require the section to be an embedded submanifold, generalizing the definitions of [38] and [11]. However, it turns out by our classification that on symmetric spaces of type I, the sections are flat and therefore closed embedded submanifolds by Corollary 2.12 of [24].

The dimension of Σ equals the cohomogeneity of the G -action and hence the tangent space $T_p\Sigma$ at a regular point $p \in \Sigma$ coincides with the normal space $N_p(G \cdot p)$ at p of the orbit through p . From this, it follows that any two sections are mapped isometrically onto each other by some group element. It has been proved in [38] that sections are totally geodesic.

In the special case where the sections are flat in the induced metric, the action is called *hyperpolar*. Examples for hyperpolar actions are given by the action of a compact Lie group on itself by conjugation, where the sections are the maximal tori. More generally, the action of an isotropy group of a symmetric space is hyperpolar, the sections being the flats of the symmetric space.

For a polar action, one can define the Weyl group by considering the normalizer of a section, i.e., of all group elements which map the section onto itself, this group acts on the section by isometries and the Weyl group is defined by factoring out the kernel of this action.

Definition 2.1. Let M be a Riemannian manifold on which the compact Lie group G acts polarly with section Σ . The (*generalized*) *Weyl group* $W_\Sigma = W_\Sigma(M, G)$ is the group $N_G(\Sigma)/Z_G(\Sigma)$, where $N_G(\Sigma) = \{g \in G \mid g \cdot \Sigma = \Sigma\}$ and $Z_G(\Sigma) = \{g \in G \mid g \cdot s = s \text{ for all } s \in \Sigma\}$ are the *normalizer* and *centralizer* of Σ in G , respectively.

Two Riemannian G -manifolds are called *conjugate* if there exists an equivariant isometry between them. In particular, the actions of two conjugate subgroups of the isometry group of a Riemannian manifold are conjugate. To study isometric actions on a Riemannian manifold,

it suffices to consider conjugacy classes of subgroups in the isometry group.

Two isometric actions of two Lie groups G and G' on a Riemannian manifold M are called *orbit equivalent* if there exists an isometry of M which maps G -orbits onto G' -orbits; they are called *locally orbit equivalent* if there is an isometry mapping connected components of G -orbits onto connected components of G' -orbits. Obvious examples of orbit equivalent actions are given by various groups acting transitively on spheres, e.g., the actions of $\mathrm{SO}(4n)$, $\mathrm{SU}(2n)$, $\mathrm{U}(2n)$, and $\mathrm{Sp}(n)$ on \mathbb{R}^{4n} are all orbit equivalent.

We use the term *subaction* for the restriction of an action of a group G to a subgroup $H \subseteq G$; in case the H -orbits coincide with the G -orbits, the H -action is called an *orbit equivalent subaction*.

A normal subgroup N of a compact Lie group $G = G' \cdot N$ acting isometrically on a Riemannian manifold is called *inessential* if the G -action restricted to G' is orbit equivalent to the G -action. An isometric action of a compact connected Lie group G on a Riemannian manifold M is called *orbit maximal* if any other isometric action of any other compact connected Lie group G' such that every G -orbit is contained in a G' -orbit is either orbit equivalent or transitive on M .

An immersed submanifold M in a symmetric space N is said to have *parallel focal structure* if the normal bundle $\nu(M)$ is globally flat and the focal data is invariant under normal parallel translation; that is, for every parallel normal field v on M , the rank of $d\eta_{v(x)}$ is locally constant on M , where the end point map $\eta: NM \rightarrow N, v \mapsto \exp(v)$ is defined to be the restriction of the exponential map to the normal bundle $\nu(M)$, see [45]. The principal orbits of a polar action on a symmetric space have parallel focal structure [19].

A submanifold with parallel focal structure is called *equifocal* if the normal bundle $\nu(M)$ is *abelian*. That is, $\exp(\nu(M))$ is contained in some totally geodesic flat subspace of N for each point $x \in M$. The principal orbits of hyperpolar actions on symmetric spaces of compact type are equifocal submanifolds, see [45], Theorem 2.1. Our Theorem 1 shows that submanifolds with parallel focal structure which arise as principal orbits of polar actions on symmetric spaces of higher rank with simple compact isometry group are in fact equifocal. We conjecture that, more generally, submanifolds with parallel focal structure in irreducible compact symmetric spaces of higher rank are equifocal, hence of codimension one or homogeneous by the result of Christ [9].

2.1. Notation. We will frequently use the following notational conventions for compact Lie groups and their representations. We view the classical Lie groups $\mathrm{SO}(n)$, $\mathrm{SU}(n)$, and $\mathrm{Sp}(n)$ as matrix Lie groups as described in [26], Ch. X, § 2.1. We assume that reducible subgroups of

the classical groups are standardly embedded, e.g., by $\mathrm{SO}(m) \times \mathrm{SO}(n)$ we denote the subgroup

$$(2.1) \quad \left\{ \left(\begin{array}{c|c} A & \\ \hline & B \end{array} \right) \middle| A \in \mathrm{SO}(m), B \in \mathrm{SO}(n) \right\} \subset \mathrm{SO}(m+n).$$

We write $H_1 \otimes H_2$ for the Kronecker product of two matrix Lie groups. When we write G_2 , we refer to an irreducible representation by orthogonal 7×7 -matrices; similarly, $\mathrm{Spin}(7)$ stands for a matrix Lie group which is the image of the 8-dimensional spin representation of $\mathrm{Spin}(7)$. By \mathbb{R}^n , \mathbb{C}^n , \mathbb{H}^n we will denote the standard representation of $\mathrm{O}(n)$, $\mathrm{U}(n)$ or $\mathrm{Sp}(n)$, respectively.

2.2. Polar representations. A well known class of examples for polar actions is given by polar representations on Euclidean space. Since the sections of polar actions are totally geodesic, they are linear subspaces in the case of polar representations and polar representations are therefore automatically hyperpolar. Polar representations are of importance for our classification since they occur as slice representations of polar actions. Let M be a Riemannian G -manifold, and let G_p be the isotropy subgroup at p . The restriction of the isotropy representation to $N_p(G \cdot p)$ is called the *slice representation* at p . Slice representations are a fundamental tool for the study of Lie group actions since they provide a means to describe the local behavior of an action in a tubular neighborhood of an orbit by a linear representation.

Slice Theorem 2.2. *Let M be a Riemannian G -manifold, let $p \in M$, and let $V = N_p(G \cdot p)$ be the normal space at p to the G -orbit through p . Then there is an equivariant diffeomorphism Ψ of a G -invariant open neighborhood around the zero section in the normal bundle $G \times_{G_p} V \rightarrow G/G_p$ onto a G -invariant open neighborhood around the orbit $G \cdot p$ such that the zero section in $G \times_{G_p} V$ is mapped to the orbit $G \cdot p$. The diffeomorphism Ψ is given by the end point map which maps any normal vector $v_q \in N_q(G \cdot p)$ to its image under the exponential map $\exp_q(v_q)$.*

Proof. See e.g., [32], p. 3. q.e.d.

It is an immediate consequence of the Slice Theorem 2.2 that the slice representation and the G -action on M have the same cohomogeneity. Slice representations are particularly useful for our classification since the polarity of an action is inherited by its slice representations.

Proposition 2.3. *Let M be a Riemannian G -manifold. If the action on N is polar, then for all $p \in M$ the slice representation at p is polar with $T_p \Sigma$ as a section, where Σ is the section of the G -action on M containing p .*

Proof. This was proved in [38], Theorem 4.6. Although in [38] the sections are assumed to be embedded submanifolds, the proof is still valid if one requires the sections only to be immersed. q.e.d.

We use the term *effectivized slice representation* to describe the representation of the isotropy group with the effectivity kernel factored out. Let us recall some known results about polar representations.

Definition 2.4. Let G be a compact Lie group and let K be a closed subgroup. By

$$\chi(G, K) = \text{Ad}_G|_K \ominus \text{Ad}_K$$

we denote the equivalence class of the isotropy representation of the homogeneous space G/K , i.e., the restriction of the adjoint representation of G to K acting on a K -invariant complement of \mathfrak{k} in \mathfrak{g} . In the special case of a symmetric pair (G, K) , see below, the (equivalence class of the) representation $\chi(G/K)$ is called an *s-representation*.

A compact subgroup K of a Lie group G is called *symmetric subgroup* if there exists an involutive automorphism of G such that $G_0^\sigma \subseteq K \subseteq G^\sigma$, where G^σ and G_0^σ denote the fixed point set of σ and its connected component, respectively. A pair (G, K) , where G is a Lie group and K a symmetric subgroup, is called a *symmetric pair*. Any Riemannian globally symmetric space M has a homogeneous presentation G/K , where G is the isometry group of M such that K is a symmetric subgroup of G . Conversely, if (G, K) is a symmetric pair, then G/K endowed with a G -invariant metric is a Riemannian globally symmetric space, see [26].

It is well known that the adjoint representations of compact Lie groups, and more generally s-representations, are polar. As far as concerns the geometry of the orbits, the converse is also true.

Theorem 2.5 (Dadok). *A representation $\rho: G \rightarrow \text{O}(n)$ of a compact Lie group is polar if and only if it is locally orbit equivalent to an s-representation, i.e., the connected components of its orbit agree with the orbits of an s-representation after a suitable isometric identification of the representation spaces.*

Proof. The proof given in [13] relies on a classification of the irreducible polar representations. See [17] for a conceptual proof in case the cohomogeneity is ≥ 3 . See [34] for an alternative proof, where a similar classification strategy as in the present work is used. q.e.d.

It is shown in Theorem 3.12 of [24] that irreducible polar representations of cohomogeneity ≥ 2 are orbit maximal when restricted to a sphere around the origin. For irreducible s-representations of cohomogeneity ≥ 2 , orbit equivalent subgroups were determined in [18]. We state the result below.

Lemma 2.6. *Let G be a connected simple compact Lie group and let K be a connected symmetric subgroup such that $\text{rk}(G/K) \geq 2$. Let $H \subseteq K$ be a closed connected subgroup. Then $\chi(G, K)$ and $\chi(G, K)|_H$ are orbit equivalent if and only if either $H = K$ or the triple (G, K, H) is as given in Table 2.*

G	K	H	Range
SO(9)	SO(2) \times SO(7)	SO(2) \times G ₂	
SO(10)	SO(2) \times SO(8)	SO(2) \times Spin(7)	
SO(11)	SO(3) \times SO(8)	SO(3) \times Spin(7)	
SU($p+q$)	S(U(p) \times U(q))	SU(p) \times SU(q)	$p \neq q$
SO($2n$)	U(n)	SU(n)	n odd
E ₆	U(1) \cdot Spin(10)	Spin(10)	

Table 2. Orbit equivalent subactions of polar representations.

2.3. Hyperpolar actions on symmetric spaces. If H, K are two symmetric subgroups of the compact Lie group G , then the action of H on G/K is hyperpolar [27]. Slightly more generally, if H is a subgroup of G such that its Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ is the fixed point set of an involution of \mathfrak{g} , then the action of H on the symmetric space G/K is hyperpolar, and we call such actions *Hermann actions*. In the special case $H = K$, we have the isotropy action of the symmetric space and the sections are just the flats of the symmetric space. It was shown in [33] that all hyperpolar actions on irreducible symmetric spaces of compact type are of cohomogeneity one or orbit equivalent to Hermann actions.

All fixed-point free Hermann actions on the symmetric spaces of type I are given by Table 3. Here α denotes the non-trivial diagram automorphism of $\text{SO}(2n)$ given by conjugation with a matrix from $\text{O}(2n) \setminus \text{SO}(2n)$ and τ an order three diagram automorphism of $\text{Spin}(8)$. The type of the Hermann action indicated in the first column refers to the type of the symmetric subgroups involved as given in Table 4, e.g., the symbol AI-II refers to the action of H on G/K , where $G/H = \text{SU}(2n)/\text{SO}(2n)$ is of type AI and $G/K = \text{SU}(2n)/\text{Sp}(n)$ is of type AII; whereas for the action of K on G/H we use the notation AII-I. For the conjugacy classes of connected symmetric subgroups in simple compact Lie groups, see [33], 3.1.1 and 3.1.2. The cohomogeneity of the actions is given in the last column.

Hyperpolar actions on compact symmetric spaces have the remarkable property that they lift under certain Riemannian submersions to actions which are again hyperpolar, cf. [24].

Proposition 2.7. *Let G be a compact simple Lie group and let $K \subset G$ be a symmetric subgroup, $M = G/K$ the corresponding symmetric space, and H a closed subgroup of G . Then the H -action on M is hyperpolar if and only if the $H \times K$ -action on G is hyperpolar.*

Proof. See [24], Proposition 2.11.

q.e.d.

Type	H	G	K	Coh.
A I-II	$\mathrm{SO}(2n)$	$\mathrm{SU}(2n)$	$\mathrm{Sp}(n)$	$n-1$
A I-III($k \leq \lfloor \frac{n}{2} \rfloor$)	$\mathrm{SO}(n)$	$\mathrm{SU}(n)$	$\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(n-k))$	k
A II-III($k \leq n$)	$\mathrm{Sp}(n)$	$\mathrm{SU}(2n)$	$\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(2n-k))$	$\lfloor \frac{k}{2} \rfloor$
AIII-III($k \leq \ell \leq \lfloor \frac{n}{2} \rfloor$)	$\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(n-k))$	$\mathrm{SU}(n)$	$\mathrm{S}(\mathrm{U}(\ell) \times \mathrm{U}(n-\ell))$	k
BD I-I($k \leq \ell \leq \lfloor \frac{n}{2} \rfloor$)	$\mathrm{SO}(k) \times \mathrm{SO}(n-k)$	$\mathrm{SO}(n)$	$\mathrm{SO}(\ell) \times \mathrm{SO}(n-\ell)$	k
C I-II($k \leq \lfloor \frac{n}{2} \rfloor$)	$\mathrm{U}(n)$	$\mathrm{Sp}(n)$	$\mathrm{Sp}(k) \times \mathrm{Sp}(n-k)$	k
C II-II($k \leq \ell \leq \lfloor \frac{n}{2} \rfloor$)	$\mathrm{Sp}(k) \times \mathrm{Sp}(n-k)$	$\mathrm{Sp}(n)$	$\mathrm{Sp}(\ell) \times \mathrm{Sp}(n-\ell)$	k
D I-III($k \leq n$)	$\mathrm{SO}(k) \times \mathrm{SO}(2n-k)$	$\mathrm{SO}(2n)$	$\mathrm{U}(n)$	$\lfloor \frac{k}{2} \rfloor$
D III-III'	$\mathrm{U}(2n)$	$\mathrm{SO}(4n)$	$\alpha(\mathrm{U}(2n))$	$n-1$
D ₄ I-I'($k \leq \ell \leq 3$)	$\mathrm{Spin}(k) \cdot \mathrm{Spin}(8-k)$	$\mathrm{Spin}(8)$	$\tau(\mathrm{Spin}(\ell) \cdot \mathrm{Spin}(8-\ell))$	$k-1$
E I-II	$\mathrm{Sp}(4)/\{\pm 1\}$	E_6	$\mathrm{SU}(6) \cdot \mathrm{Sp}(1)$	4
E I-III	$\mathrm{Sp}(4)/\{\pm 1\}$	E_6	$\mathrm{Spin}(10) \cdot \mathrm{U}(1)$	2
E I-IV	$\mathrm{Sp}(4)/\{\pm 1\}$	E_6	F_4	2
E II-III	$\mathrm{SU}(6) \cdot \mathrm{Sp}(1)$	E_6	$\mathrm{Spin}(10) \cdot \mathrm{U}(1)$	2
E II-IV	$\mathrm{SU}(6) \cdot \mathrm{Sp}(1)$	E_6	F_4	1
E III-IV	$\mathrm{Spin}(10) \cdot \mathrm{U}(1)$	E_6	F_4	1
E V-VI	$\mathrm{SU}(8)/\{\pm 1\}$	E_7	$\mathrm{SO}'(12) \cdot \mathrm{Sp}(1)$	4
E V-VII	$\mathrm{SU}(8)/\{\pm 1\}$	E_7	$\mathrm{E}_6 \cdot \mathrm{U}(1)$	3
E VI-VII	$\mathrm{SO}'(12) \cdot \mathrm{Sp}(1)$	E_7	$\mathrm{E}_6 \cdot \mathrm{U}(1)$	2
E VIII-IX	$\mathrm{SO}'(16)$	E_8	$\mathrm{E}_7 \cdot \mathrm{Sp}(1)$	4
F I-II	$\mathrm{Sp}(3) \cdot \mathrm{Sp}(1)$	F_4	$\mathrm{Spin}(9)$	1

Table 3. Hermann actions.

It can be shown using Proposition 4.1 that polar actions on compact symmetric spaces have this lifting property only if they are hyperpolar. In particular, if we lift the known polar actions on symmetric spaces to the groups, we do *not* obtain any examples of polar actions besides the hyperpolar ones. Another remarkable property of hyperpolar actions is that they are orbit maximal on irreducible symmetric spaces of compact type.

Proposition 2.8. *Let $M = G/K$ be a connected irreducible symmetric space of compact type and $H \subset L \subset G$ be closed connected subgroups. If the H -action on M is hyperpolar, then the L -action on M is transitive or orbit equivalent to the H -action.*

Proof. See [23], Corollary D.

q.e.d.

The non-orbit maximal examples of polar actions found by Podestà and Thorbergsson [39] show that Proposition 2.8 does not directly generalize to polar actions, see below. However, it is a consequence of our classification that polar actions on symmetric spaces of rank ≥ 2 with simple compact isometry group are orbit maximal.

2.4. Polar actions on rank one symmetric spaces. Polar actions on rank one symmetric spaces have been classified by Podestà and Thorbergsson [39]. The hyperpolar, i.e., cohomogeneity one, actions on these spaces had before been classified in [29], [43], [14] and [31]. The results can be summarized as follows. The classification of polar actions on spheres (and real projective spaces) follows from [13], since every polar action on the sphere is given as the restriction of a polar representation to the sphere. The isotropy representations of Hermitian symmetric spaces of real dimension $2n + 2$ induce polar actions on $\mathbb{C}P^n$ and all polar actions on $\mathbb{C}P^n$ are orbit equivalent to actions obtained in this fashion. Similarly, all polar actions on $\mathbb{H}P^n$ come from isotropy representations of products of quaternion-Kähler symmetric spaces, with the additional restriction that all factors but one must be of rank one. While all these polar actions arise from polar actions on the sphere, the actions on the Cayley plane $\mathbb{O}P^2 = F_4/\text{Spin}(9)$ do not have such an interpretation. The maximal connected subgroup $SU(3) \cdot SU(3) \subset F_4$ acts polarly on the Cayley plane with cohomogeneity two. The groups $Sp(3) \cdot Sp(1)$, $Sp(3) \cdot U(1)$, $Sp(3)$ and $\text{Spin}(9)$ act with cohomogeneity one. In addition, there are three polar actions of cohomogeneity two with a fixed point of the following subgroups of $\text{Spin}(9)$:

$$\text{Spin}(8), \quad \text{SO}(2) \cdot \text{Spin}(7), \quad \text{Spin}(3) \cdot \text{Spin}(6).$$

In particular, polar actions on rank one symmetric spaces are not orbit maximal in general.

3. Symmetric spaces and their totally geodesic submanifolds

In the following we will collect some useful facts about symmetric spaces and their totally geodesic submanifolds. Sections of polar actions are totally geodesic submanifolds and it will be shown in Theorem 5.4 that the sections of a non-trivial polar action on an irreducible compact symmetric space are locally isometric to Riemannian products whose factors are spaces of constant curvature. We give an upper bound on the dimension of such submanifolds in Lemma 3.3. For the proof of Theorem 5.4, which is essentially a consequence of the Splitting Theorem 5.2, we will need the characterization of totally geodesic hypersurfaces in reducible symmetric spaces given in Corollary 3.5, because the Weyl group of a polar action is generated by reflections in totally geodesic hypersurfaces. We will conclude this section by recalling a well known characterization of maximal subgroups in the classical groups.

Every symmetric space M may be presented as G/K , where G is the isometry group of M and K is a symmetric subgroup of G . Conversely, if (G, K) is a symmetric pair, then G/K is a symmetric space if it is equipped with an appropriate metric. A Riemannian manifold is an irreducible Riemannian symmetric space of compact type if and only if it is isometric to

- either G/K , where G is a simple, compact, connected Lie group and K a symmetric subgroup of G (*symmetric space of type I*),
- or a simple, compact, connected Lie group equipped with a biinvariant metric (*symmetric space of type II*).

The local isometry classes of the symmetric spaces of type I are given by Table 4. By $SO'(2n)$ we denote the image of a half-spin representation of $\text{Spin}(2n)$.

Type	G/K	Rank	Dimension
A I	$SU(n)/SO(n)$	$n - 1$	$\frac{1}{2}(n - 1)(n + 2)$
A II	$SU(2n)/Sp(n)$	$n - 1$	$(n - 1)(2n + 1)$
A III	$SU(p + q)/S(U(p) \times U(q))$	$\min(p, q)$	$2pq$
B D I	$SO(p + q)/SO(p) \times SO(q)$	$\min(p, q)$	pq
C I	$Sp(n)/U(n)$	n	$n(n + 1)$
C II	$Sp(p + q)/Sp(p) \times Sp(q)$	$\min(p, q)$	$4pq$
D III	$SO(2n)/U(n)$	$\lfloor \frac{n}{2} \rfloor$	$n(n - 1)$
E I	$E_6/(Sp(4)/\{\pm 1\})$	6	42
E II	$E_6/SU(6) \cdot Sp(1)$	4	40
E III	$E_6/Spin(10) \cdot U(1)$	2	32
E IV	E_6/F_4	2	26
E V	$E_7/(SU(8)/\{\pm 1\})$	7	70
E VI	$E_7/SO'(12) \cdot Sp(1)$	4	64
E VII	$E_7/E_6 \cdot U(1)$	3	54
E VIII	$E_8/SO'(16)$	8	128
E IX	$E_8/E_7 \cdot Sp(1)$	4	112
F I	$F_4/Sp(3) \cdot Sp(1)$	4	28
F II	$F_4/Spin(9)$	1	16
G	$G_2/SO(4)$	2	8

Table 4. Symmetric spaces of type I.

The global isometry classes of symmetric spaces are described by the following theorem, which will be needed for the proof of the Splitting Theorem 5.2.

Theorem 3.1. *Let M be a simply connected Riemannian symmetric space with decomposition $M = M_0 \times M_1 \times \dots \times M_t$ into Euclidean and*

irreducible parts. Define $G = V \times I(M_1)_0 \times \dots \times I(M_t)_0$ where V is the vector group of pure translations of the Euclidean space M_0 . Define $\Delta = V \times \Delta_1 \times \dots \times \Delta_t$ where Δ_i ($i > 0$) is the centralizer of $I(M_i)_0$ in $I(M_i)$. Then G is the group generated by all transvections of M , and Δ is the centralizer of G in $I(M)$. In particular, the symmetric spaces covered by M are just the manifolds M/Γ where Γ is a discrete subgroup of Δ .

The group Δ_i is trivial if M_i is noncompact, and finite if M_i is compact. In particular, the discrete subgroups of Δ are just the subgroups $\Gamma \subseteq \Delta$ with discrete projection on the vector group V .

Proof. See [48], Ch. 8, Sec. 3. q.e.d.

Totally geodesic submanifolds of symmetric spaces correspond to Lie triple systems.

Proposition 3.2. *Let M be a Riemannian globally symmetric space and let $p_0 \in M$. Let $G = I(M)_0$ and let $K = G_{p_0}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where \mathfrak{k} is the Lie algebra of K , and where we identify $\mathfrak{p} = T_{p_0}M$ as usual. Let $\sigma_*: \mathfrak{g} \rightarrow \mathfrak{g}$ be the automorphism of \mathfrak{g} which acts on \mathfrak{k} as $\text{id}_{\mathfrak{k}}$ and on \mathfrak{p} as $-\text{id}_{\mathfrak{p}}$. The totally geodesic submanifolds of M containing p_0 are in one-to-one correspondence with the Lie triple systems $\mathfrak{s} \subseteq \mathfrak{p}$; i.e., if $\mathfrak{s} \subseteq \mathfrak{p}$ is a Lie triple system, then $\exp(\mathfrak{s}) \subseteq M$ is a totally geodesic submanifold and, conversely, if $S \subseteq M$ is a totally geodesic submanifold, then $T_{p_0}S \subseteq \mathfrak{p}$ is a Lie triple system.*

Moreover, for any Lie triple system $\mathfrak{s} \subset \mathfrak{p}$, define $\mathfrak{g}' = \mathfrak{s} + [\mathfrak{s}, \mathfrak{s}]$ and $\mathfrak{k}' = [\mathfrak{s}, \mathfrak{s}]$; then \mathfrak{g}' is the Lie subalgebra of \mathfrak{g} generated by \mathfrak{s} , \mathfrak{g}' is invariant under σ_* and $\mathfrak{k}' = \mathfrak{g}' \cap \mathfrak{k}$. Let G' and K' be the connected Lie subgroups of G with Lie algebras \mathfrak{g}' and \mathfrak{k}' , respectively. Then (G', K') is a symmetric pair and G' acts transitively on $\exp(\mathfrak{s})$.

Proof. See [26], Ch. IV, § 7. q.e.d.

Lemma 3.3. *Let (G, K) be a symmetric pair such that $M = G/K$ is a Riemannian symmetric space of compact type and let $\Sigma \subseteq M$ be a totally geodesic submanifold whose universal cover is a product of spheres. Then $\dim(\Sigma) \leq \text{rk}(G) + \text{rk}(K)$.*

Proof. Let (G_Σ, K_Σ) be the symmetric pair corresponding to Σ ; we have $\mathfrak{g}_\Sigma = \mathfrak{g}_\Sigma^1 \oplus \dots \oplus \mathfrak{g}_\Sigma^m$, where $\mathfrak{g}_\Sigma^i \cong \mathfrak{so}(n_i + 1)$. Let $\mathfrak{k}_\Sigma^i \subseteq \mathfrak{g}_\Sigma^i$ such that $\mathfrak{k}_\Sigma^i \cong \mathfrak{so}(n_i)$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as usual. By Proposition 3.2, we may assume $\mathfrak{k}_\Sigma^i \subseteq \mathfrak{k}$. Now choose maximal abelian subalgebras $\mathfrak{a}_i \subseteq \mathfrak{g}_\Sigma^i$ as follows. If n_i is even, then $\text{rk}(\mathfrak{k}_\Sigma^i) = \text{rk}(\mathfrak{g}_\Sigma^i)$ and we may choose $\mathfrak{a}^i = \mathfrak{a}_i^{\mathfrak{k}} \subseteq \mathfrak{k}_\Sigma^i$. If n_i is odd, then we may choose $\mathfrak{a}^i \subseteq \mathfrak{g}_\Sigma^i$ such that $\mathfrak{a}_i = \mathfrak{a}_i^{\mathfrak{p}} \oplus \mathfrak{a}_i^{\mathfrak{k}}$ where $\mathfrak{a}_i^{\mathfrak{p}} \subseteq \mathfrak{p}$ is one-dimensional and $\mathfrak{a}_i^{\mathfrak{k}} \subseteq \mathfrak{k}_\Sigma^i$. Let $\mathfrak{a}^{\mathfrak{k}} = \bigoplus_{i=1}^m \mathfrak{a}_i^{\mathfrak{k}}$ and $\mathfrak{a}^{\mathfrak{p}} = \bigoplus_{n_i \equiv 1(2)} \mathfrak{a}_i^{\mathfrak{p}}$. Then we have $\dim(\Sigma) = 2 \cdot \dim(\mathfrak{a}^{\mathfrak{k}}) + \dim(\mathfrak{a}^{\mathfrak{p}})$. Since $\mathfrak{a}^{\mathfrak{k}} \oplus \mathfrak{a}^{\mathfrak{p}}$ is an

abelian subalgebra of \mathfrak{g} and $\mathfrak{a}^{\mathfrak{k}}$ is an abelian subalgebra of \mathfrak{k} , it follows that $\dim(\Sigma) \leq \text{rk}(G) + \text{rk}(K)$. q.e.d.

The estimate on the dimension given by the Lemma above is not optimal in all cases. See [8] for classifications of totally geodesic submanifolds in symmetric spaces.

The following theorem, which characterizes maximal totally geodesic submanifolds of reducible symmetric spaces, is an analogue of Theorem 15.1 in [15], which characterizes maximal subalgebras of semisimple Lie algebras. We give a proof which is similar to the proof in [15].

Theorem 3.4. *Let S be a connected simply connected symmetric space with decomposition $S = S_0 \times S_1 \times \dots \times S_k$ such that S_1, \dots, S_k are irreducible and S_0 is of Euclidean type. Let V be a maximal totally geodesic submanifold of S , (i.e., if there is a totally geodesic submanifold W such that $V \subseteq W \subseteq S$ then either $V = W$ or $W = S$). Let $p = (p_0, \dots, p_k) \in V$. Then either there is an index $i \in \{0, \dots, k\}$ and a totally geodesic submanifold $\tilde{V} \subset S_i$ such that*

$$V = S_0 \times \dots \times S_{i-1} \times \tilde{V} \times S_{i+1} \times \dots \times S_k,$$

or there are two factors S_i and S_j ($i \neq j$) and a map $\phi: S_i \rightarrow S_j$ which is an isometry up to scaling such that

$$V = \prod_{\substack{\ell=1 \\ \ell \neq i, j}}^k S_\ell \times \{(x, \phi(x)) | x \in S_i\}.$$

Proof. Let $G = I(S)$ and let $K = I(S)_p$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition associated with the symmetric space $S = G/K$. Let $G_i = I(S_i)$ and let $K_i = I(S_i)_{p_i}$ such that $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$ are Cartan decompositions corresponding to the irreducible factors $S_i = G_i/K_i$. Since V is a totally geodesic submanifold, we have that $\nu = T_p V \subseteq \mathfrak{p}$ is a Lie triple system by Proposition 3.2. Obviously, the projection $\pi_i(\nu)$ onto each of the summands \mathfrak{p}_i is again a Lie triple system. Now there are two cases:

Either there is an index $i \in \{0, \dots, k\}$ such that $\text{pr}_i(V) \neq S_i$, where $\text{pr}_i: S \rightarrow S_i$ denotes the canonical projection onto S_i . Then $\text{pr}_i(V)$ is a totally geodesic submanifold in S_i and there is a maximal totally geodesic submanifold $\tilde{V} \subset S_i$ containing $\text{pr}_i(V)$. Thus, $S_0 \times \dots \times S_{i-1} \times \tilde{V} \times S_{i+1} \times \dots \times S_k$ is a totally geodesic submanifold of S containing V and which is, by maximality, equal to V .

Or $\pi_i(\nu) = \mathfrak{p}_i$ for all $i = 0, \dots, k$, where the Lie algebra epimorphisms $\pi_i: \mathfrak{g} \rightarrow \mathfrak{g}_i$ are given by the canonical projections. In this case, it follows that there are at least two indices $i, j \in \{0, \dots, k\}$ such that \mathfrak{p}_i and \mathfrak{p}_j are both not contained in ν . Define $\nu^* = \nu \cap (\mathfrak{p}_i \oplus \mathfrak{p}_j) \neq \nu$. This is a Lie triple system in \mathfrak{p} , since it is the intersection of two Lie triple systems.

Hence, $\bigoplus_{\substack{\ell=0 \\ \ell \neq i,j}}^k \mathfrak{p}_\ell \oplus \nu^*$ is a Lie triple system in \mathfrak{p} which contains ν and is different from \mathfrak{p} ; thus, by maximality, is the tangent space $T_p V$.

It remains now to study the Lie triple system $\nu^* \subset \mathfrak{p}_i \oplus \mathfrak{p}_j$. By Proposition 3.2, it follows that the Lie algebra $\mathfrak{g}' = \nu^* \oplus [\nu^*, \nu^*]$ generated by ν^* is the Lie algebra of a group $G' \subset G_i \times G_j$ acting transitively on the totally geodesic submanifold V^* of $S_i \times S_j$, which is the exponential image of $\nu^* \subset T_{(p_i, p_j)}(S_i \times S_j)$.

We show that $\mathfrak{g}' \cap \mathfrak{g}_i$ is an ideal in \mathfrak{g}_i : Let $x \in \mathfrak{g}_i$, $y \in \mathfrak{g}' \cap \mathfrak{g}_i$. There is a $z \in \mathfrak{g}'$ such that $\pi_i(z) = x$, and it follows that $[x, y] = [z, y] \in \mathfrak{g}' \cap \mathfrak{g}_i$. By the same argument, $\mathfrak{g}' \cap \mathfrak{g}_j$ is an ideal in \mathfrak{g}_j .

Let us assume for the moment that $i, j \neq 0$. Since $\pi_i(\nu) = \mathfrak{p}_i$ and $\pi_j(\nu) = \mathfrak{p}_j$, we have that $\pi_i(\mathfrak{g}') = \mathfrak{g}_i$ and $\pi_j(\mathfrak{g}') = \mathfrak{g}_j$, since S_i and S_j are irreducible symmetric spaces. Since they are the Lie algebras of isometry groups of irreducible symmetric spaces, the \mathfrak{g}_i are either simple or the direct sum of two isomorphic simple ideals $\mathfrak{h}_i \oplus \mathfrak{h}_i$ (in case S_i is of type II). Therefore the ideal $\mathfrak{g}' \cap \mathfrak{g}_i$ is either zero, equal to \mathfrak{g}_i or equal to \mathfrak{h}_i . The last case is impossible, since \mathfrak{g}' has to be invariant under the action of the Cartan involution of $\mathfrak{h}_i \oplus \mathfrak{h}_i$, which is given by $(x, y) \mapsto (y, x)$; the case $\mathfrak{g}' \cap \mathfrak{g}_i = \mathfrak{g}_i$ is also impossible, since \mathfrak{p}_i is not contained in ν^* . Now we can show that \mathfrak{g}_i and \mathfrak{g}_j are isomorphic: Let $x \in \mathfrak{g}_i$. Then there is an element $y \in \mathfrak{g}_j$ such that $(x, y) \in \mathfrak{g}'$; but this element is uniquely defined, since otherwise \mathfrak{g}_j would have a non-trivial intersection with \mathfrak{g}' . The map $\mathfrak{g}_i \rightarrow \mathfrak{g}_j$ we defined in this way is easily seen to be a Lie algebra isomorphism, and the subalgebra \mathfrak{g}' is given by the diagonal embedding of $\mathfrak{g}' \rightarrow \mathfrak{g}' \oplus \mathfrak{g}' \cong \mathfrak{g}_i \oplus \mathfrak{g}_j$. It remains to be shown that the spaces S_i and S_j are isometric up to scaling: This follows from the requirement that the Cartan involution corresponding to $\mathfrak{g}_i \oplus \mathfrak{g}_j = (\mathfrak{k}_i \oplus \mathfrak{k}_j) \oplus (\mathfrak{p}_i \oplus \mathfrak{p}_j)$ must leave the diagonally embedded subalgebra \mathfrak{g}' invariant and is thus of the form $(x, y) \mapsto (\sigma(x), \sigma(y))$, where σ is an involution of $\mathfrak{g}_i \cong \mathfrak{g}_j$.

Finally, assume $i = 0$, $j \in \{1, \dots, k\}$. This case can be included in the above proof if we further split \mathfrak{p}_0 into the direct sum of $\mathfrak{g}' \cap \mathfrak{p}_0$ plus a complementary subspace. Then one is again in the situation that \mathfrak{g} can be written as a direct sum of ideals, all of which have either trivial intersection with the Lie algebra generated by ν or are contained in this Lie algebra. Then the same type of argument leads to the contradiction that an abelian Lie algebra is isomorphic to one of $\mathfrak{g}_1, \dots, \mathfrak{g}_k$. q.e.d.

Corollary 3.5. *Let S be a connected simply connected symmetric space with decomposition $S = S_0 \times S_1 \times \dots \times S_k$ such that S_1, \dots, S_k are irreducible and S_0 is of Euclidean type. Let H be a totally geodesic hypersurface of S . Let $p = (p_0, \dots, p_k) \in H$. Then there is an index $i \in \{0, \dots, k\}$ and a totally geodesic hypersurface $\tilde{H} \subset S_i$ such that*

$$H = S_0 \times \dots \times S_{i-1} \times \tilde{H} \times S_{i+1} \times \dots \times S_k.$$

Proof. Obviously, a totally geodesic hypersurface is a maximal totally geodesic submanifold, so we may apply Theorem 3.4. Irreducible non-flat symmetric spaces are at least of dimension two; thus the second possibility in the assertion of Theorem 3.4 does not occur here, since this would lead to submanifolds of codimension at least two. q.e.d.

The following facts on the maximal connected subgroups of the classical groups can be proven by standard arguments from the representation theory of compact Lie groups, see e.g., [16]. It should be remarked that some of subgroups of $\mathrm{SO}(n)$, $\mathrm{SU}(n)$ or $\mathrm{Sp}(n)$ given by irreducible representations of simple groups of corresponding (real, complex or quaternionic) type are not maximal connected; see [16] for complete lists of inclusions.

Proposition 3.6. *Let K be a connected proper subgroup of $\mathrm{SO}(n)$. Then there is an automorphism α of $\mathrm{SO}(n)$ such that $\alpha(K)$ is contained in one of the following subgroups of $\mathrm{SO}(n)$:*

- (i) $\mathrm{SO}(k) \times \mathrm{SO}(n - k)$, $1 \leq k \leq n - 1$
- (ii) $\mathrm{SO}(p) \otimes \mathrm{SO}(q)$, $pq = n$, $3 \leq p \leq q$
- (iii) $\mathrm{U}(k)$, $2k = n$
- (iv) $\mathrm{Sp}(p) \cdot \mathrm{Sp}(q)$, $4pq = n \neq 4$,

or K is a simple irreducible subgroup $K = \varrho(H) \subset \mathrm{SO}(n)$, where H is a simple compact Lie group and ϱ is an irreducible representation of H of real type such that $\deg \varrho = n$.

Proposition 3.7. *Let K be a connected proper subgroup of $\mathrm{SU}(n)$. Then there is an automorphism α of $\mathrm{SU}(n)$ such that $\alpha(K)$ is contained in one of the following subgroups of $\mathrm{SU}(n)$:*

- (i) $\mathrm{SO}(n)$
- (ii) $\mathrm{Sp}(m)$, $2m = n$
- (iii) $\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(n - k))$, $1 \leq k \leq n - 1$
- (iv) $\mathrm{SU}(p) \otimes \mathrm{SU}(q)$, $pq = n$, $p \geq 3$, $q \geq 2$,

or K is a simple irreducible subgroup $K = \varrho(H) \subset \mathrm{SU}(n)$, where H is a simple compact Lie group and ϱ is an irreducible representation of H of complex type such that $\deg \varrho = n$.

Proposition 3.8. *Let K be a connected proper subgroup of $\mathrm{Sp}(n)$. Then there is an automorphism α of $\mathrm{Sp}(n)$ such that $\alpha(K)$ is contained in one of the following subgroups of $\mathrm{Sp}(n)$:*

- (i) $\mathrm{U}(n)$,
- (ii) $\mathrm{Sp}(k) \times \mathrm{Sp}(n - k)$, $1 \leq k \leq n - 1$
- (iii) $\mathrm{SO}(p) \otimes \mathrm{Sp}(q)$, $pq = n$, $p \geq 3$, $q \geq 1$,

or K is a simple irreducible subgroup $K = \varrho(H) \subset \mathrm{Sp}(n)$, where H is a simple compact Lie group and ϱ is an irreducible representation of H of quaternionic type such that $\deg \varrho = 2n$.

4. Criteria for polarity

The following is a generalization of the criterion for hyperpolarity given in [24]. Note that we do not require the sections to be embedded submanifolds here. Hyperpolar actions are characterized by the property that the Lie triple system ν in Proposition 4.1 is abelian.

Proposition 4.1. *Let G be a connected compact Lie group, $K \subset G$ a symmetric subgroup, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition. Let $H \subseteq G$ be a closed subgroup. Let k be the cohomogeneity of the H -action on G . Then the following are equivalent.*

- (i) *The H -action on G/K is polar w.r.t some Riemannian metric induced by an $\text{Ad}(G)$ -invariant scalar product on \mathfrak{g} .*
- (ii) *For any $g \in G$ such that gK lies in a principal orbit of the H -action on G/K , the subspace $\nu = g^{-1}N_{gK}(H \cdot gK) \subseteq \mathfrak{p}$ is a k -dimensional Lie triple system such that the Lie algebra $\mathfrak{s} = \nu \oplus [\nu, \nu]$ generated by ν is orthogonal to $\text{Ad}(g^{-1})\mathfrak{h}$.*
- (iii) *The normal space $N_{eK}(H \cdot eK) \subseteq \mathfrak{p}$ contains a k -dimensional Lie triple system ν such that the Lie algebra $\mathfrak{s} = \nu \oplus [\nu, \nu]$ generated by ν is orthogonal to \mathfrak{h} .*

Proof. Let $g \in G$ be such that gK lies in a principal orbit of the H -action on G/K . Then the action of $g^{-1}Hg$ on G/K has a principal orbit containing eK and the equivalence of (i) and (ii) follows from [20], Proposition, p. 193.

Assume now that condition (i) holds. Let Σ be a section of the polar H -action on G/K such that $eK \in \Sigma$. Let $\nu = T_{eK}\Sigma \subseteq \mathfrak{p}$, let $\mathfrak{s} = \nu \oplus [\nu, \nu]$, and let S be the connected subgroup of G corresponding to \mathfrak{s} . Since S acts transitively on Σ , there is an element $s \in S$ such that the point sK lies in a principal orbit of the H -action on G/K . Now it follows from (ii) that $\text{Ad}(s^{-1})\mathfrak{h}$ is orthogonal to the Lie algebra generated by $s^{-1}N_{sK}(H \cdot sK)$, which coincides with \mathfrak{s} . Since $\text{Ad}(s^{-1})$ leaves \mathfrak{s}^\perp invariant, we have that \mathfrak{h} is orthogonal to \mathfrak{s} and (iii) follows.

We will now show that if (iii) holds, then $\Sigma = \exp(\nu) \subseteq G/K$ meets the orbits orthogonally. Let $sK \in \Sigma$, where s is an arbitrary element of the Lie group S corresponding to the Lie algebra generated by ν . The tangent space of the H -orbit through sK is orthogonal to $T_{sK}\Sigma$ if and only if $s^{-1}\mathfrak{h}s \perp \nu$. But since the adjoint representation of G restricted to S leaves the orthogonal complement of \mathfrak{s} invariant, $s^{-1}\mathfrak{h}s$ is perpendicular to \mathfrak{s} . Thus the H -action on G/K is polar. q.e.d.

As an immediate consequence of this criterion, the problem of classifying polar actions on G/K is reduced to a problem on the Lie algebra level. We conclude this section with the simple observation that a polar action restricted to an invariant totally geodesic submanifold is polar.

Lemma 4.2. *Let G be a compact Lie group acting polarly on a connected Riemannian manifold N . Let $M \subseteq N$ be a connected totally geodesic submanifold which is invariant under the G -action. Then the G -action on M is polar.*

Proof. Let $\Sigma \subseteq N$ be a section of the G -action on N . Let Σ_0 be a connected component of $\Sigma \cap M$. Then the totally geodesic submanifold $\Sigma_0 \subseteq M$ obviously meets the G -orbits in M orthogonally at every intersection point. Furthermore, since M is connected, any two orbits of the G -action on M can be joined by a shortest geodesic that meets the principal G -orbits orthogonally and is hence contained in Σ after conjugation with a group element. This geodesic is now also contained in M , since M is totally geodesic. This proves that Σ_0 meets all G -orbits in M . q.e.d.

5. Sections and Weyl group actions

Let us first recall some known properties of the Weyl group.

Lemma 5.1 (Thorbergsson, Podestà). *Let M be a simply connected symmetric space on which a compact, connected Lie group G acts polarly and nontrivially. Let Σ be a section of the polar action and let $p \in \Sigma$ be such that the orbit through p is singular. Then there is a totally geodesic hypersurface H in Σ passing through p and consisting of singular points; moreover, there exists a non-trivial element $g \in W_\Sigma$ which fixes H pointwisely.*

The set of singular points in Σ is a union of finitely many totally geodesic hypersurfaces $\{H_i\}_{i \in I}$ in Σ ; the Weyl group W_Σ is generated by reflections in the hypersurfaces $\{H_i\}_{i \in I}$. Let $\tilde{\Sigma}$ be the universal covering of Σ and let $\{P_j\}_{j \in J}$ be the collection of all lifts of all the totally geodesic hypersurfaces $\{H_i\}_{i \in I}$ in Σ . Let \hat{W}_Σ be the subgroup of the isometry group of $\tilde{\Sigma}$ which is generated by the reflections in the hypersurfaces $\{P_j\}_{j \in J}$. Then \hat{W}_Σ is a Coxeter group and W_Σ is a quotient group of \hat{W}_Σ .

Proof. See [39], Lemma 1A.4 or [19], Section 2.3 for a more general statement. q.e.d.

The following splitting theorem is a generalization of Lemma 1A.2 in [39], where $\tilde{\Sigma}_1$ is a point and the hypothesis is equivalent to a trivial Weyl group action. We consider the weaker hypothesis that the section of a polar action is locally a product such that the Weyl group acts trivially on one factor.

Splitting Theorem 5.2. *Let N be a compact connected Riemannian symmetric space on which a connected compact Lie group G acts polarly. Assume the universal covering $\tilde{\Sigma}$ of a section Σ decomposes as*

a Riemannian product $\tilde{\Sigma} = \tilde{\Sigma}_1 \times \tilde{\Sigma}_2$ and the action of \hat{W}_Σ on $\tilde{\Sigma}$ descends to an action on $\tilde{\Sigma}_1$ such that

$$w \cdot (p, q) = (w \cdot p, q) \quad \text{for all } w \in \hat{W}_\Sigma, p \in \tilde{\Sigma}_1, q \in \tilde{\Sigma}_2.$$

Then the universal cover of N is a Riemannian product isometric to $\tilde{M} \times \tilde{\Sigma}_2$, where $M = G \cdot \Sigma_1$ and Σ_1 is the image of $\tilde{\Sigma}_1$ under the covering map $\tilde{\Sigma} \rightarrow \Sigma$.

Proof. Let Σ be a section and $p \in \Sigma$ be an arbitrary point of this section. For $i = 1, 2$, let $\Sigma_i = \Sigma_i(p)$ be the totally geodesic submanifolds of Σ corresponding to $\tilde{\Sigma}_i$ (uniquely determined by Theorem 3.1) such that $p \in \Sigma_i$.

First we show that the isotropy group G_p acts trivially on $T_p\Sigma_2$. Consider the slice representation of G_p on $V = N_p(G \cdot p)$ which is polar by Proposition 2.3, with section $T_p\Sigma = T_p\Sigma_1 \oplus T_p\Sigma_2$. Now consider the Weyl group W' of this polar linear representation; it coincides with $(W_\Sigma)_p$. Its representation space decomposes into a sum of irreducible modules and one trivial module and the section $T_p\Sigma$ decomposes accordingly, see [13]. It follows from the hypothesis that W' acts trivially on the linear subspace $T_p\Sigma_2$. Since irreducible polar representations have irreducible Weyl groups, it follows that G_p acts trivially on $T_p\Sigma_2$.

We will now show that the set $M(p) = G \cdot \Sigma_1$ is an embedded submanifold of N . By the Slice Theorem 2.2 there is an equivariant diffeomorphism Ψ of a G -invariant open neighborhood around the zero section in the normal bundle $G \times_{G_p} V \rightarrow G/G_p$ onto a G -invariant open neighborhood around the orbit $G \cdot p$ such that the zero section in $G \times_{G_p} V$ is mapped to the orbit $G \cdot p$. The diffeomorphism Ψ is given by the end point map which maps any normal vector $v_q \in V = N_q(G \cdot p)$ to its image under the exponential map $\exp_q(v_q)$.

Since Σ_2 is a totally geodesic submanifold of N , we have $\exp_p(T_p\Sigma_2) = \Sigma_2$. The subspace $T_p\Sigma_1 \subseteq V$ is fixed by the Weyl group W' of the slice representation and hence fixed by G_p . Hence the orthogonal complement S of $T_p\Sigma_2$ in V is a linear subspace invariant under the polar representation of G_p on V . Therefore, S defines a smooth subbundle of the normal bundle $G \times_{G_p} V$.

But since $T_p\Sigma_1$ is the section of the G_p -representation on S , we have $S = G_p \cdot T_p\Sigma_1$. From the fact that Ψ is an equivariant diffeomorphism it follows now that the elements of the subbundle defined by S are mapped into the set $G \cdot \Sigma_1$. This shows that, in a neighborhood of p , the subset $M(p) = G \cdot \Sigma_1 \subseteq N$ is a smooth submanifold of codimension $\dim(\Sigma_2)$. Thus we see that the symmetric space N is foliated by the totally geodesic submanifolds $\{g \cdot \Sigma_2(p)\}_{g \in G, p \in \Sigma}$ with integrable normal bundle whose integral manifolds are given by $\{M(p)\}_{p \in \Sigma}$.

It follows from Theorem A of [4] that the universal cover \tilde{N} of N is topologically a product diffeomorphic to $\tilde{M} \times \tilde{\Sigma}_2$ such that the projection of \tilde{N} on the factor $\tilde{\Sigma}_2$ is a Riemannian submersion. We have just shown that the horizontal distribution of this Riemannian submersion is integrable. Since the sectional curvature of N is nonnegative, it follows from Theorem 1.3 of [46] that the fibers of this Riemannian submersion are totally geodesic. We conclude that \tilde{N} is a Riemannian product isometric to $\tilde{M} \times \tilde{\Sigma}_2$. q.e.d.

Corollary 5.3. *Let N be an irreducible Riemannian symmetric space of compact type on which a compact Lie group G acts polarly and non-trivially. Then the G -action on N has a singular orbit.*

Proof. Assume there is no singular orbit. Then by Lemma 5.1, the Weyl group W_Σ acts trivially on Σ . Hence it follows from Splitting Theorem 5.2 that \tilde{N} is a Riemannian product $\tilde{M} \times \tilde{\Sigma}$, where M is a G -orbit. But this is a contradiction to the irreducibility of N . q.e.d.

The following theorem is a generalization of Proposition 1B.1 of [39], where it was proved that the section of a polar action on a compact rank one symmetric space has constant curvature.

Theorem 5.4. *Let N be an irreducible compact simply connected symmetric space on which a compact Lie group G acts polarly and non-trivially with section Σ . Then Σ is covered by a Riemannian product of spaces that have constant curvature.*

Proof. Let $\tilde{\Sigma} = \tilde{\Sigma}_1 \times \tilde{\Sigma}_2$ be a decomposition of the universal covering $\tilde{\Sigma}$ of Σ such that Σ_1 is a Riemannian product of spaces of constant curvature and Σ_2 is either a point or a Riemannian product of irreducible symmetric spaces of non-constant curvature. The section Σ contains a union of finitely many totally geodesic hypersurfaces $\{H_i\}_{i \in I}$ such that the Weyl group W_Σ is generated by the reflections in the hypersurfaces $\{H_i\}_{i \in I}$. In view of Corollary 3.5, and since it is well known that the only irreducible symmetric spaces containing totally geodesic hypersurfaces are those of constant curvature, it is clear that the hypothesis of Splitting Theorem 5.2 is fulfilled and we conclude that Σ_2 is a point. q.e.d.

It follows from Theorem 5.4 and Lemma 3.3 that the cohomogeneity of a polar action on an irreducible symmetric space G/K is less or equal $\text{rk}(G) + \text{rk}(K)$. For hermitian symmetric spaces G/K the upper bound on the cohomogeneity can be further improved; see Proposition 5.5 below. These dimension bounds are essential for our classification, since they reduce the classification problem to a finite number of cases.

Proposition 5.5. *Let H be a compact Lie group acting polarly on a compact Kähler manifold M . Then the cohomogeneity of the H -action on M is less or equal $\text{rk}(H)$.*

Proof. By the Equivalence Theorem [30], see also Theorem 1.4 in [40], the cohomogeneity of the H -action is equal to the difference between the rank of H and the rank of a regular isotropy subgroup of H . q.e.d.

We have the following lower bounds on the dimension of groups acting polarly on the classical symmetric spaces.

Proposition 5.6. *Let H be a connected compact Lie group acting polarly and non-trivially on a symmetric space M . Assume $3 \leq k \leq n - 3$, $2 \leq \ell \leq n - 2$ and let $d = \dim(H)$.*

- (i) *If $M = \mathbb{G}_k(\mathbb{R}^n)$, then $d \geq 2n - 9$.*
- (ii) *If $M = \mathbb{G}_\ell(\mathbb{C}^n)$, then $d \geq 3n - 7$.*
- (iii) *If $M = \mathbb{G}_\ell(\mathbb{H}^n)$, then $d \geq 6n - 16$.*
- (iv) *If $M = \text{SO}(n)/\text{U}(\frac{n}{2})$, then $d \geq \frac{n^2}{4} - n$.*
- (v) *If $M = \text{SU}(n)/\text{SO}(n)$, then $d \geq \frac{n^2}{2} - n$.*
- (vi) *If $M = \text{SU}(n)/\text{Sp}(\frac{n}{2})$, then $d \geq \frac{n^2}{2} - 2n$.*
- (vii) *If $M = \text{Sp}(n)/\text{U}(n)$, then $d \geq n^2$.*

Proof. Follows from Proposition 5.5 in case of the spaces $\mathbb{G}_\ell(\mathbb{C}^n)$, $\text{SO}(n)/\text{U}(\frac{n}{2})$, and $\text{Sp}(n)/\text{U}(n)$, which are Hermitian symmetric, and from Theorem 5.4 and Lemma 3.3 otherwise. q.e.d.

6. Polar subactions

In this section, we will introduce our main tool for classifying polar actions through studying slice representations. The basic observation is the following maximality property of linear polar actions, see [35], Theorem 6.

Theorem 6.1. *Let $G \subset \text{SO}(n)$ be a closed connected subgroup which acts irreducibly on \mathbb{R}^n and non-transitively on the sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Let $H \subseteq G$ be a closed connected subgroup $\neq \{e\}$ that acts polarly on \mathbb{R}^n . Then the H -action and the G -action on \mathbb{R}^n are orbit equivalent.*

The proof of the above theorem relies on [41]. As an immediate consequence of Theorem 6.1, we have the following, cf. [7], Theorem 2.2.

Corollary 6.2. *Let X be a strongly isotropy irreducible Riemannian homogeneous space. Assume a connected compact Lie group H acts polarly on X such that the H -action has a one-dimensional orbit $H \cdot p$ or a fixed point $p \in X$. Then the space X is locally symmetric. Furthermore, X is a rank-one symmetric space or the action of H is orbit equivalent to the action of the connected component of the isotropy group of X at p .*

Proof. Assume first that $p \in X$ is a fixed point of the H -action on X . Let K be the connected component of the isotropy group of p . The isotropy representation of K on $T_p X$ restricted to H is polar by Proposition 2.3. If the action of K on the unit sphere in $T_p X$ is transitive, then the space X is rank-one symmetric. If K does not act transitively on the sphere, then the linear H -action on $T_p X$ is orbit equivalent to the K -action by Theorem 6.1. In particular they have the same cohomogeneity; hence the K -action on $T_p X$ is polar. It now follows that the principal orbits of the H -action agree with those of the K -action on X and the orbit equivalence of the two actions follows from Proposition 4.1, since the principal orbits of a hyperpolar action determine all other orbits. In case X is compact, the symmetry follows from [35], since then one may assume that X is a homogeneous space of a simple compact Lie group, see [47], Chapter I.1. Non-compact strictly isotropy irreducible Riemannian homogeneous spaces are symmetric by [47].

Now assume $p \in X$ is such that $\dim(H \cdot p) = 1$. If $H \cdot p$ is a regular orbit, it follows that a section $\Sigma \subset X$ is a totally geodesic hypersurface and hence X is locally isometric to a space of constant curvature. Assume now that $H \cdot p$ is a singular orbit, hence the slice representation of H_p on $N_p(H \cdot p)$ is nontrivial and polar by Proposition 2.3. However, since $T_p(H \cdot p)$ is one-dimensional, the isotropy representation of H_p on $T_p X = T_p(H \cdot p) \oplus N_p(H \cdot p)$ is polar. It now follows from Theorem 6.1 that the irreducible isotropy representation of M at p is orbit equivalent to the reducible H_p -action on $T_p M$, a contradiction. q.e.d.

In particular, we may restrict our attention to actions without fixed point in the following. The full classification of connected Lie groups acting polarly with a fixed point on the irreducible symmetric spaces of higher rank follows immediately from Lemma 2.6. As the proof of Corollary 6.2 shows, one obtains the same result also under the weaker hypothesis that the linear action of H on the tangent space $T_p X$ is polar.

Let G be a connected compact Lie group acting isometrically on a Riemannian manifold. We say the action of G on M is *polarity minimal* if there is no closed connected subgroup $H \subset G$ which acts nontrivially and polarly on M and such that the H -action is not orbit equivalent to the G -action. Note that a polarity minimal action can be polar or non-polar. We give various sufficient conditions for an orthogonal representation to be polarity minimal in the following proposition.

Proposition 6.3. *Let $\rho: G \rightarrow O(V)$ be a representation of the compact connected Lie group G . Then ρ is polarity minimal if one of the following holds.*

- (i) *The representation ρ is irreducible of cohomogeneity ≥ 2 .*

- (ii) *The representation space V is the direct sum of two equivalent G -modules.*
- (iii) *The representation space V contains a G -invariant submodule W such that the G -representation on W is almost effective, non-polar, and polarity minimal.*

Proof. Part (i) is a just a reformulation of Theorem 6.1. Assume now V is the direct sum of two equivalent G -modules; then the representation ρ restricted to any closed connected subgroup $H \subseteq G$ which acts nontrivially on V will have two equivalent nontrivial submodules; it then follows from [33], Lemma 2.9 that H acts non-polarly on V ; this proves part (ii). To prove part (iii), assume there is a closed connected subgroup H of G acting polarly on V . Since the G -action on the subspace W is non-polar and polarity minimal, it follows that H acts trivially on W . But W is an almost effective representation, thus H acts trivially on all of V . q.e.d.

While we do not have an *a priori* proof that polar actions on irreducible compact symmetric spaces of higher rank are orbit maximal, the following proposition gives various sufficient conditions under which one can show that certain non-polar actions are polarity minimal. In fact, this is our main tool to exclude subactions, and it will be used frequently in the sequel.

Lemma 6.4. *Let G be compact Lie group and $K \subset G$ be symmetric subgroup such that $M = G/K$ is an irreducible symmetric space and let $H \subset G$ be a closed subgroup. The action of H on M is non-polar and polarity minimal if there is a non-polar polarity minimal submodule $V \subseteq N_p(H \cdot p)$ of the slice representation at p such that one of the following holds.*

- (i) *M is Hermitian symmetric and $\dim(V) > \text{rk}(H)$.*
- (ii) *$\dim(V) > s(M)$, where $s(M)$ is the maximal dimension of a totally geodesic submanifold of M locally isometric to a product of spaces with constant curvature, cf. Lemma 3.3.*
- (iii) *$V \subseteq \mathfrak{p} = T_p M$ (where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as usual such that \mathfrak{k} is the Lie algebra of $K = G_p$) contains a Lie triple system corresponding to an irreducible symmetric space of nonconstant curvature, e.g., an irreducible symmetric space of higher rank.*
- (iv) *The isotropy group $H \cap K$ acts almost effectively on V and $\text{rk}(H \cap K) = \text{rk}(H)$.*

Proof. Assume a closed connected subgroup $U \subseteq H$ acts polarly on M . Consider the isotropy group at U_p of the U -action on M . Since $U_p \subseteq H_p$, the action of U_p on the normal space $N_p(U \cdot p)$ leaves the subspace V invariant. By Proposition 2.3, the slice representation of U_p on $N_p(U \cdot p)$ is polar; in particular, the U_p -action on V is polar. Since

the action of H_p on V is polarity minimal and non-polar, it follows that the action of the connected component $(U_p)_0$ on V is trivial. Hence V is contained in the section of the polar U_p -action on $N_p(U \cdot p)$, and thus V is tangent to a section Σ of the U -action on M ; in particular, $\dim(\Sigma) \geq \dim(V)$. Part (i) now follows from Lemma 5.5. Parts (ii) and (iii) follow from Theorem 5.4. If $\text{rk}(H \cap K) = \text{rk}(H)$ and $H \cap K$ acts almost effectively on V , then any closed subgroup $U \subseteq H$ with $\dim U > 0$ will have an intersection $U \cap K \subseteq H \cap K$ of positive dimension with K ; but $U \cap K$ acts on V non-polarly since V is polarity minimal; this proves (iv). q.e.d.

7. Hermann actions of higher cohomogeneity

In the remaining part of the paper, we will carry out the classification. We begin with subactions of Hermann actions whose cohomogeneity is ≥ 2 .

To study actions of reducible groups on the Grassmannians we will need the following technical lemma. Let us first introduce some notation. Let $G \subseteq \text{Gl}(n, \mathbb{R})$. Let V be a linear subspace of \mathbb{R}^n . Then we define the *normalizer* of V in G as $N_G(V) = \{g \in G \mid g(V) = V\}$, and similarly by $Z_G(V) = \{g \in G \mid g|_V = \text{id}_V\}$ the *centralizer* of V in G . Clearly, $N_G(V)$ is a subgroup of G , and since the elements of $N_G(V)$ leave V invariant, the group $N_G(V)$ acts on V . The kernel of this representation is the normal subgroup $Z_G(V) \subseteq N_G(V)$. The group $N_G(V)$ is the isotropy subgroup G_V of the G -action on the Grassmannian $\mathbb{G}_{\dim V}(\mathbb{R}^n)$ of $(\dim V)$ -dimensional linear subspaces in \mathbb{R}^n .

Lemma 7.1. *Let $H \subset \text{SO}(n)$ be a closed connected proper subgroup.*

- (i) *If for any 8-dimensional subspace $V \subseteq \mathbb{R}^n$ the natural action of the connected component of $N_H(V)/Z_H(V)$ on V is equivalent to the 8-dimensional spin representation of $\text{Spin}(7)$ or the standard representation of $\text{SO}(8)$, then $n = 8$ and $H \cong \text{Spin}(7)$.*
- (ii) *If for any 7-dimensional subspace $V \subseteq \mathbb{R}^n$ the natural action of the connected component of $N_H(V)/Z_H(V)$ on V is equivalent to the 7-dimensional irreducible representation of G_2 or the standard representation of $\text{SO}(7)$, then either $n = 7$ and $H \cong G_2$ or $n = 8$ and $H \cong \text{Spin}(7)$.*

Proof. We first show that in both cases the group H acts transitively on the unit sphere in \mathbb{R}^n . Let $p, q \in \mathbb{R}^n$ be two unit vectors and let V be linear subspace of \mathbb{R}^n containing p and q such that V is 8- or 7-dimensional, respectively. Then it follows from the hypothesis that $N_H(V)/Z_H(V)$ acts transitively on the unit sphere in the space V ; thus there is an element in H which maps p to q . This shows that H acts transitively on the unit sphere in \mathbb{R}^n and hence the pair (H, n) is one of

the following, see Table 7 in [36].

H	$U(m)$ $m \geq 2$	$SU(m)$ $m \geq 2$	$Sp(\ell) \cdot Sp(1)$ $\ell \geq 2$	$Sp(\ell) \cdot U(1)$ $\ell \geq 2$	$Sp(\ell)$ $\ell \geq 2$	$Spin(7)$	$Spin(9)$	G_2
n	$2m$	$2m$	4ℓ	4ℓ	4ℓ	8	16	7

It is easy to see that the first five groups do not have the property described in the hypothesis. For the groups $Spin(7)$ and G_2 the statement is either trivial or follows from the well known fact $S^7 = Spin(7)/G_2$.

It remains the case of $H = Spin(9)$ acting on \mathbb{R}^{16} . To prove the assertion for $\dim V = 8$, it suffices to exhibit an isotropy subgroup of the $Spin(9)$ -action on $\mathbb{G}_8(\mathbb{R}^{16})$ not containing $Spin(7)$ as a Lie subgroup. First choose an 8-dimensional subspace $\tilde{V} \subset \mathbb{R}^{16}$ such that the subgroup $Spin(8)$, acting by a representation equivalent to the sum of the two half-spin representations on \mathbb{R}^{16} , stabilizes \tilde{V} . Since $Spin(8) \subset Spin(9)$ is maximal connected, it coincides with the connected component of the isotropy group $H_{\tilde{V}}$ of the H -action on the Grassmannian $\mathbb{G}_8(\mathbb{R}^{16})$. Thus the H -orbit through \tilde{V} is 8-dimensional. We will determine the slice representation of the H -action at \tilde{V} . The group $(H_{\tilde{V}})_0 \cong Spin(8)$ acts on the tangent space $T_{\tilde{V}}\mathbb{G}_8(\mathbb{R}^{16})$ by the tensor product of the two half-spin representations of $Spin(8)$. By Weyl's dimension formula, this representation contains an irreducible summand which is 56-dimensional and must therefore coincide with the normal space $N_{\tilde{V}}(H \cdot \tilde{V})$. This shows that \tilde{V} lies in a singular orbit of the H -action on $\mathbb{G}_8(\mathbb{R}^{16})$. By [28], the principal isotropy subgroups of this slice representation are finite and we conclude that for generic subspaces $V \subset \mathbb{R}^{16}$ the group $N_H(V)/Z_H(V)$ is finite.

Similarly, to prove the assertion for $H = Spin(9)$ and $\dim V = 7$, choose a 7-dimensional subspace $\tilde{V} \subset \mathbb{R}^{16}$ such that $Spin(7) \subset Spin(9)$ stabilizes \tilde{V} . Using an analogous argument as in the case $k = 8$, we see that the 48-dimensional slice representation at \tilde{V} of the H -action on $\mathbb{G}_7(\mathbb{R}^{16})$ has finite principal isotropy subgroups, and hence for a generic 7-dimensional subspace $V \subset \mathbb{R}^{16}$ the group $N_H(V)/Z_H(V)$ is finite. q.e.d.

Lemma 7.2. *Let H, G, K be as in the following table, where $2 \leq k, \ell \leq \frac{n}{2}$.*

H	G/K
$S(U(k) \times U(n - k))$	$SU(n)/S(U(\ell) \times U(n - \ell))$
$SO(k) \times SO(n - k)$	$SO(n)/SO(\ell) \times SO(n - \ell)$
$Sp(k) \times Sp(n - k)$	$Sp(n)/Sp(\ell) \times Sp(n - \ell)$

Let U be a connected subgroup of H . Then the action of U on G/K is polar if either $U = H$ or U is conjugate to one of the following

subgroups, where in each case the U -action on G/K is orbit equivalent to the H -action. In particular, the U -action on G/K is hyperpolar.

U	G	Range
$G_2 \times G_2$	$SO(14)$	$\ell = 2$
$G_2 \times Spin(7)$	$SO(15)$	$\ell = 2$
$Spin(7) \times Spin(7)$	$SO(16)$	$\ell = 2, 3$
$G_2 \times SO(n-7)$	$SO(n)$	$\ell = 2, n \geq 9$
$Spin(7) \times SO(n-8)$	$SO(n)$	$\ell = 2, 3, n \geq 10$
$SU(k) \times SU(n-k)$	$SU(n)$	$(k, \ell) \neq (\frac{n}{2}, \frac{n}{2})$

Proof. To prove the lemma, we compute certain slice representations, cf. Section 2.3 in [33]. We assume in the following that the maximal reducible groups are standardly embedded as block diagonal matrices, cf. (2.1). Assume first $k \leq \ell$. We compute a slice representation of the action of $H = SO(k) \times SO(n-k)$ on $G/K = SO(n)/SO(\ell) \times SO(n-\ell)$. The connected component of the isotropy group is the group $(H \cap K)_0 = SO(k) \times SO(\ell-k) \times SO(n-\ell)$; it acts on the normal space

$$N_{eK}(H \cdot eK) = \left\{ \left(\begin{array}{ccc} 0 & 0 & M \\ 0 & 0 & 0 \\ -M^t & 0 & 0 \end{array} \right) \middle| M \in \mathbb{R}^{k \times n-\ell} \right\} \subset \mathfrak{so}(n)$$

by the tensor product of the two standard representations of the first and the last factor, i.e., $SO(k) \otimes SO(n-\ell)$. Assume $U \subseteq H$ is a closed subgroup acting polarly on G/K . By Lemma 2.6 the connected component of the isotropy group $U \cap K$ of the U -action must contain the product of the first and last factor $L = SO(k) \times SO(n-\ell) \subset H \cap K$, except possibly in cases $k = 2, 3$ and $n-\ell = 7, 8$, which will be treated below. Since this argument also holds for any conjugate subgroup hUh^{-1} , $h \in H$, it follows that U contains hLh^{-1} , for all $h \in H$, and hence U contains the subgroup generated by $\{h x h^{-1} \mid h \in H, x \in L\}$, which is the minimal normal subgroup of H containing L and we conclude $H = U$.

Using an analogous argument in the case of $H = S(U(k) \times U(n-k))$ acting on $G/K = SU(n)/S(U(\ell) \times U(n-\ell))$, we see that the only polar subaction is the action of $U = SU(k) \times SU(n-k)$, except in the case of $U = SU(k) \times SU(k)$ acting on $G/K = SU(2k)/S(U(k) \times U(k))$, where the slice representation of the U -action is non-polar, see Lemma 2.6. The same argument also works for the actions on the quaternionic Grassmannians and for the case $\ell \leq k$.

It remains to study the case where a slice representation of the U -action is given by the first three rows of Lemma 2.6. It follows from

Lemma 7.1 that this can only happen if U is obtained from H by replacing an $\mathrm{SO}(7)$ -factor with G_2 or replacing an $\mathrm{SO}(8)$ -factor with $\mathrm{Spin}(7)$. A dimension count shows that the actions obtained in this fashion are orbit equivalent to the respective H -action. q.e.d.

Action type	Effectivized slice representation	Kernel
A I-II	$\mathrm{SU}(n) \times \mathrm{SU}(n) / \Delta \mathrm{SU}(n)$	$\mathrm{U}(1)$
A I-III	$\mathrm{SO}(n) / \mathrm{SO}(k) \times \mathrm{SO}(n - k)$	
A II-III, k even	$\mathrm{Sp}(n) / \mathrm{Sp}(\frac{k}{2}) \times \mathrm{Sp}(n - \frac{k}{2})$	
A II-III, k odd	$\mathrm{Sp}(n - 1) / \mathrm{Sp}(\frac{k-1}{2}) \times \mathrm{Sp}(n - \frac{k+1}{2})$	$\mathrm{U}(1)$
A III-III	$\mathrm{SU}(k + n - \ell) / \mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(n - \ell))$	$\mathrm{SU}(\ell - k)$
BD I-I	$\mathrm{SO}(k + n - \ell) / \mathrm{SO}(k) \times \mathrm{SO}(n - \ell)$	$\mathrm{SO}(\ell - k)$
C I-II	$\mathrm{SU}(n) / \mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(n - k))$	$\mathrm{U}(1)$
C II-II	$\mathrm{Sp}(k + n - \ell) / \mathrm{Sp}(k) \times \mathrm{Sp}(n - \ell)$	$\mathrm{Sp}(\ell - k)$
D I-III, k even	$\mathrm{SU}(n) / \mathrm{S}(\mathrm{U}(\frac{k}{2}) \times \mathrm{U}(n - \frac{k}{2}))$	$\mathrm{U}(1)$
D I-III, k odd	$\mathrm{SU}(n - 1) / \mathrm{S}(\mathrm{U}(\frac{k-1}{2}) \times \mathrm{U}(n - \frac{k-1}{2}))$	$\mathrm{U}(1)$
D III-III'	$\mathrm{SO}(4n - 4) / \mathrm{U}(2n - 1)$	
D_4 I-I', $k=\ell=3$	$\mathrm{G}_2 / \mathrm{SO}(4)$	
E I-II	$\mathrm{F}_4 / \mathrm{Sp}(3) \cdot \mathrm{Sp}(1)$	
E I-III	$\mathrm{Sp}(4) / \mathrm{Sp}(2) \times \mathrm{Sp}(2)$	
E I-IV	$\mathrm{SU}(6) / \mathrm{Sp}(3)$	$\mathrm{Sp}(1)$
E II-III	$\mathrm{SU}(6) / \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(4))$	$\mathrm{Sp}(1)$
E II-IV	$\mathrm{Sp}(4) / \mathrm{Sp}(3) \times \mathrm{Sp}(1)$	
E III-IV	$\mathrm{F}_4 / \mathrm{Spin}(9)$	
E V-VI	$\mathrm{SU}(8) / \mathrm{S}(\mathrm{U}(4) \times \mathrm{U}(4))$	
E V-VII	$\mathrm{SU}(8) / \mathrm{Sp}(4)$	
E VI-VII	$\mathrm{SU}(8) / \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(6))$	
E VIII-IX	$\mathrm{SO}(16) / \mathrm{U}(8)$	
F I-II	$\mathrm{Sp}(3) / \mathrm{Sp}(2) \times \mathrm{Sp}(1)$	$\mathrm{Sp}(1)$

Table 5. Slice representations of Hermann actions.

Theorem 7.3. *Let G be a connected simple compact Lie group and let H and K be two non-conjugate connected symmetric subgroups of G such that the cohomogeneity r of the H -action on G/K is ≥ 2 . Let $U \subseteq H$ be a closed connected nontrivial subgroup acting polarly on G/K . Then the action of U on G/K is orbit equivalent to the hyperpolar H -action on G/K .*

Furthermore, $U \neq H$ if and only if U is as described in Lemma 7.2 or the triple (U, G, K) is one of

$$\begin{aligned} & (\mathrm{SU}(2n - 2k - 1) \times \mathrm{SU}(2k + 1), \mathrm{SU}(2n), \mathrm{Sp}(n)); \\ & (\mathrm{SU}(n), \mathrm{Sp}(n), \mathrm{Sp}(k) \times \mathrm{Sp}(n - k)); \\ & (\mathrm{SU}(n), \mathrm{SO}(2n), \mathrm{SO}(k) \times \mathrm{SO}(2n - k)), k < n; \\ & (\mathrm{SU}(n), \mathrm{SO}(2n), \alpha(\mathrm{U}(n))); \\ & (\mathrm{Spin}(10), \mathrm{E}_6, \mathrm{SU}(6) \cdot \mathrm{Sp}(1)); \\ & (\mathrm{E}_6, \mathrm{E}_7, \mathrm{SO}'(12) \cdot \mathrm{Sp}(1)). \end{aligned}$$

Proof. To prove the theorem, we use the explicit knowledge of slice representations of Hermann actions as given in Tables 3 and 5. The first table is a list of all Hermann actions (i.e., a list of all pairs (H, K) of non-conjugate symmetric subgroups of the simple compact Lie groups G up to automorphisms of G); Table 5 contains information about one irreducible slice representation of each action; slice representations of Hermann actions are s-representations by Lemma 11.1 and so each representation is described by a symmetric space G'/K' whose isotropy representation $\chi(G', K')$ is equivalent to the slice representation on the Lie algebra level; in the third column, the (local isomorphism type of the) kernel of the slice representation is given. It is straightforward to determine these slice representations for actions on the classical symmetric spaces; for the exceptional symmetric spaces one may use the technique described in Remark 10.1, cf. also [33], Prop. 3.5. Note that actually two different actions are given in each row of the table, i.e., the action of H on G/K and the action of K on G/H ; they have the same isotropy subgroups and slice representations.

Assume now that H and K are symmetric subgroups of the simple compact Lie group G , and $U \subseteq H$ is a closed connected subgroup acting polarly on G/K and such that the hyperpolar action of H on G/K is of cohomogeneity $r \geq 2$. The subactions of the types A III-III, BD I-I, and C II-II were treated in Lemma 7.2.

The slice representations given by the table are irreducible and non-transitive on the sphere, since we assume the cohomogeneity is ≥ 2 . Thus we may apply Theorem 6.1. The isotropy group of the U -action at $p = eK \in G/K$ is $U_p = U \cap H_p$. The representation of $H_p = H \cap K$ on the normal space $N_p(H \cdot p)$ restricted to U_p occurs as a submodule in the slice representation of U_p on $N_p(U \cdot p)$, and is therefore polar. By Theorem 6.1, the U_p -action on $V = N_p(H \cdot p)$ is either orbit equivalent to the H_p -action or trivial.

We first show that the slice representation of H_p restricted to U_p is non-trivial. If it is trivial, then V is contained in the tangent space of a section through p and we obtain a contradiction with Theorem 5.4, since $V \subset \mathfrak{g}$ is a Lie triple system corresponding to a totally geodesic

submanifold of G/K which is isometric to an irreducible symmetric space of higher rank; see the proof of Lemma 11.1. We may therefore assume the U_p -action on V is locally orbit equivalent to the irreducible polar representation of H_p . From Table 5, we see that we may assume the slice representation of H_p on V is not equivalent to one of the first three items in Lemma 2.6, except in the case of AI-III, which will be treated separately.

Then it follows that the group U_p contains the (component of the) isotropy group H_p or a subgroup as described in the 4th, 5th, and 6th item of Lemma 2.6. In these cases there exists a uniquely determined connected subgroup $L \subset H_p$ which is minimal with respect to the property that the L -action on V is orbit equivalent to the H_p -action. Note that this argument actually shows that for any $h \in H$ also hUh^{-1} contains the subgroup $L \subset H_p$, and hence U contains all groups $h^{-1}Lh$ conjugate to L in H . We conclude that U contains the subgroup \hat{L} generated by $\{h\ell h^{-1} \mid \ell \in L, h \in H\}$, i.e., the minimal normal subgroup of H containing L . (In the first three items of Lemma 2.6, an orbit equivalent subgroup is only unique up to conjugation.)

Subgroups of codimension one. Let us first consider the case where $U \subset H$ is a subgroup of codimension one, i.e., $H = U \cdot U(1)$, then we have that either U acts transitively on the H -orbit through p , in which case the U -action and the H -action are orbit equivalent, or U acts with cohomogeneity one on the orbit $H \cdot p$, in which case we arrive at a contradiction since a section Σ through p of the U -action contains the flat section Σ_0 of the H -action, on whose tangent space $T_p\Sigma_0$ the Weyl group of the irreducible slice representation still acts irreducibly when restricted to the Weyl group of the U_p -representation. So Σ would be either flat, contradicting Proposition 2.8, or an irreducible symmetric space of dimension $r + 1$ and rank $r \geq 2$, which does not exist.

Subactions of Hermann actions on exceptional symmetric spaces. Assume the subgroup $U \subset H$ acts polarly on the symmetric space G/K . One can see from Table 3 that the group H has either one or two simple factors if it is semisimple, or it is the product $H = H' \cdot U(1)$ of a one-dimensional abelian and a simple factor. Since U contains the nontrivial normal subgroup \hat{L} of H , it follows that $U = H$ if H is simple; if $H = H' \cdot U(1)$, then U contains H' (since $\dim \hat{L} > 1$) and the U -action is orbit equivalent to the H -action by the argument above, since then $U \subset H$ is a subgroup of codimension one. In those cases where H is a product of two simple factors, comparison of the Tables 4 and 5 shows that in each case, except for $H = \mathrm{SU}(6) \cdot \mathrm{SU}(2)$, $G/K = \mathrm{E}_6/\mathrm{Spin}(10) \cdot \mathrm{U}(1)$, the normal subgroup \hat{L} contains both simple factors of H and it follows that $H = U$. Consider the action of $H = \mathrm{SU}(6) \cdot \mathrm{SU}(2)$ on $G/K = \mathrm{E}_6/\mathrm{Spin}(10) \cdot \mathrm{U}(1)$; in this case it follows

from the data given in Table 5 only that \hat{L} contains the $SU(6)$ -factor of H . An explicit calculation as described in Remark 10.1 shows that the embedding of the connected component of the isotropy group $(H \cap K)_0 = U(1) \cdot SU(4) \cdot SU(2) \cdot SU(2)$ into $H = SU(6) \cdot SU(2)$ is such that the $SU(2)$ -factor in the kernel of the slice representation lies in the $SU(6)$ -factor of H , and the other $SU(2)$ -factor of $(H \cap K)_0$, which acts nontrivially on the slice, coincides with the $SU(2)$ -factor of H . From this it follows that the actions $SU(6)$ or $SU(6) \cdot U(1)$ on G/K have a slice representation with two equivalent nontrivial modules, and are therefore not polar.

Subactions of Hermann actions on classical symmetric spaces.

The cases A I-II, A II-III, C I-II, D I-III, D III-III', and D_4 I-I' can be handled in a similar way as the subactions on the exceptional spaces. One can see from Table 5 that \hat{L} contains every simple factor of H . For the case of D_4 I-I', i.e., subactions of $Spin(5) \cdot Spin(3) \cong Sp(2) \cdot Sp(1)$ on $SO(8)/SO(3) \times SO(5)$, the slice representation was explicitly computed in [33], p. 592-593, and it follows that \hat{L} is not contained in one of the simple factors of H , thus $H = U$.

It remains to study the case A I-III. For the slice representation of this action there are in some cases orbit equivalent polar subgroups as given in the first three items of Lemma 2.6; otherwise the argument is as above. Assume $H = SO(n)$, $G = SU(n)$, $K = S(U(k) \times U(n-k))$, where $(n, k) = (9, 2)$, $(10, 2)$ or $(11, 3)$. Let us first consider the H -action on G/K . The connected component of the isotropy subgroup at eK is $SO(k) \times SO(n-k)$. It follows that U must contain the group given in the right column of the table in Lemma 2.6, and it follows from Lemma 7.4 below that either $U = H$ or $U \subseteq H_p$, but in the latter case the U -action on G/K has a fixed point. Finally, consider the K -action on G/H , i.e., assume a closed connected subgroup $U \subset K = S(U(k) \times U(n-k))$ acts polarly on $SU(n)/SO(n)$; it follows by the arguments above that U contains a subgroup L conjugate to $SO(2) \times G_2$, if $K = S(U(2) \times U(7))$, or $SO(k) \times Spin(7)$, if $K = S(U(k) \times U(8))$; all possibilities for the group U are given by Lemma 7.4. It follows that the slice representation $V|_{U_p} \oplus \chi(K, U)|_{U_p}$ of the U -action on G/H is non-polar by [1] if U does not contain both simple factors of K ; thus the codimension of U in K is at most one and we conclude that the U -action is orbit equivalent to the K -action.

To prove the last part of the theorem, one can easily determine all proper closed subgroups U of H whose action on G/K is orbit equivalent to the H -action on G/K by using the information from Table 5. q.e.d.

For the proof of Theorem 7.3 we used the following simple lemma.

Lemma 7.4. *For the following inclusions of compact Lie groups $A \subset B \subset C$, the intermediate subgroups B are unique in the following sense: If $B' \subset C$ is a closed connected subgroup such that $A \subsetneq B' \subsetneq C$, then*

$B' = B$.

$$\begin{aligned} \mathbb{G}_2 &\subset \mathrm{SO}(7) \subset \mathrm{SU}(7); \\ \mathrm{Spin}(7) &\subset \mathrm{SO}(8) \subset \mathrm{SU}(8); \\ \mathrm{SO}(k) \times \mathbb{G}_2 &\subset \mathrm{SO}(k) \times \mathrm{SO}(7) \subset \mathrm{SO}(7+k), \quad k \in \mathbb{N}; \\ \mathrm{SO}(k) \times \mathrm{Spin}(7) &\subset \mathrm{SO}(k) \times \mathrm{SO}(8) \subset \mathrm{SO}(8+k), \quad k \in \mathbb{N}. \end{aligned}$$

Proof. It is easily checked in each case that the representation $\chi(C, A)$ splits into the irreducible modules $\chi(B, A)$ and $\chi(C, B)|_A$. Note that $\chi(\mathrm{SU}(7), \mathrm{SO}(7))|_{\mathbb{G}_2}$ is equivalent to the irreducible 27-dimensional representation of \mathbb{G}_2 and $\chi(\mathrm{SU}(8), \mathrm{SO}(8))|_{\mathrm{Spin}(7)}$ is equivalent to the irreducible 35-dimensional representation of $\mathrm{Spin}(7)$, cf. Table 1, p. 364 of [16]. q.e.d.

8. Actions of non-simple irreducible groups

In the following, we will assume that G is a simple classical compact Lie group $G = \mathrm{SO}(n)$, $\mathrm{SU}(n)$ or $\mathrm{Sp}(n)$ and K is a symmetric subgroup such that $\mathrm{rk}(G/K) \geq 2$. We will classify all closed connected subgroups $H \subset G$ such that H acts polarly on G/K .

The symmetric quotient spaces of the simple classical compact Lie groups of rank ≥ 2 that are not locally isometric to one of the Grassmannians $\mathbb{G}_k(\mathbb{R}^n)$, $\mathbb{G}_k(\mathbb{C}^n)$, $\mathbb{G}_k(\mathbb{H}^n)$ are the following:

$$(8.1) \quad \begin{aligned} &\mathrm{SO}(2m)/\mathrm{U}(m), \quad m \geq 5; \\ &\mathrm{SU}(m)/\mathrm{SO}(m), \quad m \geq 3, \quad m \neq 4; \\ &\mathrm{SU}(2m)/\mathrm{Sp}(m), \quad m \geq 3; \\ &\mathrm{Sp}(m)/\mathrm{U}(m), \quad m \geq 3. \end{aligned}$$

(Note that $\mathrm{SO}(8)/\mathrm{U}(4)$ is locally isometric to $\mathbb{G}_2(\mathbb{R}^8)$, $\mathrm{SU}(4)/\mathrm{SO}(4)$ is locally isometric to $\mathbb{G}_3(\mathbb{R}^6)$ and $\mathrm{Sp}(2)/\mathrm{U}(2)$ is locally isometric to $\mathbb{G}_3(\mathbb{R}^5)$, cf. [26], Ch. X, §6.4). In the sequel we will refer to these spaces as “*structure spaces*” since they can be interpreted as: a space of complex structures on \mathbb{R}^{2n} , spaces of real or quaternionic structures on \mathbb{C}^n or \mathbb{C}^{2m} , respectively, and a space of complex structures on \mathbb{H}^n .

We will first consider the maximal subgroups $H_1 \subset G$; for classical groups G , they are given by Propositions 3.6, 3.7, and 3.8. Note that for $\mathrm{SO}(n)$ the subgroups (i) and (iii), for $\mathrm{SU}(n)$ the subgroups (i), (ii), (iii) and for $\mathrm{Sp}(n)$ the subgroups (i) and (ii) are symmetric, and thus the actions of these groups are Hermann actions. The remaining types of subgroups are either given by tensor product representations or are simple irreducible subgroups. We will also study certain subactions of cohomogeneity one or transitive Hermann actions

Henceforth we will refer to the following maximal connected subgroups of the classical Lie groups (cf. Section 3) as (*maximal*) *tensor product subgroups*:

$$(8.2) \quad \begin{aligned} H = \mathrm{SO}(p) \otimes \mathrm{SO}(q) &\subset G = \mathrm{SO}(pq), & p \geq 3, q \geq 3; \\ H = \mathrm{SU}(p) \otimes \mathrm{SU}(q) &\subset G = \mathrm{SU}(pq), & p \geq 3, q \geq 2; \\ H = \mathrm{SO}(p) \otimes \mathrm{Sp}(q) &\subset G = \mathrm{Sp}(pq), & p \geq 3, q \geq 1; \\ H = \mathrm{Sp}(p) \otimes \mathrm{Sp}(q) &\subset G = \mathrm{SO}(4pq), & p \geq 2, q \geq 1. \end{aligned}$$

Proposition 8.1 (Tensor product groups on “structure spaces”). *Let G be a simple compact classical Lie group, let H_1 be a maximal tensor product subgroup of G as in (8.2), let K be a structure subgroup as in (8.1), and let $H \subseteq H_1$ be a closed connected subgroup acting nontrivially on G/K . Then the H -action on G/K is not polar.*

Proof. There are a few exceptions remaining not excluded by Proposition 5.6:

$$(8.3) \quad \begin{aligned} H = \mathrm{SU}(3) \otimes \mathrm{SU}(2) &\text{ acting on } \mathrm{SU}(6)/\mathrm{Sp}(3) = G/K; \\ \mathrm{SU}(4) \otimes \mathrm{SU}(2) &\text{ acting on } \mathrm{SU}(8)/\mathrm{Sp}(4); \\ \mathrm{Sp}(3) \otimes \mathrm{Sp}(1) &\text{ acting on } \mathrm{SO}(12)/\mathrm{U}(6). \end{aligned}$$

We will apply Lemma 6.4 to show that none of the actions 8.3 can have a polar subaction. We use the information collected in Table 2 of [35] to determine slice representations. In the case of the $H = \mathrm{SU}(3) \otimes \mathrm{SU}(2)$ -action on $G/K = \mathrm{SU}(6)/\mathrm{Sp}(3)$, a slice representation is $\mathrm{Ad}_{\mathrm{SO}(3)} \otimes \mathrm{Ad}_{\mathrm{SU}(2)}$, which is polar. However, an explicit calculation shows that the normal space to a regular orbit is not a Lie triple system; thus by Proposition 4.1, the action is not polar. Now let $U \subset H$ be closed connected proper subgroup acting polarly on G/K . By Theorem 6.1 and Lemma 2.6, the above slice representation restricted to $(U \cap K)_0$ is either trivial, leading to a contradiction by Proposition 5.6 since the slice is 9-dimensional, or equivalent to the action of $H \cap K$, in which case U must contain $\mathrm{SO}(3) \cdot \mathrm{SU}(2)$. But since $\mathrm{SO}(3) \subset \mathrm{SU}(3)$ is maximal connected we have that the U -action on G/K has a fixed point and is non-polar by Corollary 6.2.

Let us consider the $\mathrm{SU}(4) \otimes \mathrm{SU}(2)$ -action on $\mathrm{SU}(8)/\mathrm{Sp}(4)$. A slice representation is $\mathrm{Ad}_{\mathrm{SO}(4)} \otimes \mathrm{Ad}_{\mathrm{SU}(2)}$, which is non-polar [1] and polarity minimal, hence we may apply Lemma 6.4 (ii). For the $\mathrm{Sp}(3) \otimes \mathrm{Sp}(1)$ -action we find a slice representation $\mathrm{P}_2(\mathrm{Sp}(3)) \otimes \mathbb{R}^2$ of $\mathrm{Sp}(3) \otimes \mathrm{U}(1)$, which is also non-polar [13] and polarity minimal; hence Lemma 6.4 (ii) also applies in this case. q.e.d.

Proposition 8.2. *Let $H_1 \subset G$ and G/K be as in Table 6 and let $H \subseteq H_1$ be a closed connected subgroup acting nontrivially on G/K . Then the H -action on G/K is not polar.*

Proof. None of the subgroups $H_1 \subset G$ fulfills the lower bound on its dimension given in Proposition 5.6, except the action of $\mathrm{S}(\mathrm{U}(2) \otimes \mathrm{U}(3)) \times \mathrm{U}(2)$ on $\mathrm{SU}(8)/\mathrm{Sp}(4)$. However, an explicit calculation shows

Type	H_1	G/K	Range
A III-I	$S((U(p)\otimes U(q))\times U(1))$	$\frac{SU(pq+1)}{SO(pq+1)}$	$p\geq 3, q\geq 2$
A III-II	$S((U(2r+1)\otimes U(2s+1))\times U(1))$	$\frac{SU(4rs+2r+2s+2)}{Sp(2rs+r+s+1)}$	$r, s\geq 1$
A III-II	$S((U(2p)\otimes U(q))\times U(2))$	$\frac{SU(2pq+2)}{Sp(pq+1)}$	$p\geq 1, q\geq 3$
A III-II	$S((U(2r+1)\otimes U(2s+1))\times U(3))$	$\frac{SU(4rs+2r+2s+4)}{Sp(2rs+r+s+2)}$	$r, s\geq 1$
C II-I	$(SO(p)\otimes Sp(q))\times Sp(1)$	$\frac{Sp(pq+1)}{U(pq+1)}$	$p\geq 3, q\geq 1$
D I-III	$SO(2r+1)\otimes SO(2s+1)$	$\frac{SO(4rs+2r+2s+2)}{U(2rs+r+s+1)}$	$r, s\geq 1$
D I-III	$(SO(2p)\otimes SO(q))\times SO(2)$	$\frac{SO(2pq+2)}{U(pq+1)}$	$p\geq 2, q\geq 3$
D I-III	$(SO(2r+1)\otimes SO(2s+1))\times SO(3)$	$\frac{SO(4rs+2r+2s+4)}{U(2sr+r+s+2)}$	$r, s\geq 1$

Table 6. Certain subactions of cohomogeneity one or transitive Hermann actions on “structure spaces”.

that the cohomogeneity of this action is 12, hence this action is non-polar and polarity minimal by Lemma 3.3. q.e.d.

Proposition 8.3 (Tensor product subgroups on Grassmannians). *Let G, H, K be as in (8.4). Assume $n = 2, \dots, \lfloor \frac{pq}{2} \rfloor$ in cases (a), (b), (c). In case (d), assume that $n = 2, \dots, 2pq$ and $n \geq 3$ if $q = 1$. Assume further that in case (d) $pq \neq 2$. Then the action of H on G/K is non-polar and polarity minimal, i.e., any nontrivial action of a closed connected subgroup $U \subseteq H$ on G/K is non-polar.*

(8.4)

	$H =$	$G/K =$	Range
(a)	$SO(p)\otimes SO(q),$	$SO(pq)/SO(n)\times SO(pq-n),$	$p\geq 3, q\geq 3;$
(b)	$SU(p)\otimes SU(q),$	$SU(pq)/S(U(n)\times U(pq-n)),$	$p\geq 3, q\geq 2;$
(c)	$SO(p)\otimes Sp(q),$	$Sp(pq)/Sp(n)\times Sp(pq-n),$	$p\geq 3, q\geq 1;$
(d)	$Sp(p)\otimes Sp(q),$	$SO(4pq)/SO(n)\times SO(4pq-n),$	$p\geq 2, q\geq 1.$

Proof. We use the slice representations which were explicitly determined in Section 2.3 of [33]. In each case, one finds a non-polar, polarity minimal, and almost effective submodule of the slice representation; by Lemma 6.3 (iii) this implies that the slice representation is polarity minimal. But then Lemma 6.4 (iii) shows that the U -action is non-polar, since the normal space $N_{eK}(H \cdot eK)$ contains a Lie triple system corresponding to an irreducible symmetric space of non-constant curvature in each case, as can be seen from the explicit description of the normal spaces in [33]. q.e.d.

Proposition 8.4. *Let $H_1 \subset G$ and G/K be as in Table 7 and let $H \subseteq H_1$ be a closed connected subgroup acting nontrivially on G/K . Then the H -action on G/K is not polar.*

Type	H_1	G/K	Range
A III-III	$S((U(p) \otimes U(q)) \times U(1))$	$\mathbb{G}_k(\mathbb{C}^{pq+1})$	$p \geq 3, q \geq 2, 2 \leq k \leq \lfloor \frac{pq+1}{2} \rfloor$
BD I-I	$SO(p) \otimes SO(q)$	$\mathbb{G}_k(\mathbb{R}^{pq+1})$	$p \geq 3, q \geq 3, 3 \leq k \leq \lfloor \frac{pq+1}{2} \rfloor$
BD I-I	$Sp(p) \otimes Sp(q)$	$\mathbb{G}_k(\mathbb{R}^{4pq+1})$	$p \geq 2, q \geq 1, 3 \leq k \leq \lfloor \frac{4pq+1}{2} \rfloor$
C II-II	$(SO(p) \otimes Sp(q)) \times Sp(1)$	$\mathbb{G}_k(\mathbb{H}^{pq+1})$	$p \geq 3, q \geq 1, 2 \leq k \leq \lfloor \frac{pq+1}{2} \rfloor$
A III-III	$SO(p) \otimes Sp(q)$	$\mathbb{G}_\ell(\mathbb{C}^{2pq})$	$p \geq 3, q \geq 1, \ell = 2, 3$
BD I-I	$U(p) \otimes U(q)$	$\mathbb{G}_3(\mathbb{R}^{2pq})$	$p \geq 3, q \geq 2$

Table 7. Certain subactions of cohomogeneity one or transitive Hermann actions on Grassmannians.

Proof. Consider the first four items in Table 7. By Lemma 11.2, the corresponding Hermann action (indicated in the first column) has a totally geodesic orbit isometric to $\mathbb{G}_k(\mathbb{C}^{pq})$, $\mathbb{G}_k(\mathbb{R}^{pq})$, $\mathbb{G}_k(\mathbb{R}^{4pq})$, and $\mathbb{G}_k(\mathbb{H}^{pq})$, respectively, on which H acts. It follows from Proposition 8.3 and Lemma 4.2 that H acts non-polarly except if $p = 3, q = 1, k = 2$ in case of the fourth action; however, in this case the normal space of a principal orbits is not a Lie triple system and H_1 acts non-polarly by Proposition 4.1; closed proper subgroups of H_1 are excluded by Proposition 5.6 (iii).

Let us now consider the last two items of Table 7. Assume H acts polarly and nontrivially on G/K . Then the H -action has a singular orbit by Corollary 5.3. As can be seen from Table 10, the almost effective slice representation of the H_1 -action on G/K occurs also as a submodule of the isotropy representation $\chi(H_1, H_1 \cap K)$ of the H_1 -orbit $H_1 \cdot eK$. Thus the H -action on G/K is non-polar by Proposition 6.3 (ii). q.e.d.

9. Subactions of simple irreducible groups

We will now study simple irreducible maximal subgroups of the classical groups acting on the classical symmetric spaces. We start with actions on the spaces (8.1).

Proposition 9.1 (Simple irreducible groups on “structure spaces”). *Let G be a simple compact classical Lie group $SO(n)$, $SU(n)$ or $Sp(n)$ and let $\rho: H \rightarrow G$ be an irreducible representation of corresponding (real, complex or quaternionic) type where H is a simple compact Lie group and such that $\rho(H)$ is a maximal connected subgroup of G . Let*

$K \subset G$ be a subgroup as in (8.1) such that $\text{rk}(G/K) \geq 2$. Then the action of any closed subgroup of $\rho(H)$ on G/K is non-polar except for the Hermann actions of subgroups conjugate to $\text{Spin}(7) \subset \text{SO}(8)$ on $\text{SO}(8)/\text{U}(4)$.

Proof. For the spaces $\text{SU}(n)/\text{SO}(n)$ and $\text{Sp}(n)/\text{U}(n)$ this follows directly from Lemmata 2.7, 2.8 of [33] and Proposition 5.6.

Let us consider the spaces $\text{SO}(n)/\text{U}(\frac{n}{2})$, $n \geq 8$. By Proposition 5.6 we have that $\dim(H) \geq \frac{n^2}{4} - n$ if H acts polarly on $\text{SO}(n)/\text{U}(\frac{n}{2})$. Hence ρ is a representation as described in Lemma 2.6 (i), (iv) of [33] and all possibilities for ρ are given in the table of [33], Lemma 2.8 (i). However, all of these subgroups $\rho(H) \subset \text{SO}(n)$ are excluded by the dimension bounds given in Proposition 5.6, except $\text{Spin}(7) \subset \text{SO}(8)$.

For the spaces $\text{SU}(n)/\text{Sp}(\frac{n}{2})$, $n \geq 3$, it follows from Proposition 5.6 that $\dim(H) \geq \frac{n^2}{2} - 2n$ for a group H acting polarly on $\text{SU}(n)/\text{Sp}(\frac{n}{2})$. Thus ρ is a representation as in Lemma 2.6 (ii), (iv) of [33] and all such representations ρ are given by the table in [33], Lemma 2.8 (ii). However, none of the simple groups there fulfills the necessary condition on its dimension given by Proposition 5.6. q.e.d.

We also need to consider certain subactions of cohomogeneity one or transitive actions.

Lemma 9.2.

- (i) Let $k \in \{1, 2, 3\}$. Let H be a simple compact connected Lie group and let $\rho: H \rightarrow \text{SO}(2n - k)$ be an irreducible representation of real type such that $\rho(H) \subset \text{SO}(2n - k)$ is maximal connected. Then any closed subgroup of $\rho(H) \times \text{SO}(k)$ acts non-polarly on $\text{SO}(2n)/\text{U}(n)$, except if ρ is equivalent to the 7-dimensional irreducible representation of G_2 and $k = 1$.
- (ii) Let $k \in \{1, 2, 3\}$. Let H be a simple compact connected Lie group and let $\rho: H \rightarrow \text{SU}(2n - k)$ be an irreducible representation of complex type such that $\rho(H) \subset \text{SU}(2n - k)$ is maximal connected. Then any closed subgroup of $\text{S}((\rho(H) \otimes \text{U}(1)) \times \text{U}(k))$ acts non-polarly on $\text{SU}(2n)/\text{Sp}(n)$.
- (iii) Let H be a simple compact connected Lie group and let $\rho: H \rightarrow \text{SU}(n)$ be an irreducible representation of complex type such that $\rho(H) \subset \text{SU}(n)$ is maximal connected. Then any closed subgroup of $\text{S}((\rho(H) \otimes \text{U}(1)) \times \text{U}(1))$ acts non-polarly on $\text{SU}(n)/\text{SO}(n)$.
- (iv) Let H be a simple compact connected Lie group and let $\rho: H \rightarrow \text{Sp}(n)$ be an irreducible representation of real type such that $\rho(H) \subset \text{Sp}(n)$ is maximal connected. Then any closed subgroup of $\rho(H) \times \text{Sp}(1)$ acts non-polarly on $\text{Sp}(n)/\text{U}(n)$.

Proof. The proof is almost literally the same as the proof of Proposition 9.1. For the cases (i) and (ii), we may use the tables in parts (i)

and (ii) of Lemma 2.8 in [33]. The only representations not excluded by this argument are the 8-dimensional spin representation of $\text{Spin}(7)$ and the 7-dimensional representation of G_2 . However, in case of the actions of $\text{Spin}(7) \times \text{SO}(2)$ and $G_2 \times \text{SO}(3)$ on $\text{SO}(10)/\text{U}(5)$, the normal space at a principal orbit is not a Lie triple system, and hence these actions are non-polar by Proposition 4.1. Closed connected subgroups of these groups can be shown to act non-polarly by the same argument or are excluded by Proposition 5.6. The action of G_2 on $\text{SO}(8)/\text{U}(4)$ is orbit equivalent to the action of $\text{Spin}(7)$ on $\text{SO}(8)/\text{U}(4)$. The statements (iii) and (iv) follow directly from Lemmata 2.7 and 2.8 of [33]. q.e.d.

We will now consider the maximal simple irreducible subgroups of the classical groups $\text{SO}(n)$, $\text{SU}(n)$, $\text{Sp}(n)$, given by irreducible representations of the real, complex, or quaternionic type, respectively, and their actions on the corresponding Grassmannians $\mathbb{G}_k(\mathbb{K}^n)$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} , of higher rank. A necessary condition for polarity on the dimension of these subgroups is given by Propositions 5.6 and 5.5. The irreducible representations of simple compact Lie groups whose degrees are sufficiently low can be obtained from Lemma 2.6 of [33], see also the tables in Lemma 2.8 and the Appendix of [33]. These representations are given by Table 8; the column marked with k_{\max} indicates the maximal rank $k \leq \lfloor \frac{n}{2} \rfloor$ for which the necessary condition for polarity of an action on $\mathbb{G}_k(\mathbb{K}^n)$ given by Proposition 5.6 is fulfilled. We only list such representations of complex or quaternionic type where $k_{\max} \geq 2$ and representations of real type where $k_{\max} \geq 3$, since polar actions on $\mathbb{G}_2(\mathbb{R}^n)$ have been classified in [40]. It turns out that there are no such representations of complex or quaternionic type.

Group	Highest Weight	Degree	Type	Description	k_{\max}
A_2	(1, 1)	8	real	adjoint	4
B_3	(0, 0, 1)	8	real	$\text{Spin}(7)$	4
B_4	(0, 0, 0, 1)	16	real	F II	4
C_3	(0, 1, 0)	14	real	A II	3
F_4	(1, 0, 0, 0)	26	real	E IV	3
G_2	(1, 0)	7	real	$\text{Aut}(\mathbb{O})$	3

Table 8. Representations of low degree.

Subactions of $\text{Ad}(\text{SU}(3))$ on $\mathbb{G}_3(\mathbb{R}^8)$ and $\mathbb{G}_4(\mathbb{R}^8)$. The group $H = \text{SU}(3)$ acts on its Lie algebra \mathfrak{h} by the adjoint representation $\text{Ad}: H \rightarrow \text{SO}(\mathfrak{h})$ and we obtain a subgroup $\text{Ad}(\text{SU}(3)) \subset \text{SO}(8)$ by identifying \mathfrak{h} with \mathbb{R}^8 . We will study the actions of this group on the Grassmannians

$\mathbb{G}_k(\mathbb{R}^8)$, $k = 3, 4$. Any closed connected proper subgroups of $SU(3)$ are of dimension ≤ 4 by Table 3 of [33] and are thus excluded by Proposition 5.6.

The maximal connected subgroup $SO(3) \subset SU(3)$ leaves a 3-dimensional subspace of \mathbb{R}^8 invariant, thus it is the connected component of an isotropy subgroup of the $\text{Ad}(SU(3))$ -action on $\mathbb{G}_3(\mathbb{R}^8)$. The slice representation contains the irreducible 7-dimensional representation of $SO(3)$ and is hence non-polar [13].

Consider now the subgroup $S(U(1) \times U(2)) \subset SU(3)$. The action of H on $\mathfrak{h} = \mathbb{R}^8$ restricted to $S(U(1) \times U(2))$ leaves the 4-dimensional linear subspace corresponding to $\mathfrak{s}(\mathfrak{u}(1) + \mathfrak{u}(2)) \subset \mathfrak{h}$ invariant. Thus, the maximal connected $S(U(1) \times U(2))$ coincides with the connected component of the stabilizer of the H -action on $\mathbb{G}_4(\mathbb{R}^8)$. Its slice representation contains an 8-dimensional irreducible representation of $S(U(1) \times U(2))$ which is non polar [13] and the H -action on $\mathbb{G}_4(\mathbb{R}^8)$ is polarity minimal by Lemma 6.4 (ii).

Subactions of $\text{Spin}(7)$ on $\mathbb{G}_3(\mathbb{R}^8)$ and $\mathbb{G}_4(\mathbb{R}^8)$. The subgroup $\text{Spin}(7)$ of $SO(8)$ gives rise to a Hermann action since its Lie algebra is the fixed point set of an involution of $\mathfrak{so}(8)$.

Subactions of $\text{Spin}(9)$ on $\mathbb{G}_3(\mathbb{R}^{16})$. The action on $\mathbb{G}_3(\mathbb{R}^{16})$ was shown not to be polar in [33], the slice representation being equivalent to a 16-dimensional non-polar irreducible representation of $\text{Sp}(1) \cdot \text{Sp}(2)$. Thus by Lemma 6.4 (ii), no subaction of the $\text{Spin}(9)$ -action on $\mathbb{G}_3(\mathbb{R}^{16})$ is polar.

Subactions of $\text{Spin}(9)$ on $\mathbb{G}_4(\mathbb{R}^{16})$. Consider the subgroup $H_0 = \text{Spin}(4) \cdot \text{Spin}(4) \subset \text{Spin}(8) \subset \text{Spin}(9)$. Its action on \mathbb{R}^{16} leaves a four-dimensional subspace V invariant. Since $H_0 \subset \text{Spin}(9)$ is a subgroup of maximal rank, it is easy to check that no other connected subgroup of $\text{Spin}(9)$ containing H_0 leaves V invariant, and thus H_0 is the connected component of an isotropy subgroup of the $\text{Spin}(9)$ -action on \mathbb{R}^{16} . The slice representation is equivalent to the sum of two 12-dimensional irreducible modules and is easily seen to be non-polar and polarity minimal.

Subactions of $\text{Sp}(3)$ on $\mathbb{G}_3(\mathbb{R}^{14})$. An isotropy subgroup of the $\text{Sp}(3)$ -action on $\mathbb{G}_3(\mathbb{R}^{14})$ is

$$SO(3) \cdot U(1) \subset U(3) \subset \text{Sp}(3).$$

Its 16-dimensional slice representation does not contain any trivial submodule and is therefore non-polar by [1]. Any proper subgroups of $\text{Sp}(3)$ can be excluded by Proposition 5.6.

Subactions of F_4 on $\mathbb{G}_3(\mathbb{R}^{26})$. The maximal connected subgroups of maximal rank in $H_1 = F_4$ are, see [37], Chapter 1, § 3.11.

$$(9.1) \quad \text{Sp}(3) \cdot \text{Sp}(1), \quad \text{SU}(3) \cdot \text{SU}(3), \quad \text{Spin}(9)$$

We will determine a slice representation for the H_1 -action on $\mathbb{G}_3(\mathbb{R}^{26})$. According to [15], Table 25, p. 199, the subgroup $\text{Spin}(9)$ acts on \mathbb{R}^{26} by the direct sum of the 9-dimensional standard representation, the 16-dimensional spin representation, and a one-dimensional trivial representation. Thus, if we further restrict this representation to the maximal connected subgroup $\text{Spin}(7) \cdot \text{SO}(2)$ of $\text{Spin}(9)$, a three-dimensional subspace W is left invariant and it follows that $\text{Spin}(7) \cdot \text{SO}(2)$ is contained in an isotropy subgroup $(H_1)_W$ of the H_1 -action on $\mathbb{G}_3(\mathbb{R}^{26})$. The subgroup $\text{Spin}(7) \cdot \text{SO}(2) \subset F_4$ is of maximal rank and it can be deduced from Table 25 of [15] that none of the groups in (9.1) leaves a three-dimensional subspace of \mathbb{R}^{26} invariant. Hence $\text{Spin}(7) \cdot \text{SO}(2)$ is the connected component of the isotropy subgroup $(H_1)_W$ and the slice representation is, by a dimension count, equivalent to $\mathbb{R}^7 \oplus 2 \cdot \mathbb{R}^2 \otimes \mathbb{R}^8$, where $\text{Spin}(7)$ acts on \mathbb{R}^8 by the spin representation, and therefore the action of F_4 on $\mathbb{G}_3(\mathbb{R}^{26})$ is non-polar and polarity minimal by Proposition 6.3 (ii) and Lemma 6.4 (ii).

Subactions of G_2 on $\mathbb{G}_3(\mathbb{R}^7)$. This is a cohomogeneity one action and its subactions will be treated in Section 12.

10. Polar actions on the exceptional spaces

In this section we will study those isometric actions on the exceptional symmetric spaces of compact type which are subactions neither of Hermann actions nor of cohomogeneity one actions. It will turn out that none of these actions is polar.

The maximal connected subgroups of the simple compact Lie groups were determined in [15], Tables 12 and 12a, p. 150–151, and Theorem 14.1, p. 231. By Theorem 5.4 and Lemma 3.3, the cohomogeneity of a polar action on a symmetric quotient G/K of a simple Lie group G is at most $\text{rk}(G) + \text{rk}(K)$. By Proposition 5.5, this estimate can be further improved for Hermitian symmetric spaces, for which the cohomogeneity is at most $\text{rk}(G)$. From this it follows by using the classification of symmetric spaces, see Table 4, that a group acting polarly on a symmetric quotient G/K with $\text{rk}(G/K) \geq 2$ of one of the simple exceptional Lie groups $G = E_6, E_7, E_8, F_4, G_2$ is at least of dimension 16, 47, 96, 20, 4, respectively. (We do need not consider the Cayley plane $F_4/\text{Spin}(9)$, since it is of rank one.) First we would like to recall a method to describe certain subgroups of a (semi)simple compact Lie group in terms of the root system, which is particularly useful for our purposes, see [22], § 8.3 and [37], Ch. 1, § 3.11.

Remark 10.1 (Borel-De Siebenthal theory). Let G be a connected compact simple Lie group. A subgroup $H \subset G$ is called a *subgroup of maximal rank* if $\text{rk}(H) = \text{rk}(G)$, i.e., H contains a maximal torus T of G . Consider the root space decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_0 + \sum_{\alpha \in R} \mathfrak{g}_{\alpha}$, where \mathfrak{g}_0

is the complexification of the maximal abelian subalgebra of \mathfrak{g} tangent to T . Since the Lie algebra $\mathfrak{h}_{\mathbb{C}}$ contains \mathfrak{g}_0 , it is a \mathfrak{g}_0 -stable subspace of $\mathfrak{g}_{\mathbb{C}}$, and it follows that $\mathfrak{h}_{\mathbb{C}} = \mathfrak{g}_0 + \sum_{\alpha \in S} \mathfrak{g}_{\alpha}$ where $S \subset R$ is a subset of the root system. Conversely, from suitable subsets $S \subset R$, one may construct the Lie algebra of a subgroup $H \subset G$ of maximal rank, see [37], Chapter 1, § 3.11. In particular, one can obtain all maximal connected subgroups of maximal rank by such a construction. These are obtained by deleting certain vertices from the extended Dynkin diagram, see [37] for details. The classification of all such subgroups up to conjugation by automorphisms of G is given in Table 5, p. 64 of [37] or in Table 12, p. 150 of [15], see also [22].

Now assume H and K are both subgroups of maximal rank in G . Then we can use the above description to obtain information about the H -action on the homogeneous space G/K , and in particular, to compute an isotropy algebra together with its slice representation. In fact, we may assume by conjugation of K with a suitable element from G that both H and K contain a maximal torus T of G . Then $\mathfrak{h}_{\mathbb{C}} = \mathfrak{g}_0 + \sum_{\alpha \in S} \mathfrak{g}_{\alpha}$ and $\mathfrak{k}_{\mathbb{C}} = \mathfrak{g}_0 + \sum_{\alpha \in S'} \mathfrak{g}_{\alpha}$ for some subsets S, S' of the root system R . In particular, the complexified isotropy algebra $(\mathfrak{h} \cap \mathfrak{k})_{\mathbb{C}}$ of the H -action on G/K at eK is spanned by the Cartan algebra \mathfrak{g}_0 and the root spaces corresponding to the roots in the intersection $S \cap S'$; hence it follows that $H \cap K$ is also a subgroup of maximal rank in G . On the other hand, the complexified normal space $(\mathfrak{h}^{\perp} \cap \mathfrak{k}^{\perp})_{\mathbb{C}}$ of the H -orbit through eK is spanned by the root spaces corresponding to the roots in $R \setminus (S \cup S')$. Since T is also a maximal torus of $H \cap K$, the roots in $R \setminus (S \cup S')$ are exactly the weights of the slice representation of $H \cap K$ on the normal space $\mathfrak{h}^{\perp} \cap \mathfrak{k}^{\perp}$. It follows [32] that the H -orbit through eK is a singular orbit, since T acts nontrivially on $\mathfrak{h}^{\perp} \cap \mathfrak{k}^{\perp}$, in fact, the slice representation does not have any trivial submodules, since the complexified normal space is spanned by root spaces corresponding to non-zero roots. In the special case $H = K$, one obtains the isotropy representation $\chi(G, K)$ by this method.

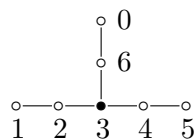
Note that if a subgroup $H \subseteq G$ is a fixed point set of an inner automorphism σ of G , i.e., $\sigma(x) = gxg^{-1}$, it is a subgroup of maximal rank, since the element $g = \exp(X)$, $X \in \mathfrak{g}$, lies in a maximal torus T of G and it follows that $\mathfrak{h}_{\mathbb{C}} = \mathfrak{g}_0 + \sum_{\{\alpha | X \in \ker \alpha\}} \mathfrak{g}_{\alpha}$ where \mathfrak{g}_0 is the complexified Lie algebra of T . (Conversely, if a subgroup of maximal rank is the fixed point set of an automorphism, then the automorphism is inner.)

Let H and K be two subgroups of maximal rank with common maximal torus T as above. If both groups are fixed point sets of involutions i.e., $H = G^{\sigma}$, $K = G^{\tau}$, then it follows that the involutions σ and τ commute, since they both act as either plus or minus identity on the root spaces of \mathfrak{g} . This shows that if σ and τ are two inner involutions

of a simple compact Lie group G , then τ is conjugate to an involution which commutes with σ , cf. [10].

10.1. Symmetric spaces of E_6 . The maximal connected non-symmetric subgroups of E_6 of dimension ≥ 16 are $SU(3) \cdot SU(3) \cdot SU(3)$ and $G_2^1 \cdot A_2^{2''}$, see [15]. (The upper indices denote the Dynkin index of subgroups and the primes are used to distinguish non-conjugate subgroups of the same Dynkin index). By a dimension count, no closed subgroup of these groups acts polarly on the spaces $E_6/(\mathrm{Sp}(4)/\{\pm 1\})$, $E_6/SU(6) \cdot \mathrm{Sp}(1)$ or $E_6/\mathrm{Spin}(10) \cdot U(1)$.

It remains to determine the polar actions on E_6/F_4 . We start with the group $H_1 = SU(3) \cdot SU(3) \cdot SU(3)$. The subgroup $SU(3) \cdot SU(3) \cdot SU(3)$ is constructed from the extended Dynkin diagram of E_6 as follows, cf. Remark 10.1.



The vertices numbered $1, \dots, 6$ correspond to the simple roots $\alpha_1, \dots, \alpha_6$ of E_6 and the vertex with number 0 represents α_0 , where $-\alpha_0$ is the maximal root. Now the group $H_1 = SU(3) \cdot SU(3) \cdot SU(3)$ arises from the extended Dynkin diagram if one deletes the central vertex 3, i.e., it is the regular subgroup whose simple roots are $\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_0$. The subgroup $F_4 \subset E_6$ is the fixed point set of the diagram automorphism σ of E_6 which maps $\alpha_1 \mapsto \alpha_5, \alpha_2 \mapsto \alpha_4, \alpha_4 \mapsto \alpha_2, \alpha_5 \mapsto \alpha_1$ and leaves α_3 and α_6 fixed. This automorphism σ also leaves α_0 fixed, since $-\alpha_0 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$, see [37], Chapter 1, § 3.11. It follows that σ also acts on H_1 , i.e., trivially on one $SU(3)$ -factor (the one whose simple roots are α_6, α_0) and by interchanging the other two $SU(3)$ -factors. Thus, $H_1 \cap F_4$ is the fixed point set H_1^σ and is hence isomorphic to $SU(3) \cdot \Delta SU(3)$, where the $\Delta SU(3)$ -factor is diagonally embedded into two of the $SU(3)$ -factors of H_1 . Let us determine the slice representation of the H_1 -action on M . It is a submodule of $\chi(E_6, H_1)$ restricted to $H_1 \cap F_4$. The real 54-dimensional isotropy representation $\chi(E_6, H_1)$ is, after complexification,

$$(10.1) \quad \left(\overset{1}{\circ} \text{---} \overset{1}{\circ} \otimes \overset{1}{\circ} \text{---} \overset{1}{\circ} \otimes \overset{1}{\circ} \text{---} \overset{1}{\circ} \right) \oplus \left(\overset{1}{\circ} \text{---} \overset{1}{\circ} \otimes \overset{1}{\circ} \text{---} \overset{1}{\circ} \otimes \overset{1}{\circ} \text{---} \overset{1}{\circ} \right),$$

see [47], Corollary 13.2; i.e., the isotropy representation is equivalent to the action of $SU(3) \cdot SU(3) \cdot SU(3)$ on $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ by the tensor product of the standard representations. If we restrict this representation to the subgroup $SU(3) \cdot \Delta SU(3)$, it splits into the irreducible modules $(\mathbb{C}^3 \otimes \mathrm{Sym}^2 \mathbb{C}^3)$ and $(\mathbb{C}^3 \otimes \Lambda^2 \mathbb{C}^3)$, where the first $SU(3)$ -factor acts on \mathbb{C}^3 and

the $\Delta\text{SU}(3)$ -factor acts on $\text{Sym}^2\mathbb{C}^3$ or $\Lambda^2\mathbb{C}^3$, respectively. A dimension count shows that the real 36-dimensional slice representation of the H_1 -action is equivalent to the first irreducible summand, hence it is non-polar [13] and polarity minimal. We conclude that the H_1 -action on G/K is non-polar and polarity minimal by Lemma 6.4 (ii).

Now consider subactions of $H_1 = G_2^1 \cdot A_2^{2''}$ on $G/K = E_6/F_4$. We determine a slice representation of the H_1 -action on G/K . First observe that F_4 contains the subgroup G_2^1 , according to Table 39 of [15], p. 233. Since the subgroup $F_4 \subset E_6$ has Dynkin index 1, it follows that $G_2^1 \subset E_6$ also has Dynkin index 1, see [15], Ch. I, § 2. By Table 25 of [15], p. 200, there is only one conjugacy class of subgroups isomorphic to G_2 of Dynkin index 1 in E_6 , and it follows that an isotropy subgroup $(H_1)_x$ of the H_1 -action on G/K contains G_2^1 . The homogeneous space G/H_1 is strongly isotropy irreducible, see [47], Theorem 3.1, p. 66; and its isotropy representation decomposes into 8 equivalent 7-dimensional irreducible modules when restricted to G_2 . Thus the dimension of the normal space $N_x(H_1 \cdot x)$ to the H_1 -orbit through x is a multiple of 7. The only possibility is a 21-dimensional slice representation which splits into 3 irreducible 7-dimensional modules when restricted to G_2^1 . By [13] and [1], such a representation is non-polar and it is polarity minimal by Lemma 6.3, part (i) or (ii), and hence we can apply Lemma 6.4 to show that no closed subgroup $H \subseteq H_1$ acts polarly on G/K .

10.2. Symmetric spaces of E_7 . The only maximal connected non-symmetric subgroup of dimension ≥ 47 is $H_1 = F_4^1 \cdot A_1^{3''}$, see [15]. By Lemma 3.3 and Proposition 5.5, respectively, no closed subgroup of H_1 acts polarly on $E_7/(\text{SU}(8)/\{\pm 1\})$ or on $E_7/E_6 \cdot U(1)$.

Let us determine an isotropy subgroup of the H_1 -action on $G/K = E_7/\text{SO}'(12) \cdot \text{Sp}(1)$. First observe that \mathfrak{h}_1 contains a subalgebra $\mathfrak{spin}(9) \subset \mathfrak{f}_4 \subset \mathfrak{h}_1$. By Table 25 of [15], p. 201, there is only one conjugacy class of subalgebras isomorphic to $\mathfrak{spin}(9)$ in \mathfrak{e}_7 and it follows that, after conjugation, this subalgebra coincides with the subalgebra $\mathfrak{spin}(9) \subset \mathfrak{spin}(12) \subset \mathfrak{k}$. Thus there is an isotropy subgroup $(H_1)_x$ of the H_1 -action on G/K whose Lie algebra contains $\mathfrak{spin}(9)$ as a subalgebra. The 64-dimensional isotropy representation of $E_7/\text{SO}'(12) \cdot \text{Sp}(1)$ decomposes into 4 copies of the 16-dimensional spin representation when restricted to $\text{Spin}(9)$. Thus the dimension of an orbit $H_1 \cdot x$ is a multiple of 16 and it follows by a dimension count that the Lie algebra of $(H_1)_x$ is isomorphic to $\mathfrak{spin}(9) \oplus \mathfrak{a}_1$. The 32-dimensional slice representation at the point x is the sum of two modules equivalent to the 16-dimensional spin representation of $\text{Spin}(9)$, hence it is non-polar and polarity minimal and thus by Lemma 6.4, the H_1 -action is non-polar and polarity minimal.

Now consider the action of $H_1 = F_4^1 \cdot A_1^{3''}$ on $G/K = E_7/E_6 \cdot U(1)$. By Table 25 of [15], p. 201, there is only one conjugacy class of subgroups isomorphic to F_4 in E_7 and it follows that an isotropy subgroup $(H_1)_x$

of the H_1 -action on G/K contains F_4 . The space G/H_1 is strongly isotropy irreducible, see [47], Theorem 3.1, and its isotropy representation decomposes into 3 equivalent 26-dimensional irreducible modules when restricted to F_4 . This shows that the dimension of the normal space $N_x(H_1 \cdot x)$ is a multiple of 26, and it follows by a dimension count that the connected component of the isotropy subgroup $(H_1)_x$ is isomorphic to $F_4 \cdot U(1)$. Thus the slice representation is non-polar and polarity minimal, and we can apply Lemma 6.4 (i) to show that no closed subgroup $H \subseteq H_1$ acts polarly on G/K .

10.3. Symmetric spaces of E_8 . The maximal connected subgroups of E_8 whose dimension is at least 96 are symmetric. Hence any polar actions on the symmetric spaces $E_8/SO'(16)$ and $E_8/E_7 \cdot Sp(1)$ are subactions of Hermann actions.

10.4. Actions on $F_4/Sp(3) \cdot Sp(1)$. All maximal connected subgroups of F_4 whose dimension is at least 20 are symmetric. Thus any polar action on the space $F_4/Sp(3) \cdot Sp(1)$ is a subaction of a Hermann action.

10.5. Actions on $G_2/SO(4)$. The maximal connected subgroups of G_2 are

$$(10.2) \quad SO(4), \quad SU(3), \quad A_1^{28},$$

where A_1^{28} is a maximal connected subgroup in G_2 of type A_1 , cf. [15]. If the group H acting on $M = G_2/SO(4)$ is contained in $SO(4)$, then the action has a fixed point. The group $SU(3)$ acts with cohomogeneity one on M . The only closed connected subgroup $H \subset SU(3)$ of dimension ≥ 4 is $S(U(1) \times U(2)) \cong U(2)$, whose action on M has a fixed point. Subgroups of rank one are ruled out by a dimension count.

11. Subactions of hyperpolar actions

To complete the classification, it remains to study subactions of cohomogeneity one and transitive actions. We will need the following lemmata to study subactions of Hermann actions. The first lemma shows that the slice representations of a Hermann action are s-representations.

Lemma 11.1 (Slice representations of Hermann actions). *Let G be a connected simple compact Lie group and let σ, τ be two involutive automorphisms of G . Let $K = G_0^\sigma$ be the connected component of the fixed point set of σ and let $H_1 = G_0^\tau$ be the connected component of the fixed point set of τ . Consider the H_1 -action on G/K . Then the exponential image $S = \exp_{eK}(N_{eK}(H_1 \cdot eK))$ of the normal space to the orbit through eK is a totally geodesic submanifold locally isometric to a symmetric space $G^{\sigma\sigma\tau}/G^\sigma \cap G^\tau$, whose isotropy representation is on the Lie algebra level equivalent to the slice representation of the H_1 -action on G/K at eK .*

Proof. The Lie algebra \mathfrak{g} of G admits the two decompositions

$$(11.1) \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h}_1 \oplus \mathfrak{m}_1,$$

where \mathfrak{p} and \mathfrak{m}_1 are the -1 -eigenspaces of σ_* and τ_* , respectively. The normal space $N_{eK}(H_1 \cdot eK) = \mathfrak{p} \cap \mathfrak{m}_1 \subseteq \mathfrak{p}$ is a Lie triple system and $\mathfrak{k} \cap \mathfrak{h}_1 \oplus \mathfrak{p} \cap \mathfrak{m}_1$ is the Lie algebra generated by $\mathfrak{p} \cap \mathfrak{m}_1$; the isotropy algebra at eK of the H_1 -action on G/K is just $\mathfrak{h}_1 \cap \mathfrak{k}$ and its action on the normal space $\mathfrak{p} \cap \mathfrak{m}_1$ agrees on the Lie algebra level with the isotropy representation of $G^{\sigma \circ \tau} / G^\sigma \cap G^\tau$, which is locally isometric to S by Proposition 3.2. q.e.d.

In the special case where the two involutions defining a Hermann action commute (possibly after conjugation), the action has a totally geodesic orbit. The pairs of involutions on the compact simple Lie groups for which this is the case have been determined in [10]. By Lemma 4.2, a polar subaction also acts polarly on this totally geodesic orbit.

Lemma 11.2 (Subactions of Hermann actions with commuting involutions). *Let G , K , and H_1 be as in Lemma 11.1. Assume in addition that $\sigma \circ \tau = \tau \circ \sigma$. Let $H \subseteq H_1$ be a closed connected subgroup acting polarly on $M = G/K$. Then the H_1 -orbit $H_1 \cdot eK = H_1/H_1 \cap K$ through $eK \in M$ is a totally geodesic submanifold and H acts polarly on the symmetric space $H_1 \cdot eK \cong H_1/H_1 \cap K$.*

Proof. Since σ and τ commute, we have the direct sum decomposition

$$(11.2) \quad \mathfrak{g} = (\mathfrak{k} \cap \mathfrak{h}_1) \oplus (\mathfrak{k} \cap \mathfrak{m}_1) \oplus (\mathfrak{p} \cap \mathfrak{h}_1) \oplus (\mathfrak{p} \cap \mathfrak{m}_1).$$

Consider now the H_1 -action on the symmetric space G/K . We can identify \mathfrak{p} with the tangent space $T_{eK}G/K$. Then $\mathfrak{h}_1 \cap \mathfrak{p}$ is the tangent space of the H_1 -orbit through the point eK . Using the Cartan relations for the decompositions (11.1), it is easy to verify that $\mathfrak{h}_1 \cap \mathfrak{p}$ is a Lie triple system. Hence, the H_1 -orbit through eK is totally geodesic by Proposition 3.2. Clearly, the action of H leaves all H_1 -orbits invariant and the polarity of the H -action on $H_1 \cdot eK$ follows from Lemma 4.2.

q.e.d.

The following lemma is just a simple reformulation of the criterion for polarity given by Proposition 4.1 in the special case of subaction of a Hermann action; it is, however, useful in particular to study polar actions on the exceptional spaces since it enables us to test for polarity on a subspace.

Lemma 11.3. *Let G , K , H_1 , H and $M = G/K$ be as in Lemma 11.1. Assume the group H acts transitively on the H_1 -orbit through eK . Then the action of H on G/K is polar if and only if the action of $H \cap K$ on $S = \exp_{eK}(N_{eK}(H_1 \cdot eK))$ is polar and $[\nu, \nu] \perp \mathfrak{h}$, where $\nu \subseteq \mathfrak{m}_1 \cap \mathfrak{p}$ is*

a normal space to a principal orbit of the slice representation of $H \cap K$ on $\mathfrak{m}_1 \cap \mathfrak{p}$.

12. Subactions of cohomogeneity one and transitive actions

To finish the proof of our classification result, it remains to consider subactions of cohomogeneity one and transitive actions. Note that polar actions on the real Grassmannians $\mathbb{G}_2(\mathbb{R}^n)$ of rank two were completely classified in [40], and we will not consider any spaces locally isometric to them here. We may also ignore all actions with a fixed point, since they are known to be hyperpolar by Corollary 6.2, see also [7], Theorem 2.2.

Proof of Theorems 1 and 2. Let G be a connected simple compact Lie group and let K be a symmetric subgroup such that $\text{rk}(G/K) \geq 2$. Assume the closed connected subgroup $H \subset G$ acts polarly on $M = G/K$.

We have already completed the classification in the case where G is an exceptional Lie group, hence it remains the case where G is one of the classical Lie groups $\text{SO}(n)$, $n \geq 7$, $\text{SU}(n)$, $n \geq 3$, or $\text{Sp}(n)$, $n \geq 2$. Then H is contained in one of the maximal connected subgroups of G as described in Propositions 3.6, 3.7, and 3.8. Thus at least one of the following holds:

- H is contained in a maximal tensor product subgroup (8.2) of G .
- H is contained in maximal connected simple irreducible subgroup of G .
- H is contained in a symmetric subgroup of G .

If the first possibility holds, then the result follows from Propositions 8.1 and 8.3 except if M is a Grassmannian and the tensor product subgroup acts with cohomogeneity one on M . The second possibility was studied in Section 9, except for subactions of cohomogeneity one or transitive actions. In the case where H is contained in symmetric subgroup H_1 of G , i.e., the H -action on M is a subaction of a Hermann action, the result follows from Theorem 7.3, under the assumption that the cohomogeneity of the H_1 -action on M is ≥ 2 . Thus it remains the case where H is a proper closed connected subgroup of $H_1 \subset G$ such that H_1 acts on M with cohomogeneity ≤ 1 .

It will turn out that all polar actions on G/K are hyperpolar, hence it follows from Corollary 2.12 of [24] that the sections are embedded submanifolds.

Subactions of “exceptional” cohomogeneity one and transitive actions. Let us first consider the case where the H_1 -action on $M = G/K$ is not of Hermann type. These cohomogeneity one and transitive actions were determined in [33], Theorem A and [36], respectively. We only consider the cases where G/K is symmetric of rank ≥ 2 ; these actions are given in Table 9.

H_1	G	K	Coh.
G_2	$SO(7)$	$SO(5) \times SO(2)$	0
G_2	$SO(7)$	$SO(4) \times SO(3)$	1
$Spin(9)$	$SO(16)$	$SO(14) \times SO(2)$	1
$Sp(n) \cdot Sp(1), n \geq 2$	$SO(4n)$	$SO(4n-2) \times SO(2)$	1
$SU(3)$	G_2	$SO(4)$	1

Table 9. “Exceptional” cohomogeneity one and transitive actions.

We examine these actions case by case. Assume first $H \subset G_2$ is acting on $G_2(\mathbb{R}^7)$ or $G_3(\mathbb{R}^7)$. Then H is contained in one of the maximal connected subgroups (10.2). Under the 7-dimensional irreducible orthogonal representation of G_2 , the first two groups $SO(4)$ and $SU(3)$ are mapped to reducible subgroups of $SO(7)$, and thus the H -action is in this case a subaction of a Hermann action. If H is contained in $SU(3)$, then the H -action is a subaction of a cohomogeneity one Hermann action of type BDI-I and will be treated on page 475. The third group can be excluded by Lemma 3.3.

Let us now consider subgroups of $Spin(9)$, acting on $G_2(\mathbb{R}^{16})$. The maximal connected subgroups of $SO(9)$ are, see Proposition 3.6 and [15]:

$$SO(8), \quad SO(7) \times SO(2), \quad SO(6) \times SO(3), \\ SO(5) \times SO(4), \quad SO(3) \otimes SO(3), \quad A_1^{60}.$$

Let $H_1 \subset Spin(9)$ such that $\pi(H_1) \subset SO(9)$ is one of the above, where $\pi: Spin(9) \rightarrow SO(9)$ is the double cover. We need to consider the image of H_1 under the spin representation $\delta: Spin(9) \rightarrow SO(16)$. We have $\delta(Spin(8)) \subset SO(8) \times SO(8)$ and $\delta(Spin(7) \cdot Spin(2)) \subset U(8)$, thus any subgroups of these are contained in symmetric subgroups of $SO(16)$; the remaining subgroups can be excluded by a dimension count, see Lemma 3.3.

We do not need to consider subactions of the $Sp(n) \cdot Sp(1)$ -action on $G_2(\mathbb{R}^{4n}) = SO(4n)/SO(4n-2) \times SO(2)$, since polar actions on these spaces have been completely classified by Podestà and Thorbergsson [40].

The last item in Table 9 was treated in Section 10.

Subactions of cohomogeneity one and transitive Hermann actions. It now remains to study subactions of cohomogeneity one and transitive Hermann actions. These Hermann actions are listed in Table 10. The column marked with $\frac{G^{\sigma\tau}}{G^\sigma \cap G^\tau}$ indicates (the local isometry type of) the symmetric space $G^{\sigma\tau}/G^\sigma \cap G^\tau$ whose isotropy representation is equivalent to one slice representation of the G^σ -action

Type	G^σ	G	G^τ	Coh.	$\frac{G^{\sigma\tau}}{G^\sigma \cap G^\tau}$
A I-III	$SO(p+1)$	$SU(p+1)$	$S(U(p) \times U(1))$	1	$\frac{SO(p+1)}{SO(p)}$
A III-II	$S(U(2n-1) \times U(1))$	$SU(2n)$	$Sp(n)$	0	$\frac{Sp(n-1) \times U(1)}{Sp(n-1) \times U(1)}$
A III-II	$S(U(2n-2) \times U(2))$	$SU(2n)$	$Sp(n)$	1	$\frac{Sp(n)}{Sp(n-1) \times Sp(1)}$
A III-II	$S(U(2n-3) \times U(3))$	$SU(2n)$	$Sp(n)$	1	$\frac{Sp(n-1) \times U(1)}{Sp(n-2) \times U(1) \times Sp(1)}$
A III-III	$S(U(a+b) \times U(1))$	$SU(a+b+1)$	$S(U(a) \times U(b+1))$	1	$\frac{S(U(a+1) \times U(b))}{S(U(a) \times U(1) \times U(b))}$
BD I-I	$SO(a+b)$	$SO(a+b+1)$	$SO(a) \times SO(b+1)$	1	$\frac{SO(a+1) \times SO(b)}{SO(a) \times SO(b)}$
C I-II	$Sp(p) \times Sp(1)$	$Sp(p+1)$	$U(p+1)$	1	$\frac{U(p+1)}{U(p) \times U(1)}$
C II-II	$Sp(a+b) \times Sp(1)$	$Sp(a+b+1)$	$Sp(a) \times Sp(b+1)$	1	$\frac{Sp(a+1) \times Sp(b)}{Sp(a) \times Sp(1) \times Sp(b)}$
D I-III	$SO(2n-1)$	$SO(2n)$	$U(n)$	0	$\frac{U(n-1)}{U(n-1)}$
D I-III	$SO(2n-2) \times SO(2)$	$SO(2n)$	$U(n)$	1	$\frac{U(n)}{U(n-1) \times U(1)}$
D I-III	$SO(2n-3) \times SO(3)$	$SO(2n)$	$U(n)$	1	$\frac{U(n-1)}{U(n-2) \times U(1)}$
E II-IV	$SU(6) \cdot Sp(1)$	E_6	F_4	1	$\frac{Sp(4)}{Sp(3) \cdot Sp(1)}$
E III-IV	$Spin(10) \cdot U(1)$	E_6	F_4	1	$\frac{F_4}{Spin(9)}$
F I-II	$Sp(3) \cdot Sp(1)$	F_4	$Spin(9)$	1	$\frac{Sp(3) \cdot Sp(1)}{Sp(2) \cdot Sp(1) \cdot Sp(1)}$

Table 10. Cohomogeneity one and transitive Hermann actions.

on G/G^τ (and of the G^τ -action on G/G^σ) by Lemma 11.1. (The presentation may be non-effective; in particular for transitive actions the space $G^{\sigma\tau}/G^\sigma \cap G^\tau$ is a noneffective presentation of a zero-dimensional space.) We only have to consider actions on symmetric spaces of rank ≥ 2 .

A III-I. Consider the action of $H_1 = S(U(p) \times U(1))$ on $G/K = SU(p+1)/SO(p+1)$, $p \geq 2$. Assume a closed connected subgroup $H \subset H_1$ acts polarly on G/K . Then H is contained in some maximal connected subgroup H_2 of H_1 . By Theorem 2.1 of [33], either $H_2 = SU(p)$ or $H_2 = S((H'_2 \otimes U(1)) \times U(1))$ where H'_2 is a maximal connected subgroup of $SU(p)$, see Proposition 3.7. In the case of the $SU(p)$ -action on G/K , an explicit calculation shows that the normal space to a principal orbit is not a Lie triple system, thus the $SU(p)$ -action on G/K is non-polar by Proposition 4.1. Thus we may restrict our attention to the second case, where we may further assume that $H'_2 \subset SU(p)$ is irreducible, since otherwise the H -action is a subaction of a Hermann action whose cohomogeneity is ≥ 2 . Assume first $H'_2 = SO(p)$. Then the H -action on G/K is non-polar by Corollary 6.2, since the H_2 -action has a one-dimensional orbit. Now assume $H'_2 = Sp(p/2)$, $p \geq 4$; then one isotropy group is $U(p/2)$ and the slice representation is the adjoint representation of $SU(p/2)$ plus the standard representation of $U(p/2)$, see Table 5, which is non-polar and polarity minimal [1]. The normal space $N_{eK}H_2 \cdot eK$ contains a Lie triple system corresponding to an irreducible symmetric space of higher rank, thus the H_2 -action on G/K is non-polar and polarity minimal by Lemma 6.4 (iii). Subgroups of $S(U(p/\ell) \otimes U(\ell)) \times U(1)$

are excluded by Proposition 8.2, simple irreducible maximal connected subgroups $H_2 \subset \mathrm{SU}(p)$ by Proposition 9.2 (iii).

A III-II. Let H be a closed connected subgroup of $H_1 = \mathrm{S}(\mathrm{U}(2n-k) \times \mathrm{U}(k))$, $k = 1, 2, 3$ acting on $G/K = \mathrm{SU}(2n)/\mathrm{Sp}(n)$, $n \geq 3$. It is well known [36] that the action of $H = \mathrm{SU}(2n-1)$ is transitive on G/K .

We will now study actions of closed connected subgroups H in $H_2 = \mathrm{S}((H'_2 \otimes \mathrm{U}(1)) \times \mathrm{U}(k))$. The cases where $H'_2 \subset \mathrm{SU}(2n-k)$ is a simple irreducible or tensor product subgroup are excluded by Lemma 9.2 and Proposition 8.2. Thus it remains to consider the case where H'_2 is a symmetric subgroup of $\mathrm{SU}(2n-k)$. Assume $H'_2 = \mathrm{SO}(2n-k)$; if $k = 1$ then $\mathrm{SO}(2n-1)$ acts on the symmetric space $M = \mathrm{SU}(2n)/\mathrm{Sp}(n)$, homogeneously presented as $M = \mathrm{SU}(2n-1)/\mathrm{Sp}(n-1)$; an isotropy subgroup of this action is $H_2 \cap K = \mathrm{U}(n-1)$, its slice representation is equivalent to the adjoint representation of $\mathrm{SU}(n-1)$ plus the standard representation on $\mathbb{C}^{n-1} = \mathbb{R}^{2n-2}$, see Table 5. This representation is non-polar [1] and polarity minimal by Proposition 6.3, and hence the H_2 -action on G/K is non-polar and polarity minimal by Lemma 6.4 (iii), since the normal space contains an irreducible Lie triple system of higher rank. If $k = 2$ or $k = 3$, then a slice representation of the H_2 -action on G/K contains a module equivalent to the isotropy representation of $\mathbb{H}\mathbb{P}^{n-1}$ or $\mathbb{H}\mathbb{P}^{n-2}$ restricted to $\mathrm{U}(n-1) \times \mathrm{Sp}(1)$ or $\mathrm{U}(n-2) \times \mathrm{Sp}(1)$, respectively. This representation contains two equivalent modules and the H -action is thus non-polar and polarity minimal by Lemma 6.4 (iii), except if $n = k = 3$, a case which can be handled by explicit calculations using the criterion in Proposition 4.1.

If $k = 2$ and $H'_2 = \mathrm{Sp}(n-1)$, then the H_2 -action has a one-dimensional orbit and the action of any closed subgroup $H \subseteq H_2$ on G/K is non-polar by Corollary 6.2. Thus we are left with the case where $H'_2 = \mathrm{U}(2n-k-\ell) \times \mathrm{U}(\ell)$. We may assume $k + \ell \leq 3$ since otherwise the H -action on G/K is a subaction of a Hermann action of cohomogeneity ≥ 2 , which were already treated in Section 7. If $k = \ell = 1$ then we obtain the cohomogeneity one actions of $H = \mathrm{S}(\mathrm{U}(2n-2) \times \mathrm{U}(1) \times \mathrm{U}(1))$ and $\mathrm{S}(\mathrm{U}(2n-2) \times \mathrm{U}(1))$ on G/K , we have already seen that no further closed proper subgroup of these groups acts polarly. In case $k + \ell = 3$, we have to consider closed connected subgroups H of $H_2 = \mathrm{S}(\mathrm{U}(2n-3) \times (H''_2 \otimes \mathrm{U}(1)))$, where $H''_2 \subset \mathrm{SU}(3)$ is a maximal connected subgroup. Since there are a number of subgroups $H \subseteq H_2$ acting with cohomogeneity two (in these cases all slice representations are polar), we have to exclude them by explicit calculations using Proposition 4.1.

In case $n = k = 3$ there are additional maximal connected subgroups, i.e.,

$$H_2 = \{(zA, z^{-1}\alpha(A)) \mid A \in \mathrm{SU}(3), z \in \mathbb{C}, |z| = 1\} \subset \mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(3)),$$

where $\alpha \in \mathrm{Aut}(\mathrm{SU}(3))$, see Theorem 2.1 of [33]. If α is an outer automorphism, e.g., given by complex conjugation, then the H_2 -action has a

one-dimensional orbit and is non-polar and polarity minimal by Corollary 6.2. If α is an inner automorphism, then a stabilizer component is $U(1) \cdot SO(3)$, and the 9-dimensional slice representation is equivalent to $\mathbb{R}^3 \oplus \mathbb{C}^1 \otimes \mathbb{R}^3$, hence non-polar [1] and polarity minimal by Proposition 6.3 (iii). Thus the H_2 -action on G/K is non-polar and polarity minimal by Lemma 6.4.

A II-III. Consider the action of $H_1 = Sp(n)$ on the space $G/K = SU(2n)/S(U(2n-2) \times U(2))$, $n \geq 2$. Let $H \subseteq H_1$ be a closed connected subgroup acting polarly on G/K . The H_1 -orbit $H_1 \cdot eK \cong Sp(n)/Sp(n-1) \times Sp(1)$ is totally geodesic and H acts polarly on this orbit by Lemma 11.2, the action being non-transitive by [36]. The H -action on $H_1 \cdot eK$ has a singular orbit by Corollary 5.3 and we may assume by conjugation of H in H_1 that eK lies in a singular orbit. From Table 10 we read off that the slice representation of the H_1 -action on G/K is equivalent to the isotropy representation of the symmetric space $H_1 \cdot eK = H_1/H_1 \cap K$. Thus the nontrivial slice representation of the H -action on $H_1 \cdot eK$, which is a submodule of the isotropy representation of $H_1/H_1 \cap K$, also occurs as a submodule of the slice representation of H_1 on G/K restricted to $H \cap K$. We conclude that the slice representation of the H -action on G/K contains two nontrivial equivalent modules and is hence non-polar by [33], Lemma 2.9.

Consider the action of $H_1 = Sp(n)$ on $G/K = SU(2n)/S(U(2n-3) \times U(3))$, $n \geq 3$. Let $H \subseteq H_1$ be a closed connected subgroup acting polarly on G/K . Then H is contained in a maximal connected subgroup H_2 of $H_1 = Sp(n)$. We may assume that H_2 is irreducible, since otherwise the H -action is a subaction of a Hermann action with cohomogeneity ≥ 2 . If $H_2 = U(n)$, then H_2 is contained after conjugation in $S(U(n) \times U(n)) \subset SU(2n)$ and the H -action is also a subaction of a Hermann action of cohomogeneity ≥ 2 . The actions of maximal connected subgroups of type $SO(q) \otimes Sp(p/q)$ have been treated in Proposition 8.4. The actions of simple irreducible subgroups $\rho(H)$, where $\rho: H \rightarrow Sp(n)$ is an irreducible representation of quaternionic type, have been excluded in Section 9.

A III-III. Consider the action of $H_1 = S(U(a+b) \times U(1))$ on the complex Grassmannian $G/K = SU(a+b+1)/S(U(a) \times U(b+1))$, $a \geq b \geq 1$, $a+b \geq 3$. Assume $H \subset H_1$ is a closed connected subgroup acting polarly on G/K . First note that the action of $SU(a+b)$ on G/K is orbit equivalent to the H_1 -action. Now assume $H \subseteq H_2 = S((H'_2 \otimes U(1)) \times U(1))$ is a closed connected subgroup acting polarly on G/K , where $H'_2 \subset SU(a+b)$ is a maximal connected subgroup. We may assume that the standard representation of $SU(a+b)$ restricted to H'_2 acts irreducibly on \mathbb{C}^{a+b} , since otherwise H_2 is contained in a subgroup of G conjugate to $S(U(k) \times U(a+b+1-k))$ for $2 \leq k \leq a+b-1$ and the H_2 -action on G/K is a subaction of a Hermann action of cohomogeneity ≥ 2 , which

have already been treated in Section 7. It follows from Lemma 11.2 that the orbit $H_1 \cdot eK$ is a totally geodesic submanifold of G/K isometric to $SU(a+b)/S(U(a) \times U(b))$ on which H acts polarly.

Assume first that H'_2 is an irreducible symmetric subgroup of $SU(a+b)$, hence conjugate to either $SO(a+b)$ or $Sp(\frac{a+b}{2})$. However, in the first case, the isotropy subgroup of the H_2 -action at eK is $S(O(a) \times O(b)) \cdot U(1)$ and its slice representation is equivalent to $(\mathbb{R}^a \otimes \mathbb{R}^b) \oplus (\mathbb{R}^a \otimes \mathbb{C}^1)$. Thus it is non-polar [1] and polarity minimal by Lemma 6.3. It follows from Table 10 that the normal space contains a Lie triple system corresponding to a totally geodesic submanifold isometric to $\mathbb{C}P^a$, thus the H_2 -action is non-polar and polarity minimal by Lemma 6.4 (iii), since $a \geq 2$. Let us now consider the case where $a+b$ is even and H'_2 is conjugate to $Sp(\frac{a+b}{2})$, and hence $a+b \geq 4$. The group H acts polarly on the totally geodesic H_1 -orbit $H_1 \cdot eK \cong SU(a+b)/S(U(a) \times U(b))$, which is of rank b . If $b \geq 2$, then the reducible slice representation is non-polar and polarity minimal by [1], and the H -action is non-polar and polarity minimal by Lemma 6.4. If $b = 1$, then H_2 acts transitively on the orbit $H_1 \cdot eK$, but an explicit calculation using Proposition 4.1 shows that the H_2 -action on G/K is non-polar. Let $H \subset H_2$ be a proper closed subgroup; then H acts non-transitively on $H_1 \cdot eK$ by [36]. If the H -action on G/K is polar, then also the H -action restricted to $H_1 \cdot eK \cong \mathbb{C}P^a$ is polar by Lemma 4.2 and it has a singular orbit $H \cdot p$ by Corollary 5.3. The normal space $\mathfrak{p} \cap \mathfrak{m}_1$, see Lemma 11.1, of the H_1 -action on G/K contains a submodule that is equivalent to the slice representation at $p \in H_1 \cdot eK$ of the H -action on $H_1 \cdot eK$ after a $U(1)$ -factor is removed from both representations. Since both modules belong to the polar slice representation of the H -action on G/K , it follows from [1] that H is at most three-dimensional, a contradiction with Proposition 5.5.

Now assume H'_2 is a non-symmetric irreducible maximal connected subgroup of $Sp(a+b)$. It follows from what we have shown so far that this can only happen if $\text{rk}(H_1 \cdot eK) = b = 1$. Assume $H'_2 = SO(p) \otimes Sp(q)$; then the H_2 -action on $H_1 \cdot eK$ is non-polar and polarity minimal by Proposition 8.3. If H'_2 is a simple irreducible maximal connected subgroup of $Sp(a+b)$, then it follows from the results of Section 9 that the action of H_2 on G/K is non-polar and polarity minimal, since if the action of H'_2 on $\mathbb{G}_k(\mathbb{H}^{a+b})$ for $2 \leq k \leq a+b-2$ is excluded by Proposition 5.6, then also the action of H_2 on $\mathbb{G}_k(\mathbb{H}^{a+b+1})$ is excluded by a dimension count.

BD I-I. Let $H_1 = SO(a+b)$, $G/K = SO(a+b+1)/SO(a) \times SO(b+1)$, $a+b \geq 6$, $a \geq b \geq 1$. Assume $H \subset H_1$ is a closed connected subgroup acting polarly on G/K . Without loss of generality we may assume that $H \subseteq SO(a+b)$ acts irreducibly on \mathbb{R}^{a+b} , since otherwise the H -action on G/K is a subaction of a Hermann action of cohomogeneity ≥ 2 , see

Section 7. By Lemma 11.2, the H_1 -orbit $H_1 \cdot eK$ is a totally geodesic submanifold isometric to $\mathrm{SO}(a+b)/\mathrm{S}(\mathrm{O}(a) \times \mathrm{O}(b))$ and H acts polarly on $H_1 \cdot eK$ (the action may be transitive).

We do not need to consider the case where $\mathrm{rk}(H_1 \cdot eK) = b = 1$, since polar actions on $\mathbb{G}_2(\mathbb{R}^{a+2})$ were completely classified in [40]. Hence we may assume $b \geq 2$. Let us first consider the cases where $H \subset H_1$ is not a symmetric subgroup. Then it follows from what we have shown so far and [36] that we have one of the following:

- $H = \mathrm{G}_2$, $a + b = 7$, $b = 2, 3$;
- $H = \mathrm{Spin}(7)$, $a + b = 8$, $b = 2, 3, 4$;
- $H = \mathrm{Spin}(9)$, $a + b = 16$, $b = 2$;
- $H = \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, $a + b = 4n$, $b = 2$;
- $H = \mathrm{U}(n)$, $a + b = 2n$.

In the case of the G_2 -actions, an explicit calculation using Proposition 4.1 shows that the actions are non-polar; subgroups (10.2) of G_2 are either reducible or are excluded by a dimension count. The actions of $\mathrm{Spin}(7)$ are orbit equivalent to the $\mathrm{SO}(8)$ -action in case $b = 2, 3$; in case $b = 4$ the action can be shown to be non-polar by an explicit calculation. Subgroups of $\mathrm{Spin}(7)$ are either contained in groups treated below or ruled out by a dimension count. The actions of $\mathrm{Spin}(9)$ and $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ can be excluded by replacing K with the conjugate subgroup $K' = \mathrm{SO}(3) \times \mathrm{SO}(a)$, the actions on $H_1 \cdot eK'$ have already been shown to be non-polar and polarity minimal. Assume now $H = \mathrm{U}(\frac{a+b}{2})$. The slice representation of the H -action at eK , as can be seen from Tables 5 and 10, contains a module equivalent to the representation of $\mathrm{U}(\lfloor \frac{a}{2} \rfloor) \times \mathrm{U}(\lfloor \frac{b}{2} \rfloor)$ on $\mathbb{C}^{\lfloor \frac{a}{2} \rfloor} \otimes \mathbb{C}^{\lfloor \frac{b}{2} \rfloor} \oplus \mathbb{C}^{\lfloor \frac{a}{2} \rfloor}$, which is non-polar [1] and polarity minimal by Lemma 6.3 since $\lfloor \frac{a}{2} \rfloor \geq 2$.

CI-II. Let H be closed connected subgroup of $H_1 = \mathrm{Sp}(p) \times \mathrm{Sp}(1)$ acting polarly on $G/K = \mathrm{Sp}(p+1)/\mathrm{U}(p+1)$, $p \geq 2$. We first observe that the actions of $\mathrm{Sp}(p)$ and $\mathrm{Sp}(p) \times \mathrm{U}(1)$ are not orbit equivalent to the H_1 -action; since the normal space at a regular orbit is not a Lie triple system, these actions are non-polar by Proposition 4.1. Now assume $H \subseteq H_2 = H'_2 \times \mathrm{Sp}(1)$, where H'_2 is a maximal connected subgroup of $\mathrm{Sp}(n)$. We may assume $H'_2 \subset \mathrm{Sp}(n)$ acts irreducibly on \mathbb{H}^n , since otherwise H_2 is a subgroup of $\mathrm{Sp}(p+1-k) \times \mathrm{Sp}(k)$, $2 \leq k \leq p-1$, see Section 7. Consider the action of $H_2 = \mathrm{U}(p) \times \mathrm{Sp}(1)$ on G/K ; then the slice representation of the isotropy subgroup $\mathrm{U}(p) \times \mathrm{U}(1)$ is equivalent to the isotropy representation of $\mathrm{Sp}(p)/\mathrm{U}(p)$ plus $\mathbb{C}^p \otimes \mathbb{C}^1$. This representation is non-polar [1] and the action of H_2 on G/K is polarity-minimal by Lemma 6.4 (iii). If $H_2 = \rho(H'_2) \times \mathrm{Sp}(1)$, where $\rho: H'_2 \rightarrow \mathrm{Sp}(p)$ is an irreducible representation of the simple compact Lie group H , then the H -action on G/K is non-polar by Lemma 9.2. Tensor product subgroups H'_2 have been excluded in Proposition 8.2.

C II-II. Let $H_1 = \mathrm{Sp}(a+b) \times \mathrm{Sp}(1)$ and let $G/K = \mathrm{Sp}(a+b+1)/\mathrm{Sp}(a) \times \mathrm{Sp}(b+1)$, $a \geq b \geq 1$, $a+b \geq 3$. First observe that the action of $\mathrm{Sp}(a+b)$ on G/K is orbit equivalent to the H_1 -action. Now assume $H \subset H_2 = H'_2 \times \mathrm{Sp}(1)$ is a closed connected subgroup acting polarly on G/K , where $H'_2 \subset \mathrm{Sp}(a+b)$ is a maximal connected subgroup. We may assume that $H'_2 \subseteq \mathrm{Sp}(a+b)$ acts irreducibly on \mathbb{H}^{a+b} , since otherwise the H -action on G/K is a subaction of a Hermann action of cohomogeneity ≥ 2 , which were examined in Section 7. It follows from Lemma 11.2 that the H_1 -orbit $H_1 \cdot eK$ is a totally geodesic submanifold isometric to $\mathrm{Sp}(a+b)/\mathrm{Sp}(a) \times \mathrm{Sp}(b)$ on which H acts polarly.

Assume H'_2 is an irreducible symmetric subgroup of $\mathrm{Sp}(a+b)$, hence conjugate to $\mathrm{U}(a+b)$. However, the isotropy subgroup of the H_2 -action at eK is $\mathrm{U}(a) \times \mathrm{U}(b) \times \mathrm{Sp}(1)$, its slice representation contains two equivalent modules; thus it is non-polar and polarity minimal by Lemma 6.3, parts (ii) and (iii). The normal space contains a Lie triple system corresponding to a totally geodesic submanifold isometric to $\mathbb{H}\mathbb{P}^a$, hence the H_2 -action is non-polar and polarity minimal by Lemma 6.4 (iii).

Now assume H'_2 is a non-symmetric irreducible maximal connected subgroup of $\mathrm{Sp}(a+b)$ acting polarly on $H_1 \cdot eK = \mathrm{Sp}(a+b)/\mathrm{Sp}(a) \times \mathrm{Sp}(b)$. It follows from what we have shown so far that this can only happen if $\mathrm{rk}(H_1 \cdot eK) = 1$, i.e., $b = 1$. Assume $H'_2 = \mathrm{SO}(p) \otimes \mathrm{Sp}(q)$. Then the H_2 -action on $H_1 \cdot eK$ is non-polar and polarity minimal by Proposition 8.3. If H'_2 is a simple irreducible subgroup of $\mathrm{Sp}(a+b)$, then it follows from the results of Section 9 that the action of H_2 on G/K is non-polar and polarity minimal, since if the action of H'_2 on $\mathbb{G}_k(\mathbb{H}^{a+b})$ for $2 \leq k \leq a+b-2$ is excluded by Proposition 5.6, then so is the action of H_2 on $\mathbb{G}_k(\mathbb{H}^{a+b+1})$.

D I-III. Let H be a closed connected subgroup of $\mathrm{SO}(n-k) \times \mathrm{SO}(k)$, $k = 1, 2, 3$ acting on $G/K = \mathrm{SO}(2n)/\mathrm{U}(n)$, $n \geq 3$. We first study actions of closed connected subgroups H in $H_2 = H'_2 \times \mathrm{SO}(k)$. The cases where $H'_2 \subset \mathrm{SO}(2n-k)$ is a simple irreducible or tensor product subgroup were excluded by Lemma 9.2 and Lemma 8.2. Let us consider the case where H'_2 is a symmetric subgroup of $\mathrm{SO}(2n-k)$. If $k = 2$ and $H'_2 = \mathrm{U}(n-1)$, then the H -action has a fixed point. Thus it remains the case where $H'_2 = \mathrm{SO}(2n-k-\ell) \times \mathrm{SO}(\ell)$. We may assume $k+\ell \leq 3$ since otherwise the H -action on G/K is a subaction of a Hermann action of cohomogeneity ≥ 2 , see Section 7. If $k = \ell = 1$ then we obtain the cohomogeneity one action of $H = \mathrm{SO}(2n-2)$ on G/K ; we have already seen that no closed proper subgroup of this group acts polarly. In the case where $k+\ell = 3$, an explicit calculation using Proposition 4.1 shows that the actions of $\mathrm{SO}(2n-3)$ and $\mathrm{SO}(2n-3) \times \mathrm{SO}(2)$ on G/K are non-polar; we have already excluded any closed subgroups of these two groups.

In case $n = 3$ there is an additional maximal connected subgroup, i.e., $\Delta\mathrm{SO}(3) = \{(g, g) \mid g \in \mathrm{SO}(3)\} \subset \mathrm{SO}(3) \times \mathrm{SO}(3)$, but its action has a fixed point.

D III-I. Let H be a closed connected subgroup of $H_1 = \mathrm{U}(n)$ acting polarly on the symmetric space $G/K = \mathrm{SO}(2n)/\mathrm{SO}(n-3) \times \mathrm{SO}(3)$, $n \geq 3$. It follows from Theorem 2.1 in [33] that the conjugacy classes of maximal connected subgroups H_2 in $\mathrm{U}(n)$ are exhausted by $\mathrm{SU}(n)$ and $H'_2 \otimes \mathrm{U}(1)$, where H'_2 runs through the maximal connected subgroups of $\mathrm{SU}(n)$; see Proposition 3.7. We observe first that the action of $\mathrm{SU}(n)$ is orbit equivalent to the $\mathrm{U}(n)$ -action. Now assume $H \subset H'_2 \otimes \mathrm{U}(1)$. We do not need to consider reducible subgroups H'_2 since they lead to subactions of Hermann actions with cohomogeneity ≥ 2 , which were treated in Section 7. Also, if $H'_2 = \mathrm{SO}(n)$, then the same argument as in Proposition 8.3 shows that the action is non-polar and polarity minimal. If $H'_2 = \mathrm{Sp}(n/2)$, then H_2 is contained in $\mathrm{Sp}(n/2) \cdot \mathrm{Sp}(1)$ and the action of any closed subgroup $H \subseteq H_2$ was shown not to be polar in Proposition 8.3. Assume now H'_2 is a tensor product subgroup $\mathrm{SU}(p) \otimes \mathrm{SU}(n/p)$. Then the action of H is non-polar according to Proposition 8.4. Finally, let H'_2 be given by an irreducible representation $\rho: H'_2 \rightarrow \mathrm{SU}(n)$ where H'_2 is a simple compact Lie group; these actions were excluded in Section 9.

E II-IV. Let $H \subseteq H_1 = \mathrm{SU}(6) \cdot \mathrm{Sp}(1)$ be a closed connected subgroup acting on $G/K = \mathrm{E}_6/\mathrm{F}_4$. Then H is contained in one of the maximal connected subgroups of $\mathrm{SU}(6) \cdot \mathrm{Sp}(1)$. By [10], we may assume the involutions of E_6 corresponding to H_1 and K commute such that the totally geodesic H_1 -orbit through eK is isometric to $\mathrm{SU}(6) \cdot \mathrm{Sp}(1)/\mathrm{Sp}(3) \cdot \mathrm{Sp}(1) \cong \mathrm{SU}(6)/\mathrm{Sp}(3)$, see Lemma 11.2. The slice representation is equivalent to the isotropy representation of $\mathrm{Sp}(4)/\mathrm{Sp}(3) \cdot \mathrm{Sp}(1)$, and this shows that the $\mathrm{Sp}(1)$ -factor is inessential for the H_1 -action. Now assume $H'_2 \subseteq \mathrm{SU}(6)$ is a symmetric subgroup and $H \subseteq H_2 = H'_2 \cdot \mathrm{Sp}(1)$ is a closed connected subgroup. If $H'_2 = \mathrm{Sp}(3)$, then the H -action on G/K has a fixed point. If $H'_2 = \mathrm{SO}(6)$, then the connected component of an isotropy subgroup is $\mathrm{U}(3) \times \mathrm{Sp}(1)$ and its slice representation is equivalent to the adjoint representation of $\mathrm{SU}(3)$ plus $\chi(\mathrm{Sp}(4), \mathrm{Sp}(3) \times \mathrm{Sp}(1))|_{\mathrm{U}(3) \times \mathrm{Sp}(1)}$, see Tables 5 and 10; it is non-polar [1] and polarity minimal by Proposition 6.3. Hence the H_2 -action on G/K is non-polar and polarity minimal by Lemma 6.4 (iii).

Now assume H'_2 is one of the groups $\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(6-k))$, $k = 1, 2, 3$. If $k = 3$, then $(H_2 \cap K)_0 \cong \mathrm{Sp}(1) \cdot \mathrm{U}(1) \cdot \mathrm{Sp}(1) \cdot \mathrm{Sp}(1)$ and the slice representation is equivalent to

$$\chi(\mathrm{Sp}(2), \mathrm{Sp}(1) \times \mathrm{Sp}(1)) \oplus \chi(\mathrm{Sp}(4), \mathrm{Sp}(3) \times \mathrm{Sp}(1))|_{\mathrm{Sp}(1) \cdot \mathrm{U}(1) \cdot \mathrm{Sp}(1) \cdot \mathrm{Sp}(1)},$$

where both modules are polar, but their sum is non-polar, see [1]. Using the results of [1], it can be directly verified that this representation is

polarity minimal, and it follows that the H_2 -action on G/K is non-polar and polarity minimal by Lemma 6.4. If $k = 2$, then $(H_2 \cap K)_0 \cong \mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \cdot \mathrm{Sp}(1)$ and the corresponding slice representation is equivalent to

$$\chi(\mathrm{Sp}(3), \mathrm{Sp}(2) \times \mathrm{Sp}(1)) \oplus \chi(\mathrm{Sp}(4), \mathrm{Sp}(3) \times \mathrm{Sp}(1))|_{\mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \cdot \mathrm{Sp}(1)},$$

hence non-polar by [1] and polarity minimal by Proposition 6.3 (ii). The H_2 -action on G/K is non-polar and polarity minimal by Lemma 6.4. Finally, if $k = 1$ then H_2 acts transitively on the H_1 -orbit through eK , and the group $(H_2 \cap K)_0 \cong \mathrm{U}(1) \times \mathrm{Sp}(2) \times \mathrm{Sp}(1)$ acts polarly on $\exp_{eK}(\mathcal{N}_{eK}(H_2 \cdot eK)) \cong \mathbb{H}\mathbb{P}^3$ by [39], cf. Lemma 11.1; however, an explicit calculation shows that $[\nu, \nu] \notin \mathfrak{h}_2$, where $\nu \subset \mathfrak{m}_1 \cap \mathfrak{p}$ is the tangent space to a section of the $H_2 \cap K$ -action, and hence the H_2 -action and the orbit equivalent action of $\mathrm{SU}(5) \cdot \mathrm{Sp}(1)$ are non-polar by Lemma 11.3. The actions of $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(5))$ and $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(5)) \cdot \mathrm{U}(1)$ are non-polar since the slice representations at eK are non-polar. Any other closed subgroups of $H_2 = \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(5)) \cdot \mathrm{Sp}(1)$ are contained in the groups treated above or excluded by Proposition 5.6.

E IV-II. Consider now closed connected subgroups H of $H_1 = \mathrm{F}_4$ acting polarly on $\mathrm{E}_6/\mathrm{SU}(6) \cdot \mathrm{Sp}(1)$. It follows from Lemma 3.3 that $\dim(H) \geq 28$. By [15], the only closed connected subgroup of F_4 of sufficient dimension is $\mathrm{Spin}(9)$. By conjugation, the subgroup $H = \mathrm{Spin}(9) \subset \mathrm{F}_4$ can be chosen such that the connected component of the isotropy group $(H \cap K)_0$ is $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)^2 \cong \mathrm{Spin}(5) \cdot \mathrm{Spin}(4)$, see Table 10. From Table 10 one sees further that the slice representation restricted to $(H \cap K)_0$ is equivalent to the isotropy representation of $\mathrm{Sp}(3)/\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ plus the isotropy representation of $\mathrm{Sp}(4)/\mathrm{Sp}(3) \cdot \mathrm{Sp}(1)$ restricted to $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$, hence it is non-polar by [1].

E III-IV. Let H be a closed connected subgroup of $H_1 = \mathrm{Spin}(10) \cdot \mathrm{U}(1)$ acting polarly on $G/K = \mathrm{E}_6/\mathrm{F}_4$. Let us first show that the action of $\mathrm{Spin}(10)$ on G/K is non-polar. Assume the converse, i.e., the action of $H = \mathrm{Spin}(10)$ on $G/K = \mathrm{E}_6/\mathrm{F}_4$ is polar. Since this action is of cohomogeneity two, it follows from Proposition 2.8 that the sections are locally isometric to a two-sphere. The H_1 -orbit through eK is totally geodesic by Lemma 11.2 and locally isometric to $\mathrm{S}^9 \times \mathrm{S}^1$, where the S^1 -factor is the orbit of the $\mathrm{U}(1)$ -factor in $H_1 = \mathrm{Spin}(10) \cdot \mathrm{U}(1)$; hence it is totally geodesic in G/K . It follows from the decomposition (11.2) that $\mathrm{T}_{eK}\mathrm{S}^1 \perp \mathrm{T}_{eK}(H \cdot eK)$, hence $\mathrm{T}_{eK}\mathrm{S}^1 \subset \mathrm{T}_{eK}\Sigma$, where Σ is a section of the H -action on G/K containing eK . Since the Lie algebra of the $\mathrm{U}(1)$ -factor in $H_1 = \mathrm{Spin}(10) \cdot \mathrm{U}(1)$ is contained in $\mathrm{T}_{eK}\Sigma \subset \mathfrak{p}$, it follows by Proposition 3.2 that this $\mathrm{U}(1)$ -factor acts on Σ as a group of transvections. Now, since this $\mathrm{U}(1)$ -action commutes with the H -action, it follows that any two points of Σ which lie in the same $\mathrm{U}(1)$ -orbit are of the same orbit type with respect to the H -action on G/K . In particular, all singular orbits of the H -action on G/K intersect Σ in the $\mathrm{U}(1)$ -orbit

that is covered by a great circle of $\tilde{\Sigma} \cong S^2$, since all reflection hyper-surfaces $\{P_j\}_{j \in J}$ have to be invariant under the $U(1)$ -action induced on $\tilde{\Sigma}$, see Lemma 5.1. However, the $U(1)$ -action on Σ , which is isometric to $\mathbb{R}P^2$ or S^2 , has at least one fixed point $p \in \Sigma$. It follows that this point $p \in G/K$ lies in a regular orbit of the H -action on G/K , but is left fixed by the $U(1)$ -factor of $H_1 = \text{Spin}(10) \cdot U(1)$. Hence the connected component of the isotropy subgroup $(H_1)_p$ is a subgroup $L \cdot U(1)$, where $L \subset \text{Spin}(10)$ is 20-dimensional — a contradiction, since the principal isotropy subgroup is $\text{Spin}(7)$.

Now we may assume that the group H acting polarly on G/K is contained in $H'_2 \cdot U(1)$, where H'_2 is a maximal connected subgroup of $\text{Spin}(10)$. It follows from Lemma 3.3 that $\dim(H) \geq 16$. This implies that H'_2 is one of the following, see Proposition 3.6:

$\text{Spin}(9)$, $\text{Spin}(8) \cdot \text{SO}(2)$, $\text{Spin}(7) \cdot \text{Spin}(3)$, $\text{Spin}(6) \cdot \text{Spin}(4)$, $U(5)$.

The actions of these groups are non-polar and polarity minimal by Lemma 6.4 (iv). If $H'_2 = \text{Spin}(9)$, then the H -action can also be shown to be non-polar by Corollary 6.2.

E IV-III. Assume H is a closed connected subgroup of F_4 acting polarly on $G/K = E_6/\text{Spin}(10) \cdot U(1)$. Proposition 5.5 implies $\dim(H) \geq 28$. By [15], the only maximal connected subgroups $H \subset F_4$ of dimension ≥ 28 is $\text{Spin}(9)$. It follows from Table 10 that the action of $\text{Spin}(9)$ leaves a point fixed.

F II-I. Let H be a closed connected subgroup of $H_1 = \text{Spin}(9)$ acting polarly on $G/K = F_4/\text{Sp}(3) \cdot \text{Sp}(1)$. Lemma 3.3 implies $\dim(H) \geq 20$. The only closed connected subgroups of $\text{Spin}(9)$ of dimension ≥ 20 are $\text{Spin}(8)$, $\text{Spin}(7) \cdot \text{SO}(2)$ and $\text{Spin}(7)$.

The subgroup $H = \text{Spin}(8) \subset \text{Spin}(9)$ may be chosen such that $(H_1 \cap K)_0 = \text{Sp}(2) \cdot \text{Sp}(1) \cong \text{Spin}(5) \cdot \text{Spin}(3)$. The group H acts with cohomogeneity one on the orbit $H_1 \cdot eK$, which is covered by $\text{Spin}(9)/\text{Spin}(5) \cdot \text{Spin}(4)$. With our choice of the subgroup $\text{Spin}(8) \subset \text{Spin}(9)$, the slice representation of the H -action on G/K at eK is equivalent (on the Lie algebra level) to the representation of $\text{Sp}(2) \cdot \text{Sp}(1) \cong \text{Spin}(5) \cdot \text{Spin}(3)$ on $\mathbb{H}^2 \otimes_{\mathbb{H}} \mathbb{H}^1 \oplus \mathbb{R}^5$, hence it is non-polar by [1].

The action of $\text{Spin}(7) \cdot \text{SO}(2)$ has an isotropy subgroup whose connected component is isomorphic to $\text{Spin}(5) \cdot \text{SO}(2) \cdot \text{SO}(2) \cong \text{Sp}(2) \cdot U(1) \cdot U(1)$ and whose slice representation is $\mathbb{R}^5 \otimes \mathbb{R}^2 \oplus \chi(\text{Sp}(3), \text{Sp}(2) \cdot \text{Sp}(1))|_{\text{Sp}(2) \cdot U(1)}$, hence it is non-polar [1]. This also shows that the $\text{Spin}(7)$ -action is non-polar. q.e.d.

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