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DYNAMICAL CONVERGENCE AND POLYNOMIAL VECTOR FIELDS

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Abstract

Let $f_n \to f_0$ be a convergent sequence of rational maps, preserving critical relations, and f_0 be geometrically finite with parabolic points. It is known that for some unlucky choices of sequences f_n, the Julia sets $J(f_n)$ and their Hausdorff dimensions may fail to converge as $n \to \infty$. Our main result here is to prove the convergence of $J(f_n)$ and H.dim $J(f_n)$ for generic sequences f_n. The same conclusion was obtained earlier, with stronger hypotheses on the sequence f_n, by Bodart-Zinsmeister and then by McMullen. We characterize those choices of f_n by means of flows of appropriate polynomial vector fields (following Douady-Estrada-Sentenac). We first prove an independent result about the (s-dimensional) length of separatrices of such flows, and then use it to estimate tails of Poincaré series. This, together with existing techniques, provides the desired control of conformal densities and Hausdorff dimensions. Our method may be applied to other problems related to parabolic perturbations.

1. Introduction

We say that a sequence of rational maps f_n converges to f_0 *algebraically* if $\deg f_n = \deg f_0$ and the coefficients of f_n (as a ratio of polynomials) can be chosen to converge to those of f_0. Algebraic convergence is equivalent to uniform convergence for the spherical metric on $\mathbb{P}^1$.

Assume $f_n \to f_0$ algebraically. Let $J(f_n)$ be the Julia set of f_n and H.dim $J(f_n)$ be its Hausdorff dimension. The question that interests us here is: do we have

$$J(f_n) \underset{n\to\infty}{\longrightarrow} J(f_0) \quad \text{and} \quad \text{H.dim } J(f_n) \underset{n\to\infty}{\longrightarrow} \text{H.dim } J(f_0)?$$

(The limit of Julia sets is for the Hausdorff topology on compact subsets of $\mathbb{P}^1$.)

If f_0 is a hyperbolic rational map, the answer is yes. But in general, one only has

$$J(f_0) \subset \liminf J(f_n) \quad \text{and} \quad \text{H.dim } J(f_0) \leq \liminf \text{H.dim } J(f_n).$$

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The typical example where those inequalities are strict is given by $f_n(z) = \lambda_n z + z^2$ with $\lambda_n \to 1$ (and $f_0(z) = z + z^2$). On the one hand, if $\mathrm{Re}(1/(1-\lambda_n))$ remains bounded (i.e., $\lambda_n \to 1$ avoiding two disks $D(1+\varepsilon, \varepsilon)$ and $D(1-\varepsilon, \varepsilon)$ with $\varepsilon > 0$), then $J(f_0) \subsetneq \liminf J(f_n)$ (see Douady [**D**], 1994) and H.dim $J(f_0) < \liminf$ H.dim $J(f_n)$ (see Douady-Sentenac-Zinsmeister [**DSZ**], 1997). On the other hand, if $\lambda_n - 1 \to 0$ avoiding a sector neighborhood of $i\mathbb{R}^+ \cup i\mathbb{R}^-$, then $J(f_n) \to J(f_0)$ and H.dim $J(f_n) \to$ H.dim $J(f_0)$ (see Bodart-Zinsmeister [**BZ**], 1996).

This last result was generalized by McMullen [**McM**], 2000, who proved the following result. Let f_0 be a geometrically finite rational map (i.e., every critical point in $J(f)$ has a finite forward orbit) and let $f_n \to f_0$ algebraically, preserving the critical relations (see the statement of Theorem A for a precise definition). For each parabolic point $\beta \in J(f_0)$, let j be the least integer such that $f_0^{\circ j}$ fixes β with multiplier 1, and let p be the number of petals of f_0 at β. Assume in addition

(a) $f_n^{\circ j}$ has a fixed point β_n converging to β with multiplier λ_n, and p simple fixed points which are symmetrically placed at the vertices of an almost regular p-side-polygon centered at β_n (see Definition 2.16 for a more rigorous statement).

Then, $J(f_n) \to J(f_0)$ and H.dim $J(f_n) \to$ H.dim $J(f_0)$ as long as

(b) $\lambda_n - 1 \to 0$ avoiding a sector neighborhood of $i\mathbb{R}^+ \cup i\mathbb{R}^-$.

In this article, we will prove the same result in its full generality, namely without the extra assumption (a). Therefore the perturbed fixed points do not have to present any symmetry, and may very well fail to be simple. In this case, the characterization of good perturbations, namely condition (b), is no longer valid. We will replace it by the notion of 'stable perturbations' in terms of some polynomial vector fields. This will take us some time to describe.

(In [**McM**], McMullen shows that condition (b) can be replaced by condition H.dim $J(f_0) > 2p(f_0)/(p(f_0)+1)$ and $\mathrm{Re}(1/(1-\lambda_n)) \to \infty$, where $p(f_0)$ denotes the maximum number of petals at a parabolic point of f_0 or one of its preimages. We do not generalize this result.)

Set $D(r) = \{z \ ; \ |z| < r\}$. Let $f_0 : D(r) \to \mathbb{C}$ be a holomorphic map in the form $f_0(z) = z + z^{p+1} + \mathcal{O}(z^{p+2})$ with $p \geq 1$; in other words f_0 has a multiple fixed point at 0. Most of our work consists in studying the dynamics of holomorphic maps $f : D(r) \to \mathbb{C}$ which are small perturbations of f_0, and in finding out which perturbations are dynamically stable with respect to f_0.

If $p = 1$, $f_0(z) = z + z^2 + \mathcal{O}(z^3)$ and if $f(z) = \lambda z + \mathcal{O}(z^2)$, $\lambda \neq 1$, is a small perturbation of f_0, then f has two simple fixed points close to 0: 0 itself and σ. The classical method is to use the Möbius transformation $z \mapsto w = z/(z-\sigma)$ to pull apart the two fixed points. And then, in a suitably normalized log-coordinate of w, our map f is conjugate to a

map close to the translation $Z \mapsto Z + 1$. In such a way, we obtain the so-called approximate Fatou coordinates and may analyse the dynamics in these coordinates.

When $p > 1$, there are too many fixed points to be pulled apart by a Möbius transformation. The advantage of Assumption (a) is that one may quotient out the symmetry and reduce the situation to the case with two fixed points. Without this assumption, we will have to take a different approach. The key idea, developed by Douady, Epstein, Oudkerk and Shishikura (among others), is to approximate f by the time-one map of the flow of $\dot{z} = f(z) - z$, or of an appropriate polynomial differential equation. More precisely, a small perturbation f of f_0 has $p + 1$ fixed points close to 0, counting multiplicities. Let P_f be the monic polynomial of degree $p + 1$ which vanishes at those points. Then one can prove that the time-one map of the flow of $\dot{z} = P_f(z)$ gives a good approximation of short term and sometimes long term iterates of f; further, the complex time coordinates of the differential equation provide excellent approximate Fatou coordinates for f (see Lemma 5.1).

One may then expect to describe different types of perturbations in terms of the corresponding flows. Following [**DES**], we say that a maximal real-time solution $\psi :]t_{\min}, t_{\max}[\to \mathbb{C}$ for the polynomial differential equation $\dot{z} = P(z)$ is a *homoclinic connection* (at infinity) if $t_{\min}$ and $t_{\max}$ are finite and $\lim_{t \to t_{\min}} \psi(t) = \lim_{t \to t_{\max}} \psi(t) = \infty$. For $\alpha \in \left]0, \frac{\pi}{2}\right[$, the polynomial P is called *α-stable* if the differential equation of the rotated vector field, $\dot{z} = e^{i\theta} P(z)$, has no homoclinic connections, for any $\theta \in]-\alpha, \alpha[$.

Let $(f_n : D(r) \to \mathbb{C})_{n \geq 1}$ be a sequence of holomorphic maps converging locally uniformly to a holomorphic map $f_0 : D(r) \to \mathbb{C}$ with $f_0(z) = z + z^{p+1} + \mathcal{O}(z^{p+2})$, $p \geq 1$. We may now define that the convergence $f_n \to f_0$ is *stable at* 0 if for n sufficiently large, the corresponding monic polynomials P_{f_n} of degree $p + 1$ are α-stable for some uniform $\alpha \in \left]0, \frac{\pi}{2}\right[$. And more generally, an algebraically convergent sequence of rational maps $f_n \to f_0$ is *stable* if for each parabolic point of f_0, there are suitable local coordinates in which the convergence is stable.

From results in [**DES**] we know already two important properties of this concept:

• First, stable perturbations are "generic". For instance, in an analytically parameterized family

$$(f_\lambda : D(r) \to \mathbb{C})_{\lambda \in \mathbb{D}} \quad \text{with} \quad f_0(z) = z + z^{p+1} + \mathcal{O}(z^{p+2}),$$

there exists a finite set of directions (the implosive directions), such that the convergence $f_\lambda \to f_0$ is stable at 0 as soon as $\lambda \to 0$ avoiding a sector neighborhood of those directions. For example, in the family $(1 - \lambda)z + z^2$, there are exactly two implosive directions, namely $i\mathbb{R}^+$ and $i\mathbb{R}^-$.

• Second, stable convergence implies convergence of Julia sets. More precisely, let f_n be a sequence of rational maps converging algebraically to f_0, such that f_0 has neither Siegel disks nor Herman rings. If the convergence is stable then $J(f_n) \to J(f_0)$.

We may now state our main result in this paper:

Theorem A. *Assume f_0 is a geometrically finite rational map (i.e., every critical point in $J(f_0)$ has a finite forward orbit), and $f_n \to f_0$ algebraically, preserving the critical relations on J_{f_0} (i.e., for every critical point $b \in J(f_0)$ satisfying $f_0^{\circ i}(b) = f_0^{\circ j}(b)$, there are critical points $b_n \to b$ for f_n, with the same multiplicity as b, also satisfying $f_n^{\circ i}(b_n) = f_n^{\circ j}(b_n)$). If the convergence is stable, then for n large enough, f_n is geometrically finite, $J(f_n) \to J(f_0)$ and* H.dim $J(f_n) \to$ H.dim $J(f_0)$.

(We will provide a self-contained proof, namely independent of results in [**DES**].)

The reason why f_0 is assumed to be geometrically finite is that in that case, there is a unique non-atomic f_0-invariant conformal measure μ_{f_0} supported on $J(f_0)$, and the dimension of μ_{f_0} is equal to H.dim $J(f_0)$ (an f_0-invariant conformal measure of dimension $\delta > 0$ is a probability measure μ on $\mathbb{P}^1$ such that $\mu(f_0(E)) = \int_E |f_0'(x)|^\delta \, d\mu$ whenever $f_0|E$ is injective). This conformal measure is called the *canonical conformal measure*.

Once we know that for n large enough, f_n is geometrically finite and that $J(f_n) \to J(f_0)$, we see that any weak accumulation point ν of the canonical conformal measures μ_{f_n} is an f_0-invariant conformal measure and is supported on $J(f_0)$. In order to prove that

$$\text{H.dim } J(f_n) \to \text{H.dim } J(f_0),$$

it is therefore enough to show that ν is non-atomic.

The proofs of Bodart-Zinsmeister and of McMullen can be both decomposed into two parts: the first one is to use appropriate Fatou coordinates to establish the convergence of tails of Poincaré series (in the terminology of McMullen), and the second is to show that this intermediate convergence implies the non-atomicity of limits of the canonical conformal measures.

The second part, as is stated in [**McM**], is still valid in our more general setting. Our only task is to prove the first part. We will first prove an independent result which concerns only polynomial vector fields.

A maximal real-time solution $\psi(t)$ of $\dot{z} = P(z)$ (for a polynomial P), with defining interval of the form $]0, t_0[$ or $]-t_0, 0[$, and with $\lim_{t\to 0} \psi(t) = \infty$, is called a *separatrix*. We have $t_0 = \infty$ in case P is α-stable for some $\alpha > 0$. We have (a more precise version will be given in Corollary 2.10):

Proposition 1.1. *Assume $(P_n)_{n\geq 1}$ is a sequence of α-stable polynomials converging algebraically to the polynomial $P_0(z) = z^{p+1}$. Assume $\psi_n :]0, +\infty[\to \mathbb{C}$ is a separatrix of $\dot{z} = P_n(z)$. Then, for any $\eta > p/(p+1)$, we have $\displaystyle\int_t^{+\infty} |\psi_n'(u)|^\eta \, du \underset{t,n\to+\infty}{\longrightarrow} 0$.*

Remark. There is an analogous result for a sequence of separatrices defined on $]-\infty, 0[$.

The main tool in proving this is to take Hausdorff limits of closures of invariant trajectories, to use appropriate renormalizations to get normal families (an idea of C. Petersen [**P**], see also [**PT**, **T**] for other applications), and to apply inequalities from hyperbolic geometry.

Once this is done, we will establish a lemma connecting the flow of $\dot{z} = P(z)$ to the iteration of $z + P(z)(1 + s(z))$ with $s(z)$ small (various forms of the lemma can be found in [**DES**, **E**, **O**]). We then use hyperbolic geometry and bounded Koebe distortion to translate Proposition 1.1 into a control of tails of Poincaré series, establishing thus their convergence and consequently Theorem A.

Most of our intermediate results will in fact require only partial stability of a polynomial vector field, and provide partial stability of the flow as well as of the dynamics.

The paper is organized as follows: In §2.1 we define stable polynomial vector fields and restate Proposition 1.1. Its proof is completed in §4. In §2.2 we define stable convergence of iterated maps. §3 contains criteria of stability of polynomial vector fields. §5.1 contains the linking lemma from the discrete dynamics to the time-one map of the flow of some polynomial vector fields. In §5.2 we prove that a stable perturbation of $z + z^{p+1}$ is well approximated by the corresponding time-one maps. We then prove in §§6.2-6.3 the absence of implosion and the continuity of Julia sets, and in §6.4 the uniform convergence of tails of Poincaré series. In §7 we recall known results about conformal measures and their relations to Hausdorff dimension of Julia sets, and prove Theorem A together with its corollary.

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2. Definitions and statements

2.1. Stable polynomial differential equations. Let $P : \mathbb{C} \to \mathbb{C}$ be a complex polynomial. Consider the holomorphic vector field $P(z)\cdot\vec{1}$. It

is associated to an autonomous ordinary differential equation $\dot{z} = P(z)$ and a meromorphic 1-form $\frac{dz}{P(z)}$. The following lemma and definition single out certain particular solutions of the differential equation, which will play important roles in the sequel of our study.

Lemma 2.1 (coordinates at ∞)**.** *Assume $p \geq 1$ and $P : z \mapsto Az^{p+1} + \mathcal{O}(z^p)$ is a polynomial of degree $p+1$. Then, the anti-derivative $\Phi_P(z) = \int_\infty^z \frac{du}{P(u)}$ is well defined and holomorphic in a neighborhood of ∞. As $z \to \infty$, $\Phi_P(z) \sim -1/(pAz^p)$. If $P_n \to P_0$ algebraically, then $\Phi_{P_n} \to \Phi_{P_0}$ uniformly in some neighborhood of ∞.*

The proof is elementary. See [**DES**, **E**] for details. Any local inverse Ψ of Φ_P satisfies $\Psi'(w) = P(\Psi(w))$, thus is a solution of the equation $\frac{dz}{dw} = P(z)$, with w complex. It follows that $\dot{z} = P(z)$ has exactly p germs of forward real-time trajectories with initial point ∞ (i.e., solutions $\gamma :]0, \varepsilon[\to \mathbb{C}$ such that $\gamma(t) \to \infty$ as $t \searrow 0$), and p germs of backward real-time trajectories with initial point ∞ (i.e., solutions $\gamma :]-\varepsilon, 0[\to \mathbb{C}$ such that $\gamma(t) \to \infty$ as $t \nearrow 0$). We call them **outgoing**, respectively **incoming**, ∞*-germs*. See Figure 1.

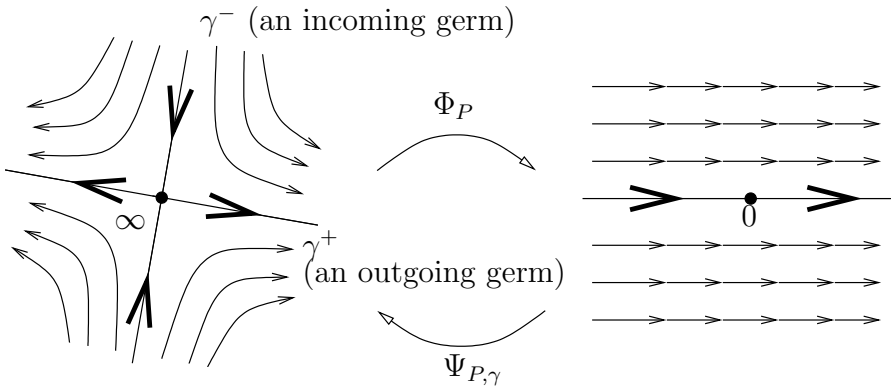

Figure 1. ∞-germs of a polynomial differential equation (with $p = 2$).

When P is monic of degree $p + 1$, there is a natural numeration of these germs by $\{\gamma_k, k \in \mathbb{Z}/2p\mathbb{Z}\}$, so that γ_k is tangential to $e^{2\pi i \frac{k}{2p}} \cdot \mathbb{R}^+$ at ∞. The germ γ_k is of outgoing (resp. incoming) type if k is odd (resp. even).

Definition 2.2. For a (polynomial,∞-germ) pair (P, γ), define $\Psi_{P,\gamma}$ to be the inverse branch of Φ_P in a sector neighborhood of 0 as follows:

- for γ^+ an outgoing ∞-germ, Ψ_{P,γ^+} is defined on $D(\varepsilon) \setminus \mathbb{R}^-$ and coincides with γ^+ on $]0, \varepsilon[$;

- for γ^- an incoming ∞-germ, Ψ_{P,γ^-} is defined on $D(\varepsilon) \setminus \mathbb{R}^+$ and coincides with γ^- on $]-\varepsilon, 0[$.

Example 0. Let $P_0(z) = z^{p+1}$. We have $\Phi_{P_0}(z) = -\dfrac{1}{pz^p} = w$ and $\Psi_{P_0,\gamma_k}(w) = \dfrac{1}{(-pw)^{\frac{1}{p}}}$, where the p-th root is chosen so that

- if k is odd, Ψ_{P_0,γ_k} extends analytically to $\mathbb{C}\setminus\mathbb{R}^-$ with $\Psi_{P_0,\gamma_k}(\mathbb{R}^+) = e^{2\pi i \frac{k}{2p}} \cdot \mathbb{R}^+$;
- if k is even, Ψ_{P_0,γ_k} extends analytically to $\mathbb{C} \setminus \mathbb{R}^+$ with $\Psi_{P_0,\gamma_k}(\mathbb{R}^-) = e^{2\pi i \frac{k}{2p}} \cdot \mathbb{R}^+$.

These local solutions have analytic extensions as more global solutions of the differential equation $\dot{z} = P(z)$. In this article, we are mainly interested by the solutions with real-time. In that case, the maximal solutions are defined on real open intervals of the form $]t_{\min}, t_{\max}[$. We do need however complex-time solutions on suitable neighborhoods of $]t_{\min}, t_{\max}[$ in order to control perturbed trajectories.

Definition 2.3 (separatrices and homoclinic connections)**.** For a polynomial differential equation $\dot{z} = P(z)$, a *trajectory* (or an *orbit*) is a maximal solution $\psi :]t_{\min}, t_{\max}[\to \mathbb{C}$. The maximal solution of an ∞-germ γ is called the γ-*separatrix*. A *homoclinic connection* is a maximal solution $\psi(t)$ with $|t_{\min}|, |t_{\max}| < \infty$ and with $\lim_{t\to t_{\min},t_{\max}} \psi(t) = \infty$.

We now come to the definition of α-stability. For $\alpha \in \left]0, \frac{\pi}{2}\right[$, let us define a sector neighborhood of $\mathbb{R}^{\pm}$ by

$$S^+(\alpha) = \{w \in \mathbb{C}^*;\ |\arg(w)| < \alpha\}$$

and

$$S^-(\alpha) = \{w \in \mathbb{C}^*;\ |\pi - \arg(w)| < \alpha\}.$$

When there is no possible confusion, we will use the notation $S(\alpha)$ instead of $S^+(\alpha)$ or $S^-(\alpha)$. The following is to be compared with the notion of *tolerant angles* in [**DES**]:

Definition 2.4 (α-stability)**.** Given a polynomial P and an ∞-germ γ, we say that P is γ-*implosive* if the γ-separatrix is a homoclinic connection.

For $\alpha \in \left]0, \frac{\pi}{2}\right[$, we say that P is (α, γ)-*stable* if $\Psi_{P,\gamma}$ extends holomorphically to the entire sector $S^+(\alpha)$ (if γ is an outgoing germ), or $S^-(\alpha)$ (if γ is an incoming germ). We will denote by $\Psi_{P,\gamma} : S^{\pm}(\alpha) \to \mathbb{C}$ this extension.

We say that P is (globally) α-*stable* if it is (α, γ)-stable for all ∞-germs γ.

It is proved in [**DES**] that P is not γ-implosive implies that it is (α, γ)-stable for some $\alpha > 0$ (see also Proposition 3.2 below). Note that when P is (α, γ)-stable, the γ-separatrix coincides with $\Psi_{P,\gamma}(S(\alpha) \cap \mathbb{R})$ and is not a homoclinic connection, and the set $\Psi_{P,\gamma}(S(\alpha))$ may be considered as a protecting neighborhood of it. Criteria of α-stability will be given in §3.2. For instance it is enough to require P to be (α, γ)-stable for every outgoing germ (or every incoming germ).

Example 1. If $P_\lambda(z) = z(z^p - \lambda)$, we have

$$\Phi_{P_\lambda}(z) = \frac{1}{p\lambda} \log\left(1 - \frac{\lambda}{z^p}\right),$$

with formal inverse

$$\Psi_{P_\lambda}(w) = \left(\frac{\lambda}{1 - e^{p\lambda w}}\right)^{\frac{1}{p}}.$$

No matter which p-th root we take, the map Ψ_{P_λ} has singularities at $\{w = \frac{2m\pi i}{p\lambda}, m \in \mathbb{Z}\}$. If $\lambda \notin i\mathbb{R}$, the polynomial P_λ is α-stable, as soon as $0 < \alpha < \min\{|\arg(i/\lambda)|, |\pi - \arg(i/\lambda)|\}$, where $\arg(i/\lambda) \in]-\pi/2, 3\pi/2[$. However, if $\lambda \in i\mathbb{R}$, P_λ is γ-implosive for every ∞-germ γ.

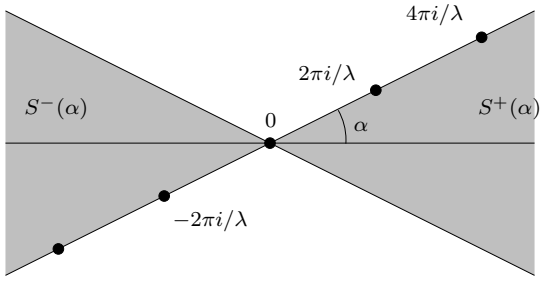

Figure 2. α-stability for $P_\lambda(z) = z(z - \lambda)$.

Example 2. If $P_\lambda(z) = z(z - \lambda)(z - 2\lambda)$, we have

$$p = 2, \quad \Phi_{P_\lambda}(z) = \frac{1}{2\lambda^2} \log \frac{1 - 2\lambda/z}{(1 - \lambda/z)^2}, \quad \Psi_{P_\lambda}(w) = \lambda \pm \left(\frac{\lambda^2}{1 - e^{2\lambda^2 w}}\right)^{1/2}.$$

The singularities of Ψ_{P_λ} are at $\{w = \frac{m\pi i}{\lambda^2}, m \in \mathbb{Z}\}$. If $\lambda^2 \in i\mathbb{R}$, P_λ is γ-implosive for every germ γ.

We now state a list of results regarding α-stable polynomial vector fields (the first of them is contained in [**DES**], but with a different proof). We will provide self-contained proofs (independent of [**DES**]) in §4.

Proposition 2.5 ([**DES**])**.** *Assume that a polynomial P is (α', γ)-stable for some ∞-germ γ and some $0 < \alpha' < \frac{\pi}{2}$. Then*

$$\Psi_{P,\gamma}(S(\alpha')) \subset \mathbb{C} \setminus \{\textit{zeros of } P\}$$

and there exists a zero $a_{P,\gamma}$ of P such that $\Psi_{P,\gamma}(w) \to a_{P,\gamma}$ uniformly as $w \to \infty$ within any sector $S(\alpha)$ with $\alpha < \alpha'$.

In particular, the γ-separatrix starts at ∞ and lands at $a_{P,\gamma}$ (called the landing point of the γ-separatrix).

Note that due to the continuity of Φ_P with respect to P near ∞ (Lemma 2.1), an algebraic convergence of polynomials $P_n \to P_0$ induces necessarily the uniform and bijective convergence of the set of ∞-germs of P_n (restricted to some common time-interval) to those of P_0. Thus it makes sense to talk about the convergence of pairs $(P_n, \gamma_n) \to (P_0, \gamma_0)$. In this case, γ_0 and all γ_n for n sufficiently large, are of the same (outgoing or incoming) type.

Proposition 2.6. *Let (P_n, γ_n) be a sequence of pairs (polynomial, ∞-germ) converging to a pair (P_0, γ_0). Assume P_n are (α', γ_n) stable for some common $\alpha' \in \left]0, \frac{\pi}{2}\right[$. Then,*

- *P_0 is (α', γ_0)-stable and*
- *for $\alpha < \alpha'$, we have $\Psi_{P_n,\gamma_n} \to \Psi_{P_0,\gamma_0}$ uniformly (not just locally uniformly) on $S(\alpha)$.*

In particular, $a_{P_n,\gamma_n} \to a_{P_0,\gamma_0}$.

Definition 2.7 (η-length of separatrices)**.** Let P be a polynomial and γ be an ∞-germ such that P is not γ-implosive. If γ is an outgoing (resp. incoming) ∞-germ, let $\psi :]0, +\infty[\to \mathbb{C}$ (resp. $\psi :]-\infty, 0[\to \mathbb{C}$) be the γ-separatrix. For $\eta > 0$ and $t > 0$, set

$$\ell_\eta(P, \gamma, t) = \int_t^{+\infty} |\psi'(u)|^\eta \, du \quad (\text{resp.} \quad \ell_\eta(P, \gamma, t) = \int_{-\infty}^{-t} |\psi'(u)|^\eta \, du).$$

They should be considered as the η-dimensional length of a portion of the underlying separatrix.

Proposition 2.8. *Assume P is a polynomial of degree $p + 1$ which is (α, γ)-stable for some ∞-germ γ and some $0 < \alpha < \frac{\pi}{2}$. Then, for all $\eta > p/(p+1)$ and all $t > 0$, the η-length $\ell_\eta(P, \gamma, t)$ is finite. It decreases with respect to t and tends to 0 as t tends to $+\infty$.*

Our main result about flows of polynomial vector fields is the following.

Proposition 2.9 (length stability of separatrices)**.** *Let (P_n, γ_n) be a sequence of pairs (polynomial, ∞-germ) converging to a pair (P_0, γ_0). Assume there is an $\alpha > 0$ independent of n such that P_n are (α, γ_n)-stable. Finally, assume $t_n \to t_0 > 0$ as $n \to \infty$. Then for $\eta > p/(p+1)$, we have $\ell_\eta(P_n, \gamma_n, t_n) \to \ell_\eta(P_0, \gamma_0, t_0) \quad as \quad n \to \infty$.*

Corollary 2.10. *Let (P_n, γ_n) be a sequence of pairs (polynomial, ∞-germ) converging to a pair (P_0, γ_0) with $P_0(z) = z^{p+1}$ and assume that the P_n are (α, γ_n)-stable for some common $\alpha \in \left]0, \frac{\pi}{2}\right[$ and all n sufficiently large. Then, for $\eta > p/(p+1)$ and for all $\varepsilon > 0$, there exist t_0 and n_0 such that for all $t \geq t_0$ and all $n \geq n_0$, we have $\ell_\eta(P_n, \gamma_n, t) < \varepsilon$.*

2.2. Stable convergence for analytic maps. Assume the sequence $(f_n : D(r) \to \mathbb{C})_{n\geq 1}$ converges locally uniformly to $f_0 : D(r) \to \mathbb{C}$ with $f_0(z) = z + z^{p+1} + \mathcal{O}(z^{p+2})$, $p \geq 1$. As $n \to +\infty$, f_n has $p+1$ fixed points, counting multiplicities, converging to 0. Let P_n be the monic polynomials of degree $p+1$ which vanish at those $p+1$ fixed points of f_n.

Let us now fix $k \in \mathbb{Z}/2p\mathbb{Z}$ odd and let $K \subset D(r)$ be a compact set such that for all $z \in K$,

- $f_0^{\circ j}(z)$ is defined for all $j \geq 0$ and $f_0^{\circ j}(z) \underset{j\to+\infty}{\overset{\neq}{\longrightarrow}} 0$ tangentially to the direction $e^{2i\pi\frac{k}{2p}}$.

Let γ_n be the ∞-germ tangent to $\mathbb{R}^+ \cdot e^{2i\pi\frac{k}{2p}}$ at ∞. Assume there is an $\alpha \in \left]0, \frac{\pi}{2}\right[$ such that P_n is (α, γ_n)-stable for all sufficiently large n. By Proposition 2.5, we know that the γ_n-separatrix lands at a fixed point a_n of f_n. The following result is essentially contained in [**DES**]. We will reprove it in §4.

Proposition 2.11. *Under the previous assumptions, for n large enough, K is contained in the basin of attraction of a_n. In other words, for all $z \in K$,*

- *$f_n^{\circ j}(z)$ is defined for all $j \geq 0$ and $f_n^{\circ j}(z) \underset{j\to+\infty}{\overset{\neq}{\longrightarrow}} a_n$.*

The next result, concerning tails of Poincaré series, can be considered as a discrete version of Corollary 2.10. Replacing f_0 with f_0^{-1} and f_n with f_n^{-1}, we obtain similar results with k odd replaced by k even and forward iterations replaced by backward iterations.

Proposition 2.12. *Under the previous assumptions, if $\delta_0 > p/(p+1)$ and $\varepsilon > 0$, there exist m_0 and n_0 such that for all $z \in K$, all $\delta \in [\delta_0, 2]$, all $m \geq m_0$ and all $n \geq n_0$, we have*

$$S_\delta(f_n, z, m) := \sum_{j=m}^{+\infty} |(f_n^{\circ j})'(z)|^\delta < \varepsilon.$$

Definition 2.13 (stable convergence)**.** We say that the convergence $f_n \to f_0$ is *stable at* 0 if there is an $\alpha \in \left]0, \frac{\pi}{2}\right[$ such that for n large enough, the polynomials P_n are α-stable.

We will prove that this notion is invariant under coordinate changes, but with probably a different α (see Lemma 4.2).

Example 1 (continued)**.** Set $f_\lambda(z) = z + z(z^p - \lambda)(1 + s_\lambda(z))$, with $(\lambda, z) \mapsto s_\lambda(z)$ holomorphic satisfying $s_\lambda(0) = 0$. Then the convergence $f_{\lambda_n} \to f_0$ is stable at 0 if and only if there is an $\alpha \in \left]0, \frac{\pi}{2}\right[$ such that the polynomials $P_{\lambda_n}(z) := z(z^p - \lambda_n)$ are α-stable for n large enough, if and only if $\lambda_n \to 0$ avoiding a sector neighborhood of $i\mathbb{R}^+ \cup i\mathbb{R}^-$.

Definition 2.14. If $(f_n : \mathbb{P}^1 \to \mathbb{P}^1)_{n\geq 1}$ is a sequence of rational maps converging algebraically to a rational map $f_0 : \mathbb{P}^1 \to \mathbb{P}^1$ and if β is a parabolic point of f_0, we say that the convergence is *stable at* β if we can find a local coordinate sending β_0 to 0 such that $f_0^{\circ j}(z) = z + z^{p+1} + \mathcal{O}(z^{p+2})$ and such that the convergence $f_n^{\circ j} \to f_0^{\circ j}$ is stable at 0 (j is the least integer such that $f_0^{\circ j}$ fixes β with multiplier 1 and p is the number of petals at β).

We say that the convergence $f_n \to f_0$ is *stable* if it is stable at all the parabolic points of f_0.

Theorem A claims essentially that modulo technical assumptions stable convergence implies convergence of dimensions. Combining Propositions 2.11 and 2.12 to results concerning Poincaré series and conformal measures in [**McM**], we obtain easily Theorem A.

Definition 2.15. We write $(f_n, \beta_n) \to (f_0, \beta_0)$ if (1) β_n (respectively β_0) is a fixed point of f_n (respectively f_0), (2) $\beta_n \to \beta_0$ and (3) $f_n \to f_0$ uniformly in some neighborhood of β_0.

Definition 2.16 (following [**McM**])**.** Assume f_0 has a multiple fixed point at β_0 with p petals. We say that the convergence $(f_n, \beta_n) \to (f_0, \beta_0)$ is *dominant* if there exists an M such that

$$|f_n^{(i)}(\beta_n)| \leq M|f_n'(\beta_n) - 1| \quad \text{for } 1 < i < p + 1.$$

Furthermore, we say that the convergence $(f_n, \beta_n) \to (f_0, \beta_0)$ is *radial* if in addition $f_n'(\beta_n) - 1$ tends to 0 avoiding a sector neighborhood of $i\mathbb{R}^+ \cup i\mathbb{R}^-$.

We will show in Lemma 3.12 that radial convergence implies stable convergence. Therefore, our Theorem A recovers as a corollary the following:

Theorem (McMullen, [**McM**])**.** *Assume f_0 is a geometrically finite rational map and $f_n \to f_0$ algebraically, preserving critical relations. For each parabolic point $\beta_0 \in J(f_0)$, let j be the least integer such that $f_0^{\circ j}$ fixes β_0 with multiplier 1 and assume $f_n^{\circ j}$ has a fixed point β_n converging to β_0 such that $(f_n^{\circ j}, \beta_n) \to (f_0^{\circ j}, \beta_0)$ radially. Then, $J(f_n) \to J(f_0)$ and* H.dim $J(f_n) \to$ H.dim $J(f_0)$.

Combining criteria of α-stability given in [**DES**] (see Proposition 3.11 below) with Theorem A, we get another important corollary.

Corollary 2.17. *Assume $\{f_\lambda\}_{\lambda\in\mathbb{D}}$ is an analytic family of rational maps such that f_0 is geometrically finite and such that the critical orbit relations in $J(f_0)$ are persistent. Then, there exists a set $L \subset \mathbb{D}$ composed of finitely many rays $e^{i\theta}\cdot]0,1[$ (called implosive directions), such that if $\lambda_n \to 0$ avoiding a sector neighborhood of the set L, then for n large enough, f_{λ_n} is geometrically finite, $J(f_{\lambda_n}) \to J(f_0)$ and* $\text{H.dim } J(f_{\lambda_n}) \to \text{H.dim } J(f_0)$.

Remark. The case $f_\lambda(z) = (1-\lambda)z + z^2$ is proved by Bodart-Zinsmeister [**BZ**], 1996. In this case the implosive directions correspond to $\arg(\lambda) = \pm\frac{\pi}{2}$.

Theorem A and Corollary 2.17 will be proved in §7.

3. Polynomial differential equations

In §3.1 and §3.2, we recall results mostly contained [**DES**], together with some easy consequences. In §3.3 we connect radial convergence to stable convergence.

3.1. Time criterion for a separatrix. Given a polynomial $P : \mathbb{C} \to \mathbb{C}$ and a point $z_0 \in \mathbb{C}$, we will denote by Ψ_{P,z_0} the solution of the differential equation $\dot z = P(z)$ such that $\Psi_{P,z_0}(0) = z_0$. By uniqueness of solutions, two such solutions coincide in a neighborhood of 0, and thus define the same germ at 0.

Lemma 3.1. *Let Q be a holomorphic germ.*

a) *Let $\psi(w)$ be a non-constant holomorphic solution of the equation $\dot z = Q(z)$, defined on a disc $D(R)$ with $R < \infty$. Then $\overline{\psi(D(R))}$ does not meet the zeros of Q.*

b) *Let $\psi(t)$ be a non-constant real-time solution of the equation $\dot z = Q(z)$, defined (at least) on a bounded interval $]a,b[$. Then $\overline{\psi(]a,b[)}$ does not meet the zeros of Q.*

Proof. a) We may assume by contradiction that $Q(0) = 0$ and $0 \in \overline{\psi(D(R))}$. Choose a sequence $w_n \in D(R)$ tending to w' such that $\psi(w_n) \to 0$. However, using the local study in [**DES**] on the various (sink, source or multiple) types of zeros of Q, one deduces easily that for n sufficiently large, ψ has an analytic extension to $w_n + D(1)$ with $\psi(w_n + D(1))$ avoiding the zeros of Q. In particular ψ has a continuous extension at w' with $\psi(w') \neq 0$. This is a contradiction.

The proof of b) is similar. q.e.d.

Proposition 3.2. *Assume P is a polynomial and $\psi :]t_{\min}, t_{\max}[$ is a maximal solution of $\dot z = P(z)$. Then,*

(a) *$\psi(t) \to \infty$ as $t \searrow t_{\min}$ (respectively as $t \nearrow t_{\max}$) if and only if $t_{\min} > -\infty$ (respectively $t_{\max} < +\infty$). In that case, $t \mapsto \psi(t+t_{\min})$*

(*respectively* $t \mapsto \psi(t+t_{\max})$) *is the* γ*-separatrix for some outgoing* (*respectively incoming*) *germ* γ.

(b) *if* $t_{\min} = -\infty$ (*respectively* $t_{\max} = +\infty$) *then either*
- ψ *is periodic and the trajectory* $\psi(\mathbb{R})$ *is a topological circle, or*
- $\psi(t)$ *tends to a zero of* P *as* $t \to -\infty$ (*respectively as* $t \to +\infty$).

(c) *If* P *is not* γ*-implosive for some* ∞*-germ* γ*, then* P *is* (α, γ)*-stable for some* $\alpha > 0$.

Please refer to [**DES**] for a proof. We do not need this result in the proof of Theorem A.

Corollary 3.3. *A maximal solution* $\psi :]t_{\min}, t_{\max}[$ *of* $\dot{z} = P(z)$ *is a homoclinic connection iff both* $t_{\min}$ *and* $t_{\max}$ *are finite, iff it contains an incoming* ∞*-germ on the one end and an outgoing* ∞*-germ on the other end.*

An example of homoclinic connection is provided by the real axis for any real polynomial which does not vanish on $\mathbb{R}$.

3.2. Criteria of α-stability of polynomial vector fields (following [DES]). Recall that under the change of variables $z = h(u)$, the equation $\dot{z} = P(z)$ is transformed into the equation $\dot{u} = P \circ h(u)/h'(u)$. The proofs of the following sequence of lemmas are fairly elementary and can be easily supplied by the reader. Details are to be found in [**DES**].

Lemma 3.4 (affine conjugacies)**.** *Assume* P *and* Q *are related by* $Q(u) = P(au + b)/a$ *with* $a \in \mathbb{C}^*$ *and* $b \in \mathbb{C}$. *Then,* P *is* α*-stable if and only if* Q *is* α*-stable. In other words, affine conjugacies preserve* α*-stability.*

Lemma 3.5 (semi-conjugacies)**.** *Assume* P *and* Q *are two polynomials which vanish at* 0 *and are related by* $Q(u) = \dfrac{1}{mu^{m-1}} P(u^m)$ *for some integer* $m \geq 2$. *Then,* P *is* α*-stable if and only if* Q *is* α*-stable. In other words, semi-conjugacies* $u \mapsto u^m$ *preserve* α*-stability.*

Lemma 3.6. *If* $\lambda \in \mathbb{C}^*$ *and if* $\psi(w)$ *is a complex solution of* $\dot{z} = P(z)$ *then* $\psi(\lambda w)$ *is a complex solution of the vector field* $\dot{z} = \lambda P(z)$.

In particular, if $k \in \mathbb{R}^*$, the trajectories of the vector field $\dot{z} = kP(z)$ and $\dot{z} = P(z)$ are the same up to re-parameterization. It follows that P is α-stable if and only if kP is α-stable.

In addition, if $\psi(w)$ is a solution of $\dot{z} = P(z)$, then $\psi(e^{i\theta}w)$ is a solution of the differential equation of the rotated vector field, $\dot{z} = e^{i\theta}P(z)$.

Lemma 3.7. *Assume* $\alpha \in \left]0, \frac{\pi}{2}\right[$. *Then,* P *is* α*-stable if and only if* $\dot{z} = e^{i\theta}P(z)$ *does not have homoclinic connections for* $\theta \in]-\alpha, \alpha[$.

We now come to a characterization of stability in terms of residues of $1/P$:

Lemma 3.8. *Assume P has a homoclinic connection $\psi :]t_{\min}, t_{\max}[\to \mathbb{C}$. Then, there is a subset X of the set of zeros of P such that*

$$2\pi i \sum_{x\in X} \operatorname{Res}\left(\frac{1}{P}, x\right) = t_{\max} - t_{\min} \in \mathbb{R}^+.$$

Lemma 3.9. *Assume $\alpha \in \left]0, \frac{\pi}{2}\right[$ and P is not α-stable. Then, there exists $R > 0$, $\theta \in]-\alpha, \alpha[$ and a subset X of the set of zeros of P such that the equation $\dot{z} = e^{i\theta}P(z)$ has a homoclinic connection $\psi :]0, R[\to \mathbb{C}$, and $2\pi i \sum_{x\in X} \operatorname{Res}\left(\frac{1}{P}, x\right) = e^{i\theta}R \in S^+(\alpha)$.*

Consequently, if $2\pi i \sum_{x\in X} \operatorname{Res}\left(\frac{1}{P}, x\right) \notin S^+(\alpha)$ for any subset X of zeros of P, then P is α-stable.

Lemma 3.10. *If P is (α, γ)-stable for every outgoing ∞-germ γ, then P is α-stable. Idem if we replace outgoing germs by incoming germs.*

We will now give a result in a slightly more general form than the original result of [**DES**] (for instance we allow P_λ to have multiple zeros), but with essentially the same proof.

Proposition 3.11 ([**DES**])**.** *Assume $(P_\lambda)_{\lambda\in\mathbb{D}}$ is an analytic family of polynomials of degree $p+1$ such that $P_0(z) = z^{p+1}$. Then, there exists a set $L \subset \mathbb{D}$ composed of finitely many rays $e^{i\theta}\cdot]0, 1[$ such that for every closed sector S avoiding L, there exists $\alpha > 0$ and $\varepsilon > 0$ such that for all $\lambda \in S \cap D(\varepsilon)$, P_λ is α-stable.*

Proof. Without loss of generality, re-parameterizing by $\lambda^{1/n}$ for some integer n if necessary, we may assume that we can follow holomorphically all the zeros of P_λ in a neighborhood of $\lambda = 0$. In that case, we can follow holomorphically all the possible sums of residues of $1/P_\lambda$ in some punctured disc $\mathbb{D}^*_\varepsilon$. There are only finitely many such sums. We denote by $(\sigma_j : \mathbb{D}^*_\varepsilon \to \mathbb{C})_{j\in J}$ those sums of residues. The functions σ_j extend meromorphically at 0. Some of them might be constant. We denote by J^* the set of indices for which σ_j is not constant. And we choose $T_0 > \max\{|2\pi\sigma_j|,\ \sigma_j \text{ is constant}\}$.

We claim that taking ε smaller if necessary, we may assume that for all $\lambda \in \mathbb{D}_\varepsilon$, and all $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, all the separatrices of $\dot{z} = e^{i\theta}P_\lambda(z)$ with initial point at ∞ are defined for a time larger than this T_0. It is enough to show that for λ close enough to 0, Φ_{P_λ} is a ramified covering above the disk $D(T_0)$, ramified only above 0. This easily follows from the fact that $\Phi_{P_\lambda} \to \Phi_{P_0}$ as $\lambda \to 0$ and that $\Phi_{P_0}(z) = -1/pz^p$.

Let us now fix $\alpha > 0$ and $\lambda \in \mathbb{D}^*_\varepsilon$ and assume P_λ is not α-stable. Then by Lemma 3.9 there are $j \in J$, $R > 0$ and $|\theta'| < \alpha$ such that $Re^{i\theta'} = 2\pi i\sigma_j(\lambda)$ and $\psi :]0, R[\to \mathbb{C}$ is a homoclinic connection for the differential equation of the rotated vector field, $\dot{z} = e^{i\theta}P(z)$.

By assumption, $R > T_0$. Hence σ_j is not constant and $j \in J^*$. The proposition follows easily by choosing for L the union of all the rays in $\mathbb{D}^*$ which are tangent at 0 to a connected component of the curve $\{\lambda \mid 2\pi i\sigma_j(\lambda) \in \mathbb{R}^+\}$ for some $j \in J^*$. Indeed, if S is a closed sector avoiding L, then we can find $\alpha > 0$ and $\varepsilon > 0$ such that for all $j \in J^*$ and all $\lambda \in D^*_\varepsilon \cap S$, $|\arg(2\pi i\sigma_j(\lambda))| > \alpha$. It follows from the above discussion that the polynomial P_λ is α-stable. q.e.d.

3.3. Radial convergence implies stable convergence.

Lemma 3.12. *Assume f_0 has a multiple fixed point with p petals at β_0 and assume $(f_n, \beta_n) \to (f_0, \beta_0)$ radially. Then, there exists a local coordinate sending β_0 to 0 such that $f_0(z) = z + z^{p+1} + \mathcal{O}(z^{p+2})$ and such that the convergence $f_n \to f_0$ is stable at 0.*

Proof. Using an affine change of coordinate, we may assume that $\beta_0 = 0$ and that $f_0(z) = z + z^{p+1} + \mathcal{O}(z^{p+2})$. We assume that the convergence $(f_n, \beta_n) \to (f_0, 0)$ is radial (this property is clearly preserved by affine changes of coordinates). We will show that for n sufficiently large, the monic polynomials P_{f_n} of degree $p+1$ which vanish at the $p+1$ fixed points of f_n close to 0 are α-stable for some uniform α.

We can write $f_0(z) = z + z^{p+1}(1 + s_0(z))$ and $f_n(z) = z + P_{f_n}(z)(1 + s_n(z))$ with $s_0(0) = 0$ and $s_n(z) \to s_0(z)$. Let x_n be a zero of P_{f_n}, i.e., a fixed point of f_n. Then, denoting $A_n \underset{n\to+\infty}{\sim} B_n$ if $A_n = B_nC_n$ and $\lim_{n\to\infty} C_n = 1$,

$$\operatorname{Res}\left(\frac{1}{P_{f_n}(z)}, x_n\right) = \operatorname{Res}\left(\frac{1 + s_n(z)}{f_n(z) - z}, x_n\right) \underset{n\to+\infty}{\sim} \operatorname{Res}\left(\frac{1}{f_n(z) - z}, x_n\right).$$

Indeed, $1 + s_n(x_n) \to 1 + s_0(0) = 1$. We will prove below that

$$\begin{cases} \operatorname{Res}\left(\frac{1}{f_n(z)-z}, x_n\right) = \frac{1}{f_n'(\beta_n)-1} & \text{if } x_n = \beta_n \\ \operatorname{Res}\left(\frac{1}{f_n(z)-z}, x_n\right) \underset{n\to+\infty}{\sim} \frac{1}{p(1-f_n'(\beta_n))} & \text{if } x_n \neq \beta_n \end{cases} \tag{3.1}$$

It follows immediately that when $1 - f_n'(\beta_n) \to 0$ avoiding a sector neighborhood of the imaginary axis, there exists an $\alpha \in \left]0, \frac{\pi}{2}\right[$ such that for all n sufficiently large and for any subset X_n of the set of zeros of P_{f_n}, we have $2i\pi \sum_{x\in X_n} \operatorname{Res}\left(\frac{1}{P_{f_n}(z)}, x\right) \notin S^+(\alpha)$. It follows from Lemma 3.9 that P_{f_n} is α-stable for n large enough.

Let us now prove (3.1) . Set $a_n = f_n'(\beta_n)$. Part $x_n = \beta_n$ is trivial. Assume now $x_n \neq \beta_n$. According to McMullen ([**McM**] Proposition

7.2), and taking a subsequence if necessary, we can find maps $\phi_n \to \phi_0$ univalent in a common neighborhood of 0, sending β_n to 0 and such that the maps $g_n = \phi_n \circ f_n \circ \phi_n^{-1}$ are of the form $g_n(z) = a_n z + z^{p+1} + \mathcal{O}(z^{p+2})$. If $y_n \neq 0$ is a fixed point of g_n, then $y_n = g_n(y_n) = a_n y_n + y_n^{p+1} + \mathcal{O}(y_n^{p+2})$, so that $y_n^p \sim (1 - a_n)$, and $g_n'(y_n) - 1 = a_n - 1 + (p+1)y_n^p + \mathcal{O}(y_n^{p+1}) \sim p(1 - a_n)$. The fixed points of g_n are simple, so are those of f_n and

$$\operatorname{Res}\left(\frac{1}{f_n(z) - z}, x_n\right) = \frac{1}{f_n'(x_n) - 1} = \frac{1}{g_n'(\phi_n(x_n)) - 1} \underset{n \to +\infty}{\sim} \frac{1}{p(1 - a_n)}.$$

q.e.d.

3.4. Lifting via $z = u^m$. The following lemma will not be used before the end of §7. Recall that the notation $(f_n, 0) \to (f_0, 0)$ means that f_n (respectively f_0) is holomorphic in a neighborhood of 0 and fixes 0 and $f_n \to f_0$ uniformly in a neighborhood of 0.

Lemma 3.13. *Assume $(f_n, 0) \to (f_0, 0)$ and $(g_n, 0) \to (g_0, 0)$ with $g_n(z)^m = f_n(z^m)$. If the convergence $f_n \to f_0$ is stable, then the convergence $g_n \to g_0$ is stable.*

Proof. Without loss of generality, conjugating with a scaling map if necessary, we may assume that $f_0(z) = z + z^{p+1} + \mathcal{O}(z^{p+2})$. Set $z = u^m$. Then, $g_0(u) = u + \frac{1}{m}u^{p+1} + \mathcal{O}(u^{p+2})$. Let us work in the coordinate $v = u/m^{1/p}$. Then, g_n is conjugate to h_n and $h_n \to h_0$ with $h_0(v) = v + v^{p+1} + \mathcal{O}(v^{p+2})$. Let P_n (respectively Q_n) be the monic polynomials which vanish at the fixed points of f_n (respectively h_n) close to 0. We assume P_n are α-stable for some uniform α. One easily checks that

$$Q_n(v) = \frac{P_n(m^{m/p}v^m)}{m^{m(p+1)/p}v^{m-1}} = \frac{1}{m^{m-1}} \cdot \frac{P_n(m^{m/p}v^m)}{m^{m/p} \cdot mv^{m-1}} =: \frac{1}{m^{m-1}} \cdot R_n(v).$$

It follows easily from lemmas 3.4 and 3.5 that $R_n(v) = \frac{P_n(m^{m/p}v^m)}{m^{m/p} \cdot mv^{m-1}}$ is α-stable. Since for all θ, $e^{i\theta}Q_n$ is a real multiple of $e^{i\theta}R_n$, we do not change the trajectories, and $e^{i\theta}Q_n$ has a homoclinic connection if and only if $e^{i\theta}R_n$ has a homoclinic connection. Thus, Q_n is α-stable. q.e.d.

4. (α, γ)-stability and length of the γ-separatrix

In this section, we will prove the propositions stated in §2.1 and some refinements. Our proof will be self-contained, in particular independent of Proposition 3.2 above. The key idea is to take Hausdorff limits of closures of invariant arcs and renormalize the map appropriately to get a normal family. We will only do the proofs for outgoing ∞-germs γ_n and γ_0. The proofs for incoming ∞-germs are similar or can be obtained by replacing P by $e^{i\pi/p}P$, which has the effects of changing the orientation on trajectories.

4.1. Proof of Proposition 2.5. Assume that P is (α', γ)-stable for some outgoing ∞-germ γ and some $\alpha' > 0$. We will prove that

$$\Psi_{P,\gamma}(S(\alpha')) \subset \mathbb{C} \setminus \{\text{zeros of } P\}$$

and there exists a zero $a_{P,\gamma}$ of P such that $\Psi_{P,\gamma}(w) \to a_{P,\gamma}$ as $w \to \infty$ within any sector $S(\alpha)$ with $0 < \alpha < \alpha'$.

By assumption $\Psi := \Psi_P$ is defined and holomorphic on $S(\alpha')$.

Claim A. $\Psi(S(\alpha'))$ intersects neither the zeros nor the incoming ∞-germs of P.

Proof. The first part is because it requires an infinite time to reach a zero (see Lemma 3.1). For the second part, assume $\Psi(w_0) \in \gamma^-$ for some incoming ∞-germ γ^- and some $w_0 \in S(\alpha)$. Then, by definition of incoming ∞-germs, the trajectory with initial point $\Psi(w_0)$ reaches ∞ at some positive finite time t_0. However, by uniqueness of solutions, this trajectory coincides with $\Psi(w_0 + t)$. The fact that Ψ is defined on a neighborhood of $w_0 + t_0$ implies that $\Psi(w_0 + t_0) \neq \infty$. This leads to a contradiction. q.e.d.

Claim B. Assume $w_n \in S(\alpha)$ with $w_n \to \infty$ and $\Psi(w_n) \to z_0$. Then either $z_0 = \infty$ or z_0 is a zero of P.

Proof. Assume by contradiction that $z_0 \neq \infty$ and $P(z_0) \neq 0$.

Set $L_n(w) = \Psi(w_n + w)$. Those maps are defined on the translated sectors $T_{-w_n}S(\alpha')$ (which eventually contain any compact set of $\mathbb{C}$ for n large enough), whose images coincide with $\Psi(S(\alpha'))$. Therefore, they form a normal family as they avoid the incoming ∞-germs of P by Claim A. So, we may take a subsequence and assume $L_n \to L_0$ locally uniformly in $\mathbb{C}$, and the limit L_0 is an entire function. As $z_0 \neq \infty$ and $P(z_0) \neq 0$, the flow of $\dot{z} = P(z)$ with initial point z_0 is well defined and non-constant. But $L_n(w)$ is the flow of $\dot{z} = P(z)$ with initial point $z_n = \Psi(w_n)$. Therefore L_0 is the flow of $\dot{z} = P(z)$ with initial point z_0, and thus non-constant. This contradicts Picard's Theorem since $L_0(\mathbb{C})$ avoids incoming ∞-germs of P. q.e.d.

Recall that $0 < \alpha < \alpha' < \frac{\pi}{2}$. Set

$$S(\alpha)_R = S(\alpha) \cap \{w \ ; \ |\mathrm{Re}(w)| > R\}.$$

Claim C. There is a zero $a(P)$ of P such that

$$\lim_{R \to +\infty} \Psi(S(\alpha)_R) = a(P).$$

Proof. For each $R > 0$, the set $\Psi(S(\alpha)_R)$ is connected and therefore has connected closure in $\overline{\mathbb{C}}$. As the intersection of nested continua is again a continuum, $\bigcap_{R>0} \overline{\Psi(S(\alpha)_R)}$ is a continuum. But it is contained in the finite set $\{\infty\} \cup \{\text{zeros of } P\}$ by Claim B. So it reduces to a single

point $a(P)$. In other words $\lim_{R\to+\infty} \Psi(S(\alpha)_R) = a(P)$. In particular $\lim_{t\to+\infty,\ t\in\mathbb{R}^+} \Psi(t) = a(P)$. This implies that $a(P) \neq \infty$, since no real-time trajectory converges to ∞ in infinite time. So $a(P)$ is a zero of P. q.e.d.

This ends the proof of Proposition 2.5.

4.2. Proof of Proposition 2.6. We must show that if $(P_n, \gamma_n) \to (P_0, \gamma_0)$ for outgoing ∞-germs $\gamma_n \to \gamma_0$ and if the P_n are all (α', γ_n) stable, then,

• P_0 is (α', γ_0)-stable and $\Psi_{P_n,\gamma_n} \to \Psi_{P_0,\gamma_0}$ uniformly on $S(\alpha)$ for any $\alpha < \alpha'$.

Set $\Psi_n = \Psi_{P_n,\gamma_n}|_{S(\alpha')}$ for simplicity. Observe that the family $\{\Psi_n : S(\alpha') \to \mathbb{C}\}$ is a normal family. Indeed, since $P_n \to P_0$ uniformly, for any incoming ∞-germ γ^- of P_0, there is a unique sequence γ_n^- of incoming ∞-germs of P_n converging to γ^-. Normality of the family $\{\Psi_n : S(\alpha') \to \mathbb{C}\}$ follows from the fact (given in Claim A) that $\Psi_n(S(\alpha'))$ avoids $\gamma_n^-(]-\varepsilon, 0[) \underset{n\to\infty}{\longrightarrow} \gamma^-(]-\varepsilon, 0[)$.

Next, the maps Ψ_n converge to Ψ_0 locally uniformly in $S(\alpha') \cap D(\varepsilon)$ for some $\varepsilon > 0$. So, any limit function of the sequence Ψ_n must coincide with Ψ_0 on an open set. By analytic continuation, there is only one such limit function $\Psi : S(\alpha') \to \mathbb{C}$ and it coincides with Ψ_0 on $S(\alpha') \cap D(\varepsilon)$. It follows that P_0 is (α, γ_0)-stable and we have the local uniform convergence of the entire sequence $\Psi_n : S(\alpha') \to \mathbb{C}$ to $\Psi_0 : S(\alpha') \to \mathbb{C}$.

We must now promote this local uniform convergence on $S(\alpha')$ to a global uniform convergence on $S(\alpha)$, $\alpha < \alpha'$.

Claim D. Assume $w_n \in S(\alpha)$ with $w_n \to \infty$ and $\Psi_n(w_n) \to z_0$. We claim that either $z_0 = \infty$ or z_0 is a zero of P_0.

Proof. This is very similar to the proof of Claim B. Assume by contradiction that $z_0 \neq \infty$ and $P_0(z_0) \neq 0$. Set $L_n(w) = \Psi_n(w_n + w)$. Those maps are defined on $T_{-w_n}S(\alpha')$ which eventually contain any compact set of $\mathbb{C}$ for n large enough. They form, as (Ψ_n), a normal family. So, we may take a further subsequence and assume $L_n \to L_0$ locally uniformly in $\mathbb{C}$, and the limit L_0 is an entire function. As $z_0 \neq \infty$ and $P_0(z_0) \neq 0$, the flow of $\dot{z} = P_0(z)$ with initial point z_0 is well defined and non-constant. But $L_n(w)$ is the flow of $\dot{z} = P_n(z)$ with initial point z_n. By local uniform convergence of P_n to P_0 we conclude that L_0 is the flow of $\dot{z} = P_0(z)$ with initial point z_0, and thus non-constant. This contradicts Picard's Theorem since $L_0(\mathbb{C})$ avoids incoming ∞-germs of P_0. q.e.d.

Claim E. Set $\Gamma_n = \overline{\Psi_n(\mathbb{R}^+)}$ and $\Gamma_0 = \overline{\Psi_0(\mathbb{R}^+)}$ (the closures are taken in $\mathbb{P}^1$). We claim that $\Gamma_n \to \Gamma_0$ in the Hausdorff topology, and $a(P_n) \to a(P_0)$.

Proof. Taking a subsequence if necessary we may assume $\Gamma_n \to \Gamma$. By local uniform convergence of Ψ_n to Ψ_0, we have, for any $R \in \mathbb{R}^+$, $\Psi_0(R) = \lim_{n\to\infty} \Psi_n(R) \in \Gamma$. So $\Gamma_0 \subset \Gamma$.

Now choose $z_0 \in \Gamma$. There is a sequence $\Psi_n(R_n) \in \Gamma_n$ tending to z_0. If $R_n \to R \in]0, +\infty[$, we have $z_0 = \Psi_0(R) \in \Gamma_0$. If $R_n \to +\infty$ by Claim D the point z_0 is either at ∞ (which is in Γ_0) or at a zero of P_0. Consequently $\Gamma \subset \Gamma_0 \cup \{\text{zeros of } P_0\}$. But Γ is compact and connected. We conclude that $\Gamma = \Gamma_0$.

Now any accumulation point of $a(P_n)$ is in Γ_0 and must be a zero of P_0, by *global* uniform convergence of P_n to P_0. So $a(P_n) \to a(P_0)$.

q.e.d.

We will now show

$$(\forall \varepsilon > 0)\ (\exists R > 0)\ (\forall n)\ (\forall w \in S(\alpha)_R) \quad |\Psi_n(w) - a(P_n)| \leq \varepsilon. \tag{4.1}$$

Assume this is not the case. Then for any $j \in \mathbb{N}$, there is n_j and $w_j \in S(\alpha)_j$ such that $|\Psi_{n_j}(w_j) - a(P_{n_j})| \geq \varepsilon_0 > 0$. By Proposition 2.5, this implies that $n_j \to \infty$ as $j \to \infty$. Set $w_j = R_j e^{i\theta_j}$. We have $R_j \to \infty$. We will prove that $\Psi_{n_j}(w_j) \to a(P_0)$ and thus get a contradiction. Set $\Lambda_j = \Psi_{n_j}(w_j + [0, +\infty[\,) \cup \{a(P_{n_j})\}$. They are compact connected. Let Λ_0 be a limit set of a subsequence. It is again connected, containing $a(P_0)$ by Claim E and is contained in a finite set by Claim D, and therefore reduces to $\{a(P_0)\}$. It follows that for any $\varepsilon > 0$, there are $R > 0, N > 0$ such that for all $n \geq N$ and all $w \in S(\alpha)_R$, we have

$$\begin{aligned} &|\Psi_n(w) - \Psi_0(w)| \\ &\leq |\Psi_n(w) - a(P_n)| + |a(P_n) - a(P_0)| + |a(P_0) - \Psi_0(w)| \\ &< 3\varepsilon. \end{aligned}$$

Uniform convergence of Φ_{P_n} to Φ_{P_0} in a neighborhood of ∞ yields uniform convergence of Ψ_n to Ψ_0 in $S(\alpha) \cap D(\varepsilon)$. We have local uniform convergence of Ψ_n to Ψ_0 in $S(\alpha)$, thus, uniform convergence on $S(\alpha) \setminus (D(\varepsilon) \cup S(\alpha)_R)$. So, if n is sufficiently large, for all $w \in S(\alpha)$, the spherical distance between $\Psi_n(w)$ and $\Psi_0(w)$ is less than 3ε.

This ends the proof of Proposition 2.6.

4.3. Non-algebraic convergence.

Proposition 4.1. *Assume P_n is a sequence of polynomials of degree $p+1$ which converge locally uniformly in $\mathbb{C}$ to some polynomial P_0 (not necessarily of degree $p+1$). Assume $P_n(0) = P_n(1) = 0$. Assume Ψ_n are holomorphic maps defined on $S(\alpha)$ such that*

$$\Psi_n(S(\alpha)) \subset \mathbb{C} \setminus \{0, 1\}, \quad \Psi_n' = P_n \circ \Psi_n,$$

$$\lim_{t\in\mathbb{R}, t\to 0} \Psi_n(t) = \infty \quad \textit{and} \quad \lim_{t\in\mathbb{R}, t\to\infty} \Psi_n(t) = 0.$$

Then, any limit function Ψ_0 of the normal family $\{\Psi_n\}$ satisfies

$$\Psi_0(S(\alpha)) \subset \mathbb{C} \setminus \{0,1\}, \quad \Psi_0' = P_0 \circ \Psi_0,$$

$$\lim_{t\in\mathbb{R},t\to 0} \Psi_0(t) = \infty \quad \textit{and} \quad \lim_{t\in\mathbb{R},t\to\infty} \Psi_0(t) = 0.$$

Furthermore, the Hausdorff limit of the separatrices $\Gamma_n = \Psi_n(]0,\infty[) \cup \{0,\infty\}$ is the separatrix $\Gamma_0 = \Psi_0(]0,\infty[) \cup \{0,\infty\}$. In particular, the separatrices Γ_n remain uniformly bounded away from 1.

Proof. As we no longer assume global uniform convergence $P_n \to P_0$ in $\mathbb{P}^1$, we cannot conclude as above that Ψ_n converges to some $\Psi_{P_0,\gamma}$ locally uniformly in $S(\alpha')$ without arguing further.

Since the maps Ψ_n take their values in $\mathbb{C} \setminus \{0,1\}$, they form a normal family. So, extracting a subsequence if necessary, we may assume that the maps Ψ_n converge to a map Ψ_0 locally uniformly in $S(\alpha')$. A priori, the map Ψ_0 could be constant (even constantly equal to 0, 1 or ∞).

Claim D′. If $\lim_{t_n \nearrow +\infty} \Psi_n(t_n) = z_0$, then either $z_0 = \infty$ or z_0 is a zero of P_0.

This is proved as Claim D. Let us now consider the opposite case $t_n \in \mathbb{R}^+$, $t_n \searrow 0$.

Claim F. If $\lim_{t_n \searrow 0} \Psi_n(t_n) = z_0$ then $z_0 = \infty$.

Proof. We proceed by contradiction and assume that $z_0 \neq \infty$. Then, starting at z_0, we can follow the real-time flow of the differential equation $\dot{z} = P_0(z)$ backwards at least during time 2ε. By local uniform convergence $P_n \to P_0$, we know that for n large enough, starting at $z_n = \Psi_n(t_n)$, we can also follow the real-time flow of the differential equation $\dot{z} = P_n(z)$ backwards at least during time ε. It follows that $t_n \geq \varepsilon$ for n large enough. q.e.d.

Set $\Gamma_n = \Psi_n(]0,+\infty[) \cup \{0,\infty\}$. It is a continuum. Extracting a further subsequence if necessary, we may assume that $\Gamma_n \to \Gamma$ for the Hausdorff topology on compact subsets of $\mathbb{P}^1$. The set Γ is connected and contains 0 and ∞. There are therefore infinitely many points in Γ which are neither zeros of P_0, nor ∞.

Claim G. If $z_0 \in \Gamma$, is neither a zero of P_0, nor ∞, then it is the image by Ψ_0 of some point $t_0 \in]0,\infty[$.

Proof. Since $z_0 \in \Gamma$, it is a limit of points $z_n = \Psi_n(t_n) \in \Gamma_n$. By Claims D' and F, we see that the sequence t_n is bounded away from 0 and ∞. Extracting a subsequence if necessary, we may assume that $t_n \to t_0 \in]0,+\infty[$. q.e.d.

Thus, we now know that Ψ_0 is not constantly equal to ∞ or to a zero of P_0 (in particular, it takes its values in $\mathbb{C}\setminus\{0,1\}$). Passing to the limit on the equation $\Psi_n' = P_n \circ \Psi_n$, we get $\Psi_0' = P_0 \circ \Psi_0$.

Let us now prove $\lim_{t\to 0} \Psi_0(t) = \infty$. Let t_j be a sequence which tends to 0 and choose n_j large enough so that $|\Psi_{n_j}(t_j) - \Psi_0(t_j)| < 1/j$. Then, the two sequences have the same limit as $j \to \infty$, and by Claim F, this limit is ∞.

Let us finally prove $\lim_{t\to+\infty} \Psi_0(t) = 0$. Applying Proposition 2.5 we know that as $t \nearrow +\infty$, $\Psi_0(t)$ has a limit A which is a zero of P_0. Since Γ connects 0 to ∞, we can find points $z_j \in \Gamma$ which are not zeros of P_0 with $\lim z_j = 0$. By Claim G, we can find a sequence $t_j \in]0,\infty[$ such that $\Psi_0(t_j) = z_j \to 0$. We necessarily have $t_j \to +\infty$ since $\Psi_0(]0,\infty[)$ avoids the zeros of P_0 and since $\Psi_0(t) \to \infty$ as $t \to 0$. Therefore, $A = 0$ and $\lim_{t\to+\infty} \Psi_0(t) = 0$. Finally, the exact same argument shows that any point on Γ is a limit of points $\Psi_0(t_j)$ with $t_j \in]0,\infty]$ and so, $\Gamma = \Psi_0(]0,\infty]) \cup \{0,\infty\}$. This completes the proof of Proposition 4.1. q.e.d.

4.4. Stable convergence and change of coordinates. Assume

$$f_0(z) = z + z^{p+1} + \mathcal{O}(z^{p+2}) \quad \text{and} \quad g_0(z) = z + z^{p+1} + \mathcal{O}(z^{p+2})$$

are conjugate by a change of coordinates $h_0 : (\mathbb{C},0) \to (\mathbb{C},0)$, i.e., $g_0 = h_0 \circ f_0 \circ h_0^{-1}$, with $h_0'(0) = 1$.

We will say that $g_n \to g_0$ is related to $f_n \to f_0$ by a coordinate change if there are univalent maps $h_n \to h_0$ such that the new sequence is obtained from the old one by conjugation, that is: $g_n = h_n \circ f_n \circ h_n^{-1}$.

Lemma 4.2. *Stable convergence is preserved by a coordinate change.*

Proof. Assume $g_n \to g_0$ is related to $f_n \to f_0$ by a coordinate change. Let $h_n \to h_0$ be the coordinate change conjugating f_n to g_n. We must show that the convergence $g_n \to g_0$ is stable at 0 if and only if the convergence $f_n \to f_0$ is stable at 0.

Let P_n (respectively Q_n) be the monic polynomials vanishing at the $p+1$ fixed points of f_n (respectively g_n) close to 0 and assume Q_n are all α-stable for some uniform α. We will show that for n sufficiently large, the polynomials P_n are all $\alpha/3$-stable. Note that $P_n \to P_0$ and $Q_n \to Q_0$ algebraically, with $P_0(z) = Q_0(z) = z^{p+1}$. It is enough to prove that for any sequence of outgoing ∞-germs γ_n for $\dot u = P_n(u)$, which converge to an outgoing ∞-germ γ_0 for $\dot u = P_0(u)$, the polynomials P_n are $(\alpha/3, \gamma_n)$-stable.

Set $\xi_0(u) := Q_0(h_0(u))/h_0'(u)$. Let us choose $\varepsilon > 0$ sufficiently small so that the h_n are all defined on $D(2\varepsilon)$ and so that $\xi_0(u)$ on $\overline{D(\varepsilon)}$ makes an angle less than $\alpha/5$ with $P_0(u)$. This is possible since $h_0(u) = u\cdot(1+o(1))$ and thus, as u tends to 0, $\xi_0(u) = u^{p+1}\cdot(1+o(1)) = P_0(u)\cdot(1+o(1))$. Next, set $\xi_n(u) := Q_n(h_n(u))/h_n'(u)$. The differential equation $\dot u =$

$\xi_n(u)$ has the same set of zeros (counting with multiplicities) as the differential equation $\dot{u} = P_n(u)$ (i.e. the set of fixed points of f_n on $D(2\varepsilon)$), so $\xi_n(u)/P_n(u)$ is holomorphic on $D(2\varepsilon)$. On the circle $C(0,\varepsilon)$, we have $\xi_n(u)/P_n(u) \to \xi_0(u)/P_0(u)$ as $n \to \infty$. Thus, by the maximum modulus principle, for n large enough and for all $u \in D(\varepsilon)$, the vector $\xi_n(u)$ makes an angle less than $\alpha/4$ with the vector $P_n(u)$.

Denote by $\eta_0 = \gamma_0$ the ∞-germ of $Q_0 = P_0$. Denote by η_n the corresponding sequence of ∞-germs of Q_n tending to η_0.

Increasing n_0 and choosing R sufficiently large, we may assume that for all $n \geq n_0$ we have $V_n := \Psi_{Q_n,\eta_n}(S^+(3\alpha/4)_R) \subset h_n(D(\varepsilon))$. Indeed, by Proposition 2.6, as $n \to \infty$, $\Psi_{Q_n,\eta_n} : S^+(3\alpha/4) \to \mathbb{C}$ converges uniformly to $\Psi_{Q_0,\eta_0} : S^+(3\alpha/4) \to \mathbb{C}$. Note that at each point $z \in \partial V_n$, the vector $Q_n(z)$ points towards the interior of V_n and makes an angle $\geq 3\alpha/4$ with ∂V_n.

Set $U_n := h_n^{-1}(V_n)$. Then, for every $u \in \partial U_n$, the vector $\xi_n(u)$ points towards the interior of U_n and makes an angle $\geq 3\alpha/4$ with ∂U_n (this is because h_n is conformal and $\dot{u} = \xi_n(u)$ is the pullback of $\dot{z} = Q_n(z)$ via $z = h_n(u)$). Since the vector $\xi_n(u)$ makes an angle less than $\alpha/4$ with the vector $P_n(u)$, we see that at each point $u \in \partial U_n$, the vector $P_n(u)$ points towards the interior of U_n and makes an angle $\geq \alpha/2$ with ∂U_n.

Choose $\theta \in]-\alpha/2, \alpha/2[$ and consider the differential equation of the rotated vector field, $\dot{u} = R_n(u) := e^{i\theta}P_n(u)$. Then, at every point $u \in \partial U_n$, the vector $R_n(u)$ points towards the interior of U_n, and thus, every orbit for $\dot{u} = R_n(u)$ which enters U_n remains in U_n and cannot form a homoclinic connection.

Fix now t_0 sufficiently large so that $h_0(\Psi_{P_0,\gamma_0}(t_0)) \in V_0$ and therefore $\Psi_{P_0,\gamma_0}(t_0) \in U_0$ (such a t_0 exists always by a local study of the pushed forward field of P_0 by h_0, see [**DES**]). It follows that for n sufficiently large, Ψ_{P_n,γ_n} extends analytically to $t_0 + S^+(\alpha/2)$ (and maps it into U_n). On the other hand, as $P_n \to P_0$ uniformly, for n sufficiently large the map Ψ_{P_n,γ_n} extends analytically to a large slit disc $D(\hat{R})\backslash\mathbb{R}^-$ containing $S^+(\alpha/3) \setminus (t_0 + S^+(\alpha/2))$. As a consequence, for n sufficiently large, Ψ_{P_n,γ_n} extends analytically to the entire sector $S^+(\alpha/3)$. q.e.d.

4.5. Length stability of separatrices.

Proof of Proposition 2.8. Assume that P is a polynomial of degree $p+1$, that γ is an outgoing ∞-germ and that P is (α,γ)-stable for some $\alpha > 0$. Let $\psi :]0,+\infty[\to \mathbb{C}$ be the γ-separatrix. We must show that for $\eta > p/(p+1)$ and $t > 0$, the η-dimensional length $\ell_\eta(P,\gamma,t) := \int_t^{+\infty} |\psi'(u)|^\eta \, du$ is finite. It is then clear that it decreases with respect to t and tends to 0 as t tends to $+\infty$.

By Proposition 2.5, the γ-separatrix lands at a zero a of P. If a is a simple zero of P, then we have $\dfrac{1}{P(z)} \underset{z\to a}{\sim} \dfrac{1}{P'(a)(z-a)}$. It follows that

for t_0 large enough, we have

$$t = t_0 + \int_{\psi(t_0)}^{\psi(t)} \frac{1}{P(z)}\, dz = \frac{1}{P'(a)} \log(\psi(t) - a) + C + o(1)$$

as $t \to +\infty$. It follows that

$$\psi(t) - a \sim K e^{P'(a)t} \quad \text{and} \quad \psi'(t) \sim K P'(a) e^{P'(a)t}$$

as $t \to +\infty$. Thus $\displaystyle\int_t^{+\infty} |\psi'(u)|^\eta \, du < \infty$ for $t > 0$.

If a is a multiple zero of P with multiplicity $p' + 1 \leq p + 1$, then

$$\frac{1}{P(z)} \underset{z\to a}{\sim} \frac{C}{(z-a)^{p'+1}}\ ,\ \psi(t) - a \underset{t\to+\infty}{\sim} \frac{C'}{t^{1/p'}} \ \text{ and } \ \psi'(t) \underset{t\to+\infty}{\sim} \frac{C''}{t^{(p'+1)/p'}}.$$

Since $p' \leq p$, we have $\eta \dfrac{p'+1}{p'} \geq \eta \dfrac{p+1}{p} > 1$ and $\displaystyle\int_t^{+\infty} |\psi'(u)|^\eta \, du < \infty$ for $t > 0$. q.e.d.

Lemma 4.3 (a uniform constant)**.** *For all $p \geq 1$ and $\alpha \in \left]0, \frac{\pi}{2}\right[$, there exists a constant C_α, such that for any degree $p+1$ polynomial P which is (α, γ)-stable, any point z on the γ-separatrix and any zero a_j of P, we have $|z - a_j| \geq C_\alpha |z - a_0|$, where a_0 is the zero of P which is the ending point of the γ-separatrix.*

Proof. We will proceed by contradiction. Without loss of generality, conjugating with a translation if necessary, we may assume that $a_0 = 0$ (see Lemma 3.4). Since multiplying P by a positive real k only changes the parametrization of the real trajectories (not their images), we may assume that the leading coefficient of P is of modulus 1.

We assume that we can find a sequence of (α, γ_n)-stable polynomials P_n of degree $p+1$ with the γ_n-separatrix Γ_n ending at 0, points $z_n \in \Gamma_n \setminus \{0\}$ and $a_n \neq 0$ with of $P_n(a_n) = 0$ such that $\lim_{n\to\infty} \frac{z_n - a_n}{z_n} = 0$. We will show that this is not possible thanks to Proposition 4.1. Note that $\Gamma_n = \Psi_n(]0, \infty[)$ for some map $\Psi_n : S(\alpha) \to \mathbb{C} \setminus \{0, a_n\}$ satisfying $\Psi_n' = P_n \circ \Psi_n$, $\lim_{t\to 0} \Psi_n(t) = \infty$ and $\lim_{t\to+\infty} \Psi_n(t) = 0$.

We first need to re-scale the situation. We factorize P_n into

$$P_n(z) = A_n z(z - a_n) \prod_{j=2}^{p} (z - a_{j,n}) \quad \text{with} \quad |A_n| = 1.$$

Extracting a subsequence if necessary, we may assume that as n tends to ∞, the ratios $a_{j,n}/a_n$ have limits in $\mathbb{P}^1$ and we let J be the set of indices $j \in \{2, \cdots, p\}$ for which the limit is finite. We then define

$$\rho_n := |a_n|^p \cdot \prod_{j \in \{2,\cdots,p\} \setminus J} \left| \frac{a_{j,n}}{a_n} \right| \quad (\ \rho_n := |a_n|^p \text{ if } J = \{2, \cdots, p\}).$$

Set

$$Q_n(z) := \frac{P_n(a_n z)}{a_n \rho_n} \text{ and } \Xi_n(w) := \frac{1}{a_n}\Psi_n\left(\frac{w}{\rho_n}\right).$$

Then Q_n is again a polynomial and

$$\Xi_n'(w) = \frac{1}{a_n\rho_n}\Psi_n'\left(\frac{w}{\rho_n}\right) = \frac{1}{a_n\rho_n}P_n\circ\Psi_n\left(\frac{w}{\rho_n}\right) = Q_n\circ\Xi_n(w).$$

Taking a subsequence if necessary, as $n\to\infty$, the polynomial Q_n converges locally uniformly in $\mathbb{C}$ to

$$Q_0(z) = \mu\cdot z(z-1)\prod_{j\in J}\left(z - \lim_{n\to\infty}\frac{a_{j,n}}{a_n}\right) \text{ or } Q_0(z) = \mu\cdot z(z-1) \text{ if } J=\emptyset$$

with $|\mu| = 1$.

Since $\rho_n > 0$, the map Ξ_n is also defined on $S(\alpha)$, and $\Lambda_n := \Xi_n(]0,+\infty[)$ is equal to Γ_n/a_n. By Proposition 4.1, we know that Λ_n remains bounded away from 1 as n tends to ∞. This shows that $z_n/a_n \in \Lambda_n$ cannot tend to 1, which gives a contradiction. q.e.d.

Proof of Proposition 2.9. We assume that (P_n,γ_n) is a sequence of pairs (polynomial, outgoing ∞-germ) converging to a pair (P_0,γ_0) and that the P_n are (α,γ_n)-stable. We let $\psi_n :]0,+\infty[\to\mathbb{C}$ be the γ_n-separatrices. Given $\eta > p/(p+1)$ and $t_n\to t_0 > 0$, we must show that

$$\int_{t_n}^{+\infty}|\psi_n'(u)|^\eta\, du \to \int_{t_0}^{+\infty}|\psi_0'(u)|^\eta\, du.$$

We will do this by using Lebesgue dominated convergence theorem.

By Proposition 2.6, we know that $\psi_n'\to\psi_0'$ uniformly on $\mathbb{R}^+$. So, we only have to find an integrable function that dominates $|\psi_n'(t)|^\eta$ for $n\geq n_0$ and $t\geq R$.

Let a_n be the landing point of the γ_n-separatrix ψ_n. By Proposition 2.6, we know that a_n tends to a_0 as $n\to+\infty$. Thus, conjugating with the translations $z\mapsto z-a_n$ (this will not change the stability, by Lemma 3.4), we may assume that $a_n = 0$, so that $\psi_n(t)\to 0$ as $t\to+\infty$ and $P_n(0) = 0$.

It follows from Lemma 4.3 (with $a_0(P_n) = 0$) that for all $n\geq 0$ and all $t > 0$, we have

(4.2)

$$|\psi_n'(t)| = |P_n(\psi_n(t))| = |A_n|\cdot|\psi_n(t)-0|\cdot\prod_{j=1}^{p}|\psi_n(t)-a_{j,n}| \geq \frac{C_\alpha^p}{2}|\psi_n(t)|^{p+1},$$

where $a_{j,n}$ are the other zeros of P_n, and A_n is the leading coefficient of P_n (with $A_n \underset{n\to\infty}{\longrightarrow} A_0$).

Set $\Psi_n := \Psi_{P_n,\gamma_n}$ (note that $\psi_n = \Psi_n|_{\mathbb{R}^+}$). By Proposition 2.5, for $\alpha' < \alpha$, if R is sufficiently large, $\Psi_0(S(\alpha')_R)\subset D(1/2)\setminus\{0\}$ and

by Proposition 2.6, if n_0 is sufficiently large, $\Psi_n(S(\alpha')_R) \subset \mathbb{D}^*$ for all $n \geq n_0$. By Schwarz lemma, with d_V denoting the hyperbolic metric of V, we have, $(\forall n \geq n_0)$ $(\forall w, w' \in S(\alpha')_R)$,

$$d_{\mathbb{D}^*}(\Psi_n(w), \Psi_n(w')) \leq d_{S(\alpha')_R}(w, w') \leq d_{S(\alpha)}(w, w').$$

The coefficient of the hyperbolic metric on $\mathbb{D}^*$ is $\frac{|dz|}{|z| \log \frac{1}{|z|}}$, and the coefficient at $t \in \mathbb{R}$ in $S(\alpha)$ is bounded from above by C_1/t. So,

$$(\forall n \geq n_0)\ (\forall t > R) \quad \frac{|\psi_n'(t)|}{|\psi_n(t)| \log \frac{1}{|\psi_n(t)|}} \leq \frac{C_1}{t}\ . \tag{4.3}$$

Recall that $\eta > \frac{p}{p+1}$. Choose $\varepsilon > 0$ small so that $\eta \frac{p+1}{p+\varepsilon} > 1$. There is a constant C_2 such that

$$(\forall x \in]0, 1[) \quad \log \frac{1}{x} < \frac{C_2}{x^\varepsilon}. \tag{4.4}$$

So, for all $n \geq N_0$ and all $t > R$, we have

$$\begin{aligned} \frac{C_\alpha^p}{2} |\psi_n(t)|^{p+1} &\overset{(4.2)}{\leq} |\psi_n'(t)| \overset{(4.3)}{\leq} \frac{C_1 |\psi_n(t)| \log \frac{1}{|\psi_n(t)|}}{t} \\ &\overset{(4.4)}{\leq} \frac{C_1 C_2 |\psi_n(t)|^{1-\varepsilon}}{t}. \end{aligned} \tag{4.5}$$

Hence, $|\psi_n(t)|^{p+\varepsilon} \leq \frac{C_3}{t}$. Thus,

$$\begin{aligned} |\psi_n'(t)|^\eta &\overset{(4.5)}{\leq} \left(\frac{C_1 C_2 |\psi_n(t)|^{1-\varepsilon}}{t} \right)^\eta \\ &= \left(\frac{C_1 C_2\, |\psi_n(t)|^{(p+\varepsilon)\frac{1-\varepsilon}{p+\varepsilon}}}{t} \right)^\eta \leq \left(\frac{C_4}{t^{\frac{p+1}{p+\varepsilon}}} \right)^\eta. \end{aligned}$$

The right hand side function is integrable since $\eta \frac{p+1}{p+\varepsilon} > 1$. q.e.d.

5. Relating iterated orbits to trajectories

5.1. Case of a single map $f(z)$ close to $z + z^{p+1}$. For a holomorphic map $g : D(r) \to \mathbb{C}$, we define $\|g\|_r := \sup_{z \in D(r)} |g(z)|$.

Let $f(z) = z + g(z) = z + (1 + s(z))Q(z)$. We do not assume that Q is a polynomial. The goal of the following lemma is to show that once $Q(z)$ is close to z^{p+1}, the time-one iterate of f, and sometimes the long term iterates of f, are well approximated by the time-one map of the flow of $\dot{z} = Q(z)$.

Lemma 5.1. *Let $g, Q, s, f : D(r) \to \mathbb{C}$ be four holomorphic functions with the following relations: $g(z) = (1 + s(z))Q(z)$ and $f(z) = z +$*

$g(z) = z + (1 + s(z))Q(z)$. *Let $\psi(w)$ be a local solution of $\dot{z} = Q(z)$ (the time w is considered to be complex). Set*

$$\varepsilon'_f := \max\{\|Q'\|_r, \|g'\|_r, \|s\|_r\} \quad \textit{and} \quad \varepsilon_f := \max\{\|g\|_r, \|Q\|_r\}.$$

A (*very much inspired by the Main Lemma in* [**O**] *and a similar estimate in* [**DES**]*, see also* [**E**]*, Lemma* 3). *Assume*

$$\varepsilon'_f \le \varepsilon' < \frac{1}{5}, \quad \varepsilon_f \le \varepsilon < \frac{r}{4} \quad \text{and} \quad \psi(w_0) \in D(r - 4\varepsilon) \setminus \{\textit{zeros of } Q\}.$$

Then $\psi(w)$ is defined at least on $w_0 + D(4)$, and this last disk contains two points $F^{\pm}(w_0)$ satisfying $\psi(F^{\pm}(w_0)) = f^{\pm}(\psi(w_0))$, $|F^{\pm}(w_0) - (w_0 \pm 1)| < 5\varepsilon'$ and $|f^{\pm}(\psi(w_0)) - \psi(\pm 1)| \le 5\varepsilon\varepsilon'$.

B *Set $S(\alpha) = S^+(\alpha)$. Assume $\frac{\pi}{2} > \alpha > 0$ and ψ is defined on an open set W. Assume further*

$$\boxed{\begin{array}{c} \varepsilon'_f \le \varepsilon' < \dfrac{\sin\alpha}{5} \quad , \quad \varepsilon_f \le \varepsilon < \dfrac{r}{4} \ ; \\ W + S(\alpha) \subset W \quad \textit{and} \quad \psi(W) \subset D(r - 4\varepsilon) \setminus \{\textit{zeros of } Q\} \end{array}} \tag{5.1}$$

(*Note that such a map ψ is often globally non-univalent.*) *Then ψ has a local inverse map ϕ and $F(w) := \phi(f(\psi(w)))$ is a globally well defined and holomorphic function on W, mapping W into W, and satisfying:*

$$\psi(F(w)) = f(\psi(w)) \quad \textit{and} \quad \sup_{j \in \mathbb{N}, w \in W} |F^j(w) - (w + j)| \le 5j\varepsilon'. \tag{5.2}$$

If W is convex the map F is also univalent. There is a similar statement for $S^-(\alpha)$ replacing the triple $(F, f, w+j)$ by $(F^{-1}, f^{-1}, w - j)$.

One may consider w as an approximate Fatou coordinate, in which the dynamics is close to the translation by 1.

Proof. **Part A.** Set $\varepsilon_1 = \|s\|_r$. We have $\varepsilon_1 < \frac{1}{5} < \frac{1}{4}$.

(I) *f is univalent on $D(r)$, its image contains $D(r-\varepsilon)$ and for all $z \in D(r-\varepsilon)$, we have $f^{\pm 1}(z) \in D(z, \varepsilon)$ and $|f(z)-z| \le \frac{5}{4}|Q(z)|$, $|f^{-1}(z)-z| < \frac{5}{3}|Q(z)|$.*

Proof. Note that $D(r)$ is convex. For $a, b \in D(r)$, $a \ne b$, we have

$$f(b) - f(a) = b - a + \int_{[a,b]} (f'(z) - 1)dz \quad \text{and} \quad |f(b) - f(a)| \ge \frac{4}{5}|b - a| \tag{5.3}$$

as $|f'(z) - 1| \le \varepsilon' < \frac{1}{5}$. So f is univalent. Evidently $\dfrac{|f(z) - z|}{|Q(z)|} = \dfrac{|g(z)|}{|Q(z)|} = |1 + s(z)| \le 1 + \varepsilon_1 < \dfrac{5}{4}$.

Let $b \in D(r-\varepsilon)$. For z on the circle $|z-b| = \varepsilon$, we have $|g(z)| < \varepsilon = |z-b|$. So, by Rouché's theorem, $z \mapsto z-b$ and $z \mapsto z-b+g(z)$ have the same number of roots in $D(b,\varepsilon)$. This shows that $f^{-1}(b)$ is well defined and $f^{-1}(b) \in D(b,\varepsilon)$. Set $z = f^{-1}(b)$. We have

$$\frac{|f^{-1}(b)-b|}{|Q(b)|} = \frac{|z-f(z)|}{|Q(f(z))|} = \frac{|g(z)|}{|Q(z)|} \cdot \frac{|Q(z)|}{|Q(f(z))|} < \frac{5}{4} \cdot \frac{|Q(z)|}{|Q(f(z))|}.$$

On the other hand,

$$\begin{aligned}\frac{|Q(z+g(z))-Q(z)|}{|Q(z)|} &\leq \sup_{\tau \in [z,z+g(z)]} |Q'(\tau)| \cdot \frac{|g(z)|}{|Q(z)|} < \varepsilon' |1+s(z)| \\ &< \frac{1}{5} \cdot \frac{5}{4} = \frac{1}{4}.\end{aligned}$$

So $\left|\dfrac{Q(f(z))}{Q(z)}\right| \geq 1 - \dfrac{1}{4} = \dfrac{3}{4}$ and $\dfrac{5}{4} \cdot \dfrac{|Q(z)|}{|Q(f(z))|} \leq \dfrac{5}{4} \cdot \dfrac{4}{3} = \dfrac{5}{3}$. This proves (I). q.e.d.

Let us now assume that $w_0 = 0$ and set $z_0 = \psi(w_0) = \psi(0)$. By assumption, $z_0 \in D(r-4\varepsilon) \setminus \{\text{zeros of } Q\}$.

(II) *The map ψ is defined at least on $\overline{D(4)} = \{|w| \leq 4\}$ with images contained in $D(r)$.*

Proof. Let $D(s)$ be the maximal disc on which ψ is defined. Since Q is holomorphic on $D(r)$, the set $\partial\psi(D(s))$ meets either the zeros of Q or $\partial D(r)$. The former case does not occur, due to Lemma 3.1. Hence there is w with $|w| < s$ such that $|\psi(w)| > 4\varepsilon + |z_0|$. Therefore

$$\begin{aligned}4\varepsilon < |\psi(w) - z_0| = |\psi(w) - \psi(0)| &\leq \sup_{\tau \in D(s)} |\psi'(\tau)| \cdot |w| \\ &= \sup_{\tau \in D(s)} |Q(\psi(\tau))| \cdot |w| \leq \varepsilon s.\end{aligned}$$

So $s > 4$. q.e.d.

Now set $\Delta := D(z_0, \frac{5}{3}|Q(z_0)|) \subset D(r)$. We have the following property.

(III) *$f^{\pm 1}(z_0) \in \Delta$ and the map $\phi(z) := \displaystyle\int_{z_0}^{z} \frac{dw}{Q(w)}$ is well defined on Δ, with image contained in $D(4)$ and with $\psi \circ \phi = id$ on Δ.*

Proof. The fact $f^{\pm 1}(z_0) \in \Delta$ follows from (I). By assumption, $|s(z)| < \frac{1}{4}$, thus, $|1+s(z)| > \frac{3}{4}$ and so, $|g(z)| > \frac{3}{4}|Q(z)|$ if $Q(z) \neq 0$. In particular, $g(z)$ has the same set of zeros as $Q(z)$. By assumption $Q(z_0) \neq 0$. For $z \in \Delta$, we have

$$|Q(z) - Q(z_0)| \leq \varepsilon' |z - z_0| \overset{z \in \Delta}{\leq} \varepsilon' \cdot \frac{5}{3}|Q(z_0)| \leq \frac{1}{5} \cdot \frac{5}{3}|Q(z_0)| = \frac{1}{3}|Q(z_0)|.$$

So,

$$|Q(z)| \geq \frac{2}{3}|Q(z_0)| \tag{5.4}$$

and

$$|g(z)| > \frac{3}{4}|Q(z)| \geq \frac{3}{4}\cdot\frac{2}{3}|Q(z_0)| = \frac{1}{2}|Q(z_0)| > 0.$$

Now $\phi(z)$ is just a primitive of $\frac{1}{Q(z)}$ on Δ with $\phi(z_0) = 0$. For $z \in \Delta$ we have

$$\begin{aligned}|\phi(z) - \phi(z_0)| \leq \sup_{\tau\in\Delta}|\phi'(\tau)|\cdot|z - z_0| &\leq \sup_{\tau\in\Delta}\frac{1}{|Q(\tau)|}\cdot\frac{5}{3}|Q(z_0)| \\ &\overset{(5.4)}{\leq} \frac{3}{2|Q(z_0)|}\cdot\frac{5|Q(z_0)|}{3} < 3.\end{aligned}$$

Thus, $\phi(\Delta) \subset D(3)$. Now $\psi \circ \phi$ is holomorphic on Δ and is locally the identity map. So it is identity on Δ, and $\phi|_\Delta$ is univalent. q.e.d.

Finally we have the following property.

(IV) *Setting $F^{\pm}(w_0) = F^{\pm}(0) := \phi(f^{\pm 1}(z_0))$, we have*

$$|F^{\pm}(0) - (\pm 1)| < 5\varepsilon'.$$

Proof. Set $w^{\pm} = F^{\pm}(0)$. To compare $w^{\pm}$ with ± 1 we need the help of the second derivative of ϕ. We have $\phi' = 1/Q$ and so, for $z \in \Delta$,

$$|\phi''(z)| = \left|-\frac{Q'(z)}{Q(z)^2}\right| \overset{(5.4)}{\leq} \frac{9\varepsilon'}{4|Q(z_0)|^2} \quad \text{and} \tag{5.5}$$

$$\begin{aligned}|w^+ - (1 + s(z_0))| &= |\phi(f(z_0)) - \phi(z_0) - \phi'(z_0)g(z_0)| \\ &\leq \frac{|f(z_0) - z_0|^2}{2}\sup_{\tau\in\Delta}|\phi''(\tau)| \\ &\overset{(I),(5.5)}{\leq} \frac{(\frac{5}{4})^2|Q(\hat{z})|^2}{2}\cdot\frac{9\varepsilon'}{4|Q(z_0)|^2} < 2\varepsilon'.\end{aligned}$$

So

$$|w^+ - 1| \leq |w^+ - (1 + s(z_0))| + \varepsilon_1 \leq 2\varepsilon' + \varepsilon_1 \leq 5\varepsilon'.$$

Similarly,

$$\begin{aligned}|w^- + (1 + s(f^{-1}(z_0)))| &= |\phi(z_0) - \phi(f^{-1}(z_0)) - \phi'(f^{-1}(z_0))g(f^{-1}(z_0))| \\ &\overset{(I)}{\leq} \frac{(\frac{5}{3})^2|Q(\hat{z})|^2}{2}\cdot\frac{9\varepsilon'}{4|Q(z_0)|^2} < 4\varepsilon'.\end{aligned}$$

Therefore, $|w^- + 1| \leq 4\varepsilon' + \varepsilon_1 \leq 5\varepsilon'$. q.e.d.

Part B. This part is an easy corollary of Part A. For any $w \in W$ we have $\psi(w) \in D(r-4\varepsilon) \setminus \{\text{zeros of } Q\}$, $\varepsilon' < \frac{1}{5}$ and $\varepsilon < \frac{r}{4}$. So $F(w)$ is well defined by Part A. To check that $F(w) \in W$, we note that $|F(w) - (w+1)| < 5\varepsilon' < \sin\alpha$. So $F(w) \in w + S(\alpha) \subset W$. The fact that F is holomorphic follows from the functional equation $\psi(F(w)) = f(\psi(w))$. The inequality about $F^j(w)$ is proved by induction. The univalency of F follows similarly as in (5.3) , by checking $|F'(w)-1| < 1$. In fact:

$$\begin{aligned}
|F'(w)-1| &\leq \left\| \frac{(Q(z)-Q(f(z)))f'(z) + Q(f(z))(f'(z)-1)}{Q(f(z))} \right\|_r \\
&\leq (1+\varepsilon') \left\| \frac{Q(z)-Q(f(z))}{Q(f(z))} \right\|_r + \varepsilon' \\
&\leq (1+\varepsilon')\varepsilon' \left\| \frac{f(z)-z}{Qf(z)} \right\|_r + \varepsilon' \\
&\overset{(I)}{\leq} (1+\varepsilon')\varepsilon' \cdot \frac{5}{4} + \varepsilon' \overset{\varepsilon'<\frac{1}{5}}{<} \frac{3}{5} < 1.
\end{aligned}$$

The case $S(\alpha) = S^-(\alpha)$ is similar. q.e.d.

5.2. Stable perturbations of $z+z^{p+1}$ implies well approximation by flow.

Definition 5.2. If $V \subset \mathbb{C}$ is a hyperbolic subset, we denote by $d_V(\cdot,\cdot)$ the hyperbolic distance in V. By convention, we set $d_V(a,b) = +\infty$ if one or two of a, b do not belong to V.

Let us now assume $r < 1$ and that the sequence $(f_n : D(r) \to \mathbb{C})_{n\geq 1}$ converges locally uniformly to $f_0 : D(r) \to \mathbb{C}$ with $f_0(z) = z + z^{p+1} + \mathcal{O}(z^{p+2})$. Let P_n be the monic polynomials of degree $p+1$ which vanish at the $p+1$ fixed points of f_n close to 0. Fix $k \in \mathbb{Z}/2p\mathbb{Z}$. Let γ_n be the ∞-germ tangent to $\mathbb{R}^+ \cdot e^{2i\pi\frac{k}{2p}}$ at ∞ and assume P_n is (α', γ_n)-stable for some $\alpha' > \alpha$ and all n sufficiently large. Set $\Psi_n := \Psi_{P_n,\gamma_n} : S(\alpha') \to \mathbb{C}$. Finally, for $z \in \mathbb{C}$, let $\psi_n(z,\cdot) :]t^-(z), t^+(z)[\to \mathbb{C}$ be the maximal real-time solution of $\dot{z} = P_n(z)$ with initial condition $\psi_n(z,0) = z$.

Lemma 5.3 (long term approximation by flow)**.** *There are $n_0 > 0$ and $R_0 > 0$ such that for all $n \geq n_0$,*

(a) *$\Psi_n(S(\alpha)_{R_0}) \subset D(r) \setminus \{$fixed points of $f_n\}$ and $\Psi_n'(S(\alpha)_{R_0}) \subset \mathbb{D}^*$;*
(b) *there is a univalent map $F_n : S(\alpha)_{R_0} \to S(\alpha)_{R_0}$ satisfying $\Psi_n \circ F_n = f_n \circ \Psi_n$;*
(c) *$F_n \longrightarrow F_0$ uniformly on $S(\alpha)_{R_0}$;*

(d)
(5.6)
$$\begin{cases} \sup\limits_{n>N, j\in\mathbb{N}, w\in S(\alpha)_R} d_{S(\alpha)_R}\left(F_n^{\circ j}(w), w+j\right) \overset{N,R\to\infty}{\longrightarrow} 0 \\ \sup\limits_{n>N, j\in\mathbb{N}, z\in\Psi_n(S(\alpha)_R)} d_{D(r)\setminus\{\textit{fixed points of } f_n\}}(f_n^{\circ j}(z), \psi_n(z,j)) \overset{N,R\to\infty}{\longrightarrow} 0 \\ \sup\limits_{n>N, j\in\mathbb{N}, z\in\Psi_n(S(\alpha)_R)} d_{\mathbb{D}^*}\left(P_n(f_n^j(z)), P_n(\psi_n(z,j))\right) \overset{N,R\to\infty}{\longrightarrow} 0. \end{cases}$$

Remark. When $P_n(0) = 0$, one may use $d_{\mathbb{D}^*}$ in the second limit of (5.6).

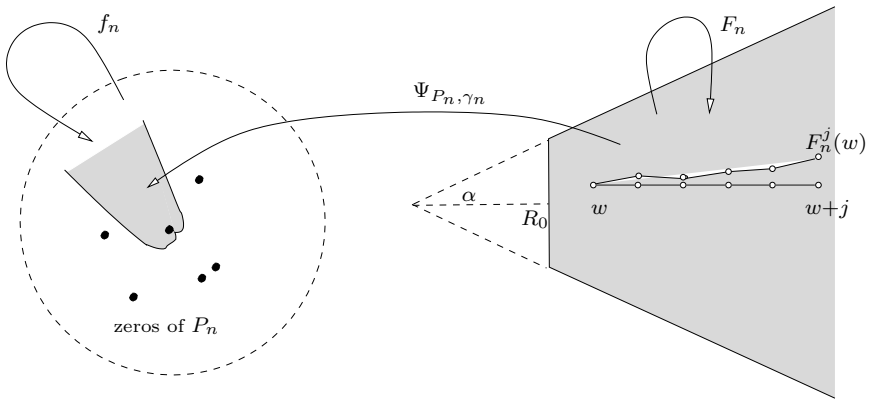

Figure 3. The lifted dynamics F_n is close to translation by 1.

Proof. We will do the proof for outgoing ∞-germs (k is even). The proof for incoming ones is similar.

We will find W such that f_n, Ψ_n and W satisfy the hypothesis (5.1) of Lemma 5.1.B for n large enough.

Proof of (a). Since $P_n(z) \to z^{p+1}$, if $\varepsilon \in]0, r[$ is small enough and n is sufficiently large, $P_n(D(\varepsilon)) \subset \mathbb{D}$. By Proposition 2.6 with $P_0(z) = z^{p+1}$, we can find $N_0 > 0$ and $R_0 > 0$ such that for any $n \geq N_0$, we have $\Psi_n(S(\alpha)_{R_0}) \subset D(\varepsilon) \setminus \{\text{zeros of } P_n\}$. Since $\Psi_n' = P_n \circ \Psi_n$ we have $\Psi_n'(S(\alpha)_{R_0}) \subset \mathbb{D}^*$. By definition, the zeros of P_n are the fixed points of f_n.

Proof of (b). For any $N \geq N_0$ and $R \geq R_0$, there is a constant $\rho_{N,R} \searrow 0$ as $N, R \to \infty$ such that

$$\Psi_n(S(\alpha)_R) \subset D(\rho_{N,R}) \setminus \{\text{zeros of } P_n\}\ ,\ \forall\ n \geq N. \tag{5.7}$$

Set

$$r(N,R) := 5 \cdot \rho_{N,R} \quad \text{and} \tag{5.8}$$

$$\rho'_{N,R} := 2\max\{\|P'_0\|_{r(N,R)}, \|g'_0\|_{r(N,R)}, \|s_0\|_{r(N,R)}\}.$$

As $P'_0(0) = g'_0(0) = s_0(0) = 0$ and $r(N,R) \searrow 0$ as $N, R \to \infty$, we have $\rho'_{N,R} \searrow 0$ as $N, R \to \infty$. We may increase N_0, R_0 if necessary so that

$$\rho'_{N,R} \le \rho'_{N_0,R_0} < \frac{\sin\alpha}{5} < \frac{1}{5}. \tag{5.9}$$

Set $\varepsilon_{f_n|D(r(N,R))} := \max\{\|g_n\|_{r(N,R)}, \|P_n\|_{r(N,R)}\}$ and

$$\varepsilon'_{f_n|D(r(N,R))} := \max\{\|P'_n\|_{r(N,R)}, \|g'_n\|_{r(N,R)}, \|s_n\|_{r(N,R)}\}$$

so that $\varepsilon'_{f_0|D(r(N,R))} = \rho'_{N,R}/2$. We know that P_n, g_n, s_n and their derivatives converge uniformly to P_0, g_0, s_0 and their derivatives in some neighborhood of 0 as $n \to \infty$. So,

$$\varepsilon'_{f_n|D(r(N,R))} \overset{n\to\infty}{\longrightarrow} \varepsilon'_{f_0|D(r(N,R))}, \quad \varepsilon_{f_n|D(r(N,R))} \overset{n\to\infty}{\longrightarrow} \varepsilon_{f_0|D(r(N,R))}.$$

There is therefore $n_0(N,R) > N$ such that for $n \ge n_0(N,R)$,

$$\varepsilon'_{f_n|D(r(N,R))} < 2 \cdot \varepsilon'_{f_0|D(r(N,R))} = \rho'_{N,R} \le \rho'_{N_0,R_0} \overset{(5.9)}{<} \frac{\sin\alpha}{5}, \tag{5.10}$$

in particular $\varepsilon'_{f_0|D(r(N,R))} < \frac{1}{10}$; and

(5.11)

$$\begin{aligned} \varepsilon_{f_n|D(r(N,R))} &< 2\varepsilon_{f_0|D(r(N,R))} \overset{\varepsilon'_{f_0}<\frac{1}{10}}{\le} 2 \cdot \tfrac{r(N,R)}{10} = \tfrac{r(N,R)}{5} \overset{(5.8)}{=} \rho_{N,R} \\ &< r(N,R)/4\ . \end{aligned}$$

Set $W = S(\alpha)_R$. We have

(5.12)

$$W + S(\alpha) \subset W, \quad \Psi_n(W) \overset{(5.7)}{\subset} D(\rho_{N,R}) \overset{(5.8)}{=} D(r(N,R) - 4\rho_{N,R}).$$

Now (5.10), (5.11), (5.12) together imply that, given any $N \ge N_0$, $R \ge R_0$, for all $n \ge n_0(N,R)$, the conditions (5.1) in Lemma 5.1.B are satisfied for $r = r(N,R)$, $f = f_n|_{D(r(N,R))}$, $W = S(\alpha)_R$, $\psi = \Psi_n|_W$, $\varepsilon' = \rho'_{N,R}$ and $\varepsilon = \rho_{N,R}$. In particular for $n_0 := n_0(N_0,R_0)$, and all $n > n_0$, there is a univalent map $F_n : S(\alpha)_{R_0} \to S(\alpha)_{R_0}$ satisfying $\Psi_n(F_n(w)) = f_n(\Psi_n(w))$.

Proof of (c). The uniform convergence of F_n towards F_0 on $S(\alpha)_{R_0}$ follows from that of f_n (by assumption) and of Ψ_n (by Proposition 2.6).

Proof of (d). Given any $N \ge n_0$ and $R \ge R_0$, and for $n \ge N$, the F_n defined on $S(\alpha)_R$ is the restriction of the F_n defined on $S(\alpha)_{R_0}$. So by (5.2), and setting $N' = n_0(N,R)$,

$$\sup_{n>N', j\in\mathbb{N}, w\in S(\alpha)_R} \frac{|F_n^{\circ j}(w) - (w+j)|}{5j} \le \rho'_{N,R}.$$

Replacing now the Euclidean metric by the hyperbolic metric, we can find $s_{N,R}$ depending in fact only on $\rho'_{N,R}$, with $s_{N,R} \to 0$ as $\rho'_{N,R} \to 0$ (therefore as $N, R \to \infty$), such that

$$\sup d_{S(\alpha)_{R_0}}(F_n^{\circ j}(w), w+j) \leq \sup d_{S(\alpha)_R}(F_n^{\circ j}(w), w+j) \leq s_{N,R}$$

where both sup are taken over the set $\{n > N',\ j \in \mathbb{N},\ w \in S(\alpha)_R\}$. Now $f_n^{\circ j} \circ \Psi_n = \Psi_n \circ F_n^{\circ j}$, and

$$\Psi_n : S(\alpha)_{R_0} \to \mathbb{D} \setminus \{\text{fixed points of } f_n\} =: U_n$$

is holomorphic. So, by Schwarz Lemma:

$$\begin{aligned} d_{U_n}(f_n^{\circ j} \circ \Psi_n(w), \Psi_n(w+j)) &= d_{U_n}(\Psi_n(F_n^{\circ j}(w)), \Psi_n(w+j)) \\ &\leq d_{S(\alpha)_{R_0}}(F_n^{\circ j}(w), w+j) \leq s_{N,R}. \end{aligned}$$

So

$$\sup_{n>N', j\in\mathbb{N},\ z\in\Psi_n(S(\alpha)_R)} d_{U_n}(f_n^{\circ j}(z), \psi_n(z,j)) \leq s_{N,R}.$$

From this one derives easily the first two limits in (5.6) . The remaining limit in (5.6) is obtained similarly, by composing with P_n, and by using $P_n \circ \Psi_n = \Psi'_n$. q.e.d.

6. Continuity of Julia sets and Poincaré series

6.1. Proof of Proposition 2.11. Assume the sequence

$$(f_n : D(r) \to \mathbb{C})_{n\geq 1}$$

converges locally uniformly to $f_0 : D(r) \to \mathbb{C}$ with

$$f_0(z) = z + z^{p+1} + \mathcal{O}(z^{p+2}), \quad p \geq 1.$$

Let P_n be the monic polynomials of degree $p+1$ which vanish at the $p+1$ fixed points of f_n close to 0. Let us fix $k \in \mathbb{Z}/2p\mathbb{Z}$ odd and let γ_n be the ∞-germ for $\dot{z} = P_n(z)$, tangent to $e^{2i\pi\frac{k}{2p}}\mathbb{R}^+$ at ∞.

Definition 6.1. We say that the convergence $f_n \to f_0$ is (α, k)-stable if for n sufficiently large, the polynomials P_n are (α', γ_n)-stable for some $\alpha' > \alpha$.

Now, let $K \subset D(r)$ be a compact set such that for all $z \in K$,

- $f_0^{\circ j}(z)$ is defined for all $j \geq 0$ and $f_0^{\circ j}(z) \xrightarrow[j\to+\infty]{\neq} 0$ tangentially to the direction $e^{2i\pi\frac{k}{2p}}$.

Let a_n be the landing point of the γ_n-separatrix of $\dot{z} = P_n(z)$. We will now show that for n large enough, K is contained in the basin of attraction of a_n, i.e., for all $z \in K$,

- $f_n^{\circ j}(z)$ is defined for all $j \geq 0$ and $f_n^{\circ j}(z) \xrightarrow[j\to+\infty]{\neq} a_n$.

Set $\Psi_n := \Psi_{P_n,\gamma_n} : S(\alpha') \to \mathbb{C}$.

Lemma 6.2. *For any $R > 0$, there are an open set L relatively compact in $S(\alpha)_R$, a compact $K' \subset \Psi_0(L)$, $n_0 > 0$ and $j_0 > 0$ such that for all $n \geq n_0$,*

$$f_n^{\circ j_0}(K) \subset K' \subset \Psi_n(L).$$

Proof. Note that $\Psi_0(S(\alpha)_R)$ is a sector neighborhood of 0 around the $e^{2\pi i \frac{k}{2p}}\mathbb{R}^+$ of opening angle α/p (see Example 0).

By the classical theory of Fatou flowers around parabolic points (which can be easily reproved using Lemma 5.1), $f_0^{\circ j}(K)$ will tend to 0 within a cusp region bounded by two curves tangential to $\mathbb{R}^+ \cdot e^{2\pi i \frac{k}{2p}}$.

Thus we can find j_0 and an open set L compactly contained in $S(\alpha)_R$, such that $f_0^{\circ j_0}(K) \subset \Psi_0(L)$. Since $f_n \to f_0$ uniformly and $\Psi_n \to \Psi_0$ uniformly on every compact subset of $S(\alpha')$, we see that for n sufficiently large, $f_n^{\circ j_0}(K) \subset K' \subset \Psi_n(L)$ for some compact set K'. q.e.d.

Proof of Proposition 2.11. By Lemma 5.3, for n and R large enough, $\Psi_n(S(\alpha)_R) \subset D(0,r) \setminus \{\text{fixed points of } f_n\}$ and there is a holomorphic map $F_n : S(\alpha)_R \to S(\alpha)_R$ such that $\Psi_n \circ F_n = f_n \circ \Psi_n$. Moreover, for all $w \in S(\alpha)_R$, the hyperbolic distance in $S(\alpha)_R$ between $F_n^{\circ j}(w)$ and $w + j$ is bounded. Thus, $F_n^{\circ j}(w)$ tends to ∞ as j tends to ∞, and so, $f_n^{\circ j}(\Psi_n(w)) = \Psi_n(F_n^{\circ j}(w)) \to a_n$ as $j \to \infty$.

The previous lemma asserts that for this R and for n large enough, $f_n^{\circ j_0}(K) \subset \Psi_n(L)$ with $L \subset S(\alpha)_R$. q.e.d.

6.2. Partial continuity of Fatou components. Let us now assume that β is a parabolic periodic point of a rational map f_0. Let l be the period of β. Then, $f_0^{\circ l}(\beta) = \beta$ and $[f_0^{\circ l}]'(\beta) = e^{2i\pi r/s}$ for some integers $r \in \mathbb{Z}$ and $s \geq 1$ co-prime. Then, $[f_0^{\circ ls}]'(\beta) = 1$ and conjugating f_0 with a Moebius transformation (non uniquely determined), we may assume that $\beta = 0$ and $f_0^{\circ ls}(z) = z + z^{p+1} + \mathcal{O}(z^{p+2})$ with $p = ms$ a multiple of s. The number p is called the number of petals of f_0 at β.

It is known that there exist p attracting petals and p repelling petals $\mathcal{P}_k$, $k \in \mathbb{Z}/2p\mathbb{Z}$, contained in a neighborhood of 0 in which f_0 is univalent, such that

- $\mathcal{P}_k$ contains $\left\{re^{i\theta};\ 0 < r < \varepsilon \text{ and } |\theta - \frac{k}{2p}| < \frac{1}{4p}\right\}$ for ε small enough,
- $f_0^{\circ l}(\mathcal{P}_k) \subset \mathcal{P}_{k+2mr}$ if k is odd; and $f_0^{\circ l}(\mathcal{P}_k) \supset \mathcal{P}_{k+2mr}$ if k is even.

The repelling petals are those for k even and the attracting petals are those for k odd. Under iteration of $f_0^{\circ l}$, the orbit of every point contained in an attracting petal converges to the parabolic fixed point.

Let us fix $k \in \mathbb{Z}/2p\mathbb{Z}$ odd and let $\mathcal{F}_k$ be the set of points whose forward orbit under iteration of f_0 intersects $\mathcal{P}_k$. The set $\mathcal{F}_k$ is a union of Fatou components contained in the attracting basin of β.

Proposition 6.3 (partial stability of Fatou components)**.** *Assume $(f_n : \mathbb{P}^1 \to \mathbb{P}^1)_{n\geq 1}$ is a sequence of rational maps converging algebraically to $f_0 : \mathbb{P}^1 \to \mathbb{P}^1$. Let β be a parabolic periodic point of f_0, let l be the period of β, $e^{2i\pi r/s}$ be the multiplier of f_0 at β and choose a coordinate on $\mathbb{P}^1$ such that $\beta = 0$ and $f^{\circ ls}(z) = z + z^{p+1} + \mathcal{O}(z^{p+2})$. If the convergence $f_n \to f_0$ is (α, k)-stable for some odd $k \in \mathbb{Z}/2p\mathbb{Z}$, then the set $\mathcal{F}_k$ contains no limit point of J_{f_n}.*

Proof. Let Q be an open subset of $\mathcal{F}_k$, relatively compact in $\mathcal{F}_k$. Choose j_0 large enough so that $f_0^{\circ j_0}(Q)$ is contained in $\mathcal{P}_k$. Choose n_0 sufficiently large so that for $n \geq n_0$, $f_n^{\circ j_0}(Q) \subset \mathcal{P}_k$. And set

$$K := \overline{\bigcup_{n\geq n_0} f_n^{\circ j_0}(Q)}.$$

By Proposition 2.11, for all n sufficiently large, K is in the attracting basin of some fixed point a_n of $f_n^{\circ ls}$ close to 0. Thus, Q is contained in the Fatou set of f_n and $Q \cap J(f_n) = \emptyset$. q.e.d.

6.3. Continuity of Julia sets.

Theorem 6.4 (see also [**DES**])**.**

(a) *Let $(f_n : \mathbb{P}^1 \to \mathbb{P}^1)_{n\geq 1}$ be a sequence of rational maps converging algebraically to a rational map $f_0 : \mathbb{P}^1 \to \mathbb{P}^1$. Then $J(f_n) \to J(f_0)$ assuming*
- *the convergence $f_n \to f_0$ is stable at each parabolic point of f_0,*
- *for each irrationally indifferent periodic point β_0 of f_0 with multiplier $e^{2i\pi\alpha_0}$ and period l_0, either*
  - *f_0 is not linearizable at β_0, or*
  - *α_0 is a Brjuno number and f_n has a l_0-periodic point β_n converging to β_0 with multiplier $e^{2i\pi\alpha_0}$, and*
- *f_0 does not have Herman rings.*

(b) *If in addition f_0 is geometrically finite and $f_n \to f_0$ preserving critical relations, then, for n sufficiently large, f_n is geometrically finite.*

Remark. If f_0 is geometrically finite, it does not have irrationally indifferent cycles nor Herman rings.

Proof.

(a) Let J' be a limit of a subsequence of $J(f_n)$. We know that $J' \supset J(f_0)$ due to the density of repelling periodic points in $J(f_0)$ and their stability. We just need to prove $J' \subset J(f_0)$.

According to the non-wandering theorem and the classification theorem of Sullivan, the Fatou set $\mathbb{P}^1 \setminus J(f_0)$ of f_0 has four types of components: components of attracting basins, components of parabolic basins,

preimages of Siegel discs and preimages of Herman rings. By assumption f_0 has no Herman rings.

Assume at first that B is a component of an attracting basin. Let $K \subset B$ be a compact connected subset. Then for n large enough, K is contained in the attracting basin of a nearby attracting cycle for f_n. Therefore $B \cap J' = \emptyset$.

Now assume that B is a component of a parabolic basin. Then it is contained in $\mathcal{F}_k$ for some odd k, in the setting of Proposition 6.3. As the convergence is (α, k)-stable at every parabolic periodic point and for every k, Proposition 6.3 implies that $B \cap J' = \emptyset$.

Finally assume that B is a preimage of a Siegel disk. Let $K \subset B$ be compact. Our assumption that the rotation number is a Brjuno number allows us to apply results in Risler [**R**]: for large n, we have $K \subset B_n$, for B_n the corresponding preimage of the perturbed Siegel disk of f_n. Therefore $K \cap J' = \emptyset$. Consequently $B \cap J' = \emptyset$.

These cases together prove that $J' \subset J(f_0)$.

(b) In this more particular setting, we have $J(f_n) \to J(f_0)$ from part (a). It remains to show that f_n are geometrically finite for large n, that is, every critical point is either in the Fatou set or preperiodic (the two cases are not mutually exclusive). Let c_n be a critical point of f_n with $c_n \overset{n\to\infty}{\longrightarrow} c$. Then c is a critical point of f_0. If $c \in \mathbb{P}^1 \setminus J(f_0)$ then $c_n \in \mathbb{P}^1 \setminus J(f_n)$ for large n. If $c \in J(f_0)$, then it is preperiodic, as well as c_n, by assumption of the persistence of critical relations. Hence f_n is geometrically finite for large n. q.e.d.

6.4. Convergence of Poincaré Series and proof of Proposition 2.12. We now come to the main estimates in the article: controlling the convergence of tails of Poincaré series. We will work under the same assumptions as in §6.1. We will show that if $\delta_0 > p/(p+1)$ and $\varepsilon > 0$, there exist m_0 and n_0 such that for all $z \in K$, all $\delta \in [\delta_0, 2]$, all $m \geq m_0$ and all $n \geq n_0$, we have $S_\delta(f_n, z, m) := \sum_{j=m}^{+\infty} |(f_n^{\circ j})'(z)|^\delta < \varepsilon$. Choose α, s, ε such that

(6.1)

$$\alpha' > \alpha > 0,\ s > 0,\ \varepsilon > 0, \quad \eta := \delta_0 e^{-s} > \frac{p}{p+1} \quad \text{and} \quad \eta \frac{p+1}{p+\varepsilon} > 1.$$

Recall that P_n is the monic polynomial which vanishes at the $p+1$ fixed points of f_n close to 0 and $P_0(z) = z^{p+1}$. By assumption all the P_n are (α', γ_n)-stable. Set again $\Psi_n = \Psi_{P_n,\gamma_n} : S(\alpha') \to \mathbb{C}$. Without loss of generality, conjugating with translations if necessary, we may assume that for all n, the separatrix $\Psi_n(\mathbb{R}^+)$ lands at 0: $\Psi_n(t) \to 0$ as $t \to +\infty$. In particular $P_n(0) = 0$.

By Lemma 5.3, we may fix N and R large such that the following two conditions hold.

Condition 1. For all $n \geq N$, $\Psi_n(S(\alpha)_R) \subset D(r)$ and the lifted dynamics $F_n : S(\alpha)_R \to S(\alpha)_R$ satisfy:

$$(\forall n \geq N)\ (\forall w \in S(\alpha)_R)\ (\forall j \in \mathbb{N}) \quad d_{S(\alpha)_R}(F_n^{\circ j}(w), w + j) \leq s/2.$$

Condition 2. For all $n \geq N$, $\Psi_n'(S(\alpha)_R) \subset \mathbb{D}^*$.

By Lemma 6.2, there are an open set L relatively compact in $S(\alpha)_R$, a compact $K' \subset \Psi_0(L)$, $n_0 > 0$ and $j_0 > 0$ such that for all $n \geq n_0$, we have $f_n^{\circ j_0}(K) \subset K' \subset \Psi_n(L)$. We can finally fix $m_0 \geq R$ large enough so that the following condition holds.

Condition 3. $(\forall w \in L)\ (\forall j \geq m_0)\ (\forall t \in [0,1])$, $d_{S(\alpha)_R}(w+j, j+t) \leq \frac{s}{2}$.

Note that if $z \in K$ and n is large enough, then $z_n' = f_n^{\circ j_0}(z) \in K'$ and for $m \geq j_0$, we have

$$\sum_{j=m}^{+\infty} |(f_n^{\circ j})'(z)|^\delta = |(f_n^{\circ j_0})'(z)|^\delta S_\delta(f_n, z_n', m - j_0).$$

Since $|(f_n^{j_0})'(z)|^\delta$ is uniformly bounded for $(z, \delta) \in K \times [\delta_0, 2]$ as n tends to ∞, in order to prove Proposition 2.12, it is enough to prove that as n and m tend to ∞, $S_\delta(f_n, z, m)$ tends to 0 uniformly with respect to $(z, \delta) \in K' \times [\delta_0, 2]$. Therefore, Proposition 2.12 follows from Corollary 2.10 and Lemma 6.5 below. q.e.d.

Lemma 6.5. *There exists a constant C such that for all $n > N$ and $m \geq m_0$, $z \in K'$ and $\delta \in [\delta_0, 2]$, we have $S_\delta(f_n, z, m) \leq C\ell_\eta(P_n, \gamma_n, m)$.*

Proof. Fix $n > N$, $z \in K'$ and $\delta \in [\delta_0, 2]$. Let $w_n \in L$ be such that $\Psi_n(w_n) = z$. Then, it follows from $\Psi_n \circ F_n^{\circ j} = f_n^{\circ j} \circ \Psi_n$ and the chain rule that

$$\begin{array}{rcl} |(f_n^{\circ j})'(z)|^\delta & = & \dfrac{1}{|\Psi_n'(w_n)|^\delta} |(F_n^{\circ j})'(w_n)|^\delta \cdot |\Psi_n'(F_n^{\circ j}(w_n))|^\delta \\ & \overset{\text{Koebe}}{\leq} & C|\Psi_n'(F_n^{\circ j}(w_n))|^\delta \end{array} \tag{6.2}$$

where the inequality is due to the bounded Koebe distortion theorem applied to the univalent maps $F_n^{\circ j}$. By Condition 1, the hyperbolic distance in $S(\alpha)_R$ between $F_n^{\circ j}(w_n)$ and $w_n + j$ is smaller than $s/2$. By Condition 3, the hyperbolic distance between $w + j$ and $j + t$ is also less than $s/2$, for all $w \in L$ and all $j \geq m_0$. Therefore, for all $j \geq m_0$ and all $t \in [0,1]$,

$$d_{S(\alpha)_R}(F_n^{\circ j}(w_n), j + t) < s.$$

Since Ψ_n' maps $S(\alpha)_R$ into $\mathbb{D}^*$ by Condition 2, it follows

$$d_{\mathbb{D}^*}(\Psi_n'(F_n^{\circ j}(w_n)), \Psi_n'(j + t)) \leq s$$

by Schwarz lemma. So,

(6.3)

$$|\Psi_n'(F_n^{\circ j}(w_n))|^\delta \overset{\text{Lem.}6.7}{\leq} |\Psi_n'(j+t)|^{\delta e^{-s}} \overset{\delta\geq\delta_0}{\leq} |\Psi_n'(j+t)|^{\delta_0 e^{-s}} = |\Psi_n'(j+t)|^\eta.$$

$$\text{Thus}\quad S_\delta(f_n,z,m) = \sum_{j=m}^{\infty} |(f_n^{\circ j})'(z)|^\delta \overset{(6.2)}{\leq} C\sum_{j=m}^{\infty} |\Psi_n'(F_n^{\circ j}(w_n))|^\delta$$

$$\overset{(6.3)}{\leq} C\sum_{j=m}^{\infty}\int_0^1 |\Psi_n'(j+t)|^\eta dt = C\int_m^\infty |\Psi_n'(t)|^\eta dt = C\ell_\eta(P_n,\gamma_n,m).$$

q.e.d.

Lemma 6.5 together with Corollary 2.10 proves Proposition 2.12.

Corollary 6.6. *Assume $(f_n : D(r)\to\mathbb{C})_{n\geq 1}$ is a sequence of univalent maps converging locally uniformly to $f_0 : D(r)\to\mathbb{C}$ with $f_0(z) = z + z^{p+1} + \mathcal{O}(z^{p+2})$. If the convergence $f_n \to f_0$ is stable, then, for any compact set $K \subset D(r)$ whose orbit converges to 0 under backward iteration of f_0, and for any $\delta_0 > p/(p+1)$, we have $\sum_{j=m}^{+\infty} |(f_n^{-j})'(z)|^\delta \underset{m,n\to\infty}{\longrightarrow} 0$ uniformly with respect to $(z,\delta)\in K\times[\delta_0,2]$.*

Proof. By assumption the convergence $f_n \to f_0$ is (α',k)-stable for some $\alpha' > 0$ and for every k, in particular for every even k. Now the compact set K can be written as the disjoint union of finitely many compact sets, each converging to 0 under backward iteration of f_0 along some repelling axis. Applying Proposition 2.12 to $f_n^{-1}\to f_0^{-1}$ for each even k gives the corollary. q.e.d.

Lemma 6.7. *For $a,b\in\mathbb{D}^*$ with $d_{\mathbb{D}^*}(a,b)\leq s$ we have $|a|\leq|b|^{e^{-s}}$.*

Proof. $s \geq d_{\mathbb{D}^*}(a,b) \underset{\text{if } |b|\leq|a|}{\geq} \displaystyle\int_{|b|}^{|a|} \frac{|dz|}{|z|\log\frac{1}{|z|}} = \log\frac{\log\frac{1}{|b|}}{\log\frac{1}{|a|}}.$ q.e.d.

7. Proof of Theorem A

Now we need to recall existing theory about conformal measures and their relation with Hausdorff dimension of Julia sets.

For f a rational map, denote by $\mathcal{E}(f)$ the set of preparabolic and precritical points in the Julia set. More precisely,

$$\mathcal{E}(f) = \Big\{ z\in J(f) \mid (\exists n\geq 0),\ f^{\circ n}(z) \text{ is a parabolic or critical point of } f\Big\}.$$

The map f is geometrically finite if and only if every point in $\mathcal{E}(f)$ is prerepelling or preparabolic.

An f-invariant conformal measure of dimension $\delta > 0$ is a probability measure μ on $\mathbb{P}^1$ such that $\mu(f(E)) = \int_E |f'(x)|^\delta \, d\mu$ whenever $f|E$ is injective.

The following result can be found in [**McM**, **DMNU**, **PU**].

Theorem 7.1. *Assume f is geometrically finite. Then, there exists a unique f-invariant conformal measure μ_f with support in $J(f) \setminus \mathcal{E}(f)$. Furthermore, the dimension δ_f of μ_f is equal to the Hausdorff dimension of $J(f)$.*

Sketch of proof. Uniqueness. Given a rational map f, one can define the *radial Julia set* $J_{rad}(f)$ as the set of points $z \in J(f)$ such that arbitrarily small neighborhoods of z can be blown up by the dynamics to disks of definite size centered at $f^{\circ n}(z)$. The radial Julia set $J_{rad}(f)$ supports at most one conformal measure (see [**DMNU**] Theorem 1.2 and [**McM**] Theorem 5.1). Moreover, the dimension of this conformal measure is always equal to the Hausdorff dimension of $J_{rad}(f)$ (see for example [**McM**] Theorem 2.1 and Theorem 5.1).

The fact that f is geometrically finite implies that $J(f) \setminus \mathcal{E}(f)$ is contained in the radial Julia set $J_{rad}(f)$ (see [**McM**] Theorem 6.5 and [**U**] Theorem 4.2). Since $\mathcal{E}(f)$ is countable, we have H.dim $J(f) =$ H.dim $J_{rad}(f)$.

Existence. In [**McM**] §4, a conformal measure is constructed with support contained in $J(f) \setminus \{\text{ preparabolic and prerepelling points }\}$ which is contained in $J(f) \setminus \mathcal{E}(f)$. q.e.d.

The following result is implicit in McMullen [**McM**], Theorem 11.2.

Theorem 7.2. *We have* H.dim $J(f_n) \to$ H.dim $J(f_0)$ *under the following assumptions:*

(A1) *f_0 is geometrically finite;*
(A2) *$f_n \to f_0$ algebraically and preserving critical relations;*
(B1) *f_n are geometrically finite;*
(B2) *$J(f_n) \to J(f_0)$;*
(B3) *(see below) the tails of the Poincaré series are uniformly small in neighborhoods of each preparabolic point in $\mathcal{E}(f_0)$.*

Sketch of proof. Conditions (A1) and (B1) imply that

$$(\mu_0, \delta_0) := (\mu_{f_0}, \delta_{f_0}) \quad \text{and} \quad (\mu_n, \delta_n) := (\mu_{f_n}, \delta_{f_n})$$

exist, with $\delta_0 =$ H.dim $J(f_0)$ and $\delta_n =$ H.dim $J(f_n)$. Assume $\mu_n \to \nu$ weakly and $\delta_n \to \delta'$ (by taking twice subsequences if necessary). Then, ν is a f_0-invariant conformal measure, with

$$\operatorname{supp}(\nu) \subset \lim_{n \to +\infty} J(f_n) \overset{(B2)}{=} J(f_0).$$

Furthermore, condition (B3) implies that $\nu(\{c\}) = 0$ for any prerepelling $c \in \mathcal{E}(f_0)$ (see below). Thus, the support of ν is contained in $J(f_0) \setminus$

$\mathcal{E}(f_0)$. It follows from Theorem 7.1 that $\nu = \mu_{f_0}$ and thus,

$$\text{H.dim } J(f_n) = \delta_n \underset{n\to\infty}{\longrightarrow} \delta' = \delta_0 = \text{H.dim } J(f_0).$$

Let us now explain why a weak limit ν of the sequence μ_n cannot charge the points of $\mathcal{E}(f_0)$.

Technique of change of coordinates. For every parabolic point c of f_0, we replace f_0 with an iterate, call it $\hat{f}_0$, so that $\hat{f}_0(c) = c$ and $\hat{f}_0'(c) = 1$. Define $\hat{f}_n$ by the same number of iterates of f_n. For a non periodic preparabolic point b, we replace again f_0 with an iterate to achieve $f_0(b) = c$ and c is a fixed point as above. We then lift the dynamics of f_0 near c to a map $\hat{f}_0 = f_0^{-1} \circ f_0 \circ f_0$. For f_n, and any $c_n \to c$ with $f_n(c_n) = c_n$, there is $b_n \to b$ such that $f_n(b_n) = c_n$. Due to our assumption that critical orbit relations in $J(f_0)$ are preserved, if b is a critical or a postcritical point, then there are unique choices of c_n and b_n such that b_n is again a critical or a postcritical point of f_n, with the same local degree. Thus we can form $\hat{f}_n = f_n^{-1} \circ f_n \circ f_n$ and obtain a sequence $(\hat{f}_n, b_n) \to (\hat{f}_0, b)$.

In any case we consider $\hat{f}_0, \hat{f}_n$ as the induced local dynamics. Note that μ_n is locally $\hat{f}_n$-conformal.

We can now give a more precise definition of (B3) as follows. Denote by p the maximal petal number of points in $\mathcal{E}(f_0)$. Given a point $b \in \mathcal{E}(f_0)$, change coordinates as above so that this point is 0 with induced local dynamics $\hat{f}_0, \hat{f}_n$. We say that the tails of the Poincaré series are uniformly small in neighborhoods of 0 if there exists $r > 0$, such that for all $\delta_0 > \frac{p}{p+1}$ and all compact set $K \subset \overline{D}(r)$ whose backward orbit under $\hat{f}_0$ remains in $\overline{D}(r)$ and tends to 0 and such that the forward orbit under $\hat{f}_0$ of every point in $J(f_0) \cap D(r)$ eventually falls into K before leaving $D(r)$,

$$(7.1) \quad (\forall \varepsilon > 0)\ (\exists m, N)\ (\forall n \geq N)\ (\forall z \in K) \quad \sum_{j=m}^{+\infty} |(\hat{f}_n^{-j})'(z)|^{\delta_{f_n}} \leq \varepsilon.$$

Now (B3) together with (A2) guarantee that for any preparabolic $c \in \mathcal{E}(f_0)$, for any $\varepsilon, m > 0$, there are $N > 0$ and a neighborhood V of c such that for $n \geq N$,

$$V \cap J(f_n) \subset \{c_n\} \cup \bigcup_{j=m}^{+\infty} \hat{f}_n^{-j}(K \cap J(f_n))$$

and $\mu_n(V) = \mu_n(V \cap J(f_n)) < \varepsilon$ (see the proof of Theorem 11.2, [**McM**]). Consequently $\nu(\{c\}) = 0$. q.e.d.

Remark. In [**McM**], conditions (B1), (B2) and (B3) are replaced by a stronger condition (B'): namely $f_n \to f_0$ dominantly and for the

corresponding multipliers λ_n, either $\lambda_n \to 1$ radially or $\lambda_n \to 1$ horocyclically with $\liminf \text{ H.dim } J(f_n) > \dfrac{2p(f_0)}{p(f_0)+1}$ (where $p(f_0)$ is the maximal possible petal number at a point of $\mathcal{E}(f_0)$).

Proof of Theorem A. Conditions (A1) and (A2) are kept as such in the hypothesis of our Theorem A. We have assumed that the convergence is stable. This implies conditions (B1) and (B2), due to Theorem 6.4 part (b). Condition (B3) for any preparabolic point $b \in \mathcal{E}(f_0)$ will follow from Corollary 6.6 with the induced local dynamics, together with the following argument.

Denote by $p(b)$ the petal number of $\hat{f}_0$ at b and by p the maximal petal number of points in $\mathcal{E}(f_0)$. Assume again $f_0(b) = c$ is a fixed point with multiplier 1. By assumption the convergence $(f_n, c_n) \to (f_0, c)$ is stable. It follows from Lemmas 4.2 and 3.13 that the convergence $(\hat{f}_n, b_n) \to (\hat{f}_0, b)$ is stable at b. By Corollary 6.6, we see that for all $\delta_0 > \frac{p(b)}{p(b)+1}$,

(7.2)

$$(\forall \varepsilon > 0)\ (\exists m, N)\ (\forall n \geq N)\ (\forall z \in K)\ (\forall \delta \in [\delta_0, 2])\quad \sum_{j=m}^{+\infty} |(\hat{f}_n^{-j})'(z)|^{\delta} \leq \varepsilon.$$

In order to get from (7.2) to (7.1), we need two more results in [**McM**] (see Theorems 3.1, 2.1, 6.1 and Proposition 11.3). At first

$$\delta_{f_0} > \frac{p}{p+1},$$

and at second,

$$\liminf \delta_{f_n} \geq \delta_{f_0}$$

(semi-continuity of the dimension). So there is

$$\delta_0 > \frac{p}{p+1} > \frac{p(b)}{p(b)+1}$$

so that $2 \geq \delta_{f_n} \geq \delta_0$ for all large n. We may thus apply (7.2) to this δ_0. The uniform control on δ in (7.2) leads to (7.1), where δ is replaced by δ_{f_n}. q.e.d.

Proof of Corollary 2.17. We just need to combine Theorem A with Proposition 3.11. q.e.d.

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