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SMOOTH s-COBORDISMS OF ELLIPTIC 3-MANIFOLDS

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Abstract

The main result of this paper states that a symplectic s-cobordism of elliptic 3-manifolds is diffeomorphic to a product (assuming a canonical contact structure on the boundary). Based on this theorem, we conjecture that a smooth s-cobordism of elliptic 3-manifolds is smoothly a product if its universal cover is smoothly a product. We explain how the conjecture fits naturally into the program of Taubes of constructing symplectic structures on an oriented smooth 4-manifold with $b_2^+ \geq 1$ from generic self-dual harmonic forms. The paper also contains an auxiliary result of independent interest, which generalizes Taubes' theorem "$SW \Rightarrow Gr$" to the case of symplectic 4-orbifolds.

1. Introduction: conjecture and main result

One of the fundamental results in topology is the so-called s-cobordism theorem, which allows one to convert topological problems into questions of algebra and homotopy theory. This theorem says that if W is a compact $(n+1)$-dimensional manifold with boundary the disjoint union of manifolds Y_1 and Y_2, then when $n \geq 5$, W is diffeomorphic, piecewise linearly homeomorphic, or homeomorphic, depending on the category, to the product $Y_1 \times [0,1]$, provided that the inclusion of each boundary component into W is a homotopy equivalence and that a certain algebraic invariant $\tau(W; Y_1) \in Wh(\pi_1(W))$, the Whitehead torsion, vanishes. (Such a W is called an s-cobordism from Y_1 to Y_2; when $\pi_1(W) = \{1\}$, the theorem is called the h-cobordism theorem, first proved by Smale.) Note that the s-cobordism theorem is trivial for the dimensions where $n \leq 1$. However, great effort has been made to understand the remaining cases where $n = 2, 3$ or 4, and the status of the s-cobordism theorem in these dimensions, for each different category, reflects the fundamental distinction between topology of low-dimensional manifolds and that of the higher dimensional ones.

For the case of $n = 2$, the s-cobordism theorem is equivalent to the original Poincaré conjecture, which asserts that a closed, simply

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connected 3-manifold is homeomorphic to the 3-sphere (cf. e.g., [**19**, **31**, **38**]). For $n = 4$, work of Freedman [**17**] yielded a topological s-cobordism theorem for W with a relatively small fundamental group, e.g., finite or polycyclic. On the other hand, Donaldson [**16**] showed that the h-cobordism theorem fails in this dimension for the smooth (and equivalently, the piecewise linear) category.

This paper is concerned with 4-dimensional s-cobordisms with boundary components homeomorphic to elliptic 3-manifolds. (An elliptic 3-manifold is one which is homeomorphic to $\mathbb{S}^3/G$ for some finite subgroup $G \subset SO(4)$ acting freely on $\mathbb{S}^3$.) Building on the aforementioned work of Freedman, the classification of topological s-cobordisms of elliptic 3-manifolds up to orientation-preserving homeomorphisms was completed in a series of papers by Cappell and Shaneson [**6**, **7**], and Kwasik and Schultz [**27**, **28**]. Their results showed that for each elliptic 3-manifold, the set of distinct topological s-cobordisms is finite, and is readily determined from the fundamental group of the 3-manifold. In particular, there are topologically nontrivial (i.e., non-product), orientable s-cobordisms in dimension four[1], and the nontriviality of these s-cobordisms is evidently related to the fundamental group of the 3-manifold. On the other hand, not much is known in the smooth category. Note that the construction of the aforementioned nontrivial s-cobordisms involves surgery on some topologically embedded 2-spheres, and it is generally a difficult problem to determine whether these 2-spheres are smoothly embedded. In particular, it is not known whether these nontrivial s-cobordisms are smoothable or not. As for smooth s-cobordisms obtained from constructions other than taking a product, examples can be found in Cappell and Shaneson [**8**, **9**] (compare also [**1**]), where the authors exhibited a family of smooth s-cobordisms W_r from $\mathbb{S}^3/Q_r$ to itself, with

$$Q_r = \{x, y \mid x^2 = y^{2^r} = (xy)^2 = -1\}$$

being the group of generalized quaternions of order 2^{r+2} (note that Q_r with $r = 1$ is the group of order 8 generated by the quaternions i, j). It has been an open question, only until recently, as to whether any of these s-cobordisms or their finite covers are smoothly nontrivial. In [**2**], Akbulut showed that the universal cover of W_r with $r = 1$ is smoothly a product. Despite this result, however, the following general questions have remained untouched.

(1) Are there any exotic smooth structures on a trivial 4-dimensional s-cobordism?
(2) Are any of the nontrivial topological 4-dimensional s-cobordisms smoothable?

[1] Existence of nontrivial, non-orientable 4-dimensional s-cobordisms, which is of a different nature, has been fairly understood, cf. [**33**], also [**26**] for a concrete example.

In this paper, we propose a program for understanding smooth s-cobordisms of elliptic 3-manifolds. At the heart of this program is the following conjecture, which particularly suggests that in the smooth category, any nontrivial s-cobordism (should there exist any) will have nothing to do with the fundamental group of the 3-manifold.

Conjecture 1.1. A smooth s-cobordism of an elliptic 3-manifold to itself is smoothly a product if and only if its universal cover is smoothly a product.

We propose two steps toward Conjecture 1.1, and undertake the first one in this paper.

In order to describe the first step, we recall some relevant definitions from symplectic and contact topology. Let Y be a 3-manifold. A contact structure on Y is a distribution of tangent planes $\xi \subset TY$ where $\xi = \ker \alpha$ for a 1-form α such that $\alpha \wedge d\alpha$ is a volume form on Y. Note that the contact manifold (Y, ξ) has a canonical orientation defined by the volume form $\alpha \wedge d\alpha$. Let (Y_i, ξ_i), $i = 1, 2$, be two contact 3-manifolds given with the canonical orientation. A symplectic cobordism from (Y_1, ξ_1) to (Y_2, ξ_2) is a symplectic 4-manifold with boundary (W, ω) such that $\partial W = Y_2 - Y_1$ (here W is given with the canonical orientation defined by $\omega \wedge \omega$), and that there exists a normal vectorfield v in a neighborhood of ∂W where $L_v \omega = \omega$, and $\xi_i = \ker(i_v \omega|_{Y_i})$ for $i = 1, 2$.

Notice that every elliptic 3-manifold is diffeomorphic to $\mathbb{S}^3/G$, where $\mathbb{S}^3 \subset \mathbb{C}^2$, for a finite subgroup $G \subset U(2)$ acting freely on $\mathbb{S}^3$ (we shall explain later in this section). The 3-manifold $\mathbb{S}^3/G$ has a canonical contact structure, i.e., the descendant of the distribution of complex lines on $\mathbb{S}^3$ under the map $\mathbb{S}^3 \to \mathbb{S}^3/G$. Furthermore, the canonical orientation from the contact structure coincides with the one induced from the canonical orientation on $\mathbb{S}^3$. With the preceding understood, the following theorem is the main result of this paper.

Theorem 1.2. *A symplectic s-cobordism from an elliptic 3-manifold $\mathbb{S}^3/G$ to itself is diffeomorphic to a product. Here G is a subgroup of $U(2)$ and $\mathbb{S}^3/G$ is given with the canonical contact structure.*

Thus, in order to prove Conjecture 1.1, it suffices to show, which is the second step, that a smooth s-cobordism of an elliptic 3-manifold to itself is symplectic if its universal cover is smoothly a product. We shall explain next how this step fits naturally into Taubes' program of constructing symplectic structures on an oriented smooth 4-manifold with $b_2^+ \geq 1$ from generic self-dual harmonic forms on the 4-manifold, cf. [**43**].

The starting point of Taubes' program is the observation that on an oriented smooth 4-manifold with $b_2^+ \geq 1$, a self-dual harmonic form for a generic Riemannian metric has only regular zeroes, which consist of a disjoint union of embedded circles in the 4-manifold. In the complement of the zero set, the 2-form defines a symplectic structure, and furthermore, given the almost complex structure in the complement which is canonically defined by the metric and the self-dual 2-form, Taubes showed that nontriviality of the Seiberg-Witten invariant of the 4-manifold implies existence of pseudoholomorphic subvarieties in the complement which homologically bound the zero set. Having said this, the basic idea of the program is to cancel the zeroes of the self-dual 2-form to obtain a symplectic form on the 4-manifold, by modifying it in a neighborhood of the pseudoholomorphic subvarieties.

As illustrated in [**43**], it is instructive to look at the case where the 4-manifold is $S^1 \times M^3$, the product of a circle with a closed, oriented 3-manifold. Let α be a harmonic 1-form on M^3 with integral periods, which has only regular zeroes for a generic metric. In that case, $\alpha = df$ for some circle-valued harmonic Morse function f on M^3. Given with such a 1-form α, one can define a self-dual harmonic form ω on $S^1 \times M^3$ for the product metric by

$$\omega = dt \wedge \alpha + *_3 \alpha,$$

where t is the coordinate function on the S^1 factor, and $*_3$ is the Hodge-star operator on M^3. The zero set of ω is regular, and can be easily identified with $\sqcup_{\{p|df(p)=0\}} S^1 \times \{p\}$. Moreover, the pseudoholomorphic subvarieties in this case are nothing but the embedded tori or cylinders in $S^1 \times M^3$ which are of the form $S^1 \times \gamma$, where γ is an orbit of the gradient flow of the Morse function f, either closed or connecting two critical points of f. With these understood, Taubes' program for the 4-manifold $S^1 \times M^3$, if done S^1-equivariantly, is nothing but to cancel all critical points of a circle-valued Morse function on M^3 to make a fibration $M^3 \to S^1$. It is well-known that there are substantial difficulties in canceling critical points in dimension 3. This seems to suggest that in general one may expect similar difficulties in implementing Taubes' program in dimension 4 as well.

With the preceding understood, our philosophy is to consider Taubes' program in a more restricted context where the 4-manifold is already symplectic, so that one may use the existing symplectic structure as a reference point to guide the cancellation of the zeroes of a self-dual harmonic form. For a model of this consideration, we look at the case of $S^1 \times M^3$ where M^3 is fibered over S^1 with fibration $f_0 : M^3 \to S^1$. Suppose the circle-valued Morse function f is homotopic to f_0. Then a generic path of functions from f to f_0 will provide a guide to cancel the critical points of f through a sequence of birth/death of critical points of Morse functions on M^3. Note that Taubes' program in this restricted sense will not help to solve the existence problem of symplectic structures in general, but it may be used to construct symplectic structures with certain special features, e.g., equivariant symplectic structures in the presence of symmetry. (Note that this last point, when applied to the case of $S^1 \times M^3$, is related to the following conjecture which still remains open: If $S^1 \times M^3$ is symplectic, M^3 must be fibered over S^1. Under some stronger conditions, the conjecture was verified in [**13**] through a different approach.) Now we consider

Problem 1.3. Let G be a finite group acting smoothly on $\mathbb{CP}^2$ which has an isolated fixed point p and an invariant embedded 2-sphere S disjoint from p, such that S is symplectic with respect to the Kähler structure ω_0 and generates the second homology. Suppose ω is a G-equivariant, self-dual harmonic form which vanishes transversely in the complement of S and p. Modify ω in the sense of Taubes [**43**], away from S and p, to construct a G-equivariant symplectic form on $\mathbb{CP}^2$.

A positive solution to Problem 1.3 will confirm Conjecture 1.1, as we shall explain next.

Let W be a s-cobordism with boundary the disjoint union of elliptic 3-manifolds Y_1, Y_2. Clearly W is orientable. We note first that for any orientations on Y_1, Y_2 induced from an orientation on W, there exists an orientation-preserving homeomorphism from Y_1 to Y_2. Such a homeomorphism may be obtained as follows. Let $h : Y_1 \to Y_2$ be the simple homotopy equivalence induced by the s-cobordism W. Then h is easily seen to be orientation-preserving for any induced orientations on Y_1, Y_2. On the other hand, as a simple homotopy equivalence between geometric 3-manifolds, h is homotopic to a homeomorphism $\hat{h} : Y_1 \to Y_2$ (cf. [**44**], and for a proof, cf. [**29**]), which is clearly also orientation-preserving.

Next we recall the fact that every finite subgroup $G \subset SO(4)$ which acts freely on $\mathbb{S}^3$ is conjugate in $O(4)$ to a subgroup of $U(2)$. In order to understand this, we fix an identification $\mathbb{R}^4 = \mathbb{C}^2 = \mathbb{H}$, where $\mathbb{C}^2$ is identified with the space of quaternions $\mathbb{H}$ as follows

$$(z_1, z_2) \mapsto z_1 + z_2 j.$$

Consequently, the space of unit quaternions $\mathbb{S}^3$ is canonically identified with $SU(2)$. Consider the homomorphism $\phi : \mathbb{S}^3 \times \mathbb{S}^3 \to SO(4)$ which is defined such that for any $(q_1, q_2) \in \mathbb{S}^3 \times \mathbb{S}^3$, $\phi(q_1, q_2)$ is the element of $SO(4)$ that sends $x \in \mathbb{R}^4 = \mathbb{H}$ to $q_1 x q_2^{-1} \in \mathbb{H} = \mathbb{R}^4$. It is easily seen that ϕ is surjective with $\ker \phi = \{(1,1), (-1,-1)\}$, where we note that the center of $\mathbb{S}^3$ consists of $\{\pm 1\}$. Let $\mathbb{S}^1 \subset \mathbb{S}^3$ be the subset consisting of elements of the form $(z, 0) \in \mathbb{C}^2 = \mathbb{H}$. Then it is easily seen that a subgroup of $SO(4)$ acts complex linearly on $\mathbb{C}^2 = \mathbb{H}$ if it lies in the image $\phi(\mathbb{S}^1 \times \mathbb{S}^3)$. Note on the other hand that one can switch the two factors of $\mathbb{S}^3$ in $\mathbb{S}^3 \times \mathbb{S}^3$ by an element of $O(4)$ which sends $x \in \mathbb{R}^4 = \mathbb{H}$ to its conjugate $\bar{x} \in \mathbb{H} = \mathbb{R}^4$. With these understood, it suffices to note that every finite subgroup of $SO(4)$ which acts freely on $\mathbb{S}^3$ is conjugate in $SO(4)$ to a subgroup of either $\phi(\mathbb{S}^1 \times \mathbb{S}^3)$ or $\phi(\mathbb{S}^3 \times \mathbb{S}^1)$, cf. Theorem 4.10 in [**39**].

Now suppose W is a smooth s-cobordism of elliptic 3-manifolds Y_1, Y_2. By combining the aforementioned two facts, it is easily seen that for any fixed orientation on W, one can choose a normal orientation near ∂W such that with respect to the induced orientations on Y_1, Y_2, there exist orientation-preserving diffeomorphisms $f_1 : Y_1 \to \mathbb{S}^3/G$, $f_2 : Y_2 \to \mathbb{S}^3/G$, where $\mathbb{S}^3 \subset \mathbb{C}^2$ and G is a finite subgroup of $U(2)$ acting freely on $\mathbb{S}^3$, and $\mathbb{S}^3/G$ is given with the canonical orientation. Call the regular neighborhood of a component of ∂W the positive end (resp. negative end) of W if it is identified by an orientation-preserving map with $(-1, 0] \times (\mathbb{S}^3/G)$ (resp. $[0, 1) \times (\mathbb{S}^3/G)$).

Lemma 1.4. *By further applying a conjugation in $SO(4)$ to the G-action on the negative end if necessary, one can fix an identification $\mathbb{R}^4 = \mathbb{C}^2 = \mathbb{H}$ and regard G canonically as a subgroup of $U(2)$, such that there exists a 2-form ω on W, which is self-dual and harmonic with respect to some Riemannian metric, and has the following properties.*

1. *There are constants $\lambda_+ > \lambda_- > 0$ for which $\omega = \lambda_+\omega_0$ on the positive end and $\omega = \lambda_-\omega_0$ on the negative end. Here ω_0 is the standard symplectic form on $\mathbb{C}^2/G$ (i.e., the descendant of $\frac{\sqrt{-1}}{2}\sum_{i=1}^{2} dz_i \wedge d\bar{z}_i$ on $\mathbb{C}^2$), and the two ends of W are identified via f_1, f_2 to the corresponding neighborhoods of $\mathbb{S}^3/G$ in $\mathbb{C}^2/G$.*
2. *The 2-form ω has only regular zeroes.*

With this understood, let $\widetilde{W}$ be the universal cover of W and $\tilde{\omega}$ be the pull-back of ω to $\widetilde{W}$. Then $\partial\widetilde{W} = \mathbb{S}^3 \sqcup \mathbb{S}^3$ and $\tilde{\omega}$ equals a constant multiple of the standard symplectic form on $\mathbb{C}^2$ near $\partial\widetilde{W}$. In particular, both ends of $\widetilde{W}$ are of contact type with respect to $\tilde{\omega}$, with one end convex and one end concave. As $\tilde{\omega}$ is invariant under the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$, we can close up $\widetilde{W}$ by collapsing each fiber of the Hopf fibration on the convex end and capping off the concave end with the standard symplectic 4-ball $\mathbb{B}^4$. The resulting smooth 4-manifold X is a homotopy $\mathbb{CP}^2$, with a smoothly embedded 2-sphere S representing a generator of $H_2(X;\mathbb{Z})$ and a self-dual harmonic form $\omega^\dagger$, which is G-equivariant with respect to the obvious G-action on X, has only regular zeroes, and obeys $\omega^\dagger|_S > 0$.

If, furthermore, the universal cover $\widetilde{W}$ is smoothly a product, then the 4-manifold X is diffeomorphic to $\mathbb{CP}^2$, with a symplectic form $\omega_0^\dagger$ such that $\omega_0^\dagger|_S > 0$. It is clear now that a positive solution to Problem 1.3, when applied to X, would yield a symplectic structure on W, making it into a symplectic s-cobordism. By Theorem 1.2, W is smoothly a product.

Remark 1.5. We add a remark here about the smooth s-cobordism W_r with $r = 1$ in the examples of Cappell and Shaneson [**8**, **9**]. According to Akbulut's theorem in [**2**], the universal cover of W_r with $r = 1$ is smoothly a product. Thus Conjecture 1.1 suggests that the s-cobordism itself is smoothly a product. It would be interesting to find out by direct means such as in Akbulut [**2**] whether W_r with $r = 1$ has an exotic smooth structure (it is known to be topologically a product by the classification of Cappell and Shaneson [**6**, **7**]).

We now turn to the technical aspect of this paper.

Despite the tremendous progress over the last two decades, topology of smooth 4-manifolds is still largely obscure as far as classification is concerned. In particular, there is lack of effective methods for determining the diffeomorphism type of a 4-manifold in a given homotopy class. However, in some rare cases and under an additional assumption that the 4-manifold is symplectic, Gromov in [**18**] showed us how to recover the diffeomorphism type using certain moduli space of pseudoholomorphic curves (if it is nonempty). Later, Taubes showed in [**42**] that in Gromov's argument, the existence of pseudoholomorphic curves may be replaced by a condition on the Seiberg-Witten invariant of the 4-manifold, which is something more manageable. As a typical example one obtains the following theorem.

Theorem 1.6 (Gromov-Taubes)**.** *Let X be a symplectic 4-manifold with the rational homology of $\mathbb{CP}^2$. Then X is diffeomorphic to $\mathbb{CP}^2$ if the Seiberg-Witten invariant of X at the 0-chamber vanishes, e.g., if X has a metric of positive scalar curvature.*

Our proof of Theorem 1.2, in a nutshell, is based on an orbifold analog of the above theorem.

More precisely, in order to prove Theorem 1.2 we extend in this paper (along with the earlier one [**11**]) Gromov's pseudoholomorphic curve techniques and Taubes' work on the Seiberg-Witten invariants of symplectic 4-manifolds to the case of 4-orbifolds. (See [**12**] for an exposition.) In particular, we prove the following theorem (see Theorem 2.2 for more details).

Theorem 1.7 (Orbifold Version of Taubes' Theorem "$SW \Rightarrow Gr$")**.** *Let (X, ω) be a symplectic 4-orbifold. Suppose E is an orbifold complex line bundle such that the corresponding Seiberg-Witten invariant (in Taubes chamber when $b_2^+(X) = 1$) is nonzero. Then for any ω-compatible almost complex structure J, the Poincaré dual of $c_1(E)$ is represented by J-holomorphic curves in X.*

Remark 1.8.

(1) The proof of Theorem 1.7 follows largely the proof of Taubes in [**42**]. However, we would like to point out that Taubes' proof

involves in a few places Green's function for the Laplacian $\Delta = d^*d$ and a covering argument by geodesic balls of uniform size. This part of the proof requires the assumption that the injective radius is uniformly bounded from below, which does not generalize to the case of orbifolds straightforwardly. Some modification or reformulation is needed here.

(2) The situation of the full version of Taubes' theorem "$SW = Gr$" is more complicated for 4-orbifolds. In fact, the proof of "$SW = Gr$" relies on a regularity result (i.e., embeddedness) of the J-holomorphic curves in Taubes' theorem "$SW \Rightarrow Gr$" for a generic almost complex structure. While this is generally no longer true for 4-orbifolds, how "regular" the J-holomorphic curves in Theorem 1.7 could be depends, in a very interesting way, on what types of singularities the 4-orbifold has. We plan to explore this issue on a future occasion.

(3) There are only a few examples of 4-manifolds which are symplectic and have a metric of positive scalar curvature. Hence the 4-manifold in Theorem 1.6 rarely occurs. On the other hand, there are numerous examples of symplectic 4-orbifolds which admit positive scalar curvature metrics. In fact, there is a class of normal complex surfaces, called log Del Pezzo surfaces, which are Kähler orbifolds with positive first Chern class. (By Yau's theorem, these surfaces admit positive Ricci curvature metrics.) Unlike their smooth counterpart, log Del Pezzo surfaces occur in bewildering abundance and complexity (cf. e.g., [**23**]). Recently, these singular surfaces appeared in the construction of Sasakian-Einstein metrics on certain 5-manifolds (including $\mathbb{S}^5$). In particular, the following question arose naturally in this context: What are the log Del Pezzo surfaces that appear as the quotient space of a fixed point free $\mathbb{S}^1$-action on $\mathbb{S}^5$? (See Kollár [**24**].) We believe that the techniques developed in this paper would be useful in answering this question.

We end this section with an outline for the proof of Theorem 1.2. First of all, note that the case where the elliptic 3-manifold is a lens space was settled in [**11**] using a different method. Hence, in this paper, we shall only consider the remaining cases, where the elliptic 3-manifold is diffeomorphic to $\mathbb{S}^3/G$ with G being a non-abelian subgroup of $U(2)$.

Let W be a symplectic s-cobordism as in Theorem 1.2. Note that near the boundary the symplectic form on W is standard, and is invariant under the obvious Seifert fibration on the boundary. We close up W by collapsing each fiber of the Seifert fibration on the convex end of W and capping off the concave end with a standard symplectic cone — a regular neighborhood of $\{0 \in \mathbb{C}^2\}/G$ in the orbifold $\mathbb{C}^2/G$ which is given with the standard symplectic structure. The diffeomorphism type of W can be easily recovered from that of the resulting symplectic 4-orbifold X. In order to determine the diffeomorphism type of X, we compare it with the "standard" 4-orbifold X_0, which is $\mathbb{B}^4/G$ with boundary $\mathbb{S}^3/G$ collapsed along the fibers of the Seifert fibration. More concretely, we consider the space $\mathcal{M}$ of pseudoholomorphic maps into X, which corresponds, under the obvious homotopy equivalence $X \to X_0$, to the family of complex lines in $\mathbb{B}^4/G$ with boundary collapsed. Using the pseudoholomorphic curve theory of 4-orbifolds developed in [**11**], one can easily show that X is diffeomorphic to X_0 provided that $\mathcal{M} \neq \emptyset$, from which Theorem 1.2 follows.

Thus the bulk of the argument is devoted to proving that $\mathcal{M} \neq \emptyset$. We follow the usual strategy of applying Taubes' theorem "$SW \Rightarrow Gr$". More concretely, the proof of $\mathcal{M} \neq \emptyset$ consists of the following three steps.

(1) Construct an orbifold complex line bundle E such that the homology class of a member of $\mathcal{M}$ is Poincaré dual to $c_1(E)$. Note that when X is smooth, a complex line bundle is determined by its Chern class in $H^2(X;\mathbb{Z})$. This is no longer true for orbifolds. In particular, we have to construct E by hand, which is given in Lemma 3.6. The explicit construction of E is also needed in order to calculate the contribution of singular points of X to the dimension $d(E)$ of the Seiberg-Witten moduli space corresponding to E, which is a crucial factor in the proof. (See Lemma 3.8.)
(2) Show that the Seiberg-Witten invariant corresponding to E is zero in the 0-chamber. This follows from the fact that the 4-orbifold X contains a 2-suborbifold C_0 which has a metric of positive curvature and generates $H_2(X;\mathbb{Q})$. Here C_0 is the image of the convex boundary component of W in X. (See Lemma 3.7.)
(3) By a standard wall-crossing argument, with the fact that $d(E) \geq 0$, the Poincaré dual of $c_1(E)$ is represented by J-holomorphic curves by Theorem 1.7. The main issue here is to show that there is a component of the J-holomorphic curves which is the image of a member of $\mathcal{M}$, so that $\mathcal{M}$ is not empty. When $c_1(E) \cdot c_1(E)$ is relatively small, one can show that this is indeed the case by using the adjunction formula in [**11**]. The key observation is that, when $c_1(E)\cdot c_1(E)$ is not small, the dimension $d(E)$ of the Seiberg-Witten moduli space is also considerably large, so that one may break the J-holomorphic curves from Theorem 1.7 into smaller components by requiring them to pass through a certain number (equaling half of the dimension $d(E)$) of specified points. It turns out that one of the resulting smaller components is the image of a member of $\mathcal{M}$, so that $\mathcal{M}$ is also nonempty in this case. This part of the proof is the content of Lemma 3.9, which is the most delicate one,

often involving a case-by-case analysis according to the type of the group G in $\mathbb{S}^3/G$.

The organization of this paper is as follows. In §2 we briefly go over the Seiberg-Witten-Taubes theory for 4-orbifolds, ending with a statement of the orbifold version of Taubes' theorem, whose proof is postponed to §4. The proof of the main result, Theorem 1.2, is given in §3. There are three appendices. Appendix A contains a brief review of the index theorem over orbifolds in Kawasaki [**22**], and a calculation for the dimension of the relevant Seiberg-Witten moduli space. Appendix B is concerned with some specific form of Green's function for the Laplacian $\Delta = d^*d$ on orbifolds, which is involved in the proof of Taubes' theorem for 4-orbifolds. In Appendix C, we give a proof of Lemma 1.4.

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2. The Seiberg-Witten-Taubes theory for 4-orbifolds

In this section, we first go over the Seiberg-Witten theory for smooth 4-orbifolds, and then we extend Taubes' work [**40**, **41**, **42**] on symplectic 4-manifolds to the orbifold setting. The discussion will be brief since the theory is parallel to the one for smooth 4-manifolds.

Let X be an oriented smooth 4-orbifold. Given any Riemannian metric on X, a $\text{Spin}^{\mathbb{C}}$ structure is an orbifold principal $\text{Spin}^{\mathbb{C}}(4)$ bundle over X which descends to the orbifold principal $SO(4)$ bundle of oriented orthonormal frames under the canonical homomorphism $\text{Spin}^{\mathbb{C}}(4) \to SO(4)$. There are two associated orbifold $U(2)$ vector bundles (of rank 2) S_+, S_- with $\det(S_+) = \det(S_-)$, and a Clifford multiplication which maps T^*X into the skew adjoint endomorphisms of $S_+ \oplus S_-$.

The Seiberg-Witten equations associated to the $\text{Spin}^{\mathbb{C}}$ structure (if there is one) are equations for a pair (A, ψ), where A is a connection on $\det(S_+)$ and ψ is a section of S_+. Recall that the Levi-Civita connection together with A defines a covariant derivative ∇_A on S_+. On the other hand, there are two maps $\sigma : S_+ \otimes T^*X \to S_-$ and $\tau : \text{End}(S_+) \to \Lambda_+ \otimes \mathbb{C}$ induced by the Clifford multiplication, with the latter being the adjoint of $c_+ : \Lambda_+ \to \text{End}(S_+)$, where Λ_+ is the orbifold bundle of self-dual 2-forms. With these understood, the Seiberg-Witten equations read

$$D_A\psi = 0 \text{ and } P_+F_A = \frac{1}{4}\tau(\psi \otimes \psi^*) + \mu,$$

where $D_A \equiv \sigma \circ \nabla_A$ is the Dirac operator, $P_+ : \Lambda^2 T^* X \to \Lambda_+$ is the orthogonal projection, and μ is a fixed, imaginary valued, self-dual 2-form which is added in as a perturbation term.

The Seiberg-Witten equations are invariant under the gauge transformations $(A, \psi) \mapsto (A - 2\varphi^{-1} d\varphi, \varphi\psi)$, where $\varphi \in C^\infty(X; S^1)$ are circle valued smooth functions on X. The space of solutions modulo gauge equivalence, denoted by M, is compact, and when $b_2^+(X) \geq 1$ and nonempty, M is a smooth orientable manifold for a generic choice of (g, μ), where g is the Riemannian metric and μ is the self-dual 2-form of perturbations. Furthermore, M contains no classes of reducible solutions (i.e., those with $\psi \equiv 0$), and letting M^0 be the space of solutions modulo the based gauge group, i.e., those $\varphi \in C^\infty(X; S^1)$ such that $\varphi(p_0) = 1$ for a fixed base point $p_0 \in X$, then $M^0 \to M$ defines a principal S^1-bundle. Let c be the first Chern class of $M^0 \to M$, $d = \dim M$, and fix an orientation of M. Then the Seiberg-Witten invariant associated to the Spin$^{\mathbb{C}}$ structure is defined as follows:

- When $d < 0$ or $d = 2n + 1$, the Seiberg-Witten invariant is zero.
- When $d = 0$, the Seiberg-Witten invariant is a signed sum of the points in M.
- When $d = 2n > 0$, the Seiberg-Witten invariant equals $c^n[M]$.

As in the case of smooth 4-manifolds, the Seiberg-Witten invariant of X is well-defined when $b_2^+(X) \geq 2$, depending only on the diffeomorphism class of X (as orbifolds). Moreover, there is an involution on the set of Spin$^{\mathbb{C}}$ structures which preserves the Seiberg-Witten invariant up to a change of sign. When $b_2^+(X) = 1$, there is a chamber structure and the Seiberg-Witten invariant also depends on the chamber where the pair (g, μ) is in. Moreover, the change of the Seiberg-Witten invariant when crossing a wall of the chambers can be similarly analyzed as in the smooth 4-manifold case.

For the purpose of this paper, we need the following wall-crossing formula. Its proof is identical to the manifold case, and hence is omitted, cf. e.g., [**25**].

Lemma 2.1. *Suppose* $b_1(X) = 0$, $b_2^+(X) = b_2(X) = 1$, *and* $c_1(S_+) \neq 0$. *Then there are two chambers for the Seiberg-Witten invariant associated to the* Spin$^{\mathbb{C}}$ *structure* $S_+ \oplus S_-$: *the* 0*-chamber where* $\int_X \sqrt{-1}\mu \wedge \omega_g$ *is sufficiently close to* 0, *and the* ∞*-chamber where* $\int_X \sqrt{-1}\mu \wedge \omega_g$ *is sufficiently close to* $+\infty$. *Here* ω_g *is a fixed harmonic* 2*-form with respect to the Riemannian metric* g *such that* $c_1(S_+) \cdot [\omega_g] > 0$. *Moreover, if the dimension of the Seiberg-Witten moduli space* M (*which is always an even number in this case*) *is non-negative, the Seiberg-Witten invariant changes by* ± 1 *when considered in the other chamber.*

Now we focus on the case where X is a symplectic 4-orbifold. Let ω be a symplectic form on X. We orient X by $\omega \wedge \omega$, and fix a ω-compatible almost complex structure J. Then with respect to the associated Riemannian metric $g = \omega(\cdot, J\cdot)$, ω is self-dual with $|\omega| = \sqrt{2}$. The set of Spin$^{\mathbb{C}}$ structures on X is nonempty. In fact, the almost complex structure J gives rise to a canonical Spin$^{\mathbb{C}}$ structure where the associated orbifold $U(2)$ bundles are $S_+^0 = \mathbb{I} \oplus K_X^{-1}$, $S_-^0 = T^{0,1}X$. Here $\mathbb{I}$ is the trivial orbifold complex line bundle and K_X is the canonical bundle $\det(T^{1,0}X)$. Moreover, the set of Spin$^{\mathbb{C}}$ structures is canonically identified with the set of orbifold complex line bundles where each orbifold complex line bundle E corresponds to a Spin$^{\mathbb{C}}$ structure whose associated orbifold $U(2)$ bundles are $S_+^E = E \oplus (K_X^{-1} \otimes E)$ and $S_-^E = T^{0,1}X \otimes E$. The involution on the set of Spin$^{\mathbb{C}}$ structures which preserves the Seiberg-Witten invariant up to a change of sign sends E to $K_X \otimes E^{-1}$.

As in the manifold case, there is a canonical (up to gauge equivalence) connection A_0 on $K_X^{-1} = \det(S_+^0)$ such that the fact $d\omega = 0$ implies that $D_{A_0}u_0 = 0$ for the section $u_0 \equiv 1$ of $\mathbb{I}$ which is considered as the section $(u_0, 0)$ in $S_+^0 = \mathbb{I} \oplus K_X^{-1}$. Furthermore, by fixing such an A_0, any connection A on $\det(S_+^E) = K_X^{-1} \otimes E^2$ is canonically determined by a connection a on E. With these understood, there is a distinguished family of the Seiberg-Witten equations on X, which is parametrized by a real number $r > 0$ and is for a triple (a, α, β), where in the equtions, the section ψ of S_+^E is written as $\psi = \sqrt{r}(\alpha, \beta)$ and the perturbation term μ is taken to be $-\sqrt{-1}(4^{-1}r\omega) + P_+F_{A_0}$. (Here α is a section of E and β a section of $K_X^{-1} \otimes E$.) Note that when $b_2^+(X) = 1$, this distinguished family of Seiberg-Witten equations (with $r \gg 0$) belongs to a specific chamber for the Seiberg-Witten invariant. This particular chamber will be referred to as the Taubes chamber.

The following is the analog of the relevant theorems of Taubes in the orbifold setting. (Its proof is postponed to §4.)

Theorem 2.2. *Let (X, ω) be a symplectic 4-orbifold. Then the following are true.*

(1) *The Seiberg-Witten invariant associated to the canonical* Spin$^{\mathbb{C}}$ *structure equals ± 1. (When $b_2^+(X) = 1$, the Seiberg-Witten invariant is in Taubes chamber.) Moreover, when $b_2^+(X) \geq 2$, the Seiberg-Witten invariant corresponding to the canonical bundle K_X equals ± 1, and for any orbifold complex line bundle E, if the Seiberg-Witten invariant corresponding to E is nonzero, then E must satisfy*

$$0 \leq c_1(E) \cdot [\omega] \leq c_1(K_X) \cdot [\omega],$$

where $E = \mathbb{I}$ or $E = K_X$ when either equality holds.

(2) *Let E be an orbifold complex line bundle. Suppose there is an unbounded sequence of values for the parameter r such that the*

corresponding Seiberg-Witten equations have a solution (a, α, β). *Then for any* ω*-compatible almost complex structure* J, *there are* J*-holomorphic curves* $C_1, C_2, \dots, C_k$ *in* X *and positive integers* $n_1, n_2, \dots, n_k$ *such that* $c_1(E) = \sum_{i=1}^k n_i PD(C_i)$. *Moreover, if a subset* $\Omega \subset X$ *is contained in* $\alpha^{-1}(0)$ *throughout, then* $\Omega \subset \cup_{i=1}^k C_i$ *also.*

(*Here* $PD(C)$ *is the Poincaré dual of the* J*-holomorphic curve* C. *See* §3 *of* [**11**] *for the definition of* J*-holomorphic curves in an almost complex* 4*-orbifold and the definition of Poincaré dual of a* J*-holomorphic curve in the* 4*-orbifold.*)

Remark 2.3. There are two typical sources for the subset Ω in the theorem. For the first one, suppose $p \in X$ is an orbifold point such that the isotropy group at p acts nontrivially on the fiber of E at p. Then $p \in \alpha^{-1}(0)$ for any solution (a, α, β), and consequently $p \in \cup_{i=1}^k C_i$. For the second one, suppose the Seiberg-Witten invariant corresponding to E is nonzero and the dimension of the moduli space M is $d = 2n > 0$. Then for any subset of distinct n points $p_1, p_2, \dots, p_n \in X$, and for any value of parameter r, there is a solution (a, α, β) such that $\{p_1, p_2, \dots, p_n\} \subset \alpha^{-1}(0)$. Consequently, we may require the J-holomorphic curves $C_1, C_2, \dots, C_k$ in the theorem to contain any given subset of less than or equal to n points in this circumstance. (The proof goes as follows. Observe that the map $(a, \alpha, \beta) \mapsto \alpha(p)$ descends to a section s_p of the complex line bundle associated to the principal S^1 bundle $M^0 \to M$, where M^0 is the moduli space of solutions modulo the based gauge group with base point p. Moreover, there are submanifolds $\Sigma_1, \Sigma_2, \dots, \Sigma_n$ of codimension 2 in M such that each Σ_i is Poincaré dual to the first Chern class c of $M^0 \to M$ and is arbitrarily close to $s_{p_i}^{-1}(0)$. Now if there were no solution (a, α, β) such that $\{p_1, p_2, \dots, p_n\} \subset \alpha^{-1}(0)$, which means that $s_{p_1}^{-1}(0) \cap s_{p_2}^{-1}(0) \cap \dots \cap s_{p_n}^{-1}(0) = \emptyset$, then one would have $\Sigma_1 \cap \Sigma_2 \cap \dots \cap \Sigma_n = \emptyset$. But this contradicts the assumption that $c^n[M]$, the Seiberg-Witten invariant, is nonzero.)

3. Proof of the main result

We begin by recalling the classification of finite subgroups of $\mathrm{GL}(2, \mathbb{C})$ without quasi-reflections, which is due to Brieskorn [**5**]. The following is a list of the non-abelian ones up to conjugations in $\mathrm{GL}(2, \mathbb{C})$.

- $\langle Z_{2m}, Z_{2m}; \widetilde{D}_n, \widetilde{D}_n \rangle$, where m is odd, $n \geq 2$, and m, n are relatively prime.
- $\langle Z_{4m}, Z_{2m}; \widetilde{D}_n, C_{2n} \rangle$, where m is even, $n \geq 2$, and m, n are relatively prime.
- $\langle Z_{2m}, Z_{2m}; \widetilde{T}, \widetilde{T} \rangle$, where m and 6 are relatively prime.

- $\langle Z_{6m}, Z_{2m}; \widetilde{T}, \widetilde{D}_2\rangle$, where m is odd and is divisible by 3.
- $\langle Z_{2m}, Z_{2m}; \widetilde{O}, \widetilde{O}\rangle$, where m and 6 are relatively prime.
- $\langle Z_{2m}, Z_{2m}; \widetilde{I}, \widetilde{I}\rangle$, where m and 30 are relatively prime.

Here $Z_k \subset \mathrm{ZL}(2,\mathbb{C})$ is the cyclic subgroup of order k in the center of $\mathrm{GL}(2,\mathbb{C})$, $C_k \subset SU(2)$ is the cyclic subgroup of order k, and $\widetilde{D}_n, \widetilde{T}, \widetilde{O}, \widetilde{I} \subset SU(2)$ are the binary dihedral, tetrahedral, octahedral and icosahedral groups of order $4n$, 24, 48 and 120 respectively, which are the double covers of the corresponding subgroups of $SO(3)$ under the canonical homomorphism $SU(2) \to SO(3)$. As for the notation $\langle H_1, N_1; H_2, N_2\rangle$, it stands for the image under $(h_1, h_2) \mapsto h_1 h_2$ of the subgroup of $H_1 \times H_2$, which consists of pairs (h_1, h_2) such that the classes of h_1 and h_2 in H_1/N_1 and H_2/N_2 are equal under some fixed isomorphism $H_1/N_1 \cong H_2/N_2$. (In the present case, the group does not depend on the isomorphism $H_1/N_1 \cong H_2/N_2$, at least up to conjugations in $\mathrm{GL}(2,\mathbb{C})$.)

We shall assume throughout that the elliptic 3-manifolds under consideration are diffeomorphic to $\mathbb{S}^3/G$ for some finite subgroup $G \subset \mathrm{GL}(2,\mathbb{C})$ listed above. (Note that a finite subgroup $G \subset U(2)$ acts freely on $\mathbb{S}^3$ if and only if G is a subgroup of $\mathrm{GL}(2,\mathbb{C})$ containing no quasi-reflections.)

Next we begin by collecting some preliminary but relevant information about the elliptic 3-manifold $\mathbb{S}^3/G$. First of all, note that G contains a cyclic subgroup of order $2m$, Z_{2m}, which is the subgroup that preserves each fiber of the Hopf fibration on $\mathbb{S}^3$. Evidently, the Hopf fibration induces a canonical Seifert fibration on $\mathbb{S}^3/G$, which can be obtained in two steps as follows. First, quotient $\mathbb{S}^3$ and the Hopf fibration by the subgroup Z_{2m} to obtain the lens space $L(2m,1)$ and the $\mathbb{S}^1$-fibration on it. Second, quotient $L(2m,1)$ and the $\mathbb{S}^1$-fibration by G/Z_{2m} to obtain $\mathbb{S}^3/G$ and the Seifert fibration on $\mathbb{S}^3/G$. It follows immediately that the Euler number of the Seifert fibration is

$$e = \frac{|Z_{2m}|}{|G/Z_{2m}|} = \frac{4m^2}{|G|}.$$

The Seifert fibration has three singular fibers, and the normalized Seifert invariant

$$(b, (a_1, b_1), (a_2, b_2), (a_3, b_3)), \text{ where } 0 < b_i < a_i,\ a_i, b_i \text{ relatively prime},$$

can be determined from the Euler number e and the induced action of G/Z_{2m} on the base of the $\mathbb{S}^1$-fibration on $L(2m,1)$. We collect these data in the following list.

- $\langle Z_{2m}, Z_{2m}; \widetilde{D}_n, \widetilde{D}_n\rangle$, $\langle Z_{4m}, Z_{2m}; \widetilde{D}_n, C_{2n}\rangle$: $(a_1, a_2, a_3) = (2, 2, n)$, and b, b_1, b_2 and b_3 are given by $b_1 = b_2 = 1$, $m = (b+1)n + b_3$.
- $\langle Z_{2m}, Z_{2m}; \widetilde{T}, \widetilde{T}\rangle$, $\langle Z_{6m}, Z_{2m}; \widetilde{T}, \widetilde{D}_2\rangle$: $(a_1, a_2, a_3) = (2, 3, 3)$, and b, b_1, b_2 and b_3 are given by $b_1 = 1$, $m = 6b + 3 + 2(b_2 + b_3)$.

- $\langle Z_{2m}, Z_{2m}; \widetilde{O}, \widetilde{O}\rangle$: $(a_1, a_2, a_3) = (2, 3, 4)$, and b, b_1, b_2 and b_3 are given by $b_1 = 1$, $m = 12b + 6 + 4b_2 + 3b_3$.
- $\langle Z_{2m}, Z_{2m}; \widetilde{I}, \widetilde{I}\rangle$: $(a_1, a_2, a_3) = (2, 3, 5)$, and b, b_1, b_2 and b_3 are given by $b_1 = 1$, $m = 30b + 15 + 10b_2 + 6b_3$.

Now let $\omega_0 = \frac{\sqrt{-1}}{2}\sum_{i=1}^{2} dz_i \wedge d\bar{z}_i$. We consider the Hamiltonian $\mathbb{S}^1$-action on $(\mathbb{C}^2, \omega_0)$ given by the complex multiplication, with the Hamiltonian function given by $\mu(z_1, z_2) = \frac{1}{2}(|z_1|^2 + |z_2|^2)$. It commutes with the action of G on $\mathbb{C}^2$, hence a Hamiltonian $\mathbb{S}^1$-action on the symplectic orbifold $(\mathbb{C}^2, \omega_0)/G$ results, with the Hamiltonian function μ' equaling $\frac{1}{2m}$ times the descendant of μ to $\mathbb{C}^2/G$. Given any $r > 0$, consider the subset $(\mu')^{-1}([0, r]) \subset \mathbb{C}^2/G$. According to [**30**], we can collapse each fiber of the $\mathbb{S}^1$-action in $(\mu')^{-1}(r) \subset (\mu')^{-1}([0, r])$ to obtain a closed symplectic 4-dimensional orbifold, which we denote by X_r. The symplectic 4-orbifold X_r contains $(\mu')^{-1}([0, r))$ as an open symplectic suborbifold, and also contains a 2-dimensional symplectic suborbifold $C_0 \equiv (\mu')^{-1}(r)/\mathbb{S}^1$. Note that $(\mu')^{-1}(r) \to C_0$ is the canonical Seifert fibration on $\mathbb{S}^3/G$ we mentioned earlier. Moreover, the Euler number of the normal bundle of C_0 in X_r equals the Euler number of the Seifert fibration on $\mathbb{S}^3/G$.

Suppose we are given with a symplectic s-cobordism W of the elliptic 3-manifold $\mathbb{S}^3/G$ to itself. Fixing a sufficiently large $r > 0$, there are $0 < r_1, r_2 < r$ such that a neighborhood of the convex end of W is symplectomorphic to a neighborhood of the boundary of $(\mu')^{-1}([0, r_1])$ and a neighborhood of the concave end is symplectomorphic to a neighborhood of the boundary of $X_r \setminus (\mu')^{-1}([0, r_2))$. We close up W by gluing $X_r \setminus (\mu')^{-1}([0, r_1))$ to the convex end and gluing $(\mu')^{-1}([0, r_2])$ to the concave end. We denote the resulting symplectic 4-orbifold by (X, ω). Note that X inherits a 2-dimensional symplectic suborbifold C_0 from $X_r \setminus (\mu')^{-1}([0, r_1))$, whose normal bundle in X has an Euler number equaling that of the Seifert fibration on $\mathbb{S}^3/G$. We fix a ω-compatible almost complex structure J on X such that C_0 is J-holomorphic and J is integrable in a neighborhood of each singular point of X. (Note that the latter is possible because of the equivariant Darboux' theorem.)

The 4-orbifold X has 4 singular points. One of them, denoted by p_0, is inherited from $(\mu')^{-1}([0, r_2])$ and has a neighborhod modeled by that of $\{0 \in \mathbb{C}^2\}/G$. The other three, denoted by p_1, p_2 and p_3, are all contained in C_0, and are of type (a_1, b_1), (a_2, b_2) and (a_3, b_3) respectively, where $\{(a_i, b_i) \mid i = 1, 2, 3\}$ is part of the normalized Seifert invariant of the Seifert fibration on $\mathbb{S}^3/G$. (Here a singular point is said of type (a, b) if the isotropy group is cyclic of order a and the action on a local uniformizing system is of weight $(1, b)$.) The Betti numbers of X are $b_1(X) = b_3(X) = 0$ and $b_2(X) = b_2^+(X) = 1$. In fact, we have $H_2(X; \mathbb{Q}) = \mathbb{Q} \cdot [C_0]$, and using the intersection product $C_0 \cdot C_0$, we may identify $H^2(X; \mathbb{Q})$ with $H_2(X; \mathbb{Q}) = \mathbb{Q} \cdot [C_0]$ canonically. Finally, using the normalized Seifert invariant and the adjunction formula (cf. [**11**], Theorem 3.1), we obtain

$$C_0 \cdot C_0 = \frac{4m^2}{|G|} \text{ and } c_1(K_X) \cdot C_0 = -\frac{4m(m+1)}{|G|},$$

where K_X is the canonical bundle of (X, J), and m is one half of the order of the subgroup Z_{2m} of G which preserves each fiber of the Hopf fibration.

With the preceding understood, we now introduce the relevant moduli space of pseudoholomorphic curves. To this end, let Σ be the orbifold Riemann sphere of one orbifold point $z_\infty = \infty$ of order $2m$. (Recall that $2m$ is the order of the cyclic subgroup $Z_{2m} \subset G$ which preserves each fiber of the Hopf fibration.) Note that Σ has a unique complex structure. The group of automorphisms of Σ, denoted by $\mathcal{G}$, is easily identified with the group of linear translations on $\mathbb{C}$.

We shall consider the space $\mathcal{M}$ of J-holomorphic maps $f : \Sigma \to X$ such that

(1) The homology class $[f(\Sigma)] \in H_2(X; \mathbb{Z})$ obeys $[f(\Sigma)] \cdot C_0 = 1$.
(2) $f(z_\infty) = p_0$, and in a local representative (f_∞, ρ_∞) of f at z_∞,

$$\rho_\infty(\mu_{2m}) = \mu_{2m} I \equiv \begin{pmatrix} \mu_{2m} & 0 \\ 0 & \mu_{2m} \end{pmatrix} \in Z_{2m}, \text{ where } \mu_k \equiv \exp(\sqrt{-1}\frac{2\pi}{k}).$$

Here the notion of maps between orbifolds is as defined in [**10**]. For the terminology used in this paper in connection with J-holomorphic maps or curves, the reader is specially referred to the earlier paper [**11**]. With these understood, we remark that $\mathcal{G}$ acts on $\mathcal{M}$ by reparametrization.

The following proposition is the central technical result of this section.

Proposition 3.1. *The space $\mathcal{M}$ is nonempty, and is a smooth 6-dimensional manifold. Moreover, the quotient space $\mathcal{M}/\mathcal{G}$ is compact.*

The proof of Proposition 3.1 will be given through a sequence of lemmas. We begin with the Fredholm theory for pseudoholomorphic curves in a symplectic 4-orbifold (X, ω).

Let (Σ, j) be an orbifold Riemann surface with a fixed complex structure j. Consider the space of C^k maps $[\Sigma; X]$ from Σ to X for some fixed, sufficiently large integer $k > 0$. By Theorem 1.4 in Part I of [**10**], the space $[\Sigma; X]$ is a smooth Banach orbifold, which we may simply assume to be a smooth Banach manifold for the sake of technical simplicity, because the relevant subset $\mathcal{M}$ in the present case is actually contained in the smooth part of $[\Sigma; X]$. There is a Banach bundle $\mathcal{E} \to [\Sigma; X]$, with a Fredholm section $\underline{L}$ defined by

$$\underline{L}(f) = df + J \circ df \circ j, \ \forall f \in [\Sigma; X].$$

The zero locus $\underline{L}^{-1}(0)$ is the set of J-holomorphic maps from (Σ, j) into X. In the present case, $\mathcal{M}$ is contained in $\underline{L}^{-1}(0)$ as an open subset with respect to the induced topology.

The index of the linearization $D\underline{L}$ at $f \in \underline{L}^{-1}(0)$ can be computed using the index theorem of Kawasaki [**22**] for elliptic operators on orbifolds, see Appendix A for a relevant review. To state the general index formula for $D\underline{L}$ (cf. Lemma 3.2.4 of [**14**]), let (Σ, j) be an orbifold Riemann surface with orbifold points z_i of order m_i, where $i = 1, 2, \ldots, l$, and let $f : \Sigma \to X$ be a J-holomorphic map from (Σ, j) into an almost complex 4-orbifold (X, J). If a local representative of f at each z_i is given by (f_i, ρ_i) where $\rho_i(\mu_{m_i})$ acts on a local uniformizing system at $f(z_i)$ by $\rho_i(\mu_{m_i}) \cdot (z_1, z_2) = (\mu_{m_i}^{m_{i,1}} z_1, \mu_{m_i}^{m_{i,2}} z_2)$, with $0 \leq m_{i,1}, m_{i,2} < m_i$ (here $\mu_k \equiv \exp(\sqrt{-1}\frac{2\pi}{k})$), then the index of $D\underline{L}$ at f is $2d$ with

$$d = c_1(TX) \cdot [f(\Sigma)] + 2 - 2g_{|\Sigma|} - \sum_{i=1}^{l} \frac{m_{i,1} + m_{i,2}}{m_i},$$

where $g_{|\Sigma|}$ is the genus of the underlying Riemann surface of Σ. In the present case, for each $f \in \mathcal{M}$, one half of the index of $D\underline{L}$ at f is

$$d = \frac{m+1}{m} + 2 - \frac{1+1}{2m} = 3.$$

Thus $\mathcal{M}$ is a 6-dimensional smooth manifold provided that it is non-empty and $\underline{L}$ is transversal to the zero section at $\mathcal{M}$.

Transversality of the Fredholm section $\underline{L}$ at its zero locus can be addressed in a similar fashion as in the case when X is a manifold. For the purpose of this paper, we shall use the following regularity criterion, which is the orbifold analog of Lemma 3.3.3 in [**35**].

Lemma 3.2. *Let $f : \Sigma \to X$ be a J-holomorphic map from an orbifold Riemann surface into an almost complex 4-orbifold. Suppose at each $z \in \Sigma$, the map f_z in a local representative (f_z, ρ_z) of f at z is embedded. Then f is a smooth point in the space of J-holomorphic maps from Σ into X provided that $c_1(T\Sigma)(\Sigma) > 0$ and $c_1(TX) \cdot [f(\Sigma)] > 0$.*

Proof. Let $E \to \Sigma$ be the pull back of TX via f. Since f_z is embedded for each $z \in \Sigma$, $T\Sigma$ is a subbundle of E, and one has the decomposition $E = T\Sigma \oplus (E/T\Sigma)$. Then a similar argument as in Lemma 3.3.3 of [**35**] shows that $D\underline{L}$ is surjective at f if both $(-c_1(T\Sigma) + c_1(K_\Sigma))(\Sigma)$ and $(-c_1(E/T\Sigma) + c_1(K_\Sigma))(\Sigma)$ are negative. (Here K_Σ is the canonical bundle of Σ.) The lemma follows easily. q.e.d.

Note that the conditions $c_1(T\Sigma)(\Sigma) > 0$ and $c_1(TX) \cdot [f(\Sigma)] > 0$ in the previous lemma are met by each $f \in \mathcal{M}$: $c_1(T\Sigma)(\Sigma) = 1 + \frac{1}{2m} > 0$, and $c_1(TX) \cdot [f(\Sigma)] = \frac{m+1}{m} > 0$. Thus for the smoothness of $\mathcal{M}$, it suffices to verify that for each $f \in \mathcal{M}$, f_z is embedded, $\forall z \in \Sigma$. This condition is verified in the next lemma. But in order to state the lemma, it proves convenient to introduce the following:

Definition. Let C be a J-holomorphic curve in X which contains the singular point p_0, and is parametrized by $f : \Sigma \to X$. We call C a *quasi-suborbifold* if the following are met.

- f induces a homeomorphism between the underlying Riemann surface and C,
- f is embedded in the complement of the singular points in X,
- a local representative (f_z, ρ_z) of f at each $z \in \Sigma$ where $f(z)$ is a singular point obeys
  (i) f_z is embedded,
  (ii) ρ_z is isomorphic if $f(z) \neq p_0$, and if $f(z) = p_0$, ρ_z (which is injective by definition) maps onto the maximal subgroup of G that fixes the tangent space of Im f_z at the inverse image of p_0 in the local uniformizing system at p_0.

We remark that in terms of the adjunction formula (cf. Theorem 3.1 of [**11**])

$$g(C) = g_\Sigma + \sum_{\{[z,z']|z\neq z', f(z)=f(z')\}} k_{[z,z']} + \sum_{z\in\Sigma} k_z,$$

a J-holomorphic curve C is a quasi-suborbifold if and only if $k_{[z,z']} = 0$ for all $[z, z']$, $k_z = 0$ for any z such that $f(z) \neq p_0$, and $k_{z_0} = \frac{1}{2m_0}(\frac{|G|}{m_0} - 1)$ where $f(z_0) = p_0$. Here m_0 is the order of $z_0 \in \Sigma$. (Compare Corollary 3.3 in [**11**], and note that $\frac{1}{2m_0}(\frac{|G|}{m_0} - 1)$ is the least of the possible values of k_{z_0}.)

Now in the lemma below, we describe what the members of $\mathcal{M}$ look like.

Lemma 3.3. *Each $f \in \mathcal{M}$ is either a (multiplicity-one) parametrization of a J-holomorphic quasi-suborbifold intersecting C_0 transversely at a smooth point, or a multiply covered map onto a J-holomorphic quasi-suborbifold intersecting C_0 at a singular point, such that the order of the singular point equals the multiplicity of f. Moreover, even in the latter case, the map f_z in a local representative (f_z, ρ_z) of f at z is embedded for all $z \in \Sigma$.*

Proof. Set $C \equiv \text{Im } f$. We first consider the case where f is not multiply covered. Under this assumption, we have

$$C \cdot C = \frac{|G|}{4m^2} \text{ and } c_1(K_X) \cdot C = -\frac{m+1}{m},$$

which implies that the virtual genus

$$g(C) = \frac{1}{2}\left(\frac{|G|}{4m^2} - \frac{m+1}{m}\right) + 1 = \frac{|G|}{8m^2} - \frac{m+1}{2m} + 1.$$

On the other hand, the orbifold genus $g_\Sigma = \frac{1}{2}(1-\frac{1}{2m})$ and $k_{z_\infty} \geq \frac{1}{4m}(\frac{|G|}{2m}-1)$. It follows easily from the adjunction formula that C is a quasi-suborbifold and C intersects C_0 transversely at a smooth point.

Now consider the case where f is multiply covered. Let $s>1$ be the multiplicity of f. Clearly C, C_0 are distinct, hence by the intersection formula (cf. [**11**], Theorem 3.2),

$$\frac{1}{s} = C \cdot C_0 = \sum_{i=1}^{3} \frac{k_i}{a_i},$$

where a_i is the order of the singular point p_i, and $k_i \geq 0$ is an integer which is nonzero if and only if $p_i \in C \cap C_0$. It follows immediately that $s \leq \frac{a_i}{k_i}$ if $k_i \neq 0$. From the possible values of (a_1, a_2, a_3), one can easily see that C intersects C_0 at exactly one singular point.

Suppose C, C_0 intersect at p_i for some $i = 1, 2$ or 3. Let $\hat{f} : \hat{\Sigma} \to X$ be a (multiplicity-one) parametrization of C by a J-holomorphic map such that f factors through a map $\varphi : \Sigma \to \hat{\Sigma}$ to $\hat{f}$, and let $\hat{f}(\hat{z}_0) = p_i$ for some $\hat{z}_0 \in \hat{\Sigma}$ whose order is denoted by $\hat{m}_0$. Set $\hat{z}_\infty \equiv \varphi(z_\infty)$. First, by the intersection formula, we get $\frac{1}{s} \geq \frac{a_i/\hat{m}_0}{a_i} = \frac{1}{\hat{m}_0}$. Hence $s \leq \hat{m}_0$. On the other hand, let $z_0 \in \Sigma$ be an inverse image of $\hat{z}_0$ under φ. Then $\hat{m}_0$ is no greater than the degree of the branched covering φ at z_0, which is no greater than the total multiplicity s. This implies that $s = \hat{m}_0$. Now we look at the point $\hat{z}_\infty$. Let m_∞ be the degree of the branched covering φ at z_∞. Then (1) the order of $\hat{z}_\infty$, denoted by $\hat{m}_\infty$, is no greater than $2mm_\infty$, and (2) $m_\infty \leq s = \hat{m}_0$. In particular, $\hat{m}_\infty \leq 2m\hat{m}_0$.

Now in the adjunction formula for C, the virtual genus

$$g(C) = \frac{|G|}{8m^2\hat{m}_0^2} - \frac{m+1}{2m\hat{m}_0} + 1,$$

and on the right hand side,

$$g_{\hat{\Sigma}} \geq \frac{1}{2}\left(1 - \frac{1}{\hat{m}_0}\right) + \frac{1}{2}\left(1 - \frac{1}{\hat{m}_\infty}\right),$$
$$k_{\hat{z}_0} \geq \frac{1}{2\hat{m}_0}\left(\frac{a_i}{\hat{m}_0} - 1\right),$$
$$k_{\hat{z}_\infty} \geq \frac{1}{2\hat{m}_\infty}\left(\frac{|G|}{\hat{m}_\infty} - 1\right).$$

If $2m\hat{m}_0 > \hat{m}_\infty$, then the adjunction formula for C gives rise to

$$\frac{|G|}{4m\hat{m}_0} < 1,$$

which is impossible because

$$1 \leq \frac{|G|}{4ma_i} < \frac{|G|}{4m\hat{m}_0}.$$

Hence $2m\hat{m}_0 = \hat{m}_\infty$. With this in hand, the adjunction formula further implies that $a_i = \hat{m}_0$ and C is a quasi-suborbifold, and the multiplicity of f equals the order of the singular point where C, C_0 intersect.

It remains to check that f_z is embedded for all $z \in \Sigma$. But this follows readily from (1) C is a quasi-suborbifold, and (2) $\varphi : \Sigma \to \hat{\Sigma}$ is a cyclic branched covering of degree s, branched at z_0, z_∞, and $\hat{m}_0 = s$, $\hat{m}_\infty = 2ms$. q.e.d.

Up to this point, we see that $\mathcal{M}$ is a 6-dimensional smooth manifold provided that it is nonempty. Next we show

Lemma 3.4. *The quotient space $\mathcal{M}/\mathcal{G}$ is compact.*

Proof. According to the orbifold version of the Gromov compactness theorem proved in [**14**], for any sequence of maps $f_n \in \mathcal{M}$, there exists a subsequence which converges to a cusp-curve after suitable reparametrization. More concretely, after reparametrization if necessary, there is a subsequence of f_n, which is still denoted by f_n for simplicity, and there are at most finitely many simple closed loops $\gamma_1, \ldots, \gamma_l \subset \Sigma$ containing no orbifold points, and a nodal orbifold Riemann surface $\Sigma' = \cup_\omega \Sigma_\omega$ obtained by collapsing $\gamma_1, \ldots, \gamma_l$, and a J-holomorphic map $f : \Sigma' \to X$, such that (1) f_n converges in C^∞ to f on any given compact subset in the complement of $\gamma_1, \ldots, \gamma_l$, (2) $[f(\Sigma')] = [f_n(\Sigma)] \in H_2(X; \mathbb{Q})$, (3) if $z_\omega \in \Sigma_\omega, z_\nu \in \Sigma_\nu$ are two distinct points (here $\Sigma_\nu = \Sigma_\omega$ is allowed) with orders m_ω, m_ν respectively, such that z_ω, z_ν are the image of the same simple closed loop collapsed under $\Sigma \to \Sigma'$, then $m_\omega = m_\nu$, and there exist local representatives $(f_\omega, \rho_\omega), (f_\nu, \rho_\nu)$ of f at z_ω, z_ν, which obey $\rho_\omega(\mu_{m_\omega}) = \rho_\nu(\mu_{m_\nu})^{-1}$, and (4) if f is constant over a component Σ_ν of Σ', then either the underlying surface of Σ_ν has nonzero genus, or Σ_ν contains at least 3 special points, where a special point is either an orbifold point inherited from Σ or any point resulted from collapsing a simple closed loop in Σ. Regarding the last point about constant components, since in the present case Σ is an orbifold Riemann sphere with only one orbifold point z_∞, any constant component in the limiting cusp-curve must be obtained by collapsing at least 2 simple closed loops, and if z_∞ is not contained, by collapsing at least 3 simple closed loops.

With the preceding understood, note that there are two possibilities: (1) none of the simple closed loops $\{\gamma_i\}$ are null-homotopic in the complement of z_0, z_∞ in Σ where $f_n(z_0) \in C_0$ and $f_n(z_\infty) = p_0$, or (2) there is a simple closed loop $\gamma \in \{\gamma_i\}$ such that γ bounds a disc D in the complement of z_0, z_∞ in Σ, such that D contains none of the simple closed loops $\gamma_1, \ldots, \gamma_l$.

Case (1). Under this assumption, it is easily seen that there are no constant components in the limiting cusp-curve. Moreover, there is a component Σ_ω such that $f_\omega \equiv f|_{\Sigma_\omega} : \Sigma_\omega \to X$ obeys the following conditions:

- There exists an orbifold point $w_\infty \in \Sigma_\omega$ of order $2m$ inherited from Σ (i.e., $w_\infty = z_\infty$) such that $f_\omega(w_\infty) = p_0$, and a local representative of f_ω at w_∞ obeys the second condition in the definition of $\mathcal{M}$.
- $f_\omega^{-1}(C_0)$ consists of only one point w_0, which is necessarily obtained from collapsing one of the simple closed loops $\gamma_1, \ldots, \gamma_l$.
- All points in $\Sigma_\omega \setminus \{w_0, w_\infty\}$ are regular, i.e., of order 1 in Σ_ω.

First of all, we show that w_0 is actually a regular point of Σ_ω. In order to see this, we only need to consider the case where $f_\omega(w_0)$ is a singular point, say p_i, for some $i = 1, 2$ or 3. (Note that w_0 is automatically a regular point if $f_\omega(w_0)$ is a smooth point.) Let (w_1, w_2) be holomorphic coordinates on a local uniformizing system at p_i, where C_0 is locally given by $w_2 = 0$, and the singular fiber of the Seifert fibration at p_i is defined by $w_1 = 0$, $|w_2| \equiv$ constant. The local $\mathbb{Z}_{a_i}$-action is given by $\mu_{a_i} \cdot (w_1, w_2) = (\mu_{a_i} w_1, \mu_{a_i}^{b_i} w_2)$. (Here (a_i, b_i) is the normalized Seifert invariant at p_i.) Let $m_0 \geq 1$ be the order of w_0, and let (f_0, ρ_0) be a local representative of f_ω at w_0, where $\rho_0(\mu_{m_0}) = \mu_{m_0}^r$ with $0 \leq r < m_0$, r, m_0 relatively prime, and $f_0(w) = (c(w^{l_1} + \cdots), w^{l_2} + \cdots)$ (note that $\mathrm{Im}\, f_\omega \neq C_0$). By a $\mathbb{Z}_{m_0}$-equivariant change of coordinates $w' \equiv w(1+\cdots)^{1/l_2}$ near $w = 0$, we may simply assume $f_0(w) = (c(w^{l_1} + \cdots), w^{l_2})$. Furthermore, $l_2 \equiv b_i r \pmod{m_0}$, so that l_2, m_0 are relatively prime. With these understood, note that the image of the link of w_0 in Σ_ω under f_ω is parametrized in the local uniformizing system by $f_0(\epsilon \exp(\sqrt{-1}\frac{2\pi}{m_0}\theta))$, $0 \leq \theta \leq 1$. Through $f_t(w) \equiv (c(1-t)(w^{l_1} + \cdots), w^{l_2})$, $0 \leq t \leq 1$, it is homotopic to $(0, \epsilon^{l_2} \exp(\sqrt{-1}\frac{2\pi l_2}{m_0}\theta))$ in the complement of C_0. It follows easily that the link of w_0 in Σ_ω under f_ω is homotopic in $\mathbb{S}^3/G$ to $\frac{l_2 a_i}{m_0}$ times of the singular fiber of the Seifert fibration at p_i, whose homotopy class in $\pi_1(\mathbb{S}^3/G) = G$ has order $2ma_i$. On the other hand, the link of w_0 in Σ_ω under f_ω is homotopic in W to the image of the inverse of the link of w_∞ in Σ_ω under f_ω. The latter's homotopy class in G is $\mu_{2m}^{-1} I \in Z_{2m}$, which implies that the former's homotopy class is an element of order $2m$ in Z_{2m} (in fact, it is $\mu_{2m}^{-1} I \in Z_{2m}$, cf. Lemma 3.5 below). This gives $\frac{l_2 a_i}{m_0} \cdot 2m = 2ma_i l$ for some $l > 0$, which contradicts the fact that l_2, m_0 are relatiely prime unless $m_0 = 1$. Therefore w_0 is a regular point of Σ_ω.

Now note that $f_\omega : \Sigma_\omega \to X$ satisfies all the conditions in the definition of $\mathcal{M}$ except for the first one, i.e., $[f_\omega(\Sigma_\omega)] \cdot C_0 = 1$, which we prove next. To see this, let $\hat{f}_\omega : \hat{\Sigma}_\omega \to X$ be the multiplicity-one parametrization of $C_\omega \equiv \mathrm{Im}\, f_\omega$ obtained by factoring f_ω through a branched covering map $\varphi : \Sigma_\omega \to \hat{\Sigma}_\omega$ of degree s. (If f_ω is not multiply covered, we simply let $\hat{\Sigma}_\omega \equiv \Sigma_\omega$, $\hat{f}_\omega \equiv f_\omega$, and $s = 1$.) Set $\hat{w}_0 \equiv \varphi(w_0) \in \hat{\Sigma}_\omega$, and let $\hat{m}_0 \geq 1$ be the order of $\hat{w}_0$ in $\hat{\Sigma}_\omega$. Then by the intersection formula, we get

$$\frac{1}{s} \geq \frac{1}{s} \cdot [f_\omega(\Sigma_\omega)] \cdot C_0 = C_\omega \cdot C_0 \geq \frac{a_i/\hat{m}_0}{a_i} = \frac{1}{\hat{m}_0}.$$

Hence $s \leq \hat{m}_0$. On the other hand, $\hat{m}_0$ is no greater than the degree of the branched covering φ at w_0, which is no greater than the total multiplicity s. This implies that $s = \hat{m}_0$ and $C_\omega \cdot C_0 = \frac{1}{s}$. Hence $[f_\omega(\Sigma_\omega)] \cdot C_0 = s \cdot C_\omega \cdot C_0 = 1$. It is clear that $f = f_\omega \in \mathcal{M}$ in this case.

Case (2). Let Σ_ω be the component obtained from the disc D that γ bounds, and let $z_0 \in \Sigma_\omega$ be the point which is the image of γ under $D \to \Sigma_\omega$. Note that f is nonconstant over Σ_ω. Set $f_\omega \equiv f|_{\Sigma_\omega}$ and $C_\omega \equiv \text{Im } f_\omega$. Since $f_n(D)$ is disjoint from C_0, either $f_\omega^{-1}(C_0)$ consists of only one point z_0, or $C_\omega = C_0$. However, the latter case can be ruled out for the following reason. Note that Σ_ω contains at most one orbifold point, and hence is simply connected as an orbifold. Consequently, the degree of the map $f_\omega : \Sigma_\omega \to C_\omega = C_0$ is at least $\frac{|G|}{2m}$, which is the order of G/Z_{2m}, the orbifold fundamental group of C_0. It follows that $[f_\omega(\Sigma_\omega)] \cdot C_0 \geq \frac{|G|}{2m} \cdot \frac{4m^2}{|G|} = 2m > 1$, which is a contradiction.

Let $\hat{f}_\omega : \hat{\Sigma}_\omega \to X$ be the multiplicity-one parametrization of C_ω obtained by factoring f_ω through a map $\varphi : \Sigma_\omega \to \hat{\Sigma}_\omega$ of degree s. (If f_ω is not multiply covered, we simply let $\hat{\Sigma}_\omega \equiv \Sigma_\omega$, $\hat{f}_\omega \equiv f_\omega$, and $s = 1$.) Set $\hat{z}_0 \equiv \varphi(z_0) \in \hat{\Sigma}_\omega$, and let m_0 be the order of $\hat{z}_0$ in $\hat{\Sigma}_\omega$.

Note that in this case Σ_ω is necessarily not the only component of Σ' over which f is nonconstant. Consequently $C_\omega \cdot C_0 < 1$, and $f_\omega(z_0)$ is a singular point of X on C_0, say p_i for some $i = 1, 2$ or 3. Let z_1, z_2 be the holomorphic coordinates on a local uniformizing system at p_i, with local group action given by $\mu_{a_i} \cdot (z_1, z_2) = (\mu_{a_i} z_1, \mu_{a_i}^{b_i} z_2)$, such that C_0 is locally given by $z_2 = 0$ and the singular fiber of the Seifert fibration on $\mathbb{S}^3/G$ at p_i is given by $z_1 = 0$, $|z_2| \equiv$ constant. (Here (a_i, b_i) is the normalized Seifert invariant at p_i.) Let (f_0, ρ_0) be a local representative of $\hat{f}_\omega$ at $\hat{z}_0$, where we write $f_0(z) = (c(z^{l_1} + \cdots), z^{l_2})$ (note that $C_\omega \neq C_0$). Then by the intersection formula, we have

$$\frac{1}{s} \geq C_\omega \cdot C_0 = \frac{(a_i/m_0) \cdot l_2}{a_i} = \frac{l_2}{m_0}.$$

On the other hand, by a similar argument, we see that the link of p_i in C_ω is homotopic to $\frac{l_2 a_i}{m_0}$ times the singular fiber of the Seifert fibration at p_i, whose homotopy class in G is of order $2ma_i$. Since $\mathbb{S}^3/G \to W$ is a homotopy equivalence, and $f_n(\gamma)$, which bounds a disc $f_n(D) \subset W$, is homotopic to s times the link of p_i in C_ω, we have $s \cdot \frac{l_2 a_i}{m_0} = 2ma_i l$ for some $l > 0$. But this contradicts the inequality $\frac{1}{s} \geq \frac{l_2}{m_0}$ we obtained earlier.

Hence Case (2) is impossible, and therefore the quotient space $\mathcal{M}/\mathcal{G}$ is compact. q.e.d.

It remains to show, in the proof of Proposition 3.1, that $\mathcal{M}$ is nonempty. This will be achieved in the following three steps:

(1) Construct an orbifold complex line bundle $E \to X$ such that $c_1(E)\cdot C_0 = 1$.
(2) Show that the associated Seiberg-Witten invariant is nonzero in Taubes chamber.
(3) Apply Theorem 2.2 (2) to produce a J-holomorphic curve C such that $C = \text{Im } f$ for some $f \in \mathcal{M}$.

For Step (1), we derive a preliminary lemma first. To state the lemma, let $h : \mathbb{S}^3/G \to \mathbb{S}^3/G$ be the simple homotopy equivalence induced by the s-cobordism W. Then there is a pair $(\hat{h}, \hat{\rho}) : (\mathbb{S}^3, G) \to (\mathbb{S}^3, G)$ where $\hat{h} : \mathbb{S}^3 \to \mathbb{S}^3$ is $\hat{\rho}$-equivariant and descends to $h : \mathbb{S}^3/G \to \mathbb{S}^3/G$. The pair $(\hat{h}, \hat{\rho})$ is unique up to conjugation by an element of G.

Lemma 3.5. *The restriction of $\hat{\rho}$ to $Z_{2m} \subset G$ is the identity map.*

Proof. Recall the double cover $\phi : \mathbb{S}^3 \times \mathbb{S}^3 \to SO(4)$, which is defined by sending $(q_1, q_2) \in \mathbb{S}^3 \times \mathbb{S}^3$ to the matrix in $SO(4)$ that sends $x \in \mathbb{R}^4 = \mathbb{H}$ to $q_1 x q_2^{-1} \in \mathbb{H} = \mathbb{R}^4$. Regard G as a subgroup of $\phi(\mathbb{S}^1 \times \mathbb{S}^3)$.

Note that as a simple homotopy equivalence, $h : \mathbb{S}^3/G \to \mathbb{S}^3/G$ is homotopic to a diffeomorphism (cf. [**44**], and for a proof, [**29**]). On the other hand, any diffeomorphism between elliptic 3-manifolds is homotopic to an isometry, cf. e.g., [**34**], hence h is homotopic to an isometry. It follows easily that $\hat{h} : \mathbb{S}^3 \to \mathbb{S}^3$ is $\hat{\rho}$-equivariantly homotopic to an isometry $\xi \in SO(4)$. In particular, $\hat{\rho}(g) = \xi g \xi^{-1}$.

Now let $\xi = \phi(q, q')$ and $g = \phi(x, y)$. Then $\hat{\rho}(g) = \phi(qxq^{-1}, q'y(q')^{-1})$. Note that for any $g \in Z_{2m} \subset G$, $g = \phi(x, 1)$ with $x = (\mu_{2m}^l, 0)$, $0 \le l \le 2m-1$. If we let $q = (w_1, w_2)$, then

$$qxq^{-1} = (|w_1|^2\mu_{2m}^l + |w_2|^2\mu_{2m}^{-l}, w_1 w_2(\mu_{2m}^{-l} - \mu_{2m}^l)).$$

Note that when $m = 1$, $qxq^{-1} = x$ so that the lemma holds trivially. For the case where $m \neq 1$, the fact that $qxq^{-1} \in \mathbb{S}^1$ implies that either w_1 or w_2 must be zero. Clearly, for any $g \in Z_{2m}$, $\hat{\rho}(g) = g$ iff $w_2 = 0$ and $\hat{\rho}(g) = g^{-1}$ iff $w_1 = 0$.

It remains to show that $\hat{\rho}(g) = g^{-1}$, $\forall g \in Z_{2m}$, is impossible. Here we need to use the assumption that W is symplectic. Let $\widetilde{W}$ be the universal cover of W. Note that the canonical bundle $K_{\widetilde{W}}$ is trivial. This gives rise to a representation $\theta : G = \pi_1(W) \to \mathbb{S}^1$, which obeys $\theta = \theta \circ \hat{\rho}$. Let $g \in Z_{2m}$ be the matrix $\mu_{2m} I$. Then $\theta(g) = \mu_{2m}^2$, which implies $\mu_m = \mu_m^{-1}$ if $\hat{\rho}(g) = g^{-1}$. But this is impossible unless $m = 2$, which occurs only when $G = \langle Z_{4m}, Z_{2m}; \widetilde{D}_n, C_{2n}\rangle$. But even in this case, $\hat{\rho}(g) \neq g^{-1}$ because otherwise, we would have $q = (0, w_2)$, which implies that $\hat{\rho}(\mu_{4m}\alpha) = \mu_{4m}^{-1}q'\alpha(q')^{-1}$ for any $\alpha \in \widetilde{D}_n$ whose class is nonzero in $\widetilde{D}_n/C_{2n}$. But $\theta(\mu_{4m}\alpha) = \mu_{2m}$ and $\theta(\mu_{4m}^{-1}q'\alpha(q')^{-1}) = \mu_{2m}^{-1}$, which contradicts $\theta = \theta \circ \hat{\rho}$ and $m = 2$. Hence the lemma. q.e.d.

Now back to Step (1) of the proof. In the following lemma, we give an explicit construction of the orbifold complex line bundle E.

Lemma 3.6. *There exists a canonically defined orbifold complex line bundle $E \to X$ such that $c_1(E) \cdot C_0 = 1$.*

Proof. Note that X is decomposed as $N \bigcup W \bigcup N_0$, where N is a regular neighborhood of C_0, which is diffeomorphic to the unit disc bundle associated to the Seifert fibration on $\mathbb{S}^3/G$, and $N_0 = \mathbb{B}^4/G$ is a regular neighborhood of the singular point p_0. The orbifold complex line bundle E will be defined by patching together an orbifold complex line bundle on each of N, W and N_0, which agree on the intersections.

The bundle on N is defined as follows. Take the complex line bundle on the complement of the singular points p_1, p_2 and p_3 in N, which is Poincaré dual to a regular fiber of N. (The regular fibers of N are so oriented that the intersection with C_0 has a + sign.) This bundle is trivial on the link of each p_i in N, so we can simply extend it over to the whole N trivially to obtain the orbifold complex line bundle on N.

The restriction of the bundle on N to $\partial N = \mathbb{S}^3/G$ is Poincaré dual to a regular fiber of the Seifert fibration. By Lemma 3.5, there exists a map $\psi : \mathbb{S}^1 \times [0,1] \to W$ such that $\psi(\mathbb{S}^1 \times \{0\})$ is a regular fiber of the Seifert fibration on $\partial N = \mathbb{S}^3/G$, and $\psi(\mathbb{S}^1 \times \{1\})$ is the image of the boundary of a generic unit complex linear disc in $\mathbb{B}^4$ under the quotient map $\partial\mathbb{B}^4 = \mathbb{S}^3 \to \mathbb{S}^3/G$. We let the bundle on W be the Poincaré dual of $\psi(\mathbb{S}^1 \times [0,1])$.

It remains to construct an orbifold complex line bundle E_0 on $N_0 = \mathbb{B}^4/G$ such that the restriction of E_0 on $\partial\mathbb{B}^4/G$ is Poincaré dual to $\psi(\mathbb{S}^1 \times \{1\})$. The resulting orbifold complex line bundle $E \to X$ clearly obeys $c_1(E) \cdot C_0 = 1$.

To this end, note that given any representation $\rho : G \to \mathbb{S}^1$, there exists an orbifold complex line bundle on N_0, which is given by the projection $(\mathbb{B}^4 \times \mathbb{C}, G) \to (\mathbb{B}^4, G)$ on the uniformizing system, where the action of G on $\mathbb{B}^4 \times \mathbb{C}$ is given by $g \cdot (z,w) = (gz, \rho(g)w)$, $\forall (z,w) \in \mathbb{B}^4 \times \mathbb{C}, g \in G$. With this understood, the definition of $E_0 \to N_0$ for the various cases of G is given below.

- $\langle Z_{2m}, Z_{2m}; \widetilde{D}_n, \widetilde{D}_n \rangle$: $\rho(h) = \mu_{2m}^{2n}$, $\rho(x) = (-1)^n$, and $\rho(y) = 1$, where $h = \mu_{2m} I \in Z_{2m}$, and x, y are the generators of $\widetilde{D}_n$ with relations $x^2 = y^n = (xy)^2 = -1$.
- $\langle Z_{4m}, Z_{2m}; \widetilde{D}_n, C_{2n} \rangle$: $\rho(h^2) = \mu_{2m}^{2n}$, $\rho(hx) = (-\mu_{2m})^n$, $\rho(y) = 1$, where $h = \mu_{4m} I \in Z_{4m}$, and x, y are the generators of $\widetilde{D}_n$ with relations $x^2 = y^n = (xy)^2 = -1$.

- $\langle Z_{2m}, Z_{2m}; \widetilde{T}, \widetilde{T}\rangle$: $\rho(h) = \mu_{2m}^{12}$, $\rho(x) = 1$, and $\rho(y) = 1$, where $h = \mu_{2m} I \in Z_{2m}$, and x, y are the generators of $\widetilde{T}$ with relations $x^2 = y^3 = (xy)^3 = -1$.
- $\langle Z_{6m}, Z_{2m}; \widetilde{T}, \widetilde{D}_2\rangle$: $\rho(h^3) = \mu_{2m}^{12}$, $\rho(x) = 1$, and $\rho(hy) = \mu_{2m}^{4}$, where $h = \mu_{6m} I \in Z_{6m}$, and x, y are the generators of $\widetilde{T}$ with relations $x^2 = y^3 = (xy)^3 = -1$.
- $\langle Z_{2m}, Z_{2m}; \widetilde{O}, \widetilde{O}\rangle$: $\rho(h) = \mu_{2m}^{24}$, $\rho(x) = 1$, and $\rho(y) = 1$, where $h = \mu_{2m} I \in Z_{2m}$, and x, y are the generators of $\widetilde{O}$ with relations $x^2 = y^4 = (xy)^3 = -1$.
- $\langle Z_{2m}, Z_{2m}; \widetilde{I}, \widetilde{I}\rangle$: $\rho(h) = \mu_{2m}^{60}$, $\rho(x) = 1$, and $\rho(y) = 1$, where $h = \mu_{2m} I \in Z_{2m}$, and x, y are the generators of $\widetilde{I}$ with relations $x^2 = y^5 = (xy)^3 = -1$.

The verification that the restriction of $E_0 \to N_0$ to ∂N_0 is Poincaré dual to $\psi(\mathbb{S}^1 \times \{1\})$ goes as follows. Fixing a generic vector $u = (u_1, u_2) \in \mathbb{C}^2$, we let f_u be the linear function on $\mathbb{C}^2$ defined by

$$f_u(z_1, z_2) \equiv u_1 z_1 + u_2 z_2.$$

The action of $g \in G$ as a 2×2 complex valued matrix on f_u is given by $g^* f_u = f_{ug}$, where $ug = (u_1, u_2)g$ is the row vector obtained from multiplying by g on the right. With this understood, consider the epimorphism $\pi : G \to \Gamma \equiv G/Z_{2m}$, where Γ is isomorphic to the corresponding subgroup (dihedral, tetrahedral, octahedral, or icosahedral) in $SO(3)$. For any $\gamma \in \Gamma$, we fix a $\hat{\gamma} \in G$ such that $\pi(\hat{\gamma}) = \gamma$. Then consider the product

$$f(z) \equiv \prod_{\gamma \in \Gamma} f_{u\hat{\gamma}}(z), \ \forall z \in \mathbb{C}^2.$$

The claim is that for any $g \in G, z \in \mathbb{C}^2$, $f(gz) = \rho(g) f(z)$, so that $z \mapsto (z, f(z))$ is an equivariant section of the G-bundle $\mathbb{B}^4 \times \mathbb{C} \to \mathbb{B}^4$, which descends to a section s of the orbifold complex line bundle $E_0 \to N_0$. The zero locus of s in ∂N_0 is the image of $f_u^{-1}(0) \cap \mathbb{S}^3$ under $\mathbb{S}^3 \to \mathbb{S}^3/G = \partial N_0$, which can be so arranged that it is actually $\psi(\mathbb{S}^1 \times \{1\})$.

So it remains to verify the claim that for any $g \in G, z \in \mathbb{C}^2$, $f(gz) = \rho(g) f(z)$. This is elementary but tedious, so we shall only illustrate it by a simple example but also with some general remarks. The details for all other cases are left out to the reader.

Consider the case $G = \langle Z_{2m}, Z_{2m}; \widetilde{D}_3, \widetilde{D}_3\rangle$. The dihedral group D_3 is generated by α, β with relations $\alpha^2 = \beta^3 = (\alpha\beta)^2 = 1$, while the binary dihedral group $\widetilde{D}_3$ is generated by x, y with relations $x^2 = y^3 = (xy)^2 = -1$. Clearly $x \mapsto \alpha$, $y \mapsto \beta$ under $\widetilde{D}_3 \to D_3$. Set $h \equiv \mu_{2m} I \in Z_{2m}$. In this case, we may take

$$f \equiv f_u f_{uy} f_{uy^2} f_{ux} f_{uyx} f_{uy^2 x}.$$

One can easily check that $f(hz) = \mu_{2m}^6 f(z)$,

$$f(xz) = f_{ux}(z)f_{uyx}(z)f_{uy^2x}(z)f_{ux^2}(z)f_{uyx^2}(z)f_{uy^2x^2}(z) = (-1)^3 f(z),$$

and similarly $f(yz) = (-1)^2 f(z) = f(z)$.

As for the general remarks, the dihedral case can be similarly handled as in the above example. For the tetrahedral case, the order of the group $\Gamma = G/Z_{2m}$ is 12, so it is not terribly complicated. For the octahedral case, the trick is to fix an explicit identification between the octahedral group O and the symmetric group S_4, e.g., $\alpha \mapsto (12), \beta \mapsto (1234)$ where α, β are generators of O with relations $\alpha^2 = \beta^4 = (\alpha\beta)^3 = 1$, and use the identification between O and S_4 to guide the manipulation of the words generated by $\hat{\alpha}$ and $\hat{\beta}$, where $\hat{\alpha}, \hat{\beta} \in \widetilde{O}$ are some fixed choice of elements which obey $\hat{\alpha} \mapsto \alpha$, $\hat{\beta} \mapsto \beta$ under $\widetilde{O} \to O$. The case of the icosahedral group is actually quite simple. The observation is that $H_1(\mathbb{S}^3/\widetilde{I}; \mathbb{Z})$ is trivial, so that any representation $\rho' : G \to \mathbb{S}^1$ obtained from $f(gz) = \rho'(g)f(z)$ has to satisfy $\rho'(g) = 1$, $\forall g \in \widetilde{I}$, because ρ' factors through $H_1(\mathbb{S}^3/G; \mathbb{Z})$. q.e.d.

Next for Step (2), we show that the Seiberg-Witten invariant corresponding to E is nonzero in Taubes chamber. First of all, we observe the following lemma.

Lemma 3.7. *The Seiberg-Witten invariant corresponding to E is zero in the* 0*-chamber.*

Proof. Decompose X as $X_1 \bigcup X_2$ where X_1 is a regular neighborhood of C_0. Note that X_1 is diffeomorphic to the unit disc bundle associated to the Seifert fibration on $\mathbb{S}^3/G$.

The lemma follows readily from the fact that X_1 has a Riemannian metric of positive scalar curvature which is a product metric near ∂X_1. Accept this fact momentarily, and suppose that the Seiberg-Witten invariant is nonzero in the 0-chamber. Then one can stretch the neck along $\partial X_1 = \partial X_2$, such that any solution of the Seiberg-Witten equations on X will yield a solution (A, ψ) on $\hat{X}_1 \equiv X_1 \cup [0, -\infty) \times \partial X_1$, where $|\psi|$ converges to zero exponentially fast along the cylindrical end of $\hat{X}_1$. Since the natural metric on $\hat{X}_1$ is of positive scalar curvature, we must have $\psi \equiv 0$ by the Weitzenböck formula. But this implies that $P_+ F_A = \frac{1}{4}\tau(\psi \otimes \psi^*) \equiv 0$, which contradicts the fact that $\frac{\sqrt{-1}}{2\pi}\int_{C_0} F_A = c_1(E^2 \otimes K_X^{-1}) \cdot C_0 \neq 0$.

As for the fact that X_1 has a metric of positive scalar curvature, here is a proof. Note that $X_1 = ((\mathbb{S}^3/G) \times D^2)/\mathbb{S}^1$, where the $\mathbb{S}^1$-action on $\mathbb{S}^3/G$ defines the Seifert fibration, and where D^2 is the unit 2-disc with the $\mathbb{S}^1$-action given by complex multiplication. Give $(\mathbb{S}^3/G) \times D^2$ a product metric such that on the factor $\mathbb{S}^3/G$, it is the metric of constant curvature which is clearly invariant under the $\mathbb{S}^1$-action, and on the factor D^2, it is an $\mathbb{S}^1$-invariant metric with nonnegative curvature which is a product metric near the boundary. Now observe that the orthogonal complement of the vector field generated by the $\mathbb{S}^1$-action on $(\mathbb{S}^3/G) \times D^2$ is an $\mathbb{S}^1$-equivariant subbundle of the tangent bundle of $(\mathbb{S}^3/G) \times D^2$, which canonically defines a Riemannian metric on X_1 through the projection $(\mathbb{S}^3/G) \times D^2 \to X_1$, making it into a Riemannian submersion in the sense of O'Neill [**36**]. It follows easily from the calculation therein that the metric on X_1 has positive scalar curvature. q.e.d.

Observe that $c_1(K_X) \cdot C_0 < 0$, so that $c_1(S_+^E) \cdot [\omega] = c_1(K_X^{-1} \times E^2) \cdot [\omega] > 0$. By the wall-crossing formula in Lemma 2.1, the Seiberg-Witten invariant corresponding to E is nonzero in Taubes chamber provided that the dimension of the corresponding moduli space of the Seiberg-Witten equations is nonnegative, which is shown in the next lemma.

Lemma 3.8. *The dimension of the Seiberg-Witten moduli space corresponding to E, denoted by $d(E)$, is given for the various cases of G in the following list.*

- $\langle Z_{2m}, Z_{2m}; \widetilde{D}_n, \widetilde{D}_n\rangle$ *or* $\langle Z_{4m}, Z_{2m}; \widetilde{D}_n, C_{2n}\rangle$*:* $d(E) = \delta + 2 + \frac{1}{2}((-1)^\delta - 1)$ *if* $m < n$*, where* $n = \delta m + r$ *with* $0 \leq r \leq m-1$*, and* $d(E) = 2$ *if* $m > n$*.*
- $\langle Z_{2m}, Z_{2m}; \widetilde{T}, \widetilde{T}\rangle$ *or* $\langle Z_{6m}, Z_{2m}; \widetilde{T}, \widetilde{D}_2\rangle$*:* $d(E) = 2$ *if* $m \neq 1$*, and* $d(E) = 8$ *if* $m = 1$*.*
- $\langle Z_{2m}, Z_{2m}; \widetilde{O}, \widetilde{O}\rangle$*:* $d(E) = 2$ *if* $m \neq 1$*, and* $d(E) = 14$ *if* $m = 1$*.*
- $\langle Z_{2m}, Z_{2m}; \widetilde{I}, \widetilde{I}\rangle$*:* $d(E) = 2$ *if* $m \neq 1, 7$*,* $d(E) = 4$ *if* $m = 7$*, and* $d(E) = 32$ *if* $m = 1$*.*

The proof of Lemma 3.8 is given in Appendix A.

Now the last step, where we apply Theorem 2.2 (2) to produce a J-holomorphic curve C such that $C = \operatorname{Im} f$ for some $f \in \mathcal{M}$.

Lemma 3.9. *The space $\mathcal{M}$ is nonempty.*

The proof of Lemma 3.9 is given at the end of this section. Accepting it for now, and hence Proposition 3.1, we shall prove Theorem 1.2 next.

Proof of Theorem 1.2. The group $\mathcal{G}$ acts on $\mathcal{M}$ smoothly (see the general discussion at the end of §3.3, Part I of [**10**]). Moreover, the action is free at any $f \in \mathcal{M}$ which is not multiply covered. At a multiply covered $f \in \mathcal{M}$ with multiplicity $a > 1$, the isotropy subgroup is $\{(\mu_a^l, 0) \mid l = 0, 1, \ldots, a-1\} \subset \mathcal{G}$ up to conjugations in $\mathcal{G}$. Here $\mathcal{G}$ is canonically identified with the group of linear translations on $\mathbb{C}$, $\{(s,t) \mid s \in \mathbb{C}^*, t \in \mathbb{C}\}$. Clearly, $\mathcal{M} \to \mathcal{M}^\dagger \equiv \mathcal{M}/\mathcal{G}$ is a smooth orbifold principle $\mathcal{G}$-bundle over a compact 2-dimensional orbifold.

Recall that the domain of each $f \in \mathcal{M}$ is the orbifold Riemann sphere Σ of one orbifold point $z_\infty \equiv \infty$ of order $2m$. We identify $\Sigma \setminus \{z_\infty\}$

canonically with $\mathbb{C}$ such that the action of $\mathcal{G}$ on $\Sigma \setminus \{z_\infty\}$ is given by linear translations on $\mathbb{C}$. We introduce the associated orbifold fiber bundle $Z \equiv \mathcal{M} \times_{\mathcal{G}} (\Sigma \setminus \{z_\infty\}) \rightarrow \mathcal{M}^\dagger$. Then as shown in our earlier paper [**11**], there is a canonically defined smooth map of orbifolds in the sense of [**10**], $\text{Ev} : Z \rightarrow X$, such that the induced map between the underlying spaces is the evaluation map $[(f, z)] \mapsto f(z)$, $\forall f \in \mathcal{M}, z \in \Sigma \setminus \{z_\infty\}$.

The map $\text{Ev} : Z \rightarrow X$ is a diffeomorphism of orbifolds onto $X \setminus \{p_0\}$. In fact, as in the proof of Lemma 4.3 in [**11**], one can show that the differential of Ev is invertible and that the induced map between the underlying spaces is onto $X \setminus \{p_0\}$. It remains to see that the induced map of Ev between the underlying spaces is injective. This is because: (1) for each $f \in \mathcal{M}$, the J-holomorphic curve Im f is a quasi-suborbifold, and (2) for any $f_1, f_2 \in \mathcal{M}$ which have different orbits in $\mathcal{M}^\dagger \equiv \mathcal{M}/\mathcal{G}$, the J-holomorphic curves Im f_1, Im f_2 intersect only at p_0. The former is proved in Lemma 3.3. To see the latter, suppose for simplicity that $f_1, f_2 \in \mathcal{M}$ are not multiply covered. Then by the intersection formula, the contribution of p_0 to the intersection product Im $f_1 \cdot$ Im f_2 is at least

$$\frac{\frac{|G|}{2m} \cdot \frac{|G|}{2m}}{|G|} = \frac{|G|}{4m^2} = c_1(E) \cdot c_1(E),$$

which implies that Im f_1, Im f_2 can not intersect at any other point. The discussion for the remaining cases is similar, so we leave the details to the reader. Hence the claim.

Let $M_0 \equiv \text{Ev}^{-1}(C_0)$ be the inverse image of C_0 in Z. Then M_0 is a suborbifold in Z. Moreover, since for each $f \in \mathcal{M}$, the J-holomorphic curve Im f intersects C_0 at exactly one point, we see that M_0 is a smooth section of the orbifold fiber bundle $Z \rightarrow \mathcal{M}^\dagger$. Consequently, we may regard Z as an orbifold complex line bundle over M_0. Note that under $\text{Ev} : Z \rightarrow X$, M_0 is mapped diffeomorphically onto $C_0 \subset X$.

One can show, by an identical argument as in [**11**], that there exists a regular neighborhood N_0 of the singular point p_0 in X, such that for any $f \in \mathcal{M}$, ∂N_0 intersects Imf transversely at a simple closed loop. It follows easily that $X \setminus \text{int}\,(N_0)$ is diffeomorphic to the associated unit disc bundle of $Z \rightarrow M_0$ via the inverse of Ev, under which C_0 is mapped diffeomorphically onto the 0-section M_0. Now observe that the s-cobordism W is diffeomorphic to $X \setminus \text{int}\,(N_0)$ with a regular neighborhood of C_0 removed. It follows easily that W is diffeomorphic to the product $(\mathbb{S}^3/G) \times [0, 1]$. q.e.d.

Proof of Lemma 3.9. The basic observation here is that if a component C_i in the Poincaré dual of $c_1(E)$ has a relatively small self-intersection $C_i \cdot C_i$, then one can easily show that $C_i = \text{Im } f$ for some $f \in \mathcal{M}$. In particular, $\mathcal{M}$ is nonempty when $c_1(E) \cdot c_1(E) = \frac{|G|}{4m^2}$ is sufficiently small. On the other hand, in the cases where $c_1(E)\cdot c_1(E)=\frac{|G|}{4m^2}$ is not small, it turns out that $d(E)$, the dimension of the Seiberg-Witten moduli space, is also considerably large, which allows us to break the Poincaré dual of $c_1(E)$ into smaller pieces by requiring it to pass through a certain number of specified points (cf. Remark 2.3).

Case 1. $|G|<4m^2$. Let $\{C_i\}$ be the set of J-holomorphic curves obtained by applying Theorem 2.2 (2) to E. The assumption $|G|<4m^2$ has the following immediate consequences: (1) C_0 is not contained in $\{C_i\}$ because $C_0\cdot C_0=\frac{4m^2}{|G|}>1$ and $c_1(E)\cdot c_1(E)=\frac{|G|}{4m^2}<1$, and (2) if we let $C_i=r_i\cdot c_1(E)$ for some $0<r_i\leq 1$, then the virtual genus

$$\begin{aligned} g(C_i) &= \frac{1}{2}(r_i^2\cdot c_1(E)\cdot c_1(E)+r_i\cdot c_1(K_X)\cdot c_1(E))+1 \\ &= \frac{1}{2}\left(r_i^2\cdot\frac{|G|}{4m^2}-r_i\cdot\frac{m+1}{m}\right)+1<1. \end{aligned}$$

As corollaries of (2), we note that for any $f_i:\Sigma_i\to X$ parametrizing C_i, $g_{|\Sigma_i|}=0$ because $g_{|\Sigma_i|}\leq g_{\Sigma_i}\leq g(C_i)<1$. (Here $g_{|\Sigma_i|}$ is the genus of the underlying Riemann surface of Σ_i.) Furthermore, note that $p_0\in\cup_i C_i$ because in the present case, the representation $\rho:G\to\mathbb{S}^1$ defined in Lemma 3.6 is nontrivial, cf. Remark 2.3. If $C\in\{C_i\}$ is a compoment containing p_0, then $f^{-1}(p_0)$ consists of only one point for any $f:\Sigma\to X$ parametrizing C. This is because for any $z'\in f^{-1}(p_0)$ with order $m'\geq 1$, the contribution from z' to g_Σ is $\frac{1}{2}(1-\frac{1}{m'})$, and $k_{z'}\geq\frac{1}{2m'}(\frac{|G|}{m'}-1)\geq\frac{1}{2m'}$. Hence the contribution from each point in $f^{-1}(p_0)$ to the right hand side of the adjunction formula for C least $\frac{1}{2}$. If there were more than one point in $f^{-1}(p_0)$, the right hand side of the adjunction formula would be no less than 1, which is a contradiction to $g(C)<1$.

With these understood, note that $d(E)\geq 2$ by Lemma 3.8, so that we may require that $\cup_i C_i$ also contains a smooth point $p\in C_0$. It follows easily, since C_0 is not contained in $\{C_i\}$, that $\{C_i\}$ consists of only one component, denoted by C, which contains both $p\in C_0$ and p_0. Let $f:\Sigma\to X$ be a parametrization of C. Then as we argued earlier, $f^{-1}(p_0)$ consists of only one point, say z_∞. Moreover, $f^{-1}(C_0)$ also consists of only one point, say z_0, because C,C_0 intersect at a smooth point p and $C\cdot C_0=1$. It follows easily that the link of p in C is homotopic in $\mathbb{S}^3/G$ to a regular fiber of the Seifert fibration, which has homotopy class $\mu_{2m}^{-1}I\in Z_{2m}$. On the other hand, $g_{|\Sigma|}=0$, so that the link of p in C is homotopic in W to the inverse of the link of p_0 in C. Hence the link of p_0 in C must have homotopy class $\mu_{2m}I\in Z_{2m}$ (cf. Lemma 3.5), from which it is easily seen that $f\in\mathcal{M}$. This proves that $\mathcal{M}$ is nonempty when $|G|<4m^2$.

Case 2. $|G| > 4m^2$. The proof will be done case by case according to the type of G.

(1) $G = \langle Z_{2m}, Z_{2m}; \widetilde{D}_n, \widetilde{D}_n\rangle$ or $\langle Z_{4m}, Z_{2m}; \widetilde{D}_n, C_{2n}\rangle$. In this case, note that $|G| > 4m^2$ is equivalent to $m < n$. We start with the following

Sublemma 3.10. *Let C be a J-holomorphic curve which intersects C_0 at only one singular point. If furthermore* (1) $C \cdot C_0 < 1$ *when the singular point in $C \cap C_0$ is of order* 2*, and* (2) *C contains p_0 when the singular point in $C \cap C_0$ is of order n, then C is the image of a member of $\mathcal{M}$.*

Proof. Let $f : \Sigma \to X$ be a parametrization of C.

First, consider the case where the singular point in $C \cap C_0$, say p_1, has order 2. Note that $C \cdot C_0 < 1$ implies that $f^{-1}(C_0)$ consists of only one point, say $z_0 \in \Sigma$, which has order 2 in Σ, and in a local representative (f_0, ρ_0) of f at z_0, $\rho_0(\mu_2) = \mu_2$ and $f_0(z) = (a(z^l + \cdots), z)$ with l odd if $a \neq 0$. In particular, the link of p_1 in C is homotopic in $\mathbb{S}^3/G$ to the singular fiber of the Seifert fibration at p_1. Now recall that $H_1(\mathbb{S}^3/G; \mathbb{Z}) = \mathbb{Z}_m \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ if n is even, and $H_1(\mathbb{S}^3/G; \mathbb{Z}) = \mathbb{Z}_{4m}$ when n is odd, where, if we let x, y be the standard generators of $\widetilde{D}_n$ with relations $x^2 = y^n = (xy)^2 = -1$, one of the factor in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ in the former case is generated by x and the other by y, and in the latter case, the generator of Z_{4m} is the class of $\mu_{2m}x$ or $\mu_{4m}x$, depending on whether m is odd or even. With this understood, note that the class in $H_1(\mathbb{S}^3/G; \mathbb{Z})$ of the link of p_1 in C projects nontrivially to the $\mathbb{Z}_2$ factor generated by x in the former case, and is a generator of Z_{4m} in the latter case. It follows easily that $f^{-1}(p_0)$ is nonempty, and there must be a $z_\infty \in f^{-1}(p_0)$, such that the pushforward of the link of z_∞ in Σ under f has a homology class in $\mathbb{S}^3/G$ which projects nontrivially onto the $\mathbb{Z}_2$ factor generated by x in the former case, and is a generator of Z_{4m} in the latter case. In any event, the order of z_∞ in Σ must be $4m$ or less, and as argued in the proof of Lemma 3.3, C is a quasi-suborbifold, and is easily seen to be the image of a member of $\mathcal{M}$.

Next we suppose that the singular point in $C \cap C_0$ is p_3, which has order n. Note that $m < n$ implies that the normalized Seifert invariant at p_3 is (n, m). Let (w_1, w_2) be a holomorphic coordinate system on a local uniformizing system at p_3, where C_0 is given locally by $w_2 = 0$, and the singular fiber of the Seifert fibration at p_3 is defined by $w_1 = 0$, $|w_2| \equiv$ constant, and the $\mathbb{Z}_n$-action is given by $\mu_n \cdot (w_1, w_2) = (\mu_n w_1, \mu_n^m w_2)$. Let $f^{-1}(C_0) = \{z_i \mid i = 1, 2, \ldots, k\}$ where each z_i has order $m_i \geq 1$, and let (f_i, ρ_i) be a local representative of f at z_i, where $\rho_i(\mu_{m_i}) = \mu_{m_i}^{r_i}$, with $0 \leq r_i < m_i$, r_i, m_i relatively prime, and $f_i(w) = (c_i(w^{l_i'} + \cdots), w^{l_i})$ such that $c_i \neq 0$ unless $m_i = n$ and $l_i = 1$. Note that f_i being ρ_i-equivariant implies that $l_i \equiv m r_i \pmod{m_i}$, and when $c_i \neq 0$, $l_i' \equiv r_i \pmod{m_i}$. By the intersection formula, the contribution from z_i to $C \cdot C_0$ is $\frac{(n/m_i)l_i}{n} = \frac{l_i}{m_i}$. Hence $C \cdot C_0 = \sum_{i=1}^{k} \frac{l_i}{m_i}$, and the virtual genus of C is

$$g(C) = \sum_{i,j=1}^{k} \frac{l_i l_j}{m_i m_j} \cdot \frac{n}{2m} - \sum_{i=1}^{k} \frac{l_i}{m_i} \cdot \frac{m+1}{2m} + 1.$$

Evidently, the contribution to $g(C)$ from each z_i is

$$L_i \equiv \frac{l_i^2}{m_i^2} \cdot \frac{n}{2m} - \frac{l_i}{m_i} \cdot \frac{m+1}{2m},$$

and the contribution from each unordered pair $[z_i, z_j]$, $i \neq j$, is

$$L_{[i,j]} \equiv \frac{l_i l_j}{m_i m_j} \cdot \frac{n}{m}.$$

On the other hand, the contribution of each z_i to the right hand side of the adjunction formula is

$$R_i \equiv \frac{1}{2}\left(1 - \frac{1}{m_i}\right) + k_{z_i},$$

and the contribution of each unordered pair $[z_i, z_j]$, $i \neq j$, is

$$R_{[i,j]} \equiv k_{[z_i,z_j]}.$$

In order to estimate k_{z_i} and $k_{[z_i,z_j]}$, we next recall some basic facts about the local self-intersection number and local intersection number of J-holomorphic curves, cf. [**11**] and the references therein.

- Let C be a holomorphic curve in $\mathbb{C}^2$ parametrized by $f(z) = (a(z^{l_1} + \cdots), z^{l_2})$, where $f : (D, 0) \to (\mathbb{C}^2, 0)$ is from a disc $D \subset \mathbb{C}$ centered at 0 such that $f|_{D \setminus \{0\}}$ is embedded. Then the local self-intersection number $C \cdot C \geq \frac{1}{2}(l_1 - 1)(l_2 - 1)$. Note that the above inequality still makes sense even if $a = 0$ in the formula for f, in which case l_1 is undefined. This is because $l_2 = 1$ by the assumption that $f|_{D \setminus \{0\}}$ is embedded.
- Let C, C' be distinct holomorphic curves in $\mathbb{C}^2$ parametrized by $f(z) = (a(z^{l_1} + \cdots), z^{l_2})$ and $f'(z) = (a'(z^{l_1'} + \cdots), z^{l_2'})$ respectively, where $f : (D, 0) \to (\mathbb{C}^2, 0)$, $f' : (D, 0) \to (\mathbb{C}^2, 0)$ are from a disc $D \subset \mathbb{C}$ centered at 0 such that $f|_{D \setminus \{0\}}$, $f'|_{D \setminus \{0\}}$ are embedded. Then the local intersection number $C \cdot C' \geq \min(l_1 l_2', l_2 l_1')$. Here $l_1 = \infty$ (resp. $l_1' = \infty$) if $a = 0$ (resp. $a' = 0$).

With the preceding understood and by the definition in [**11**], we have

$$k_{z_i} \geq \frac{1}{2m_i}\left((l_i - 1)(l_i' - 1) + \left(\frac{n}{m_i} - 1\right) l_i l_i'\right),$$

$$k_{[z_i,z_j]} \geq \frac{1}{n} \cdot \frac{n}{m_i} \cdot \frac{n}{m_j} \cdot \min(l_i l_j', l_j l_i').$$

(Note that the right hand side of the first inequality still makes sense even when l_i' is undefined, because in this case, $l_i = 1$ and $n = m_i$ must be true.)

Next we shall compare L_i with R_i and $L_{[i,j]}$ with $R_{[i,j]}$. To this end, we write $l_i' = r_i + t_i m_i$ and $mr_i = l_i + s_i m_i$. Here $t_i \geq 0$, and $s_i \geq 0$ if $l_i < m_i$. When $l_i = m_i$, we must have $l_i = m_i = 1$ and $r_i = 0$. In this case, $s_i = -1$ and $l_i' = t_i \geq 1$. It follows easily that $\min(l_i l_j', l_j l_i') \geq \frac{l_i l_j}{m}$, hence $R_{[i,j]} \geq L_{[i,j]}$ for all $i \neq j$. To compare L_i with R_i, we note that

$$\begin{aligned} k_{z_i} &\geq \frac{1}{2m_i}\left((l_i-1)(l_i'-1) + \left(\frac{n}{m_i} - 1\right) l_i l_i'\right) \\ &= \frac{1}{2m_i}\left(1 - l_i - (r_i + t_i m_i) + \frac{n l_i}{m_i}\left(\frac{l_i + s_i m_i}{m} + t_i m_i\right)\right) \\ &= \frac{1}{2m_i} + \frac{n l_i^2}{2m_i^2 m} - \frac{l_i(m+1)}{2m_i m} + \frac{n l_i - m_i}{2m_i m}(s_i + t_i m), \end{aligned}$$

which easily gives $R_i - L_i \geq \frac{1}{2} + \frac{nl_i - m_i}{2m_i m}(s_i + t_i m)$.

With the above estimates in hand, now observe that $f^{-1}(p_0)$ is not empty by the assumption, so that, as we argued earlier, the contribution of $f^{-1}(p_0)$ to the right hand side of the adjunction formula is at least $\frac{1}{2}$. It follows easily that $f^{-1}(C_0)$ contains only one point (i.e., $k = 1$), and that either $m_1 = n$ with $l_1 = 1$, or $s_1 + t_1 m = 0$, which means either $s_1 = t_1 = 0$, or $m = t_1 = -s_1 = 1$ with $m_1 = l_1 = 1$. Moreover, $g_{|\Sigma|} = 0$, and $f^{-1}(p_0)$ contains only one point, say z_∞, with the contribution from z_∞ to the right hand side of the adjunction formula being exactly $\frac{1}{2}$. The last point particularly implies that the order of z_∞ is $\frac{1}{2}|G| = 2mn$.

The case where $s_1 = t_1 = 0$ but $nl_1 \neq m_1$ or $l_1 = m_1 = 1$ can be ruled out as follows. Consider $s_1 = t_1 = 0$ but $nl_1 \neq m_1$ first. Note that $l_1 = mr_1$ must be true in this case. As we have seen earlier, the link of p_3 in C is homotopic in $\mathbb{S}^3/G$ to $l_1 \cdot \frac{n}{m_1} = \frac{r_1 mn}{m_1}$ times the singular fiber of the Seifert fibration at p_3, which has order $2mn$. It is easily seen that the order d of the link of p_3 is either divisible by m_1, in which case r_1 is even, or divisible by $2m_1$, in which case r_1 is odd. In any event, $d < 2mn$ if $nl_1 \neq m_1$. On the other hand, $g_{|\Sigma|} = 0$ implies that the link of p_0 in C is homotopic, in the s-cobordism W and through C, to the inverse of the link of p_3 in C. But the homotopy class of the link of p_0 has the same order in $\pi_1(\mathbb{S}^3/G)$ as the order of z_∞, which is $2mn$. This is a contradiction. The discussion for the case where $l_1 = m_1 = 1$ is similar. In this case, the link of p_3 is homotopic in $\mathbb{S}^3/G$ to n times of the singular fiber, hence has order at most $2m$, which is less than $2mn$. This is also a contradiction. Hence $m_1 = n$ and $l_1 = 1$, and the adjunction formula implies that C is a quasi-suborbifold. Moreover, it is easily seen that C is the image of a member of $\mathcal{M}$. Hence Sublemma 3.10. q.e.d.

Now back to the proof. Let $\{C_i\}$ be the J-holomorphic curves which are obtained by applying Theorem 2.2 (2) to E. Set $N \equiv \frac{d(E)}{2}$ when $m \neq 1$ and $N \equiv \frac{d(E)}{2} - 1$ when $m = 1$. Then by Remark 2.3, we can specify any N distinct smooth points $q_1, \ldots, q_N \in X \setminus C_0$ and require that $q_1, \ldots, q_N \in \cup_i C_i$, and moreover, $p_0 \in \cup_i C_i$. Now we let $q_1 \equiv q_{1,j}$ be a sequence of points converging to a smooth point q in C_0, while keeping $q_2, \ldots, q_N$ fixed, and let $\{C_i^{(j)}\}$ be the corresponding sequence of (sets of) J-holomorphic curves. By passing to a subsequence if necessary, we may assume that the number of components in $\{C_i^{(j)}\}$ is independent of j, and each $C_i^{(j)}$ is parametrized by a J-holomorphic map $f_{i,j} : \Sigma_i \rightarrow X$ from an orbifold Riemann surface independent of j (note that the complex structure on Σ_i is allowed to vary). This follows readily from the fact that $C_i^{(j)} \cdot C_0 \geq \frac{1}{n}$, and that the virtual genus $g(C_i^{(j)})$, hence the corresponding orbifold genus, is uniformly bounded from above. By the Gromov compactness theorem (cf. [**14**]), each $f_{i,j}$ converges to a cusp-curve $f_i' : \Sigma_i' \rightarrow X$. The upshot here is that we can always manage to have C_0 contained in $\cup_i \mathrm{Im}\ f_i'$, or else $\mathcal{M}$ is nonempty. Accepting this momentarily, we note as a consequence that

$$C \cdot C_0 \leq 1 - C_0 \cdot C_0 = 1 - \frac{m}{n},$$

where $C \subset \cup_i \mathrm{Im}\ f_i'$ is any component containing p_0. By letting $q_2, \ldots,$ q_N converge to a smooth point in C_0 one by one, we have at the end

$$C \cdot C_0 \leq 1 - N \cdot \frac{m}{n}$$

for any component C in the limiting cusp-curve that contains p_0. Now observe that $N \geq \frac{\delta+1}{2}$ when $m \neq 1$, where $n = \delta m + r$ with $0 \leq r \leq m-1$, and $N \geq \frac{n-1}{2}$ when $m = 1$. It follows easily that $C \cdot C_0 < \frac{1}{2}$ when $m \neq 1$ and $C \cdot C_0 \leq \frac{1}{2} + \frac{1}{2n}$ when $m = 1$. Clearly, C can only intersect C_0 at one singular point, because otherwise, we would have $C \cdot C_0 \geq \frac{1}{2} + \frac{1}{n}$. By Sublemma 3.10, $\mathcal{M}$ is nonempty.

It remains to show that $C_0 \subset \cup_i \mathrm{Im}\ f_i'$. Note that if the component $C^{(j)} \in \{C_i^{(j)}\}$ which contains $q_{1,j}$ intersect C_0 at a singular point, which is the case when $C^{(j)} \cdot C_0 < 1$, then it is clear that C_0 must be one of the component in the limiting cusp-curve of $\{C^{(j)}\}$. If $C^{(j)}$ intersects C_0 at a smooth point q_j, then $\lim_{j\rightarrow\infty} q_j = q = \lim_{j\rightarrow\infty} q_{1,j}$ must hold if C_0 is not contained in the limiting curve, and moreover, the limiting curve must contain only one nonconstant component, which intersects C_0 at q transversely. In this case we let $q \equiv q_k$ be a sequence of smooth points on C_0 that converges to the singular point $p_3 \in C_0$ of order n.

Let C_k be the corresponding J-holomorphic curves, which we assume to be parametrized by $f_k : \Sigma \rightarrow X$ from a fixed orbifold Riemann surface without loss of generality. If the limiting curve $f' : \Sigma' \rightarrow X$ of $\{C_k\}$ intersects C_0 only at p_3, then $\mathcal{M}$ is nonempty by Sublemma 3.10. If the limiting curve f' has a nonconstant component, denoted by $f_\nu \equiv f'|_{\Sigma_\nu} : \Sigma_\nu \rightarrow X$, which intersects C_0 at a singular point of order 2, say p_1, then there must be a simple closed loop $\gamma \subset \Sigma$ collapsed to p_1 during the convergence. Note that there are two distinct points $z_\nu, z_\omega \in \Sigma'$, where $z_\nu \in \Sigma_\nu$ with $f_\nu(z_\nu) = p_1$, which are the images of γ under the collapsing $\Sigma \rightarrow \Sigma'$. Let Σ_ω be the component of Σ' which contains z_ω (here $\Sigma_\omega = \Sigma_\nu$ is allowed). Then one of the following must be true: (a) either f' is nonconstant on Σ_ω, or f' is constant on Σ_ω but there exists a component Σ'_ω and a point $z'_\omega \neq z_\nu$ such that f' is nonconstant on Σ'_ω and $f'(z'_\omega) = p_1$; (b) the simple closed loop γ bounds a sub-surface Γ in Σ which contains no orbifold points and is of nonzero genus, such that f' is constant on every component in the image of $\Gamma \subset \Sigma \rightarrow \Sigma'$. However, the latter case can be ruled out as follows. According to the Gromov compactness theorem (cf. [**14**]), if we fix a sufficiently small $\epsilon > 0$, then there exists a regular neighborhood of γ in Σ, identified with $\gamma \times [-1, 1]$, such that (1) $\gamma \times \{-1\}$ is mapped to the link of z_ν of radius ϵ in Σ_ν under $\Sigma \rightarrow \Sigma'$, (2) f_k converges to f' in C^∞ on $\gamma \times \{-1, 1\}$, (3) the diameter of $f_k(\gamma \times [-1, 1])$ is less than ϵ when k is sufficiently large, and (4) because f' is constant on every component in the image of $\Gamma \subset \Sigma \rightarrow \Sigma'$, the diameter of $f_k(\Gamma \setminus \gamma \times [0, 1))$ is also less than ϵ when k is sufficiently large. Let $U(10\epsilon)$ be the regular neighborhood of p_1 in X of radius 10ϵ. Then it is clear that when k is sufficiently large, $f_k(\Gamma) \subset U(10\epsilon) \setminus \{p_1\}$ and $f_k(\gamma) = f_k(\partial\Gamma)$ is homotopic in $U(10\epsilon) \setminus \{p_1\}$ to the push-forward of the link of z_ν in Σ_ν under f_ν. But this is impossible because (1) f_ν is clearly not multiply covered, and (2) the link of p_1 in Im f_ν, which is the push-forward of the link of z_ν in Σ_ν under f_ν because of (1), is not null-homologous in $U(10\epsilon) \setminus \{p_1\} \cong \mathbb{RP}^3 \times (0, 1]$. Hence the latter case (i.e., case (b)) is ruled out. On the other hand, the former case (i.e., case (a)) is also impossible if C_0 is not contained in $\cup_i \mathrm{Im}\, f'_i$. This is because each of z_ν and z_ω (or z'_ω) will contribute at least $\frac{1}{2}$ to $[f(\Sigma')] \cdot C_0$, and with the contribution from p_3, we would have $[f(\Sigma')] \cdot C_0 > 1$, which is a contradiction. Hence we can always manage to have $C_0 \subset \cup_i \mathrm{Im}\, f'_i$, or else $\mathcal{M}$ is nonempty.

(2) $G = \langle Z_{2m}, Z_{2m}; \widetilde{T}, \widetilde{T}\rangle$ or $\langle Z_{6m}, Z_{2m}; \widetilde{T}, \widetilde{D}_2\rangle$. Note that $|G| > 4m^2$ is equivalent to $m < 6$, which means that $m = 1$ or 5 in the former case, and $m = 3$ in the latter case.

First, observe that if C is a J-holomorphic curve parametrized by $f : \Sigma \rightarrow X$ such that $p_0 \in C$, then each $z \in f^{-1}(p_0)$ will contribute at least $\frac{1}{2} + \frac{1}{6m}$ to the right hand side of the adjunction formula. To see this, let m_0 be the order of z. Then the total contribution of z is

$$\frac{1}{2}\left(1-\frac{1}{m_0}\right)+k_z \geq \frac{1}{2}\left(1-\frac{1}{m_0}\right)+\frac{1}{2m_0}\left(\frac{24m}{m_0}-1\right)$$
$$=\frac{1}{2}+\frac{1}{2m_0}\left(\frac{24m}{m_0}-2\right).$$

Note that $m_0 \leq 6m$. Hence the claim.

We consider the case where $m = 3$ or 5 first. Let $\{C_i\}$ be the J-holomorphic curves obtained by applying Theorem 2.2 (2) to E. Note that $p_0 \in \cup_i C_i$, and because $d(E) = 2$, we can specify a smooth point $q \in C_0$ and require that $q \in \cup_i C_i$. We claim that there exists a J-holomorphic curve $\hat{C}$ such that $\hat{C} \cdot C_0 \leq \frac{1}{2}$ and $p_0 \in \hat{C}$.

Let $C \in \{C_i\}$ be a component containing p_0. If C does not contain q, then the component in $\{C_i\}$ which contains q must be C_0. As a consequence, $C \cdot C_0 \leq 1 - C_0 \cdot C_0 = 1 - \frac{m}{6} \leq \frac{1}{2}$.

Now suppose $q \in C$. Then C must be the only component in $\{C_i\}$, and C, C_0 intersect transversely at the smooth point q. We let $q \equiv q_k$ be a sequence of smooth points on C_0 converging to the singular point $p_3 \in C_0$ of order 3, and let C_k be the corresponding J-holomorphic curves, which we assume without loss of generality to be parametrized by $f_k : \Sigma \to X$ from a fixed orbifold Riemann surface. Note that $g_{|\Sigma|} = 0$ because $g(C_k) = \frac{5-m}{2m} + 1 \leq 1 + \frac{1}{3}$, and because C_k contains p_0 so that $f_k^{-1}(p_0)$ contributes at least $\frac{1}{2}$ to the right hand side of the adjunction formula. Similarly, $f_k^{-1}(p_0)$ contains at most two points. Let $z_0 = f_k^{-1}(q_k)$. Note that z_0 is a regular point of Σ.

By the Gromov compactness theorem, a subsequence of $\{f_k\}$, after reparametrization if necessary, will converge either in C^∞ to $f : \Sigma \to X$, or to a cusp-curve $f : \Sigma' \to X$. If the convergence is in C^∞, then f must be multiply covered, because otherwise, we will have $k_{z_0} \geq \frac{1}{2}(\frac{3}{1} - 1) = 1$, and together with the contribution of $f^{-1}(p_0)$ which is at least $\frac{1}{2}$, it would imply that the right hand side of the adjunction formula is no less than $1 + \frac{1}{2}$, which is greater than the left hand side $g(\mathrm{Im}\ f) = \frac{5-m}{2m} + 1 \leq 1 + \frac{1}{3}$. This is a contradiction. For a multiply covered f, it is clear that $\hat{C} \equiv \mathrm{Im}\ f$ is a J-holomorphic curve which obeys $\hat{C} \cdot C_0 \leq \frac{1}{2}$ and $p_0 \in \hat{C}$.

Now suppose f_k converges to a cusp-curve $f : \Sigma' \to X$. For technical convenience, we shall regard z_0 as a marked point so that z_0 will not lie on a collapsing simple closed loop during the Gromov compactification (cf. [**14**]). Let Σ_ν be the component of Σ' which contains z_0. First, we consider the case where f is nonconstant over Σ_ν. Note that under this assumption, if $\mathrm{Im}\ f|_{\Sigma_\nu} \neq C_0$, one can easily show, because $z_0 \in \Sigma_\nu$ is a regular point, that $[f(\Sigma_\nu)] \cdot C_0 = 1$. This implies that Σ_ν is the only component of Σ' over which f is nonconstant. In particular, $p_0 \in \operatorname{Im} f|_{\Sigma_\nu}$ because Σ' is connected. Then as we argued in the preceding paragraph, f must be multiply covered over Σ_ν, and $\operatorname{Im} f|_{\Sigma_\nu}$ is the J-holomorphic curve that we are looking for. If $\operatorname{Im} f|_{\Sigma_\nu} = C_0$, then $\hat{C} \cdot C_0 \leq 1 - \frac{m}{6} \leq \frac{1}{2}$ for any (nonconstant) component $\hat{C}$ in $\operatorname{Im} f$ that contains p_0. Now suppose f is constant over Σ_ν. Then it follows easily, because $g_{|\Sigma|} = 0$ and because no simple closed loops bounding a disc $D \subset \Sigma$ will collapse if $f_k(D)$ lies in the complement of C_0 and p_0 (cf. case (2) in the proof of Lemma 3.4), it follows easily that each $f_k^{-1}(p_0)$ contains exactly two points $z_\infty^{(1)}$, $z_\infty^{(2)}$, and there are simple closed loops $\gamma_1, \gamma_2 \subset \Sigma$, each bounding a disc $D \subset \Sigma$, such that (1) D contains exactly one of $z_\infty^{(1)}, z_\infty^{(2)}$ but not z_0, (2) no simple closed loops in D collapsed, (3) D is mapped to a component Σ_ω under $\Sigma \rightarrow \Sigma'$ such that $f|_{\Sigma_\omega} \neq$ constant. It is clear that one of these two components of $\operatorname{Im} f$, which all contains p_0, will have intersection product with C_0 no greater than $\frac{1}{2}$.

Let C be a J-holomorphic curve such that $p_0 \in C$ and $C \cdot C_0 \leq \frac{1}{2}$, which we have just shown to exist, and let $f : \Sigma \rightarrow X$ be a parametrization of C. Note that $g(C) < 1$, so that $g_{|\Sigma|} = 0$ and $f^{-1}(p_0)$ consists of only one point z_∞. Moreover, it follows easily from $C \cdot C_0 \leq \frac{1}{2}$ that $f^{-1}(C_0)$ consists of only one point also, and either $C \cdot C_0 = \frac{1}{2}$ or $\frac{1}{3}$, and if p_i of order a_i for some $i = 1, 2$ or 3 is the singular point where C, C_0 intersect, then the link of p_i in C is homotopic in $\mathbb{S}^3/G$ to the singular fiber of the Seifert fibration at p_i, which has order $2ma_i$ in $\pi_1(\mathbb{S}^3/G) = G$. It implies that the order of z_∞ is also $2ma_i$ because $g_{|\Sigma|} = 0$, and the adjunction formula implies that C is a quasi-suborbifold, which is easily seen to be the image of a member of $\mathcal{M}$.

It remains to consider the case where $m = 1$. We will show in this case that there is also a J-holomorphic curve C such that $p_0 \in C$ and $C \cdot C_0 \leq \frac{1}{2}$. To see this, let $\{C_i\}$ be the J-holomorphic curves obtained by applying Theorem 2.2 (2) to E. Since $d(E) = 8$ in the present case, we can specify any 3 distinct smooth points q_1, q_2, q_3, where $q_1 \in C_0$ and $q_2, q_3 \in X \setminus C_0$, and require that they are contained in $\cup_i C_i$, and moreover, we require $p_0 \in \cup_i C_i$ also. Let $q_2 \equiv q_{2,j}$ be a sequence of points converging to a smooth point $q_2' \in C_0$ such that $q_2' \neq q_1$, and we denote by $\{C_i^{(j)}\}$ the corresponding sequence of J-holomorphic curves. If $C_0 \in \{C_i^{(j)}\}$, then as we have seen earlier, the components containing $q_{2,j}$ will converge to a cusp-curve during which a component will be split off that goes to C_0. At this stage, the component $\hat{C}$ in the limiting cusp-curve which contains p_0 obeys $\hat{C} \cdot C_0 \leq 1 - 2C_0 \cdot C_0$. We claim that this is also true even when C_0 is not contained in $\{C_i^{(j)}\}$. To see this, note that under this assumption, there is only one component, denoted by $C^{(j)}$, in $\{C_i^{(j)}\}$, which intersects C_0 transversely at q_1. Let $f_j : \Sigma \to X$ be a parametrization of $C^{(j)}$, which is from a fixed orbifold Riemann surface by passing to a subsequence if necessary. Since $q_2' = \lim_{j\to\infty} q_{2,j} \neq q_1$, f_j has to converge to a cusp-curve $f : \Sigma' \to X$. Let Σ_ν be a component of Σ' such that $f_\nu \equiv f|_{\Sigma_\nu}$ is nonconstant and $q_1 \in \text{Im } f_\nu$. Then since q_1 is a smooth point, and $\text{Im } f$ contains another smooth point $q_2' = \lim_{j\to\infty} q_{2,j} \neq q_1$, we see that $\text{Im } f_\nu = C_0$ must be true. If the degree of $f_\nu : \Sigma_\nu \to X$ is at least 2, then the claim is clearly true. Suppose $f_\nu : \Sigma_\nu \to X$ is of degree 1. Then Σ_ν contains three orbifold points z_1, z_2, z_3, with $f_\nu(z_i) = p_i$ for $i = 1, 2, 3$. Note that z_1, z_2, z_3 must be the result of collapsing 3 simple closed loops under $\Sigma \to \Sigma'$. Consequently, there are components Σ_i of Σ', where $i = 1, 2$ and 3, such that $\Sigma_i \neq \Sigma_\nu$ and there are $z_i' \in \Sigma_i$ satisfying $f(z_i') = p_i$ (here $\Sigma_i = \Sigma_{i'}$ is allowed). The key observation is that one of $C^{(i)} \equiv \text{Im } f_{\Sigma_i}$ must be either C_0 or constant, because otherwise, one has $c_1(E) \cdot C_0 \geq \sum_{i=1}^3 C^{(i)} \cdot C_0 \geq \sum_{i=1}^3 \frac{1}{a_i} > 1$ (here a_i is the order of p_i, with $a_1 = 2$, $a_2 = a_3 = 3$), which is a contradiction. But as we have seen earlier, none of $C^{(i)}$ is constant. (Suppose $C^{(i)}$ is constant for some $i = 1, 2$ or 3; then z_i' must be the image of a collapsing simple closed loop γ which bounds a sub-surface $\Gamma \subset \Sigma$ such that for sufficiently large j, $f_j(\Gamma)$ lies in a regular neighborhood of p_i with p_i removed, which is diffeomorphic to the product of a lens space with $(0, 1]$. But on the other hand, $f_j(\gamma) = f_j(\partial\Gamma)$ is homotopic in the punctured neighborhood of p_i to the link of p_i in C_0, which is not null-homologous in the punctured neighborhood. This is a contradiction.) Hence one of $C^{(i)}$ is C_0, and the claim follows. Now we let q_3 converge to a smooth point in C_0, and at the end, it follows easily that the component C in the limiting cusp-curve which contains p_0 obeys $C \cdot C_0 \leq 1 - 3C_0 \cdot C_0 = \frac{1}{2}$.

We shall prove that such a J-holomorphic curve C is the image of a member of $\mathcal{M}$ also. Let $f : \Sigma \to X$ be a parametrization of C. We write $C = \frac{r}{2} \cdot c_1(E)$ for some $0 < r \leq 1$. Then $g(C) = \frac{1}{2}(r^2 \cdot \frac{6}{4} - r \cdot \frac{1+1}{2}) + 1 \leq \frac{1}{4} + 1$. Now observe that each $z \in f^{-1}(C_0)$ will contribute at least $\frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4}$ to the right hand side of the adjunction formula, and each $z' \in f^{-1}(p_0)$ will contribute at least $\frac{1}{2} + \frac{1}{6m} = \frac{1}{2} + \frac{1}{6}$. It follows easily that $f^{-1}(p_0)$ consists of only one point, and $g_{|\Sigma|} = 0$. Finally, observe that $f^{-1}(C_0)$ also consists of only one point, because each point in $f^{-1}(C_0)$ will contribute at least $\frac{1}{3}$ to $C \cdot C_0$. It is easily seen from the adjunction formula that C is a quasi-suborbifold, and moreover, it is the image of a member of $\mathcal{M}$.

(3) $G = \langle Z_{2m}, Z_{2m}; \widetilde{O}, \widetilde{O} \rangle$. In this case, $|G| > 4m^2$ implies that $m = 1, 5, 7$ or 11. The proof is similar, although modification is needed at a few places.

First of all, observe that the largest order of an element in G is $8m$, so that if C is a J-holomorphic curve parametrized by $f:\Sigma\rightarrow X$ with $p_0\in C$, then any point in $f^{-1}(p_0)$ will contribute at least $\frac{1}{2}+\frac{1}{4m}$ to the right hand side of the adjunction formula.

Consider the case where $m\neq 1$ first. Let $\{C_i\}$ be the J-holomorphic curves obtained by applying Theorem 2.2 (2) to E, where $p_0\in\cup_i C_i$, and a specified smooth point $q\in C_0$ is also contained in $\cup_i C_i$. This time we claim that there exists a J-holomorphic curve $\hat{C}$ such that $p_0\in\hat{C}$ and $\hat{C}\cdot C_0\leq\frac{7}{12}$, which is clearly true when q is not contained in the same component in $\{C_i\}$ with p_0.

Now suppose $C\in\{C_i\}$ is a component containing both p_0 and q. Then C must be the only component in $\{C_i\}$, and C, C_0 intersect transversely at the smooth point q. We let $q\equiv q_k$ be a sequence of smooth points on C_0 converging to the singular point $p_3\in C_0$ of order 4, and let C_k be the corresponding J-holomorphic curves parametrized by $f_k:\Sigma\rightarrow X$ from a fixed orbifold Riemann surface. Note that this time $g_{|\Sigma|}\leq 1$, because $g(C_k)=\frac{11-m}{2m}+1\leq 1+\frac{3}{5}$, and moreover, $g_{|\Sigma|}=1$ only when $m=5$ and $f^{-1}(p_0)$ consists of only one point. In general, $f_k^{-1}(p_0)$ contains at most two points. Let $z_0=f_k^{-1}(q_k)$, which is a regular point of Σ.

The proof goes in the same way as in the preceding case, except at the end we need to consider the following scenario caused by the possibility that $g_{|\Sigma|}=1$. More concretely, let $f:\Sigma'\rightarrow X$ be the limiting cusp-curve of f_k, and let Σ_ν be the component of Σ' containing z_0. We need to consider the situation where $g_{|\Sigma|}=1$ and f is constant on Σ_ν. Note that if f is constant on Σ_ν, then one of the following must be true: (a) Σ_ν is an orbifold Riemann sphere obtained from collapsing two simple closed loops in Σ, and (b) Σ_ν is an orbifold torus obtained from collapsing one simple closed loop in Σ. Let Σ_ω be the component of Σ' which contains z_∞. Then in the latter case, Σ_ω is obtained by collapsing one simple closed loop, and hence f must be nonconstant on Σ_ω. It can be easily shown that in this case there is a J-holomorphic curve C such that $p_0\in C$ and $C\cdot C_0\leq\frac{7}{12}$. Now consider the former case. If f is constant on Σ_ω, then Σ_ω must be obtained from collapsing two simple closed loops, and it is easy to see that there will be at least two components of Σ' over which f is nonconstant. It is clear that one of these components will give the J-holomorphic curve that we are looking for. Suppose f is nonconstant on Σ_ω. Then one can easily show that we are done in either one of the following cases: there is a constant component other than Σ_ν, in which case Σ_ω is obtained from collapsing one simple closed loop, or there is a nonconstant component other than Σ_ω, which will break away with at least $\frac{5}{12}$ or $\frac{1}{2}$ of the homology. So it remains to consider the case where Σ_ω is the only component of Σ' other than Σ_ν, in which case Σ_ω is an orbifold Riemann sphere obtained from collapsing two simple closed loops. We are done if $f_\omega \equiv f|_{\Sigma_\omega}$ is multiply covered, so we assume that f_ω is not multiply covered. Set $C_\omega \equiv \operatorname{Im} f_\omega$. Then note that $f_\omega^{-1}(C_0) = \{z_0^{(1)}, z_0^{(2)}\}$, both of which are sent to p_3 of order 4 under f_ω. By the intersection formula as we have seen earlier, the contribution of each $z_0^{(i)}$, $i = 1, 2$, to $C_\omega \cdot C_0$ can be written as $\frac{l_i}{m_i}$, where m_i is the order of $z_0^{(i)}$ and l_i, m_i are relatively prime. It is clear that either $\frac{l_1}{m_1} = \frac{l_2}{m_2} = \frac{1}{2}$, or one of them is $\frac{1}{4}$ and the other is $\frac{3}{4}$, because $C_\omega \cdot C_0 = 1$. In the former case, the right hand side of the adjunction formula receives at least 2 for the contribution from $f_\omega^{-1}(C_0)$, which is more than the left hand side $g(C_\omega) = \frac{11-m}{2m} + 1 = 1 + \frac{3}{5}$. This is a contradiction. As for the latter case, note that when $m = 5$, the normalized Seifert invariant at p_3 is $(4, 1)$ (recall the relation $m = 12b + 6 + 4b_2 + 3b_3$). Assume without loss of generality that $\frac{l_1}{m_1} = \frac{3}{4}$. Then it follows easily that a local representative (f_1, ρ_1) of f_ω at $z_0^{(1)}$ must obey $\rho_1(\mu_4) = \mu_4^3$ and $f_1(w) = (c(w^l + \cdots), w^3)$ for some $c \neq 0$ and some positive interger l satisfying $l \equiv 3 \pmod 4$. With this understood, it follows that

$$k_{z_0^{(1)}} \geq \frac{1}{2m_1}(l-1)(3-1) \geq \frac{1}{2} \qquad \text{and}$$
$$k_{[z_0^{(1)}, z_0^{(2)}]} \geq \frac{1}{4}\min(l, 3l') \geq \frac{3}{4},$$

which is easily seen as a contradiction to the adjunction formula. In any event, there exists a J-holomorphic curve C such that $p_0 \in C$ and $C \cdot C_0 \leq \frac{7}{12}$.

Next we show that such a J-holomorphic curve C is the image of a member of $\mathcal{M}$. First, note that if C, C_0 intersect at more than one point, then $C \cap C_0 = \{p_2, p_3\}$ of order 3 and 4, and one can easily show that this is impossible using the adjunction formula. (In this case, $g(C) = \frac{1}{2}((\frac{7}{12})^2 \cdot \frac{12}{m} - \frac{7}{12} \cdot \frac{m+1}{m}) + 1 \leq \frac{127}{120}$, but the right hand side of the adjunction formula is at least $\frac{1}{2}(1 - \frac{1}{3}) + \frac{1}{2}(1 - \frac{1}{4}) + \frac{1}{2} + \frac{1}{4m} > \frac{29}{24}$.) Second, suppose C, C_0 intersect only at the singular point p_i of order a_i. Then one can easily show, as we did earlier, that C is the image of a member of $\mathcal{M}$ if $C \cdot C_0 = \frac{1}{a_i}$. With this understood, it remains to rule out the case where C, C_0 intersect at p_3 of order 4 but $C \cdot C_0 = \frac{1}{2}$. To this end, let $f : \Sigma \to X$ be a parametrization of C. Then there are two possibilities: either $f^{-1}(p_3)$ consists of two points, or it contains only one point. In any case, one can easily show that the contribution of $f^{-1}(p_3)$ to the right hand side of the adjunction formula is at least $\frac{1}{2}$. With the contribution of at least $\frac{1}{2} + \frac{1}{4m}$ from $f^{-1}(p_0)$, the right hand side of the adjunction formula is greater than 1. But on the left hand side, $g(C) = \frac{1}{2}((\frac{1}{2})^2 \cdot \frac{12}{m} - \frac{1}{2} \cdot \frac{m+1}{m}) + 1 = \frac{5-m}{4m} + 1 \leq 1$, which is a contradiction. Hence C is the image of a member of $\mathcal{M}$, and the case where $m \neq 1$ is done.

Finally, consider the case of $m = 1$. Let $\{C_i\}$ be the J-holomorphic curves obtained by applying Theorem 2.2 (2) to E. This time $d(E) = 14$, so we can specify any 6 distinct smooth points $q_1, q_2, \ldots, q_6$, where $q_1 \in C_0$ and $q_2, \ldots, q_6 \in X \setminus C_0$, and require that they are contained in $\cup_i C_i$; and moreover, we require $p_0 \in \cup_i C_i$ also. We let $q_2, \ldots, q_6$ converge to a smooth point $q \neq q_1$ in C_0 one by one, and as we have argued earlier, we obtain at the end a J-holomorphic curve C such that $p_0 \in C$ and $C \cdot C_0 \leq 1 - 6 C_0 \cdot C_0 = \frac{1}{2}$. However, in order to show that such a curve C is the image of a member of $\mathcal{M}$, we actually need to obtain a sharper estimate that $C \cdot C_0 < \frac{1}{2}$. To this end, for each $k = 2, \ldots, 6$, we let $\{C_i^{(k)}\}$ be the non-C_0 components in the limiting cusp-curve as q_k converges to a smooth point in C_0, and let $\alpha_k \equiv \sum_i n_i^{(k)} C_i^{(k)} \cdot C_0$, where $n_i^{(k)}$ is the multiplicity of $C_i^{(k)}$. If the said sharper estimate does not hold, then we must have $\alpha_2 = \frac{10}{12}$, $\alpha_3 = \frac{9}{12}$, $\alpha_4 = \frac{8}{12}$, $\alpha_5 = \frac{7}{12}$, and $\alpha_6 = \frac{6}{12} = \frac{1}{2}$. We will show that this is impossible. To see this, note that $\alpha_4 = \frac{8}{12} = \frac{2}{3}$. It follows easily that every component in $\{C_i^{(4)}\}$ must intersect C_0 only at the singular point p_2 of order 3. On the other hand, as q_5 converges to a smooth point in C_0, there must be a component Σ_ν in the limiting cusp-curve $f : \Sigma' \rightarrow X$ such that $\mathrm{Im}\, f|_{\Sigma_\nu} = C_0$. The key point here is that the degree of $f|_{\Sigma_\nu} : \Sigma_\nu \rightarrow C_0$ is at least 2. Suppose to the contrary that the degree is 1. Then Σ_ν must contain three orbifold points z_1, z_2, z_3 such that $f(z_i) = p_i$ for $i = 1, 2$ and 3. Moreover, the points z_1, z_3, which are sent to the singular points p_1, p_3 of order 2 and 4 under the map f, must be the images of some collapsing simple closed loops. This implies that for each of p_1 and p_3, there exists a nonconstant component in $\{C_i^{(5)}\}$ which intersects C_0 at it. But this is impossible because it would imply that $\alpha_5 \geq \frac{1}{2} + \frac{1}{4} = \frac{9}{12}$, which is a contradiction. Hence there exists a J-holomorphic curve C such that $p_0 \in C$ and $C \cdot C_0 < \frac{1}{2}$. It is easy to show that C is the image of a member of $\mathcal{M}$. Hence the case where $m = 1$.

(4) $G = \langle Z_{2m}, Z_{2m}; \widetilde{I}, \widetilde{I} \rangle$. In this case, $|G| > 4m^2$ implies that $m = 1, 7, 11, 13, 17, 19, 23$ or 29. We shall divide them into three groups: $m \geq 11$, $m = 7$, and $m = 1$.

First of all, since each element of G has order no greater than $10m$, we see that for any J-holomorphic curve C parametrized by f, a point in $f^{-1}(p_0)$ will contribute at least $\frac{1}{2} + \frac{1}{2m}$ to the right hand side of the adjunction formula.

Consider first the case where $m \geq 11$. By a similar argument, one can show that there exists a J-holomorphic curve C such that $p_0 \in C$ and $C \cdot C_0 \leq \frac{19}{30}$. The only part in the proof that is not so straightforward is to rule out the possibility, in the case where $m=11$ or 13, of having a J-holomorphic curve C' which obeys (1) $p_0\in C'$, (2) $C'\cdot C_0=1$, (3) C' is parametrized by $f:\Sigma\rightarrow X$ such that $f^{-1}(C_0)=\{z_1,z_2\}$ with both $f(z_1)$, $f(z_2)$ being the singular point p_3 of order 5. First, suppose $m=11$. In this case, the normalized Seifert invariant at p_3 is $(5,1)$. It follows easily that a local representative (f_i,ρ_i) of f at z_i, where $i=1,2$, must be of the form $\rho_i(\mu_5)=\mu_5^{l_i}$, $f_i(w)=(c_i(w^{l_i'}+\cdots),w^{l_i})$, where $l_i'\equiv l_i \pmod 5$ when $c_i\neq 0$, which is the case unless $l_i=1$. Moreover, $l_1+l_2=5$ because $C'\cdot C_0=1$. With these data, one can easily show that $k_{z_1}+k_{z_2}\geq \frac{2}{5}$ and $k_{[z_1,z_2]}\geq\frac{4}{5}$. Consequently, the right hand side of the adjunction formula is at least $\frac{1}{2}(1-\frac{1}{5})+\frac{1}{2}(1-\frac{1}{5})+\frac{2}{5}+\frac{4}{5}+\frac{1}{2}=2+\frac{1}{2}$. But the left hand side is $g(C')=\frac{1}{2}(\frac{30}{m}-\frac{m+1}{m})+1=\frac{29-m}{2m}+1=\frac{20}{11}$, which is a contradiction. As for the case where $m=13$, the normalized Seifert invariant at p_3 is $(5,3)$. By a similar argument, one can show that in this case, the right hand side of the adjunction formula is greater than 2, which is also a contradiction.

The next order of business is to show that a J-holomorphic curve C with $p_0\in C$ and $C\cdot C_0\leq\frac{19}{30}$ must be the image of a member of $\mathcal{M}$. First of all, observe that C,C_0 can not intersect at more than one point. This is because if otherwise, the two points of intersection must be p_2,p_3 of order 3 and 5 because $C\cdot C_0\leq\frac{19}{30}$. But in this case, the right hand side of the adjunction formula is greater than $\frac{1}{2}(1-\frac{1}{3})+\frac{1}{2}(1-\frac{1}{5})+\frac{1}{2}=1+\frac{7}{30}$, while the left hand side is $g(C)=\frac{1}{2}((\frac{8}{15})^2\cdot\frac{30}{m}-\frac{8}{15}\cdot\frac{m+1}{m})+1=1+\frac{32}{11}\cdot\frac{1}{30}$, which is a contradiction. Second, if C,C_0 intersect only at the singular point p_i of order a_i for some $i=1,2$ or 3 and $C\cdot C_0=\frac{1}{a_i}$, then C is the image of a member of $\mathcal{M}$ as we argued earlier. With these understood, it is easy to see that there are only two other possibilities that we need to rule out: $C\cdot C_0=\frac{2}{5}$ or $C\cdot C_0=\frac{3}{5}$, where in both cases, C,C_0 intersect only at p_3. Let $f:\Sigma\rightarrow X$ be a parametrization of C. First, it is fairly easy to rule out the possibility that $f^{-1}(p_3)$ and $f^{-1}(p_0)$ may contain more than one point. Moreover, observe also that $g_{|\Sigma|}=0$. Now let $z_0=f^{-1}(p_3)$ and $z_\infty=f^{-1}(p_0)$. Note that in both cases, the order of z_0 is 5. If $C\cdot C_0=\frac{2}{5}$, then as we have seen earlier, the link of p_3 in C is homotopic in $\mathbb{S}^3/G$ to 2 times of the singular fiber of the Seifert fibration at p_3, which has order $10m$ in G. This implies, since $g_{|\Sigma|}=0$, that the order of z_∞ is no greater than $5m$. As a consequence, the right hand side of the adjunction formula is no less than $\frac{1}{2}(1-\frac{1}{5})+\frac{1}{2}(1-\frac{1}{5m})+\frac{1}{10m}(\frac{120m}{5m}-1)=\frac{11}{5m}+\frac{9}{10}$. But on the left hand side, $g(C)=\frac{1}{2}((\frac{2}{5})^2\cdot\frac{30}{m}-\frac{2}{5}\cdot\frac{m+1}{m})+1=\frac{11}{5m}+\frac{4}{5}$, which is a contradiction. The case where $C\cdot C_0=\frac{3}{5}$ is more involved. First, let $h\equiv\mu_{2m}I\in Z_{2m}$ and let x,y be the generators of $\widetilde{I}$ with relations $x^2=y^5=(xy)^3=-1$. Then the homotopy class of the singular fiber of the Seifert fibration at p_3 is represented by γ^{-1}, where $\gamma=h^{-t}y^s$ for some positive integers s, t satisfying $sm - 5t = 1$. The action of γ on $\mathbb{C}^2$ is given, in suitable coordinates, by $\gamma \cdot (z_1, z_2) = (\mu_{10m} z_1, \mu_{10m}^k z_2)$ with $k \equiv -sm - 5t \pmod{10m}$. With these understood, observe that the link of p_3 in C is homotopic in $\mathbb{S}^3/G$ to 3 times of the singular fiber of the Seifert fibration at p_3, and consequently, since $g_{|\Sigma|} = 0$, there are holomorphic coordinates z_1, z_2 on a local uniformizing system at p_0, such that a local representative of f at z_∞ is given by (f_∞, ρ_∞), with $\rho_\infty(\mu_{10m})$ acting by $\rho_\infty(\mu_{10m}) \cdot (z_1, z_2) = (\mu_{10m}^3 z_1, \mu_{10m}^{3k} z_2)$, and $f_\infty(z) = (c_1(z^{l_1} + \cdots), c_2(z^{l_2} + \cdots))$, where both $c_1, c_2 \neq 0$ because both l_1, l_2 can not be 1. Because of this, we have relations $l_1 \equiv 3 \pmod{10m}$ and $l_2 \equiv 3k \equiv -3(sm + 5t) \pmod{10m}$. The latter particularly implies that $l_2 \geq 3$. With these understood, we have $k_{z_\infty} \geq \frac{1}{20m}((3-1)(3-1) + (\frac{120m}{10m} - 1) \cdot 3^2) = \frac{103}{20m}$. With this estimate, it is easy to see that the right hand side of the adjunction formula is at least $\frac{9}{10} + \frac{51}{10m}$. However, the left hand side is $g(C) = \frac{1}{2}((\frac{3}{5})^2 \cdot \frac{30}{m} - \frac{3}{5} \cdot \frac{m+1}{m}) + 1 = \frac{51}{10m} + \frac{7}{10}$, which is a contradiction. This finishes the proof that a J-holomorphic curve C with $p_0 \in C$ and $C \cdot C_0 \leq \frac{19}{30}$ must be the image of a member of $\mathcal{M}$, and the case where $m \geq 11$ is done.

For the next case where $m = 7$, we begin with the following observation. Let C_i, $i = 0, 1, \ldots, k$, be J-holomorphic curves with multiplicities n_i such that $\sum_{i=0}^k n_i C_i \cdot C_0 = 1$. Here C_i with $i = 0$ stands for the distinguished J-holomorphic curve C_0, and we allow $n_0 = 0$, which simply means that C_0 is not included. Note that on the one hand, $\sum_{i \neq 0} n_i C_i \cdot C_0 = 1 - n_0 C_0 \cdot C_0 = 1 - \frac{7n_0}{30}$, and on the other hand, $\sum_{i \neq 0} n_i C_i \cdot C_0 = \frac{c_1}{2} + \frac{c_2}{3} + \frac{c_3}{5}$ for some non-negative integers c_1, c_2, c_3, where at least one of them must be 0 because $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} > 1$. It follows easily that either $n_0 = 0$, or $n_0 = 2$ with $\sum_{i \neq 0} n_i C_i \cdot C_0 = \frac{1}{3} + \frac{1}{5}$. With this understood, note that $d(E) = 4$ when $m = 7$, so that we can specify any 2 distinct smooth points q_1, q_2, with $q_1 \in C_0$ and $q_2 \in X \setminus C_0$, such that q_1, q_2 are contained in the J-holomorphic curves $\{C_i\}$ obtained by applying Theorem 2.2 (2) to E, where we note that $p_0 \in \cup_i C_i$ also. It follows easily from the preceding observation that as q_2 converges to a smooth point $q_2' \in C_0$ where $q_2' \neq q_1$, $\{C_i\}$ will have only one component, which intersects C_0 transversely at q_1. Let C be the J-holomorphic curve and let $f : \Sigma \to X$ be a parametrization of C. First, the virtual genus of C is $g(C) = \frac{1}{2}(\frac{30}{7} - \frac{7+1}{7}) + 1 = 2 + \frac{4}{7}$, from which it follows that $g_{|\Sigma|} \leq 2$. However, we shall need a sharper estimate that $g_{|\Sigma|} \leq 1$, which is obtained as follows. Observe that by the adjunction formula, if $g_{|\Sigma|} = 2$, then $f^{-1}(p_0)$ must contain only one point, denoted by z_∞, which must have order $10m = 70$. Moreover, if we let (f_∞, ρ_∞) be a local representative of f at z_∞, then f_∞ must also be embedded. Now suppose the action of $\rho_\infty(\mu_{70})$ on a local uniformizing system at p_0 is given by $\rho_\infty(\mu_{70}) \cdot (z_1, z_2) = (\mu_{70}^{m_1} z_1, \mu_{70}^{m_2} z_2)$ in some holomorphic coordinates z_1, z_2. Write $f_\infty(z) = (c_1(z^{l_1} + \cdots), c_2(z^{l_2} + \cdots))$ where $c_1 \neq 0$ (resp. $c_2 \neq 0$) unless $l_2 = 1$ (resp. $l_1 = 1$). Then for any $i = 1, 2$, we have $l_i \equiv m_i \pmod{70}$ as long as $c_i \neq 0$. It follows easily, since f_∞ is embedded, that one of m_1, m_2 must equal 1. On the other hand, the index formula for the linearization $D\underline{L}$ at f gives rise to

$$\begin{aligned} & c_1(TX) \cdot [f(\Sigma)] + 2 - 2g_{|\Sigma|} - \frac{m_1 + m_2}{70} \\ &= \frac{7+1}{7} + 2 - 2g_{|\Sigma|} - \frac{m_1 + m_2}{70} \in \mathbb{Z}. \end{aligned}$$

If we write $\rho_\infty(\mu_{70}) = h^k y^l$, then $\frac{m_1+m_2}{70} = \frac{k}{7}$ because $\widetilde{I} \subset SU(2)$. Here $h = \mu_{14} I \in Z_{2m} = Z_{14}$ and $y \in \widetilde{I}$ with eigenvalues μ_{10}, μ_{10}^{-1}, and without loss of generality, we assume that $0 \leq k \leq 6$. As a consequence, we obtain $k = 1$ and $\rho_\infty(\mu_{70}) = hy^l$. It follows easily that $m_1 \equiv 5 + 7l \pmod{70}$ and $m_2 \equiv 5 - 7l \pmod{70}$, and from this one can easily check that $m_1 \neq 1$, $m_2 \neq 1$ for any l. This is a contradiction, hence $g_{|\Sigma|} \leq 1$. With this in hand, we let $q_2 \equiv q_{2,j}$ be a sequence of points converging to a smooth point $q_2' \neq q_1$ on C_0, and denote by C_j the corresponding J-holomorphic curves, and by $f_j : \Sigma \to X$ a parametrization of C_j, which is assumed to be from a fixed orbifold Riemann surface without loss of generality. As we argued earlier, f_j will converge to a cusp-curve $f : \Sigma' \to X$ such that a component Σ_ν of Σ' is mapped to C_0 under f, over which f is a map of degree at least 2. By the observation made at the beginning of this paragraph, we see that the degree of $f|_{\Sigma_\nu} : \Sigma_\nu \to X$ is exactly 2, and moreover, the remaining component(s) in the limiting cusp-curve must intersect C_0 at exactly two singular points, p_2 of order 3 and p_3 of order 5, each contributing $\frac{1}{3}$ and $\frac{1}{5}$ to the intersection product. Now we observe that since $f|_{\Sigma_\nu} : \Sigma_\nu \to X$ is of degree 2, there must be at least two orbifold points in Σ_ν, one of order 3 and the other of order 5, which are all obtained by collapsing simple closed loops in Σ. Furthermore, if Σ_ν contains exactly 2 orbifold points, then we must also have $g_{|\Sigma_\nu|} \neq 0$. Since $g_{|\Sigma|} \leq 1$, it is not hard to see that $g_{|\Sigma|}$ must equal 1 and each $f_j^{-1}(p_0)$ consists of two points, and that there are exactly two non-C_0 components, denoted by C_1, C_2, in the limiting cusp-curve Im f, such that $p_0 \in C_1 \cap C_2$ and $C_1 \cdot C_0 = \frac{1}{3}$, $C_2 \cdot C_0 = \frac{1}{5}$ (or the other way). It follows easily that both C_1, C_2 are the image of a member of $\mathcal{M}$. Hence the case where $m = 7$.

Finally, we consider the case where $m = 1$. Since $d(E) = 32$, we can specify any 15 distinct smooth points $q_1, q_2, \ldots, q_{15}$, where $q_1 \in C_0$ and $q_2, \ldots, q_{15} \in X \setminus C_0$, such that the J-holomorphic curves $\{C_i\}$ obtained from Theorem 2.2 (2) contain these points as well as the singular point p_0. We then let q_k, $k = 2, \ldots, 15$, converge one by one to a smooth point in C_0 which is different than q_1. If we denote by α_k the intersection product with C_0 of the non-C_0 components (counted with multiplicity) in the limiting cusp-curve at each stage, then we have, as in the previous cases, that $\alpha_2 \leq \frac{28}{30}$ and $\alpha_k - \alpha_{k+1} \geq \frac{1}{30}$ for $k = 2, \ldots, 14$. The key observation for the present case is that each $\alpha_k = \frac{c_1^{(k)}}{2} + \frac{c_2^{(k)}}{3} + \frac{c_3^{(k)}}{5}$ for some integers $c_i^{(k)} \geq 0$, $i = 1, 2, 3$, where at least one of $c_i^{(k)}$ is zero. With this understood, note, for instance, that $\frac{23}{30}$ can not be written in the above form, and therefore it can not be realized as α_k for any k. In fact, a simple inspection like this shows that the following is the only possibility for the values of α_k:

$$\alpha_2 = \frac{28}{30}, \ldots, \alpha_6 = \frac{24}{30}, \alpha_7 = \frac{22}{30}, \ldots, \alpha_9 = \frac{20}{30}, \alpha_{10} = \frac{18}{30},$$
$$\alpha_{11} = \frac{16}{30}, \alpha_{12} = \frac{15}{30}, \alpha_{13} = \frac{12}{30}, \alpha_{14} = \frac{10}{30} \text{ and } \alpha_{15} = \frac{6}{30} = \frac{1}{5}.$$

(In fact, using the adjunction formula, one can explicitly recover the process of degeneration of the J-holomorphic curves, i.e., understanding how at each stage a component carrying the correct amount of homology splits off during the convergence. But these details are not needed here for the proof, so we leave them to the reader as an exercise.) In particular, we obtain at the last stage a J-holomorphic curve C such that $p_0 \in C$ and $C \cdot C_0 = \frac{1}{5}$. It follows easily that C is the image of a member of $\mathcal{M}$. Hence the case where $m = 1$.

The proof of Lemma 3.9 is thus completed. q.e.d.

4. Proof of Taubes' theorems for 4-orbifolds

For the assertions in Theorem 2.2 (1), observe that Taubes' proof (cf. e.g., [**21**]) works in the orbifold setting without changing a word.

The rest of this section is occupied by a proof of Theorem 2.2 (2). We shall follow the proof of Taubes in [**42**], indicating along the way where modifications to Taubes' proof are necessary in the orbifold setting, and how to implement them.

Basic estimates. Section 2 in Taubes [**42**] is concerned with the following estimates:

- $|\alpha| \leq 1 + zr^{-1}$
- $|\beta|^2 \leq zr^{-1}((1 - |\alpha|^2) + r^{-2})$
- $|P_\pm F_a| \leq (4\sqrt{2})^{-1} r(1 + zr^{-1/2})(1 - |\alpha|^2) + z$
- $|\nabla_a \alpha|^2 + r|\nabla'_A \beta|^2 \leq zr(1 - |\alpha|^2) + z$.

Here $z > 0$ is a constant solely determined by $c_1(E)$ and the Riemannian metric, and r is sufficiently large. The principal tool for obtaining these estimates is to apply the maximum principle to the various differential inequalities derived from the Seiberg-Witten equations. Another important ingredient is the total energy bound in Lemma 2.6 of [**42**]:

$$\left|\frac{r}{4}\int_X |1-|\alpha|^2| - 2\pi[\omega]\cdot c_1(E)\right| \leq zr^{-1}.$$

These are all valid in the orbifold setting. However, in the estimate for $|P_-F_a|$ (specifically (2.35) in the proof of Lemma 2.7 in [**42**]), Green's function for the Laplacian d^*d is also involved. Here additional care is needed in the orbifold case because even on a compact, closed Riemannian orbifold, the injectivity radius at each point is not uniformly bounded from below by a positive constant due to the presence of orbifold points.

Green's function for the Laplacian on orbifolds is discussed in Appendix B. Given that, let's recall that the part in the proof of Lemma 2.7 in Taubes [**42**] which involves Green's function is to derive the following estimate (cf. (2.35) in [**42**]) for the function q_1':

$$q_1' \leq \frac{z\cdot R\cdot \sup(|P_-F_a|)}{r^{1/2}}$$

where q_1' satisfies $\frac{1}{2}d^*dq_1' + \frac{r}{4}|\alpha|^2 q_1' = R\cdot\sup(|P_-F_a|)\cdot|1-|\alpha|^2|$. In the present case, we apply Theorem 1 in Appendix B to q_1',

$$q_1'(x) = \mathrm{Vol}\,(X)^{-1}\int_X q_1' + \int_X G(x,\cdot)\Delta q_1'.$$

Now observe that in the first term, $\int_X q_1'$ is bounded by

$$\int_X |1-|\alpha|^2| q_1' + \int_X |\alpha|^2 q_1' \leq z\left(\frac{\sup(q_1')}{r} + \frac{R\cdot\sup(|P_-F_a|)}{r^2}\right)$$

because $|\frac{r}{4}\int_X |1-|\alpha|^2| - 2\pi[\omega]\cdot c_1(E)| \leq zr^{-1}$ and $\frac{r}{4}\int_X |\alpha|^2 q_1' = \int_X R\cdot \sup(|P_-F_a|)\cdot|1-|\alpha|^2|$. As for the second term, suppose $q_1'(x_0) = \sup(q_1')$ for some $x_0\in X$, and recall Theorem 1 (3) in Appendix B that one may write $G(x_0,y) = G_0(x_0,y) + G_1(x_0,y)$. Thus $\int_X G(x_0,\cdot)\Delta q_1'$ is bounded by

$$\int_X G_0(x_0,\cdot)(2R\cdot\sup(|P_-F_a|)\cdot|1-|\alpha|^2|) \\ + \int_X G_1(x_0,\cdot)(2R\cdot\sup(|P_-F_a|)\cdot|1-|\alpha|^2|).$$

The last term above is bounded by $\frac{z\cdot R\cdot\sup(|P_-F_a|)}{r}$, and for the first term, recall that there is a uniformizing system $(\widehat{U},G,\pi)$ such that $G_0(x_0,y)$ is supported in $\pi(\widehat{U})$ and $G_0\circ\pi$ equals $\sum_{h\in G}\widehat{G_0}(h\cdot\hat{x_0},\hat{y})$ for some $\hat{x_0}\in\pi^{-1}(x_0)$ with $\widehat{G_0}(\hat{x_0},\hat{y})$ satisfying $|\widehat{G_0}(\hat{x_0},\hat{y})| \leq \frac{z}{\hat{d}(\hat{x_0},\hat{y})^2}$. Moreover, $\widehat{U}$ contains a closed ball of radius $\delta_0>0$ centered at $\hat{x_0}$. Now when

$r \geq \delta_0^{-4}$, the first term $\int_X G_0(x_0, \cdot)(2R \cdot \sup(|P_- F_a|) \cdot |1 - |\alpha|^2|)$, which equals

$$\int_{\widehat{U}} \widehat{G_0}(\hat{x_0}, \cdot)((2R \cdot \sup(|P_- F_a|) \cdot |1 - |\alpha|^2|) \circ \pi),$$

is bounded by

$$z \cdot R \cdot \sup(|P_- F_a|) \cdot r^{-1/2} + z \cdot R \cdot \sup(|P_- F_a|) \cdot r^{1/2} \int_X |1 - |\alpha|^2|$$

by writing the integration over $\widehat{U}$ as the sum of integration over the closed ball $B_{\hat{x_0}}(r^{-1/4}) \subset \widehat{U}$ of radius $r^{-1/4} \leq \delta_0$ centered at $\hat{x_0}$, and its complement in $\widehat{U}$ as in [**42**]. It is easily seen that the estimate for q_1' follows immediately for r sufficiently large.

Monotonicity formula and refined estimate for $|P_- F_a|$. Recall that the monotonicity formula in Section 3 of Taubes [**42**] is for the purpose of estimating the growth rate of the local energy $\frac{r}{4}\int_B |1 - |\alpha|^2|$, where B is a geodesic ball of radius s centered at a given point, against the radius of the ball s. In this part of the argument, the radius s is required to satisfy an inequality $\frac{1}{2r^{1/2}} \leq s \leq \frac{1}{z}$. Thus again, because the injectivity radius has no positive uniform lower bound on the orbifold X, a reformulation for the definition of local energy is needed.

More precisely, we shall fix the set $\mathcal{U}$ of finitely many uniformizing systems and the constant $\delta_0 > 0$ as described in Theorem 1 (3) in Appendix B. Given that, for any $p \in X$, we choose a uniformizing system $(\widehat{U}, G, \pi) \in \mathcal{U}$ for p as described therein, and define the local energy at p to be

$$\mathcal{E}(p, s) = \frac{r}{4} \int_B |1 - |\alpha|^2|,$$

where B is the geodesic ball of radius $s \leq \delta_0$ in $\widehat{U}$ centered at some $\hat{p} \in \pi^{-1}(p)$, and by abusing the notation, the function $|1 - |\alpha|^2| \circ \pi$ on $\widehat{U}$ is still denoted by $|1 - |\alpha|^2|$. It is easily seen that $\mathcal{E}(p, s)$ is well-defined, i.e., $\mathcal{E}(p, s)$ is independent of the choice of $(\widehat{U}, G, \pi) \in \mathcal{U}$ and $\hat{p} \in \pi^{-1}(p)$.

With the preceding understood, the relevant argument in Taubes [**42**] can be quoted to establish the corresponding estimates in the present case:

- $\mathcal{E}(p, s) \leq zs^2$ for all $p \in X$, and
- $\mathcal{E}(p, s) \geq \frac{1}{z+1} s^2$ when $|\alpha(p)| < 1/2$,

where $z > 0$ is a constant depending only on $c_1(E)$ and the Riemannian metric, r is sufficiently large, and $\frac{1}{2r^{1/2}} \leq s \leq \frac{1}{z}$. (cf. Prop. 3.1 in [**42**].)

Now we discuss the refined estimate for $|P_- F_a|$ (cf. Prop. 3.4 in [**42**]). Here the argument involves Green's function as well as a ball covering procedure using geodesic balls. Hence Taubes' original proof in [**42**] needs to be modified in the present case.

Recall that the key to the refinement is Lemma 3.5 in Taubes [**42**] where a smooth function u on X is constructed which obeys

(1) $|u| \leq z$,
(2) $\frac{1}{2}d^*du \geq r$ where $|\alpha| < 1/2$,
(3) $|d^*du| \leq z \cdot r$.

Here $z > 0$ is a constant depending only on $c_1(E)$ and the Riemannian metric. The strategy for the present case is to follow the proof of Lemma 3.5 in Taubes [**42**] to construct, for each uniformizing system $(\widehat{U}, G, \pi) \in \mathcal{U}$, a function $u_{\widehat{U}}$ on X, and define $u \equiv \sum u_{\widehat{U}}$.

To be more concrete, let $(\widehat{U}, G, \pi)$ be any element in $\mathcal{U}$. Recall that (cf. Theorem 1 in Appendix B) $\widehat{U}$ is a geodesic ball of radius $\delta(p_i)$, the injectivity radius at p_i for some $p_i \in X$, and $G = G_{p_i}$, the isotropy group at p_i. Moreover, the open subset $\widehat{U}' \subset \widehat{U}$ is the concentric ball of radius $\delta_i = N^{-1}\delta(p_i)$, and if we denote by $\widehat{U}_0$ the concentric ball of radius $N^{-1}\delta_i = N^{-1} \cdot \text{radius}(\widehat{U}')$, then the set $\{\pi(\widehat{U}_0)\}$ is an open cover of X. Here N is a fixed integer no less than $12 = 3 \cdot \dim X$.

With the preceding understood, let V be the region in $\widehat{U}_0$ where $|\alpha| < 1/2$. Then Lemma 3.6 in Taubes [**42**] is valid here. To be more precise, there is a set $\{B_i\}$ of geodesic balls in $\widehat{U}$ of radius $r^{-1/2} \leq \delta_0$ having the following properties: (1) each B_i is centered at a point $\hat{p}_i \in V$, the region in $\widehat{U}_0$ where $|\alpha| < 1/2$, (2) $\{B_i\}$ covers V, (3) the number of balls, $\#\{B_i\}$, is bounded by $z \cdot r$ as r grows, (4) the concentric balls of only half radius (i.e., $\frac{1}{2}r^{-1/2}$) are disjoint, and furthermore in the present case, (5) the set of centers $\{\hat{p}_i\}$ of the balls is invariant under the action of G.

Now observe that Lemma 3.7 in Taubes [**42**] is valid for the set of balls $\{B_i\}$. Thus there exists a set of concentric balls $\{\widetilde{B_i}\}$ of radius $z \cdot r^{-1/2}$ for some constant $z > 1$ such that $\text{Volume}((\widehat{U} \setminus V') \cap \widetilde{B_i}) \geq \text{Volume}(B_i)$, where V' is the region in $\widehat{U}$ where $|\alpha| < 3/4$. Here we choose r sufficiently large so that each $\widetilde{B_i}$ is contained in the ball of radius $\delta = \delta_0 + \text{radius}(\widehat{U}_0)$ which has the same center of $\widehat{U}_0$.

As in Taubes [**42**], we let $s_i, \tilde{s}_i$ be the characteristic functions of B_i and $(\widehat{U} \setminus V') \cap \widetilde{B_i}$. Then as in [**42**], there is a κ_i, with bound $z^{-1} < \kappa_i < z$, such that

$$\int_{\widehat{U}} (s_i - \kappa_i \tilde{s}_i) = 0.$$

Note that the function $\sum_i (s_i - \kappa_i \tilde{s}_i)$ on $\widehat{U}$ is invariant under the action of G and is compactly supported in the ball of radius $\delta = \delta_0 + \text{radius}(\widehat{U}_0)$ which has the same center of $\widehat{U}_0$. Thus $\sum_i (s_i - \kappa_i \tilde{s}_i)$ descends to a function $f_{\widehat{U}}$ on X by defining $f_{\widehat{U}} \equiv 0$ outside $\pi(\widehat{U})$. With the preceding understood, the function $u_{\widehat{U}}$ is the unique solution to

$$\frac{1}{2}d^*du_{\widehat{U}} = r \cdot f_{\widehat{U}} \text{ and } \int_X u_{\widehat{U}} = 0.$$

(By suitably smoothing $f_{\widehat{U}}$, as indicated in [**42**], one may arrange to have $u_{\widehat{U}}$ smooth.)

The following properties of $u_{\widehat{U}}$ are straightforward as in [**42**]:

- $|d^*du_{\widehat{U}}| \le z \cdot r$.
- $\frac{1}{2}d^*du_{\widehat{U}} \ge 0$ where $|\alpha| < 1/2$, and $\frac{1}{2}d^*du_{\widehat{U}} \ge r$ in $\pi(V)$.

Thus to furnish Lemma 3.5 in Taubes [**42**] with $u \equiv \sum u_{\widehat{U}}$, it suffices to show that

$$|u_{\widehat{U}}| \le z$$

for a constant $z > 0$ which is independent of r.

To this end, we invoke Theorem 1 in Appendix B to obtain

$$u_{\widehat{U}}(p) = 2r \cdot \int_X G(p,\cdot)f_{\widehat{U}} = 2r \cdot \int_X G_0(p,\cdot)f_{\widehat{U}} + 2r \cdot \int_X G_1(p,\cdot)f_{\widehat{U}}.$$

Note that $G_1(p,q)$ is C^1 on $X \times X$, so that

$$2r \cdot \int_X G_1(p,\cdot)f_{\widehat{U}} \le z_1 \cdot \frac{2r}{|G|}\sum_i \text{Volume}(\widetilde{B_i}) \le z_1 \cdot \frac{2r}{|G|} \cdot \#\{B_i\} \cdot \frac{z_2}{r^2} \le z_3,$$

which is an r-independent constant. As for $2r \cdot \int_X G_0(p,\cdot)f_{\widehat{U}}$, note that $\int_X G_0(p,\cdot)f_{\widehat{U}} = 0$ if $p \in X \setminus \pi(\widehat{U}')$ because $\{q \mid G_0(p,q) \neq 0\} \subseteq \{q \mid d(p,q) \le (4+1)\delta_0\}$ (cf. Theorem 1 in Appendix B). Hence by fixing a $\hat{p} \in \pi^{-1}(p)$ for any given $p \in \pi(\widehat{U}')$, we have

$$2r \cdot \int_X G_0(p,\cdot)f_{\widehat{U}} = \sum_i 2r \cdot \int_{\widehat{U}} \widehat{G_0}(\hat{p},\cdot) \cdot (s_i - \kappa_i\tilde{s}_i).$$

If we set $u_i(\hat{p}) = 2r \cdot \int_{\widehat{U}} \widehat{G_0}(\hat{p},\cdot)\cdot(s_i - \kappa_i\tilde{s}_i)$, then as in Taubes [**42**], u_i satisfies

$$|u_i(\hat{p})| \le z \text{ when } \hat{d}(\hat{p},\hat{p}_i) \le \frac{z}{r^{1/2}}, \quad \text{and}$$
$$|u_i(\hat{p})| \le \frac{z}{r^{3/2}\hat{d}(\hat{p},\hat{p}_i)^3} \text{ when } \hat{d}(\hat{p},\hat{p}_i) > \frac{z}{r^{1/2}}.$$

Recall that, here, $\hat{p}_i \in V$ is the center of the ball B_i.

To complete the proof, we observe that Lemma 3.8 in Taubes [**42**] is valid here, that is, for any $\hat{p} \in \widehat{U}'$, the number $N(n)$ of balls in $\{B_i\}$ whose center $\hat{p}_i$ satisfies $\hat{d}(\hat{p},\hat{p}_i) \le n \cdot r^{-1/2}$ obeys $N(n) \le z \cdot n^2$. (Here n is any positive integer.) If we let $\Omega(n)$ be the set of indices i for the balls B_i whose center $\hat{p}_i$ obeys $(n-1)\cdot r^{-1/2} < \hat{d}(\hat{p},\hat{p}_i) \le n \cdot r^{-1/2}$, then as in [**42**],

$$\left|2r \cdot \int_X G_0(p,\cdot) f_{\widehat{U}}\right| \le \sum_i |u_i(\hat{p})| = \sum_{n\ge 1}\sum_{i\in\Omega(n)} |u_i(\hat{p})| \le z_1 + \sum_{n\ge 1} z_2 \cdot \frac{N(n)-N(n-1)}{n^3} \le z_3$$

for a constant $z_3 > 0$ which is independent of r.

The local structure of $\alpha^{-1}(0)$ *and exponential decay estimates.* The discussion in Section 4 of Taubes [**42**] extends to the present case almost word by word, except for the exponential decay estimates

$$|q(x)| \le z \cdot r \cdot \exp\left(-\frac{1}{z} r^{1/2} d(x,\alpha^{-1}(0))\right)$$

for $q \in \{r(1-|\alpha|^2), r^{3/2}\beta, F_a, r^{1/2}\nabla_a\alpha, r\nabla'_A\beta\}$, where d is the distance function.

The part that needs modification is the construction of a comparison function h (cf. (4.19) in [**42**]) which obeys

- $\frac{1}{2}d^*dh + \frac{r}{32}h \ge 0$ where $d(\cdot,\alpha^{-1}(0)) \ge zr^{-1/2}$.
- $h \ge r$ where $d(\cdot,\alpha^{-1}(0)) = zr^{-1/2}$.
- $h \le z_1 \cdot r \cdot \exp(-\frac{1}{z_1} r^{1/2} d(\cdot,\alpha^{-1}(0)))$ where $d(\cdot,\alpha^{-1}(0)) \ge zr^{-1/2}$.

Here $z, z_1 > 1$ are r-independent constants.

Modification is needed here at least for one reason: the construction of $h(x)$ involves a ball covering argument by geodesic balls of radius of size $r^{-1/2}$, along with the local energy growth rate estimates, i.e., Prop. 3.1 in [**42**]. On the other hand, observe that the construction of comparison function in [**42**] does not require the compactness of the underlying manifold. The compactness enters only when the maximum principle is applied. Hence Taubes' argument in [**42**] should in principle work here also.

More concretely, we shall construct for each uniformizing system $(\widehat{U}, G, \pi) \in \mathcal{U}$ (cf. Theorem 1 in Appendix B) a smooth function $h_{\widehat{U}} > 0$ on X which obeys

- $\frac{1}{2}d^*dh_{\widehat{U}} + \frac{r}{32}h_{\widehat{U}} \ge 0$ where $d(\cdot,\alpha^{-1}(0)\cap\pi(\widehat{U}')) \ge zr^{-1/2}$.
- $h_{\widehat{U}} \ge r$ where $d(\cdot,\alpha^{-1}(0)\cap\pi(\widehat{U}')) = zr^{-1/2}$.
- $h_{\widehat{U}} \le z_1 \cdot r \cdot \exp(-\frac{1}{z_1} r^{1/2} d(\cdot,\alpha^{-1}(0)\cap\pi(\widehat{U}')))$ where $d(\cdot,\alpha^{-1}(0)\cap\pi(\widehat{U}')) \ge zr^{-1/2}$.

Here $z, z_1 > 1$ are r-independent constants. Accepting this, we may take $h \equiv \sum h_{\widehat{U}}$ for the comparison function.

To define $h_{\widehat{U}}$, use Lemma 3.6 in [**42**] to find a maximal set $\{\hat{p}_i\} \subset \alpha^{-1}(0)\cap\widehat{U}'$ such that (1) the geodesic balls with centers $\{\hat{p}_i\}$ and radius

$r^{-1/2}$ are disjoint, (2) the set $\{\hat{p}_i\}$ is invariant under the action of G. Then set $\hat{h} \equiv \sum_i H_i$ where

$$H_i(\hat{p}) = \frac{\rho(\hat{d}(\hat{p}, \hat{p}_i))}{\hat{d}(\hat{p}, \hat{p}_i)^2} \exp\left(-\frac{1}{c} r^{1/2} \cdot \hat{d}(\hat{p}, \hat{p}_i)\right) + c \cdot \exp\left(-\frac{\delta_0}{2c} r^{1/2}\right).$$

Here $\hat{d}$ is the distance function on $\widehat{U}$, $\rho(t)$ is a fixed cut-off function which equals 1 for $t \leq \frac{\delta_0}{2}$ and equals zero for $t \geq \delta_0$. Moreover, r is sufficiently large, and $c > 1$ is a fixed, sufficiently large, r-independent constant (cf. (4.17) in [**42**]). Note that $\hat{h} \equiv \sum_i H_i$ is smooth, positive, and invariant under the action of G, and is constant outside a compact subset in $\widehat{U}$. Hence $\hat{h}$ descends to a smooth, positive function on X, which is defined to be $h_{\widehat{U}}$. The claimed properties of $h_{\widehat{U}}$ follow essentially as in Taubes [**42**]. (Cf. Lemma 4.6 and (4.18) in [**42**].)

Convergence to a current (Section 5 of Taubes [**42**]). First of all, note that generalization of the basic theory of currents on smooth manifolds (cf. e.g., [**15**]) to the orbifold setting is straightforward. In particular, note that a differential form or a differentiable chain in an orbifold (as introduced in [**11**]) naturally defines a current.

Having said this, for any given sequence of solutions (a_n, α_n, β_n) to the Seiberg-Witten equations with the values of the parameter r unbounded, we define as in Taubes [**42**] a sequence of currents $\mathcal{F}_n$ by

$$\mathcal{F}_n(\eta) = \frac{\sqrt{-1}}{2\pi} \int_X F_{a_n} \wedge \eta, \ \forall \eta \in \Omega^2(X).$$

As in [**42**], the mass norm of $\{\mathcal{F}_n\}$ is uniformly bounded, thus there is a subsequence, still denoted by $\{\mathcal{F}_n\}$ for simplicity, which weakly converges to a current $\mathcal{F}$, namely,

$$\mathcal{F}(\eta) = \lim_{n \to \infty} \mathcal{F}_n(\eta), \ \forall \eta \in \Omega^2(X).$$

The current $\mathcal{F}$ is closed, and is Poincaré dual to $c_1(E)$ in the sense that

$$\mathcal{F}(\eta) = c_1(E) \cdot [\eta]$$

for all closed 2-forms η.

As for the support of $\mathcal{F}$, which, by definition, is the intersection of all the closed subsets of X such that the evaluation of $\mathcal{F}$ on any 2-form supported in the complement of the closed subset is zero, we proceed as follows. We fix the set $\mathcal{U}$ of finitely many uniformizing systems in Theorem 1 of Appendix B, and for each $(\widehat{U}, G, \pi) \in \mathcal{U}$, we run Taubes' argument on $\widehat{U}$. More concretely, for each integer $N \geq 1$ and each index n with $r_n > z^2 \cdot (256)^N$, we find a maximal set $\Lambda'_n(N)_{\widehat{U}}$ of disjoint geodesic balls in $\widehat{U}$ with centers in $\alpha_n^{-1}(0) \cap \text{closure}(\widehat{U}')$ and radius 16^{-N} such that the centers of the balls in $\Lambda'_n(N)_{\widehat{U}}$ are invariant under the action of G, and for any two uniformizing systems $(\widehat{U}_i, G_i, \pi_i) \in \mathcal{U}$,

$i = 1, 2$, the centers of the balls in $\Lambda_n'(N)_{\widehat{U}_i}$ which are in the domain or range of a transition map between the two uniformizing systems are invariant under the transition map. Then proceeding as in Taubes [**42**], we find a nested set $\{U(N)_{\widehat{U}}\}_{N\geq 1}$ for each $(\widehat{U}, G, \pi) \in \mathcal{U}$, which satisfies

$$\hat{d}\left(U(N+1)_{\widehat{U}}, \widehat{U} \setminus U(N)_{\widehat{U}}\right) \geq \frac{3}{2}16^{-N}.$$

We define $C_{\widehat{U}} \equiv \bigcap_N U(N)_{\widehat{U}}$, and define C to be the set of orbits of $\bigsqcup C_{\widehat{U}} \subset \bigsqcup \widehat{U}$ in X. It is clear as in [**42**] that the support of $\mathcal{F}$ is contained in C, and $\mathcal{F}$ is of type $1-1$.

As for the Hausdorff measure of C, first of all, we say that a subset of X has a finite m-dimensional Hausdorff measure if for every uniformizing system of X, the inverse image of the subset in that uniformizing system has a finite m-dimensional Hausdorff measure. Equivalently, a subset of X has a finite m-dimensional Hausdorff measure if the inverse image of the subset in $\widehat{U}'$ for each $(\widehat{U}, G, \pi) \in \mathcal{U}$ has a finite m-dimensional Hausdorff measure. Having said this, the subset C has a finite 2-dimensional Hausdorff measure because each $C_{\widehat{U}}$ does, as argued in Taubes [**42**].

Finally, the local intersection number. We simply apply the relevant definition and discussion in Taubes [**42**] to $C_{\widehat{U}}$ in $\widehat{U}$ for each $(\widehat{U}, G, \pi) \in \mathcal{U}$.

Representing $\mathcal{F}$ by J-holomorphic curves. Section 6 of Taubes [**42**] deals with the regularity of the subset C in the manifold case, where the main conclusions are: (1) each regular point in C has a neighborhood which is an embedded, J-holomorphic disc (cf. Lemma 6.11 in [**42**]), (2) the singular points in C are isolated (cf. Lemmas 6.17, 6.18 in [**42**]), and their complement in C is diffeomorphic to a disjoint union of $[1, \infty) \times S^1$ when restricted in a small neighborhood of each singular point (courtesy of Lemma 6.3 in [**42**]). The arguments for these results are local in nature, hence applicable to $C_{\widehat{U}} \subset \widehat{U}$ for each $(\widehat{U}, G, \pi) \in \mathcal{U}$.

With the preceding understood, particularly that the subset $C_{\widehat{U}} \cap \widehat{U}'$ has the said regularity properties for each $(\widehat{U}, G, \pi) \in \mathcal{U}$, we now analyze the subset C of X, which is the set of orbits of $\bigsqcup C_{\widehat{U}} \subset \bigsqcup \widehat{U}$ in X. To this end, note that the isotropy subgroup $G_{\hat{p}}$ of a point $\hat{p} \in \widehat{U}$ falls into two types: type A if $G_{\hat{p}}$ fixes a complex line through $\hat{p}$ (which is in fact a J-holomorphic submanifold), or type B if $G_{\hat{p}}$ only fixes $\hat{p}$ itself. By the unique continuation property of J-holomorphic curves (cf. e.g., [**35**]), it follows easily that there is a subset $C_{\widehat{U},s} \subset C_{\widehat{U}} \cap \widehat{U}'$ of isolated points, such that the complement of $C_{\widehat{U},s}$, denoted by $C_{\widehat{U},r}$, consists of regular points and is modeled on a disjoint union of $[1, \infty) \times S^1$ in a small neighborhood of each point in $C_{\widehat{U},s}$, and furthermore, $C_{\widehat{U},r}$ is the disjoint union of $C^{(1)}_{\widehat{U},r}$ and $C^{(2)}_{\widehat{U},r}$, where $C^{(1)}_{\widehat{U},r}$ consists of points of trivial isotropy subgroups and $C^{(2)}_{\widehat{U},r}$ consists of points of type A isotropy subgroups. In particular, $C_{\widehat{U},r}$ is a J-holomorphic submanifold in $\widehat{U}$. It is easy to see that the quotient space $C_{\widehat{U},r}/G \subset X$ has the structure of an open Riemann surface with a set of isolated points removed, and since $\{\pi(\widehat{U}')\}$ is an open cover of X, there is a subset $C_0 \subset C$ which has the structure of a closed Riemann surface with a set of isolated, hence finitely many (since C is compact) points removed. The restriction of the inclusion $C \hookrightarrow X$ to each component $C_{0,i}$ of C_0 extends to a continuous map $f_i : \Sigma_i \to X$, none of which is multiply covered, where Σ_i is the closed Riemann surface obtained by closing up $C_{0,i}$. We define $C_i \equiv f_i(\Sigma_i)$. Clearly $C = \cup_i C_i$. Note that C_0 is the disjoint union of $C_0^{(1)}$ and $C_0^{(2)}$, where the former is covered by $\{C^{(1)}_{\widehat{U},r}\}$ and the latter by $\{C^{(2)}_{\widehat{U},r}\}$. The set $\{C_i\}$ is correspondingly a disjoint union of two subsets, $\{C_i^{(1)}\}$ and $\{C_i^{(2)}\}$. We will show next that $\{C_i^{(1)}\}$ are type I J-holomorphic curves and $\{C_i^{(2)}\}$ are type II J-holomorphic curves (in the sense of [**11**]). Moreover, there are positive integers n_i such that

$$c_1(E) = \sum_i n_i \cdot PD(C_i).$$

(Note that for a type II J-holomorphic curve, the Poincaré dual $PD(C)$ differs from the usual one by a factor, see [**11**] for details.)

To this end, consider the étale topological groupoid Γ that defines the orbifold structure on X whose space of units is $\bigsqcup \widehat{U}'$. There is a canonical orbispace structure on each $C_{0,i}$, making it into a suborbispace $f_i' : C_{0,i} \hookrightarrow X$, which is obtained by restricting Γ to $\bigsqcup C_{\widehat{U},i}$, where $C_{\widehat{U},i}$ is the inverse image of $C_{0,i}$ under $\pi : \widehat{U}' \to \widehat{U}'/G$ (cf. [**10**]). Because $C_{\widehat{U},r}$ (the inverse image of C_0 in $\widehat{U}'$) is modeled on a disjoint union of $[1,\infty) \times S^1$ in a small neighborhood of each point in $C_{\widehat{U},s}$ (the inverse image of $C \setminus C_0$ in $\widehat{U}'$), it is easily seen that the orbispace structure on $C_{0,i}$ extends uniquely to define an orbifold structure on Σ_i, making it into an orbifold Riemann surface. Moreover, the map $f_i' : C_{0,i} \hookrightarrow X$ extends uniquely to a map $\hat{f}_i : \Sigma_i \to X$ between orbifolds in the sense of [**10**], which is J-holomorphic and defines C_i as a J-holomorphic curve in X in the sense of [**11**]. Clearly $\{C_i^{(1)}\}$ are of type I and $\{C_i^{(2)}\}$ are of type II according to the definitions in [**11**].

The positive integers n_i are assigned to C_i as follows. At each point $\hat{p} \in C_{\widehat{U},i}$, there is an embedded J-holomorphic disc D in $\widehat{U}$ which intersects $C_{\widehat{U},i}$ transversely at $\hat{p}$. It is shown in [**42**] that $\lim_{n\to\infty} \frac{\sqrt{-1}}{2\pi} \int_D F_{a_n}$, denoted by $n(\hat{p})$, exists and is a positive integer. Moreover, $n(\hat{p})$ is locally constant, hence it depends on C_i only. We define $n_i \equiv n(\hat{p})$, $\forall \hat{p} \in C_{\widehat{U},i}$. (cf. Prop 5.6 and the discussion before Lemma 6.9 in [**42**].)

With n_i so defined, now for any 2-form η on X, we write $\eta = \sum \eta_{\widehat{U}}$ by a partition of unity subordinate to the cover $\{\pi(\widehat{U})\}$, and observe that

$$\begin{aligned}\mathcal{F}(\eta) &= \lim_{n\to\infty} \mathcal{F}_n(\eta) = \lim_{n\to\infty}\left(\sum \frac{1}{|G|}\int_{\widehat{U}'} \frac{\sqrt{-1}}{2\pi} F_{a_n} \wedge \eta_{\widehat{U}}\right)\\ &= \sum_i n_i \cdot \left(\sum \frac{1}{|G|}\int_{C_{\widehat{U},i}} \eta_{\widehat{U}}\right) = \sum_i n_i \cdot \int_{\Sigma_i} \hat{f}_i^* \eta.\end{aligned}$$

Thus $c_1(E) = \sum_i n_i \cdot PD(C_i)$.

Finally, as in Taubes [**42**], if a subset $\Omega \subset X$ is contained in $\alpha_n^{-1}(0)$ for all n, then Ω is contained in $C = \cup_i C_i$ also.

Appendix A. Dimension of the Seiberg-Witten Moduli Space

We begin with a brief review on the index theorem in Kawasaki [**22**].

Let X be an orbifold (compact and connected), and P be an elliptic operator over X. In order to state the index theorem, we first introduce the space $\widetilde{X} \equiv \{(p,(g)_{G_p}) \mid p \in X, g \in G_p\}$, where G_p is the isotropy group at p, and $(g)_{G_p}$ is the conjugacy class of g in G_p. The following properties of $\widetilde{X}$ are easily verified (cf. [**22**], compare also [**14**]).

- $\widetilde{X}$ has a canonical orbifold structure, with a canonical map $i : \widetilde{X} \rightarrow X$: let (V_p, G_p) be a local uniformizing system at $p \in X$; then $(V_p^g, Z_p(g))$, where $V_p^g \subset V_p$ is the fixed-point set of $g \in G_p$ and $Z_p(g) \subset G_p$ is the centralizer of g, is a local uniformizing system at $(p,(g)_{G_p}) \in \widetilde{X}$, and the map $i : \widetilde{X} \rightarrow X$ is defined by the collection of embeddings $\{(V_p^g, Z_p(g)) \hookrightarrow (V_p, G_p) \mid p \in X, g \in G_p\}$.
- $\widetilde{X}$ is a disjoint union of compact, connected orbifolds of various dimensions, containing the orbifold X as the component $\{(p,(1)_{G_p}) \mid p \in X\}$: $\widetilde{X} = \bigsqcup_{(g)\in T} X_{(g)}$ with $X_{(1)} = X$, where $T = \{(g)\}$ is the set of equivalence classes of $(g)_{G_p}$ with the equivalence relation $\sim$ defined as follows: $(h)_{G_q} \sim (g)_{G_p}$ if q is contained in a local uniformizing system centered at p and $h \mapsto g$ under the natural injective homomorphism $G_q \rightarrow G_p$ which is defined only up to conjugation by an element of G_p.

We remark that the orbifold structure on $\widetilde{X}$ is in a more general sense that the group action in a local uniformizing system is not required to be effective. For such an orbifold, we shall use the following convention: the fundamental class of the orbifold, whenever it exists, equals the fundamental class of the underlying space divided by the order of the isotropy group at a smooth point (cf. [**14**] and §2 in [**11**]).

Next we describe the characteristic classes involved in the index theorem. Let $u = [\sigma(P)]$ be the class of the principal symbol of the elliptic operator P in the K-theory of TX. Then the pullback of u via the differential of the map $i : X_{(g)} \to X$, denoted by $u_{(g)}$, is naturally decomposed as $\oplus_{0\leq\theta<2\pi} u_{(g),\theta}$ where $u_{(g),\theta}$ is the restriction of $u_{(g)}$ to the $\exp(\sqrt{-1}\theta)$-eigenbundle of $g \in (g)$. We set

$$\mathrm{ch}_{(g)} u_{(g)} \equiv \sum_{\theta} \exp(\sqrt{-1}\theta)\mathrm{ch}\; u_{(g),\theta} \in H_c^*(TX_{(g)};\mathbb{C}).$$

On the other hand, the normal bundle N_p^g of $V_p^g \hookrightarrow V_p$ patches up to define an orbifold vector bundle $N_{(g)} \to X_{(g)}$, and the decomposition $N_p^g = N_p^g(-1) \oplus_{0<\theta<\pi} N_p^g(\theta)$, where $N_p^g(-1)$, $N_p^g(\theta)$ are the (-1)-eigenspace and $\exp(\sqrt{-1}\theta)$-eigenspace of g respectively, defines a natural decomposition of orbifold vector bundles $N_{(g)} = N_{(g)}(-1) \oplus_{0<\theta<\pi} N_{(g)}(\theta)$.

Now let R, S_θ be the characteristic classes over $\mathbb{C}$ of the orthogonal group and unitary group, which are defined by the power series

$$\left\{\prod_i \left(\frac{1+\exp x_i}{2}\right)\left(\frac{1+\exp(-x_i)}{2}\right)\right\}^{-1},$$

$$\left\{\prod_i \left(\frac{1-\exp(y_i+\sqrt{-1}\theta)}{1-\exp(\sqrt{-1}\theta)}\right)\left(\frac{1-\exp(-y_i-\sqrt{-1}\theta)}{1-\exp(-\sqrt{-1}\theta)}\right)\right\}^{-1}$$

respectively. We set

$$I_{(g)} \equiv R(N_{(g)}(-1)) \prod_{0<\theta<\pi} \frac{S_\theta(N_{(g)}(\theta))\tau(TX_{(g)}\otimes_{\mathbb{R}}\mathbb{C})}{\det(1-(g)|_{N_{(g)}})} \in H^*(X_{(g)};\mathbb{C}),$$

where $\tau = \prod_i x_i(1-\exp(-x_i))^{-1}$ is the Todd class, and $\det(1-(g)|_{N_{(g)}})$ is the constant function on $X_{(g)}$ which equals $\det(1-g|_{N_p^g})$ at $(p,(g)_{G_p}) \in X_{(g)}$. Note that when X is almost complex, $N_{(g)}$ is an orbifold complex vector bundle, and there is a compatible decomposition $N_{(g)} = \oplus_{0<\theta<2\pi} N_{(g)}(\theta)$. In this case, it is easily seen that

$$I_{(g)} = \prod_{0<\theta<2\pi} \frac{S_\theta(N_{(g)}(\theta))\tau(TX_{(g)}\otimes_{\mathbb{R}}\mathbb{C})}{\det(1-(g)|_{N_{(g)}})} \in H^*(X_{(g)};\mathbb{C}).$$

Theorem (Kawasaki [**22**]).

$$\text{index } P = \sum_{(g)\in T} (-1)^{\dim X_{(g)}} \langle \mathrm{ch}_{(g)} u_{(g)} \cdot I_{(g)}, [TX_{(g)}]\rangle$$

where $u = [\sigma(P)]$. (*Here the orientation for the fundamental class* $[TX_{(g)}]$ *is given according to the* (*now standard*) *convention in Atiyah-Singer* [**3**].)

For the rest of the appendix, we shall consider specifically the case where X is an almost complex 4-orbifold (which is in the classical sense that the local group actions are effective), and where P is either the Dirac operator associated to a Spin$^{\mathbb{C}}$ structure on X, or the de Rham operator, or the signature operator.

First, the index of the Dirac operator associated to a Spin$^{\mathbb{C}}$ structure on X. Recall that the almost complex structure of X defines a canonical Spin$^{\mathbb{C}}$ structure $S_+ \oplus S_-$, $S_+ = \mathbb{I} \oplus K_X^{-1}$, $S_- = TX$, where $\mathbb{I}$ is the trivial orbifold complex line bundle, K_X is the canonical bundle, and TX is the tangent bundle which, with the given almost complex structure, is viewed as an orbifold $\mathbb{C}^2$-bundle. Any other Spin$^{\mathbb{C}}$ structure has the form $(S_+ \oplus S_-) \otimes E$ for an orbifold complex line bundle E over X.

Let $P = P^E_{Dirac} : C^\infty(S_+ \otimes E) \to C^\infty(S_- \otimes E)$ be the Dirac operator. We shall determine the contribution to the index of P from each component $X_{(g)}$ of $\widetilde{X}$. Let $l = \dim_{\mathbb{C}} X_{(g)}$. Then $l = 0, 1, 2$, where $X_{(g)} = p/G_p$ for a singular point $p \in X$ when $l = 0$, $X_{(g)}$ is 2-dimensional, pseudo-holomorphic when $l = 1$, and $X_{(g)} = X_{(1)} = X$ when $l = 2$. In any event, the orbifold principal $U(2)$-bundle associated to the almost complex structure reduces to an orbifold principal $U(l) \times U(2-l)$-bundle when restricted to $X_{(g)}$ via the map $i : X_{(g)} \to X$, and there is an orbifold principal H-bundle F over $X_{(g)}$, $H \equiv U(l) \times U(2-l) \times U(1) \subset U(3)$, such that $TX_{(g)} = F \times_H \mathbb{C}^l$ and $E|_{X_{(g)}} = F \times_H \mathbb{C}$, where $\mathbb{C}^l, \mathbb{C}$ are H-modules via $\mathbb{C}^l = \mathbb{C}^l \times \{0\} \subset \mathbb{C}^3$ and $\mathbb{C} = \{0\} \times \mathbb{C} \subset \mathbb{C}^3$. Moreover, let $M_+ = (\mathbb{I} \oplus \Lambda^2\mathbb{C}^2) \otimes \mathbb{C}$, $M_- = \mathbb{C}^2 \otimes \mathbb{C}$ be the H-modules where $\mathbb{I}$ is the 1-dimensional trivial module, $\mathbb{C}^2 = \mathbb{C}^2 \times \{0\} \subset \mathbb{C}^3$ and $\mathbb{C} = \{0\} \times \mathbb{C} \subset \mathbb{C}^3$. Then $S_+ \otimes E|_{X_{(g)}} = F \times_H M_+$ and $S_- \otimes E|_{X_{(g)}} = F \times_H M_-$, so that $u_{(g)}$, the pullback of the symbol class of P via the differential of the map $i : X_{(g)} \to X$, is an elliptic symbol class associated to the H-structure (cf. Atiyah-Singer [**3**]).

There is a linear action of $g \in (g)$ on M_+ and M_- associated to the bundles $TX|_{X_{(g)}}, E|_{X_{(g)}}$. Let $M_+ = \oplus_{0 \le \theta < 2\pi} M_{+,\theta}$, $M_- = \oplus_{0 \le \theta < 2\pi} M_{-,\theta}$ be the corresponding decompositions into $\exp(\sqrt{-1}\theta)$-eigenspaces. Let $\psi : H^*(X_{(g)}; \mathbb{C}) \to H^*_c(TX_{(g)}; \mathbb{C})$ be the Thom isomorphism. Then the contribution to the index of P from $X_{(g)}$ is

$$
\begin{aligned}
&(-1)^{2l} \langle \mathrm{ch}_{(g)} u_{(g)} \cdot I_{(g)}, [TX_{(g)}] \rangle \\
&= (-1)^l \langle \psi^{-1}(\mathrm{ch}_{(g)} u_{(g)}) \cdot I_{(g)}, [X_{(g)}] \rangle \\
&= (-1)^l \frac{\sum_\theta (\exp(\sqrt{-1}\theta) \mathrm{ch}\ M_{+,\theta} - \exp(\sqrt{-1}\theta) \mathrm{ch}\ M_{-,\theta})}{x_1 \cdots x_l} (F) I_{(g)} [X_{(g)}]
\end{aligned}
$$

where $x_1 \cdots x_l = 1$ when $l = 0$.

In the above formula, the symbol class of the Dirac operator contributes through the modules M_+, M_-. Similarly, in order to determine the index formulae for the other geometric differential operators on X, it suffices to write down the corresponding modules.

Let's look at the de Rham operator $d + d^*$, whose index is the Euler characteristic $\chi(X)$. The modules are $N_+ = (\mathbb{I} \oplus \Lambda^2\mathbb{R}^4 \oplus \Lambda^4\mathbb{R}^4) \otimes_{\mathbb{R}} \mathbb{C}$ and $N_- = (\mathbb{R}^4 \oplus \Lambda^3\mathbb{R}^4) \otimes_{\mathbb{R}} \mathbb{C}$. Because of the almost complex structure, $\mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^2 \oplus \overline{\mathbb{C}}^2$, and if we set $W = \Lambda^2(\mathbb{C}_1 \oplus \overline{\mathbb{C}_2})$ where $\mathbb{C}^2 = \mathbb{C}_1 \oplus \mathbb{C}_2$, then we may rewrite $N_+ = 4\mathbb{I} \oplus \Lambda^2\mathbb{C}^2 \oplus \Lambda^2\overline{\mathbb{C}}^2 \oplus W \oplus \overline{W}$, $N_- = 2(\mathbb{C}^2 \oplus \overline{\mathbb{C}}^2)$.

For the signature operator whose index is $\operatorname{sign}(X)$, the signature of X, the modules are $Q_+ = \Lambda^2_+\mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{C}$ and $Q_- = \Lambda^2_-\mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{C}$. With the almost complex structure, we may rewrite $Q_+ = \mathbb{I} \oplus \Lambda^2\mathbb{C}^2 \oplus \Lambda^2\overline{\mathbb{C}}^2$ and $Q_- = \mathbb{I} \oplus W \oplus \overline{W}$.

With these preparations, we give a formula in the following proposition for the dimension $d(E)$ of the moduli space of Seiberg-Witten equations associated to the Spin$^{\mathbb{C}}$-structure given by an orbifold complex line bundle E.

Proposition. *$d(E) = 2 \cdot \operatorname{index} P^E_{\text{Dirac}} - \frac{1}{2}(\chi(X) + \operatorname{sign}(X)) = I_0 + I_1 + I_2$ where*

$$I_0 = \sum_{\{(g)|\, \dim_{\mathbb{C}} X_{(g)}=0\}} \frac{2(\exp(\sqrt{-1}\theta_{E,(g)}) - 1)}{(1 - \exp(-\sqrt{-1}\theta_{1,(g)}))(1 - \exp(-\sqrt{-1}\theta_{2,(g)}))}[X_{(g)}]$$

$$\begin{aligned} I_1 = \sum_{\{(g)|\, \dim_{\mathbb{C}} X_{(g)}=1\}} \Bigg(&\frac{2\exp(\sqrt{-1}\theta_{E,(g)})c_1(E)}{1 - \exp(-\sqrt{-1}\theta_{(g)})} \\ &+ \frac{(\exp(\sqrt{-1}\theta_{E,(g)}) - 1)c_1(TX_{(g)})}{1 - \exp(-\sqrt{-1}\theta_{(g)})} \\ &- \frac{2\exp(-\sqrt{-1}\theta_{(g)})(\exp(\sqrt{-1}\theta_{E,(g)}) - 1)c_1(N_{(g)})}{(1 - \exp(-\sqrt{-1}\theta_{(g)}))^2}\Bigg)[X_{(g)}] \end{aligned}$$

and

$$I_2 = (c_1^2(E) - c_1(E)c_1(K_X))[X].$$

In the above equations, $\exp(\sqrt{-1}\theta_{E,(g)})$ denotes the eigenvalue of $g \in (g)$ acting on the orbifold complex line bundle $E|_{X_{(g)}}$, $\exp(\sqrt{-1}\theta_{1,(g)})$ and $\exp(\sqrt{-1}\theta_{2,(g)})$ denote the eigenvalues of $g \in (g)$ acting on the normal bundle $N_{(g)}$ of $X_{(g)}$ when $\dim_{\mathbb{C}} X_{(g)} = 0$, and $\exp(\sqrt{-1}\theta_{(g)})$ denotes the eigenvalue of $g \in (g)$ acting on $N_{(g)}$ when $\dim_{\mathbb{C}} X_{(g)} = 1$. Moreover, $X_{(g)}$ and X are given with the canonical orientation as almost complex orbifolds.

Proof. We first consider the case when the orbifold complex line bundle $E = \mathbb{I}$ is trivial, and show that $2\cdot\text{index } P^{\mathbb{I}}_{\text{Dirac}} - \frac{1}{2}(\chi(X)+\text{sign}(X)) = 0$.

To this end, set $M^0_+ = \mathbb{I} \oplus \Lambda^2\mathbb{C}^2$ and $M^0_- = \mathbb{C}^2$, and let $M^0_+ = \oplus_{0\le\theta<2\pi} M^0_{+,\theta}$ and $M^0_- = \oplus_{0\le\theta<2\pi} M^0_{-,\theta}$ be the decompositions into $\exp(\sqrt{-1}\theta)$-eigenspaces. Then the contribution to $\frac{1}{2}(\chi(X) + \text{sign}(X))$ that comes from $X_{(g)}$ is

$$(-1)^l \frac{\sum_\theta(\exp(\sqrt{-1}\theta)\text{ch}\, M^0_{+,\theta} - \exp(\sqrt{-1}\theta)\text{ch}\, M^0_{-,\theta})}{x_1\cdots x_l}(F_0)I_{(g)}[X_{(g)}]+$$
$$(-1)^l \frac{\sum_\theta(\exp(-\sqrt{-1}\theta)\text{ch}\, \overline{M^0_{+,\theta}} - \exp(-\sqrt{-1}\theta)\text{ch}\, \overline{M^0_{-,\theta}})}{x_1\cdots x_l}(F_0)I_{(g)}[X_{(g)}],$$

where $l = \dim_{\mathbb{C}} X_{(g)}$, and F_0 is the orbifold principal $U(l)\times U(2-l)$-bundle over $X_{(g)}$ which is the reduction when restricted to $X_{(g)}$ of the orbifold principal $U(2)$-bundle over X associated to the almost complex structure.

Observe that only the terms of degree $2l$ in

$$\sum_\theta(\exp(\sqrt{-1}\theta)\text{ch}\ M^0_{+,\theta} - \exp(\sqrt{-1}\theta)\text{ch}\ M^0_{-,\theta})$$

could possibly make a contribution, which is invariant under $x_i \mapsto -x_i$. Moreover, $\theta \mapsto -\theta$ under $(g) \mapsto (g^{-1})$, and $I_{(g)} = I_{(g^{-1})}$ under the identification $X_{(g)} = X_{(g^{-1})}$. Hence the following two expressions

$$(-1)^l \frac{\sum_\theta(\exp(-\sqrt{-1}\theta)\text{ch}\, \overline{M^0_{+,\theta}} - \exp(-\sqrt{-1}\theta)\text{ch}\, \overline{M^0_{-,\theta}})}{x_1\cdots x_l}(F_0)I_{(g)}[X_{(g)}],$$
$$(-1)^l \frac{\sum_\theta(\exp(\sqrt{-1}\theta)\text{ch}\, M^0_{+,\theta} - \exp(\sqrt{-1}\theta)\text{ch}\, M^0_{-,\theta})}{x_1\cdots x_l}(F_0)I_{(g^{-1})}[X_{(g^{-1})}]$$

are equal, from which it follows easily that

$$2\cdot\text{index } P^{\mathbb{I}}_{\text{Dirac}} - \frac{1}{2}(\chi(X)+\text{sign}(X)) = 0.$$

For the general case, notice that

$$2\cdot\text{index } P^{E}_{\text{Dirac}} - \frac{1}{2}(\chi(X)+\text{sign}(X)) = 2(\text{index } P^{E}_{\text{Dirac}} - \text{index } P^{\mathbb{I}}_{\text{Dirac}}).$$

The formula follows easily from direct evaluation of the right hand side.

q.e.d.

Proof of Lemma 3.8. The dimension $d(E)$ of the Seiberg-Witten moduli space corresponding to E equals

$$2 \cdot \text{index } P^E_{\text{Dirac}} - \frac{1}{2}(\chi(X) + \text{sign } (X)).$$

According to the proposition, it is the sum of $c_1(E) \cdot c_1(E) - c_1(K_X) \cdot c_1(E)$ with a term contributed by the singular point p_0, which can be written as $\frac{1}{|G|}\sum_{g\in G, g\neq 1}\chi(g)$, with

$$\chi(g) = \frac{2(\rho(g)-1)}{(1-\exp(-\sqrt{-1}\theta_{1,g}))(1-\exp(-\sqrt{-1}\theta_{2,g}))},$$

where $\rho : G \to \mathbb{S}^1$ is the representation given in Lemma 3.6, and $\exp(\sqrt{-1}\theta_{i,g})$, $i = 1, 2$, are the two eigenvalues of $g \in G \subset U(2)$. The evaluation of this term constitutes the main task in the proof, which will be done case by case according to the type of G.

(1) $G = \langle Z_{2m}, Z_{2m}; \widetilde{D}_n, \widetilde{D}_n\rangle$. Let $h = \mu_{2m} I \in Z_{2m}$, and let x, y be generators of $\widetilde{D}_n$ satisfying $x^2 = y^n = (xy)^2 = -1$. Then $G \setminus \{1\} = \Lambda_1 \sqcup \Lambda_2 \sqcup \Lambda_3$, where $\Lambda_1 = \sqcup_{l=0}^{n-1}\Lambda_1^{(l)}$ with $\Lambda_1^{(l)} = \{h^k y^l \mid k = 1, 2, \ldots, 2m-1\}$, $\Lambda_2 = \{h^k x y^l \mid k = 0, 1, \ldots, 2m-1, l = 0, 1, \ldots, n-1\}$, and $\Lambda_3 = \{y^l \mid l = 1, 2, \ldots, n-1\}$. Note that $\chi(g) = 0$ for any $g \in \Lambda_3$.

We first calculate $\sum_{g\in\Lambda_1}\chi(g)$. To this end, we set, for each $s = 1, 2, \ldots, 2n$, $S_{l,s}(t) \equiv \sum_{k=1}^{2m-1}\frac{(\mu_{2m}^k\mu_{2n}^{-l})^s}{1-\mu_{2m}^{-k}\mu_{2n}^{-l}t}$. Introduce $[s]$ such that $s \equiv [s] \pmod{2m}, 0 \leq [s] \leq 2m-1$. Then

$$\begin{aligned}
S_{l,s}(t) &= \sum_{k=1}^{2m-1}(\mu_{2m}^k\mu_{2n}^{-l})^s\sum_{j=0}^{\infty}(\mu_{2m}^{-k}\mu_{2n}^{-l})^j t^j \\
&= \sum_{j=0}^{\infty}\mu_{2n}^{-ls-lj}\sum_{k=1}^{2m-1}(\mu_{2m}^{s-j})^k t^j \\
&= -\sum_{j=0}^{\infty}\mu_{2n}^{-ls-lj}t^j + 2m\mu_{2n}^{-ls-l[s]}t^{[s]}\sum_{j=0}^{\infty}(\mu_{2n}^{-l}t)^{2mj} \\
&= -\frac{\mu_{2n}^{-ls}}{1-\mu_{2n}^{-l}t} + 2m\frac{\mu_{2n}^{-ls-l[s]}t^{[s]}}{1-(\mu_{2n}^{-l}t)^{2m}}.
\end{aligned}$$

We consider separately when $l = 0$ or $l \neq 0$.

$$\begin{aligned}\sum_{s=1}^{2n} S_{0,s}(1) = \sum_{s=1}^{2n} S_{0,s}(t)|_{t=1} &= \sum_{s=1}^{2n}\left(-\frac{1}{1-t}+\frac{2mt^{[s]}}{1-t^{2m}}\right)\Big|_{t=1}\\ &= \sum_{s=1}^{2n}\frac{-\sum_{i=0}^{2m-1} t^i + 2mt^{[s]}}{1-t^{2m}}\Big|_{t=1} = \sum_{s=1}^{2n}\frac{-\sum_{i=0}^{2m-1} i + 2m[s]}{-2m}\\ &= \sum_{s=1}^{2n}\left(\frac{1}{2}(2m-1)-[s]\right).\end{aligned}$$

For each $l \neq 0$, note that $\sum_{s=1}^{2n}(\mu_{2n}^{-l})^s = 0$, so that $\sum_{s=1}^{2n} S_{l,s}(1) = \sum_{s=1}^{2n}\frac{2m\mu_{2n}^{-ls-l[s]}}{1-(\mu_{2n}^{-l})^{2m}}$. Introduce j_s which is uniquely defined by $0 \le j_s \le n-1$ and $s+[s]+2mj_s \equiv 0 \pmod{2n}$. Then

$$\begin{aligned}S(t) \equiv \sum_{s=1}^{2n}\sum_{l=1}^{n-1}\frac{\mu_{2n}^{-ls-l[s]}}{1-\mu_{2n}^{-2ml}t} &= \sum_{s=1}^{2n}\sum_{l=1}^{n-1}\sum_{j=0}^{\infty}(\mu_{2n}^{-2ml})^j t^j\\ &= \sum_{s=1}^{2n}\sum_{j=0}^{\infty}\sum_{l=1}^{n-1}\left(\mu_{2n}^{-(s+[s]+2mj)}\right)^l t^j = \sum_{s=1}^{2n}\left(\sum_{j=0}^{\infty}-t^j + nt^{j_s}\sum_{j=0}^{\infty}t^{nj}\right)\\ &= \sum_{s=1}^{2n}\left(-\frac{1}{1-t}+\frac{nt^{j_s}}{1-t^n}\right).\end{aligned}$$

Similarly, $S(1) = \sum_{s=1}^{2n}(\frac{1}{2}(n-1)-j_s)$, and

$$\sum_{l=1}^{n-1}\sum_{s=1}^{2n} S_{l,s}(1) = 2mS(1) = \sum_{s=1}^{2n}(m(n-1)-2mj_s).$$

With these preparations, now observe that if we set $S_l \equiv \frac{1}{2}\sum_{g\in\Lambda_1^{(l)}}\chi(g)$, then

$$\begin{aligned}S_l &= \sum_{k=1}^{2m-1}\frac{(\mu_{2m}^k\mu_{2n}^{-l})^{2n}-1}{(1-\mu_{2m}^{-k}\mu_{2n}^{-l})(1-\mu_{2m}^{-k}\mu_{2n}^{l})}\\ &= \sum_{s=1}^{2n}\sum_{k=1}^{2m-1}\frac{(\mu_{2m}^k\mu_{2n}^{-l})^s}{1-\mu_{2m}^{-k}\mu_{2n}^{-l}} = \sum_{s=1}^{2n} S_{l,s}(1).\end{aligned}$$

Hence

$$\sum_{g\in\Lambda_1}\chi(g) = 2\sum_{l=0}^{n-1} S_l = \sum_{s=1}^{2n}((2mn-1)-2([s]+2mj_s)).$$

Next we calculate $\sum_{g\in\Lambda_2}\chi(g)$. First of all,

$$\sum_{g\in\Lambda_2}\chi(g)=\sum_{l=0}^{n-1}\sum_{k=0}^{2m-1}\frac{2((\mu_{2m}^k)^{2n}(-1)^n-1)}{(1-\mu_{2m}^{-k}\sqrt{-1})(1-\mu_{2m}^{-k}(\sqrt{-1})^{-1})}$$
$$=\sum_{s=1}^{2n}\sum_{l=0}^{n-1}\sum_{k=0}^{2m-1}\frac{2(\mu_{2m}^k\sqrt{-1})^s}{1-\mu_{2m}^{-k}\sqrt{-1}}.$$

Set $S_s(t)\equiv\sum_{k=0}^{2m-1}\frac{(\mu_{2m}^k\sqrt{-1})^s}{1-(\mu_{2m}^{-k}\sqrt{-1})t}$, $s=1,2,\ldots,2n$. Then

$$S_s(t)=\sum_{k=0}^{2m-1}(\mu_{2m}^k\sqrt{-1})^s\sum_{j=0}^{\infty}(\mu_{2m}^{-k}\sqrt{-1})^jt^j$$
$$=\sum_{j=0}^{\infty}\sum_{k=0}^{2m-1}(\mu_{2m}^{s-j})^k(\sqrt{-1})^{s+j}t^j$$
$$=2m\sum_{j=0}^{\infty}(\sqrt{-1})^{s+[s]+2mj}\cdot t^{[s]+2mj}$$
$$=\frac{2m(\sqrt{-1})^{s+[s]}\cdot t^{[s]}}{1-(\sqrt{-1}\cdot t)^{2m}},$$

and

$$\sum_{g\in\Lambda_2}\chi(g)=n\sum_{s=1}^{2n}2S_s(1)=2mn\sum_{s=1}^{2n}(\sqrt{-1})^{s+[s]}.$$

In order to evaluate $\sum_{g\in\Lambda_1}\chi(g)$ and $\sum_{g\in\Lambda_2}\chi(g)$, we consider the cases where $m>n$ and $m<n$ separately.

Suppose $m>n$. In this case, we have $[s]=s$ for any $s=1,2,\ldots,2n$. Furthermore, $s=[s]$ and m,n being relatively prime imply that $s\mapsto j_s$ is a surjective, two to one correspondence from $\{1,2,\ldots,2n\}$ to $\{0,1,\ldots,n-1\}$. It then follows easily from these observations that

$$\sum_{g\in\Lambda_1}\chi(g)=4n(m-n-1),\ \sum_{g\in\Lambda_2}\chi(g)=0.$$

Hence

$$d(E)=c_1(E)\cdot c_1(E)-c_1(K_X)\cdot c_1(E)+\frac{1}{|G|}\sum_{g\neq1}\chi(g)$$
$$=\frac{n}{m}+\frac{m+1}{m}+\frac{1}{4mn}\cdot 4n(m-n-1)=2.$$

Now consider the case where $m<n$. We introduce δ,r satisfying $n=\delta m+r$, $0\leq r\leq m-1$. Then a simple inspection shows that

$$\sum_{g\in\Lambda_2}\chi(g)=2mn((-1)^{\delta}-1).$$

In order to evaluate $\sum_{g\in\Lambda_1}\chi(g)$, we introduce, for each $s=1,2,\ldots,2n$, k_s which obeys $s+[s]+2mj_s=2nk_s$. Then one can easily check that k_s satisfies $1\le k_s\le m$. Now observe that for any $l=0,1,\ldots,2\delta-1$, $s\mapsto k_s$ is injective, hence surjective, if $lm+1\le s\le lm+m$. Summing up the equations $s+[s]+2mj_s=2nk_s$ from $s=1$ to $s=2\delta m$, we have

$$\sum_{s=1}^{2\delta m}([s]+2mj_s)=\sum_{s=1}^{2\delta m}2nk_s-\sum_{s=1}^{2\delta m}s=\delta m(2nm+2r-1).$$

If $m\neq 1$, we need to consider the rest of the values of s, $s=2\delta m+1,\ldots,2\delta m+2r$. For this part, observe that $j_r=0$, $j_{2r}=\delta$, and for any $1\le s\le r-1$, we have the relation $2m(j_s+j_{2r-s})=2n(k_s+k_{2r-s}-2)$, which implies that $j_s+j_{2r-s}=n$. Thus

$$\sum_{s=2\delta m+1}^{2n}([s]+2mj_s)=r(2r+1)+2m(rn-n+\delta).$$

Putting things all together, we have

$$\sum_{g\in\Lambda_1}\chi(g)=4mn-4n(r+1).$$

Finally, when $m<n$, we have

$$\begin{aligned}d(E)&=\frac{n}{m}+\frac{m+1}{m}+\frac{1}{4mn}(4mn-4n(r+1)+2mn((-1)^{\delta}-1))\\&=\delta+2+\frac{1}{2}((-1)^{\delta}-1).\end{aligned}$$

(2) $G=\langle Z_{4m},Z_{2m};\widetilde{D}_n,C_{2n}\rangle$. Let $h=\mu_{4m}I\in Z_{4m}$, and let x,y be generators of $\widetilde{D}_n$ satisfying $x^2=y^n=(xy)^2=-1$. Introduce $\bar h=h^2$, $\bar x=hx$, and $\bar y=y$. Then $G\setminus\{1\}=\Lambda_1\sqcup\Lambda_2\sqcup\Lambda_3$, where $\Lambda_1=\{\bar h^k\bar y^l\mid k=1,2,\ldots,2m-1,l=0,1,\ldots,n-1\}$, $\Lambda_2=\{\bar h^k\bar x\bar y^l\mid k=0,1,\ldots,2m-1,l=0,1,\ldots,n-1\}$, and $\Lambda_3=\{\bar y^l\mid l=1,2,\ldots,n-1\}$. Again, we have $\chi(g)=0$ for any $g\in\Lambda_3$.

Note that $\sum_{g\in\Lambda_1}\chi(g)$ is the same as in the previous case, so we only need to evaluate $\sum_{g\in\Lambda_2}\chi(g)$, for which a similar calculation shows that

$$\sum_{g\in\Lambda_2}\chi(g)=2mn\sum_{s=1}^{2n}\mu_{4m}^{s-[s]}(\sqrt{-1})^{s+[s]}.$$

A simple inspection, with the fact that m is even this time, shows that $\sum_{g\in\Lambda_2}\chi(g)=0$ when $m>n$, and when $m<n$,

$$\sum_{g\in\Lambda_2}\chi(g)=2mn((-1)^{\delta}-1),\ \text{where } n=\delta m+r, 0\le r\le m-1.$$

By the same calculation, $d(E)=2$ if $m>n$, and $d(E)=\delta+2+\frac{1}{2}((-1)^{\delta}-1)$ if $m<n$.

(3) $G = \langle Z_{2m}, Z_{2m}; \widetilde{T}, \widetilde{T}\rangle$. Let $h = \mu_{2m}I \in Z_{2m}$, and let x, y be generators of $\widetilde{T}$ satisfying $x^2 = y^3 = (xy)^3 = -1$. Then $\sum_{g\neq 1}\chi(g) = S_0 + S_1 + S_2$, where $S_0 = \sum_{k=1}^{2m-1}\chi(h^k)$,

$$S_1 = \sum_{g=y,xy,x^{-1}yx,yx}\sum_{l=1}^{2}\sum_{k=0}^{2m-1}\chi(h^k g^l),$$

and

$$S_2 = \sum_{g=x,y^{-1}xy,y^{-2}xy^2}\sum_{k=0}^{2m-1}\chi(h^k g).$$

Let $[s]$ be defined by $s \equiv [s]$ and $0 \leq [s] \leq 2m-1$. Then a similar calculation shows that

$$S_0 = \sum_{s=1}^{12}(2m-1-2[s])$$

$$S_1 = 16m\sum_{l=1}^{2}\sum_{s=1}^{12}\frac{\mu_6^{(s+[s])l}}{1-\mu_3^{ml}}$$

$$S_2 = 6m\sum_{s=1}^{12}\mu_4^{s+[s]}.$$

When $m > 6$, we have $S_0 = 24(m-7)$, $S_1 = S_2 = 0$, so that

$$d(E) = \frac{6}{m} + \frac{m+1}{m} + \frac{1}{24m}\cdot 24(m-7) = 2.$$

When $m < 6$, then either $m = 1$ or $m = 5$. For $m = 1$, $S_0 = S_1 = S_2 = 0$, and $d(E) = 8$. For $m = 5$, $S_0 = 12$, $S_1 = 0$, and $S_2 = -60$, which gives $d(E) = 2$.

(4) $G = \langle Z_{6m}, Z_{2m}; \widetilde{T}, \widetilde{D}_2\rangle$. Let $h = \mu_{2m}I \in Z_{2m}$, and let x, y be generators of $\widetilde{T}$ satisfying $x^2 = y^3 = (xy)^3 = -1$. Introduce $\bar{h} = h^3$, $\bar{x} = x$, and $\bar{y} = hy$. Then $\sum_{g\neq 1}\chi(g) = S_0 + S_1 + S_2$, where $S_0 = \sum_{k=1}^{2m-1}\chi(\bar{h}^k)$,

$$S_1 = \sum_{g=\bar{y},\bar{x}\bar{y},\bar{x}^{-1}\bar{y}\bar{x},\bar{y}\bar{x}}\sum_{l=1}^{2}\sum_{k=0}^{2m-1}\chi(\bar{h}^k g^l),$$

and

$$S_2 = \sum_{g=\bar{x},\bar{y}^{-1}\bar{x}\bar{y},\bar{y}^{-2}\bar{x}\bar{y}^2}\sum_{k=0}^{2m-1}\chi(\bar{h}^k g).$$

A similar calculation, with the fact that m is divisible by 3, shows that

$$S_0 = \sum_{s=1}^{12}(2m-1-2[s])$$

$$S_1 = 16m\sum_{l=1}^{2}\sum_{s=1}^{12}\frac{\mu_6^{(s+[s])l}\mu_{6m}^{(s-[s])l}}{1-\mu_3^{-l}}$$

$$S_2 = 6m\sum_{s=1}^{12}\mu_4^{s+[s]}.$$

When $m > 6$, we have $S_0 = 24(m-7)$, $S_1 = S_2 = 0$, so that $d(E) = 2$. When $m < 6$, then $m = 3$, and in this case, $S_0 = S_2 = 0$ and $S_1 = -96$, which also gives $d(E) = 2$.

(5) $G = \langle Z_{2m}, Z_{2m}; \widetilde{O}, \widetilde{O}\rangle$. Let $h = \mu_{2m}I \in Z_{2m}$, and let x, y be generators of $\widetilde{O}$ satisfying $x^2 = y^4 = (xy)^3 = -1$. Recall that $\widetilde{O}$ is the union of three cyclic subgroups of order 8 generated by y, xyx and y^2x, four cyclic subgroups of order 6 generated by xy, yx, y^3xy^2 and y^2xy^3, and six cyclic subgroups of order 4 generated by $x, yxy^3, y^2xy^2, y^3xy^2x, xy^2xy^3$ and y^2xy^3x, where these subgroups only intersect at $\{1,-1\}$. Consequently, we have $\sum_{g\neq 1}\chi(g) = S_0 + S_1 + S_2 + S_3$, where $S_0 = \sum_{k=1}^{2m-1}\chi(h^k)$,

$$S_1 = \sum_g\sum_{l=1}^{3}\sum_{k=0}^{2m-1}\chi(h^kg^l), \text{ where } g \text{ has order } 8$$

$$S_2 = \sum_g\sum_{l=1}^{2}\sum_{k=0}^{2m-1}\chi(h^kg^l), \text{ where } g \text{ has order } 6$$

$$S_3 = \sum_g\sum_{k=0}^{2m-1}\chi(h^kg), \text{ where } g \text{ has order } 4.$$

A similar calculation shows that

$$S_0 = \sum_{s=1}^{24}(2m-1-2[s])$$

$$S_1 = 12m\sum_{l=1}^{3}\sum_{s=1}^{24}\frac{\mu_8^{(s+[s])l}}{1-\mu_4^{ml}}$$

$$S_2 = 16m\sum_{l=1}^{2}\sum_{s=1}^{24}\frac{\mu_6^{(s+[s])l}}{1-\mu_3^{ml}}$$

$$S_3 = 12m \sum_{s=1}^{24} \mu_4^{s+[s]}.$$

When $m > 12$, $S_0 = 48(m-13)$ and $S_1 = S_2 = S_3 = 0$, so that $d(E) = 2$. When $m < 12$, then $m = 1, 5, 7$ or 11. A direct calculation shows that $d(E) = 2$ in all this cases except for $m = 1$, for which $d(E) = 14$. Below we list the results of S_0, S_1, S_2 and S_3 for the sake of records.

- $m = 1$: $S_0 = S_1 = S_2 = S_3 = 0$.
- $m = 5$: $S_0 = 16$, $S_1 = -240$, $S_2 = -160$, and $S_3 = 0$.
- $m = 7$: $S_0 = 20$, $S_1 = 84$, $S_2 = -224$, and $S_3 = -168$.
- $m = 11$: $S_0 = 36$, $S_1 = 132$, $S_2 = 0$, and $S_3 = -264$.

(6) $G = \langle Z_{2m}, Z_{2m}; \widetilde{I}, \widetilde{I} \rangle$. Let $h = \mu_{2m} I \in Z_{2m}$, and let x, y be generators of $\widetilde{I}$ satisfying $x^2 = y^5 = (xy)^3 = -1$. Then $\sum_{g \neq 1} \chi(g) = S_0 + S_1 + S_2 + S_3$, where $S_0 = \sum_{k=1}^{2m-1} \chi(h^k)$, and

$$S_1 = \sum_g \sum_{l=1}^{4} \sum_{k=0}^{2m-1} \chi(h^k g^l), \ g \text{ is one of the six elements of order 10}$$

$$S_2 = \sum_g \sum_{l=1}^{2} \sum_{k=0}^{2m-1} \chi(h^k g^l), \ g \text{ is one of the ten elements of order 6}$$

$$S_3 = \sum_g \sum_{k=0}^{2m-1} \chi(h^k g), \ g \text{ is one of the fifteen elements of order 4.}$$

A similar calculation shows that

$$S_0 = \sum_{s=1}^{60} (2m - 1 - 2[s])$$

$$S_1 = 24m \sum_{l=1}^{4} \sum_{s=1}^{60} \frac{\mu_{10}^{(s+[s])l}}{1 - \mu_5^{ml}}$$

$$S_2 = 40m \sum_{l=1}^{2} \sum_{s=1}^{60} \frac{\mu_6^{(s+[s])l}}{1 - \mu_3^{ml}}$$

$$S_3 = 30m \sum_{s=1}^{60} \mu_4^{s+[s]}.$$

When $m > 30$, $S_0 = 120(m-31)$ and $S_1 = S_2 = S_3 = 0$, so that $d(E) = 2$. When $m < 30$, then $m = 1, 7, 11, 13, 17, 19, 23$ or 29. A direct calculation shows that $d(E) = 2$ for all cases except for $m = 1$, in which case $d(E) = 32$, and for $m = 7$, in which case $d(E) = 4$. We record the calculation for S_0, S_1, S_2 and S_3 below.

- $m = 1$: $S_0 = S_1 = S_2 = S_3 = 0$.
- $m = 7$: $S_0 = 32$, $S_1 = -672$, $S_2 = -560$, and $S_3 = 0$.
- $m = 11$: $S_0 = 64$, $S_1 = -1584$, $S_2 = -880$, and $S_3 = 0$.
- $m = 13$: $S_0 = 128$, $S_1 = -1248$, $S_2 = -1040$, and $S_3 = 0$.
- $m = 17$: $S_0 = 156$, $S_1 = -816$, $S_2 = 0$, and $S_3 = -1020$.
- $m = 19$: $S_0 = 308$, $S_1 = 912$, $S_2 = -1520$, and $S_3 = -1140$.
- $m = 23$: $S_0 = 420$, $S_1 = 0$, $S_2 = 0$, and $S_3 = -1380$.
- $m = 29$: $S_0 = 108$, $S_1 = 1392$, $S_2 = 0$, and $S_3 = -1740$.

q.e.d.

Appendix B. Green's Function for the Laplacian on Orbifolds

We shall follow the relevant discussion in Chapter 4 of Aubin [**4**] for Green's function on compact Riemannian manifolds.

Let (X, g) be a compact, closed, oriented n-dimensional Riemannian orbifold. For any $p, q \in X$, we define the distance between p and q, denoted by $d(p, q)$, to be the infinimum of the lengths of all piecewise C^1 paths connecting p and q. Then (X, d) is a complete metric space. Moreover, there is a geodesic γ between p, q such that $d(p, q) = \text{length}(\gamma)$. (A C^1 path $f : [a, b] \to X$ is called a (parametrized) geodesic in X if f is locally lifted to a geodesic in a uniformizing system.) Observe that when p, q are both in a uniformized open subset U and the geodesic γ with $d(p, q) = \text{length}(\gamma)$ is also contained in U, then γ may be lifted to a geodesic in $\widehat{U}$, where $(\widehat{U}, G)$ uniformizes U. This implies that in the said circumstance,

$$d(p, q) = \min_{\{h_1, h_2 \in G\}} \hat{d}(h_1 \cdot \hat{p}, h_2 \cdot \hat{q})$$

where $\hat{p}, \hat{q}$ are any inverse image of p, q in $\widehat{U}$, and $\hat{d}$ is the distance function on $\widehat{U}$ (note that $\widehat{U}$ has a Riemannian metric canonically determined by g). On the other hand, for each $p \in X$, there is a $\delta(p) > 0$, called the injectivity radius at p, such that for any $0 < \delta \leq \delta(p)$, the set $U_p(\delta) = \{q \in X \mid d(p, q) < \delta\}$ is uniformized by $(\widehat{U_p(\delta)}, G_p)$ where $\widehat{U_p(\delta)}$ is a convex geodesic ball of radius δ centered at the inverse image of p and G_p is the isotropy group at p acting linearly on $\widehat{U_p(\delta)}$. We point out that $\delta(p) \to 0$ as $p \to q$ for any q with $|G_q| > |G_p|$. In particular, there is no positive uniform lower bound for the injectivity radius on an orbifold with a nonempty orbifold-point set.

Theorem 1. *Let $\Delta = d^*d$ be the Laplacian on (X, g) and let $n \equiv \dim X \geq 2$. There exists $G(p, q)$, a Green's function for the Laplacian which has the following properties:*

(1) *For all C^2 functions φ on X,*

$$\varphi(p) = \mathrm{Vol}(X)^{-1} \int_X \varphi + \int_X G(p, \cdot)\Delta\varphi,$$

where $\mathrm{Vol}(X)$ *is the volume of* X*.*

(2) $G(p,q)$ *is a smooth function on* $X \times X$ *minus the diagonal.*

(3) *There exists a decomposition* $G(p,q) = G_0(p,q) + G_1(p,q)$ *such that*
   - $G_1(p,q)$ *is continuous in both variables and* C^2 *in* q*.*
   - *There exist a* $\delta_0 > 0$ *and a set* $\mathcal{U}$ *of finitely many uniformizing systems on* X *with the following significance: For any* $p \in X$*, there is a uniformizing system* $(\widehat{U}, G, \pi) \in \mathcal{U}$ *and a* G*-invariant open subset* $\widehat{U}' \subset \widehat{U}$*, such that* (i) $p \in \pi(\widehat{U}')$, (ii) *the support of the function* $q \mapsto G_0(p,q)$ *is contained in* $\pi(\widehat{U})$ *(more precisely,* $\{q \mid G_0(p,q) \neq 0\} \subseteq \{q \mid d(p,q) \leq (n+1)\delta_0)$*, and* (iii) $\widehat{U}$ *contains the closed ball of radius* δ_0 *centered at any* $\hat{p} \in \pi^{-1}(p)$*. Moreover, the function* $G_0(p,q)$ *is the descendant of* $\sum_{h\in G} \widehat{G_0}(h \cdot \hat{p}, \hat{q})$ *for some function* $\widehat{G_0}(\hat{p}, \hat{q})$*, which is,* (i) *continuous on* $\widehat{U}' \times \widehat{U}$ *minus the subset* $\{(\hat{p}, \hat{q}) \mid \hat{p} = \hat{q}\}$, (ii) C^2 *in* $\hat{q}$*,* (iii) *invariant under the diagonal action of* G*, and* (iv) *satisfying the following estimates for a constant* $z > 0$*:*

$$|\widehat{G_0}(\hat{p}, \hat{q})| \leq z(1 + |\log \hat{d}(\hat{p}, \hat{q})|) \quad \text{for } n = 2 \quad \text{and}$$
$$|\widehat{G_0}(\hat{p}, \hat{q})| \leq \frac{z}{\hat{d}(\hat{p}, \hat{q})^{n-2}} \quad \text{for } n > 2, \text{ with}$$
$$|\nabla_{\hat{q}}\widehat{G_0}(\hat{p}, \hat{q})| \leq \frac{z}{\hat{d}(\hat{p}, \hat{q})^{n-1}},$$
$$|\nabla^2_{\hat{q}}\widehat{G_0}(\hat{p}, \hat{q})| \leq \frac{z}{\hat{d}(\hat{p}, \hat{q})^{n}}.$$

   (Here $\hat{d}$ *is the distance function on* $\widehat{U}$*.)*

(4) *There exists a constant* C *such that* $G(p,q) \geq C$*. Green's function is defined up to a constant, so one may arrange so that* $G(p,q) \geq 1$*.*

(5) *The map* $q \mapsto \int_X G(p,q)$ *is constant. One may choose to have* $\int_X G(p,q) = 0$*.*

(6) *Green's function is symmetric:* $G(p,q) = G(q,p)$*.*

Proof.

Choose finitely many points $p_i \in X$ such that $X = \bigcup_i U_{p_i}(N^{-1}\delta_i)$ with $\delta_i \equiv N^{-1}\delta(p_i)$, where $\delta(p_i)$ is the injectivity radius at p_i and N is any fixed integer which is no less than $3n$ (recall $n = \dim X$). The set $\mathcal{U}$ is chosen to be the set of uniformizing system of $U_{p_i}(\delta(p_i))$. Set $\delta_0 \equiv \min_i \delta_i$.

Now given any $p \in X$, suppose p is contained in $U' \equiv U_{p_i}(\delta_i)$ for some i. We denote by $(\widehat{U}', G', \pi')$ the uniformizing system of U', and by $(\widehat{U}, G, \pi)$ the uniformizing system of $U \equiv U_{p_i}(\delta(p_i))$, which is an element of $\mathcal{U}$ by definition. Note that $G' = G = G_{p_i}$. With these understood, we define a function $\widehat{H_0}(\hat{p}, \hat{q})$ on $\widehat{U}' \times \widehat{U}$ for $\hat{p} \neq \hat{q}$, such that

$$\widehat{H_0}(\hat{p}, \hat{q}) = -(2\pi)^{-1}\rho(r)\log r \quad \text{for } n = 2 \quad \text{and}$$
$$\widehat{H_0}(\hat{p}, \hat{q}) = [(n-2)\omega_{n-1}]^{-1}\rho(r) r^{2-n} \quad \text{for } n > 2,$$

where $r = \hat{d}(\hat{p}, \hat{q})$, $\rho(r)$ is a fixed cut-off function which equals zero when $r \geq \delta_0$, and ω_{n-1} is the volume of the unit sphere in $\mathbb{R}^n$. It is clear that $\widehat{H_0}(\hat{p}, \hat{q})$ is invariant under the diagonal action of G. We define

$$\widehat{H}(\hat{p}, \hat{q}) = \sum_{h \in G} \widehat{H_0}(h \cdot \hat{p}, \hat{q}),$$

which is invariant under the action of $G \times G$. Let $H(p, q)$ be the descendant of $\widehat{H}(\hat{p}, \hat{q})$, which is defined on $U' \times U$ for $p \neq q$. We extend $H(p, q)$ over $q \in X$ by zero. Now observe that if we use a different element of $\{U_{p_i}(\delta_i)\}$ for U', we end up with the same function $q \mapsto H(p, q)$. Hence we obtain a function $H(p, q)$ on $X \times X$ minus the diagonal. Note that $\{q \mid H(p, q) \neq 0\} \subseteq \{q \mid d(p, q) \leq \delta_0\}$.

Define $\Gamma_1(p, q) = -\Delta_q H(p, q)$ and for each $1 \leq j \leq n$ define inductively

$$\Gamma_{j+1}(p, q) = \int_X \Gamma_j(p, \cdot)\Gamma_1(\cdot, q).$$

We note that $\{q \mid \Gamma_j(p, q) \neq 0\} \subseteq \{q \mid d(p, q) \leq j \cdot \delta_0\}$. Now suppose $p \in U'$. If we let $\widehat{\Gamma}_1(\hat{p}, \hat{q}) = -\Delta_{\hat{q}} \widehat{H_0}(\hat{p}, \hat{q})$ and for each $1 \leq j \leq n$, define

$$\widehat{\Gamma}_{j+1}(\hat{p}, \hat{q}) = \int_{\widehat{U}} \widehat{\Gamma}_j(\hat{p}, \cdot)\widehat{\Gamma}_1(\cdot, \hat{q})$$

inductively, then each $\widehat{\Gamma}_j$ is invariant under the diagonal action of G. Moreover, each $\Gamma_j(p, q)$ is the descendant of $\sum_{h \in G} \widehat{\Gamma}_j(h \cdot \hat{p}, \hat{q})$.

As in Aubin [**4**] (cf. Prop. 4.12 in [**4**]), $\widehat{\Gamma}_{n+1}(\hat{p}, \hat{q})$ is C^1 on $\widehat{U}' \times \widehat{U}$. Hence $\Gamma_{n+1}(p, q)$ is C^1 on $X \times X$. Fix each $p \in X$, we solve the Laplacian equation

$$\Delta_q F(p, q) = \Gamma_{n+1}(p, q) - \mathrm{Vol}\,(X)^{-1}.$$

(Note that for each $p \in X$, $\int_X (\Gamma_{n+1}(p, \cdot) - \mathrm{Vol}\,(X)^{-1}) = 0$, cf. [**4**].) Then we define

$$G(p, q) = H(p, q) + \sum_{j=1}^{n} \int_X \Gamma_j(p, \cdot) H(\cdot, q) + F(p, q),$$

and by adding an appropriate constant to $F(p,q)$, we arrange to have for all $p \in X$

$$\int_X G(p,\cdot) = 0.$$

One can argue as in Aubin [**4**] that $G(p,q)$ is a Green's function for the Laplacian which has the properties described in the theorem. Particularly, in the decomposition

$$G(p,q) = G_0(p,q) + G_1(p,q)$$

in Thoerem 1 (3), we have $G_0(p,q) = H(p,q) + \sum_{j=1}^n \int_X \Gamma_j(p,\cdot)H(\cdot,q)$ and $G_1(p,q) = F(p,q)$. (Note that $\{q \mid G_0(p,q) \neq 0\} \subseteq \{q \mid d(p,q) \leq (n+1)\delta_0.\}$) Moreover, for any $p \in U'$, $G_0(p,q)$ is clearly the descendant of $\sum_{h\in G} \widehat{G_0}(h\cdot\hat{p},\hat{q})$ where

$$\widehat{G_0}(\hat{p},\hat{q}) = \widehat{H_0}(\hat{p},\hat{q}) + \sum_{j=1}^n \int_{\widehat{U}} \widehat{\Gamma}_j(\hat{p},\cdot)\widehat{H_0}(\cdot,\hat{q}).$$

The estimates

$$\begin{aligned}
|\widehat{G_0}(\hat{p},\hat{q})| &\leq z(1+|\log \hat{d}(\hat{p},\hat{q})|) \quad \text{for } n=2 \quad \text{and}\\
|\widehat{G_0}(\hat{p},\hat{q})| &\leq \frac{z}{\hat{d}(\hat{p},\hat{q})^{n-2}} \quad \text{for } n>2, \text{ with}\\
|\nabla_{\hat{q}}\widehat{G_0}(\hat{p},\hat{q})| &\leq \frac{z}{\hat{d}(\hat{p},\hat{q})^{n-1}},\\
|\nabla^2_{\hat{q}}\widehat{G_0}(\hat{p},\hat{q})| &\leq \frac{z}{\hat{d}(\hat{p},\hat{q})^{n}}
\end{aligned}$$

in Thoerem 1 (3) follow immediately from the definition of $\widehat{H_0}(\hat{p},\hat{q})$.

q.e.d.

Appendix C. Proof of Lemma 1.4

Various transversality theorems for harmonic forms on a compact, closed Riemannian manifold were proved in Honda [**20**]. The machinery developed therein can be properly adapted to deal with the present situation.

First of all, we recall the relevant discussion in [**20**] regarding the case of self-dual harmonic forms on a (compact, closed) 4-manifold. Suppose M is a smooth 4-manifold with $b_2^+(M) \neq 0$. Let $\mathrm{Met}^l(M)$ be the space of C^l-Hölder metrics on M for a sufficiently large non-integer l, let Q^+ be the space of pairs (ω, g) where $g \in \mathrm{Met}^l(M)$ and ω is a nontrivial self-dual g-harmonic form, and let $\bigwedge^{2,+} \to \mathrm{Met}^l(M)\times M$ be the vector bundle whose fiber at (g,x) is $\bigwedge_g^{2,+}(T_x^*M)$, the space of 2-forms at x which is self-dual with respect to g. Then the transversality of the following evaluation map

$$\mathrm{ev} : Q^+ \times M \to \bigwedge^{2,+}, \quad ((\omega, g), x) \mapsto (\omega(x), (g, x))$$

to the zero section was studied in [**20**]. The relevant results are summarized below. For any $((\omega, g), x) \in Q^+ \times M$ where $\omega(x) = 0$, it was shown that the differential

$$(\mathrm{ev}_x)_* : T_{(\omega,g)}Q^+ \to \bigwedge_g^{2,+}(T_x^*M), \quad (v, h) \mapsto v(x)$$

is surjective. Here $T_{(\omega,g)}Q^+$ is the tangent space of Q^+ at (ω, g), which consists of pairs (v, h), where $h \in C^l(\mathrm{Sym}^2(T^*M))$ and v is a 2-form, self-dual at x with respect to g and satisfying the equation

$$\Delta_g v + \frac{d}{dt}(\Delta_{g+th})|_{t=0}\omega = 0.$$

(Here Δ_g is the Laplacian associated to a metric g.) As a consequence, for a generic metric a nontrivial self-dual harmonic form has only regular zeroes, which consist of a disjoint union of embedded circles in M.

In order to adapt the argument to the present situation, we recall some relevant details about the surjectivity of $(ev_x)_* : T_{(\omega,g)}Q^+ \to \bigwedge_g^{2,+}(T_x^*M)$. Suppose $(v, h) \in T_{(\omega,g)}Q^+$ and v is orthogonal to the space of g-harmonic 2-forms. Then one can solve for v from h by

$$\begin{aligned} v(x) &= -(\Delta_g)^{-1}\left(\frac{d}{dt}(\Delta_{g+th})|_{t=0}\omega\right)(x) \\ &= \pm \int_M \langle dd^*G_g(x, y), *(D_h*)\omega(y)\rangle_g, \end{aligned}$$

where $G_g(x, y)$ is the Green's function for Δ_g, and D_h* is shorthand for $\frac{d}{dt}(*_{g+th})|_{t=0}$. (Here the Hodge star $*$ and the integration are with respect to the metric g.) For any $x \in M$ with $\omega(x) = 0$, one considers the map $\Psi_x : C^l(Sym^2(T^*M)) \to \bigwedge_g^{2,+}(T_x^*M)$ where

$$\Psi_x : h \mapsto v(x) = \pm \int_M \langle dd^*G_g(x, y), *(D_h*)\omega(y)\rangle_g.$$

Then it is clear that the surjectivity of $(ev_x)_*$ is a consequence of that of Ψ_x.

To explain the basic ingredients in the proof of surjectivity of Ψ_x, we first introduce some notations. For any $0 \neq u \in \mathbb{R}^4$, let $R_u : \mathbb{R}^4 \to \mathbb{R}^4$ be the reflection in u, and let $\bigwedge^2 R_u : \bigwedge^2(\mathbb{R}^4) \to \bigwedge^2(\mathbb{R}^4)$ be the induced isomorphism. For any skew-symmetric 4×4 matrix A and symmetric 4×4 matrix H, let $i_A(H) = HA + AH - \frac{1}{2}tr(H) \cdot A$. Then the proof of surjectivity of Ψ_x goes roughly as follows.

- Assume g is flat. Then for any $y \neq x$ nearby, a direct calculation shows that $\langle dd^* G_g(x,y), \cdot\rangle_g = \frac{C}{|y-x|^4} \cdot \bigwedge^2 R_{y-x}$ for a constant $C \neq 0$, and that $*(D_h *)\omega(y) = i_{\omega(y)}(h)$ where $\omega(y)$, h are regarded canonically as a skew-symmetric and a symmetric 4×4 matrix respectively. Moreover, one can verify that $\bigwedge^2 R_u : \bigwedge^{2,\pm}(\mathbb{R}^4) \rightarrow \bigwedge^{2,\mp}(\mathbb{R}^4)$, and that when $\omega(y) \neq 0$ and is self-dual, the image of $i_{\omega(y)}$ is the space of anti-self-dual 2-forms. The surjectivity of Ψ_x follows in this case by letting h be a δ-function like element centered at y.
- In general, use the asymptotic expension of the Green's function $G_g(x,y)$ near the diagonal, whose leading term is the Green's function for a flat metric, to reduce the proof to the previous case.

With these preparations, we now give a proof of Lemma 1.4.

Let W be an oriented smooth s-cobordism of elliptic 3-manifolds as in Lemma 1.4. We attach a semi-cylinder $[0,+\infty) \times (\mathbb{S}^3/G)$ to the positive end of W and cone-off the negative end by $\mathbb{B}^4/G$. The resulting space, denoted by $\hat{W}$, is an orbifold with one isolated singular point p_0 and a cylindrical end over $\mathbb{S}^3/G$. We shall fix a Riemannian metric g_0 on $\hat{W}$, which is flat near p_0 and is the product metric $dt^2 + h_0$ on the semi-cylinder $[0,+\infty) \times (\mathbb{S}^3/G)$. Here h_0 stands for the standard metric on $\mathbb{S}^3/G$ which has a constant sectional curvature of 1. Fixing a $T > 1$ and a sufficiently large non-integer l, we will consider $\mathrm{Met}^l_T(\hat{W})$, the space of C^l-Hölder metrics on $\hat{W}$ which equals g_0 on $[T,+\infty) \times (\mathbb{S}^3/G)$.

We fix an identification $\mathbb{R}^4 = \mathbb{C}^2 = \mathbb{H}$ so that G as a subgroup of $\phi(\mathbb{S}^1 \times \mathbb{S}^3)$ is canonically regarded as a subgroup of $U(2)$. Let z_1, z_2 be the standard coordinates on $\mathbb{C}^2$, and let $\tilde{\alpha}$ be the pull-back of the 1-form $\sqrt{-1}\sum_{i=1}^2 (z_i d\bar{z}_i - \bar{z}_i dz_i)$ to $\mathbb{S}^3$. Then it is easy to check that $\tilde{\alpha}$ obeys $d\tilde{\alpha} = 2(*\tilde{\alpha})$ with respect to the standard metric on $\mathbb{S}^3$, and consequently, $d(\exp(2t)\tilde{\alpha})$ is a self-dual 2-form on $\mathbb{R} \times \mathbb{S}^3$ with respect to the corresponding product metric. Note that $\tilde{\alpha}$ is invariant under the action of G. Let α be the descendant of $\tilde{\alpha}$ to $\mathbb{S}^3/G$.

Proposition. *For each $g \in \mathrm{Met}^l_T(\hat{W})$, there is a unique self-dual g-harmonic 2-form ω_g which has the following properties.*

1. *On $[T,+\infty) \times (\mathbb{S}^3/G)$, $\omega_g - d(\exp(2t)\alpha) = d\alpha_t$ where α_t is a 1-form on $\mathbb{S}^1/G$ such that α_t and $\frac{d}{dt}\alpha_t$ converge to zero exponentially fast as $t \rightarrow +\infty$.*
2. *For a generic g, ω_g has only regular zeroes in the complement of the singular point p_0 and $[T,+\infty) \times (\mathbb{S}^3/G)$.*
3. *For a generic g which is sufficiently close to g_0 near p_0, $\omega_g(p_0) \neq 0$.*

Assuming the validity of the proposition, we obtain the 2-form ω claimed in Lemma 1.4 as follows. Consider the g_0-harmonic 2-form ω_{g_0} first. Since $\omega_{g_0} - d(\exp(2t)\alpha)$ converges to zero exponentially fast as $t \to +\infty$, and note that $d(\exp(2t)\alpha)$ is a symplectic form, there is a $\tau_0 > 0$ such that ω_{g_0} is symplectic on $[\tau_0, +\infty) \times (\mathbb{S}^3/G)$. We fix a $T \geq \tau_0$; then for all $g \in \mathrm{Met}_T^l(\hat{W})$ sufficiently close to g_0, ω_g is symplectic on $[T, +\infty) \times (\mathbb{S}^3/G)$. We pick such a g which is generic. Then by (2) and (3) of the proposition, ω_g has only regular zeroes, which are in the complement of p_0 and $[T, +\infty) \times (\mathbb{S}^3/G)$. In particular, $\omega_g(p_0) \neq 0$. Let $\hat{\omega}_g$ be the 2-form obtained from ω_g by replacing $\omega_g = d(\exp(2t)\alpha + \alpha_t)$ with $d(\exp(2t)\alpha + (1-\rho_\tau)\alpha_t)$ on the cylindrical end of $\hat{W}$, where ρ_τ is a cut-off function for a sufficiently large $\tau \geq T+2$ which equals 1 on $t \geq \tau$ and equals 0 on $t \leq \tau - 1$. Then $\hat{\omega}_g = d(\exp(2t)\alpha)$ on the cylindrical end of $\hat{W}$ where $t \geq \tau$, and $\hat{\omega}_g(p_0) = \omega_g(p_0) \neq 0$, so that by the equivariant Darboux' theorem $\hat{\omega}_g$ is equivalent near p_0 to the standard symplectic form on $\mathbb{B}^4/G$. It is clear that $\hat{\omega}_g$ yields a 2-form ω on the s-cobordism W as described in Lemma 1.4.

Proof of Proposition. For any $g \in \mathrm{Met}_T^l(\hat{W})$, we denote by $\bigwedge_g^{2,+}$ and $\bigwedge_g^1$ the associated bundle of self-dual 2-forms and 1-forms on $\hat{W}$ respectively. Consider the subspaces $\mathcal{E}_g$, $\mathcal{F}_g$ of the (weighted) Sobolev spaces $H_1^2(\bigwedge_g^{2,+})$, $H_0^2(\bigwedge_g^1)$, where $\mathcal{E}_g$ is the closure of self-dual 2-forms which equal $dt \wedge \alpha_t + *_3\alpha_t$ on the cylindrical end with $d_3^*\alpha_t = 0$ for $t \geq T$, and $\mathcal{F}_g$ is the closure of co-closed 1-forms which can be written as $f_t dt + \beta_t$ on the cylindrical end with $f_t = 0$ and $d_3^*\beta_t = 0$ when $t \geq T$. (Here $*_3$ is the Hodge star and d_3^* is the co-exterior differential on $\mathbb{S}^3/G$, both with respect to the standard metric h_0.) Then the differential operator $*_g d : \mathcal{E}_g \to \mathcal{F}_g$, which is of form $\frac{d}{dt} - *_3 d_3$ on the cylindrical end, defines a Fredholm operator, cf. [**32**]. (Note that there exist no harmonic 1-forms on $\mathbb{S}^3/G$, so that we can choose $\delta = 0$ in the weight factor $\exp(\delta t)$ of the weighted Sobolev spaces.)

Lemma 1. *$*_g d : \mathcal{E}_g \to \mathcal{F}_g$ has a trivial kernel and cokernel.*

Assuming the validity of Lemma 1, we obtain the self-dual g-harmonic form ω_g for each $g \in \mathrm{Met}_T^l(\hat{W})$ as follows. Let β be a self-dual 2-form on $\hat{W}$ obtained by multiplying $d(\exp(2t)\alpha)$ with a cut-off function which equals 1 on $t \geq 1$. Then $*_g d\beta \in \mathcal{F}_g$, and hence there exists a $u_g \in \mathcal{E}_g$ such that $*_g du_g = - *_g d\beta$. We set $\omega_g \equiv u_g + \beta$. It is clear that ω_g is self-dual g-harmonic. To show that ω_g has the property in (1) of the proposition, we note that on $[T, +\infty) \times (\mathbb{S}^3/G)$, $*_g du_g = - *_g d\beta = 0$, which implies that there is a α_t, with $\omega_g - d(\exp(2t)\alpha) = u_g = d\alpha_t$, such that α_t and $\frac{d}{dt}\alpha_t$ converge to zero exponentially fast as $t \to +\infty$ (see the proof of Lemma 1 below). Finally, observe that such a ω_g is unique.

To prove (2) and (3) of the proposition, we consider the evaluation maps

$$\mathrm{ev}_q : (\omega_g, g) \mapsto \omega_g(q) \in \bigwedge_g^{2,+}(T^*\hat{W}_q), \text{ where } q \in \hat{W} \setminus [T, +\infty) \times (\mathbb{S}^3/G).$$

Suppose $\omega_g(q) = 0$. Then as in [**20**], the differential of ev_q is given by $(\mathrm{ev}_q)_*(h) = v(q) \in \bigwedge_g^{2,+}(T^*\hat{W}_q)$, where (v, h) obeys

$$\Delta_g v + \frac{d}{dt}(\Delta_{g+th})|_{t=0}\omega_g = 0.$$

We shall prove first that for any $q \neq p_0$, $(ev_q)_*$ is surjective, which gives (2) of the proposition by a standard argument.

To this end, for any $\tau > T + 2$, we set $W_\tau \equiv \hat{W} \setminus (\tau, +\infty) \times (\mathbb{S}^3/G)$, and let DW_τ be the double of W_τ, which is given with the natural metric and orientation. Denote by $\Delta_\tau = d^*d + d^*d$ the Laplacian on DW_τ, and let γ_τ be the first eigenvalue of Δ_τ on the space of L^2 self-dual 2-forms on DW_τ. We will need the following lemma.

Lemma 2. *There exist a $\tau_0 > T + 2$ and a constant $c > 0$ such that $\gamma_\tau \geq c$ for all $\tau \geq \tau_0$.*

Assuming the validity of Lemma 2, we fix a $\tau \geq \tau_0 + 10$, and decompose $v = v_1 + v_2$ with $v_1 \equiv (1 - \rho_\tau)v$ and $v_2 \equiv \rho_\tau v$, where ρ_τ is a cut-off function which equals 1 on $t \geq \tau$. Then we have $\Delta_g v_1 = -\frac{d}{dt}(\Delta_{g+th})|_{t=0}\omega_g - \Delta_g v_2$. Note that h is supported in $\hat{W} \setminus (T, +\infty) \times (\mathbb{S}^3/G)$, so that the above equation may be regarded as an equation on DW_τ because $\Delta_g v_2 = \Delta_g v = 0$ on $t \geq \tau$. Moreover, $|\Delta_g v_2| \leq c \cdot \exp(-\delta\tau)$ for some constants $c > 0$ and $\delta > 0$, where c is a multiple of the C^0-norm of v on $\hat{W} \setminus (T, +\infty) \times (\mathbb{S}^3/G)$, hence is bounded by a multiple of the norm of h via the standard elliptic estimates. Therefore $|\Delta_g v_2| \leq C \cdot \exp(-\delta\tau) \cdot ||h||$. Letting G_τ be the Green's function for the Laplacian Δ_τ on DW_τ (the existence of the Green's function G_τ on the orbifold DW_τ is a straightforward generalization of that in the compact manifold case, cf. e.g., [**37**]), then for any $q \in \hat{W} \setminus [T, +\infty) \times (\mathbb{S}^3/G)$,

$$v(q) = v_1(q) = \pm \int_{DW_\tau} \langle dd^* G_\tau(q, y), *(D_h*)\omega_g(y)\rangle_g - (\Delta_\tau)^{-1}(\Delta_g v_2),$$

where $D_h* = \frac{d}{dt}(*_{g+th})|_{t=0}$. By Lemma 2 and the standard elliptic estimates, the last term $(\Delta_\tau)^{-1}(\Delta_g v_2)$ in the above equation is bounded by a multiple of $\exp(-\delta\tau) \cdot ||h||$, and hence can be neglected by taking τ sufficiently large. The surjectivity of $(\mathrm{ev}_q)_* : h \mapsto v(q)$ for $q \neq p_0$ follows from the surjectivity of

$$\Psi_q : h \mapsto \pm \int_{DW_\tau} \langle dd^* G_\tau(q, y), *(D_h*)\omega_g(y)\rangle_g$$

as in [**20**], which we have recalled at the beginning.

It remains to prove (3) of the proposition, i.e., for a generic metric g which is sufficiently close to g_0, ω_g does not vanish at the singular point p_0. To this end, we consider g_0 first, and assume $\omega_{g_0}(p_0) = 0$ (otherwise the claim is trivially true). Identify a local uniformizing system at p_0 with $(\mathbb{B}^4, G)$, which is given with a flat metric. Then the bundle of self-dual 2-forms has a local basis $\omega_0, \omega_1, \omega_2$, where $\omega_0 = \sqrt{-1}\sum_i dz_i \wedge d\bar{z}_i$, $\omega_1 = \mathrm{Re}\,(dz_1 \wedge dz_2)$ and $\omega_2 = \mathrm{Im}\,(dz_1 \wedge dz_2)$. Note that ω_0 is invariant under the action of G. With this understood, we claim that for g_0, $(ev_{p_0})_*$ is transverse to the subspace spanned by ω_1, ω_2. To see this, we pick a y sufficiently close to p_0 such that $\omega_{g_0}(y) \neq 0$, and denote by $y_0, y_1, \ldots, y_N$ the inverse images of y in $\mathbb{B}^4$. Then according to [**20**] as we recalled earlier, there exists a $h_0 \in \mathrm{Sym}^2\mathbb{R}^4$ such that $\bigwedge^2 R_{y_0} \circ i_{\omega_{g_0}(y_0)}(h_0) = \omega_0$. Let h_i, $i = 0, 1, \ldots, N$, be the orbit of h_0 under the action of G; then because ω_0 is invariant under the action of G, we have $\bigwedge^2 R_{y_i} \circ i_{\omega_{g_0}(y_i)}(h_i) = \omega_0$ for $i = 0, 1, \ldots, N$. Now observe that $(ev_{p_0})_* : h \mapsto v(p_0)$ is given by the equation (with $\tau \gg 0$)

$$(\mathrm{ev}_{p_0})_*(h) = \pm \int_{DW_\tau} \langle dd^* G_\tau(p_0, y), *(D_h *)\omega_{g_0}(y)\rangle_{g_0} - (\Delta_\tau)^{-1}(\Delta_{g_0} v_2).$$

It is clear that we can use $\{h_i\}$ to define a G-equivariant section $h \in C^\infty(\mathrm{Sym}^2 T^*\mathbb{B}^4)$, which is supported in a small neighborhood of $\{y_i\}$, such that the projection of $(\mathrm{ev}_{p_0})_*(h)$ to the ω_0 factor is nonzero. Hence for g_0, $(\mathrm{ev}_{p_0})_*$ is transverse to the subspace spanned by ω_1, ω_2, so is it for any g sufficiently close to g_0. As a corollary, let Z be the subbundle spanned by ω_1, ω_2 over a sufficiently small, G-invariant neighborhood of $0 \in \mathbb{B}^4$. Then for any generic metric g sufficiently close to g_0, $\omega_g^{-1}(Z)$ is a 3-dimensional manifold in $\mathbb{B}^4$ which is invariant under the action of G. If $0 \in \omega_g^{-1}(Z)$, then the tangent space of $\omega_g^{-1}(Z)$ at $0 \in \mathbb{B}^4$ is invariant under the action of G, which is possible only when $G = \{1, -1\}$. Hence when $G \neq \{1, -1\}$, $\omega_g(p_0)$ is not contained in Z, and therefore $\omega_g(p_0) \neq 0$.

When $G = \{1, -1\}$, note that all $\omega_0, \omega_1, \omega_2$ are invariant under the action of G. The above argument then shows that $(\mathrm{ev}_{p_0})_*$ is surjective for g_0. It follows easily that for a generic metric g which is sufficiently close to g_0, if $\omega_g(p_0) = 0$, then one of the components of $\omega_g^{-1}(0)$ in $\hat{W}$ is a compact 1-dimensional manifold with boundary, whose boundary consists of the singular point p_0. This is a contradiction. Hence the proposition. q.e.d.

The rest of the appendix is occupied by the proofs of Lemma 1 and Lemma 2.

Proof of Lemma 1. We first prove that the kernel of $*_g d : \mathcal{E}_g \to \mathcal{F}_g$ is trivial. By elliptic regularity, it suffices to show that if $\omega \in \mathcal{E}_g$ is smooth and satisfies $*_g d\omega = 0$, then $\omega = 0$.

First of all, note that $\omega = dt \wedge \alpha_t + *_3\alpha_t$ on the cylindrical end with $d_3^*\alpha_t = 0$ for $t \geq T$. Moreover, the equation $*_g d\omega = 0$ is equivalent to $\frac{d}{dt}\alpha_t - *_3 d_3\alpha_t = 0$. If we write $\alpha_t = \sum_i f_i(t)\alpha_i$, where $\{\alpha_i\}$ is a complete set of eigenforms for the self-adjoint operator $*_3 d_3$ on the space of L^2 co-closed 1-forms on $\mathbb{S}^3/G$, with $*_3 d_3 \alpha_i = \lambda_i \alpha_i$, then the functions $f_i(t)$ satisfy $f_i'(t) - \lambda_i f_i(t) = 0$. It follows easily, since ω has a bounded L^2-norm, that each $f_i(t) = c_i \exp(\lambda_i t)$ for some constant c_i with $\lambda_i < 0$. Set $\delta \equiv \min_i |\lambda_i| > 0$. Then $|\alpha_t| \leq c\exp(-\delta t)$ for a constant $c > 0$.

Since $H^2_{dR}(\hat{W}) = 0$, $*_g d\omega = 0$ implies that there exists a 1-form γ on $\hat{W}$ such that $d\gamma = \omega$. We write $\gamma = f_t dt + g_t$ on the cylindrical end, then f_t, g_t satisfy

$$\frac{d}{dt} g_t = d_3 f_t + \alpha_t \text{ and } d_3 g_t = *_3\alpha_t.$$

Set $\bar{f}_t \equiv \int_{t_0}^t f_s ds$ and $\bar{\alpha}_t \equiv \int_{t_0}^t \alpha_s ds$. Then $g_t = d_3\bar{f}_t + \bar{\alpha}_t + \text{constant}$. With this understood, we have

$$0 = \int_{\hat{W}} d^*d\gamma \wedge *\gamma = \int_{\hat{W}} d\gamma \wedge *d\gamma \pm \lim_{t\to+\infty} \int_{\{t\}\times(\mathbb{S}^3/G)} d\gamma \wedge \gamma,$$

where $d\gamma \wedge \gamma = *_3\alpha_t \wedge g_t = *_3\alpha_t \wedge (d_3\bar{f}_t + \bar{\alpha}_t + \text{constant})$. Since $\alpha_t \to 0$ exponentially fast along the cylindrical end, and $*_3\alpha_t = d_3 g_t$, it follows easily that $\lim_{t\to+\infty} \int_{\{t\}\times(\mathbb{S}^3/G)} d\gamma \wedge \gamma = 0$, which implies $\omega = d\gamma = 0$.

Next we show that the cokernel of $*_g d : \mathcal{E}_g \to \mathcal{F}_g$ is trivial. To this end, note that the formal adjoint of $*_g d$ is d^+. By elliptic regularity, it suffices to prove that for any smooth 1-form $\theta \in \mathcal{F}_g$, $d^+\theta = 0$ implies $\theta = 0$. Note that on the cylindrical end, $\theta = f_t dt + \beta_t$ where $f_t = 0$ and $d_3^*\beta_t = 0$ when $t \geq T$. Thus $d^+\theta = 0$ implies that $\frac{d}{dt}\beta_t + *_3 d_3\beta_t = 0$. Similarly, there exists a $\delta > 0$, such that $|\theta| = |\beta_t| \leq c\exp(-\delta t)$ for some constant $c > 0$. As a consequence, since $d^+\theta = 0$, we have

$$\int_{\hat{W}} d\theta \wedge *d\theta = -\int_{\hat{W}} d\theta \wedge d\theta = -\lim_{t\to+\infty} \int_{\{t\}\times(\mathbb{S}^3/G)} \theta \wedge d\theta = 0,$$

and hence $d\theta = 0$. Now $H^1_{dR}(\hat{W}) = 0$ implies that $\theta = df$ for some smooth function f on $\hat{W}$. On the cylindrical end, $\theta = df = \frac{\partial f}{\partial t} \cdot dt + d_3 f$, so that when $t \geq T$, $\frac{\partial f}{\partial t} = 0$. This implies that f is bounded on $\hat{W}$, and therefore

$$\int_{\hat{W}} |df|^2 = \int_{\hat{W}} \langle d^*df, f\rangle = \int_{\hat{W}} \langle d^*\theta, f\rangle = 0,$$

which implies $\theta = df = 0$. This proves the lemma. q.e.d.

Proof of Lemma 2. Suppose to the contrary that there exists a sequence $\tau_n \to +\infty$ for which $\gamma_{\tau_n} \to 0$. Let ω_n be a corresponding sequence of self-dual 2-forms on DW_{τ_n} such that $\Delta_{\tau_n}\omega_n = \gamma_{\tau_n}\omega_n$.

To fix the notation, we identify the cylindrical neck of DW_{τ_n} with $[T, 2\tau_n - T] \times (\mathbb{S}^3/G)$. We denote by W_1, W_2 the two components of $DW_{\tau_n} \setminus (T+1, 2\tau_n - T - 1) \times (\mathbb{S}^3/G)$. With this understood, we rescale each ω_n and pass to a subsequence if necessary, so that the following conditions hold:

$$\int_{W_2} |\omega_n|^2 \leq \int_{W_1} |\omega_n|^2 = 1.$$

By the interior elliptic estimates and the fact that $\Delta_{\tau_n}\omega_n = \gamma_{\tau_n}\omega_n$ with γ_{τ_n} bounded, there exists a constant $M_0 > 0$, such that $|\omega_n| \leq M_0$ holds on $W_1 \setminus (T + \frac{1}{2}, T] \times (\mathbb{S}^3/G)$ and $W_2 \setminus [2\tau_n - T - 1, 2\tau_n - T - \frac{1}{2}) \times (\mathbb{S}^3/G)$.

On the other hand, if we write $\omega_n = dt \wedge \alpha_{n,t} + *_3\alpha_{n,t}$ and set

$$f_n(t) \equiv \int_{\{t\}\times(\mathbb{S}^3/G)} |\omega_n|^2 = \int_{\mathbb{S}^3/G} |\alpha_{n,t}|^2$$

on the cylindrical neck $[T, 2\tau_n - T] \times (\mathbb{S}^3/G)$, then $\Delta_{\tau_n}\omega_n = \gamma_{\tau_n}\omega_n$ implies that

$$-\frac{d^2}{dt^2}\alpha_{n,t} + \Delta_3\alpha_{n,t} = \gamma_{\tau_n}\alpha_{n,t}$$

(here $\Delta_3 = d_3^* d_3 + d_3 d_3^*$ is the Laplacian on $\mathbb{S}^3/G$), and consequently, we have

$$\begin{aligned}
\frac{d^2 f_n}{dt^2} &= \int_{\mathbb{S}^3/G} \frac{d^2}{dt^2}|\alpha_{n,t}|^2 \\
&= 2\left(\int_{\mathbb{S}^3/G} \left\langle \frac{d^2}{dt^2}\alpha_{n,t}, \alpha_{n,t} \right\rangle + \int_{\mathbb{S}^3/G} \left|\frac{d}{dt}\alpha_{n,t}\right|^2\right) \\
&= 2\left(\int_{\mathbb{S}^3/G} \langle (\Delta_3 - \gamma_{\tau_n})\alpha_{n,t}, \alpha_{n,t}\rangle + \int_{\mathbb{S}^3/G} \left|\frac{d}{dt}\alpha_{n,t}\right|^2\right) > 0
\end{aligned}$$

for sufficiently large $n > 0$. By the maximum principle, f_n will attain its maximum at the end point $t = T$ or $2\tau_n - T$. This implies that for any $a \in [T, 2\tau_n - T - 1]$, the integral $\int_{[a,a+1]\times(\mathbb{S}^3/G)} |\omega_n|^2$ is uniformly bounded, hence by the interior elliptic estimates, $|\omega_n| \leq M_1$ for some constant $M_1 > 0$ on the cylindrical neck of DW_{τ_n}. Setting $M \equiv \max(M_0, M_1)$, then $|\omega_n| \leq M$ on DW_{τ_n}.

By the standard elliptic estimates, there exists a subsequence of ω_n (still denoted by ω_n for simplicity), and a self-dual 2-form ω on $\hat{W}$, such that $\omega_n \to \omega$ in C^∞ on any given compact subset of $\hat{W}$. In particular, the 2-form ω obeys (1) $\int_{\hat{W}\setminus[T+1,+\infty)\times(\mathbb{S}^3/G)} |\omega|^2 = 1$, and (2) $\Delta_g\omega = 0$ and $|\omega| \leq M$ on $\hat{W}$. The lemma is proved by observing that (2) above implies that $\omega = 0$, which contradicts (1) above. q.e.d.

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