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OPEN SETS OF MAXIMAL DIMENSION IN COMPLEX HYPERBOLIC QUASI-FUCHSIAN SPACE

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Abstract

Let π_1 be the fundamental group of a closed surface Σ of genus $g > 1$. One of the fundamental problems in complex hyperbolic geometry is to find all discrete, faithful, geometrically finite and purely loxodromic representations of π_1 into $\mathrm{SU}(2,1)$, (the triple cover of) the group of holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$. In particular, given a discrete, faithful, geometrically finite and purely loxodromic representation ρ_0 of π_1, can we find an open neighbourhood of ρ_0 comprising representations with these properties. We show that this is indeed the case when ρ_0 preserves a totally real Lagrangian plane.

1. Introduction

Let Σ be a closed surface of genus $g > 1$ and let $\pi_1 = \pi_1(\Sigma)$ denote its fundamental group. A specific choice of generators for π_1 is called a *marking*. The collection of marked representations of π_1 into a Lie group G up to conjugation will be denote $\mathrm{Hom}(\pi_1, G)/G$. We give $\mathrm{Hom}(\pi_1, G)/G$ the compact-open topology. This enables us to make sense of what it means for two representations to be close. In the cases we consider, the compact-open topology is equivalent to the l^2-topology on the relevant matrix group. Our main interest in this paper will be the case where $G = \mathrm{SU}(2,1)$ but, before we consider this case, we motivate our discussion by reviewing the better known cases when G is $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$.

Suppose that $\rho : \pi_1 \longrightarrow \mathrm{SL}(2,\mathbb{R})$ is a discrete and faithful representation of π_1. Then $\rho(\pi_1)$ is called *Fuchsian*. Also, $\rho(\pi_1)$ is necessarily geometrically finite and totally loxodromic (if Σ had punctures then this condition would be replaced with type-preserving, which requires that an element of $\rho(\pi_1)$ is parabolic if and only if it represents a peripheral curve). The group $\mathrm{SL}(2,\mathbb{R})$ is a double cover of the group of orientation

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preserving isometries of the hyperbolic plane. The quotient of the hyperbolic plane by $\rho(\pi_1)$ naturally corresponds to a hyperbolic structure on Σ. The collection of distinct, marked Fuchsian representations, up to conjugacy within $\mathrm{SL}(2,\mathbb{R})$, is the *Teichmüller space* of Σ, denoted $\mathcal{T} = \mathcal{T}(\Sigma) \subset \mathrm{Hom}\big(\pi_1,\, \mathrm{SL}(2,\mathbb{R})\big)/\mathrm{SL}(2,\mathbb{R})$. This has been studied extensively and is known to be a ball of real dimension $6g-6$. It also has a structure of a complex Banach manifold and is equipped with a Kähler metric (the well known Weil-Petersson metric) of negative holomorphic sectional curvature.

Instead of considering representations of π_1 into $\mathrm{SL}(2,\mathbb{R})$, we may consider representations to $\mathrm{SL}(2,\mathbb{C})$. If such a representation ρ is discrete, faithful, geometrically finite and totally loxodromic then $\rho(\pi_1)$ is *quasi-Fuchsian* (again in the presence of punctures purely loxodromic should be replaced with type-preserving). The collection of distinct, marked quasi-Fuchsian representations, up to conjugation in $\mathrm{SL}(2,\mathbb{C})$ is called *quasi-Fuchsian space* $\mathcal{Q} = \mathcal{Q}(\Sigma) \subset \mathrm{Hom}\big(\pi_1,\, \mathrm{SL}(2,\mathbb{C})\big)/\mathrm{SL}(2,\mathbb{C})$. A quasi-Fuchsian representation corresponds to a three dimensional hyperbolic structure on an interval bundle over Σ. According to a celebrated theorem of Bers [**2**], $\mathcal{Q}$ may be identified with the product of two copies of Teichmüller space, and so has dimension $12g-12$. Furthermore, $\mathcal{Q}$ has a rich geometrical and analytic structure. It is a complex manifold of dimension $6g-6$ and it is endowed with a hyper-Kähler metric whose induced complex symplectic form is the complexification of the Weil-Petersson metric on $\mathcal{T}$.

Motivated by these two examples, one may consider representations of π_1 into $\mathrm{SU}(2,1)$ up to conjugation, that is $\mathrm{Hom}\big(\pi_1, \mathrm{SU}(2,1)\big)/\mathrm{SU}(2,1)$. A representation in $\mathrm{Hom}\big(\pi_1, \mathrm{SU}(2,1)\big)/\mathrm{SU}(2,1)$ is said to be *complex hyperbolic quasi-Fuchsian* if it is discrete, faithful, geometrically finite and totally loxodromic (for surfaces with punctures the last condition should be type-preserving, see [**17**]). The group $\mathrm{SU}(2,1)$ is a triple cover of the holomorphic isometry group of complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$. Thus such a representation corresponds to a complex hyperbolic structure on a disc bundle over Σ.

We remark that if $\rho : \pi_1 \longrightarrow \mathrm{SU}(2,1)$ is totally loxodromic and $\rho(\pi_1)$ neither fixes a point of $\partial\mathbf{H}^2_{\mathbb{C}}$ nor preserves a totally geodesic subspace of $\mathbf{H}^2_{\mathbb{C}}$, then $\rho(\pi_1)$ is automatically discrete, see Corollary 4.5.2 of [**4**]. This constrasts with the case of representations to $\mathrm{SL}(2,\mathbb{C})$. In our definition of complex hyperbolic quasi-Fuchsian we have included the conditions that such a representation should be both discrete and totally loxodromic. We have chosen to do so both for clarity and to emphasise the similarity with the classical case of quasi-Fuchsian representations in $\mathrm{SL}(2,\mathbb{C})$. In our proof we verify discreteness directly.

Bowditch has discussed notions of geometrical finiteness for variable negative curvature in [**3**]. In particular, if Γ is a discrete subgroup of

$\mathrm{SU}(2,1)$ and $\Omega \subset \partial\mathbf{H}^2_{\mathbb{C}}$ is the domain of discontinuity of Γ then consider the orbifold $M_C(\Gamma) = \big(\mathbf{H}^2_{\mathbb{C}} \cup \Omega\big)/\Gamma$. Bowditch defines Γ to have property F1, that is Γ is *geometrically finite in the first sense*, if $M_C(\Gamma)$ has only finitely many topological ends, each of which is a parabolic end. In our context, Γ will be totally loxodromic and so will have property F1 provided $M_C(\Gamma)$ is a closed manifold.

The space of all marked complex hyperbolic quasi-Fuchsian representations, up to conjugacy, will be called *complex hyperbolic quasi-Fuchsian space* $\mathcal{Q}_{\mathbb{C}} = \mathcal{Q}_{\mathbb{C}}(\Sigma) \subset \mathrm{Hom}\big(\pi_1,\ \mathrm{SU}(2,1)\big)/\mathrm{SU}(2,1)$. Compared to Teichmüller space and quasi-Fuchsian space, relatively little is known about complex hyperbolic quasi-Fuchsian space $\mathcal{Q}_{\mathbb{C}}$.

There are two ways to make a Fuchsian representation act on $\mathbf{H}^2_{\mathbb{C}}$. These correspond to the two types of totally geodesic, isometric embeddings of the hyperbolic plane into $\mathbf{H}^2_{\mathbb{C}}$. Namely, totally real Lagrangian planes, which may be thought of as copies of $\mathbf{H}^2_{\mathbb{R}}$, and complex lines, which may be thought of as copies of $\mathbf{H}^1_{\mathbb{C}}$. If a discrete, faithful representation ρ is conjugate to a representation $\rho : \pi_1 \longrightarrow \mathrm{SO}(2,1) < \mathrm{SU}(2,1)$ then it preserves a Lagrangian plane and is called $\mathbb{R}$-*Fuchsian.* If a discrete, faithful representation ρ is conjugate to a representation $\rho : \pi_1 \longrightarrow \mathrm{S}\big(\mathrm{U}(1) \times \mathrm{U}(1,1)\big) < \mathrm{SU}(2,1)$ then it preserves a complex line and is called $\mathbb{C}$-*Fuchsian.* There is an important invariant of a representation $\rho : \pi_1 \longrightarrow \mathrm{SU}(2,1)$ called the *Toledo invariant* denoted $\tau(\rho)$. The main properties of the Toledo invariant are

(i) τ varies continuously with ρ,
(ii) $2-2g \le \tau(\rho) \le 2g-2$, see [**6**],
(iii) $\tau(\rho) \in 2\mathbb{Z}$, see [**15**],
(iv) ρ is $\mathbb{C}$-Fuchsian if and only if $|\tau(\rho)| = 2g-2$, see [**20**],
(v) if ρ is $\mathbb{R}$-Fuchsian then $\tau(\rho) = 0$, see [**15**].

Further properties of complex hyperbolic representations of surface groups which refer to the Toledo invariant are

(vi) for each even integer t with $2-2g \le t \le 2g-2$ there exists a discrete, faithful representation ρ of π_1 with $\tau(\rho) = t$, see [**15**],
(vii) if $\tau(\rho_1) = \tau(\rho_2)$ then ρ_1 and ρ_2 lie in the same component of $\mathrm{Hom}\big(\pi_1, \mathrm{SU}(2,1)\big)/\mathrm{SU}(2,1)$, see [**22**].

We remark that in the case where Σ has cusps then, in fact, $\tau(\rho)$ is a real number in the interval $\big[\chi(\Sigma),\ -\chi(\Sigma)\big]$ and for any real number t in this interval there exists a discrete, faithful representation ρ of $\pi_1(\Sigma)$ with $\tau(\rho) = t$, see [**17**]. Moreover, Dutenhefner and Gusevskii [**7**] have constructed an example of a discrete, faithful, type-preserving representation of the fundamental group of a particular punctured surface whose limit set is a wild knot. This means that it cannot be in the same component of the space of discrete faithful representations as a Fuchsian representation. It may well be possible to extend this example to the case of closed surfaces, which would lead to questions about the number of components of complex hyperbolic quasi-Fuchsian space (Xia's result [**22**], given in (vii) above, does not involve discreteness).

An immediate consequence of (i) and (iii) is that τ is locally constant and, together with (iv), implies that given a $\mathbb{C}$-Fuchsian representation ρ_0 any nearby representation ρ_t is also $\mathbb{C}$-Fuchsian. This result is known as the Toledo-Goldman rigidity theorem [**20**], [**13**]. In fact, the components of $\mathrm{Hom}\big(\pi_1, \mathrm{SU}(2,1)\big)/\mathrm{SU}(2,1)$ with $|\tau| = 2g-2$ have dimension $8g-6$ and the other components have dimension $16g-16$ (see Theorem 6 of [**13**]).

In this paper we begin with any $\mathbb{R}$-Fuchsian representation ρ_0 and we consider nearby representations ρ_t in $\mathrm{Hom}\big(\pi_1, \mathrm{SU}(2,1)\big)/\mathrm{SU}(2,1)$. Our main result is:

Theorem 1.1. *Let Σ be a closed surface of genus g with fundamental group $\pi_1 = \pi_1(\Sigma)$. Let $\rho_0 : \pi_1 \longrightarrow \mathrm{SU}(2,1)$ be an $\mathbb{R}$-Fuchsian representation of π_1. Then there exists an open neighbourhood $U = U(\rho_0)$ of ρ_0 in $\mathrm{Hom}\big(\pi_1, \mathrm{SU}(2,1)\big)/\mathrm{SU}(2,1)$ so that any representation ρ_t in U is complex hyperbolic quasi-Fuchsian (that is discrete, faithful, geometrically finite and totally loxodromic).*

This theorem may be thought of as an instance of structural stability, see Sullivan [**19**]. However, it is not clear how to generalise the details of Sullivan's method from subgroups of $\mathrm{SL}(2,\mathbb{C})$ to subgroups of $\mathrm{SU}(2,1)$. Therefore we use a different method.

An immediate consequence of Theorem 1.1 is:

Corollary 1.2. *There are open sets of dimension $16g-16$ in $\mathcal{Q}_{\mathbb{C}}(\Sigma)$.*

Up to now, families of complex hyperbolic quasi-Fuchsian groups have only been constructed by varying a particular geometrical construction, see for example [**16**], [**17**], [**9**], [**10**], [**11**], [**18**]. By contrast, in this paper we only use the hypothesis that ρ_t and ρ_0 are nearby representations. From this information we must make a geometrical construction of a fundamental domain. To go from algebra to geometry (and back again) we use the theorem of Falbel and Zocca [**12**], Theorem 2.1. We prove Theorem 1.1 by first constructing a fundamental domain $\boldsymbol{\Delta}_0$ in $\mathbf{H}^2_{\mathbb{C}}$ for $\rho_0(\pi_1)$ and then showing that for any other representation ρ_t sufficiently close to ρ_0 we may construct a fundamental domain $\boldsymbol{\Delta}_t$ for $\rho_t(\pi_1)$. By sufficiently close, we mean that there exists an $\epsilon > 0$ so that the generators of $\rho_t(\pi_1)$ are ϵ-close to the generators of $\rho_0(\pi_1)$ in the l^2-topology on $\mathrm{SU}(2,1)$.

Constructing fundamental domains in complex hyperbolic space is challenging because, unlike the case of constant curvature, there are no totally geodesic real hypersurfaces. Thus, before constructing a fundamental polyhedron we must choose the class of real hypersurfaces containing its faces. The most usual method of constructing a fundamental domain in complex hyperbolic space involves domains whose boundary is made up of pieces of bisectors. In particular, this is the case for the construction of Dirichlet domains. This idea goes back to Giraud and was developed further by Mostow and Goldman (see [**14**] and the references therein), and see [**16**], [**17**] for other examples of fundamental domains bounded by bisectors. Other classes of hypersurfaces used to build fundamental domains are $\mathbb{C}$-spheres [**12**] and $\mathbb{R}$-spheres [**18**] (for the relationship between $\mathbb{C}$-spheres and $\mathbb{R}$-spheres see [**11**]).

Since bisectors are rather badly adapted to $\mathbb{R}$-Fuchsian representations, we have chosen to introduce a new class of hypersurfaces. Just as bisectors are foliated by slices that are complex lines so our hypersurfaces are foliated by Lagrangian planes. These hypersurfaces resemble a pack of (infinitely many) playing cards, each Lagrangian plane representing a card. Therefore we call we call such hypersurfaces *packs*. The boundaries of packs are foliated by $\mathbb{R}$-circles and so are closely related to Schwartz' $\mathbb{R}$-spheres [**18**] and examples of packs (with no twist) were introduced by Will [**21**], who calls them $\mathbb{R}$-balls. Both Schwartz and Will use these objects to construct fundamental domains. The relationship between bisectors and packs is an example of the duality, which resembles mirror symmetry, between complex and real objects in complex hyperbolic space, see the discussion in the introduction to [**11**]. The polyhedra $\boldsymbol{\Delta}_0$ and $\boldsymbol{\Delta}_t$ we construct have boundaries that are made up of pieces of packs. In order to show that $\rho_t(\pi_1)$ is complex hyperbolic quasi-Fuchsian we give a version of Poincaré's polyhedron theorem, Theorem 4.2, for such polyhedra (this should be compared with [**8**]).

2. Preliminaries

2.1. Complex Hyperbolic Space. Let $\mathbb{C}^{2,1}$ be the vector space $\mathbb{C}^3$ with the Hermitian form of signature $(2,1)$ given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* J \mathbf{z} = z_1\overline{w}_1 + z_2\overline{w}_2 - z_3\overline{w}_3.$$

Its matrix is

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Consider the following subspaces of $\mathbb{C}^{2,1}$:

$$V_- = \Big\{ \mathbf{z} \in \mathbb{C}^{2,1} \ : \ \langle \mathbf{z}, \mathbf{z} \rangle < 0 \Big\},$$
$$V_0 = \Big\{ \mathbf{z} \in \mathbb{C}^{2,1} - \{0\} \ : \ \langle \mathbf{z}, \mathbf{z} \rangle = 0 \Big\}.$$

Let $\mathbb{P} : \mathbb{C}^{2,1} - \{0\} \longrightarrow \mathbb{C}P^2$ be the canonical projection onto complex projective space. Then *complex hyperbolic space* $\mathbf{H}^2_{\mathbb{C}}$ is defined to be $\mathbb{P}V_-$ and its boundary $\partial\mathbf{H}^2_{\mathbb{C}}$ is $\mathbb{P}V_0$. Specifically, $\mathbb{C}^{2,1} - \{0\}$ may be covered with three charts H_1, H_2, H_3 where H_j comprises those points in $\mathbb{C}^{2,1} - \{0\}$ for which $z_j \neq 0$. It is clear that V_- and V_0 are both contained in H_3. The canonical projection from H_3 to $\mathbb{C}^2$ is given by $\mathbb{P}(\mathbf{z}) = (z_1/z_3,\, z_2/z_3)$. Therefore we can write $\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}(V_-)$ and $\partial\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}(V_0)$ as

$$\mathbf{H}^2_{\mathbb{C}} = \left\{(z_1,\, z_2) \in \mathbb{C}^2 \;:\; |z_1|^2 + |z_2|^2 < 1\right\},$$
$$\partial\mathbf{H}^2_{\mathbb{C}} = \left\{(z_1,\, z_2) \in \mathbb{C}^2 \;:\; |z_1|^2 + |z_2|^2 = 1\right\}.$$

In other words, $\mathbf{H}^2_{\mathbb{C}}$ is the unit ball in $\mathbb{C}^2$ and likewise $\partial\mathbf{H}^2_{\mathbb{C}}$ is the unit sphere S^3.

Conversely, given a point z of $\mathbb{C}^2 = \mathbb{P}(H_3) \subset \mathbb{C}P^2$ we may lift $z = (z_1, z_2)$ to a point $\mathbf{z}$ in $H_3 \subset \mathbb{C}^{2,1}$, called the *standard lift* of z, by writing $\mathbf{z}$ in non-homogeneous coordinates as

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix}.$$

The *Bergman metric* on $\mathbf{H}^2_{\mathbb{C}}$ is defined by the distance function ρ given by the formula

$$\cosh^2\left(\frac{\rho(z,w)}{2}\right) = \frac{\langle \mathbf{z}, \mathbf{w}\rangle \langle \mathbf{w}, \mathbf{z}\rangle}{\langle \mathbf{z}, \mathbf{z}\rangle \langle \mathbf{w}, \mathbf{w}\rangle} = \frac{\left|\langle \mathbf{z}, \mathbf{w}\rangle\right|^2}{|\mathbf{z}|^2|\mathbf{w}|^2}$$

where $\mathbf{z}$ and $\mathbf{w}$ in V_- are the standard lifts of z and w in $\mathbf{H}^2_{\mathbb{C}}$ and $|\mathbf{z}| = \sqrt{-\langle \mathbf{z}, \mathbf{z}\rangle}$. Alternatively,

$$ds^2 = -\frac{4}{\langle \mathbf{z}, \mathbf{z}\rangle^2} \det \begin{bmatrix} \langle \mathbf{z}, \mathbf{z}\rangle & \langle d\mathbf{z}, \mathbf{z}\rangle \\ \langle \mathbf{z}, d\mathbf{z}\rangle & \langle d\mathbf{z}, d\mathbf{z}\rangle \end{bmatrix}.$$

The holomorphic sectional curvature of $\mathbf{H}^2_{\mathbb{C}}$ equals to -1 and its real sectional curvature is pinched between -1 and $-1/4$.

There are no totally geodesic, real hypersurfaces of $\mathbf{H}^2_{\mathbb{C}}$, but there are two kinds of totally geodesic 2-dimensional subspaces of complex hyperbolic space (see Section 3.1.11 of [**14**]), namely:

(i) complex lines L, which have constant curvature -1, and
(ii) totally real Lagrangian planes R, which have constant curvature $-1/4$.

Both of these subspaces are isometrically embedded copies of the hyperbolic plane.

2.2. Isometries. Let $\mathrm{U}(2,1)$ be the group of unitary matrices for the Hermitian form $\langle\cdot,\cdot\rangle$. Each such matrix A satisfies the relation $A^{-1} = JA^*J$ where $A^* = \overline{A}^T$.

The full group of holomorphic isometries of complex hyperbolic space is the *projective unitary group* $\mathrm{PU}(2,1) = \mathrm{U}(2,1)/\mathrm{U}(1)$, where $\mathrm{U}(1) = \{e^{i\theta}I, \theta \in [0, 2\pi)\}$ and I is the 3×3 identity matrix. For our purposes we shall consider instead the group $\mathrm{SU}(2,1)$ of matrices which are unitary with respect to $\langle\cdot,\cdot\rangle$, and have determinant 1. Therefore $\mathrm{PU}(2,1) = \mathrm{SU}(2,1)/\{I, \omega I, \omega^2 I\}$, where ω is a non real cube root of unity, and so $\mathrm{SU}(2,1)$ is a 3-fold covering of $\mathrm{PU}(2,1)$.

Every complex line L is the image under some $A \in \mathrm{SU}(2,1)$ of the complex line where the first coordinate is zero. The subgroup of $\mathrm{SU}(2,1)$ stabilising this particular complex line is thus the group of block diagonal matrices $\mathrm{S}\big(\mathrm{U}(1)\times\mathrm{U}(1,1)\big) < \mathrm{SU}(2,1)$. Similarly, every Lagrangian plane is the image under some element of $\mathrm{SU}(2,1)$ of the Lagrangian plane $R_{\mathbb{R}}$ where both coordinates are real, called the standard real Lagrangian plane. This is preserved by the subgroup of $\mathrm{SU}(2,1)$ comprising matrices with real entries, that is $\mathrm{SO}(2,1) < \mathrm{SU}(2,1)$.

Holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$ are classified as follows.

(i) An isometry is *loxodromic* if it fixes exactly two points of $\partial\mathbf{H}^2_{\mathbb{C}}$.
(ii) An isometry is *parabolic* if it fixes exactly one point of $\partial\mathbf{H}^2_{\mathbb{C}}$.
(iii) An isometry is *elliptic* if it fixes at least one point of $\mathbf{H}^2_{\mathbb{C}}$.

The complex conjugation map $\iota_{\mathbb{R}} : (z_1, z_2) \longmapsto (\overline{z}_1, \overline{z}_2)$ is an involution of $\mathbf{H}^2_{\mathbb{C}}$ fixing the standard real Lagrangian plane $R_{\mathbb{R}}$. It too is an isometry. Indeed any anti-holomorphic isometry of $\mathbf{H}^2_{\mathbb{C}}$ may be written as $\iota_{\mathbb{R}}$ followed by some element of $\mathrm{PU}(2,1)$. Any Lagrangian plane may be written as $R = B(R_{\mathbb{R}})$ for some $B \in \mathrm{SU}(2,1)$ and so $\iota = B\iota_{\mathbb{R}}B^{-1}$ is an anti-holomorphic isometry of $\mathbf{H}^2_{\mathbb{C}}$ fixing R.

Falbel and Zocca [**12**] have used involutions fixing Lagrangian planes to give the following characterisation of elements of $\mathrm{SU}(2,1)$:

Theorem 2.1. *Any element C of $\mathrm{SU}(2,1)$ may be written as $C = \iota_1 \circ \iota_0$ where ι_0 and ι_1 are involutions fixing Lagrangian planes R_0 and R_1 respectively. Moreover*

(i) *$C = \iota_1 \circ \iota_0$ is loxodromic if and only if R_0 and R_1 are disjoint;*
(ii) *$C = \iota_1 \circ \iota_0$ is parabolic if and only if R_0 and R_1 intersect in exactly one point of $\partial\mathbf{H}^2_{\mathbb{C}}$;*
(iii) *$C = \iota_1 \circ \iota_0$ is elliptic if and only if R_0 and R_1 intersect in at least one point of $\mathbf{H}^2_{\mathbb{C}}$.*

We conclude this section by considering the case where C is loxodromic in more detail. Since elements of $\mathrm{SU}(2,1)$ preserve the Hermitian form, it is not hard to show that if μ is an eigenvalue of $A \in \mathrm{SU}(2,1)$ then so is $\overline{\mu}^{-1}$ (Lemma 6.2.5 of [**14**]). From this fact we find that A is loxodromic if and only if one of its eigenvalues μ satisfies $|\mu| > 1$. In particular, if $|\mathrm{tr}(A)| > 3$ then A is loxodromic. We will use this fact repeatedly. Goldman gives a more precise statement in Theorem 6.2.4 of [**14**], but we will not need this level of detail.

If $C \in \mathrm{SU}(2,1)$ is loxodromic then one of its eigenvalues is $\mu = e^{\delta - i\phi}$ where $\delta > 0$ and $\phi \in (-\pi, \pi]$. Hence another eigenvalue of C is $\overline{\mu}^{-1} = e^{-\delta - i\phi}$. Since $\det(C) = 1$, its third eigenvalue must be $e^{2i\phi}$. Therefore

$\mathrm{tr}(C) = 2\cosh(\delta)e^{-i\phi} + e^{2i\phi}$. The eigenvectors corresponding to μ and $\overline{\mu}^{-1}$ span a complex line L in $\mathbf{H}^2_{\mathbb{C}}$. This line is called the *complex axis* of C and is written $L = L_C = \mathrm{Ax}(C)$. In fact we may write

$$C = Q \begin{bmatrix} \cosh(\delta)e^{-i\phi} & 0 & \sinh(\delta)e^{-i\phi} \\ 0 & e^{2i\phi} & 0 \\ \sinh(\delta)e^{-i\phi} & 0 & \cosh(\delta)e^{-i\phi} \end{bmatrix} Q^{-1} \tag{2.1}$$

for some $Q \in \mathrm{SU}(2,1)$. If C lies in $\mathrm{SO}(2,1)$ and corresponds to a a loxodromic isometry of the hyperbolic plane then $\phi = 0$ and so $\mathrm{tr}(C) = 2\cosh(\delta)+1$ is real and greater than 3. (If $\phi = \pi$ then C corresponds to a hyperbolic glide reflection on $\mathbf{H}^2_{\mathbb{R}}$ and $\mathrm{tr}(C) = -2\cosh(\delta) + 1 < -1$.)

Lemma 2.2. *Suppose that $C \in \mathrm{SU}(2,1)$ is loxodromic with eigenvalues $e^{\delta-i\phi}$, $e^{-\delta-i\phi}$, $e^{2i\phi}$ then C may be written as*

$$C = e^{2i\phi}I + \sinh(\delta)e^{-i\phi}E + \big(\cosh(\delta)e^{-i\phi} - e^{2i\phi}\big)E^2$$

for some matrix E satisfying $E^3 = E$ and $JE^*J = -E$.

Proof. This immediately follows from (2.1) writing

$$E = Q \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} Q^{-1}.$$

q.e.d.

The group $\mathrm{SU}(2,1)$ is a topological space equipped with the compact-open topology. This is equivalent to the l^2-topology on $\mathrm{SU}(2,1) \subset \mathbb{C}^9$ and thus $\mathrm{SU}(2,1)$ is a Hilbert space with inner product given by

$$\langle\!\langle A, B\rangle\!\rangle = \Re\big(\mathrm{tr}(AB^*)\big)$$

for every $A, B \in \mathrm{SU}(2,1)$. Let $\|\cdot\|$ denote the respective l^2-norm. We note that for every $A \in \mathrm{SU}(2,1)$ we have $\|A\| = \|A^{-1}\|$.

The Lie algebra $\mathfrak{su}(2,1)$ of the complex Lie group $\mathrm{SU}(2,1)$ consists of matrices D satisfying the relations $JD^*J = -D$ and $\mathrm{tr}(D) = 0$. Actually, every element of $\mathfrak{su}(2,1)$ is a zero trace matrix of the form

$$D = \begin{bmatrix} D' & z^* \\ z & i\theta \end{bmatrix},$$

where $D' \in \mathfrak{u}(2)$, $\theta \in \mathbb{R}$, z is a vector in $\mathbb{C}^2$ and z^* is its Hermitian transpose, see page 103 of [**14**]. In what follows, $\mathfrak{su}(2,1)$ will also be considered as a Hilbert space equipped with the l^2-norm. The mapping

$$\exp(D) = \sum_{n=0}^{\infty} \frac{D^n}{n!}$$

takes $D \in \mathfrak{su}(2,1)$ to $\mathrm{SU}(2,1)$ and is called the *exponential mapping*.

Lemma 2.3. *Suppose that C and E are as given in Lemma 2.2. Then $C = \exp(D)$ where $D = 2i\phi I + \delta E - 3i\phi E^2$. Moreover, $E \in \mathfrak{su}(2,1)$.*

Proof. From the proof of Lemma 2.2 it is clear that E and $2iI - 3iE^2$ are in $\mathfrak{su}(2,1)$. Using $E^3 = E$ and expanding we obtain

$$\begin{aligned}\exp(D) &= \exp(2i\phi I + \delta E - 3i\phi E^2)\\ &= \exp(\delta E)\exp(2i\phi I - 3i\phi E^2)\\ &= \Big(I + \sinh(\delta)E + \big(\cosh(\delta) - 1\big)E^2\Big)\Big(e^{2i\phi}I + e^{-i\phi}E^2 - e^{2i\phi}E^2\Big)\\ &= e^{2i\phi}I + \sinh(\delta)e^{-i\phi}E + \big(\cosh(\delta)e^{-i\phi} - e^{2i\phi}\big)E^2.\end{aligned}$$

q.e.d.

2.3. Projection onto Lagrangian planes. In this section we consider totally real Lagrangian planes R. We discuss orthogonal projection Π_R onto R and its fibres $\Pi^{-1}(z)$. First, following Goldman, we give a formula for the midpoint of two points of complex hyperbolic space.

Proposition 2.4. *Let $\mathbf{z}$, $\mathbf{w}$ be any points of $V_- \subset \mathbb{C}^{2,1}$ and $z = \mathbb{P}\mathbf{z}$, $w = \mathbb{P}\mathbf{w}$ be the corresponding points of $\mathbf{H}^2_{\mathbb{C}}$. Let*

$$\mathbf{m} = \frac{1}{|\mathbf{z}|}\,\mathbf{z} - \frac{\langle \mathbf{z},\, \mathbf{w}\rangle}{\big|\langle \mathbf{z},\, \mathbf{w}\rangle\big|\,|\mathbf{w}|}\,\mathbf{w}. \tag{2.2}$$

Then $\mathbf{m} \in V_-$ and, writing $m = \mathbb{P}\mathbf{m}$, we have $\rho(m,z) = \rho(m,w) = \rho(z,w)/2$.

If m is as defined in Proposition 2.4 then we call m the *midpoint* of z and w (see Exercise 3.1.4 of [**14**]).

Proof. First, observe that

$$\begin{aligned}\langle \mathbf{m},\, \mathbf{m}\rangle &= -2 - \frac{2\big|\langle \mathbf{z},\, \mathbf{w}\rangle\big|}{|\mathbf{z}|\,|\mathbf{w}|}\\ &= -2\Big(1 + \cosh\big(\rho(z,w)/2\big)\Big) = -4\cosh^2\Big(\rho(z,w)/4\Big).\end{aligned}$$

Thus $\mathbf{m} \in V_-$ and $m = \mathbb{P}\mathbf{m} \in \mathbf{H}^2_{\mathbb{C}}$ and we write $|\mathbf{m}| = \sqrt{-\langle \mathbf{m},\, \mathbf{m}\rangle} = 2\cosh\big(\rho(z,w)/4\big)$. Moreover,

$$\begin{aligned}\langle \mathbf{m},\, \mathbf{z}\rangle &= \frac{\langle \mathbf{z},\, \mathbf{z}\rangle}{|\mathbf{z}|} - \frac{\langle \mathbf{z},\, \mathbf{w}\rangle\langle \mathbf{w}, \mathbf{z}\rangle}{|\langle \mathbf{z},\, \mathbf{w}\rangle|\,|\mathbf{w}|}\\ &= -|\mathbf{z}| - \frac{|\langle \mathbf{z},\, \mathbf{w}\rangle|}{|\mathbf{w}|}\\ &= -|\mathbf{z}|\Big(1 + \cosh\big(\rho(z,w)/2\big)\Big)\\ &= -|\mathbf{z}|\,2\cosh^2\Big(\rho(z,w)/4\Big)\\ &= -|\mathbf{z}|\,|\mathbf{m}|\cosh\Big(\rho(z,w)/4\Big).\end{aligned}$$

Therefore

$$\cosh\Bigl(\rho(m,z)/2\Bigr) = \frac{\bigl|\langle \mathbf{m},\,\mathbf{z}\rangle\bigr|}{|\mathbf{z}|\,|\mathbf{m}|} = \cosh\Bigl(\rho(z,w)/4\Bigr).$$

Similarly

$$\begin{aligned}\langle \mathbf{m},\,\mathbf{w}\rangle &= \frac{\langle \mathbf{z},\,\mathbf{w}\rangle}{|\mathbf{z}|} - \frac{\langle \mathbf{z},\,\mathbf{w}\rangle\langle \mathbf{w},\mathbf{w}\rangle}{|\langle \mathbf{z},\,\mathbf{w}\rangle|\,|\mathbf{w}|} \\ &= \frac{\langle \mathbf{z},\,\mathbf{w}\rangle}{|\langle \mathbf{z},\,\mathbf{w}\rangle|}\left(|\mathbf{w}| + \frac{|\langle \mathbf{z},\,\mathbf{w}\rangle|}{|\mathbf{z}|}\right) \\ &= \frac{\langle \mathbf{z},\,\mathbf{w}\rangle}{|\langle \mathbf{z},\,\mathbf{w}\rangle|}\,|\mathbf{w}|\,|\mathbf{m}|\cosh\Bigl(\rho(z,w)/4\Bigr)\end{aligned}$$

and so

$$\cosh\Bigl(\rho(m,w)/2\Bigr) = \frac{\bigl|\langle \mathbf{m},\,\mathbf{w}\rangle\bigr|}{|\mathbf{w}|\,|\mathbf{m}|} = \cosh\Bigl(\rho(z,w)/4\Bigr).$$

Hence $\rho(m,z) = \rho(m,w) = \rho(z,w)/2$ as required. q.e.d.

We use Proposition 2.4 to derive a formula for the orthogonal projection onto a Lagrangian plane R (see Section 3.3.6 of [**14**]). Let ι_R denote the (anti-holomorphic) reflection in R. Then the orthogonal projection $\Pi_R(z)$ of any $z \in \mathbf{H}^2_{\mathbb{C}}$ onto R is defined to be the midpoint m of the points z and $\iota_R(z)$. That is, if $\mathbf{z} \in V_-$ is a lift of z then

$$\Pi_R(z) = \mathbb{P}\left(\frac{1}{|\mathbf{z}|}\,\mathbf{z} - \frac{\langle \mathbf{z},\,\iota_R(\mathbf{z})\rangle}{|\langle \mathbf{z},\,\iota_R(\mathbf{z})\rangle|\,|\iota_R(\mathbf{z})|}\,\iota_R(\mathbf{z})\right).$$

Proposition 2.5. *Let R be a Lagrangian plane stabilised by the subgroup G_R of $\mathrm{SU}(2,1)$. Then, for every $A \in G_R$*

$$A \circ \Pi_R = \Pi_R \circ A.$$

Consequently, if $w \in R$,

$$\Pi_R^{-1}\bigl(A(w)\bigr) = A\left(\Pi_R^{-1}(w)\right).$$

Proof. Let $z \in \mathbf{H}^2_{\mathbb{C}}$. Then, $\Pi_R(z) = m$ is the midpoint of z and $\iota(z)$. Hence

$$\rho\bigl(A(z), A(m)\bigr) = \rho(z,m) = \rho\bigl(\iota(z), m\bigr) = \rho\bigl(A\iota(z), A(m)\bigr).$$

Also

$$\rho\bigl(A(z), A\iota(z)\bigr) = \rho\bigl(z, \iota(z)\bigr) = 2\rho(m,z) = 2\rho\bigl(A(m), A(z)\bigr).$$

Thus $A(m)$ is the midpoint of $A(z)$ and $A\iota(z)$. But since $A\iota(z) = \iota A(z)$ we see that

$$\Pi_R\bigl(A(z)\bigr) = A(m) = A\bigl(\Pi_R(z)\bigr).$$

Now suppose that $w \in R$ and choose any z with $\Pi_R(z) = w$. Then

$$A(w) = A\Pi_R(z) = \Pi_R A(z).$$

Thus $A(z) \in {\Pi_R}^{-1}A(w)$ and so $A{\Pi_R}^{-1}(w) \subset {\Pi_R}^{-1}A(w)$. Similarly if z' is chosen so that $\Pi_R(z') = A(w)$ then

$$w = A^{-1}\Pi_R(z') = \Pi_R A^{-1}(z')$$

and so $z' \in A{\Pi_R}^{-1}(w)$. Hence ${\Pi_R}^{-1}A(w) \subset A{\Pi_R}^{-1}(w)$. q.e.d.

We consider the special case where R is the standard real Lagrangian plane $R_{\mathbb{R}}$, that is

$$R_{\mathbb{R}} = \mathbf{H}^2_{\mathbb{R}} = \big\{(z_1, z_2) \in \mathbf{H}^2_{\mathbb{C}} : \Im(z_1) = \Im(z_2) = 0\big\}$$

and we denote orthogonal projection onto $R_{\mathbb{R}}$ by $\Pi_{\mathbb{R}}$. Consider a point $z = (z_1, z_2) \in \mathbf{H}^2_{\mathbb{C}}$. Then reflection $\iota_{\mathbb{R}}$ in $R_{\mathbb{R}}$ is given by

$$\iota_{\mathbb{R}}(z) = \overline{z} = \big(\overline{z}_1, \overline{z}_2\big).$$

Following Goldman (page 108 of [**14**]), we write

$$\eta(z)^2 = -\langle \mathbf{z}, \iota_{\mathbb{R}}\mathbf{z}\rangle = 1 - {z_1}^2 - {z_2}^2.$$

Observe that $0 < 1 - |z_1|^2 - |z_2|^2 \le \Re\big(1 - {z_1}^2 - {z_2}^2\big) = \Re\big(\eta(z)^2\big)$, and in particular, $\eta(z)^2 \neq 0$.

Applying (2.2) we find that the midpoint $m = (m_1, m_2)$ of z and $\iota_{\mathbb{R}}(z)$ is given by

$$m_k = \frac{\big|\eta(z)^2\big| z_k + \eta(z)^2 \overline{z}_k}{\big|\eta(z)^2\big| + \eta(z)^2} = 2\big|\eta(z)^2\big| \frac{\Re\Big(\overline{z}_k\big(\big|\eta(z)^2\big| + \eta(z)^2\big)\Big)}{\Big|\big|\eta(z)^2\big| + \eta(z)^2\Big|},$$

for $k = 1$, 2. Clearly, m lies on $R_{\mathbb{R}}$, and if $z \in R_{\mathbb{R}}$, then $\Pi_{\mathbb{R}}(z) = z$.

Corollary 2.6. $\Pi_{\mathbb{R}}$ *is real analytic.*

The subgroup of $\mathrm{SU}(2,1)$ stabilising $R_{\mathbb{R}}$ comprises those matrices with all real entries, that is $\mathrm{SO}(2,1)$ the isometry group of the hyperbolic plane. Proposition 2.5 immediately implies that $\Pi_{\mathbb{R}}$ commutes with all elements of $\mathrm{SO}(2,1)$.

Proposition 2.7. *If* $R_{\mathbb{R}}$ *is the standard real Lagrangian plane*

$$R_{\mathbb{R}} = \big\{(z_1, z_2) \in \mathbf{H}^2_{\mathbb{C}} : \Im(z_1) = \Im(z_2) = 0\big\},$$

then ${\Pi_{\mathbb{R}}}^{-1}(0,0)$ *is the purely imaginary Lagrangian plane*

$$R_{\mathbb{J}} = \big\{(z_1, z_2) \in \mathbf{H}^2_{\mathbb{C}} : \Re(z_1) = \Re(z_2) = 0\big\}.$$

Proof. If z_1 and z_2 are both purely imaginary then $\eta(z)^2 = 1 - z_1^2 - z_2^2$ is a positive real number. It is clear from the above construction that

$$m_1 = \Re(\overline{z}_1) = 0, \quad m_2 = \Re(\overline{z}_2) = 0.$$

Thus the Lagrangian plane $R_{\mathbb{J}}$ is contained in ${\Pi_{\mathbb{R}}}^{-1}(0,0)$.

Conversely, the set $\Pi_{\mathbb{R}}^{-1}(0,0)$ is the collection of points $(z_1, z_2) \in \mathbf{H}^2_{\mathbb{C}}$ satisfying

$$\big|\eta(z)^2\big| z_1 + \eta(z)^2 \overline{z}_1 = \big|\eta(z)^2\big| z_2 + \eta(z)^2 \overline{z}_2 = 0.$$

When z_1 and z_2 are both non-zero, these two equations are equivalent to

$$\frac{{z_1}^2}{|z_1|^2} = \frac{{z_2}^2}{|z_2|^2} = \frac{-\eta(z)^2}{\left|\eta(z)\right|^2}.$$

Writing ${z_1}^2 = |z_1|^2 e^{i\phi}$ and ${z_2}^2 = |z_2|^2 e^{i\phi}$ we obtain

$$\left|\eta(z)\right|^2 = -\eta(z)^2 e^{-i\phi} = -\big(1 - {z_1}^2 - {z_2}^2\big)e^{-i\phi} = -e^{-i\phi} + |z_1|^2 + |z_2|^2.$$

Therefore $e^{i\phi} \in \mathbb{R}$. Since $|z_1|^2 + |z_2|^2 < 1$ we see that $e^{i\phi} = -1$. Thus z_1 and z_2 are both purely imaginary. When one of z_1 or z_2 is zero, a similar argument shows that the other one is purely imaginary (or zero). Thus ${\Pi_{\mathbb{R}}}^{-1}(0,0)$ is contained in the Lagrangian plane $R_{\mathbb{J}}$. q.e.d.

Using the fact that $\mathrm{SU}(2,1)$ acts transitively on the set of Lagrangian planes in $\mathbf{H}^2_{\mathbb{C}}$ we immediately have:

Corollary 2.8. *Let w be any point on the Lagrangian plane R. Then ${\Pi_R}^{-1}(w)$ is a Lagrangian plane.*

Corollary 2.9. *For every Lagrangian plane R, the orthogonal projection Π_R is real analytic.*

3. Packs

In this section we introduce real analytic 3-(real) dimensional submanifolds of complex hyperbolic space which are foliated by Lagrangian planes. These submanifolds can be considered as the counterparts of *bisectors*. (For an extensive treatment of the latter, see [**14**]).

Let C be a loxodromic map in $\mathrm{SU}(2,1)$ given, as in Lemma 2.2, by

$$C = e^{2i\phi} I + \sinh(\delta)e^{-i\phi}E + \big(\cosh(\delta)e^{-i\phi} - e^{2i\phi}\big)E^2 = \exp(D)$$

where $E \in \mathfrak{su}(2,1)$ satisfies $E^3 = E$. For any $x \in \mathbb{R}$ define C^x by

$$C^x = e^{2ix\phi} I + \sinh(x\delta)e^{-xi\phi}E + \big(\cosh(x\delta)e^{-xi\phi} - e^{2ix\phi}\big)E^2 = \exp(xD).$$

Observe that C^x has the same eigenvectors as C, but its eigenvalues are the eigenvalues of C raised to the xth power. Hence we immediately see that C^x is a loxodromic element of $\mathrm{SU}(2,1)$ for all $x \in \mathbb{R} - \{0\}$ and $C^0 = I$. Moreover, for any integer n it is clear that C^n agrees with the usual notion of the nth power of C. This justifies the use of a superscript.

Proposition 3.1. *Let R_0 and R_1 be disjoint Lagrangian planes in $\mathbf{H}^2_{\mathbb{C}}$ and let ι_0 and ι_1 be the respective inversions. Consider $C = \iota_1\iota_0$ (which is loxodromic map by Theorem 2.1) and its powers C^x for each $x \in \mathbb{R}$. Then:*

(i) *ι_x defined by $C^x = \iota_x\iota_0$ is inversion in a Lagrangian plane $R_x = C^{x/2}(R_0)$.*

(ii) *R_x intersects the complex axis L_C of C orthogonally in a geodesic γ_x.*
(iii) *The geodesics γ_x are the leaves of a foliation of L_C.*
(iv) *For each $x \neq y \in \mathbb{R}$, R_x and R_y are disjoint.*

Proof. Since $\iota_0 C \iota_0 = C^{-1}$ we also have $\iota_0 C^x \iota_0 = C^{-x}$. Thus $\iota_x = C^x \iota_0$ has order 2 and so is involution in a Lagrangian plane. Then inversion in $C^{x/2}(R_0)$ is

$$\begin{aligned} C^{x/2}\iota_0 C^{-x/2} &= (\iota_{x/2}\iota_0)\iota_0(\iota_0\iota_{x/2}) = \iota_{x/2}\iota_0\iota_{x/2} \\ &= (\iota_{x/2}\iota_0)^2\iota_0 = (C^{x/2})^2\iota_0 = C^x\iota_0 = \iota_x. \end{aligned}$$

Part (i) follows by construction. Likewise, parts (ii) and (iii) follow immediately. Finally, $\iota_x\iota_0 = C^x$ and $\iota_y\iota_0 = C^y$ and so $\iota_x\iota_y = C^x C^{-y} = C^{x-y}$ which is loxodromic, proving (iv). q.e.d.

Definition 3.2. Given disjoint Lagrangian planes R_0 and R_1, then for each $x \in \mathbb{R}$ let R_x be the Lagrangian plane constructed in Proposition 3.1. Define

$$P = P(R_0, R_1) = \bigcup_{x \in \mathbb{R}} R_x.$$

Then P is a real analytic 3-submanifold which we call the *pack* determined by R_0 and R_1. We call $\gamma = \mathrm{Ax}(\iota_1\iota_0)$ the *spine* of P and the Lagrangian planes R_x for $x \in \mathbb{R}$ the *slices* of P.

Observe that P contains L, the complex line containing γ, the spine of P. We remark that packs are analogous to bisectors, but with Lagrangian planes for slices rather than complex lines. The following proposition is obvious from the construction and emphasises the similarity between bisectors and packs (compare it with Section 5.1.2 of [**14**]). The definition of packs associated to loxodromic maps C that preserve a Lagrangian plane (that is with $\phi = 0$) was given by Will [**21**].

Proposition 3.3. *Let P be a pack. Then P is homeomorphic to a 3-ball whose boundary lies in $\partial\mathbf{H}^2_{\mathbb{C}}$. Moreover, $\mathbf{H}^2_{\mathbb{C}} - P$, the complement of P, has two components, each homeomorphic to a 4-ball.*

We remark that the boundary of P contains the boundary of the complex line L and is foliated by the boundaries of the Lagrangian planes R_x. Since it is also homeomorphic to a sphere, it is an example of an $\mathbb{R}$-sphere (hybrid sphere), see [**11**, **18**].

Let L be a complex line and Π_L be orthogonal projection onto L. Let γ be a geodesic contained in L. Then, following Mostow, the bisector with spine γ is the inverse image of γ under Π_L. Moreover, each slice of this bisector is the inverse image of a point of γ under Π_L, and is a complex line. Following Will, Section 6.1.1 of [**21**], we now show that performing the same construction but for a Lagrangian plane R gives a pack.

Proposition 3.4. *Suppose that the geodesic γ lies on a totally real plane R. Then the set*

$$P(\gamma) = \Pi_R^{-1}(\gamma) = \bigcup_{z\in\gamma} \Pi_R^{-1}(z).$$

is the pack determined by the Lagrangian planes $R_0 = \Pi_R^{-1}(z_0)$ and $R_1 = \Pi_R^{-1}(z_1)$ for any distinct points z_0, $z_1 \in \gamma$. Moreover, for each $z \in \gamma$, the Lagrangian plane $\Pi_R^{-1}(z)$ is a slice of $P(\gamma)$.

Proof. Choose any points z_0 and z_1 on γ and let $R_0 = \Pi_R^{-1}(z_0)$ and $R_1 = \Pi_R^{-1}$. The involutions ι_0 and ι_1 fixing R_0 and R_1 preserve R. Hence $C = \iota_1\iota_0$ is a loxodromic map with real trace and commuting with Π_R. Since the axis of C is γ, any point on z on γ has the form $z = C^x(z_0)$ for some $x \in \mathbb{R}$. The result follows. q.e.d.

4. Poincaré's Theorem

Definition 4.1. Let Γ be a discrete group of complex hyperbolic isometries. A subset $\mathbf{\Delta}$ of $\mathbf{H}^2_{\mathbb{C}}$ is called a *fundamental domain* for Γ if the following hold.

(i) $\mathbf{\Delta}$ is a domain in $\overline{\mathbf{H}}^2_{\mathbb{C}}$, that is an open connected set;
(ii) $\mathbf{\Delta} \cap A(\mathbf{\Delta}) = \emptyset$ for all $A \in \Gamma \setminus \{I\}$;
(iii) $\bigcup_{A\in\Gamma} A(\overline{\mathbf{\Delta}}) = \mathbf{H}^2_{\mathbb{C}}$;
(iv) the complex hyperbolic volume of $\partial\mathbf{\Delta}$ is 0.

In this section we establish a Poincaré's theorem suitable for our purposes, compare [**8**]. Let P be a pack then the complement of P consists of two *half-spaces*. We consider polyhedra $\mathbf{\Delta}$ obtained by intersecting finitely many such half-spaces.

A natural cell decomposition exists for the closure of $\mathbf{\Delta}$ given by the intersections of the defining packs. The cells of this decomposition are called the *faces* of $\mathbf{\Delta}$. The codimension 1 faces are called the *sides* and will be denoted S. The codimension 2 faces are called the *edges* of $\mathbf{\Delta}$ and will be denoted R (because in the applications each edge R will be a Lagrangian plane).

We also consider the action of Γ on $\partial\mathbf{H}^2_{\mathbb{C}}$. To each codimension k face of $\mathbf{\Delta}$ whose closure meets $\partial\mathbf{H}^2_{\mathbb{C}}$, we associate a codimension k subset of $\partial\mathbf{H}^2_{\mathbb{C}}$ called the *ideal boundary* of the face. To construct the ideal boundary, take the closure of a face and then remove the union of the closures of all lower dimensional faces. The ideal boundary is the intersection of what remains with $\partial\mathbf{H}^2_{\mathbb{C}}$. For example, if an edge R is a Lagrangian plane then its ideal boundary is its boundary in $\partial\mathbf{H}^2_{\mathbb{C}}$, which is an $\mathbb{R}$-circle.

Given such a polyhedron $\mathbf{\Delta}$, we wish to establish conditions so that the group Γ generated by the identifications of the sides of $\mathbf{\Delta}$, is discrete and that $\mathbf{\Delta}$ is a fundamental domain for Γ in $\mathbf{H}^2_{\mathbb{C}}$.

4.1. Side conditions. *The sides of* $\mathbf{\Delta}$ *are paired by elements of* $\mathrm{SU}(2,1)$: for each side S of $\mathbf{\Delta}$, there is a side S' (not necessarily distinct from S) and an element $A_S \in \mathrm{SU}(2,1)$ such that:

(S.1) $A_S(S) = S'$,
(S.2) $A_{S'} = A_S^{-1}$,
(S.3) $A_S(\mathbf{\Delta}) \cap \mathbf{\Delta} = \emptyset$,
(S.4) $A_S(\overline{\mathbf{\Delta}}) \cap \overline{\mathbf{\Delta}} = S'$.

The isometries A_S are called the *side pairing transformations* of $\mathbf{\Delta}$. Let Γ be the group generated by these transformations. We allow S and S' to be the same side, that is a side may be mapped to itself. In this case condition (S.2) requires us to impose ${A_S}^2 = 1$, which is called a *reflection relation*. (This will not arise in our applications.)

We require that $\mathbf{\Delta}$ is defined by intersecting finitely many half spaces determined by non-tangent packs. This gives two more side conditions:

(S.5) *The polyhedron* $\mathbf{\Delta}$ *has only finitely many sides* S, *each side has only finitely many edges.*
(S.6) *There exists* $\delta > 0$ *so that each pair of disjoint sides is at least a distance* δ *apart.*

4.2. Edge conditions.

(E.1) *Each edge of* $\mathbf{\Delta}$ *is a complete submanifold of* $\mathbf{H}^2_{\mathbb{C}}$ *homeomorphic to an open ball of real dimension* 2.

Start with an edge R_1 which lies on the boundary of two sides, call one of them S_1. Then there is a side S_1', and a side pairing transformation A_1, with $A_1(S_1) = S_1'$.

Set $R_2 = A_1(R_1)$. Like R_1, the edge R_2 lies on the boundary of exactly two sides, one of them is S_1', call the other S_2. Again, there is a side S_2', and a side pairing transformation A_2, with $A_2(S_2) = S_2'$. Following this process gives rise to a sequence R_j of edges, a sequence A_j of side pairing transformations, and a sequence (S_j, S_j') of pairs of sides.

Since $\mathbf{\Delta}$ has a finite number of sides, the sequence of edges has to be periodic and hence all three sequences are periodic. Let k be the least period so that all three sequences are periodic with period k. The cyclically ordered sequence of edges $R_1, \dots, R_k$, is called a *cycle* of edges; k is the *period* of the cycle. Observe that

$$A_k \circ \cdots \circ A_1(R_1) = R_1.$$

The element $B = A_k \circ \cdots \circ A_1$ is called the *cycle transformation at the edge* R_1.

Given a cycle transformation B as above and a positive integer m, define a sequence of mk elements B_0, B_1, $\dots$, B_{mk}, of Γ as follows:

$$\begin{array}{llll} B_0 = 1, & B_1 = A_1, & \ldots & B_{k-1} = A_{k-1} \circ \cdots \circ A_1, \\ B_k = B, & B_{k+1} = A_1 \circ B, & \ldots & B_{2k-1} = A_{k-1} \circ \cdots \circ A_1 \circ B, \\ \vdots & \vdots & & \vdots \\ B_{mk-k} = B^{m-1}, & B_{mk-k+1} = A_1 \circ B^{m-1}, & \ldots & B_{mk-1} = A_{k-1} \circ \cdots \circ A_1 \circ B^{m-1}. \end{array}$$

We define $\mathcal{F}(B)$ to be the following family of polyhedra:

$$\mathcal{F}(B) = \left\{ \mathbf{\Delta},\, B_1^{-1}(\mathbf{\Delta}),\, \ldots,\, B_{mk-1}^{-1}(\mathbf{\Delta}) \right\}.$$

(E.2) *For each edge R, the restriction to R of the cycle element $B = B_R$ at R is the identity, and there is a positive integer m so that $B^m = 1$, that is, B has order m. Moreover, the polyhedra of the family $\mathcal{F}(B)$ fit together without overlap, and their closures fill out a closed neighbourhood of the edge R.*

The relations in Γ of the form $B^m = 1$, are called the *cycle relations*.

We now give the main theorem of this section. See also Theorem 3.2 of [**17**] for a similar theorem for polyhedra whose faces are contained in bisectors (and which may have tangencies between the faces).

Theorem 4.2. *Assume that the finite sided polyhedron $\mathbf{\Delta} \subset \mathbf{H}^2_{\mathbb{C}}$ with side pairing transformations A_i satisfies all conditions* (S.1) *to* (S.6), (E.1) *and* (E.2). *Then:*

(i) *the group Γ generated by these transformations is discrete;*
(ii) *$\mathbf{\Delta}$ is a fundamental domain for Γ;*
(iii) *the reflection relations and cycle relations form a complete set of relations for Γ;*
(iv) *Γ contains no parabolic elements;*
(v) *Γ is geometrically finite.*

The proof of (i), (ii) and (iii) follows that in Epstein and Petronio, Theorem 4.14 [**8**]. Observe that, using Remark 3.24 of [**8**] our conditions (S.5) and (S.6) imply the condition (Metric) of Epstein and Petronio. If Γ had a parabolic element then, necessarily its fixed point would lie in the closure of $\mathbf{\Delta}$, compare Theorem 10.3.2 of [**1**], and this fixed point would lie on the ideal boundary of (at least) two disjoint faces of $\mathbf{\Delta}$. This contradicts (S.6). It is clear that the the ideal boundary of $\mathbf{\Delta}$ is contained in the region of discontinuity $\Omega(\Gamma)$ for the action of Γ on $\partial\mathbf{H}^2_{\mathbb{C}}$. Also, using (S.4) each point on the ideal boundary of a side S has a neighbourhood covered by the ideal boundaries of S, $\mathbf{\Delta}$ and ${A_S}^{-1}(\mathbf{\Delta})$. Hence it too is contained in $\Omega(\Gamma)$. Finally, using (E.2) a similar argument shows that each point in the ideal boundary of an edge R is also in $\Omega(\Gamma)$. Therefore $(\mathbf{H}^2_{\mathbb{C}} \cup \Omega)/\Gamma$ is a closed manifold. Using definition F1 of Bowditch [**3**] (see the discussion in the introduction) we see that Γ is geometrically finite.

Corollary 4.3. *Suppose that the group Γ from Theorem* 4.2 *is a representation of π_1, the fundamental group of a surface of genus g.*

Suppose that the cycle relations and reflection relations in (iii) *introduce no new relations. Then* Γ *is complex hyperbolic quasi-Fuchsian.*

We note that if π_1 is the fundamental group of a punctured surface then (S.6) does not hold.

5. Proof of the main theorem

5.1. A fundamental polyhedron for an $\mathbb{R}$-Fuchsian group. Let Σ be a closed surface of genus $g > 1$ and let ρ_0 be any $\mathbb{R}$-Fuchsian representation of π_1, the fundamental group of Σ. We denote the image of ρ_0 by $\Gamma_0 = \rho_0(\pi_1) < \mathrm{SU}(2,1)$. Without loss of generality, we suppose that Γ_0 preserves $R_{\mathbb{R}}$ and so $\Gamma_0 < \mathrm{SO}(2,1)$. Consider the action of Γ_0 on $R_{\mathbb{R}}$ and let Δ_0 be a fundamental hyperbolic polygon for this action with $4g$ sides $s^{(1)}, \ldots, s^{(4g)}$. Let $v^{(1)}, \ldots, v^{(4g)}$ denote the vertices of Δ_0. We adopt the convention that $s^{(k)}$ has endpoints $v^{(k)}$ and $v^{(k+1)}$ and superscripts are taken mod $4g$. Conjugating if necessary, we suppose that $v^{(1)}$ is the origin o. By construction, there are $4g$ elements of Γ_0, denoted $A_0^{(1)}, \ldots, A_0^{(4g)}$ that pair the sides of Δ according to the following rules:

(i) For $j = 0, \ldots, g-1$ the map $A_0^{(4j+1)}$ sends the side $s^{(4j+1)}$ to the side $s^{(4j+3)}$ and the map $A_0^{(4j+2)}$ sends the side $s^{(4j+2)}$ to the side $s^{(4j+4)}$. Thus $A_0^{(4j+1)} = \big(A_0^{(4j+3)}\big)^{-1}$ and $A_0^{(4j+2)} = \big(A_0^{(4j+4)}\big)^{-1}$.

(ii) There are no reflection relations and only one cycle relation:

$$\prod_{j=0}^{g-1} A_0^{(4j+2)}\Big(A_0^{(4j+1)}\Big)^{-1}\Big(A_0^{(4j+2)}\Big)^{-1}A_0^{(4j+1)} = I. \tag{5.1}$$

For this polygon, it is straightforward to verify that side conditions analogous to (S.1) to (S.6) are satisfied. In this case, each codimension 2 face is a point, namely one of $v^{(1)}, \ldots, v^{(4g)}$. This condition replaces (E.1). With this change, (E.2) is also satisfied. Thus we could have used the classical Poincaré polygon theorem to verify that Δ_0 is a fundamental domain for the action of Γ_0 on $R_{\mathbb{R}}$. Moreover, as (5.1) generates all relations in π_1 we see that ρ_0 is faithful. In particular, Γ_0 has no elliptic elements. Since there are no tangencies between faces of Δ_0 we also see that Γ_0 contains no parabolics. Hence it is totally loxodromic.

Let $\boldsymbol{\Delta}_0 = \Pi_{\mathbb{R}}^{-1}(\Delta_0)$ be the inverse image of the polygon Δ_0 under projection onto $R_{\mathbb{R}}$ (see Section 6.1.1 of [**21**] where Will constructs fundamental domains for $\mathbb{R}$-Fuchsian triangle groups and punctured torus groups in a similar way). We claim that $\boldsymbol{\Delta}_0$ satisfies the conditions (S.1) to (S.6), (E.1) and (E.2). Thus Theorem 4.2 will imply that $\boldsymbol{\Delta}_0$ is a fundamental domain for the action of Γ_0 on $\mathbf{H}^2_{\mathbb{C}}$. We now show how to check the conditions. The edges of $\boldsymbol{\Delta}_0$ are the Lagrangian planes $R_0^{(k)} = \Pi_{\mathbb{R}}^{-1}(v^{(k)})$. In particular, $R_0^{(1)} = \Pi_{\mathbb{R}}^{-1}(o) = R_{\mathbb{J}}$. Thus condition (E.1) is satisfied. The sides of $\mathbf{\Delta}_0$ are $S_0^{(k)} = \Pi_{\mathbb{R}}^{-1}(v^{(k)})$ for $k = 0, \ldots, 4g$. These are each pieces of the pack $P_0^{(k)}$ determined by the Lagrangian planes $R_0^{(k)}$ and $R_0^{(k+1)}$.

By Proposition 2.5, $\Pi_{\mathbb{R}}$ commutes with any element of $\mathrm{SO}(2,1)$, and so for $j = 0, \ldots, g-1$ the map $A_0^{(4j+1)}$ sends the side $S_0^{(4j+1)}$ to the side $S_0^{(4j+3)}$ and the map $A_0^{(4j+2)}$ sends the side $S_0^{(4j+2)}$ to the side $S_0^{(4j+4)}$. Thus the side conditions (S.1) to (S.6) are automatically satisfied. The condition (E.2) is therefore satisfied: there is only one cycle of vertices and the cycle transformation is given by (5.1) with $m = 1$. Using Poincaré's theorem, Theorem 4.2, we see that $\mathbf{\Delta}_0$ is indeed a fundamental domain for Γ_0.

By construction, for any $k = 1, \ldots, 4g$ the edge $R_0^{(k)}$ is the image of $R_0^{(1)} = R_{\mathbb{J}}$ under some fixed word in the generators $A_0^{(1)}, \ldots, A_0^{(4g)}$. In fact this word comprises the last n letters of the relation (5.1) for some n. We denote this word by $B_0^{(k)}$. For example $B_0^{(1)}$ is the identity, $B_0^{(4)} = A_0^{(1)}$ and $B_0^{(3)} = \big(A_0^{(2)}\big)^{-1} A_0^{(1)}$. There is a homotopy class of loops $\beta_k \in \pi_1$ so that $B_0^{(k)} = \rho_0(\beta_k)$. Clearly $B_0^{(k)}$ is loxodromic for each k. So there is a constant $K > 0$ so that $\mathrm{tr}(B_0^{(k)}) \geq 3 + K > 3$ for all k.

5.2. The variation of the polyhedron. Let $\Gamma_t = \rho_t(\pi_1) < \mathrm{SU}(2,1)$ be a point in the representation variety $\mathrm{Hom}\big((\pi_1, \mathrm{SU}(2,1)\big)/\mathrm{SU}(2,1)$. We will only consider representations that are close to Γ_0. To make this notion precise, for $k = 2, \ldots, 4g$ let $B_t^{(k)} = \rho_t(\beta_k)$ (here $\beta_k \in \pi_1$ is the homotopy class of loops for which $\rho_0(\beta_k) = B_0^{(k)}$ as described above). Then, given $\epsilon = \epsilon(t) > 0$ the representation ρ_t is said to be *ϵ-close to ρ_0* if for each $k = 2, \ldots, 4g$ we have

$$\left\| B_t^{(k)} - B_0^{(k)} \right\| < \epsilon$$

measured using the l^2-norm on $\mathrm{SU}(2,1)$. In the same way, for $k = 1, \ldots, 4g$ let α_k be the homotopy class of loops in π_1 so that $A_0^{(k)} = \rho_0(\alpha_k)$. Then we define $A_t^{(k)} = \rho_t(\alpha_k)$.

Our goal will be to show that there exists an ϵ depending only on ρ_0 so that all representations ρ_t that are ϵ-close to ρ_0 are complex hyperbolic-quasi-Fuchsian. In order to achieve this goal we will construct a domain $\mathbf{\Delta}_t$ and use Theorem 4.2 to show that $\mathbf{\Delta}_t$ is a fundamental domain for $\Gamma_t = \rho_t(\pi_1)$. Moreover, this will also imply that ρ_t is faithful, and Γ_t is totally loxodromic and geometrically finite. In other words, Γ_t is complex hyperbolic quasi-Fuchsian.

We begin by constructing the edges of $\mathbf{\Delta}_t$. Let $R_t^{(1)} = R_{\mathbb{J}}$, the totally imaginary Lagrangian plane. For $k = 2, \ldots, 4g$ we define $R_t^{(k)}$ to be the Lagrangian plane

$$R_t^{(k)} = B_t^{(k)}\big(R_t^{(1)}\big) = B_t^{(k)}(R_{\mathbb{J}}). \tag{5.2}$$

We will prove the following theorem in the next section.

Theorem 5.1. *There exists $\epsilon_1 = \epsilon_1(\rho_0) > 0$ so that if $\epsilon < \epsilon_1$ then the Lagrangian planes $R_t^{(1)}, \ldots, R_t^{(4g)}$ are disjoint.*

Suppose that the disjoint Lagrangian planes $R_0^{(k)}$ and $R_0^{(k+1)}$ are edges of $\mathbf{\Delta}_0$ in the boundary of the side $S_0^{(k)}$. Then we define the corresponding side $S_t^{(k)}$ of $\mathbf{\Delta}_t$ as follows. From Theorem 5.1 we see that the Lagrangian planes $R_t^{(k)}$ and $R_t^{(k+1)}$ are disjoint, and so determine a pack $P_t^{(k)}$. Define the side $S_t^{(k)}$ to be that part of $P_t^{(k)}$ lying between $R_t^{(k)}$ and $R_t^{(k+1)}$. We emphasise that once we have defined the Lagrangian planes $R_t^{(k)}$, the construction of $S_t^{(k)}$ is canonical. Thus, since the side pairing maps match the edges $R_t^{(k)}$ they automatically match the sides $S_t^{(k)}$. We will prove this theorem in the next section.

Theorem 5.2. *There exists $\epsilon_2 = \epsilon_2(\rho_0)$ with $0 < \epsilon_2 < \epsilon_1$ so that for all $\epsilon < \epsilon_2$ we have:*

(i) *the sides $S_t^{(1)}, \ldots, S_t^{(4g)}$ only intersect in the Lagrangian planes $R_t^{(1)}, \ldots, R_t^{(4g)}$;*
(ii) *the combinatorial pattern of this intersection is the same as that for the faces of $\mathbf{\Delta}_0$;*
(iii) *there is a $\lambda > 0$ so that disjoint sides of $\mathbf{\Delta}_t$ are at least a distance λ apart.*

We claim that $\mathbf{\Delta}_t$ satisfies the conditions of Poincaré's theorem, and so is a fundamental domain for Γ_t. First, observe that the following facts follow immediately from our construction:

(i) For $j = 0, \ldots, g-1$, the map $A_t^{(4j+1)}$ sends the side $S_t^{(4j+1)}$ to the side $S_t^{(4j+3)}$ and $A_t^{(4j+2)}$ sends $S_t^{(4j+2)}$ to the side $S_t^{(4j+4)}$. So (S.1) is satisfied.
(ii) $A_t^{(4j+1)} = \big(A_0^{(4j+3)}\big)^{-1}$ and $A_t^{(4j+2)} = \big(A_0^{(4j+4)}\big)^{-1}$ so (S.2) is satisfied.
(iii) Using Theorem 5.2(i) and (ii) together with the separation properties of packs, Proposition 3.3, we see that (S.3), (S.4) and (S.5) are satisfied.
(iv) Using Theorem 5.2(iii) we see that (S.6) is satisfied.
(v) From Theorem 5.1 we immediately obtain condition (E.1).

(vi) Again, use of Theorem 5.2 (i) and (ii) and Proposition 3.3, shows that (E.2) is satisfied. Furthermore, there are no reflection relations and only one cycle relation:

$$\prod_{j=0}^{g-1} A_t^{(4j+2)}\big(A_t^{(4j+1)}\big)^{-1}\big(A_t^{(4j+2)}\big)^{-1}A_t^{(4j+1)} = I.$$

Hence $\mathbf{\Delta}_t$ satisfies the conditions of Poincaré's theorem and thus Γ_t is discrete, totally loxodromic and is geometrically finite. Moreover by (vi) above we immediately see that Γ_t is a faithful representation of π_1. This has proved our main theorem subject to verifying Theorems 5.1 and 5.2 which we do in the next section.

6. The Technical Results

6.1. Some preparatory results. The following simple lemmas are going to be needed below.

Lemma 6.1. *Let A and B be $m\times m$ matrices. Then for each positive integer $n \geq 1$,*

(i) $|\mathrm{tr}(A)| \leq \sqrt{m}\|A\|$,
(ii) $\|AA^T - BB^T\| \leq \|A-B\|^2 + 2\|B\|\|A-B\|$,
(iii) $\big\|(A+B)^n - B^n\big\| \leq (\|A\|+\|B\|)^n - \|B\|^n$.

Proof. Writing $A=(a_{ij})$ and $B=(b_{ij})$ where i and j run from 1 to m, we have

$$|\mathrm{tr}(A)| \leq \sum_{i=1}^{m}|a_{ii}| \leq \sqrt{m}\left(\sum_{i=1}^{m}|a_{ii}|^2\right)^{\frac{1}{2}} \leq \sqrt{m}\|A\|,$$

where we have used the inequality $(x_1+\cdots+x_m)^2 \leq m(x_1^2+\cdots+x_m^2)$.

The proof of (ii) only uses standard properties of matrix norms:

$$\begin{aligned}\|AA^T - BB^T\| &\leq \|AA^T - AB^T\| + \|AB^T - BB^T\|\\ &\leq \|A\|\|A-B\| + \|A-B\|\|B\|\\ &\leq \big(\|A-B\|+\|B\|\big)\|A-B\| + \|A-B\|\|B\|\\ &= \|A-B\|^2 + 2\|B\|\|A-B\|.\end{aligned}$$

The proof of (iii) is by induction. The case $n=1$ is obvious. Suppose now that

$$\big\|(A+B)^{n-1} - B^{n-1}\big\| \leq \big(\|A\|+\|B\|\big)^{n-1} - \|B\|^{n-1}.$$

Then,

$$
\begin{aligned}
&\|(A+B)^n - B^n\| \\
&= \big\|(A+B)^{n-1}(A+B) - B^{n-1}B\big\| \\
&= \big\|(A+B)^{n-1}A + \big((A+B)^{n-1} - B^{n-1}\big)B\big\| \\
&\le \|A+B\|^{n-1}\|A\| + \big\|(A+B)^{n-1} - B^{n-1}\big\|\|B\| \\
&\le \big(\|A\|+\|B\|\big)^{n-1}\|A\| + \Big(\big(\|A\|+\|B\|\big)^{n-1} - \|B\|^{n-1}\Big)\|B\| \\
&= \big(\|A\|+\|B\|\big)^{n} - \|B\|^{n}.
\end{aligned}
$$

q.e.d.

Lemma 6.2. *If A and B are 3×3 matrices so that $JA^*J = -A$ and $JB^*J = B$ then*

$$\|A+B\|^2 = \|A\|^2 + \|B\|^2.$$

Proof. It suffices to show that $\langle\langle A, B\rangle\rangle = 0$ and the result will follow from Pythagoras' theorem. In fact,

$$
\begin{aligned}
\langle\langle A, B\rangle\rangle &= \Re\big(\mathrm{tr}(AB^*)\big) = -\Re\big(\mathrm{tr}(JA^*BJ)\big) \\
&= -\Re\big(\mathrm{tr}(BA^*)\big) = -\langle\langle B, A\rangle\rangle = -\langle\langle A, B\rangle\rangle.
\end{aligned}
$$

q.e.d.

Lemma 6.3. *Let $U = (0, +\infty)\times(-\pi,\pi)$. For each $\delta + i\phi \in U$ let $\tau : U \to \mathbb{C}$ be the function*

$$\tau(\delta,\phi) = 2\cosh(\delta)e^{-i\phi} + e^{2i\phi}.$$

Then τ is continuously differentiable and invertible everywhere on U.

Moreover, suppose that V is a closed, convex subset of the image of U under τ. Then there exists a constant M so that for all δ_1, δ_2, ϕ_1 and ϕ_2 for which $\tau(\delta_1,\phi_1)$ and $\tau(\delta_2,\phi_2)$ lie in V we have

$$
\begin{aligned}
|\delta_1 - \delta_2| &\le M\big|\tau(\delta_1,\phi_1) - \tau(\delta_2,\phi_2)\big|, \\
|\phi_1 - \phi_2| &\le M\big|\tau(\delta_1,\phi_1) - \tau(\delta_2,\phi_2)\big|.
\end{aligned}
$$

Proof. It is clear that τ is continuously differentiable on the whole of U. Write

$$
\begin{aligned}
u(\delta,\phi) &= 2\cosh(\delta)\cos(\phi) + \cos(2\phi), \\
v(\delta,\phi) &= -2\cosh(\delta)\sin(\phi) + \sin(2\phi).
\end{aligned}
$$

Then $\tau(\delta,\phi) = u(\delta,\phi) + iv(\delta,\phi)$. The Jacobian of these functions is

$$
\begin{aligned}
J(u,v) &= \det\begin{pmatrix} u_\delta & u_\phi \\ v_\delta & v_\phi \end{pmatrix} \\
&= \det\begin{pmatrix} 2\sinh(\delta)\cos(\phi) & -2\cosh(\delta)\sin(\phi) - 2\sin(2\phi) \\ -2\sinh(\delta)\sin(\phi) & -2\cosh(\delta)\cos(\phi) + 2\cos(2\phi) \end{pmatrix} \\
&= -4\sinh(\delta)\big(\cosh(\delta) - \cos(3\phi)\big),
\end{aligned}
$$

which is negative on the whole of U and so τ is invertible there.

In particular, we can express δ and ϕ as functions $\delta(u,v)$ and $\phi(u,v)$ of u and v. Using the mean value theorem in two variables (see for example equation (38) on page 67 of [**5**]) we find that when $\tau_1 = u_1 + iv_1$ and $\tau_2 = u_2 + iv_2$ lie in a convex domain in $\mathbb{C}$ then there exist constants M_δ and M_ϕ so that

$$\begin{aligned}\big|\delta(u_1,v_1)-\delta(u_2,v_2)\big| &\le M_\delta|u_1-u_2+iv_1-iv_2| = M_\delta|\tau_1-\tau_2|,\\ \big|\phi(u_1,v_1)-\phi(u_2,v_2)\big| &\le M_\phi|u_1-u_2+iv_1-iv_2| = M_\phi|\tau_1-\tau_2|.\end{aligned}$$

Taking M to be the larger of M_δ and M_ϕ gives the result. q.e.d.

Lemma 6.4. *Let $\iota_{\mathbb{J}}$ be reflection in $R_{\mathbb{J}}$, the standard purely imaginary Lagrangian plane. Then $\iota_{\mathbb{J}}B^{-1}\iota_{\mathbb{J}} = B^T$ for any $B \in \mathrm{SU}(2,1)$.*

Proof. Writing

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$$

we have

$$\begin{aligned}\iota_{\mathbb{J}}B^{-1}\iota_{\mathbb{J}}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} &= \iota_{\mathbb{J}}B^{-1}\begin{bmatrix} -\overline{z}_1 \\ -\overline{z}_2 \\ \overline{z}_3 \end{bmatrix} = \iota_{\mathbb{J}}\begin{bmatrix} -\overline{a}\overline{z}_1 - \overline{d}\overline{z}_2 - \overline{g}\overline{z}_3 \\ -\overline{b}\overline{z}_1 - \overline{e}\overline{z}_2 - \overline{h}\overline{z}_3 \\ \overline{c}\overline{z}_1 + \overline{f}\overline{z}_2 + \overline{j}\overline{z}_3 \end{bmatrix}\\ &= \begin{bmatrix} az_1 + dz_2 + gz_3 \\ bz_1 + ez_2 + hz_3 \\ cz_1 + fz_2 + jz_3 \end{bmatrix} = B^T\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}.\end{aligned}$$

q.e.d.

This leads to the following result which uses Theorem 2.1 to give an algebraic criterion for when two Lagrangian planes are disjoint.

Lemma 6.5. *Let B, $\widetilde{B}$ be any elements of $\mathrm{SU}(2,1)$. The Lagrangian planes $B(R_{\mathbb{J}})$ and $\widetilde{B}(R_{\mathbb{J}})$ are disjoint if and only if $\widetilde{C}C^{-1}$ is loxodromic, where $C = BB^T$ and $\widetilde{C} = \widetilde{B}\widetilde{B}^T$.*

Proof. The involution fixing $B(R_{\mathbb{J}})$ is

$$\iota = B\iota_{\mathbb{J}}B^{-1} = BB^T\iota_{\mathbb{J}} = \iota_{\mathbb{J}}(B^{-1})^TB^{-1} = \iota_{\mathbb{J}}(BB^T)^{-1},$$

where we have used Lemma 6.4. That is $\iota = C\iota_{\mathbb{J}} = \iota_{\mathbb{J}}C^{-1}$. Likewise the involution fixing $\widetilde{B}(R_{\mathbb{J}})$ is $\widetilde{\iota} = \widetilde{C}\iota_{\mathbb{J}} = \iota_{\mathbb{J}}\widetilde{C}^{-1}$. Using Theorem 2.1 we see that the Lagrangian planes are disjoint if and only if the product of the involutions ι and $\widetilde{\iota}$ is loxodromic. But

$$\widetilde{\iota}\iota = \big(\widetilde{C}\iota_{\mathbb{J}}\big)\big(\iota_{\mathbb{J}}C^{-1}\big) = \widetilde{C}C^{-1}.$$

The result follows immediately. q.e.d.

6.2. The edges are disjoint. In order to prove Theorem 5.1, we must show that for each pair of distinct $j, k = 1, \ldots, 4g$ the Lagrangian planes $R_t^{(j)} = B_t^{(j)}(R_{\mathbb{J}})$ and $R_t^{(k)} = B_t^{(k)}(R_{\mathbb{J}})$ are disjoint. We know that $R_0^{(j)} = B_0^{(j)}(R_{\mathbb{J}})$ and $R_0^{(k)} = B_0^{(k)}(R_{\mathbb{J}})$ are disjoint and a distance $\lambda_0 > 0$ apart. Thus Theorem 5.1 is a consequence of the following result:

Proposition 6.6. *Suppose that B_0, $\widetilde{B}_0 \in \mathrm{SO}(2,1)$ and that the Lagrangian planes $B_0(R_{\mathbb{J}})$ and $\widetilde{B}_0(R_{\mathbb{J}})$ are a distance λ_0 apart. There exists $\epsilon > 0$ so that for all B_t and $\widetilde{B}_t$ in $\mathrm{SU}(2,1)$ with $\|B_t - B_0\| < \epsilon$ and $\|\widetilde{B}_t - \widetilde{B}_0\| < \epsilon$ then $B_t(R_{\mathbb{J}})$ and $\widetilde{B}_t(R_{\mathbb{J}})$ are disjoint and a distance λ_t apart where $\cosh(\lambda_t) \geq \cosh(\lambda_0) - O(\epsilon)$. Here $O(\epsilon)$ is a function only depending on ϵ, B_0 and $\widetilde{B}_0$.*

Proof. Using Lemma 6.5, we see that the product of the involutions in $B_0(R_{\mathbb{J}})$ and $\widetilde{B}_0(R_{\mathbb{J}})$ is $\widetilde{C}_0 C_0^{-1}$ where $C_0 = B_0 B_0^T$ and $\widetilde{C}_0 = \widetilde{B}_0 \widetilde{B}_0^T$. Thus $\widetilde{C}_0 C_0^{-1}$ is loxodromic and, moreover,

$$\mathrm{tr}(\widetilde{C}_0 C_0^{-1}) = 2\cosh\Big(\rho\big(B_0(o), \widetilde{B}_0(o)\big)\Big) + 1 = 2\cosh(\lambda_0) + 1$$

Similarly, we must show that $\widetilde{C}_t C_t^{-1}$ is loxodromic. We estimate $\big|\mathrm{tr}(\widetilde{C}_t C_t^{-1})\big|$. From the triangle inequality,

$$\begin{aligned}\big|\mathrm{tr}(\widetilde{C}_t C_t^{-1})\big| &\geq \big|\mathrm{tr}(\widetilde{C}_0 C_0^{-1})\big| - \big|\mathrm{tr}(\widetilde{C}_t C_t^{-1} - \widetilde{C}_0 C_0^{-1})\big| \\ &\geq 2\cosh(\lambda_0) + 1 - \sqrt{3}\big\|\widetilde{C}_t C_t^{-1} - \widetilde{C}_0 C_0^{-1}\big\|,\end{aligned}$$

where we have used Lemma 6.1 (i). However, using Lemma 6.1 (ii) we see that

$$\begin{aligned}&\big\|\widetilde{C}_t C_t^{-1} - \widetilde{C}_0 C_0^{-1}\big\| \\ &\leq \big\|\widetilde{C}_t - \widetilde{C}_0\big\|\,\big\|C_t - C_0\big\| + \big\|\widetilde{C}_0\big\|\,\big\|C_t - C_0\big\| + \big\|\widetilde{C}_t - \widetilde{C}_0\big\|\,\big\|C_0\big\| \\ &\leq \Big(\big\|\widetilde{B}_t - \widetilde{B}_0\big\|^2 + 2\big\|\widetilde{B}_0\big\|\,\big\|\widetilde{B}_t - \widetilde{B}_0\big\|\Big)\Big(\big\|B_t - B_0\big\|^2 + 2\big\|B_0\big\|\,\big\|B_t - B_0\big\|\Big) \\ &\quad + \big\|\widetilde{B}_0\big\|^2\Big(\big\|B_t - B_0\big\|^2 + 2\big\|B_0\big\|\,\big\|B_t - B_0\big\|\Big) \\ &\quad + \big\|B_0\big\|^2\Big(\big\|\widetilde{B}_t - \widetilde{B}_0\big\|^2 + 2\big\|\widetilde{B}_0\big\|\,\big\|\widetilde{B}_t - \widetilde{B}_0\big\|\Big).\end{aligned}$$

Hence $\big\|\widetilde{C}_t C_t^{-1} - \widetilde{C}_0 C_0^{-1}\big\| = O(\epsilon)$ and so $\big|\mathrm{tr}(\widetilde{C}_t C_t^{-1})\big| > 3$. Thus $\widetilde{C}_t C_t^{-1}$ is loxodromic.

Finally, $\mathrm{tr}(\widetilde{C}_t C_t^{-1}) = 2\cosh(\lambda_t)e^{-i\psi_t} + e^{2i\psi_t}$ and so

$$2\cosh(\lambda_t) + 1 \geq \big|\mathrm{tr}(\widetilde{C}_t C_t^{-1})\big| \geq \big|\mathrm{tr}(\widetilde{C}_0 C_0^{-1})\big| - O(\epsilon) = 2\cosh(\lambda_0) + 1 - O(\epsilon)$$

as required. q.e.d.

6.3. The sides of $\mathbf{\Delta}_t$. We consider the following situation. Let $B_0 \in \mathrm{SO}(2,1)$ be a loxodromic element of $\mathrm{SU}(2,1)$. We have already seen that the Lagrangian planes $R_{\mathbb{J}}$ and $B_0(R_{\mathbb{J}})$ are disjoint and $C_0 = B_0 B_0^T$ is loxodromic. Let P_0 be the pack determined by $R_{\mathbb{J}}$ and $B_0(R_{\mathbb{J}})$. Then, from Proposition 3.1, we see that the slices of P_0 are the Lagrangian planes $C_0^{x/2}(R_{\mathbb{J}})$ for $x \in \mathbb{R}$. Since $B_0(R_{\mathbb{J}}) = C_0^{1/2}(R_{\mathbb{J}})$ (see Proposition 3.1 (i) with $x = 1$), we see that the slices of P_0 in the side S_0 are the Lagrangian planes $C_0^{x/2}(R_{\mathbb{J}})$ for $x \in [0,1]$.

We want to consider the pack determined by $R_{\mathbb{J}}$ and $B_t(R_{\mathbb{J}})$ where B_t is in $\mathrm{SU}(2,1)$ and $\|B_t - B_0\| < \epsilon$. We saw in Proposition 6.6 (with $B_0 = I$) that we may choose ϵ so that the Lagrangian planes $R_{\mathbb{J}}$ and $B_t(R_{\mathbb{J}})$ are disjoint. Thus we may define P_t to be the pack P_t determined by $R_{\mathbb{J}}$ and $B_t(R_{\mathbb{J}})$. Writing, $C_t = B_t B_t^T$, we see that the slices of the side S_t contained in P_t are the Lagrangian planes $C_t^{x/2}(R_{\mathbb{J}})$ for $x \in [0,1]$, which are all disjoint.

Our next goal will be to show that $\big\|C_t^{x/2} - C_0^{x/2}\big\| = O(\epsilon)$ for $x \in [0,1]$, which is Corollary 6.14 below. Following Lemma 2.2, we write

$$C_t = e^{2i\phi_t} I + \sinh(\delta_t) e^{-i\phi_t} E_t + \big(\cosh(\delta_t) e^{-i\phi_t} - e^{2i\phi_t}\big) E_t^2, \tag{6.1}$$

$$C_0 = I + \sinh(\delta_0) E_0 + \big(\cosh(\delta) - 1\big) E_0^2 \tag{6.2}$$

for some E_t, E_0 in $\mathfrak{su}(2,1)$ with $E_t^3 = E_t$, $E_0^3 = E_0$. Furthermore, using Lemma 6.1 (ii),

$$\|C_t - C_0\| \le \|B_t - B_0\|^2 + 2\|B_0\|\,\|B_t - B_0\| = O(\epsilon).$$

Lemma 6.7. *If C_0 and C_t are given by equations* (6.1) *and* (6.2) *and $\|B_t - B_0\| < \epsilon$, then $|\delta_t - \delta_0| = O(\epsilon)$ and $|\phi_t| = O(\epsilon)$.*

Proof. Write $\big|\mathrm{tr}(C_t) - \mathrm{tr}(C_0)\big| = \eta$. Then,

$$\eta = \big|\mathrm{tr}(C_t - C_0)\big| \le \sqrt{3}\|C_t - C_0\| = O(\epsilon)$$

using Lemma 6.1 (i). We write $\mathrm{tr}(C_0) = 2\cosh(\delta_0) + 1 = \tau(\delta_0, 0)$ and $\mathrm{tr}(C_t) = 2\cosh(\delta_t) e^{-i\phi_t} + e^{2i\phi_t} = \tau(\delta_t, \phi_t)$ where $\tau(\delta, \phi)$ is the function of Lemma 6.3. The closed η-ball centred at $\tau(\delta_0, 0)$ is a convex subset of the image of U under τ. From Lemma 6.3 there is a positive constant M so that

$$|\delta_t - \delta_0| \le M\big|\tau(\delta_t, \phi_t) - \tau(\delta_0, \phi_0)\big| = M\big|\mathrm{tr}(C_t) - \mathrm{tr}(C_0)\big| = M\eta = O(\epsilon).$$

Similarly $|\phi_t| = |\phi_t - 0| \le M\eta = O(\epsilon)$ and our assertion is proved.

q.e.d.

Lemma 6.8. *If C_t and C_0 are given by equations* (6.1) *and* (6.2) *with $\|C_t - C_0\| = O(\epsilon)$, then $\|E_t\|$ is bounded by terms involving $\|E_0\|$, δ_0 and ϵ.*

Proof. Writing C_t and C_0 in the form of (6.1) and (6.2) we have

$$\|C_t - C_0\| = \Big\| e^{2i\phi_t} I + \sinh(\delta_t) e^{-i\phi_t} E_t + \big(\cosh(\delta_t) e^{-i\phi_t} - e^{2i\phi_t}\big) E_t^2 - I - \sinh(\delta_0) E_0 - \big(\cosh(\delta_0) - 1\big) E_0^2 \Big\|.$$

Thus, using the triangle inequality,

$$\begin{aligned} &\big|\cosh(\delta_t) e^{-i\phi_t} - e^{2i\phi_t}\big| \, \|E_t\|^2 \\ &\le \|C_t - C_0\| + \big\|(e^{2i\phi_t} - 1) I + \sinh(\delta_t) e^{-i\phi_t} E_t \\ &\qquad - \sinh(\delta_0) E_0 - (\cosh(\delta_0) - 1) E_0\big\| \\ &\le \sinh(\delta_t)\|E_t\| + \|C_t - C_0\| + 4\sin^2(\phi)\|I\| \\ &\quad + \sinh(\delta_0)\|E_0\| + \big(\cosh(\delta_0) - 1\big)\|E_0\|^2. \end{aligned}$$

Since $\|C_t - C_0\|$, $|\delta_t - \delta_0|$ and $|\phi_t|$ are all $O(\epsilon)$ this shows that

$$\|E_t\|^2 \le \frac{\sinh(\delta_0)\|E_t\| + \sinh(\delta_0)\|E_0\| + \big(\cosh(\delta_0) - 1\big)\|E_0\|^2}{\cosh(\delta_0) - 1} + O(\epsilon).$$

Therefore

$$\|E_t\| \le \frac{\sinh(\delta_0)}{\cosh(\delta_0) - 1} + \|E_0\| + O(\epsilon).$$

This proves the result. q.e.d.

Let D_t and D_0 be defined by $C_t = \exp(D_t)$ and $C_0 = \exp(D_0)$. Using Lemma 2.3, we see that D_t and D_0 have the form

$$D_t = 2i\phi_t I + \delta_t E_t - 3i\phi_t E_t^2, \qquad D_0 = \delta_0 E_0. \tag{6.3}$$

As in the proof of Lemma 2.3 we may decompose

$$C_t = \Big(I + \sinh(\delta_t) E_t + \big(\cosh(\delta_t) - 1\big) E_t^2\Big)\Big(e^{2i\phi_t} I + e^{-i\phi_t} E_t^2 - e^{2i\phi_t} E_t^2\Big).$$

We write

$$\Delta_t = \Big(I + \sinh(\delta_t) E_t + \big(\cosh(\delta_t) - 1\big) E_t^2\Big) = \exp(\delta_t E_t), \tag{6.4}$$

$$\Phi_t = \Big(e^{2i\phi_t} I + e^{-i\phi_t} E_t^2 - e^{2i\phi_t} E_t^2\Big) = \exp(2i\phi_t I - 3i\phi_t E_t^2). \tag{6.5}$$

Lemma 6.9. *Suppose that Δ_t and Φ_t are given by* (6.4) *and* (6.5) *with $|\phi_t| = O(\epsilon)$. Then*
(i) $\|\Phi_t - I\| = O(\epsilon)$,
(ii) $\|\Delta_t - C_0\| = O(\epsilon)$.

Proof. First, we have

$$\begin{aligned} \|\Phi_t - I\| &= \big\|(e^{2i\phi_t} - 1) I + (e^{-i\phi_t} - e^{2i\phi_t}) E_t^2\big\| \\ &\le 2\sin|\phi_t| \, \|I\| + 2\sin|3\phi_t/2| \, \|E_t\|^2 = O(\epsilon). \end{aligned}$$

This proves (i). We also have

$$\|\Delta_t - C_0\| = \big\|(C_t - C_0) - \Delta_t(\Phi_t - I)\big\| \le \|C_t - C_0\| + \|\Delta_t\|\|\Phi_t - I\| = O(\epsilon)$$

where we have used the fact that $\|\Delta_t\|$ is bounded, which follows from $\|E_t\|$ being bounded and $|\delta_t - \delta_0| = O(\epsilon)$. q.e.d.

Lemma 6.10. *If E_t and E_0 are as in* (6.1) *and* (6.2) *then*

$$\big\|\sinh(\delta_t)E_t - \sinh(\delta_0)E_0\big\| = O(\epsilon),$$
$$\big\|\big(\cosh(\delta_t) - 1\big)E_t^2 - \big(\cosh(\delta_0) - 1\big)E_0^2\big\| = O(\epsilon).$$

Proof. Since $JE_t^*J = -E_t$ and $JE_0^*J = -E_0$ we have $J(E_t^2)^*J = E_t^2$ and $J(E_0^2)^*J = E_0^2$. From Lemma 6.2, we see that

$$\begin{aligned}\|\Delta_t - C_0\|^2 &= \Big\|\sinh(\delta_t)E_t - \sinh(\delta_0)E_0 \\ &\qquad + \big(\cosh(\delta_t) - 1\big)E_t^2 - \big(\cosh(\delta_0) - 1\big)E_0^2\Big\|^2 \\ &= \big\|\sinh(\delta_t)E_t - \sinh(\delta_0)E_0\big\|^2 \\ &\quad + \Big\|\big(\cosh(\delta_t) - 1\big)E_t^2 - \big(\cosh(\delta_0) - 1\big)E_0^2\Big\|^2.\end{aligned}$$

Thus the result follows since $\|\Delta_t - C_0\| = O(\epsilon)$. q.e.d.

Lemma 6.11. *With E_t and E_0 as in equations* (6.1) *and* (6.2) *then* $\|\delta_t E_t - \delta_0 E_0\| = O(\epsilon)$.

Proof.

$$\begin{aligned}\|\delta_t E_t - \delta_0 E_0\| &\le |\delta_t - \delta_0|\,\|E_0\| + |\delta_t|\,\|E_t - E_0\| \\ &\le |\delta_t - \delta_0|\,\|E_0\| + \big|\sinh(\delta_t)\big|\,\|E_t - E_0\| \\ &\le |\delta_t - \delta_0|\,\|E_0\| + \big|\sinh(\delta_t) - \sinh(\delta_0)\big|\,\|E_0\| \\ &\quad + \big\|\sinh(\delta_t)E_t - \sinh(\delta_0)E_0\big\|.\end{aligned}$$

This is $O(\epsilon)$ using Lemmas 6.7 and 6.10. q.e.d.

Lemma 6.12. *If D_t and D_0 are given by* (6.3) *and $\|B_t - B_0\| < \epsilon$, then*

$$\|D_t - D_0\| = O(\epsilon).$$

Proof. Using Lemmas 6.7, 6.8 and 6.11 we have:

$$\begin{aligned}\|D_t - D_0\| &= \|\delta_t E_t - \delta_0 E_0 + 2i\phi_t I - 3i\phi_t E_t^2\| \\ &\le \|\delta_t E_t - \delta_0 E_0\| + |\phi_t|\,\|2I - 3E_t^2\| = O(\epsilon).\end{aligned}$$

q.e.d.

Now, consider the expansions

$$(6.6)\quad C_t^x = \exp(xD_t) = \sum_{n=0}^{\infty} \frac{x^n}{n!} D_t^n, \qquad C_0^x = \exp(xD_0) = \sum_{n=0}^{\infty} \frac{x^n}{n!} D_0^n,$$

where $x \in \mathbb{R}$ and C_t and C_0 have the forms (6.1) and (6.2).

Lemma 6.13. *If C_t^x and C_0^x are given by* (6.6), *then*

$$\|C_t^x - C_0^x\| \le \exp\big(x\|D_0\|\big)\Big(\exp\big(x\|D_t - D_0\|\big) - 1\Big).$$

Proof. We use (6.6) to write $C_t^x - C_0^x$ as an infinite series. Using the triangle inequality, Lemma 6.1 (iii) and the fact that the norm is sub-multiplicative, we have

$$\begin{aligned}\|C_t^x - C_0^x\| &\le \sum_{n=1}^{\infty} \frac{x^n}{n!}\Big\|D_t^n - D_0^n\Big\| \\ &= \sum_{n=1}^{\infty} \frac{x^n}{n!}\Big\|\big(D_0 + (D_t - D_0)\big)^n - D_0^n\Big\| \\ &\le \sum_{n=1}^{\infty} \frac{x^n}{n!}\Big(\big(\|D_0\| + \|D_t - D_0\|\big)^n - \|D_0\|^n\Big) \\ &= \exp\big(x\|D_0\| + x\|D_t - D_0\|\big) - \exp\big(x\|D_0\|\big).\end{aligned}$$

q.e.d.

Lemma 6.13 and Lemma 6.12 immediately induce the following.

Corollary 6.14. *If $\|B_t - B_0\| < \epsilon$ and $x \in [0,1]$ then $\|C_t^{x/2} - C_0^{x/2}\| = O(\epsilon)$.*

6.4. The sides are disjoint. We consider two sides of our polyhedron $\boldsymbol{\Delta}_t$. As above we may take one of these sides to be S_t with edges $R_{\mathbb{J}}$ and $B_t(R_{\mathbb{J}})$. Writing $C_t = B_t B_t^T$, the slices of S_t are the Lagrangian planes $C_t^{x/2}(R_{\mathbb{J}})$ for $x \in [0,1]$, which are all disjoint.

We need to consider a second side. First consider the pack $\widetilde{P}_t$ determined by disjoint Lagrangian planes $R_{\mathbb{J}}$ and $\widetilde{B}_t(R_{\mathbb{J}})$. The slices of $\widetilde{P}_t$ are $\widetilde{C}_t^{y/2}(R_{\mathbb{J}})$ for $y \in \mathbb{R}$, where $\widetilde{C}_t = \widetilde{B}_t\widetilde{B}_t^T$. The image of $\widetilde{P}_t$ under $\widehat{B}_t$ is the pack $\widehat{P}_t$ determined by the Lagrangian planes $\widehat{B}_t(R_{\mathbb{J}})$ and $\widehat{B}_t\widetilde{B}_t(R_{\mathbb{J}})$. Its slices are the Lagrangian planes $\widehat{B}_t\widetilde{C}_t^{y/2}(R_{\mathbb{J}})$ for $y \in \mathbb{R}$. If $\widehat{B}_t(R_{\mathbb{J}})$ and $\widehat{B}_t\widetilde{B}_t(R_{\mathbb{J}})$ are edges of $\boldsymbol{\Delta}_t$ bounding a side $\widehat{S}_t$ contained in $\widehat{P}_t$, then the slices of $\widehat{S}_t$ are the Lagrangian planes $\widehat{B}_t\widetilde{C}_t^{y/2}(R_{\mathbb{J}})$ for $y \in [0,1]$.

In order to show that the non-adjacent sides S_t and $\widehat{S}_t$ are disjoint, it suffices to show that each pair of slices is disjoint. Using Lemma 6.5, this is equivalent to showing that

$$\big(\widehat{B}_t\widetilde{C}_t^{y/2}\big)\big(\widehat{B}_t\widetilde{C}_t^{y/2}\big)^T\big(C_t^{x/2}(C_t^{x/2})^T\big)^{-1} = \widehat{B}_t C_t^y \widehat{B}_t^T C_t^{-x}$$

is loxodromic (we have used $C_t^T = C_t$ to show that $(C_t^x)^T = C_t^x$ and so on). Notice that if we apply a further $B_t^{(j)}$ to both S_t and $\widehat{S}_t$, this only conjugates $\widehat{B}_t C_t^y \widehat{B}_t^T C_t^{-x}$ by $B_t^{(j)}$. Thus our assumption that one of the edges of S_t is $R_{\mathbb{J}}$ involves no loss of generality.

Consider the corresponding sides S_0 and $\widehat{S}_0$ of $\boldsymbol{\Delta}_0$. Denote their slices $C_0^{x/2}(R_{\mathbb{J}})$ and $\widehat{B}_0 \widetilde{C}_0^{y/2}(R_{\mathbb{J}})$ respectively. We assume that S_0 and $\widehat{S}_0$ are disjoint and so they are a distance $\lambda_0 > 0$ apart.

Proposition 6.15. *Suppose that S_0 and $\widehat{S}_0$ are sides of $\boldsymbol{\Delta}_0$ a distance $\lambda_0 > 0$ apart. There exists $\epsilon > 0$ so that for all B_t, $\widetilde{B}_t$ and $\widehat{B}_t$ in $\mathrm{SU}(2,1)$ with $\|B_t - B_0\| < \epsilon$, $\|\widetilde{B}_t - \widetilde{B}_0\| < \epsilon$ and $\|\widehat{B}_t - \widehat{B}_0\| < \epsilon$, the sides S_t and $\widehat{S}_t$ are disjoint and a distance λ_t apart, where $\cosh(\lambda_t) \geq \cosh(\lambda_0) - O(\epsilon)$.*

Proof. This will follow from Proposition 6.6 provided we can show that

$$\left\|C_t^{x/2} - C_0^{x/2}\right\| = O(\epsilon) \quad \text{and} \quad \left\|\widehat{B}_t \widetilde{C}_t^{y/2} - \widehat{B}_0 \widetilde{C}_0^{y/2}\right\| = O(\epsilon).$$

The first of these is Corollary 6.14. Moreover,

$$\begin{aligned}\left\|\widehat{B}_t \widetilde{C}_t^{y/2} - \widehat{B}_0 \widetilde{C}_0^{y/2}\right\| \leq & \left\|\widehat{B}_t - \widehat{B}_0\right\| \left\|\widetilde{C}_t^{y/2} - \widetilde{C}_0^{y/2}\right\| \\ & + \left\|\widehat{B}_0\right\| \left\|\widetilde{C}_t^{y/2} - \widetilde{C}_0^{y/2}\right\| + \left\|\widehat{B}_t - \widehat{B}_0\right\| \left\|\widetilde{C}_0^{y/2}\right\|\end{aligned}$$

Thus the second also follows from Corollary 6.14. q.e.d.

We now consider a pair of adjacent sides S_t and $\widetilde{S}_t$ of $\boldsymbol{\Delta}_t$. Using the discussion above, we may assume that these sides are S_t determined by $R_{\mathbb{J}}$ and $B_t(R_{\mathbb{J}})$ and $\widetilde{S}_t$ determined by $R_{\mathbb{J}}$ and $\widetilde{B}_t(R_{\mathbb{J}})$. Writing $C_t = B_t B_t^T$ and $\widetilde{C}_t = \widetilde{B}_t \widetilde{B}_t^T$, from Proposition 6.5 it suffices to show that $\widetilde{C}_t^y C_t^{-x}$ is loxodromic for all $(x,y) \in [0,1] \times [0,1]$. Observe that if x or y is zero then we already have the result from Proposition 3.1 (iv). Thus it suffices to consider $(x,y) \in (0,1] \times (0,1]$. Again, our assumption that S_t and $\widetilde{S}_t$ intersect in $R_{\mathbb{J}}$ involves no loss of generality.

The sides S_t and $\widetilde{S}_t$ are deformations of sides S_0 and $\widetilde{S}_0$ of $\boldsymbol{\Delta}_0$, which intersect in $R_{\mathbb{J}}$ by hypothesis. Thus we know that $\widetilde{C}_0^y C_0^{-x}$ is loxodromic for all $(x,y) \in (0,1] \times (0,1]$.

If x and y are not both small then there exists $\lambda_0 > 0$ so that the slices $C_0^{x/2}(R_{\mathbb{J}})$ and $\widetilde{C}_0^{y/2}(R_{\mathbb{J}})$ are a distance at least λ_0 apart. Choosing x and y so that this λ_0 is large compared to ϵ, we can use Proposition 6.6 to show that the slices $C_t^{x/2}(R_{\mathbb{J}})$ and $\widetilde{C}_t^{y/2}(R_{\mathbb{J}})$ are disjoint:

Proposition 6.16. *Suppose that S_0 and $\widetilde{S}_0$ are sides of $\boldsymbol{\Delta}_0$ with slices $C_0^{x/2}(R_{\mathbb{J}})$ and $\widetilde{C}_0^{y/2}(R_{\mathbb{J}})$. Given $\eta > 0$ there exists $\epsilon > 0$ so that for all B_t and $\widetilde{B}_t$ in $\mathrm{SU}(2,1)$ with $\|B_t - B_0\| < \epsilon$ and $\|\widetilde{B}_t - \widetilde{B}_0\| < \epsilon$ then for all $(x,y) \in \big((0,1] \times (0,1]\big) - \big((0,\eta] \times (0,\eta]\big)$ the slices $C_t^{x/2}(R_{\mathbb{J}})$ and $\widetilde{C}_t^{y/2}(R_{\mathbb{J}})$ of S_t and $\widetilde{S}_t$ are disjoint.*

Proof. This will again follow from Proposition 6.6. We can find λ_0 (depending on η) so that for all $(x,y) \in \big((0,1]\times(0,1]\big) - \big((0,\eta]\times(0,\eta]\big)$ the slices $C_0^{x/2}(R_{\mathbb{J}})$ and $\widetilde{C}_0^{y/2}(R_{\mathbb{J}})$ are a distance λ_0 apart. We also know from Corollary 6.14 that

$$\big\|C_t^{x/2} - C_0^{x/2}\big\| = O(\epsilon) \quad \text{and} \quad \big\|\widetilde{C}_t^{x/2} - \widetilde{C}_0^{x/2}\big\| = O(\epsilon).$$

q.e.d.

Finally, when x and y are both small we need a different argument.

Proposition 6.17. *Suppose that S_0 and $\widetilde{S}_0$ are sides of $\mathbf{\Delta}_0$ with slices $C_0^{x/2}(R_{\mathbb{J}})$ and $\widetilde{C}_0^{y/2}(R_{\mathbb{J}})$. There exists $\eta > 0$ and $\epsilon > 0$ so that for all B_t and $\widetilde{B}_t$ in $\mathrm{SU}(2,1)$ with $\|B_t - B_0\| < \epsilon$ and $\|\widetilde{B}_t - \widetilde{B}_0\| < \epsilon$ the slices $C_t^{x/2}(R_{\mathbb{J}})$ and $\widetilde{C}_t^{y/2}(R_{\mathbb{J}})$ of S_t and $\widetilde{S}_t$ are disjoint for all $(x,y) \in (0,\eta]\times(0,\eta]$.*

Proof. We will show that $\widetilde{C}_t^{y}C_t^{-x}$ is loxodromic for all $(x,y) \in (0,\eta]\times(0,\eta]$. The result will then follow from Lemma 6.5.

Let δ_0 be the distance between $R_{\mathbb{J}}$ and $B_0(R_{\mathbb{J}})$ and let $\widetilde{\delta}_0$ that between $R_{\mathbb{J}}$ and $\widetilde{B}_0(R_{\mathbb{J}})$. Then

$$\begin{aligned}
\mathrm{tr}(C_0) &= 2\cosh\Big(\rho\big(o, B_0(o)\big)\Big) + 1 = 2\cosh(\delta_0) + 1,\\
\mathrm{tr}(\widetilde{C}_0) &= 2\cosh\Big(\rho\big(o, \widetilde{B}_0(o)\big)\Big) + 1 = 2\cosh(\widetilde{\delta}_0) + 1.
\end{aligned}$$

Hence for $x \in (0,1]$ and $y \in (0,1]$ we have

$$\mathrm{tr}\big(C_0^x\big) = 2\cosh(x\delta_0) + 1, \quad \mathrm{tr}\big(\widetilde{C}_0^y\big) = 2\cosh(y\widetilde{\delta}_0) + 1.$$

Let $\lambda_{x,y}$ be the hyperbolic distance between $C_0^{x/2}(o) = \Pi_{\mathbb{R}}C_0^{x/2}(R_{\mathbb{J}})$ and $\widetilde{C}_0^{y/2}(o) = \Pi_{\mathbb{R}}\widetilde{C}_0^{y/2}(R_{\mathbb{J}})$. Also let ψ be the angle at $o = \Pi_{\mathbb{R}}(R_{\mathbb{J}})$ in the hyperbolic plane between the geodesic arcs $\Pi_{\mathbb{R}}(S_0)$ and $\Pi_{\mathbb{R}}(\widetilde{S}_0)$. Then using plane hyperbolic trigonometry, see page 24 of [**14**], for the Lagrangian plane $R_{\mathbb{R}}$ which has curvature $-1/4$, we see that

$$\begin{aligned}
\cosh(\lambda_{x,y}/2) &= \cosh(x\delta_0/2)\cosh(y\widetilde{\delta}_0/2)\\
&\qquad - \sinh(x\delta_0/2)\sinh(y\widetilde{\delta}_0/2)\cos(\psi).
\end{aligned}$$

By omitting subscripts, we take the second order expansion

$$\begin{aligned}
C^x &= \exp(xD) = I + \frac{x}{1!}D + \frac{x^2}{2!}D^2 + \text{ higher order terms},\\
\widetilde{C}^y &= \exp(y\widetilde{D}) = I + \frac{y}{1!}\widetilde{D} + \frac{y^2}{2!}\widetilde{D}^2 + \text{ higher order terms},
\end{aligned}$$

where the higher order terms involve multiples of x^3, respectively y^3, and higher powers. We assume that η is sufficiently small that we may neglect these higher order terms in what follows. Then, since $\mathrm{tr}(D) = \mathrm{tr}(\widetilde{D}) = 0$, we have

$$\mathrm{tr}(\widetilde{C}^y C^{-x}) = 3 + \frac{x^2}{2}\mathrm{tr}(D^2) - xy\mathrm{tr}(D\widetilde{D}) + \frac{y^2}{2}\mathrm{tr}(\widetilde{D}^2) + \text{ higher order terms}$$

where the higher order terms are of the form $x^a y^b$ with $a+b \geq 3$.

On the other hand, we have

$$\begin{aligned}
\mathrm{tr}(\widetilde{C}_0^y C_0^{-x}) &= 1 + 2\cosh(\lambda_{x,y}) \\
&= 4\cosh^2(\lambda_{x,y}/2) - 1 \\
&= 4\Big(\cosh(x\delta_0/2)\cosh(y\widetilde{\delta}_0/2) \\
&\qquad - \sinh(x\delta_0/2)\sinh(y\widetilde{\delta}_0/2)\cos(\psi)\Big)^2 - 1 \\
&= 3 + x^2\delta_0^2 + y^2\widetilde{\delta}_0^2 - 2xy\delta_0\widetilde{\delta}_0\cos(\psi) + \text{ higher order terms.}
\end{aligned}$$

where the last line was obtained by expanding into Taylor series. Again the higher order terms are multiples of $x^a y^b$ with $a+b \geq 3$. We already know from (6.3) that $\mathrm{tr}(D_0^2) = 2\delta_0^2$ and $\mathrm{tr}(\widetilde{D}_0^2) = 2\widetilde{\delta}_0^2$. By comparing these two expressions for $\mathrm{tr}(\widetilde{C}_0^y C_0^{-x})$ and equating coefficients, we see that $\mathrm{tr}(D_0\widetilde{D}_0) = 2\delta_0\widetilde{\delta}_0\cos(\psi)$.

Consider the quadratic form

$$q_0(x,y) = x^2\mathrm{tr}(D_0^2) - 2xy\mathrm{tr}(D_0\widetilde{D}_0) + y^2\mathrm{tr}(\widetilde{D}_0^2).$$

Its discriminant d_0 is

$$\begin{aligned}
d_0 &= \mathrm{tr}(D_0^2)\mathrm{tr}(\widetilde{D}_0^2) - \mathrm{tr}^2(D_0\widetilde{D}_0) \\
&= 4\delta_0^2\widetilde{\delta}_0^2 - 4\delta_0^2\widetilde{\delta}_0^2\cos^2(\psi) = 4\delta_0^2\widetilde{\delta}_0^2\sin^2(\psi) > 0.
\end{aligned}$$

Thus $q_0(x,y)$ is positive definite.

Similarly, consider

$$q_t(x,y) = x^2\mathrm{tr}(D_t^2) - 2xy\mathrm{tr}(D_t\widetilde{D}_t) + y^2\mathrm{tr}(\widetilde{D}_t^2)$$

with discriminant

$$d_t = \mathrm{tr}(D_t^2)\mathrm{tr}(\widetilde{D}_t^2) - \mathrm{tr}^2(D_t\widetilde{D}_t).$$

If we can show that for small x and y that $q_t(x,y)$ is positive definite then $\mathrm{tr}(\widetilde{C}_t^y C_t^{-x})$ will be bounded away from 3, which will prove our result. It suffices to show that $|d_t - d_0| \leq O(\epsilon)$, where $O(\epsilon)$ is a positive function of ϵ. Indeed,

$$\begin{aligned}
|d_t - d_0| &\leq \big|\mathrm{tr}(D_t^2)\mathrm{tr}(\widetilde{D}_t^2) - \mathrm{tr}(D_0^2)\mathrm{tr}(\widetilde{D}_0^2)\big| + \big|\mathrm{tr}^2(D_t\widetilde{D}_t) - \mathrm{tr}^2(D_0\widetilde{D}_0)\big| \\
&\leq \big|\mathrm{tr}(D_t^2)\mathrm{tr}(\widetilde{D}_t^2) - \mathrm{tr}(D_0^2)\mathrm{tr}(\widetilde{D}_0^2)\big| + \big|\mathrm{tr}(D_t\widetilde{D}_t) - \mathrm{tr}(D_0\widetilde{D}_0)\big|^2 \\
&\quad + 2\big|\mathrm{tr}(D_0\widetilde{D}_0)\big|\,\big|\mathrm{tr}(D_t\widetilde{D}_t) - \mathrm{tr}(D_0\widetilde{D}_0)\big|.
\end{aligned}$$

We have $\left|\mathrm{tr}(D_0\widetilde{D}_0)\right| = \left|2\delta_0\widetilde{\delta}_0\cos(\psi)\right| \le 2\delta_0\widetilde{\delta}_0$. We estimate the other terms. In the first place

$$\begin{aligned}&\left|\mathrm{tr}(D_t^2)\mathrm{tr}(\widetilde{D}_t^2) - \mathrm{tr}(D_0^2)\mathrm{tr}(\widetilde{D}_0^2)\right|\\ &= \left|(2\delta_t^2 - 6\phi_t^2)(2\widetilde{\delta}_t^2 - 6\widetilde{\phi}_t^2) - 4\delta_0^2\widetilde{\delta}_0^2\right|\\ &\le 4\left|\delta_t^2\widetilde{\delta}_t^2 - \delta_0^2\widetilde{\delta}_0^2\right| + 12\delta_t^2\widetilde{\phi}_t^2 + 12\widetilde{\delta}_t^2\phi_t^2 + 36\phi_t^2\widetilde{\phi}_t^2\end{aligned}$$

which is $O(\epsilon)$ due to Lemma 6.7. On the other hand, using Lemma 6.1,

$$\begin{aligned}&\left|\mathrm{tr}(D_t\widetilde{D}_t) - \mathrm{tr}(D_0\widetilde{D}_0)\right|\\ &\le \sqrt{3}\left\|D_t\widetilde{D}_t - D_0\widetilde{D}_0\right\|\\ &\le \sqrt{3}\left\|D_t - D_0\right\|\,\left\|\widetilde{D}_t - \widetilde{D}_0\right\| + \sqrt{3}\left\|D_0\right\|\,\left\|\widetilde{D}_t - \widetilde{D}_0\right\|\\ &\quad + \sqrt{3}\left\|D_t - D_0\right\|\,\left\|\widetilde{D}_0\right\|.\end{aligned}$$

Using Lemma 6.12 we see that this is also $O(\epsilon)$ and thus our assertion is proved. q.e.d.

Combining Propositions 6.15, 6.16 and 6.17 proves Theorem 5.2.

References

[1] A.F. Beardon, *The Geometry of Discrete Groups*, Springer, 1983, MR 0698777, Zbl 0528.30001.

[2] L. Bers, *Spaces of Kleinian groups*, Several Complex Variables I, Lecture Notes in Mathematics **155** (1970) 9–34, MR 0271333, Zbl 0211.10602.

[3] B.H. Bowditch, *Geometrical finiteness with variable negative curvature*, Duke Math. J. **77** (1995) 229–274, MR 1317633, Zbl 0877.57018.

[4] S.S. Chen & L. Greenberg, *Hyperbolic spaces*, Contributions to Analysis, Academic Press, 1974, 49–87, MR 0377765, Zbl 0295.53023.

[5] R. Courant & F. John, *Introduction to Calculus and Analysis*, Volume 2, Wiley, 1974, MR 1016380, Zbl 0294.26003.

[6] A. Domic & D. Toledo, *The Gromov norm of the Kähler class of symmetric domains*, Math. Ann. **276** (1987) 425–432, MR 0875338, Zbl 0595.53061.

[7] F. Dutenhefner & N. Gusevskii, *Complex hyperbolic Kleinian groups with limit set a wild knot*, Topology **43** (2004) 677–696, MR 2041637, Zbl 1051.30038.

[8] D. B. A. Epstein & C. Petronio, *An exposition of Poincaré's polyhedron theorem*, L'Enseignement Mathématique **40** (1994) 113–170, MR 1279064, Zbl 0808.52011.

[9] E. Falbel & P.-V. Koseleff; *Flexibility of the ideal triangle groups in complex hyperbolic geometry*, Topology **39** (2000) 1209–1223, MR 1783854, Zbl 0977.32018.

[10] ———, *Rigidity and flexibility of triangle groups in complex hyperbolic geometry*, Topology **41** (2002) 767–786, MR 1905838, Zbl 1005.32018.

[11] E. Falbel & J.R. Parker, *The moduli space of the modular group in complex hyperbolic geometry*, Invent. Math. **152** (2003) 57–88, MR 1965360.

[12] E. Falbel & V. Zocca, *A Poincaré's polyhedron theorem for complex hyperbolic geometry*, J. Reine Angew. Math. **516** (1999) 133–158, MR 1724618, Zbl 0944.53042.

[13] W.M. Goldman, *Representations of fundamental groups of surfaces*, Geometry and Topology, eds. J. Alexander & J. Harer, Lecture Notes in Mathematics **1167** (1985) 95–117, MR 0827264, Zbl 0575.57027.

[14] ———, *Complex Hyperbolic Geometry*, Oxford University Press, 1999, MR 1695450, Zbl 0939.32024.

[15] W.M. Goldman, M.E. Kapovich, & B. Leeb, *Complex hyperbolic manifolds homotopy equivalent to a Riemann surface*, Comm. Anal. Geom. **9** (2001) 61–95, MR 1807952, Zbl 0982.32024.

[16] W.M. Goldman & J.R. Parker, *Complex hyperbolic ideal triangle groups*, J. Reine Angew. Math. **425** (1992) 71–86, MR 1151314, Zbl 0739.53055.

[17] N. Gusevskii & J.R. Parker, *Complex hyperbolic quasi-Fuchsian groups and Toledo's invariant*, Geometriae Dedicata, **97** (2003) 151–185, MR 2003696, Zbl 1042.57023.

[18] R.E. Schwartz, *Degenerating the complex hyperbolic ideal triangle groups*, Acta Math., **186** (2001) 105–154, MR 1828374, Zbl 0998.53050.

[19] D. Sullivan, *Quasiconformal homeomorphisms and dynamics* II: *Structural stability implies hyperbolicity for Kleinian groups*, Acta Math. **155** (1985) 243–260, MR 0806415, Zbl 0606.30044.

[20] D. Toledo, *Representations of surface groups in complex hyperbolic space*, J. Differential Geometry **29** (1989) 125–133, MR 0978081, Zbl 0676.57012.

[21] P. Will, *The punctured torus and Lagrangian triangle groups in* $\mathrm{PU}(2,1)$, to appear in J. Reine Angew. Math..

[22] E.Z. Xia, *The moduli of flat* $\mathrm{PU}(2,1)$ *structures on Riemann surfaces*, Pacific J. Maths. **195** (2000) 231–256, MR 1781622, Zbl 1014.32010.

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