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ISOMETRY-INVARIANT GEODESICS AND NONPOSITIVE DERIVATIONS OF THE COHOMOLOGY

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Abstract

We introduce a new class of artinian weighted complete intersections, by abstracting the essential features of $\mathbb{Q}$-cohomology algebras of equal rank homogeneous spaces of compact connected Lie groups. We prove that, on a 1-connected closed manifold M with $H^*(M, \mathbb{Q})$ belonging to this class, every isometry has a non-trivial invariant geodesic, for any metric on M. We use $\mathbb{Q}$-surgery to construct large classes of new examples for which the above result may be applied.

1. Introduction and statement of main results

1.1. The isometry-invariant geodesics problem. Let M^m be a connected, complete, Riemannian m-manifold. It is well-known that the study of isometry-invariant geodesics on M is an important topic in Riemannian geometry. Recall that a geodesic curve, $\gamma\colon \mathbb{R} \to M$, is called a-invariant, where $a \in \mathrm{Isom}(M)$, if there exists a period t such that $a(\gamma(x)) = \gamma(x+t)$, for any x. When $a = \mathrm{id}$, one recovers the classical notion of closed geodesic. For non-compact or non-simply connected manifolds, there are simple examples of isometries without non-trivial (i.e., non-constant) invariant geodesics. Take for instance the rotation of angle $\frac{\pi}{2}$ in the euclidean plane, and the induced isometry on the flat 2-torus; see Grove's survey [**7**, Section 1].

It is therefore natural to confine our attention, from now on, to 1-connected, closed, manifolds M^m. The basic problem [**7**, Problem 5.1] is to prove or disprove the following:

$$\text{(1.1)} \quad \text{every Riemannian isometry } a \text{ on } M \text{ has a non-trivial } a\text{-invariant geodesic.}$$

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No counter-examples are known yet. The main direction in attacking this problem so far was to find topological conditions on M guaranteeing the ideal property (1.1), a viewpoint also adopted in this paper. For instance, strong existence results for closed geodesics were obtained, under simple topological hypotheses on M, by Sullivan and Vigué-Poirrier in [**20**], using rational homotopy methods. In the general case, a fundamental homotopical criterion, for a given isometry a, is due to Grove [**6**, Theorem 1.6(1)]. It says that a has non-trivial invariant geodesics, as soon as $\mathrm{id} - \pi_q(a)\colon \pi_q(M) \to \pi_q(M)$ is not an isomorphism on homotopy groups, for some q. It follows that if $\pi_q(a) \otimes \mathbb{Q}$ has prime power order, for some q such that $\pi_q(M) \otimes \mathbb{Q} \neq 0$, then a-invariant geodesics exist; see [**6**, Corollary 1.7]. It also follows that, if the rank of $\pi_q(M)$ is odd, for some q, then property (1.1) holds; see [**6**, Corollary 1.8(1)].

Further refinements of Grove's above-mentioned theorem led to various answers to the central question (1.1). Recall that a 1-connected finite complex S is (rationally) *elliptic* if $\dim_{\mathbb{Q}} \pi_*(S) \otimes \mathbb{Q} < \infty$, and (rationally) *hyperbolic* otherwise. It is known that the second class is vastly bigger than the first. On the other hand, the first class contains many interesting manifolds, such as all (1-connected) homogeneous spaces. Major contributions to the isometry-invariant geodesics problem (1.1) were obtained by Grove and Halperin in [**8**, Theorem A and Theorem 2]. Their results are as follows. If a is without non-trivial invariant geodesics, then M^m is elliptic and m is even; if moreover the Euler characteristic $\chi(M)$ is non-zero, then the Poincaré polynomial of M is equal to the Poincaré polynomial of $S \times S$, with S elliptic.

An interesting family of elliptic manifolds M^m, with m even and $\chi(M) \neq 0$, is provided by the homogeneous spaces $M = G/K$, with G and K compact and connected, and of the same rank. It is known that for this family, property (1.1) holds, when K is a maximal torus ([**15**, Theorem 1.4(i)]) or G is simple ([**16**, Corollary 1.4(i)]).

1.2. Complexes which look like homogeneous spaces. Let $\mathcal{A}$ be a *weighted, artinian, complete intersection* ($WACI$), i.e., a commutative graded $\mathbb{Q}$-algebra of the form

$$\mathcal{A} = \mathbb{Q}[x_1, \dots, x_n]/\mathcal{I}, \tag{1.2}$$

where the variables x_i have positive even weights, $w_i := \mid x_i \mid$, and the ideal $\mathcal{I}$ is generated by a *regular sequence*,

$$\mathcal{I} = (f_1, \dots, f_n), \tag{1.3}$$

of n weighted-homogeneous polynomials, f_i (or, equivalently, $\dim_{\mathbb{Q}} \mathcal{A} < \infty$).

We are going to introduce a special class of $WACI$'s, with an eye for applications in Riemannian geometry.

One may speak about the *pseudo-homotopy groups* of a finitely generated, graded-commutative $\mathbb{Q}$-algebra, $\mathcal{A}$, supposed to be 1-connected (that is, $\mathcal{A}^0 = \mathbb{Q}$ and $\mathcal{A}^1 = 0$):

$$\pi_*^*(\mathcal{A}) = \bigoplus_{i\geq 0, j>1} \pi_i^j(\mathcal{A})\,. \tag{1.4}$$

These algebraic invariants of $\mathcal{A}$ are finite-dimensional graded $\mathbb{Q}$-vector spaces, $\pi_i(\mathcal{A}) := \oplus_j \pi_i^j(\mathcal{A})$, for all $i \geq 0$. See Halperin and Stasheff [**10**].

When $\mathcal{A}$ is a $WACI$, as above, one knows that $\pi_{>1}(\mathcal{A}) = 0$. Moreover, for any 1-connected CW-complex, S, such that $H^*(S, \mathbb{Q}) = \mathcal{A}^*$, as graded algebras, one has the following topological interpretation (see Sullivan [**19**]):

$$\begin{cases} \pi_0^*(\mathcal{A}) = \mathrm{Hom}_{\mathbb{Q}}(\pi_{*=\mathrm{even}}(S) \otimes \mathbb{Q}, \mathbb{Q}), \quad \text{and} \\ \pi_1^*(\mathcal{A}) = \mathrm{Hom}_{\mathbb{Q}}(\pi_{*=\mathrm{odd}}(S) \otimes \mathbb{Q}, \mathbb{Q}). \end{cases} \tag{1.5}$$

Assuming $\mathcal{A} \neq \mathbb{Q}$, that is, $\pi_1^*(\mathcal{A}) \neq 0$, set now:

$$k_{\mathcal{A}} := \max\{k \mid \pi_1^{2k-1}(\mathcal{A}) \neq 0\}. \tag{1.6}$$

For $p \in \mathbb{Z}$, denote by

$$\mathrm{Der}^p(\mathcal{A}) = \Big\{\theta \in \mathrm{Hom}_{\mathbb{Q}}(\mathcal{A}^*, \mathcal{A}^{*+p}) \ \mid\ \theta(ab) = \theta(a)\cdot b + (-1)^{pq} a \cdot \theta(b), \\ \forall a \in \mathcal{A}^q, b \in \mathcal{A}^*\Big\}$$

the degree p homogeneous derivations of $\mathcal{A}^*$. (Note that $\mathrm{Der}^*(\mathbb{Q}) = 0$.)

Definition 1.3. We shall say that the $WACI$ $\mathcal{A}$ is *simple* if:

(i) $\mathrm{Der}^{<0}(\mathcal{A}) = 0$.
(ii) $\dim_{\mathbb{Q}} \mathrm{Der}^0(\mathcal{A}) = 1$.
(iii) $\dim_{\mathbb{Q}} \pi_1^{2k_{\mathcal{A}}-1}(\mathcal{A}) = 1$.

Definition 1.4. A 1-connected CW-complex S is said to be *homologically homogeneous* (HH) if

$$H^*(S, \mathbb{Q}) = \bigotimes_{j\in J} \mathcal{A}_j^*,$$

as graded algebras, where the index set J is finite and the algebra $\mathcal{A}_j$ is a simple $WACI$, for every $j \in J$.

Both definitions are motivated by geometry. Firstly, it is well-known that the $\mathbb{Q}$-cohomology algebra of an equal rank homogeneous space, G/K, of compact connected Lie groups, is a $WACI$; see Borel [**3**]. Moreover, $G/K = \prod_{j\in J} G_j/K_j$, where in addition each Lie group G_j is simple. Now, for G simple and $\mathcal{A}^* = H^*(G/K, \mathbb{Q})$, properties (i) and (ii) from Definition 1.3 were proved by Shiga and Tezuka in [**18**]. Property (iii) follows from [**16**, Theorem 1.3], via (1.5). Therefore, all equal rank homogeneous spaces G/K mentioned above are HH, in the sense of our

Definition 1.4. Note that HH manifolds are elliptic – see (1.5) – and even-dimensional, with $\chi \neq 0$, see [**9**].

We exhibit in Section 4 (Theorems 4.4 and 4.8) families of new examples of HH closed manifolds. Their $\mathbb{Q}$-cohomology algebras are actually simple $WACI$'s, not isomorphic to the $\mathbb{Q}$-cohomology algebra of any equal rank homogeneous space G/K, with G not necessarily simple. We first use tools from our previous work [**17**], to obtain simple $WACI$'s from 1-dimensional *reduced* weighted complete intersections. Rational surgery methods, due to Sullivan [**19**] and Barge [**1**], enable us to realize them by closed manifolds.

1.5. Isometry-invariant geodesics on HH manifolds. To the best of our knowledge, there has been no progress about the basic question (1.1), since the Grove–Halperin paper [**8**], excepting the families of homogeneous spaces treated in [**15**] and [**16**]. Our main result below significantly enlarges the class of manifolds M for which (1.1) holds.

Theorem 1.6. *If the closed manifold M is homologically homogeneous* (*in the sense of Definition* 1.4), *then, given an arbitrary metric on M, there is a non-trivial a-invariant geodesic, for every isometry, a.*

Our approach is based on Grove's theorem from [**6**] quoted above, and rational homotopy theory methods. Along the way, we shall also clarify the structure of the automorphism group of certain related graded algebras, $\mathcal{B}^*$. More precisely, assume that $\mathcal{B}^* = \otimes_{j \in J} \mathcal{B}_j^*$, where J is finite, and each connected, commutative, evenly-graded $\mathbb{C}$-algebra $\mathcal{B}_j^*$ is artinian and satisfies properties (i) and (ii) from Definition 1.3 (over $\mathbb{C}$). Then, the connected component of 1 in the (linear algebraic) group of graded algebra automorphisms of $\mathcal{B}^*$ is an algebraic torus, $(\mathbb{C}^*)^r$, where $r = \mid J \mid$. See Corollary 2.7.

We finally mention that there are simple examples, where none of the above known results yield existence of isometry-invariant geodesics, while our Theorem 1.6 does. Let M be the equal rank homogeneous space $(\mathbb{CP}^2)^{\times 4}$. Let $a \in \mathrm{Diff}(M)$ be the order 6 diffeomorphism defined by $a(p_1, p_2, p_3, p_4) = (\bar{p_1}, \bar{p_4}, \bar{p_2}, \bar{p_3})$, where the bar denotes the map induced by complex conjugation. Fix any metric on M and average it over the powers of a to get a new metric, for which $a \in \mathrm{Isom}(M)$. If a' is another isometry of M, homotopic to a, does a' have non-trivial invariant geodesics? By Theorem 1.6, the answer is yes. See Example 2.12 for details.

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2. Proof of the main result

The proof of our Theorem 1.6 is based on the following geometric result, due to Grove [**6**, Theorem 1.6].

Theorem 2.1. *Let M be a 1-connected, closed, Riemannian manifold and let $a\colon M \to M$ be an isometry. If there is no non-trivial a-invariant geodesic, then a has exactly one fixed point, and* $\mathrm{id} - \pi_*(a) : \pi_*(M) \to \pi_*(M)$ *is an isomorphism.*

Assuming $H^*(M, \mathbb{Q}) = \otimes_{j\in J}\mathcal{A}_j^*$ (see Definition 1.4), set $k_j = k_{\mathcal{A}_j}$, for $j \in J$, and $k = \max\{k_j \mid j \in J\}$. We will derive a contradiction, for $* = 2k-1$, by showing that $\mathrm{id} - \pi_{2k-1}(a) : \pi_{2k-1}(M) \to \pi_{2k-1}(M)$ cannot be an isomorphism, for any self-homotopy equivalence, $a : M \to M$.

2.2. Dual homotopy representations. Rational homotopy theory methods [**19**] may be used to get information on $\pi_{2k-1}(a) \otimes \mathbb{Q}$. Tools from the representation theory of linear algebraic groups (see e.g., [**11**]) will provide additional information, so we will use $\mathbb{C}$-coefficients. Our goal is to prove that one cannot have simultaneously $\det(\pi_{2k-1}(a) \otimes \mathbb{C}) = \pm 1$, and $\det(\mathrm{id} - \pi_{2k-1}(a) \otimes \mathbb{C}) = \pm 1$, where a is an arbitrary self-homotopy equivalence of M. (Compare with the proof of Corollary 1.8(1) from [**6**].)

Set $\mathcal{A}^* = H^*(M, \mathbb{Q})$. Since M is HH, $\mathcal{A}$ is a $WACI$. As shown in [**19**], this implies that the $\mathbb{C}$-minimal model of M, $(\mathcal{M}, d)$, has the following properties. It is a bigraded algebra of the form $\mathcal{M} = \mathbb{C}[Z_0] \otimes \bigwedge Z_1$, where $Z_0 = \mathbb{C} - \mathrm{span}\{x_1, \dots, x_n\}$, with x_i of even upper degree, and $Z_1 = \mathbb{C} - \mathrm{span}\{y_1, \dots, y_n\}$, with y_i of odd upper degree. The differential d is homogeneous, with upper degree $+1$ and lower degree -1. More precisely, $dx_i = 0$ and $dy_i \in \mathbb{Q}[x_1, \dots, x_n]$ is a polynomial with no linear part, for all i. Moreover, $H_+(\mathcal{M}, d) = 0$, and $Z_j^* = \pi_j^*(\mathcal{A}) \otimes \mathbb{C}$; see [**10**].

Denote by $\mathcal{E}$ the group of self-homotopy equivalences of M. By (1.5), the natural action of $\mathcal{E}$ on $\pi_{2k-1}(M)$ gives rise to a dual homotopy (anti)representation,

$$\tau\colon \mathcal{E} \longrightarrow GL(Z_1^{2k-1})\,. \tag{2.1}$$

We will use the above representation to show that $\det(\mathrm{id} - \tau(a)) \neq \pm 1$, for all $a \in \mathcal{E}$. This will yield the desired contradiction, thereby proving Theorem 1.6 from the Introduction.

Denote by $\mathcal{I} \subset \mathbb{C}[Z_0]$ the ideal generated by $\{dy_i\}_{1\le i\le n}$, and by $\mathbb{C}^+[Z_0]$ the strictly positively graded part of $\mathbb{C}[Z_0]$.

Lemma 2.3. *The differential d induces a vector space isomorphism,*

$$\Psi\colon Z_1^{2k-1} \xrightarrow{\sim} [\mathcal{I}/\mathbb{C}^+[Z_0]\cdot\mathcal{I}]^{2k}.$$

Proof. The surjectivity of Ψ is immediate. To check injectivity, assume that

$$\sum_{i=1}^{n} c_i dy_i = \sum_{i=1}^{n} p_i dy_i,$$

with $c_i \in \mathbb{C}$ and $p_i \in \mathbb{C}^+[Z_0]$. Since $H_1(\mathcal{M}) = 0$, we infer that $\sum_{i=1}^n (c_i - p_i) y_i \in d(\mathcal{M}_2)$. Plainly, $d(\mathcal{M}_2) \subset \mathbb{C}^+[Z_0] \otimes Z_1$, hence $c_i = 0$, for all i. q.e.d.

Denote by $\mathrm{Aut}_{\mathbb{C}}(\mathcal{A})$ the group of graded algebra automorphisms of $\mathcal{A}^* \otimes \mathbb{C}$. It is a linear algebraic group, defined over $\mathbb{Q}$. There is a cohomology (anti)representation,

$$h\colon \mathcal{E} \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathcal{A}), \tag{2.2}$$

which associates to $a \in \mathcal{E}$ the cohomology automorphism $H^*(a, \mathbb{C})$.

Our next goal is to relate (2.1) and (2.2), by constructing an algebraic representation,

$$\nu\colon\ \mathrm{Aut}_{\mathbb{C}}(\mathcal{A}) \longrightarrow GL(Z_1^{2k-1}), \tag{2.3}$$

called the *pseudo-homotopy representation.* To do this, we first consider the linear algebraic group $\mathrm{Aut}_{\mathcal{I}}(\mathbb{C}[Z_0])$, consisting of those graded algebra automorphisms, α, of $\mathbb{C}[Z_0]$, which leave the ideal $\mathcal{I}$ invariant. From Lemma 2.3, we get a natural algebraic representation,

$$\nu\colon\ \mathrm{Aut}_{\mathcal{I}}(\mathbb{C}[Z_0]) \longrightarrow GL(Z_1^{2k-1}). \tag{2.4}$$

Note that $\mathcal{A}^* \otimes \mathbb{C} = H^*(\mathcal{M}, d) = H_0^*(\mathcal{M}, d) = \mathbb{C}[Z_0]/\mathcal{I}$. It follows that $\mathrm{Aut}_{\mathbb{C}}(\mathcal{A})$ is the quotient of $\mathrm{Aut}_{\mathcal{I}}(\mathbb{C}[Z_0])$ by the subgroup of those elements α having the property that

$$\alpha(x_i) \equiv x_i \quad (\text{modulo } \mathcal{I})\,, \quad \text{for all} \quad i. \tag{2.5}$$

It is easy to infer from (2.5) that α induces the identity on $\mathcal{I}/\mathbb{C}^+[Z_0] \cdot \mathcal{I}$, by resorting to the minimality property of $(\mathcal{M}, d)$. Thus, (2.4) factors to give the desired representation, (2.3).

Lemma 2.4. *The representations* (2.1), (2.2) *and* (2.3) *are related by:*

$$\tau \quad = \quad \nu \circ h.$$

Proof. For $a \in \mathcal{E}$, denote by $\widehat{a}\colon (\mathcal{M}, d) \to (\mathcal{M}, d)$ the Sullivan minimal model of a. It is a differential graded algebra (DGA) automorphism, with the property that $H^*(\widehat{a}) = h(a)$.

Recall from [**19**] that $\tau(a) = \pi^{2k-1}(\widehat{a})$, where $\pi^{2k-1}(\widehat{a})\colon Z_1^{2k-1} \to Z_1^{2k-1}$ denotes the map induced by the restriction of $\widehat{a}$ to Z_1^{2k-1}, modulo decomposable elements of $\mathcal{M}^+$.

To compute $\nu(h(a))$, we need to lift $H^*(\widehat{a})$ to a graded algebra map, $\mathbb{C}[Z_0] \to \mathbb{C}[Z_0]$, leaving $\mathcal{I}$ invariant. To do this, we may proceed as follows. Start by noting that $\widehat{a}$ increases lower degrees. It is enough to check this on algebra generators of $\mathcal{M}$, which is obvious on Z_0; on Z_1, this follows from the fact that $\widehat{a}(y_i)$ has odd upper degree, for all i. Write then $\widehat{a} = \sum_{j \geq 0} \widehat{a}_j$, with $\widehat{a}_j \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{M}_*, \mathcal{M}_{*+j})$. An easy lower degree argument, together with the bihomogeneity property of d, show that $\widehat{a}_0$ is a bigraded differential algebra map. Set $\alpha := \widehat{a}_0\colon \mathbb{C}[Z_0] \to$

$\mathbb{C}[Z_0]$. Since $H_+(\mathcal{M}, d) = 0$, it is easily seen that $\alpha \in \mathrm{Aut}_{\mathcal{I}}(\mathbb{C}[Z_0])$ lifts $H^*(\widehat{a}) \in \mathrm{Aut}_{\mathbb{C}}(\mathcal{A})$.

To finish our proof, let us check that

$$d\pi^{2k-1}(\widehat{a})(y_i) \equiv \widehat{a}_0(dy_i), \quad \text{modulo} \quad \mathbb{C}^+[Z_0] \cdot \mathcal{I}, \tag{2.6}$$

for y_i of (upper) degree $2k-1$; see Lemma 2.3. Note that$\pi^{2k-1}(\widehat{a})(y_i)$ has odd upper degree; hence, $\pi^{2k-1}(\widehat{a})(y_i) \equiv \widehat{a}_0(y_i)$, modulo $\mathbb{C}^+[Z_0] \otimes Z_1$. Applying d, we get (2.6). q.e.d.

2.5. Derivations and the structure of $\mathrm{Aut}_{\mathbb{C}}(\mathcal{A})$**.** Denote by $\mathrm{Aut}^1_{\mathbb{C}}(\mathcal{A})$ the connected component of $1 \in \mathrm{Aut}_{\mathbb{C}}(\mathcal{A})$. We are going to exploit conditions (i) and (ii) from Definition 1.3, to give a complete description of the algebraic group $\mathrm{Aut}^1_{\mathbb{C}}(\mathcal{A})$.

In this subsection, all graded-commutative algebras $\mathcal{B}^*$ will be supposed to be connected, finite-dimensional over a field $\mathbb{K}$, and evenly-graded. The examples, we have in mind are $\mathcal{B} = \mathcal{A} \otimes \mathbb{C}$, where $\mathcal{A}$ is a $WACI$. For $p \in \mathbb{Z}$, recall that

$$\mathrm{Der}^p(\mathcal{B}) := \{\theta \in \mathrm{Hom}_{\mathbb{K}}(\mathcal{B}^*, \mathcal{B}^{*+p}) \mid \theta(ab) = \theta a \cdot b + a \cdot \theta b, \forall a, b \in \mathcal{B}\}.$$

Recall also that $\mathrm{Der}^*(\mathcal{B})$ has a natural graded Lie algebra structure. The Lie bracket of $\theta \in \mathrm{Der}^p$ and $\theta' \in \mathrm{Der}^q$ is defined by: $[\theta, \theta'] = \theta\theta' - (-1)^{pq}\theta'\theta$. We will be particularly interested in the Lie algebra $\mathrm{Der}^0(\mathcal{B})$. In our basic examples, one knows that $\mathrm{Der}^0(\mathcal{A} \otimes \mathbb{C})$ is the Lie algebra of $\mathrm{Aut}^1_{\mathbb{C}}(\mathcal{A})$; see [**11**, 12.5 and 13.2]. We first explore the consequences of the vanishing of $\mathrm{Der}^{<0}$ on the structure of Der^0.

Lemma 2.6. *Assume that* $\mathrm{Der}^{<0}(\mathcal{B}_j) = 0$*, for* $j = 1, \ldots, r$*. Then, the Lie algebra* $\mathrm{Der}^0(\otimes_{j=1}^r \mathcal{B}_j)$ *is isomorphic to the direct product* $\prod_{j=1}^r \mathrm{Der}^0(\mathcal{B}_j)$.

Proof. Consider $\theta \in \mathrm{Der}^p(\mathcal{B}_1 \otimes \mathcal{B}_2)$, with $\mathcal{B}_1$, $\mathcal{B}_2$ and p arbitrary. Pick $\{b_\beta\}_\beta$, a homogeneous $\mathbb{K}$-basis of $\mathcal{B}_2$, comprising 1. Write $\theta_{|\mathcal{B}_1 \otimes 1} = \sum_\beta \theta_\beta \otimes b_\beta$. A straightforward computation shows that $\theta_\beta \in \mathrm{Der}^{p-|b_\beta|}(\mathcal{B}_1)$, for every β. This shows inductively that $\mathrm{Der}^{<0}(\otimes_{j=1}^r \mathcal{B}_j) = 0$, if $\mathrm{Der}^{<0}(\mathcal{B}_j) = 0$, for all j.

Let again $\mathcal{B}_1$ and $\mathcal{B}_2$ be arbitrary. It is easy to check that the map

$$\Phi\colon \mathrm{Der}^0(\mathcal{B}_1) \times \mathrm{Der}^0(\mathcal{B}_2) \longrightarrow \mathrm{Der}^0(\mathcal{B}_1 \otimes \mathcal{B}_2), \tag{2.7}$$

defined by sending (θ_1, θ_2) to $\theta_1 \otimes \mathrm{id} + \mathrm{id} \otimes \theta_2$, is a Lie algebra monomorphism. We claim that Φ is an isomorphism, whenever $\mathrm{Der}^{<0}(\mathcal{B}_1) = \mathrm{Der}^{<0}(\mathcal{B}_2) = 0$. To check surjectivity, pick $\theta \in \mathrm{Der}^0(\mathcal{B}_1 \otimes \mathcal{B}_2)$ arbitrary. By the discussion from the preceding paragraph, $\theta_{|\mathcal{B}_1 \otimes 1} = \theta_1 \otimes 1$, and $\theta_{|1 \otimes \mathcal{B}_2} = 1 \otimes \theta_2$, with $\theta_j \in \mathrm{Der}^0(\mathcal{B}_j)$, for $j = 1, 2$. We infer that $\theta = \theta_1 \otimes \mathrm{id} + \mathrm{id} \otimes \theta_2$, which proves our claim. By making repeated use of the isomorphism (2.7), one verifies inductively the assertion of the Lemma. q.e.d.

Now, let $\mathbb{K}$ be algebraically closed, of characteristic zero. For arbitrary $\mathcal{B}_1, \dots, \mathcal{B}_r$, one has an algebraic group morphism, with source the algebraic r-torus, $(\mathbb{K}^*)^r$,

$$\rho\colon (\mathbb{K}^*)^r \longrightarrow \mathrm{Aut}^1\left(\bigotimes_{j=1}^{r} \mathcal{B}_j\right), \tag{2.8}$$

defined by $\rho(t) = \otimes_{j=1}^r \rho_j(t_j)$, for $t = (t_1, \dots, t_r)$, where each *grading automorphism* $\rho_j(t_j)$ acts on $\mathcal{B}_j^q$ as $t_j^q \cdot \mathrm{id}$.

Our next result, which extends Corollary (*ii*) from [**17**], will provide an important step in the proof of Theorem 1.6.

Corollary 2.7. *Assume that each* $\mathcal{B}_j$ *satisfies properties* (i) *and* (ii) *from Definition* 1.3 (*over* $\mathbb{K}$)*. Then, the above morphism* ρ *is onto, with finite kernel. In particular,* $\mathrm{Aut}^1(\otimes_{j=1}^r \mathcal{B}_j)$ *is an* r*-torus.*

Proof. Property (ii) forces $\mathcal{B}_j \neq \mathbb{K}$, for all j. This readily implies that the kernel of ρ is finite. Together with Lemma 2.6, this in turn guarantees the surjectivity of ρ, given that $\mathrm{Der}^0(\otimes_{j=1}^r \mathcal{B}_j)$ is the Lie algebra of $\mathrm{Aut}^1(\otimes_{j=1}^r \mathcal{B}_j)$ [**11**], via a dimension argument. The last assertion of the Corollary follows from [**11**, 16.1–2]. q.e.d.

2.8. A convenient $\mathbb{Q}$-basis. To prove Theorem 1.6, we have seen, at the beginning of §2.2, that it is enough to show that one cannot have simultaneously $\det \tau(a) = \pm 1$ and $\det(\mathrm{id} - \tau(a)) = \pm 1$, where $a \in \mathcal{E}$ is an arbitrary self-homotopy equivalence of M. To this end, the major step consists in finding a $\mathbb{Q}$-basis of $\pi_1^{2k-1}(\mathcal{A})$, the $\mathbb{Q}$-form of Z_1^{2k-1}, with respect to which $\tau(a)$ has a particularly simple matrix, for *every* $a \in \mathcal{E}$. This will be done by resorting to the last property from Definition 1.3, and then using Corollary 2.7.

Set $J' := \{j \in J \mid k_j = k\}$. One knows that

$$\pi_1^{2k-1}(\mathcal{A}) = \oplus_{j \in J} \pi_1^{2k-1}(\mathcal{A}_j);$$

see [**10**], [**19**]. With the notation from Section 2.2, pick $y_j \in \pi_1^{2k-1}(\mathcal{A}_j)$, $y_j \neq 0$, for $j \in J'$. Then, obviously $\{y_j\}_{j \in J'}$ will be a basis of the $\mathbb{Q}$-form of Z_1^{2k-1}.

Lemma 2.9. *For any* $a \in \mathcal{E}$*, there exist* $\{\lambda_j \in \mathbb{Q}\}_{j \in J'}$*, and a permutation* σ *of* J'*, such that*

$$\tau(a) y_j = \lambda_j y_{\sigma(j)}, \quad \textit{for} \quad j \in J'.$$

Proof. By Lemma 2.4, it suffices to prove the above statement, replacing $\tau(a)$ with $\nu(\alpha)$, where $\alpha \in \mathrm{Aut}_{\mathbb{C}}(\mathcal{A})$ is arbitrary, and without demanding the rationality of the λ-vector. (For $\alpha = h(a)$, the rationality property easily follows, since $\tau(a)$ respects by construction the $\mathbb{Q}$-structure.)

Since $\mathrm{Aut}_{\mathbb{C}}(\mathcal{A})$ obviously normalizes $\mathrm{Aut}^1_{\mathbb{C}}(\mathcal{A})$, the map on characters induced by conjugation by α^{-1} permutes the weights of the restriction of the linear representation ν to $\mathrm{Aut}^1_{\mathbb{C}}(\mathcal{A})$, and $\nu(\alpha)$ permutes the corresponding weight spaces; see [**11**, 11.4]. Due to the surjectivity of ρ from (2.8), these weights coincide with the weights of $\nu\rho$, and the corresponding weight spaces are equal; see again [**11**, 11.4].

At the same time, the structure of the representation $\nu\rho$ is easy to describe. Indeed, we claim that, for $j \in J'$, y_j is a weight vector of $\nu\rho$, with weight t_j^{2k}. This may be checked without difficulty, starting from the fact that $(\mathcal{M}, d)$, the minimal model of the DGA $(\mathcal{A} \otimes \mathbb{C}, d = 0)$, splits as $\otimes_{j\in J}(\mathcal{M}_j, d_j)$, where $(\mathcal{M}_j, d_j)$ is the minimal model of $(\mathcal{A}_j \otimes \mathbb{C}, d = 0)$, according to [**19**].

We infer that the (distinct) weights of $\nu\rho$ are $\{t_j^{2k}\}_{j\in J'}$, with one-dimensional corresponding weight spaces, spanned by $\{y_j\}_{j\in J'}$. The desired form of $\nu(\alpha)$ is thus obtained. q.e.d.

2.10. End of proof of Theorem 1.6. We may now compute the characteristic polynomial of $\tau(a)$, $P(z)$, as follows. Set $s =\mid J' \mid$, let $\sigma_1, \dots, \sigma_m$ be the associated cycles of σ, with lengths $s_1, \dots, s_m$, and put $\gamma_i := \prod_{j\in \mathrm{supp}(\sigma_i)} \lambda_j$, for $i = 1, \dots, m$. Then,

$$P(z) \quad = \quad \prod_{i=1}^{m}(z^{s_i} - \gamma_i). \tag{2.9}$$

We claim that,

$$\gamma_i \in \mathbb{Z}, \quad \text{for all} \quad i. \tag{2.10}$$

Granting the claim for the moment, we may quickly finish the proof of Theorem 1.6. Supposing $\det(\tau(a)) = \pm 1$ and $\det(\mathrm{id} - \tau(a)) = \pm 1$, for some $a \in \mathcal{E}$, we infer that both $P(0)$ and $P(1)$ are equal to ± 1. Due to (2.10), this is impossible, and we are done.

Coming back to the claim, let us remark that the polynomial $P(z)$ from (2.9) actually belongs to $\mathbb{Z}[z]$, since $\tau(a)$ also preserves the natural $\mathbb{Z}$-structure of $\pi_1^{2k-1}(\mathcal{A})$; see (1.5). The following elementary lemma will thus complete our proof.

Lemma 2.11. *Let $P(z)$ be given by* (2.9)*. If $P(z) \in \mathbb{Z}[z]$, then $\gamma_i \in \mathbb{Z}$, for all i.*

Proof. Write $\gamma_i = \frac{u_i}{v_i}$, with u_i and v_i relatively prime. We have

$$v_1 \cdots v_m P(z) = \prod_{i=1}^{m}(v_i z^{s_i} - u_i).$$

Applying the Gauss Lemma (see e.g., [**12**, p. 127]) in the above equality, we infer that $v_1 \cdots v_m = \pm 1$, since the content of the monic polynomial $P(z)$ is 1. q.e.d.

Example 2.12. Let us come back to the example given at the end of Paragraph 1.5. Note first that non-trivial a-invariant geodesics exist, since $\mathrm{Fix}(a) = (\mathbb{RP}^2)^{\times 2}$; see Theorem 2.1. However, this is not enough to infer existence also for a', since plainly the size of fixed points of homotopic isometries may be different.

One cannot apply the results from [**15**] and [**16**], since M up to homotopy type is neither G/T^r, with T a maximal torus, nor G/K, with G simple. To verify the first claim, we may suppose, by classical Lie theory [**3**], that G is semisimple. Consequently [**19**], the minimal model of G/T is of the form $(\mathbb{Q}[X] \otimes \wedge Y, d)$, with $dx_i = 0$ $(1 \leq i \leq r)$ on the degree 2 $\mathbb{Q}$-basis of X, and $dy_i = f_i \in \mathbb{Q}[X]$ $(1 \leq i \leq r)$ on the odd-degree $\mathbb{Q}$-basis of Y. Here $f_1, \ldots, f_r$ is a system of fundamental polynomial invariants of the Weyl group of G. Among them, there must be at least one quadric, see [**4**]. Hence, $\pi_3(G/T) \otimes \mathbb{Q} \neq 0$. On the other hand, the minimal model of M is $(\mathbb{Q}[X] \otimes \wedge Y, d)$, with $\mid x_i \mid = 2$, $dx_i = 0$ $(1 \leq i \leq 4)$, and $\mid y_i \mid = 5$, $dy_i = x_i^3$ $(1 \leq i \leq 4)$. Therefore, $\pi_3(M) \otimes \mathbb{Q} = 0$. To see that M cannot have the homotopy type of G/K, with G simple, just note that all homotopy groups of M have even rank, and compare to Definition 1.3(iii).

As already noted before, M is elliptic, even-dimensional, with $\chi(M) \neq 0$. Plainly, $M = S^{\times 2}$, with S elliptic. Thus, the only remaining possibility to get existence of invariant geodesics for a', by using known results, would be to resort to Corollary 1.7 from [**6**]; see Paragraph 1.1. For this, we need to compute the order of $\pi_q(a') \otimes \mathbb{Q} = \pi_q(a) \otimes \mathbb{Q}$, on non-zero rational homotopy groups of M.

To begin with, note that conjugation induces $-\mathrm{id}$ on $H^2(\mathbb{CP}^2, \mathbb{Q})$. Use the minimal model of $\mathbb{CP}^2$ (generated by x in degree 2 and y in degree 5, with $dx = 0$ and $dy = x^3$) and Lemma 2.4, to infer that conjugation acts by $-\mathrm{id}$ on $\pi_*(\mathbb{CP}^2) \otimes \mathbb{Q}$. We know that $\pi_q(M) \otimes \mathbb{Q} \neq 0$ only for $q = 2$ or 5. Using the dual $\mathbb{Q}$-bases provided by (1.5), $\{x_i^*\}_{1 \leq i \leq 4}$ and $\{y_i^*\}_{1 \leq i \leq 4}$ respectively, it is straightforward to check that $\pi_q(a) \otimes \mathbb{Q}$ acts as follows: $u_1^* \mapsto -u_1^*, u_2^* \mapsto -u_3^*, u_3^* \mapsto -u_4^*, u_4^* \mapsto -u_2^*$, where u stands for $x(y)$, if $q = 2(5)$. Clearly, both automorphisms have order 6.

Nevertheless, the characteristic polynomial of $\pi_5(a') \otimes \mathbb{Q}$ is $(z+1)(z^3 + 1)$, see (2.9). In particular, $\det(\mathrm{id} - \pi_5(a') \otimes \mathbb{Q}) = 4$. Hence, $\mathrm{id} - \pi_5(a')$ cannot be an isomorphism.

3. Reduced complete intersection and smoothing

To construct new examples of application of Theorem 1.6, we will proceed in two steps. We will first present a general way of obtaining simple $WACI$'s. Secondly, we will review the obstruction theory associated to the problem of realizing Artinian graded $\mathbb{Q}$-algebras by smooth manifolds.

3.1. Reduced complete intersection. The construction of simple $WACI$'s is based on our previous work [**17**]. We start by recalling the relevant facts from [**17**] (see the Corollary on p. 597 and Lemma 2.1(ii), loc. cit.).

Theorem 3.2. *Let* $\{f_i \in \mathbb{Q}[x_1, \ldots, x_n]\}_{1 \leq i \leq n}$ *be weighted-homogeneous polynomials, where the variables* x_i *have positive even weights,* w_i*; order the* f*-weights increasingly:* $0 <| f_1 |\leq \cdots \leq| f_{n-1} |\leq| f_n |$*. Set* $\mathcal{A} := \mathbb{Q}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ *and* $\mathcal{B} := \mathbb{Q}[x_1, \ldots, x_n]/(f_1, \ldots, f_{n-1})$*.*

1) *If* $\mathcal{A}$ *is a* $WACI$*,* $\mathcal{B}$ *is reduced* (*over* $\mathbb{C}$) *and* $| f_{n-1} |<| f_n |$*, then* $\mathrm{Der}^{<0}(\mathcal{A}) = 0$ *and* $\dim_{\mathbb{Q}} \mathrm{Der}^0(\mathcal{A}) = 1$*.*
2) *Assume that* $f_i = \sum_{j=1}^n \gamma_{ij} x_j^{a_j}$*, for* $i = 1, \ldots, n-1$*, where the* $\mathbb{Q}$*-matrix* (γ_{ij}) *has maximal rank, and* $w_i a_i = d$*, for all* i*. If any non-zero solution of the linear system*
$$\sum_{j=1}^n \gamma_{ij} v_j = 0, \quad i = 1, \ldots, n-1,$$
has $v_i \neq 0$*, for all* i*, then* $\mathcal{B}$ *is reduced.*

Here is a first application of the above results.

Proposition 3.3. *Let* $\mathcal{A}$ *be a* $WACI$*, as in* (1.2)–(1.3)*, where the* f*-weights satisfy* $0 <| f_1 |\leq \cdots \leq| f_{n-1} |<| f_n |$*, and the polynomial* f_n *has no linear part. If* $\mathbb{Q}[x_1, \ldots, x_n]/(f_1, \ldots, f_{n-1})$ *is reduced* (*over* $\mathbb{C}$)*, then* $\mathcal{A}$ *is simple.*

Proof. Conditions (i) and (ii) follow directly from Theorem 3.2(1).

To check the last condition (iii) from Definition 1.3, one has just to recall from [**19**] the recipe for computing π_1^* of an arbitrary $WACI$. Set $X := \mathbb{Q} - \mathrm{span}\,\{x_1, \ldots, x_n\}$ and $Y := \mathbb{Q} - \mathrm{span}\,\{y_1, \ldots, y_n\}$, with $| y_i |=| f_i | -1$. Then:

$$\pi_1^*(\mathcal{A}) = \ker (L), \tag{3.1}$$

where $L : Y \to X$ sends y_i to the linear part of f_i. q.e.d.

3.4. The smoothing problem. Let $\mathcal{A}^*$ be a 1-connected *Poincaré duality algebra* (PDA) over $\mathbb{Q}$, of formal dimension m. That is, a finite-dimensional graded-commutative $\mathbb{Q}$-algebra with $\mathcal{A}^1 = 0$, $\mathcal{A}^m \cong \mathbb{Q}$ and $\mathcal{A}^{>m} = 0$, for which the multiplication gives a duality pairing, $\mathcal{A}^i \otimes \mathcal{A}^j \to \mathbb{Q}$, for $i + j = m$. We shall say that $\mathcal{A}$ is *smoothable* if $\mathcal{A}^* \cong H^*(M^m, \mathbb{Q})$, as graded $\mathbb{Q}$-algebras, where M is a 1-connected closed smooth m-manifold. A natural source of 1-connected PDA's of even formal dimension is provided by $WACI$'s; see [**9**, Theorem 3].

Let $\mathcal{A}$ be an arbitrary 1-connected PDA, of formal dimension m. If $\mathcal{A}$ is smoothable, we may pick an orientation class, $[M] \in H_m(M, \mathbb{Z})$, which defines an algebraic orientation, $\omega \in \mathcal{A}^m \setminus \{0\}$. Likewise, the Pontrjagin classes $p_i(M)$ define a total algebraic Pontrjagin class, $q = 1 + \sum_{i \geq 1} q_i$, with $q_i \in \mathcal{A}^{4i}$. If $m = 4k$, ω and q may be used to describe the following three well-known obstructions to smoothability (see [**14**]).

By construction, $\{\langle q_1^{e_1} \cdots q_k^{e_k}, \omega\rangle \mid \sum_{i=1}^k ie_i = k\}$ are the *Pontrjagin numbers* of a closed oriented smooth $4k$-manifold.

Secondly, the non-degenerate quadratic form $\varphi(\mathcal{A}, \omega)$, defined on $\mathcal{A}^{2k}$ by Poincaré duality and the orientation, must be a sum of signed squares, over $\mathbb{Q}$ (the *integrality property*).

Finally, the *signature formula* must hold:

$$\sigma(\varphi(\mathcal{A}, \omega)) \quad = \quad \langle L_k(q_1, \ldots, q_k), \omega\rangle, \tag{3.2}$$

where σ denotes the signature and L_k is the k-th Hirzebruch polynomial.

It turns out that these obstructions completely control the smoothing problem, as explained in the proposition below, which is an immediate consequence of a basic $\mathbb{Q}$-surgery result due to Sullivan [**19**, Theorem 13.2] (see also [**1**, Theorem 8.2.2]).

Proposition 3.5. *Let $\mathcal{A}$ be a 1-connected $\mathbb{Q}$–PDA, of formal dimension m. If $m \neq 4k$, then $\mathcal{A}$ is smoothable. If $m = 4k$, then $\mathcal{A}$ is smoothable if and only if there exist $\omega \in \mathcal{A}^{4k}\setminus\{0\}$ and $\sum_{i\geq 1} q_i \in \oplus_{i\geq 1}\mathcal{A}^{4i}$, which verify the above-mentioned properties, concerning Pontrjagin numbers, integrality, and the signature formula.*

Proof. Realize $\mathcal{A}$ by a 1-connected, $\mathbb{Q}$-local, Poincaré complex S. Everything follows then by applying the results quoted from [**19**] and [**1**] to S, except the fact that, for $m = 4$, smoothability holds, as soon as the obstructions are satisfied. In this very easy case, the Poincaré quadratic form on $\mathcal{A}^2$ is a sum of r squares minus a sum of s squares, by integrality. Then plainly $\mathcal{A}^* = H^*((\#_r\mathbb{CP}^2)\#(\#_s\overline{\mathbb{CP}^2}), \mathbb{Q})$ (where $\overline{\mathbb{CP}^2}$ denotes $\mathbb{CP}^2$ with the opposite complex orientation), for $r + s > 0$, and $\mathcal{A}^* = H^*(S^4, \mathbb{Q})$, for $r + s = 0$. q.e.d.

4. *HH* manifolds which are not homogeneous

According to the signature of their cohomology algebras, our new HH manifolds fall into two classes. We begin with the (easier) zero signature case.

4.1. Split simple algebras. There is a large freedom of choice of (discrete) parameters for our construction. More precisely: pick any $n \geq 2$, $k \geq 1$, then arbitrary positive even weights, $\{w_i\}_{1\leq i\leq n}$, and integers $a_i \geq 2$ such that $w_i a_i = d$, for $i = 1, \ldots, n$. Set:

$$\begin{cases} f_1 = x_1^{a_1} - x_2^{a_2} \\ \cdots \\ f_{n-1} = x_{n-1}^{a_{n-1}} - x_n^{a_n} \\ f_n = x_n^{2ka_n}. \end{cases} \tag{4.1}$$

Proposition 4.2. *Define* $\mathcal{A} := \mathbb{Q}[x_1, \dots, x_n]/(f_1, \dots, f_n)$*, where* $f_1, \dots, f_n$ *are given by* (4.1) *above. Then:*

1) $\mathcal{A}$ *is a simple* $WACI$.
2) $\mathcal{A}$ *is a* 1*-connected Poincaré duality algebra, with even formal dimension.*
3) *The non-trivial odd pseudo-homotopy groups of* $\mathcal{A}$ *are:*

$$\begin{cases} \pi_1^{d-1}(\mathcal{A}) = \mathbb{Q}^{n-1}, & \text{and} \\ \pi_1^{2kd-1}(\mathcal{A}) = \mathbb{Q}. \end{cases}$$

Proof. Part (1) will follow from Proposition 3.3, as soon as we check that $\mathcal{A}$ is a $WACI$, and $\mathcal{B} := \mathbb{Q}[x_1, \dots, x_n]/(f_1, \dots, f_{n-1})$ is reduced. The condition on $\mathcal{A}$ is immediately verified, since plainly $\{f_1 = \cdots = f_n = 0\} = \{0\}$. The property of $\mathcal{B}$ is guaranteed by Theorem 3.2(2). Part (3) is a direct consequence of (3.1). For Part (2), see [**9**, Theorem 3]. q.e.d.

Smoothability will follow from the next well-known lemma, whose proof is included for the reader's convenience.

Lemma 4.3. *Let* $\mathcal{A}$ *be a* 1*-connected, commutative, Poincaré duality algebra over* $\mathbb{Q}$*, with even formal dimension. Assume that there is a* $\mathbb{Q}$*-subspace,* $N \subset \mathcal{A}$*, with* $\dim_{\mathbb{Q}} \mathcal{A} = 2 \dim_{\mathbb{Q}} N$*, and such that* $N \cdot N = 0$*, where the dot denotes the Poincaré inner product. Then there is a* 1*-connected, closed, smooth manifold* M*, with zero signature, such that* $\mathcal{A}^* = H^*(M, \mathbb{Q})$*, as graded algebras.*

Proof. Immediate, using Proposition 3.5. When the formal dimension of $\mathcal{A}$ is $4k$, we may use any orientation, $\omega \in \mathcal{A}^{4k} \setminus \{0\}$, and the trivial Pontrjagin class, $q = 1$. Now, Poincaré duality implies that the inner product space $(\mathcal{A}, \cdot)$ equals $(\mathcal{A}^{2k}, \cdot)$, in the Witt group $W(\mathbb{Q})$. On the other hand, our assumptions on N mean precisely that $(\mathcal{A}, \cdot)$ equals zero in $W(\mathbb{Q})$. See [**13**, I.6–7]. In particular, $\sigma(\mathcal{A}^{2k}, \cdot) = 0$. With these remarks, it is now easy to check the three obstructions to smoothing. q.e.d.

Theorem 4.4. *Let* $\mathcal{A}$ *be a simple* $WACI$ *as in* Proposition 4.2. *Then:*

1) $\mathcal{A}^*$ *is the* $\mathbb{Q}$*-cohomology algebra of a* 1*-connected, closed smooth manifold,* M*. The signature of* M *is zero.*
2) *If* $n \geq 4$*,* $\mathcal{A}^*$ *is not isomorphic to the cohomology algebra of any equal rank homogeneous space,* G/K.

Proof. Part (1). According to Lemma 4.3, we are done, as soon as we find an isotropic subspace, $N \subset \mathcal{A}$, having the appropriate dimension. This in turn may be done in the following way.

Set $\mathcal{B} = \mathbb{Q}[y_1, \dots, y_n]/(y_1 - y_2, \dots, y_{n-1} - y_n, y_n^{2k})$, where $\mid y_i \mid = d$, for all i. Plainly, $\mathcal{B} = \mathbb{Q}[y_0]/(y_0^{2k})$, with $\mid y_0 \mid = d$. Consider the graded algebra map, $\phi : \mathcal{B}^* \to \mathcal{A}^*$, which sends y_i to $x_i^{a_i}$, for $i = 1, \dots, n$. A Poincaré series argument shows that $\mathcal{A}^*$ is isomorphic to $\mathcal{B}^* \otimes F^*$, as a graded $\mathcal{B}^*$-module, where $F^* = \bigotimes_{i=1}^n \mathbb{Q}[x_i]/(x_i^{a_i})$. Indeed, one may extend ϕ to a surjective map of graded $\mathcal{B}^*$-modules,

$$\Phi \colon \mathcal{B}^* \otimes F^* \twoheadrightarrow \mathcal{A}^*, \tag{4.2}$$

by sending the elements of the monomial $\mathbb{Q}$-basis of F^*, $\{x_1^{c_1} \cdots x_n^{c_n} \mid 0 \le c_i < a_i \,, \forall i\}$, to their classes modulo $\mathcal{I}$. By Corollary 2 on p. 198 from [**9**], the Poincaré series of $\mathcal{A}^*$ is

$$(1 - t^d)^{n-1} \, (1 - t^{2kd}) \cdot \prod_{i=1}^{n} (1 - t^{w_i})^{-1},$$

which is equal to the Poincaré series of $\mathcal{B}^* \otimes F^*$. Therefore, Φ from (4.2) above is actually an isomorphism, which will be used to identify $\mathcal{A}^*$ with $\mathcal{B}^* \otimes F^*$.

Set now $V := \mathbb{Q}-\mathrm{span}\,\{y_0^s \mid 0 \le s < k\} \subset \mathcal{B}$, and $N = V \otimes F$. Clearly, $\dim_{\mathbb{Q}} \mathcal{A} = 2 \dim_{\mathbb{Q}} N$. The proof ends by checking that $N \cdot N = 0$, which is straightforward.

Part (2). Assume that $\mathcal{A}^* = H^*(G/K, \mathbb{Q})$, as graded algebras. Write $G/K = \prod_{i=1}^r G_i/K_i$, with G_i simple, for all i. Compute Der^0-dimensions to infer that G must be simple, since $\mathcal{A}$ is a simple $WACI$; see Proposition 4.2(1) and Lemma 2.6.

By our assumption, $\pi_1^*(\mathcal{A}) = \pi_1^*(H^*(G/K, \mathbb{Q}))$, as graded vector spaces. One also knows [**3**] that $H^*(G/K, \mathbb{Q})$ is a $WACI$ for which the weights of the variables are equal to 2 times the degrees of the fundamental polynomial invariants of the Weyl group W_K, and likewise for the degrees of the defining relations of $H^*(G/K, \mathbb{Q})$.

Case by case inspection of the tables in [**4**], reveals that the degree list of the fundamental W_G-invariants contains no element of multiplicity > 2, if G is simple. For $n - 1 > 2$, this contradicts the computation from Proposition 4.2(3); see (3.1). q.e.d.

4.5. Simple algebras with non-zero signature. The non-zero signature smoothing problem is more subtle. For instance, the answer may depend not only on the algebra structure, but also on the weights, as seen in the following example. Let $\mathcal{A}_k$ be $\mathbb{Q}[x]/(x^3)$, where the weight of x is $2k$, with k odd. It is easy to check that each $\mathcal{A}_k$ is a simple $WACI$, with formal dimension $4k$ and signature ± 1. If $\mathcal{A}_k$ is smoothable, then we infer from [**1**, Proposition B.5] (see also [**2**]) that $\sigma(\mathcal{A}_k)$ must be a multiple of $2^{2k-1} - 1$. Thus, $\mathcal{A}_k^*$ is not smoothable, for $k > 1$, while $\mathcal{A}_1^* = H^*(\mathbb{CP}^2, \mathbb{Q})$ is obviously smoothable.

Let us consider the Eisenbud–Levine $WACI$'s from [**5**, p. 24], with generators $\{x_i\}_{1\leq i\leq n}$ of weight 2, and defining relations

$$\begin{cases} f_1 = x_1^2 - x_n^2 \\ \dots \\ f_{n-1} = x_{n-1}^2 - x_n^2 \\ f_n = x_1 \cdots x_n. \end{cases} \tag{4.3}$$

Given a $\mathbb{Q}$-PDA, $\mathcal{A}$, of formal dimension $4k$, we shall denote in the sequel by $(\mathcal{A}^{2k}, \cdot_\omega) \in W(\mathbb{Q})$ the symmetric inner product space over $\mathbb{Q}$ obtained from the multiplication map, $\mathcal{A}^{2k} \otimes \mathcal{A}^{2k} \to \mathcal{A}^{4k}$, via a choice of orientation, $\omega \in \mathcal{A}^{4k} \setminus \{0\}$.

Proposition 4.6. *If $\mathcal{A}$ is defined by* (4.3), *where $n \geq 3$, then:*

1) *$\mathcal{A}$ is a simple $WACI$.*
2) *$\mathcal{A}$ is a 1-connected Poincaré duality algebra, with formal dimension $4(n-1)$.*
3) *There is an orientation, $\omega \in \mathcal{A}^{4(n-1)} \setminus \{0\}$, such that $(\mathcal{A}^{2(n-1)}, \cdot_\omega) \in W(\mathbb{Z})$ and $\sigma(\mathcal{A}^{2(n-1)}, \cdot_\omega) = 2^{n-1}$.*
4) *The non-trivial odd pseudo-homotopy groups of $\mathcal{A}$ are:*

$$\begin{cases} \pi_1^3(\mathcal{A}) = \mathbb{Q}^{n-1}, \quad \text{and} \\ \pi_1^{2n-1}(\mathcal{A}) = \mathbb{Q}. \end{cases}$$

Proof. Part (1) follows from Proposition 3.3, via 3.2(2). Parts (2) and (4) are direct consequences of [**9**, Theorem 3], and formula (3.1) respectively.

For the proof of Part (3), we will need a convenient $\mathbb{Q}$-basis of $\mathcal{A}^*$. To obtain it, we first set $\mathcal{B} := \mathbb{Q}[y_1, \dots, y_n]/(y_1 - y_n, \dots, y_{n-1} - y_n) = \mathbb{Q}[y_0]$, where $\mid y_i \mid = 4$, for all i, and define a graded algebra map, $\phi\colon \mathcal{B} \to \mathbb{Q}[x_1, \dots, x_n]/(f_1, \dots, f_{n-1})$, by $\phi(y_i) = x_i^2$, for $i = 1, \dots, n$. As in the proof of Theorem 4.4(1), ϕ extends in an obvious way to a surjective map of graded $\mathcal{B}^*$-modules,

$$\Phi\colon \mathcal{B}^* \otimes F^* \twoheadrightarrow \mathbb{Q}[x_1, \dots, x_n]/(f_1, \dots, f_{n-1})\,, \tag{4.4}$$

where $F^* = \otimes_{i=1}^n \mathbb{Q}[x_i]/(x_i^2)$.

We claim that the Poincaré series of $\mathbb{Q}[x_1, \dots, x_n]/(f_1, \dots, f_{n-1})$ is equal to $(1-t^4)^{n-1}(1-t^2)^{-n}$. This may be seen as follows. We know that f_i is a non-zero divisor in $\mathbb{Q}[x_1, \dots, x_n]/(f_1, \dots, f_{i-1})$, for $i = 1, \dots, n-1$, since $(f_1, \dots, f_n)$ is a regular sequence. Now, if $\mathcal{C}^*$ is a commutative graded algebra of finite type over a field, and $f \in \mathcal{C}^k$ is a non-zero divisor, it is a standard fact that the Poincaré series of $\mathcal{C}$ and $\mathcal{C}/(f)$, $\mathcal{C}(t)$ and $\mathcal{C}/(f)(t)$ respectively, are related by: $\mathcal{C}/(f)(t) = (1-t^k) \cdot \mathcal{C}(t)$. (Use the short exact sequence associated to the multiplication by f, acting on $\mathcal{C}$.) This gives our claim, by induction.

Clearly, the Poincaré series of $\mathcal{B}^* \otimes F^*$ is also equal to $(1-t^4)^{n-1}(1-t^2)^{-n}$. In this way, we see that the above map (4.4) is an isomorphism. We will use it to identify $\mathcal{A}^*$ with a quotient of $\mathcal{B}^* \otimes F^*$. To be more precise, we will need the following notation. For $I = \{i_1 < \cdots < i_k\} \subset \{1, \ldots, n\}$, we put $x_I := x_{i_1} \cdots x_{i_k} \in F^{2k}$, and we denote by $\overline{I}$ the complement of I in $\{1, \ldots, n\}$. It is easy to check in $\mathcal{B}^* \otimes F^*$ the equalities

$$f_n \cdot x_I = y_0^{|I|} \otimes x_{\overline{I}}, \quad \forall I;$$

they imply that the images of the monomials

$$\{y_0^r \otimes x_I \mid 0 \leq r <| \overline{I} |\} \tag{4.5}$$

give a $\mathbb{Q}$-basis of $\mathcal{A}$. Using this basis, we will show that, if one takes $\omega = y_0^{n-1}$, then $(\mathcal{A}, \cdot_\omega) \in W(\mathbb{Z})$ and $\sigma(\mathcal{A}, \cdot_\omega) = 2^{n-1}$, which clearly proves Part (3), via Poincaré duality.

It is not difficult to check that $(\mathcal{A}, \cdot_\omega)$ splits as an orthogonal direct sum, $\bigoplus_{\overline{I} \neq \emptyset} \mathcal{A}_I$, where $\mathcal{A}_I$ denotes the $\mathbb{Q}$-span of the monomials (4.5) with I fixed. For fixed I, it is equally easy to check that the inner product of $y_0^r \otimes x_I$ and $y_0^s \otimes x_I$ equals:

$$\begin{cases} 1, & \text{if} \quad r + s = n - 1- \mid I \mid; \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

One may use (4.6) to further decompose $\mathcal{A}_I$ into a sum of hyperbolic planes and one-dimensional summands belonging to $W(\mathbb{Z})$, thus checking that $(\mathcal{A}, \cdot_\omega) \in W(\mathbb{Z})$. Moreover,

$$\sigma(\mathcal{A}_I) = \begin{cases} 0, & \text{if} \quad n - 1- \mid I \mid \quad \text{is odd;} \\ 1, & \text{if} \quad n - 1- \mid I \mid \quad \text{is even,} \end{cases}$$

which proves that $\sigma(\mathcal{A}, \cdot_\omega) = 2^{n-1}$. q.e.d.

Remark 4.7. Let $\mathcal{A}$ be a $\mathbb{Q}$-PDA, of formal dimension $4k$, with orientation $\omega \in \mathcal{A}^{4k} \setminus \{0\}$. Then: the quadratic form $\varphi(\mathcal{A}, \omega)$ has the integrality property described in Paragraph 3.4 if and only if $(\mathcal{A}^{2k}, \cdot_\omega) \in W(\mathbb{Z})$; see [**13**, Corollary IV.2.6]. We should also point out that this integrality result is the key fact from our Proposition 4.6(3) above (the value of the signature was computed in [**5**], using a different method).

Theorem 4.8. *Let $\mathcal{A}$ be a simple $WACI$ as in* Proposition 4.6. *Then:*

1) *$\mathcal{A}^*$ is the $\mathbb{Q}$-cohomology algebra of a 1-connected, smooth closed manifold, M. The signature of M equals $\pm 2^{n-1}$.*
2) *$\mathcal{A}^*$ is not isomorphic to $H^*(G/K, \mathbb{Q})$, for any equal rank homogeneous space, G/K.*

Proof. Part (1). We have to show that $\mathcal{A}$ is smoothable. The integrality condition is satisfied; see Remark 4.7.

To check the other two $\mathbb{Q}$-surgery obstructions from Proposition 3.5, we will construct the total Pontrjagin class, q, as follows. Set $k = n-1$. Let $p(\mathbb{CP}^{2k}) = 1 + \sum_{i=1}^{k} c_i u^{2i}$ be the total Pontrjagin class of $\mathbb{CP}^{2k}$, where $u \in H^2(\mathbb{CP}^{2k}, \mathbb{Q})$ is the canonical generator. We will take

$$q := 1 + \sum_{i=1}^{k} 2^i c_i y_0^i \in \bigoplus_{i \geq 0} \mathcal{A}^{4i} \tag{4.7}$$

(see (4.5)). It readily follows from (4.7) that $\mathcal{A}$ has the same Pontrjagin numbers as $2^k \mathbb{CP}^{2k}$. Therefore, the Hirzebruch signature formula (3.2) is also verified for $\mathcal{A}$, since $\sigma(\mathcal{A}^{2k}, \cdot_\omega) = 2^k$, by Proposition 4.6(3); see [**14**]. Hence, $\mathcal{A}$ is smoothable.

Part (2). The same argument as in the proof of Theorem 4.4(2) gives the result, for $n \geq 4$. For $n = 3$, one may notice that the degree list of $\pi_1^*(\mathcal{A})$ is $(3, 3, 5)$, with $5 \not\equiv -1 \pmod 4$, and resort again to the tables in [**4**]. q.e.d.

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