

# ON THE HOMOTOPY TYPES OF KÄHLER MANIFOLDS AND THE BIRATIONAL KODAIRA PROBLEM

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## Abstract

In the paper [8], we constructed examples of compact Kähler manifolds which do not have the homotopy type of a projective complex manifold. They were, however, obtained by blowing-up certain complex tori, which are themselves deformation equivalent to complex projective manifolds. Thus it remained possible that in higher dimension, a birational version of Kodaira's theorem, saying that a compact Kähler surface deforms to a projective surface, still holds.

We construct in this paper compact Kähler manifolds, no smooth birational model of which, however, has the homotopy type of a projective manifold. Thus the possibility mentioned above is excluded, even at the topological level.

## 0. Introduction

In small dimensions, it is known that compact Kähler manifolds are deformation equivalent to smooth projective complex varieties. In dimension 2, this follows from the following theorem:

**Theorem 1** (Kodaira [6]). *Any compact Kähler surface admits small deformations which are projective.*

The so-called Kodaira problem left open by this result asked whether more generally any compact Kähler manifold can be deformed to a projective complex manifold.

Recently, we solved this question negatively by constructing, in any dimension  $n \geq 4$ , examples of compact Kähler manifolds, which do not deform to projective complex manifolds, as a consequence of the following stronger statement concerning the topology of Kähler compact manifolds:

**Theorem 2** (Voisin, [8]). *In any dimension  $n \geq 4$ , there are examples of compact Kähler manifolds which do not have the homotopy type of projective complex manifolds.*

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Let us give a simple idea of the starting point of the construction of [8], leading quickly to the construction of a rigid Kähler compact manifold which is not projective, and thus is a counterexample to the Kodaira problem. We consider a complex torus  $T$  which admits an endomorphism  $\phi_T$  such that:

- 1) The pair  $(T, \phi_T)$  is rigid,
- 2)  $T$  is not projective (in fact, the presence of  $\phi_T$  will prevent  $T$  from being projective, see section 1).

In order to construct from this a rigid Kähler compact manifold which is not projective, we first consider the product  $T \times T$ . Of course it has many deformations, those which kill the product structure, those which preserve the product structure but where the two factors become different, and finally those which keep the structure of a self-product, but loosing the existence of  $\phi_T$ . In order to construct from it a Kähler manifold whose deformations will identify to those deformations of  $T$  preserving  $\phi_T$ , we use Kodaira's Theorem [7] saying that the exceptional divisor of the blow-up of a subvariety  $V \subset W$  is stable under small deformations, so that deformations of the blow-up of  $W$  along  $V$  correspond to deformations of  $W$  preserving the subvariety  $V$ . Thus we blow-up the subvarieties of  $T \times T$  which provide all the structures above: namely, the  $T \times 0$  and  $0 \times T$  which guarantee that we have a product, the diagonal, which guarantees that the two factors in the product are isomorphic, and the graph of  $\phi_T$  which guarantees the presence of the endomorphism  $\phi_T$ , once one knows that our torus is of the form  $T \times T$ . The resulting Kähler compact manifold  $X$  is then rigid, as its only deformations must come from a deformation of  $T \times T$  preserving all the structures above, that is to deformations of  $T$  preserving  $\phi_T$ .

The construction above may seem rather artificial, as it uses blow-ups to make rigid a Kähler manifold which was not rigid at all, and indeed admits arbitrarily small projective deformations.

The following question, which was asked of me by N. Buchdahl, F. Campana, S.-T. Yau, and can be considered as a birational version of the Kodaira problem, is thus quite natural:

**Question.** Let  $X$  be a compact Kähler manifold. Does there exist a smooth bimeromorphic model  $X'$  of  $X$  which deforms to a projective complex manifold?

There are two reasons to believe that this statement could be the higher dimensional generalization of Kodaira's theorem 1. The first is that our construction described above shows that in higher dimension, there are compact Kähler manifolds which do not deform to a projective complex manifold, while a birational model of them does. This is not possible in dimension 2 by Kodaira's Theorem.

A deeper reason is the fact that birational geometry in higher dimension is much more complicated than in dimension 2. In fact, birational transformations do not affect deeply the cohomology ring of a surface, while in higher dimensions, the cohomology of a codimension  $\geq 2$  subvariety  $Z \subset V$  might be non trivial and thus strongly affect the cohomology ring of the variety  $V$  under blowing-up along  $Z$ .

The reason why we mention this effect on the cohomology ring, is that this is the study of the cohomology ring of the variety  $X$  described above, together with arguments from Hodge theory, which allowed us to prove that this  $X$  satisfies the much stronger conclusion of Theorem 2, as explained below.

In this paper, we not only show that the answer to the birational Kodaira problem is again no, but also prove the following much stronger topological result:

**Theorem 3.** *In any even dimension  $\geq 10$ , there exist compact Kähler manifolds  $X$ , such that no smooth compact bimeromorphic model  $X'$  of  $X$  has the homotopy type of a projective complex manifold.*

In this statement, we can in fact replace “homotopy type” with “rational homotopy type”, or “rational cohomology ring” (see Theorem 5). Indeed, the whole discussion deals with the Hodge structure on rational cohomology and the (non)-existence of polarizations on them.

Let us give a sketch of the main ideas leading to the proof of Theorem 3. As in [8], we want to show that a certain Kähler compact manifold  $X'$  (which will be any birational model of the  $X$  constructed in section 1) does not have the rational cohomology algebra of a projective manifold, and this is done by showing that, for any Kähler compact manifold  $Y$  with rational cohomology algebra  $H^*(Y, \mathbb{Q})$  isomorphic to the one of  $X'$ , the Hodge structure on the rational cohomology group  $H^2(Y, \mathbb{Q})$  cannot be polarized, namely, there does not exist a rational Hodge classe  $c$  of degree 2 on  $Y$  such that the form

$$q_c(\alpha, \beta) := \int_Y c^{n-2} \alpha \beta$$

polarizes the Hodge structure on  $H^2(Y, \mathbb{Q})$  (cf. [9] I, chapter 7). To get this, the key point is to show that the rational cohomology ring of  $Y$ , which has to be compatible with the Hodge decomposition, forces the existence of an endomorphism of Hodge structures

$$\psi : H^2(Y, \mathbb{Q}) \rightarrow H^2(Y, \mathbb{Q})$$

which acts in an irreducible way. (In application, we will actually work with a certain sub-Hodge structure, rather than with the whole of  $H^2$ .) The endomorphism  $\psi$  will be deduced from the cup-product with certain rational cohomology classes, which will be proved to be Hodge classes on  $Y$  (cf. Lemma 11).

Finally, if one knows that the Hodge structure on  $H^2(Y, \mathbb{Q})$  cannot be trivial, for example by the Hodge index theorem, one concludes from the existence of  $\psi$  that  $H^2(Y, \mathbb{Q})$  carries no Hodge class, and thus cannot be polarized. (Actually, this argument will lead to the conclusion that no Hodge class of degree 2 on  $Y$  can polarize a certain sub-Hodge structure of  $H^2(Y, \mathbb{Q})$  constructed in Proposition 3.)

A key ingredient to get the existence of  $\psi$  is Deligne's Lemma 9, which allows to construct certain rational sub-Hodge structures of a rational Hodge structure  $A^k$ , which is the degree  $k$  piece of an algebra  $A^*$  endowed with weight  $l$  rational Hodge structures on each  $A^l$ , compatible with the product. This argument will be used in several places, to exhibit certain sub-Hodge structures of  $H^2(Y, \mathbb{Q})$ ; one of them will be shown to be made of Hodge classes, thus leading to the construction of the endomorphism  $\psi$  above.

Up to this point, the shape of the argument described above is the same as the one used in the last section of [8], where we constructed *simply connected* examples of Kähler compact manifolds satisfying the conclusion of Theorem 2.

There are two new geometric ingredients here. First of all, we replace the blow-up construction (which allowed us to get a complicated cohomology ring) by the construction of a certain  $\mathbb{P}^1 \times \mathbb{P}^1$ -bundle  $q : X \rightarrow B$  on a Kähler compact basis  $B$ . This  $\mathbb{P}^1 \times \mathbb{P}^1$ -bundle carries interesting classes, namely the classes  $c_1(L_i)$  where  $L_i := pr_i^*(\mathcal{O}_{\mathbb{P}^1}(1))$ , and they will play an essential role in the construction of the endomorphism  $\psi$  above. Our strategy will be to apply Deligne's Lemma to show first that for any birational model  $X'$  of  $X$  and any compact Kähler manifold  $Y$  with rational cohomology algebra isomorphic to that of  $X'$ ,  $q^*H^2(B)$  is a sub-Hodge structure of  $H^2(Y)$  (see Proposition 1 and its consequence Proposition 3). The classes  $c_1(L_i)$  will then be shown to provide the endomorphism  $\psi$  of  $q^*H^2(B)$  needed to deduce that the Hodge structure on  $H^2(Y)$  cannot be polarized.

Of course, if we were in algebraic geometry, these Hodge classes  $c_1(L_i)$  would not survive birational transformations, and would not give interesting information on the cohomology of all birational models of  $X$ , as a  $\mathbb{P}^1 \times \mathbb{P}^1$ -bundle is always birationally trivial. The very important point here is that, as the basis and total space of our  $\mathbb{P}^1 \times \mathbb{P}^1$ -bundle are Kähler manifolds which contain very few closed complex analytic subsets, there are few birational transformations allowed, so that the classes  $c_1(L_i)$  above are essentially preserved under birational transformations (cf. Proposition 2).

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### 1. Construction of examples

We start as in [8], namely, we consider  $n$ -dimensional complex tori  $T$  admitting an endomorphism

$$\phi_T : T \rightarrow T$$

satisfying the following property (\*). We can write  $T$  as  $\Gamma_{\mathbb{C}}/(\Gamma \oplus \Gamma')$ , where  $\Gamma$  is a rank  $2n$  lattice,  $\Gamma_{\mathbb{C}} = \Gamma \otimes \mathbb{C}$  and  $\Gamma'$  is a complex subspace of  $\Gamma_{\mathbb{C}}$  of rank  $n$  such that

$$\Gamma' \oplus \bar{\Gamma}' = \Gamma_{\mathbb{C}}.$$

Let  $\phi$  be the endomorphism  $\phi_{T^*}$  of  $H_1(T, \mathbb{Z}) = \Gamma$ . Clearly  $\Gamma'$  has to be stable under  $\phi_{\mathbb{C}}$ , so an eigenvalue of  $\phi$  cannot be real, unless it has multiplicity  $> 1$ . The condition (\*) is the following:

(\*) *The characteristic polynomial of  $\phi$  (which has integer coefficients), has  $2n$  distinct roots (the eigenvalues of  $\phi$ ) and its Galois group over  $\mathbb{Q}$  acts as the symmetric group of  $2n$  letters on them.*

In the sequel, we will need to assume that the dimension  $n$  of  $T$  is at least 4. We make now the following construction. Let  $\hat{T}$  be the dual torus of  $T$ , namely

$$\hat{T} = \Gamma_{\mathbb{C}}^*/(\Gamma^* \oplus \Gamma'^{\perp}).$$

Geometrically,  $\hat{T}$  is the torus

$$\text{Pic}^0(T) = H^1(T, \mathbb{C})/(H^{1,0}(T) \oplus H^1(T, \mathbb{Z}))$$

which is the group of topologically trivial holomorphic line bundles on  $T$  up to isomorphism.

There exists on  $T \times \hat{T}$  the so-called Poincaré line bundle  $\mathcal{P}$  which is uniquely characterized by the following properties:

- For any  $t \in \hat{T}$  parameterizing a line bundle  $L_t$  on  $T$ , we have

$$L_t \cong \mathcal{P}|_{T \times t}.$$

- The restriction  $\mathcal{P}|_{0 \times \hat{T}}$  is trivial.

In fact  $\mathcal{P}$  is constructed as follows : first of all, its first Chern class

$$c_1(\mathcal{P}) \in NS(T \times \hat{T}) := H^{1,1}(T \times \hat{T}) \cap H^2(T \times \hat{T}, \mathbb{Z})$$

is the identity

$$\begin{aligned} \text{id}_{H^1(\hat{T})} &\in \text{Hom}(H^1(\hat{T}, \mathbb{Z}), H^1(\hat{T}, \mathbb{Z})) \\ &= H^1(T, \mathbb{Z}) \otimes H^1(\hat{T}, \mathbb{Z}) \subset H^2(T \times \hat{T}, \mathbb{Z}), \end{aligned}$$

which is easily seen to be of Hodge type  $(1, 1)$ . Next the uniqueness of  $\mathcal{P}$  is forced by the conditions

$$\mathcal{P}|_{0 \times \hat{T}} \cong \mathcal{O}_{\hat{T}}, \quad \mathcal{P}|_{T \times 0} \cong \mathcal{O}_T.$$

Next, because  $T$  admits the endomorphism  $\phi_T$ , we also have the line bundle

$$\mathcal{P}_\phi := (\phi, \text{Id})^* \mathcal{P}.$$

We now make the following construction: Over  $T \times \widehat{T}$ , consider the rank 2 vector bundles

$$E = \mathcal{P} \oplus \mathcal{P}^{-1}, \quad E_\phi = \mathcal{P}_\phi \oplus \mathcal{P}_\phi^{-1}$$

and the corresponding associated projective bundles  $\mathbb{P}(E)$ ,  $\mathbb{P}(E_\phi)$ . The two commuting involutions  $(-\text{Id}, \text{Id})$  and  $(\text{Id}, -\text{Id})$  of  $T \times \widehat{T}$  lift to commuting involutions  $i, \hat{i}$ , resp.  $i_\phi, \hat{i}_\phi$  acting on  $E$  resp.  $E_\phi$ , since we have isomorphisms

$$\begin{aligned} (-\text{Id}, \text{Id})^* \mathcal{P} &\cong \mathcal{P}^{-1}, & (\text{Id}, -\text{Id})^* \mathcal{P} &\cong \mathcal{P}^{-1}, \\ (-\text{Id}, \text{Id})^* \mathcal{P}_\phi &\cong \mathcal{P}_\phi^{-1}, & (\text{Id}, -\text{Id})^* \mathcal{P}_\phi &\cong \mathcal{P}_\phi^{-1}, \end{aligned}$$

which can be made canonical by a choice of trivialization

$$\mathcal{P}|_{(0,0)} \cong \mathbb{C},$$

$(0,0)$  being a fixed point of both  $(\text{Id}, -\text{Id})$  and  $(-\text{Id}, \text{Id})$ .

The compact Kähler manifold we shall consider is the following: We start with the fibered product

$$\mathbb{P}(E) \times_{T \times \widehat{T}} \mathbb{P}(E_\phi).$$

It admits the commuting involutions

$$(i, i_\phi), (\hat{i}, \hat{i}_\phi)$$

over  $(-\text{Id}, \text{Id})$ ,  $(\text{Id}, -\text{Id})$  respectively. The quotient  $Q$  of  $\mathbb{P}(E) \times_{T \times \widehat{T}} \mathbb{P}(E_\phi)$  by the group  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  generated by these involutions is singular along the non free locus of this action, but the quotient admits a Kähler compact desingularization. For example, one can start by desingularizing the quotient  $\mathbb{P}(E) \times_{T \times \widehat{T}} \mathbb{P}(E_\phi) / (i, i_\phi)$  by blowing-up the fixed locus of  $(i, i_\phi)$  and then taking the quotient of the blown-up variety by the natural involution which lifts  $(i, i_\phi)$ . The result is smooth Kähler and by naturality  $(\hat{i}, \hat{i}_\phi)$  acts on it as an involution. Then one can desingularize in the same way the quotient of this new variety by  $(\hat{i}, \hat{i}_\phi)$ .

Our compact Kähler manifold  $X$  will be any Kähler desingularization of this quotient.

Note that, if  $K$  is the Kummer variety of  $T$ , namely the desingularization of the quotient of  $T$  by the  $-\text{Id}$  involution, obtained by blowing-up the images of the 2-torsion points of  $T$ , and similarly  $\widehat{K}$  is the Kummer variety of  $\widehat{T}$ , then over  $K_0 \times \widehat{K}_0$ ,  $X$  is a  $\mathbb{P}^1 \times \mathbb{P}^1$ -bundle, where  $K_0$  is the open set  $T_0 / \pm \text{Id}$  of  $K$ , with

$$T_0 := T \setminus \{2\text{-torsion points}\},$$

and similarly for  $\widehat{K}_0$ .

The next sections will be devoted to the proof of the following Theorem:

**Theorem 4.** *Let  $X'$  be any compact complex manifold bimeromorphically equivalent to  $X$ . Then  $X'$  does not have the homotopy type of a complex projective manifold.*

## 2. Some results on the cohomology ring of $X'$

We plan to show in fact the following slightly stronger result:

**Theorem 5.** *Let  $X'$  be any compact complex manifold bimeromorphically equivalent to  $X$ , and let  $Y$  be a Kähler compact manifold. Assume there is an isomorphism of graded algebras:*

$$\gamma : H^*(Y, \mathbb{Q}) \cong H^*(X', \mathbb{Q}).$$

*Then  $Y$  is not projective.*

In other words, Theorem 4 is true for rational homotopy type rather than homotopy type, since it is known that the rational homotopy type of a compact simply connected Kähler manifold is determined by its rational cohomology algebra (see [4]).

This section will be devoted to the study of the cohomology ring of any compact complex manifold  $X'$  given as in Theorem 5. The proof of Theorem 5 will be given in the next section, following the same line as [8], section 3.

Recall that  $X$  admits a holomorphic map

$$q : X \rightarrow (T/\pm \text{Id}) \times (\widehat{T}/\pm \text{Id}),$$

obtained by composing the desingularization map

$$X \rightarrow \mathbb{P}(E) \times_{T \times \widehat{T}} \mathbb{P}(E_\phi) / \langle (i, i_\phi), (\hat{i}, \hat{i}_\phi) \rangle$$

with the natural map

$$\mathbb{P}(E) \times_{T \times \widehat{T}} \mathbb{P}(E_\phi) / \langle (i, i_\phi), (\hat{i}, \hat{i}_\phi) \rangle \rightarrow T \times \widehat{T} / \langle (-\text{Id}, \text{Id}), (\text{Id}, -\text{Id}) \rangle.$$

For simplicity of notations, we shall assume in the sequel that our  $X$  in section 1 has been chosen so that  $q$  extends to a holomorphic map

$$\bar{q} : X \rightarrow K \times \widehat{K},$$

which can always be achieved by a bimeromorphic transformation.

**Lemma 1.** *Let  $\psi : X' \dashrightarrow X$  be any bimeromorphic map. Then  $q \circ \psi$  is holomorphic.*

*Proof.* The complex manifold  $T \times \widehat{T}$  does not contain any closed complex curve. Indeed, it suffices to prove this for  $T$  or  $\widehat{T}$ . Now, the cohomology class  $[C]$  of such a curve would be a non zero Hodge class of degree  $2n - 2$  on  $T$ , resp.  $\widehat{T}$ , or equivalently, a non-zero Hodge class in

$H^2(\widehat{T}, \mathbb{Q})$ , resp.  $H^2(T, \mathbb{Q})$ . But in [8], Remark 3, we proved that the existence of  $\phi_T$ , resp.  $\phi_{\widehat{T}}$  satisfying property (\*) of section 1 prevents the existence of such a Hodge class.

It follows that the quotient  $(T/\pm \text{Id}) \times (\widehat{T}/\pm \text{Id})$  does not contain any rational curve, and by desingularization of meromorphic maps with value in compact complex manifolds, this is enough to conclude that  $q \circ \psi$  has to be holomorphic. q.e.d.

It follows from Lemma 1 that  $H^*(X', \mathbb{Q})$  contains a subalgebra

$$A^* := (q \circ \psi)^* H^*((T/\pm \text{Id}) \times (\widehat{T}/\pm \text{Id}), \mathbb{Q})$$

which is isomorphic to  $H^*((T/\pm \text{Id}) \times (\widehat{T}/\pm \text{Id}), \mathbb{Q})$ . Note that this last algebra is isomorphic to

$$H^*(T/\pm \text{Id}, \mathbb{Q}) \otimes H^*(\widehat{T}/\pm \text{Id}, \mathbb{Q}),$$

and that  $H^*(T/\pm \text{Id}, \mathbb{Q}) = H^{\text{even}}(T, \mathbb{Q})$  (and similarly for  $\widehat{T}$ ).

We shall denote by  $A_1^*$ , resp.  $A_2^*$ , the subalgebra  $(q \circ \psi)^* H^*((T/\pm \text{Id}) \times 0, \mathbb{Q})$ , resp.  $(q \circ \psi)^* H^*(0 \times (\widehat{T}/\pm \text{Id}), \mathbb{Q})$ .

Next, we note that the cohomology of  $X'$  in degree 2 is generated over  $\mathbb{Q}$  by  $A^2$  and by degree 2 Hodge classes. Indeed, this is true for  $X$ , because  $X$  contains a Zariski open set which is a  $\mathbb{P}^1 \times \mathbb{P}^1$ -bundle over  $K_0 \times \widehat{K}_0$ , and this implies easily that  $H^{2,0}(X) = H^0(X, \Omega_X^2)$  is equal to

$$\bar{q}^* H^{2,0}(K \times \widehat{K}) = q^* H^{2,0}((T/\pm \text{Id}) \times (\widehat{T}/\pm \text{Id})).$$

Next, this property is invariant under meromorphic transformations; hence if it is true for  $X$ , it is true for  $X'$ .

Now let  $D \subset H^2(X', \mathbb{Q})$  be the subspace generated by degree 2 Hodge classes. So we have

$$(2.1) \quad H^2(X', \mathbb{Q}) = D \oplus A^2,$$

because, by [8], Remark 3, we know that the presence of the endomorphism  $\phi_T$  of  $T$  satisfying property (\*) of section 1 implies that  $H^2(T, \mathbb{Q})$  has no non-zero Hodge class, and similarly for  $\widehat{T}$ . Furthermore, we have by definition

$$(2.2) \quad A^2 = A_1^2 \oplus A_2^2.$$

For  $\alpha \in H^2(X', \mathbb{C})$ , let

$$\alpha = \alpha_D + \alpha', \quad \alpha' = \alpha_1 + \alpha_2,$$

be its decompositions given by (2.1), (2.2).

A key role will be played by the following Proposition 1:

We consider the algebraic subset  $Z \subset H^2(X', \mathbb{C})$  defined as

$$Z = \{\alpha \in H^2(X', \mathbb{C}), \alpha^2 = 0 \text{ in } H^4(X', \mathbb{C})\}.$$

$Z$  contains the algebraic subsets  $Z_1, Z_2$  defined as

$$Z_1 = \{\alpha \in H^2(X', \mathbb{C}), \alpha_2 = 0, \alpha_1^2 = \alpha_1 \alpha_D = \alpha_D^2 = 0 \text{ in } H^4(X', \mathbb{C})\},$$



resp.

$$Z_2 = \left\{ \alpha \in H^2(X', \mathbb{C}), \alpha_1 = 0, \alpha_2^2 = \alpha_2 \alpha_D = \alpha_D^2 = 0 \text{ in } H^4(X', \mathbb{C}) \right\}.$$

**Proposition 1.** *Any irreducible component of  $Z_1$  (resp.  $Z_2$ ) containing*

$$Z_{1,0} := Z_1 \cap \{\alpha, \alpha_D = 0\}$$

(resp.  $Z_{2,0} := Z_2 \cap \{\alpha, \alpha_D = 0\}$ ) *is an irreducible component of  $Z$ .*

*Proof.* The condition  $\alpha^2 = 0$  writes as

$$(2.3) \quad \alpha'^2 + 2\alpha_D \alpha' + \alpha_D^2 = 0.$$

Now we observe that  $\alpha_D^2$  belongs to  $Hdg^4(X') \otimes \mathbb{C}$ , where

$$Hdg^4(X') := H^4(X', \mathbb{Q}) \cap H^{2,2}(X').$$

Similarly, because the Hodge structure on  $D$  is trivial, that is purely of type  $(1, 1)$ ,  $\alpha_D \alpha'$  belongs to  $N_2 H^4(X') \otimes \mathbb{C}$ , where  $N_2 H^4(X')$  is the maximal rational sub-Hodge structure of  $H^4(X', \mathbb{Q})$  which is of Hodge level 2.

(Here we recall that the level of a weight  $k$  Hodge structure  $H$ ,  $H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$ , is the integer

$$\text{Max} \{p - q, H^{p,q} \neq 0\}.$$

Thus a level 2 sub-Hodge structure of a weight 4 Hodge structure is a sub-Hodge structure which has no  $(4, 0)$ -term.)

Equation (2.3) thus implies that  $\alpha'^2$  belongs to  $N_2 A_{\mathbb{Q}}^4 \otimes \mathbb{C}$ , where again  $N_2$  means that we consider the maximal rational sub-Hodge structure of level 2.

Next we have the Künneth decomposition:

$$(2.4) \quad A_{\mathbb{Q}}^4 = A_{1\mathbb{Q}}^4 \oplus A_{1\mathbb{Q}}^2 \otimes A_{2\mathbb{Q}}^2 \oplus A_{2\mathbb{Q}}^4,$$

which is a decomposition into sub-Hodge structures of weight 4. We have the following:

**Lemma 2.**  *$A_{1\mathbb{Q}}^4$  and  $A_{2\mathbb{Q}}^4$  do not contain non trivial sub-Hodge structure of Hodge level 2.*

*Proof.* We use the fact that  $n \geq 4$ , so that  $A_{1\mathbb{Q}}^4$  and  $A_{2\mathbb{Q}}^4$  are of Hodge level 4. Next we use the assumption (\*) satisfied by  $\phi$  to conclude that  $\phi_T^*$  acts in an irreducible way on  $\bigwedge^4 H^1(T, \mathbb{Q}) = A_{1\mathbb{Q}}^4$ , and since the action is via morphisms of Hodge structures, it must preserve  $N_2 A_{1\mathbb{Q}}^4$ . Hence, because  $N_2 A_{1\mathbb{Q}}^4 \neq A_{1\mathbb{Q}}^4$ , we conclude that  $N_2 A_{1\mathbb{Q}}^4 = 0$  and similarly  $N_2 A_{2\mathbb{Q}}^4 = 0$ . q.e.d.

From the fact that  $\alpha'^2 = \alpha_1^2 + 2\alpha_1\alpha_2 + \alpha_2^2 \in N_2A_{\mathbb{Q}}^4 \otimes \mathbb{C}$ , from the decomposition (2.4) into sub-Hodge structures and from Lemma 2, we conclude that

$$\alpha_1^2 = 0, \alpha_2^2 = 0.$$

Thus our initial equation (2.3) becomes

$$(2.5) \quad 2\alpha_1\alpha_2 + 2\alpha_D\alpha' + \alpha_D^2 = 0.$$

This equation implies as already noticed that  $\alpha_1\alpha_2$  belongs to the space

$$N_2(A_{1\mathbb{Q}}^2 \otimes A_{2\mathbb{Q}}^2) \otimes \mathbb{C}.$$

In fact one can say more: indeed, note that the Hodge structure on

$$D \cdot A_{\mathbb{Q}}^2 + D^2 \subset H^4(X', \mathbb{Q})$$

is the quotient of a direct sum of Hodge structures of level 2 isomorphic either to  $A_{1\mathbb{Q}}^2$  or to  $A_{2\mathbb{Q}}^2$  or to a trivial Hodge structure.

Thus condition (2.5) implies that  $\alpha_1\alpha_2$  has in fact to belong to the space

$$N'_2(A_{1\mathbb{Q}}^2 \otimes A_{2\mathbb{Q}}^2) \otimes \mathbb{C},$$

where  $N'_2$  means *the maximal sub-Hodge structure of level 2, which is a subquotient of a sum of copies of  $A_{1\mathbb{Q}}^2$  or  $A_{2\mathbb{Q}}^2$  or a trivial Hodge structure.*

On the other hand, the Hodge structures on  $A_{1\mathbb{Q}}^2$  or  $A_{2\mathbb{Q}}^2$  are simple, that is they do not contain any non-trivial sub-Hodge structure. To see this last point, assume that there is a proper non-zero simple sub-Hodge structure

$$H \subset H^2(T, \mathbb{Q}).$$

As the endomorphism of Hodge structure  $\phi_T^*$  acts transitively on  $H^2(T, \mathbb{Q})$ , it follows that  $H^2(T, \mathbb{Q})$  must then be isomorphic to a sum of copies of  $H$ ,

$$\exists k > 1, H^2(T, \mathbb{Q}) \cong H^k.$$

But then  $H^2(T, \mathbb{Q})$  admits a projector which is an endomorphism of Hodge structure. This contradicts the fact, noted at the end of the proof of Lemma 4 below, that the algebra  $End H^2(T, \mathbb{Q})$  is generated by  $\phi_T^*$ , and thus does not contain projectors by condition (\*).

Note also that the Hodge structures on  $A_{1\mathbb{Q}}^2$  and  $A_{2\mathbb{Q}}^2$  are not isomorphic, as shown by Lemma 3 below.

Thus it follows that  $N'_2(A_{1\mathbb{Q}}^2 \otimes A_{2\mathbb{Q}}^2)$  is in fact equal to the maximal sub-Hodge structure of level 2 of  $A_{1\mathbb{Q}}^2 \otimes A_{2\mathbb{Q}}^2$ , which is a sum of copies of  $A_{1\mathbb{Q}}^2$  or  $A_{2\mathbb{Q}}^2$  or a trivial Hodge structure.

We have the following Lemma:

**Lemma 3.** *There are no non zero morphisms of Hodge structures (of bidegree (1, 1)) from  $A_{1\mathbb{Q}}^2$  or  $A_{2\mathbb{Q}}^2$  to  $A_{1\mathbb{Q}}^2 \otimes A_{2\mathbb{Q}}^2$ .*

Admitting this Lemma, we conclude that in fact  $\alpha_1\alpha_2$  has to belong to

$$\text{Hdg}(A_{1\mathbb{Q}}^2 \otimes A_{2\mathbb{Q}}^2) \otimes \mathbb{C},$$

where  $\text{Hdg}$  means *the subspace of rational Hodge classes*. We have next the following Lemma, the proof of which we shall also postpone:

**Lemma 4.** *There are (up to a coefficient) finitely many elements*

$$\beta \in \text{Hdg}(A_{1\mathbb{Q}}^2 \otimes A_{2\mathbb{Q}}^2) \otimes \mathbb{C}$$

*which are of rank 1, that is of the form  $\alpha_1\alpha_2$  as above.*

We then conclude as follows: from the above analysis, we conclude that for  $\alpha \in Z$ , we have either  $\alpha_1 \neq 0$ ,  $\alpha_2 \neq 0$  and then  $\alpha_1\alpha_2$  has to be proportional to one of the finitely many  $\beta$  of Lemma 4, or one of  $\alpha_1$ , or  $\alpha_2$  has to be 0.

We claim that in this last case,  $\alpha$  belongs to  $Z_2$  or  $Z_1$  respectively. Indeed, we know that in any case

$$\alpha_1^2 = 0, \alpha_2^2 = 0.$$

Assume  $\alpha_2 = 0$ . Equation (2.5) thus becomes:

$$2\alpha_D\alpha_1 + \alpha_D^2 = 0.$$

But this implies that

$$\alpha_1\alpha_D = 0, \alpha_D^2 = 0.$$

Indeed,  $\alpha_D^2$  belongs to  $\text{Hdg}^4(X') \otimes \mathbb{C}$  while  $\alpha_1\alpha_D$  belongs to the space

$$N_2''H^4(X', \mathbb{Q}) \otimes \mathbb{C},$$

defined as the maximal sub-Hodge structure of  $H^4(X', \mathbb{Q})$  isomorphic to a subquotient of some power of  $A_{1\mathbb{Q}}^2$ . By the same simplicity argument as before,  $N_2''H^4(X', \mathbb{Q}) \otimes \mathbb{C}$  is also the maximal sub-Hodge structure of  $H^4(X', \mathbb{Q})$  isomorphic to some power of  $A_{1\mathbb{Q}}^2$ . But the intersection

$$\text{Hdg}^4(X') \cap N_2''H^4(X', \mathbb{Q})$$

has to be zero, since there is no non zero Hodge class in  $A_{1\mathbb{Q}}^2$ . Thus also

$$\text{Hdg}^4(X') \otimes \mathbb{C} \cap N_2''H^4(X', \mathbb{Q}) \otimes \mathbb{C}$$

has to be 0. Hence we proved that  $2\alpha_D\alpha_1 + \alpha_D^2 = 0$  implies that  $\alpha_1\alpha_D = 0$ ,  $\alpha_D^2 = 0$ .

In conclusion, we proved that  $Z$  is the set-theoretic union of  $Z_1$ ,  $Z_2$ , and of a set which projects to a finite set of lines in  $A_{1\mathbb{C}}^2$  and  $A_{2\mathbb{C}}^2$ .

Let now  $Z_1'$  be an irreducible component of  $Z_1$  which contains  $Z_{1,0} \subset A_{1\mathbb{C}}^2$ . Suppose it is not an irreducible component of  $Z$ . This means that there exists an irreducible component  $Z'$  of  $Z$  containing  $Z_1'$ , not contained in  $Z_1$ , such that  $Z' \setminus Z_1'$  is dense in  $Z'$ . So  $Z' \setminus Z_1'$  has to project, via the projection to  $A_{1\mathbb{C}}^2$ , to a locally closed subvariety of  $A_{1\mathbb{C}}^2$  whose closure contains  $Z_{1,0}$ . As  $\dim Z_{1,0} \geq 4n - 3$ , because  $A_1^*$  is

isomorphic to  $\bigwedge^{\text{even}} \Gamma^*$ , this contradicts the fact that  $Z' \setminus Z'_1$  has to be contained in the union of  $Z_2$ , which projects to 0 in  $A_{1\mathbb{C}}^2$ , and of a set which projects to a finite union of lines in  $A_{1\mathbb{C}}^2$ . Thus Proposition 1 is proved, assuming Lemmas 3 and 4. q.e.d.

*Proof of Lemma 3.* Recall that

$$A_{1\mathbb{Q}}^2 = \bigwedge^2 H^1(T, \mathbb{Q}) \cong \bigwedge^2 \Gamma_{\mathbb{Q}}^*,$$

$$A_{2\mathbb{Q}}^2 = \bigwedge^2 H^1(\widehat{T}, \mathbb{Q}) \cong \bigwedge^2 \Gamma_{\mathbb{Q}}.$$

We have the endomorphisms  $\phi_T, \phi_{\widehat{T}}$  acting respectively on the complex tori  $T$  and  $\widehat{T}$ , and the induced action  $\phi_T^*, \phi_{\widehat{T}}^*$  on  $H^2(T, \mathbb{Q})$ , resp.  $H^2(\widehat{T}, \mathbb{Q})$ , identify to  $\wedge^{2t}\phi, \wedge^2\phi$  respectively.

Let  $\lambda_1, \dots, \lambda_{2n}$  be the  $2n$ -eigenvalues of  $\phi$  on  $\Gamma_{\mathbb{C}}$ . Let  $e_1, \dots, e_{2n}$  be a corresponding basis of eigenvectors of  $\Gamma_{\mathbb{C}}$ , and let  $e_i^*$  be the dual basis of  $\Gamma_{\mathbb{C}}^*$ . We choose the ordering in such a way that  $\Gamma'$  (see section 1) is generated by  $e_1, \dots, e_n$ . In other words,  $e_i \in H^1(\widehat{T}, \mathbb{C})$  have Hodge type  $(1, 0)$  for  $i \leq n$  and  $e_i^* \in H^1(T, \mathbb{C})$  have Hodge type  $(1, 0)$  for  $i > n$ .

We want to study the Hodge classes in

$$A_{1,\mathbb{Q}}^{2*} \otimes A_{1,\mathbb{Q}}^2 \otimes A_{2,\mathbb{Q}}^2$$

$$= \bigwedge^2 \Gamma_{\mathbb{Q}} \otimes \bigwedge^2 \Gamma_{\mathbb{Q}}^* \otimes \bigwedge^2 \Gamma_{\mathbb{Q}},$$

which we consider as a weight 6 Hodge structure, so the classes we search are the rational classes of Hodge type  $(3, 3)$ .

This space

$$S := \text{Hdg}\left(\bigwedge^2 \Gamma_{\mathbb{Q}} \otimes \bigwedge^2 \Gamma_{\mathbb{Q}}^* \otimes \bigwedge^2 \Gamma_{\mathbb{Q}}\right)$$

is stable under the action of the three commuting morphisms of Hodge structures

$$\wedge^2\phi \otimes \text{Id} \otimes \text{Id}, \text{Id} \otimes \wedge^{2t}\phi \otimes \text{Id}, \text{Id} \otimes \text{Id} \otimes \wedge^2\phi.$$

It follows that the complexified space  $S_{\mathbb{C}}$  is generated by eigenvectors for these actions, namely elements of the form

$$(2.6) \quad e_i \wedge e_j \otimes e_k^* \wedge e_l^* \otimes e_r \wedge e_s.$$

For  $a, b, c \in \mathbb{Z}$ , consider the endomorphism

$$\Phi_{abc} := (\wedge^2\phi)^a \otimes (\wedge^{2t}\phi)^b \otimes (\wedge^2\phi)^c$$

of  $\bigwedge^2 \Gamma_{\mathbb{Q}} \otimes \bigwedge^2 \Gamma_{\mathbb{Q}}^* \otimes \bigwedge^2 \Gamma_{\mathbb{Q}}$ .  $\Phi_{abc}$  is diagonal in the basis given by the elements (2.6), with corresponding eigenvalues

$$(\lambda_i \lambda_j)^a (\lambda_k \lambda_l)^b (\lambda_r \lambda_s)^c.$$

The Galois group of the field  $K = \mathbb{Q}[\lambda_1, \dots, \lambda_{2n}]$  over  $\mathbb{Q}$  acts on the  $\lambda_i$  and has to leave stable the set  $E_{abc}$  of eigenvalues of  $\Phi_{abc}$  on  $S$ , since  $S$  is defined over  $\mathbb{Q}$ . On the other hand, we know that this Galois group is the symmetric group  $\mathfrak{S}_{2n}$  on  $2n$  letters acting on the  $\lambda_i$ 's. Thus we conclude that if

$$(\lambda_i \lambda_j)^a (\lambda_k \lambda_l)^b (\lambda_r \lambda_s)^c \in E_{abc},$$

and  $\sigma \in \mathfrak{S}_{2n}$ , then also

$$(\lambda_{\sigma(i)} \lambda_{\sigma(j)})^a (\lambda_{\sigma(k)} \lambda_{\sigma(l)})^b (\lambda_{\sigma(r)} \lambda_{\sigma(s)})^c \in E_{abc}.$$

But for an adequate choice of  $a, b, c$ , the map

$$(\{i, j\}, \{k, l\}, \{r, s\}) \mapsto (\lambda_i \lambda_j)^a (\lambda_k \lambda_l)^b (\lambda_r \lambda_s)^c$$

is injective. Thus we conclude that if (2.6) belongs to  $S_{\mathbb{C}}$ , so does

$$(2.7) \quad e_{\sigma(i)} \wedge e_{\sigma(j)} \otimes e_{\sigma(k)}^* \wedge e_{\sigma(l)}^* \otimes e_{\sigma(r)} \wedge e_{\sigma(s)}.$$

As  $S_{\mathbb{C}}$  is contained in the (3, 3)-part of

$$A_{1, \mathbb{C}}^{2*} \otimes A_{1, \mathbb{C}}^2 \otimes A_{2, \mathbb{C}}^2$$

we see that (2.7) has to be of Hodge type (3, 3) for any permutation  $\sigma \in \mathfrak{S}_{2n}$ .

But as  $n \geq 4$ , it is immediate that we can always find  $\sigma$  in such a way that (2.7) has Hodge type (4, 2) (eg choose  $i, j, r, s$  in  $\{1, \dots, n\}$ ).

Thus an element (2.6) in

$$\text{Hdg} \left( \bigwedge^2 \Gamma_{\mathbb{Q}} \otimes \bigwedge^2 \Gamma_{\mathbb{Q}}^* \otimes \bigwedge^2 \Gamma_{\mathbb{Q}} \right) \otimes \mathbb{C}$$

does not exist, which proves the lemma.

q.e.d.

*Proof of Lemma 4.* We study the space

$$\text{Hdg} (A_{1, \mathbb{Q}}^2 \otimes A_{2, \mathbb{Q}}^2) \otimes \mathbb{C}$$

exactly as in the previous proof. The space  $A_{1, \mathbb{Q}}^2 \otimes A_{2, \mathbb{Q}}^2$  identifies to

$$\bigwedge^2 \Gamma_{\mathbb{Q}}^* \otimes \bigwedge^2 \Gamma_{\mathbb{Q}}$$

as Hodge structures, where the  $e_i^* \in \Gamma_{\mathbb{C}}^*$  have Hodge type (1, 0) for  $i > n$ , while the  $e_i \in \Gamma_{\mathbb{C}}$  have Hodge type (1, 0) for  $i \leq n$ .

Again, the space  $S := \text{Hdg}(A_{1, \mathbb{Q}}^2 \otimes A_{2, \mathbb{Q}}^2) \otimes \mathbb{C}$ , being stable under  $\wedge^{2t} \phi \otimes \text{Id}$  and  $\text{Id} \otimes \wedge^2 \phi$ , has to be generated by eigenvectors for both of these commuting endomorphisms, that is elements of the form :

$$e_i^* \wedge e_j^* \otimes e_k \wedge e_l.$$

Because this space is defined over  $\mathbb{Q}$ , we conclude as in the previous proof that it has to be stable under the action of  $\mathfrak{S}_{2n}$ , which means that for any permutation  $\sigma$  of  $1, \dots, 2n$ ,

$$e_{\sigma(i)}^* \wedge e_{\sigma(j)}^* \otimes e_{\sigma(k)} \wedge e_{\sigma(l)}$$

has to be of type  $(2, 2)$ . Now this implies that up to permuting  $i$  and  $j$ , one must have  $i = k, j = l$ . Indeed, if the four indices are distinct, by changing them by some  $\sigma \in \mathfrak{S}_{2n}$ , we may arrange that  $e_{\sigma(i)}^* \wedge e_{\sigma(j)}^* \otimes e_{\sigma(k)} \wedge e_{\sigma(l)}$  has Hodge type  $(4, 0)$ , and if eg  $i = k$  but  $j \neq l$ , by changing them by some  $\sigma \in \mathfrak{S}_{2n}$ , we may arrange that  $e_{\sigma(i)}^* \wedge e_{\sigma(j)}^* \otimes e_{\sigma(i)} \wedge e_{\sigma(l)}$  has Hodge type  $(3, 1)$ . Hence we have proved that  $Hdg(A_{1\mathbb{Q}}^2 \otimes A_{2\mathbb{Q}}^2) \otimes \mathbb{C}$  is contained in the space generated by the elements  $e_i^* \wedge e_j^* \otimes e_i \wedge e_j$ . (In fact it has to be equal to the space generated by these elements, which is nothing but the algebra generated over  $\mathbb{C}$  by  $\phi_{\widehat{T}}^*$ , and is indeed generated by Hodge classes.)

As  $\phi_{\widehat{T}}^*$  is diagonalizable with distinct eigenvalues, it is then clear that this  $\mathbb{C}$ -algebra contains (up to a scalar) only finitely elements of rank 1, namely the elements above. q.e.d.

Proposition 1 is now fully proved. Our next technical Lemma will be the following:

**Lemma 5.** *Let  $D_1 \subset D$  be defined as*

$$D_1 = \{\alpha_D \in D, \alpha_D \alpha_1 = 0 \text{ in } H^4(X', \mathbb{Q}), \forall \alpha_1 \in A_{1\mathbb{Q}}^2\}.$$

*Then, if  $\alpha_D \in D \otimes \mathbb{C}$  satisfies  $\alpha_D \alpha_1 = 0$  for one non zero  $\alpha_1 \in A_{1\mathbb{C}}^2$ , one has  $\alpha_D \in D_1 \otimes \mathbb{C}$ .*

*Proof.* First of all, note that if

$$\psi' : X'' \rightarrow X'$$

is a proper surjective holomorphic map, with  $X''$  smooth, and the result is true for  $X''$ , with  $D$  replaced by the space  $Hdg^2(X'')$  and  $A_{1\mathbb{Q}}^2$  by  $\psi'^* A_{1\mathbb{Q}}^2$ , then it is also true for  $X'$ .

Indeed, such a map  $\psi'$  induces an injective morphism  $\psi'^*$  of cohomology algebras, which sends  $D$  in the space of Hodge classes of degree 2 on  $X''$ .

Recall now that  $X'$  is bimeromorphic to a quotient of the  $\mathbb{P}^1 \times \mathbb{P}^1$ -bundle

$$W := \mathbb{P}(E) \times_{T \times \widehat{T}} \mathbb{P}(E_\phi) \xrightarrow{\tilde{q}} T \times \widehat{T}$$

over  $T \times \widehat{T}$ . Hence there is a dominant meromorphic map from  $W$  to  $X'$ . Note that this map is invariant under the two involutions  $(i, i_\phi)$  and  $(\hat{i}, \hat{i}_\phi)$  acting on  $W$ .

Using Hironaka's desingularization theorem and the previous observation, we can thus reduce to the case where  $X'$  is deduced from  $W$

by a sequence of blow-ups, and where the  $D$  is replaced with the set of Hodge classes generated by exceptional divisors and by Hodge classes on  $W$  invariant under  $(i, i_\phi)$  and  $(\hat{i}, \hat{i}_\phi)$ . We first prove that the result is true for  $W$ , with  $D$  the space of Hodge classes on  $W$  invariant under  $(i, i_\phi)$  and  $(\hat{i}, \hat{i}_\phi)$ , and  $A_1$  replaced with  $\tilde{q}^*pr_1^*H^2(T)$ . The cohomology of degree 2 of  $W$  is a free module over the cohomology of  $T \times \hat{T}$  generated by  $H^*(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Q})$ . The space of degree 2 Hodge classes  $D$  on  $W$  invariant under  $(i, i_\phi)$  and  $(\hat{i}, \hat{i}_\phi)$  is the sum of two spaces, namely  $D_0$  which has rank 2 and is isomorphic by restriction to  $H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Q})$  and  $D_1$  which is isomorphic via  $\tilde{q}^*$  to the set of degree 2 Hodge classes in  $H^2(T \times \hat{T}, \mathbb{Q})$  invariant under  $(\text{Id}, -\text{Id})$  and  $(-\text{Id}, \text{Id})$ . We saw already that  $D_1 = 0$ . We conclude from this that the product map

$$D \otimes \tilde{q}^*pr_1^*(H^2(T, \mathbb{Q}) \rightarrow H^4(W, \mathbb{Q}))$$

is injective, so that there is in fact nothing to prove for  $W$ .

It remains now only to prove that if the statement is true for  $W$ , it is true for any complex manifold obtained by successive blow-ups of  $W$  along smooth centers. This is proved by induction on the number of blow-ups. Assume it is true for  $W_i$  and let  $\tau : W_{i+1} \rightarrow W_i$  be the blow-up of a smooth irreducible center  $Z \subset W_i$ . Then the set of degree 2 Hodge classes  $D_{i+1}$  on  $W_{i+1}$  is generated by  $\tau^*D_i$  and the class  $e_Z$  of the exceptional divisor  $E_Z$ . Now, the study of the cohomology ring of  $W_{i+1}$  (see [9] I, 7.3.3) shows that if there is an equality

$$e_Z\tau^*\alpha = 0, \quad \text{modulo } \tau^*H^*(W_i, \mathbb{C})$$

then in fact  $e_Z\tau^*\alpha = 0$  in  $H^*(W_{i+1}, \mathbb{C})$ .

Now suppose there is a relation  $\alpha_D\alpha = 0$  in  $H^*(W_{i+1}, \mathbb{C})$ , where  $\alpha_D \in D_{i+1} \otimes \mathbb{C}$  and  $\alpha \in \tilde{q}^*pr_1^*H^2(T, \mathbb{C})$ . Writing

$$\alpha_D = \mu e_Z + \alpha'_D,$$

where  $\mu \in \mathbb{C}$  and  $\alpha'_D \in \tau^*D_i \otimes \mathbb{C}$ , we conclude using the previous remark that

$$\mu e_Z\alpha = 0 \text{ in } H^*(W_{i+1}, \mathbb{C}),$$

that is, either  $\mu = 0$ , in which case we can apply the result for  $W_i$ , or

$$e_Z\alpha = 0 \text{ in } H^*(W_{i+1}, \mathbb{C}).$$

Since multiplication by the Hodge class  $e_Z$  is a morphism of Hodge structures from  $H^2(T, \mathbb{Q})$  to  $H^4(W_{i+1}, \mathbb{Q})$ , its kernel is a sub-Hodge structure of  $H^2(T, \mathbb{Q})$ . So this map is either injective or 0, since the Hodge structure on  $H^2(T, \mathbb{Q})$  is simple, as already noticed before.

The conclusion is that, if there is one non-zero  $\alpha$  satisfying  $\alpha_D\alpha = 0$  in  $H^*(W_{i+1}, \mathbb{C})$  with a coefficient  $\mu \neq 0$ , we find that  $e_Z\alpha' = 0$  in  $H^*(W_{i+1}, \mathbb{C})$ , for any  $\alpha' \in \tilde{q}^*pr_1^*H^2(T \times 0, \mathbb{Q})$ , and that furthermore the equality  $\alpha_D\alpha = 0$  reduces to the equality  $\alpha'_D\alpha = 0$ , which holds already in  $H^*(W_i, \mathbb{C})$ . Hence the result is proved by induction. q.e.d.

We will need also the following result.

**Lemma 6.**

a) For any  $d \in D \otimes \mathbb{C}$ ,  $\beta \in A_{\mathbb{C}}^{4n-2} \subset H^{4n-2}(X', \mathbb{C})$ , one has

$$d^3 \beta = 0 \text{ in } H^{4n+4}(X', \mathbb{C}) = \mathbb{C}.$$

b) The complex subspace  $D \otimes \mathbb{C} \subset H^2(X', \mathbb{C})$  is an irreducible component of the algebraic set

$$(2.8) \quad Z' = \{d \in H^2(X', \mathbb{C}), d^3 \beta = 0, \forall \beta \in A_{\mathbb{C}}^{4n-2}\}.$$

*Proof.*  $D$  is made of Hodge classes. So for any  $d \in D$ , the map

$$\alpha \mapsto d^3 \alpha \in H^{4n+4}(X', \mathbb{Q}) = \mathbb{Q}$$

is a Hodge class in  $(A_{\mathbb{Q}}^{4n-2})^* = A_{\mathbb{Q}}^2$ . But we already know that  $A_{\mathbb{Q}}^2$  has no non zero Hodge classes. This proves a).

Let  $Z'_1 \subset H^2(X', \mathbb{C})$  be an irreducible component of the algebraic subset  $Z'$  of (2.8) containing strictly  $D \otimes \mathbb{C}$ . Choose any point  $d \in D \otimes \mathbb{C}$  and let

$$D'_{\mathbb{C}} := T_{Z'_1, d} \subset H^2(X', \mathbb{C}).$$

Since

$$D \oplus A_{\mathbb{Q}}^2 = H^2(X', \mathbb{Q}),$$

and  $D \otimes \mathbb{C} \subset D'_{\mathbb{C}}$ , where the inclusion is strict, there must be a non-zero element depending on  $d$

$$(2.9) \quad \alpha_d \in D'_{\mathbb{C}} \cap A_{\mathbb{C}}^2.$$

As  $\alpha_d \in T_{Z'_1, d}$ , it has to satisfy the property that for any  $\beta \in A_{\mathbb{C}}^{4n-2}$ , one has

$$(2.10) \quad d^2 \alpha_d \beta = 0 \text{ in } H^{4n+4}(X', \mathbb{C}).$$

We get a contradiction as follows: since  $X'$  is in the class  $\mathcal{C}$ , that is bimeromorphically equivalent to a Kähler compact manifold, and the map

$$q \circ \psi : X' \rightarrow (T / \pm \text{Id}) \times (\widehat{T} / \pm \text{Id})$$

is dominating with 4-dimensional fiber, there is a  $\mu \in H^2(X', \mathbb{C})$  such that

$$(q \circ \psi)_* \mu^2 \neq 0 \text{ in } H^0((T / \pm \text{Id}) \times (\widehat{T} / \pm \text{Id}), \mathbb{C}) \cong \mathbb{C}.$$

(Here we should work with  $K \times \widehat{K}$  and desingularize the map

$$q \circ \psi : X' \dashrightarrow K \times \widehat{K}$$

to be more rigorous on the definition of  $(q \circ \psi)_*$ .)

Now, write  $\mu = d_1 + \mu'$ , with  $d_1 \in D \otimes \mathbb{C}$  and  $\mu' \in A_{\mathbb{C}}^2$ . Then

$$\mu^2 = \mu'^2 + 2d_1 \mu' + d_1^2,$$

so that for any  $\alpha \in A^2$ ,  $\beta \in A^{4n-2}$ ,

$$\mu^2 \alpha \beta = (\mu'^2 + 2d_1 \mu' + d_1^2) \alpha \beta = d_1^2 \alpha \beta.$$



Choose for  $d$  the element  $d_1$  above, and introduce  $\alpha_{d_1}$  as in (2.9). Now, because  $H^2((T/\pm\text{Id}) \times (\widehat{T}/\pm\text{Id}), \mathbb{C})$  and  $H^{4n-2}((T/\pm\text{Id}) \times (\widehat{T}/\pm\text{Id}), \mathbb{C})$  are dual via the cup-product and the isomorphism

$$H^{4n}((T/\pm\text{Id}) \times (\widehat{T}/\pm\text{Id}), \mathbb{C}) = \mathbb{C},$$

there exists a  $\beta \in H^{4n-2}((T/\pm\text{Id}) \times (\widehat{T}/\pm\text{Id}), \mathbb{C})$  such that

$$\alpha_{d_1}\beta \neq 0 \text{ in } H^{4n}((T/\pm\text{Id}) \times (\widehat{T}/\pm\text{Id}), \mathbb{C}).$$

Thus, as  $(q \circ \psi)_*\mu^2 = 1$ , we have:

$$\mu^2\alpha_{d_1}\beta \neq 0 \text{ in } H^{4n+4}(X', \mathbb{C}).$$

But we have just seen that

$$\mu^2\alpha_{d_1}\beta = d_1^2\alpha_{d_1}\beta.$$

The left hand side is non zero, while the right hand side vanishes by (2.10), which proves b) by contradiction. q.e.d.

We conclude this section with the proof of a proposition concerning the geometry of the bimeromorphic map  $\psi : X' \dashrightarrow X$ , which will be essential in the sequel. Recall that we proved that the meromorphic map

$$q \circ \psi : X' \dashrightarrow (T/\pm\text{Id}) \times (\widehat{T}/\pm\text{Id})$$

is in fact holomorphic. Let  $X'_0 := (q \circ \psi)^{-1}(K_0 \times \widehat{K}_0)$ .

**Proposition 2.** *There exists a dense Zariski open set  $U \subset K_0 \times \widehat{K}_0$  such that denoting*

$$X'_U := (q \circ \psi)^{-1}(U), \quad X_U := q^{-1}(U),$$

*the induced meromorphic map*

$$\psi : X'_U \dashrightarrow X_U$$

*is holomorphic.*

In order to prove this proposition, we need to establish a few lemmas saying that  $T \times \widehat{T}$  and  $\mathbb{P}(E) \times_{T \times \widehat{T}} \mathbb{P}(E_\phi)$  contain very few closed analytic subsets. They will be needed also later on in section 3.

**Lemma 7.** *The only closed irreducible positive dimensional proper analytic subsets of  $T \times \widehat{T}$  are of the form  $x \times \widehat{T}$ ,  $x \in T$ , or  $T \times y$ ,  $y \in \widehat{T}$ .*

*Proof.* Indeed, note first that  $T$  and  $\widehat{T}$  do not contain positive dimensional proper analytic subsets. This is because they both are simple tori which are not projective (see [8]), as guaranteed by the existence of  $\phi_T$  and  $\phi_{\widehat{T}}$ .

It follows that if  $Z \subset T \times \widehat{T}$  is positive dimensional proper irreducible and not of the above form, then it must be étale over both  $T$  and  $\widehat{T}$  which implies that the rational Hodge structures on  $H^1(T, \mathbb{Q})$  and  $H^1(\widehat{T}, \mathbb{Q})$

are isomorphic. But this is not the case, as a consequence of Lemma 3.  
q.e.d.

**Lemma 8.** *The only irreducible proper closed analytic subsets of  $\mathbb{P}(E)$  which dominate  $T \times \widehat{T}$  are the images  $\Sigma_1, \Sigma_2$  of the two natural sections  $\sigma_1, \sigma_2$  of  $\mathbb{P}(E)$  corresponding to the splitting*

$$E = \mathcal{P} \oplus \mathcal{P}^{-1},$$

and similarly for  $\mathbb{P}(E_\phi)$ .

*Proof.* Indeed, let  $Z \subset \mathbb{P}(E)$  be an hypersurface dominating  $T \times \widehat{T}$ . Let us denote by  $e : Z \rightarrow T \times \widehat{T}$  the generically finite map. Note that because of the description above of the proper analytic subsets of  $T \times \widehat{T}$ ,  $Z$  has to contain a dense Zariski open set  $Z_0$  which is an étale cover of a Zariski open set  $U \subset T \times \widehat{T}$ , where the complementary set of  $U$  is an union of analytic subsets of the form  $x \times \widehat{T}$  or  $T \times y$ .

Next  $Z$  induces a section of the induced  $\mathbb{P}^1$ -bundle  $\mathbb{P}(E)_Z := e^*\mathbb{P}(E)$ . Such a section is given by a line bundle  $\mathcal{L}$  over  $Z$  and a surjective map

$$E^* = e^*\mathcal{P} \oplus e^*\mathcal{P}^{-1} \rightarrow \mathcal{L}.$$

If one of the two induced maps

$$e^*\mathcal{P} \rightarrow \mathcal{L}, \quad e^*\mathcal{P}^{-1} \rightarrow \mathcal{L}$$

is zero, then  $Z$  has to be contained in  $\Sigma_1$  or  $\Sigma_2$ . Otherwise, we find that both  $e^*\mathcal{P}^{-1} \otimes \mathcal{L}$  and  $e^*\mathcal{P} \otimes \mathcal{L}$  have non-zero sections. Note that, because  $Z_0$  is an étale cover of an open set of  $T \times \widehat{T}$  whose complementary set has codimension  $\geq 2$ , some power  $\mathcal{L}^{\otimes k}$ ,  $k > 0$  is equal to  $e^*(\mathcal{K})$  on  $Z_0$ , for some line bundle  $\mathcal{K}$  on  $T \times \widehat{T}$ . Furthermore, both line bundles

$$e^*\mathcal{P}^{-k} \otimes \mathcal{L}^{\otimes k} = e^*(\mathcal{P}^{-k} \otimes \mathcal{K})$$

and

$$e^*\mathcal{P}^{\otimes k} \otimes \mathcal{L}^{\otimes k} = e^*(\mathcal{P}^{\otimes k} \otimes \mathcal{K})$$

have non-zero sections on  $Z_0$ . It then follows that for some  $m > 0$ , there are non-zero sections of

$$\mathcal{P}^{-km} \otimes \mathcal{K}^{\otimes m}, \quad \mathcal{P}^{\otimes km} \otimes \mathcal{K}^{\otimes m}$$

on the open set  $U$ , hence on  $T \times \widehat{T}$  itself. But since  $T \times \widehat{T}$  does not contain hypersurfaces, these sections do not vanish anywhere, from which one concludes that  $\mathcal{P}^{-km}$  is isomorphic to  $\mathcal{P}^{km}$ , which is not true since there cohomology classes are different. This proves the Lemma for  $\mathbb{P}(E)$  and the result for  $\mathbb{P}(E_\phi)$  follows.  
q.e.d.

**Corollary 1.**

a) *The only irreducible codimension 1 analytic subsets  $Z$  of*

$$\mathbb{P}(E) \times_{T \times \widehat{T}} \mathbb{P}(E_\phi)$$

*which dominate  $T \times \widehat{T}$  are of the form  $pr_1^{-1}\Sigma_i$ ,  $i = 1, 2$  or  $pr_2^{-1}\Sigma_i^\phi$ ,  $i = 1, 2$ .*

b) *The only irreducible codimension 2 analytic subsets of*

$$\mathbb{P}(E) \times_{T \times \widehat{T}} \mathbb{P}(E_\phi)$$

*which dominate  $T \times \widehat{T}$  are complete intersections*

$$pr_1^{-1}\Sigma_i \cap pr_2^{-1}\Sigma_j^\phi, \quad i = 1, 2, \quad j = 1, 2.$$

*Proof.* a) Let  $\mathcal{L} := \mathcal{O}(Z)$ , and let  $H = pr_1^*(\mathcal{O}_{\mathbb{P}(E)}(1))$ . Then we have

$$\mathcal{L} = H^{\otimes \alpha} \otimes pr_2^*\mathcal{K},$$

for some line bundle  $\mathcal{K}$  on  $\mathbb{P}(E_\phi)$ . Thus we have

$$R^0 pr_{2*}\mathcal{L} = \text{Sym}^\alpha(\pi^*E^*) \otimes \mathcal{K} = \text{Sym}^\alpha(\pi^*\mathcal{P} \oplus \pi^*\mathcal{P}^{-1}) \otimes \mathcal{K},$$

where  $\pi : \mathbb{P}(E_\phi) \rightarrow T \times \widehat{T}$  is the structural map. Here  $\alpha$  has to be non negative, as  $\mathcal{L}$  has a non zero section.

The non zero section of  $\mathcal{L}$  defining  $Z$  thus gives rise to sections  $\sigma_{\gamma\gamma'}$  of

$$\pi^*\mathcal{P}^{\otimes \gamma} \otimes \pi^*\mathcal{P}^{-\gamma'} \otimes \mathcal{K},$$

for  $\gamma \geq 0$ ,  $\gamma' \geq 0$ ,  $\gamma + \gamma' = \alpha$ .

Note that only one  $\sigma_{\gamma\gamma'}$  can be non zero. Indeed, by Lemma 8, the divisors of  $\sigma_{\gamma\gamma'}$  have to be combinations of  $\Sigma_{1\phi}$  and  $\Sigma_{2\phi}$  and the two line bundles  $\mathcal{O}(\Sigma_{1\phi})$ ,  $\mathcal{O}(\Sigma_{2\phi})$  are combinations of  $\pi^*\mathcal{P}_\phi$  and  $\mathcal{O}_{\mathbb{P}(E_\phi)}(1)$ . Thus, if two sections  $\sigma_{\gamma\gamma'}$  were non zero, then we would get a non-trivial relation between  $\pi^*\mathcal{P}_\phi$ ,  $\pi^*\mathcal{P}$  and  $\mathcal{O}_{\mathbb{P}(E_\phi)}(1)$  on  $\mathbb{P}(E_\phi)$ , which is not possible.

Thus there is only one non zero section  $\sigma_{\gamma\gamma'}$ . There are now two possibilities: if the divisor  $D_{\gamma\gamma'}$  of  $\sigma_{\gamma\gamma'}$  is non-empty, then as  $Z$  is irreducible and contains  $pr_2^{-1}D_{\gamma\gamma'}$ ,  $Z$  must be a pull-back, and Lemma 8 gives the result.

Next if the divisor  $D_{\gamma\gamma'}$  of  $\sigma_{\gamma\gamma'}$  is empty, one concludes that the line bundle  $\mathcal{K}$  is a pull-back:

$$\mathcal{K} = \pi^*\mathcal{K}'$$

for some line bundle  $\mathcal{K}'$  on  $T \times \widehat{T}$ . But then,  $\mathcal{L}$  is also a pull-back:

$$\mathcal{L} = pr_1^*\mathcal{L}'$$

for some line bundle  $\mathcal{L}'$  on  $\mathbb{P}(E)$ , and thus  $Z$  is equal to  $pr_1^{-1}(Z')$ , for some  $Z' \subset \mathbb{P}(E)$ . Lemma 8 gives then the result.

The proof of b) is obtained by projecting codimension 2 subsets of  $\mathbb{P}(E) \times_{T \times \widehat{T}} \mathbb{P}(E_\phi)$  to  $\mathbb{P}(E)$  and  $\mathbb{P}(E_\phi)$ . q.e.d.

The results above give us correspondingly the description of the codimension 1 and codimension 2 analytic subsets of

$$Q := \mathbb{P}(E) \times_{T \times \widehat{T}} \mathbb{P}(E_\phi) / \langle (i, i_\phi), (\hat{i}, \hat{i}_\phi) \rangle$$

which dominate  $K \times \widehat{K}$ .

Namely they are the image in  $Q$  of the subvarieties described above.

One interesting point is that the two hypersurfaces  $pr_1^{-1}\Sigma_1, pr_1^{-1}\Sigma_2$  descend to only one irreducible hypersurface

$$(2.11) \quad \Sigma \subset Q,$$

because the two factors in the splitting  $E = \mathcal{P} \oplus \mathcal{P}^{-1}$  are exchanged under  $i$ , so that  $pr_1^{-1}\Sigma_1, pr_1^{-1}\Sigma_2$  are permuted by  $\langle (i, i_\phi), (\hat{i}, \hat{i}_\phi) \rangle$ . For the same reason,  $pr_2^{-1}\Sigma_1^\phi, pr_2^{-1}\Sigma_2^\phi$  give rise to only one hypersurface  $\Sigma_\phi$ .

Similarly the 4 codimension 2 subvarieties

$$pr_1^{-1}\Sigma_i \cap pr_2^{-1}\Sigma_j^\phi, \quad i = 1, 2, \quad j = 1, 2$$

descend to only one irreducible subvariety  $W$  of  $Q$ , because they are permuted by the group  $\langle (i, i_\phi), (\hat{i}, \hat{i}_\phi) \rangle$ .

Thus  $Q$ , and hence  $X$  contain only one irreducible codimension 2 subvariety  $W$  which dominates  $K \times \widehat{K}$ .

*Proof of Proposition 2.* The proof is now immediate from the analysis above. Starting from  $X$ , the only modifications which we can do, whose center dominates  $K \times \widehat{K}$ , is to blow-up  $W$ , because in a quadric, there is no contractible curve. In the blown-up variety, we have as divisors the exceptional divisors, the proper transforms of the divisors  $\Sigma, \Sigma_\phi$  and they are the only one. Furthermore, the only codimension 2 closed analytic subset dominating  $K \times \widehat{K}$  is the union of two copies of  $W$ , indexed by the choice of one of the divisors  $\Sigma, \Sigma_\phi$ , since  $W = \Sigma \cap \Sigma_\phi$ . The same situation happens each time we blow-up one copy of  $W$  appearing in the previous step.

The key point is now the following: If the map  $\psi : X' \dashrightarrow X$  is not defined over the generic point of  $K \times \widehat{K}$ , which we can see as a birational map between surface bundles over the generic point of  $K \times \widehat{K}$ , then after a finite sequence of blow-ups of  $X$  along codimension 2 subsets dominating  $K \times \widehat{K}$ , some divisor  $D \subset \widetilde{X}$  in the blown-up variety must be generically contractible over  $K \times \widehat{K}$ , that is be made of a disjoint union of rational curves of self-intersection  $-1$  in the generic surface  $\widetilde{X}_t$ , while this divisor  $D$  projects to a divisor in  $X$ . This follows from the factorization of birational map between surfaces (see [1]).

But as this divisor dominates  $K \times \widehat{K}$ , it must be one of those described above, that is a proper transform of  $\Sigma, \Sigma_\phi$ . The contradiction comes from the fact that after the blow-up of  $W$ , the proper transforms of  $\Sigma$  and  $\Sigma_\phi$  are families of rational curves of self-intersection  $-2$ , and this

self-intersection can only decrease after further blow-ups. On the other hand, if we do not blow-up anything, these divisors are families of curves of self-intersection 0, which do not contract. q.e.d.

### 3. Proof of Theorem 5

In this section, we assume the hypotheses of Theorem 5, namely,  $X'$  is bimeromorphically equivalent to  $X$ , and  $Y$  is a compact Kähler manifold such that there exists an isomorphism

$$\gamma : H^*(Y, \mathbb{Q}) \cong H^*(X', \mathbb{Q})$$

of graded algebras. We want to prove that  $Y$  cannot be projective.

Our argumentation will be based on the analysis of the algebra  $H^*(X', \mathbb{Q})$  made in the previous section, and on the following Lemma 9 due to Deligne (see [3], [8], section 3) which was already heavily used in the last section of [8].

Let  $B^*$  be a finite dimensional graded algebra over  $\mathbb{Q}$  and assume that each  $B^k$  carries a rational Hodge structure, compatible with the product, i.e. the product map

$$B^k \otimes B^l \rightarrow B^{k+l}$$

is a morphism of Hodge structures. Let  $Z \subset B_{\mathbb{C}}^k$  be an algebraic subset defined by homogeneous equations which can be formulated using only the product structure on  $B^*$ . We have in mind, e.g.,

$$Z = \{\alpha \in B_{\mathbb{C}}^k, \alpha^2 = 0\}$$

or, which will be also used in the sequel,  $Z' = \text{Sing } Z$ , for  $Z$  as above.

**Lemma 9** (Deligne). *For  $Z$  as above, let  $Z_1 \subset Z$  be an union of irreducible reduced components of  $Z$ . Assume that the  $\mathbb{C}$ -vector space  $\langle Z_1 \rangle$  generated by  $Z_1$  is defined over  $\mathbb{Q}$ , that is  $\langle Z_1 \rangle = Z_{1\mathbb{Q}} \otimes \mathbb{C}$ , for some  $\mathbb{Q}$ -vector space  $Z_{1\mathbb{Q}} \subset B_{\mathbb{Q}}^k$ . Then  $Z_{1\mathbb{Q}}$  is a rational sub-Hodge structure of  $B_{\mathbb{Q}}^k$ .*

Our first step is the following (notations are as in the previous section):

**Proposition 3.** *Let  $X', Y, \gamma$  be as above. Then  $\gamma^{-1}(A_{1\mathbb{Q}}^2)$  and  $\gamma^{-1}(A_{2\mathbb{Q}}^2)$  are rational sub-Hodge structures of  $H^2(Y, \mathbb{Q})$ .*

*Proof.* We give the proof for  $\gamma^{-1}(A_{1\mathbb{Q}}^2)$ ; the proof for  $\gamma^{-1}(A_{2\mathbb{Q}}^2)$  is identical.

We have only to explain how to recover the space  $A_{1\mathbb{C}}^2$  as generated by a certain algebraic subset of  $H^2(X', \mathbb{C})$  defined using only the algebra structure on  $H^*(X', \mathbb{C})$ , since then, via  $\gamma$ , we will recover similarly  $\gamma^{-1}(A_{1\mathbb{C}}^2) \subset H^2(Y, \mathbb{C})$  and then by Deligne's Lemma 9, we will know that  $\gamma^{-1}(A_{1\mathbb{Q}}^2)$  is a sub-Hodge structure of  $H^2(Y, \mathbb{Q})$ .

We first use Proposition 1. It says that the irreducible components of the algebraic subset

$$Z_1 = \{\alpha_1 + d, d \in D_{\mathbb{C}}, \alpha_1 \in A_{1\mathbb{C}}^2, \alpha_1^2 = 0, d^2 = 0, \alpha_1 d = 0\}$$

containing the algebraic subset

$$Z_{1,0} := \{\alpha \in A_{1\mathbb{C}}^2, \alpha^2 = 0\}$$

are irreducible components of

$$Z = \{\alpha \in H^2(X', \mathbb{C}), \alpha^2 = 0\}.$$

Next Lemma 5 says us that if we denote by  $D_1$  the  $\mathbb{Q}$ -vector subspace of  $H^2(X', \mathbb{Q})$  defined as

$$D_1 := \{d \in D, d\alpha = 0, \forall \alpha \in A_{1\mathbb{Q}}^2\},$$

the condition

$$\alpha_1 d = 0 \text{ in } H^4(X', \mathbb{C})$$

for some

$$0 \neq \alpha_1 \in A_{1\mathbb{C}}^2, d \in D_{\mathbb{C}},$$

implies that  $d \in D_{1\mathbb{C}} := D_1 \otimes \mathbb{C}$ .

Using this Lemma, we conclude that the following algebraic subset of  $H^2(X', \mathbb{C})$ ,

$$Z'_1 = \{\alpha_1 + d, d \in D_{1\mathbb{C}}, \alpha_1 \in A_{1\mathbb{C}}^2, \alpha_1^2 = 0, d^2 = 0\},$$

also satisfies the property that its irreducible components containing  $Z_{1,0}$  are irreducible components of  $Z$ . Note now that the vector space  $A_{1\mathbb{C}}^2$  is defined over  $\mathbb{Q}$  and generated by its algebraic subset  $Z_{1,0}$ , because  $A_1^*$  is the exterior algebra  $\bigwedge^{\text{even}} \Gamma_{\mathbb{Q}}^*$ .

Thus, it remains only to show how to recover  $Z_{1,0}$  from  $Z'_1$ . This is done as follows. Let

$$D'_{1\mathbb{C}} \subset D_{1\mathbb{C}}$$

be the complex vector space generated by the algebraic subset

$$Z_{D_1} := \{d \in D_{1\mathbb{C}}, d^2 = 0\}.$$

$D'_{1\mathbb{C}}$  is defined over  $\mathbb{Q}$ , that is

$$D'_{1\mathbb{C}} = D'_1 \otimes \mathbb{C}$$

for some rational subspace  $D'_1 \subset H^2(X', \mathbb{Q})$ , because  $D_{1\mathbb{C}}$  is, and  $Z_{D_1}$  is defined over  $\mathbb{Q}$ .

If  $D'_1 = 0$ , there is nothing to say because then  $Z'_1 = Z_{1,0}$ . In general, the formula defining  $Z'_1$  shows that it is the “join” of  $Z_{D_1}$  and  $Z_{1,0}$  in  $D'_1 \oplus A_1^2$ .

Assume first that  $Z_{D_1} \neq D'_{1\mathbb{C}}$ . In this case we recover  $Z_{1,0}$  as a component of the singular locus of  $Z'_1$  because the join of two algebraic sets admits one of these algebraic sets as an union of component of its singular locus unless the other one is linear. So in this case, we recover  $Z_{1,0}$  from the algebra structure of  $H^*(X', \mathbb{C})$  and this is finished.

It remains only to exclude the possibility that

$$(3.12) \quad D'_1 \neq 0, Z_{D_1} = D'_{1\mathbb{C}}.$$

This is done by the following argument : assume (3.12) holds. As  $D'_1$  is a  $\mathbb{Q}$ -vector space, there would be in particular a non zero real element  $d \in D \subset H_{\mathbb{R}}^{1,1}(X')$  such that

$$d^2 = 0, d\alpha = 0, \forall \alpha \in A_{1\mathbb{R}}^2.$$

But there exists also a non-zero

$$\alpha \in A_{1\mathbb{R}}^{1,1} := H_{\mathbb{R}}^{1,1}(X') \cap A_{1\mathbb{R}}^2$$

such that  $\alpha^2 = 0$ . It follows that the rank 2 real vector space

$$B := \langle d, \alpha \rangle \subset H_{\mathbb{R}}^{1,1}(X')$$

satisfies the property:

$$\forall \gamma \in B, \gamma^2 = 0.$$

But this contradicts the Hodge index theorem (cf. [9] I, 6.3.2) because  $X'$  is dominated by a Kähler compact manifold, and it follows that for some element  $c \in H^{4n}(X', \mathbb{R})$ , the intersection form

$$u \mapsto cu^2 \in H^{4n+4}(X', \mathbb{R}) = \mathbb{R}$$

has only one positive sign on  $H_{\mathbb{R}}^{1,1}(X')$ , and hence cannot admit a rank 2 real isotropic subspace. Thus (3.12) leads to a contradiction, and the proposition is proved. q.e.d.

**Corollary 2.** *With the same assumptions and notations, the sub-space*

$$\gamma^{-1}(D) \subset H^2(Y, \mathbb{Q})$$

*is a rational sub-Hodge structure.*

*Proof.* We use Lemma 6, b), which says that  $D \otimes \mathbb{C}$  is an irreducible component of the set

$$Z' = \{d \in H^2(X', \mathbb{C}), d^3\beta = 0, \forall \beta \in A_{\mathbb{C}}^{4n-2}\}.$$

It follows that  $\gamma^{-1}(D) \otimes \mathbb{C}$  is an irreducible component of the set

$$\gamma^{-1}(Z') = \{d \in H^2(Y, \mathbb{C}), d^3\beta = 0, \forall \beta \in \gamma^{-1}(A_{\mathbb{C}}^{4n-2})\}.$$

But we know as a consequence of Proposition 3 that  $\gamma^{-1}(A_{\mathbb{C}}^{4n-2})$  is a rational sub-Hodge structure of  $H^{4n-2}(Y, \mathbb{Q})$ . Indeed, it is equal to the degree  $4n-2$  piece of the subalgebra generated by  $\gamma^{-1}(A^2)$  and  $\gamma^{-1}(A^2)$  is a rational sub-Hodge structure of  $H^2(Y, \mathbb{Q})$ .

It follows that its annihilator

$$\gamma^{-1}(A_{\mathbb{C}}^{4n-2})_0 = \{\delta \in H^6(Y, \mathbb{Q}), \delta\beta = 0, \forall \beta \in \gamma^{-1}(A_{\mathbb{C}}^{4n-2})\}$$

is also a rational sub-Hodge structure of  $H^{4n-2}(Y, \mathbb{Q})$ .

Hence there is an induced rational Hodge structure on the quotient

$$H^6(Y, \mathbb{Q})/\gamma^{-1}(A_{\mathbb{C}}^{4n-2})_0$$

and we can apply Deligne's Lemma 9 to the product

$$H^2(Y, \mathbb{Q})^{\otimes 3} \rightarrow H^6(Y, \mathbb{Q})/\gamma^{-1}(A^{4n-2})_0,$$

which is compatible with the induced Hodge structure: Indeed, for this product, we have that  $\gamma^{-1}(D) \otimes \mathbb{C}$  is an irreducible component of the set

$$Z'' = \{\delta \in H^2(Y, \mathbb{C}), \delta^3 = 0\}.$$

As  $\gamma^{-1}(D)$  is a rational subspace of  $H^2(Y, \mathbb{Q})$ , Lemma 9 says that it is a rational sub-Hodge structure of  $H^2(Y, \mathbb{Q})$ . q.e.d.

*Proof of Theorem 5.* The isomorphism of graded algebras

$$\gamma : H^*(Y, \mathbb{Q}) \cong H^*(X', \mathbb{Q})$$

must be compatible up to a coefficient with Poincaré duality, which is given by the cup-product and isomorphisms

$$H^{4n+4}(X', \mathbb{Q}) = \mathbb{Q}, \quad H^{4n+4}(Y, \mathbb{Q}) = \mathbb{Q}.$$

As  $\gamma^{-1}(A^2)$  is a rational sub-Hodge structure of  $H^2(Y, \mathbb{Q})$ , so is

$$(3.13) \quad \gamma^{-1}(A^{4n-4}) \subset H^{4n-4}(Y, \mathbb{Q}),$$

because it is equal to the degree  $4n - 4$  piece of the subalgebra of  $H^*(Y, \mathbb{Q})$  generated by  $\gamma^{-1}(A^2)$ .

Now, the map which is Poincaré dual to the inclusion

$$A^{4n-4} \subset H^{4n-4}(X', \mathbb{Q})$$

is the map

$$\begin{aligned} (q \circ \psi)_* : H^8(X', \mathbb{Q}) &\rightarrow H^4((T/\langle \pm \text{Id} \rangle) \times (\widehat{T}/\langle \pm \text{Id} \rangle), \mathbb{Q}) \\ &\cong A_{1\mathbb{Q}}^4 \oplus A_{1\mathbb{Q}}^2 \otimes A_{2\mathbb{Q}}^2 \oplus A_{2\mathbb{Q}}^4, \end{aligned}$$

where the last isomorphism is given by the Künneth decomposition. We shall denote by

$$\kappa : H^4((T/\langle \pm \text{Id} \rangle) \times (\widehat{T}/\langle \pm \text{Id} \rangle), \mathbb{Q}) \rightarrow A_{1\mathbb{Q}}^2 \otimes A_{2\mathbb{Q}}^2$$

the Künneth projector given by the decomposition above.

Applying  $\gamma^{-1}$ , we thus get a projection

$$H^8(Y, \mathbb{Q}) \rightarrow \gamma^{-1}(A_{1\mathbb{Q}}^4) \oplus \gamma^{-1}(A_{1\mathbb{Q}}^2) \otimes \gamma^{-1}(A_{2\mathbb{Q}}^2) \oplus \gamma^{-1}(A_{2\mathbb{Q}}^4)$$

which must be a morphism of Hodge structures as its transpose (3.13) is. Composing further with the projection (conjugate via  $\gamma$  to  $\kappa$ )

$$\gamma^{-1}(A_{1\mathbb{Q}}^4) \oplus \gamma^{-1}(A_{1\mathbb{Q}}^2) \otimes \gamma^{-1}(A_{2\mathbb{Q}}^2) \oplus \gamma^{-1}(A_{2\mathbb{Q}}^4) \rightarrow \gamma^{-1}(A_{1\mathbb{Q}}^2) \otimes \gamma^{-1}(A_{2\mathbb{Q}}^2),$$

which is also a morphism of Hodge structures because  $\gamma^{-1}(A_{1\mathbb{Q}}^2)$  and  $\gamma^{-1}(A_{2\mathbb{Q}}^2)$  are sub-Hodge structures of  $H^2(Y, \mathbb{Q})$ , we get finally a morphism of Hodge structures

$$H^8(Y, \mathbb{Q}) \rightarrow \gamma^{-1}(A_{1\mathbb{Q}}^2) \otimes \gamma^{-1}(A_{2\mathbb{Q}}^2).$$



Restricting it to the sub-Hodge structure  $\gamma^{-1}(D)^4 = \gamma^{-1}(D^4) \subset H^8(Y, \mathbb{Q})$  generated by  $\gamma^{-1}(D)$ , we finally get a morphism of rational Hodge structures

$$\pi_\gamma : \gamma^{-1}(D^4) \rightarrow \gamma^{-1}(A_{1\mathbb{Q}}^2) \otimes \gamma^{-1}(A_{2\mathbb{Q}}^2),$$

which is conjugate via  $\gamma$  to the restriction of  $\kappa \circ (q \circ \psi)_*$  to  $D^4$ .

We have now the following two lemmas:

**Lemma 10.** *The image of*

$$\kappa \circ (q \circ \psi)_* : D^4 \rightarrow A_{1\mathbb{Q}}^2 \otimes A_{2\mathbb{Q}}^2$$

*contains*

$$\text{Id} \in \text{Hom}(A_{1\mathbb{Q}}^2, A_{1\mathbb{Q}}^2) \cong A_{1\mathbb{Q}}^2 \otimes A_{2\mathbb{Q}}^2$$

*and*

$$\phi^* = \wedge^{2t} \phi \in \text{Hom}(A_{1\mathbb{Q}}^2, A_{1\mathbb{Q}}^2) \cong A_{1\mathbb{Q}}^2 \otimes A_{2\mathbb{Q}}^2.$$

Let now  $\Pi_\gamma$  be the image of  $\pi_\gamma$ .

$$\Pi_\gamma = \gamma^{-1} \otimes \gamma^{-1}(\text{Im } \kappa \circ (q \circ \psi)_*) \subset \gamma^{-1}(A_{1\mathbb{Q}}^2) \otimes \gamma^{-1}(A_{2\mathbb{Q}}^2).$$

**Lemma 11.**

a) *The generic element of  $\Pi_\gamma$  is non-degenerate.*

*(Here we see  $u \in \gamma^{-1}(A_{1\mathbb{Q}}^2) \otimes \gamma^{-1}(A_{2\mathbb{Q}}^2)$  as an element of*

$$\text{Hom}(\gamma^{-1}(A_{2\mathbb{Q}}^2)^*, \gamma^{-1}(A_{1\mathbb{Q}}^2))$$

*and non-degenerate means invertible.)*

b) *The  $\mathbb{Q}$ -vector subspace  $\Pi'_\gamma$  of  $\text{End}(\gamma^{-1}(A_{2\mathbb{Q}}^{2*}))$  generated by the  $u^{-1} \otimes v$ ,  $u$  non-degenerate in  $\Pi_\gamma$ , consists of Hodge classes in  $\text{End}(\gamma^{-1}(A_{1\mathbb{Q}}^{2*}))$ , (relative to the Hodge structures on  $\gamma^{-1}(A_{2\mathbb{Q}}^2)$  induced by the Hodge structure on  $H^2(Y, \mathbb{Q})$ ).*

Assuming these Lemmas, the proof is now concluded as follows.

The two Lemmas together imply that the Hodge structure on  $\gamma^{-1}(A_{2\mathbb{Q}}^{2*})$  admits an endomorphism conjugate to  $\phi_T^* = \wedge^{2t} \phi$ . Hence dually the Hodge structure on  $\gamma^{-1}(A_{2\mathbb{Q}}^2)$  admits a morphism conjugate to  $\wedge^2 \phi$ .

The proof concludes then exactly as in [8], 3.2: As the action of  $\wedge^2 \phi$  is irreducible on  $A_{2\mathbb{Q}}^2$ , the above implies that either the Hodge structure on  $\gamma^{-1}(A_{2\mathbb{Q}}^2)$  is trivial or it does not contain any Hodge class. The first case is excluded by a Hodge index argument.

Next, working symmetrically with  $A_{1\mathbb{Q}}^2$ , we conclude similarly that the Hodge structure on  $\gamma^{-1}(A_{2\mathbb{Q}}^2)$  does not contain any Hodge class.

Thus it follows from Corollary 2 that the only degree 2 Hodge classes on  $Y$  are contained in  $\gamma^{-1}(D)$ .

But we look now at the intersection form

$$q_d = \int_Y d^{4n} \alpha \beta$$

for  $d \in \gamma^{-1}(D)$ , and we conclude that it is zero on  $\gamma^{-1}(A_{1\mathbb{Q}}^2)$ , because the same is true for  $D$  and  $A_{1\mathbb{Q}}^2$  on  $X'$ . Thus for no degree 2 Hodge class  $d$  on  $Y$ , the sub-Hodge structure  $\gamma^{-1}(A_{1\mathbb{Q}}^2) \subset H^2(Y, \mathbb{Q})$  can be polarized by  $q_d$ . Thus by [9], I, 6.3.2,  $Y$  cannot be projective. q.e.d.

*Proof of Lemma 10.* We first reduce to the case where  $X' = X$ : First of all, using Lemma 7, we conclude that for any non-empty Zariski open set  $U$  of  $K_0 \times \widehat{K}_0$ , the restriction map

$$H^4(K_0 \times \widehat{K}_0, \mathbb{Q}) = H^4((T/\pm \text{Id}) \times (\widehat{T}/\pm \text{Id}), \mathbb{Q}) \rightarrow H^4(U, \mathbb{Q})$$

is an isomorphism. Now we have the commutative diagram:

$$(3.14) \quad \begin{array}{ccc} D^4 \subset H^8(X', \mathbb{Q}) & \xrightarrow{\text{rest}_U} & H^8(X'_U, \mathbb{Q}) \\ (q \circ \psi)_* \downarrow & & \downarrow (q \circ \psi)_*^U \\ H^4((T/\pm \text{Id}) \times (\widehat{T}/\pm \text{Id}), \mathbb{Q}) & \xrightarrow{\text{rest}_U} & H^4(U, \mathbb{Q}). \end{array}$$

We use now Proposition 2 which says that the meromorphic map  $\psi$  is well defined on a Zariski open set  $X'_U$  as above. We thus have a commutative diagram:

$$\begin{array}{ccc} D_{X|X_U}^4 & \xrightarrow{\psi_U^*} & D_{X'_U}^4 \\ q_{U*} \downarrow & & \downarrow (q \circ \psi)_{U*} \\ H^4(U, \mathbb{Q}) & \cong & H^4(U, \mathbb{Q}), \end{array}$$

where  $q_U, (q \circ \psi)_U$  denote the restrictions of  $q, q \circ \psi$  to  $X_U, X'_U$  respectively. We used here the fact that degree 2 Hodge classes on  $X$ , restricted to  $X_U$ , pull-back via  $\psi_U$  to degree 2 Hodge classes on  $X'$ , restricted to  $X'_U$ , which follows from the fact that  $\psi$  is meromorphic.

Writing for  $X$  the same diagram as (3.14), we conclude that it suffices to prove the result for  $X$ .

Next, we look at the following Cartesian diagram:

$$\begin{array}{ccc} \tilde{q} : \mathbb{P}(E)_0 \times_{T_0 \times \widehat{T}_0} \mathbb{P}(E_\phi)_0 & \rightarrow & T_0 \times \widehat{T}_0 \\ & & e \downarrow \\ q : X_0 & \rightarrow & K_0 \times \widehat{K}_0 \end{array},$$

where the lower indices 0 denote the restrictions of the projective bundles to  $T_0 \times \widehat{T}_0$ , the vertical maps denoted by  $e$  are the quotient maps, and the induced map

$$H^4(K_0 \times \widehat{K}_0, \mathbb{Q}) \rightarrow H^4(T_0 \times \widehat{T}_0, \mathbb{Q})$$

is injective. Here  $X_0$  is the Zariski open set of  $X$  which is the smooth part of the quotient  $Q$ . Arguing as before, we see that we can replace

$X$  by  $X_0$ , and then  $X_0$  by its étale cover  $\mathbb{P}(E)_0 \times_{T_0 \times \widehat{T}_0} \mathbb{P}(E_\phi)_0$ . Thus the result for  $X$  follows from the following formulas (3.15):

Let  $\Sigma, \Sigma_\phi$  be the two divisors of (2.11), and let  $s, s_\phi \in Hdg^2(X, \mathbb{Q})$  be their cohomology classes. Then we have

$$(3.15) \quad \begin{aligned} \tilde{q}_*(e^*(s^3 s_\phi)) &= 16\text{Id} \in \text{Hom}(H^2(T_0, \mathbb{Q}), H^2(T_0, \mathbb{Q})) \\ &= H^2(T_0, \mathbb{Q}) \otimes H^2(\widehat{T}_0, \mathbb{Q}), \\ \tilde{q}_*(e^*(s s_\phi^3)) &= 16\phi^* \in \text{Hom}(H^2(T_0, \mathbb{Q}), H^2(T_0, \mathbb{Q})) \\ &= H^2(T_0, \mathbb{Q}) \otimes H^2(\widehat{T}_0, \mathbb{Q}). \end{aligned}$$

This is computed as follows: let  $s_1, s_2$  be the classes of the divisors  $\Sigma_1, \Sigma_2$  of  $\mathbb{P}(E) \times_{T_0 \times \widehat{T}_0} \mathbb{P}(E_\phi)$  given by the decomposition  $E = \mathcal{P} \oplus \mathcal{P}^{-1}$  and similarly let  $s_1^\phi, s_2^\phi$  be the classes of the divisors  $\Sigma_1^\phi, \Sigma_2^\phi$  of  $\mathbb{P}(E) \times_{T_0 \times \widehat{T}_0} \mathbb{P}(E_\phi)$  given by the decomposition  $E_\phi = \mathcal{P}_\phi \oplus \mathcal{P}_\phi^{-1}$ . Then we have

$$e^*(s) = s_1 + s_2, \quad e^*(s_\phi) = s_1^\phi + s_2^\phi.$$

Let  $h, h_\phi$  be respectively  $c_1(\mathcal{O}_{\mathbb{P}(E)}(1)), c_1(\mathcal{O}_{\mathbb{P}(E_\phi)}(1))$ , or rather their pull-backs to the fibered product  $\mathbb{P}(E) \times_{T_0 \times \widehat{T}_0} \mathbb{P}(E_\phi)$ . Let  $p, p_\phi$  be the classes  $c_1(\mathcal{P}), c_1(\mathcal{P}_\phi)$ . Then we have

$$\begin{aligned} s_1 &= \tilde{q}^* p - h, \quad s_2 = -\tilde{q}^* p - h, \\ s_1^\phi &= \tilde{q}^* p_\phi - h_\phi, \quad s_2^\phi = -\tilde{q}^* p_\phi - h_\phi. \end{aligned}$$

Thus

$$e^*(s) = -2h, \quad e^*(s_\phi) = -2h_\phi,$$

and

$$e^*(s^3 s_\phi) = 16h^3 h_\phi, \quad e^*(s_\phi^3 s) = 16h_\phi^3 h.$$

Applying  $\tilde{q}_*$  we conclude that

$$\tilde{q}_*(e^*(s^3 s_\phi)) = -16c_2(E), \quad \tilde{q}_*(e^*(s_\phi^3 s)) = -16c_2(E_\phi).$$

As  $E = \mathcal{P} \oplus \mathcal{P}^{-1}$ , and  $E_\phi = \mathcal{P}_\phi \oplus \mathcal{P}_\phi^{-1}$ , it follows that

$$c_2(E) = -p^2, \quad c_2(E_\phi) = -p_\phi^2.$$

Finally, since  $p$  identifies to

$$\begin{aligned} \text{Id} &\in \text{Hom}(H^1(\widehat{T}, \mathbb{Q}), H^1(\widehat{T}, \mathbb{Q})) \\ &= H^1(T, \mathbb{Q}) \otimes H^1(\widehat{T}, \mathbb{Q}) \subset H^2(T \times \widehat{T}, \mathbb{Q}), \end{aligned}$$

we get that  $p^2 = \frac{1}{16}\tilde{q}_*(e^*(s^3 s_\phi))$  identifies to

$$\begin{aligned} \text{Id} &\in \text{Hom}(H^2(T, \mathbb{Q}), H^2(T, \mathbb{Q})) \\ &= H^2(T, \mathbb{Q}) \otimes H^2(\widehat{T}, \mathbb{Q}) \subset H^4(T \times \widehat{T}, \mathbb{Q}), \end{aligned}$$

and similarly  $p_\phi^2 = \frac{1}{16} \tilde{q}_*(e^*(ss_\phi^3))$  identifies to

$$\begin{aligned} \wedge^2 \phi &\in \text{Hom}(H^2(\widehat{T}, \mathbb{Q}), H^2(\widehat{T}, \mathbb{Q})) \\ &= H^2(T, \mathbb{Q}) \otimes H^2(\widehat{T}, \mathbb{Q}) \subset H^4(T \times \widehat{T}, \mathbb{Q}). \end{aligned}$$

Thus (3.15) is proved, which concludes the proof of the Lemma. q.e.d.

*Proof of Lemma 11.* The first statement is obvious by Lemma 10.

Next, because we proved that  $\Pi_\gamma$  is a sub-Hodge structure of

$$\gamma^{-1}(A_{1\mathbb{Q}}^2) \otimes \gamma^{-1}(A_{2\mathbb{Q}}^2),$$

it follows that the space  $\Pi'_\gamma$  is a sub-Hodge structure of  $\text{End}(\gamma^{-1}(A_{2\mathbb{Q}}^{2*}))$ , and thus, so is the sub-algebra of  $\text{End}(\gamma^{-1}(A_{2\mathbb{Q}}^{2*}))$  generated by  $\Pi'_\gamma$ . On the other hand,  $\Pi'_\gamma$  is conjugate via  ${}^t\gamma$  to the corresponding subspace of  $\text{End}(A_{2\mathbb{Q}}^{2*})$ , defined similarly starting from  $Im \kappa \circ (q \circ \psi)_{*1D^4}$ . As  $D^4$  is made of degree 8 Hodge classes on  $X'$ , this last subspace is contained in the space of endomorphisms of Hodge structures of  $A_{2\mathbb{Q}}^{2*}$ , which has been computed to be equal to the algebra generated by  $\phi_{\widehat{T}^*} = \wedge^2 \phi$  (see proof of Lemma 4).

The key point is that because  $\wedge^2 \phi$  is diagonalizable, this algebra tensored with  $\mathbb{C}$  has no nilpotent element. It follows that  $\Pi'_\gamma \otimes \mathbb{C}$  has no nilpotent element. But as  $\Pi'_\gamma$  is a sub-Hodge structure of  $\text{End}(\gamma^{-1}(A_{2\mathbb{Q}}^{2*}))$ , it follows that it is pure of type  $(0, 0)$ , that is made of Hodge classes, because elements of type  $(-k, k)$ ,  $k > 0$  are nilpotent. q.e.d.

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