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THE EXISTENCE OF SUPERSYMMETRIC STRING THEORY WITH TORSION

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1. The system proposed by Strominger

In their proposed compactification of superstrings [4], Candelas, Horowitz, Strominger and Witten took the matric product of a maximal symmetric four dimensional spacetime M with a six dimensional Calabi–Yau vacua X as the ten dimensional spacetime; they identified the Yang–Mills connection with the SU(3) connection of the Calabi–Yau metric and set the dilaton to be a constant. To make this theory compatible with the standard grand unified field theory, Witten [28] and Horava–Witten [20] proposed to use higher rank bundles for strong coupled heterotic string theory so that the gauge groups can be SU(4) or SU(5). Mathematically, this approach relies on Uhlenbeck–Yau's theorem on constructing Hermitian–Yang–Mills connections on stable bundles [27]. Many authors, including Friedman, Morgan and Witten [18]; Donagi, Ovrat, Pantev and Reinbacher [12]; Andreas [1], Kachru [21] and others, have worked on this subject since then.

In [24], Strominger analyzed heterotic superstring background with spacetime sypersymmetry and non-zero torsion by allowing a scalar "warp factor" to multiply the spacetime metric. He considered a ten dimensional spacetime that is the product $M\times X$ of a maximal symmetric four dimensional spacetime M and an internal space X; the metric on $M\times X$ takes the form

$$e^{2D(y)}\begin{pmatrix} g_{ij}(y) & 0\\ 0 & g_{\mu\nu}(x)\end{pmatrix},\qquad x\in X,\quad y\in M;$$

the connection on an auxiliary bundle is Hermitian–Yang–Mills over X:

$$F\wedge\omega^2=0,\quad F^{2,0}=F^{0,2}=0$$

associated to the hermitian form $\omega=\frac{\sqrt{-1}}{2}g_{i\bar{j}}dz^i d\bar{z}^j$. In this system, following the convention that $d_c=\sqrt{-1}(\bar\partial-\partial)$, the physical relevant

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quantities are

$$h = \frac{1}{2} d_c \omega,$$

$$\phi = \frac{1}{8} \log \|\Omega\| + \phi_0,$$

for a constant ϕ_0 and

$$g^0_{ij} = e^{2\phi_0} \|\Omega\|^{\frac{1}{4}} g_{ij}.$$

The spacetime supersymmetry forces $D(y)$ to be the dilaton field.

In order for such ansatze to provide a supersymmetric configuration, one introduces a Majorana–Weyl spinor ϵ so that

$$\delta\phi^0_j = \nabla^0_j \epsilon^0 + \frac{1}{48} e^{2\phi} \big(\gamma^0_j H^0 - 12 h^0_j\big) \epsilon^0 = 0,$$

$$\delta\lambda^0 = \nabla^0 \phi \epsilon^0 + \frac{1}{24} e^{2\phi} h^0 \epsilon^0 = 0,$$

and

$$\delta\chi^0 = e^{\phi} F_{ij} \Gamma^{0ij} \epsilon^0 = 0.$$

Here, ψ^0 is the gravitano, λ^0 is the dilatino, χ^0 is the gluino, ϕ is the dilaton and h is the Kalb–Ramond filed strength obeying

$$dh = \operatorname{tr} R \wedge R - \operatorname{tr} F \wedge F.$$

(For details of this discussion, please consult [**24**, **25**]). By suitably transforming these quantities, Strominger showed that in order to achieve space–time supersymmetry the internal six manifold X must be a complex manifold with a non-vanishing holomorphic three form Ω; the Hermitian form ω must obey

$$\partial\bar\partial\omega = \sqrt{-1} \operatorname{tr} F \wedge F - \sqrt{-1} \operatorname{tr} R \wedge R$$

and

$$d^*\omega = d_c \log \|\Omega\|.$$

Accordingly, he proposed to solve the system

$$F \wedge \omega^2 = 0; \tag{1.1}$$

$$F^{2,0} = F^{0,2} = 0; \tag{1.2}$$

$$\partial\bar\partial\omega = \sqrt{-1} \operatorname{tr} F \wedge F - \sqrt{-1} \operatorname{tr} R \wedge R; \tag{1.3}$$

and

$$d^*\omega = d_c \ln \|\Omega\|. \tag{1.4}$$

These are solutions of superstrings with torsions that allows non-trivial dilaton fields and Yang–Mills fields (The equation (1.3) in [**24**] has $\frac{1}{30} \operatorname{tr} F \wedge F$; this is because he worked with principle bundles and the trace is that of its adjoint bundle.). Here, Ω is a nowhere vanishing holomorphic three form on the complex threefold X; ω is the Hermitian form and R is the curvature tensor of the Hermitian metric ω; F is the curvature of a vector bundle E; and tr is the trace of the endomorphism bundle of either E or TX.

In [**24**], Strominger found some solutions to this system for U(1) principle bundles. In this paper, we shall give the first irreducible non-singular solution of the supersymmetric system of Strominger for U(4) and U(5) principle bundles. We obtain our solutions by perturbing around the Calabi–Yau vacua paired with the gauge field on the tangent bundle of X.

In more concrete terms, we take a smooth Calabi–Yau threefold (X,ω) and a reducible Yang–Mills connection (metric) H on $TX\oplus\mathbb{C}_X^{\oplus r}$; (ω,H) is a reducible solution to Strominger's system. For any small deformations $\bar\partial_s$ of the holomorphic structure of $TX\oplus\mathbb{C}_X^{\oplus r}$, we derive a sufficient condition for (1.1)–(1.4) being solvable for $(X,\bar\partial_s)$: it is that the Kodaira–Spencer class of the family $\bar\partial_s$ at $s=0$ satisfies certain non-degeneracy condition (see Theorem 4.3). After that, we will construct examples of Calabi–Yau threefolds that admit small deformations of $TX\oplus\mathbb{C}_X^{\oplus r}$ satisfying this requirement. This provides the first example of regular irreducible solution to Strominger's system.

In the next paper, we would like to understand the non-perturbative theory and hope to formulate a global structure theorem of the moduli space of these fields.

It was speculated by M. Reid that all Calabi–Yau manifolds can be deformed to each other through conifold transition. To achieve this goal, it is inevitable that we must work with non-Kahler manifolds. We hope that such non-Kahler manifolds will adopt the Strominger structures. We shall come back to this in the second paper.

2. Solving Hermitian-Einstein equation by perturbation

In this section, we will solve the usual Hermitian–Yang–Mills system using perturbation method. Let (E,D_s'') be a smooth family of holomorphic vector bundles on an n-dimensional Kahler manifold (X,ω). Suppose H_0 is a Hermitian–Yang–Mills metric on (E,D_0''); we ask under what condition can we extend H_0 to a smooth family of Hermitian–Yang–Mills metrics H_s on (E,D_s'')? When H_0 is irreducible, the answer is affirmative. The case when H_0 is reducible is more subtle. Let (E_1,D_1'') and (E_2,D_2'') be two degree zero slope-stable vector bundles on X. By a theorem of Uhlenbeck–Yau, both (E_1,D_1'') and (E_2,D_2'') admit Hermitian–Yang–Mills metrics H_1 and H_2. The direct sum of their scalar multiples $H_1 \oplus e^t H_2$ is a Hermitian–Yang–Mills metric on

$$(E, D_0'') \triangleq (E_1 \oplus E_2, D_1'' \oplus D_2'').$$

Suppose we are given a smooth deformation of holomorphic structures D_s'' of D_0'', then the Kodaira–Spensor class identifies the first order deformation of the family D_s'' at 0 to an element

$$\kappa \in H^1_{\bar\partial}(X, E^\vee \otimes E)$$

in the Dolbeault cohomology of the $\bar\partial$-operator D_0''. Because $D_0'' = D_1'' \oplus D_2''$, the above cohomology space decomposes into direct sum

$$\oplus_{i,j=1}^2 H^1_{\bar\partial}(X, E_i^\vee \otimes E_j).$$

We let $\kappa_{ij} \in H^1_{\bar\partial}(X, E_i^\vee \otimes E_j)$ be its associated components under this decomposition.

Theorem 2.1. *Suppose κ_{12} and κ_{21} are non-zero, then there is a unique t so that for sufficiently small s, the metric $H_0(t) = H_1 \oplus e^t H_2$ extends to a family of Hermitian–Yang–Mills-metrics H_s on (E, D_s'').*

We will prove this theorem by applying implicit function theorem to the elliptic system of the Hermitian–Yang–Mills metrics of (E, D_s'').

To begin with, equation (1.2) holds for any hermitian connections of holomorphic vector bundles. Now, let H be a hermitian metric on E and $F_{s,H}$ be its the hermitian curvature on (E, D_s''). The Hermitian–Yang–Mills equation for (E, D_s''), which has degree zero, then becomes

$$F_{s,H} \wedge \omega^{n-1} = 0. \tag{2.1}$$

The linearization of (2.1) is self-adjoint and has two-dimensional kernel and cokernel. In case $\wedge^r(E, D_0'') \cong \mathbb{C}_X$, we can normalize H so that its induced metric on $\wedge^r E \cong \mathbb{C}_X$ is the constant one metric. Then, the linearization of the restricted system has one dimensional kernel and cokernel. We suppose the cokernel is spanned by $J \cdot \omega^n$. Then, for small s, the implicit function theorem supplies us a one dimensional family of solutions $H_{s,t}$, of determinant one, to the system (2.1) modulo the linear span of $J \cdot \omega^n$:

$$F_{s,H_{s,t}} \wedge \omega^{n-1} \equiv 0 \mod J \cdot \omega^n. \tag{2.2}$$

To prove the theorem, it remains to show that we can find a function $t = \rho(s)$ so that

$$F_{s,H_{s,\rho(s)}} \wedge \omega^{n-1} = 0. \tag{2.3}$$

For this, we will look at the functional

$$r(s,t) = \frac{\sqrt{-1}}{2} \int \operatorname{tr}\big(F_{s,H_{s,t}} \cdot J\big) \wedge \omega^{n-1}$$

and investigate the derivatives $\dot r(0,t)=\frac{d}{ds}r(s,t)|_{s=0}$. Since $r(0,t)\equiv 0$, the first order derivatives $\dot r(0,t)$ are independent of the parameterizations (s,t). By a direct calculation, they all vanish. Thus, we are forced to work at the second order derivatives $\ddot r(0,t)$; they are of the form

$$\ddot r(0,t)=e^{-\alpha t}A-e^{\alpha t}B,\quad A,B\ge 0.$$

In case κ_{12} and κ_{21} are non-zero, A and B become positive; hence, we can find a function $t=\rho(s)$ so that $\lim_{s\to 0}\rho(s)=\frac{1}{2\alpha}\ln(A/B)$ and

$$r(s,\rho(s))=0.$$

This shall prove the existence theorem. Since later we will adopt this approach to solve Strominger's system, we shall provide its detail here as a warm up.

We begin with the basic objects: the vector bundle, its holomorphic structure and its curvature. We let (X,ω) be a Kahler manifold of dimension n; we let (E_1,D_1'') and (E_2,D_2'') be two degree zero slope stable holomorphic vector bundles of ranks r_1 and r_2; we let $\langle\,,\rangle_1$ and $\langle\,,\rangle_2$ be the Hermitian–Yang–Mills metrics of (E_1,D_1'') and (E_2,D_2''). For simplicity, we assume $\wedge^{r_i}(E_i,D_i'')\cong\mathbb{C}_X$ and pick $\langle\,,\rangle_i$ so that its induced metric on $\wedge^{r_i}E_i\cong\mathbb{C}_X$ is the constant 1 metric. Under this arrangement, the $\langle\,,\rangle_i$ are unique. We then let $E=E_1\oplus E_2$, of rank $r=r_1+r_2$, and endow it with the holomorphic structure $D_1''\oplus D_2''$ and the reference hermitian metric $\langle\,,\rangle=\langle\,,\rangle_1\oplus\langle\,,\rangle_2$.

Next, we let D_s'' be a smooth family of holomorphic structures on E so that $D_0''=D_1''\oplus D_2''$. D_s'' relates to D_0'' by a global section $A_s\in\Omega^{0,1}(\operatorname{End}E)$:

$$D_s''=D_0''+A_s;$$

the hermitian connection $D_s=D_s'+D_s''$ of $(E,D_s'',\langle,\rangle)$ relates to the hermitian connection D_0 of $(E,D_0'',\langle,\rangle)$ via

$$D_s=(D_0''+A_s)+(D_0'-A_s^*);$$

the hermitian curvature of D_s becomes

$$F_s=F_0+(D_0''+D_0')(A_s-A_s^*)-(A_s-A_s^*)\wedge(A_s-A_s^*). \tag{2.4}$$

Here, A_s^* is the hermitian adjoint of A_s under $\langle\,,\rangle$.

It is instructive to express them in local coordinates. Let $e_1,\cdots,e_n$ be a (local) orthonormal basis of $(E,\langle\,,\rangle)$. We define the connection form $\Gamma_{s,\alpha\beta}$ of D_s'':

$$D_s''e_\alpha=\Gamma_{s,\alpha\beta}e_\beta;$$

then, the matrix

$$A_s=(A_{s,\alpha\beta})=(\Gamma_{s,\alpha\beta}-\Gamma_{0,\alpha\beta}).$$

For any local section $v = \sum x_\alpha e_\alpha$, written in the matrix form $v = \mathbf{x}\,\mathbf{e}^t$ with $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{e} = (e_1, \cdots, e_n)$ being row vectors, the differentiation

$$D''_s v = \big(\bar\partial \mathbf{x} + \mathbf{x}(\Gamma_{s,\alpha\beta})\big)\mathbf{e}^t = (\bar\partial \mathbf{x} + \mathbf{x}(\Gamma_{0,\alpha\beta}))\mathbf{e}^t + (\mathbf{x}A_s)\mathbf{e}^t = D''_0 v + vA_s.$$

In case φ is a section of $E^\vee \otimes E$, a local computation shows that

$$D''_s\varphi = D''_0\varphi - [A_s, \varphi] \quad \text{and} \quad D'_s\varphi = D'_0\varphi + [A^*_s, \varphi].$$

This works for endomorphism-valued p and q-forms A and B if we follow the convention $[A, B] = A \wedge B - (-1)^{pq} B \wedge A$.

Lemma 2.2. *Let (E, D''_s) be a family of holomorphic structures on a vector bundle E over a Kahler manifold (X, ω). Then, there is a family of gauge transformations $g_s \in \Omega^0(\operatorname{End} E)$, $g_0 = \mathrm{id}$, so that the first order derivative $\frac{d}{ds} g^*_s D''_s$ is D''_0-harmonic.*

Proof. First, we can find $\mu \in \Omega^0(\operatorname{End} E)$ so that $\dot D''_0 + D''_0\mu$ is D''_0-harmonic. We then choose a family of gauge transformation g_s so that $\frac{d}{ds} g^*_s D''_s = \dot D''_0 + D''_0\mu$. g_s is the desired family of gauge transformations. q.e.d.

As a corollary, we can choose the family $D''_s = D''_0 + A_s$ so that $\dot A_0$ is D''_0-harmonic with respect to the Kahler form ω.

Solving Hermitian Yang–Mills connections involves working with other hermitian metrics of E. We let $\mathcal{H}(E)_1$ be the space of all hermitian metrics on E whose induced metrics on $\wedge^r E \cong \mathbb{C}_X$ are the constant one metric. Once we have the reference metric $\langle\, ,\rangle$, the space $\mathcal{H}(E)_1$ is isomorphic to the space of determinant one pointwise positive definite $\langle\, ,\rangle$-hermitian symmetric endomorphisms of E via

$$\langle u, v\rangle_H = \langle uH, v\rangle.$$

In this paper, we shall use such H to represent its associated hermitian metric.

Given a hermitian metric H, its hermitian connection $D_{s,H}$ is

$$D_{s,H} = (D'_s + D'_s H \cdot H^{-1}) + D''_s;$$

its curvature is

$$F_{s,H} = F_s + D''_s(D'_s H \cdot H^{-1}).$$

The Hermitian–Yang–Mills connections of (E, D''_s) are hermitian metrics $H \in \mathcal{H}(E)_1$ making

$$\tilde L_s(H) = \big(F_s + D''_s(D'_s H \cdot H^{-1})\big) \wedge \omega^{n-1}$$

vanish. Because H induces the constant one metric on $\wedge^r E$, $\operatorname{tr} F_{s,H}$, which is the curvature of $(\wedge^r E, \det H)$, is zero. Hence, $\tilde L_s(H)$ is traceless

H-hermitian antisymmetric. To make it $\langle\,,\rangle$-hermitian anti-symmetric instead, we form the operator

$$L_s(H) = H^{-1/2} \cdot \tilde{L}_s(H) \cdot H^{1/2} : \mathcal{H}(E)_1 \longrightarrow \Omega^{2n}_{\mathbb{R}}(\mathfrak{su}E). \tag{2.5}$$

It takes value in the vector bundle $\mathfrak{su}E$ of traceless hermitian anti-symmetric endomorphisms of $(E, \langle\,,\rangle)$.

Next, we let I_i be the identity endomorphism of E_i, viewed as an endomorphism of E. Because both E_1 and E_2 are degree zero slope stable and H_1 and H_2 are their Hermitian–Yang–Mills metrics, the solutions to $L_0(H) = 0$ are

$$H_{0,t} = e^{t/r_2} I_1 \oplus e^{-t/r_2} I_2, \quad t \in \mathbb{R}. \tag{2.6}$$

Further, using $\delta H = H_{0,t}^{-1/2} \delta h H_{0,t}^{-1/2}$, which is an isomorphism of the tangent space of $\mathcal{H}(E)_1$ at $H_{0,t}$ with the space of sections of the vector bundle $\mathfrak{Her}^0 E$ of traceless hermitian symmetric endomorphisms of $(E, \langle\,,\rangle)$, the linearization of L_0 at $H_{0,t}$ becomes

$$\delta L_0(H_{0,t})(\delta h) = D_0'' D_0' \delta h \wedge \omega^{n-1} : \Omega^0(\mathfrak{Her}^0 E) \longrightarrow \Omega^{2n}_{\mathbb{R}}(\mathfrak{su}E). \tag{2.7}$$

Because (E, D_0'') is a direct sum of two distinct stable vector bundles, the kernel and the cokernel of δL_0 are both one-dimensional, of which the cokernel is spanned by

$$J = \frac{\sqrt{-1}}{r_2} I_1 \cdot \omega^n \oplus -\frac{\sqrt{-1}}{r_1} I_2 \cdot \omega^n,$$

independent of t. To apply the implicit function theorem, we take the projection

$$P : \Omega^{2n}_{\mathbb{R}}(\mathfrak{su}E) \longrightarrow \Omega^{2n}_{\mathbb{R}}(\mathfrak{su}E)/\mathbb{R} \cdot J$$

and look at the composite

$$P \circ L_s : \mathcal{H}(E)_1 \longrightarrow \Omega^{2n}_{\mathbb{R}}(\mathfrak{su}E)/\mathbb{R} \cdot J.$$

Because the cokernel of $P \circ \delta L_0$ at $H_{0,t}$ is 0, for small s, the operator $P \circ L_s$ is an open operator near $H_{0,t}$. Further, because the linearization of $P \circ L_s$ has index one at $H_{0,t}$, the solution space V_s of $P \circ L_s = 0$ is a one-dimensional smooth manifold near $H_{0,t}$ and the union $\cup_s V_s$ is a smooth two dimensional manifold near $H_{0,t}$. Since V_0 is parameterized by the line $\mathbb{R}$ via the solutions (2.6), we can extend this parameterization to V_s near $H_{0,t}$ so that (s, t) provides a coordinate chart of $\cup_s V_s$. We let $H_{s,t}$ be the solution to $P \circ L_s = 0$ associated to $(s, t) \in V_s$. This way, to solve $L_s(H) = 0$ it suffices to find the vanishing loci of the function

$$r(s,t) = \sqrt{-1} \int_X \operatorname{tr}\bigl(L_s(H_{s,t}) \cdot I_1\bigr) \in \mathbb{R}.$$

We will show that there is a function $t = \rho(s)$ so that $r(s, \rho(s)) = 0$. Because $r(0,t) = 0$, the first step is to investigate the sign of the derivatives of $r(s,t)$ of s at $s = 0$. Recall that

$$D''_s H_{s,t} = D''_0 H_{s,t} - [A_s, H_{s,t}] \quad \text{and} \quad D'_s H_{s,t} = D''_0 H_{s,t} + [A^*_s, H_{s,t}];$$

hence,

$$\frac{d}{ds} D''_s H_{s,t} = D''_s \dot{H}_{s,t} - [\dot{A}_s, H_{s,t}] \quad \text{and} \quad \frac{d}{ds} D'_s H_{s,t} = D'_s \dot{H}_{s,t} + [\dot{A}^*_s, H_{s,t}].$$

Therefore, following the convention that $\dot{f}(s,t) = \frac{d}{ds} f(s,t)$ and $\dot{f}(0,t) = \dot{f}(s,t)|_{s=0}$,

$$\frac{d}{ds} L_s(H_{s,t}) = \dot{F}_s - [\dot{A}_s, D_s H_{s,t} \cdot H^{-1}_{s,t}] + D''_s \varphi_{s,t} \qquad \text{with}$$
$$\varphi_{s,t} = \frac{d}{ds}(D'_s H_{s,t} \cdot H^{-1}_{s,t}).$$

We have the following useful easy observation:

Lemma 2.3. *Let $\mu_1 \in \Omega^{1,0}(E_1^\vee \otimes E_2)$, let $\mu_2 \in \Omega^{0,1}(E_1^\vee \otimes E_2)$, and let $\psi \in \Omega^0(E_2^\vee \otimes E_1)$ be a smooth section.*

1. *Suppose $D''_0 \psi = 0$, then* $\int_X \mathrm{tr}(D''_0 \mu \cdot \psi) \wedge \omega^{n-1} = 0.$
2. *Suppose $D''^*_0 \mu_2 = 0$, then* $\int_X \mathrm{tr}(\mu_2 \cdot D'_0 \psi) \wedge \omega^{n-1} = 0.$

Proof. The two identities follow directly from the Stokes' formula. First, because $D''_0 \psi = 0$, because μ is a $(1,0)$-form and because ω is a Kahler form on X,

$$\begin{aligned} \int_X \mathrm{tr}(D''_0 \mu_1 \cdot \psi) \wedge \omega^{n-1} &= \int_X \bar{\partial}\big(\mathrm{tr}(\mu_1 \cdot \psi) \wedge \omega^{n-1}\big) \\ &= \int_X d\big(\mathrm{tr}(\mu_1 \cdot \psi) \wedge \omega^{n-1}\big) = 0. \end{aligned}$$

This proves the first part. As to the second part, a direct computation shows that

$$0 = \mathrm{tr}(D''^*_0 \mu_2 \cdot \phi) \cdot \omega^n = -2n \, \mathrm{tr}(D'_0 \mu_2 \cdot \phi) \cdot \omega^{n-1}.$$

The identity follows immediately. q.e.d.

We now evaluate $\dot{r}(0,t)$ and $\ddot{r}(0,t)$. First, we show that

$$\frac{d}{ds} L_s(H_{s,t})|_{s=0} = \frac{d}{ds} \tilde{L}_s(H_{s,t})|_{s=0} = 0. \tag{2.8}$$

Because $F_{0,H_{0,t}} \wedge \omega^{n-1} = 0$, the first identity holds automatically. We now look at the second identity. By definition, there is a function $c(s,t)$ with $c(0,t)=0$ so that

$$c(s,t)J = L_s(H_{s,t}) = H_{s,t}^{-1/2} F_{s,H_{s,t}} H_{s,t}^{1/2}.$$

Taking derivative of s at $s=0$ gives us

$$\dot{c}(0,t)J = \frac{d}{ds}\big(H_{s,t}^{-1/2} F_{s,H_{s,t}} H_{s,t}^{1/2} \wedge \omega^{n-1}\big)|_{s=0} = H_{0,t}^{-1/2} \dot{F}_{0,H_{0,t}} H_{0,t}^{1/2} \wedge \omega^{n-1}.$$

Using the explicit form of $H_{0,t}$, $A_0 = 0$, $\dot{F}_0 = D_0'\dot{A}_0 - D_0''\dot{A}_0^*$ and $DH_{0,t} = 0$, we obtain

$$\begin{aligned} &\frac{d}{ds}\int_X \operatorname{tr}\big(H_{s,t}^{-1/2} F_{s,H_{s,t}} H_{s,t}^{1/2} \wedge \omega^{n-1} \cdot I_1\big)|_{s=0} \\ &= \int_X \operatorname{tr}\big((D_0'\varphi + D_0''\varphi')\cdot I_1\big) \wedge \omega^{n-1} \end{aligned}$$

for some smooth sections φ and φ'. The right-hand side of the above identity is zero by Lemma 2.3; thus,

$$\int_X \dot{c}(0,t)\operatorname{tr}\big(J\cdot I_1\big) = 0,$$

which forces $\dot{c}(0,t)=0$. This proves (2.8), and $\dot{r}(0,t)=0$ for all t.

We next compute $\ddot{r}(0,t)$. Because of (2.8),

$$\frac{d^2}{ds^2} L_s(H_{s,t})|_{s=0} = H_{0,t}^{-1/2} \frac{d^2}{ds^2}\tilde{L}_s(H_{s,t})|_{s=0} H_{0,t}^{1/2}.$$

A direct computation shows that

$$\frac{d^2}{ds^2}\tilde{L}_s(H_{s,t})|_{s=0} = \ddot{F}_0 - 2[\dot{A}_0, [\dot{A}_0^*, H_{0,t}]H_{0,t}^{-1}] - 2[\dot{A}_0, D_0'\dot{H}_{0,t}\cdot H_{0,t}^{-1}] + D_0''\dot{\varphi}_{0,t} \tag{2.9}$$

with

$$\begin{aligned} \varphi_{s,t} = \frac{d}{ds}(D_s' H_{s,t}\cdot H_{s,t}^{-1}) &= (\dot{D}_s' H_{s,t} + D_s'\dot{H}_{s,t})H_{s,t}^{-1} \\ &= [\dot{A}_s^*, H_{s,t}]H_{s,t}^{-1} + D_s'\dot{H}_{s,t}\cdot H_{s,t}^{-1}. \end{aligned} \tag{2.10}$$

Because $H_{0,t}$ commutes with I_1,

$$\begin{aligned}\ddot{r}(0,t) = \sqrt{-1}\Big(&\int_X \operatorname{tr}\big(\ddot{F}_0 \cdot I_1\big) \wedge \omega^{n-1} \\ &- 2\int_X \operatorname{tr}\big([\dot{A}_0, [\dot{A}_0^*, H_{0,t}]H_{0,t}^{-1}] \cdot I_1\big) \wedge \omega^{n-1} \\ &- 2\int_X \operatorname{tr}\big([\dot{A}_0, D_0'\dot{H}_{0,t} \cdot H_{0,t}^{-1}] \cdot I_1\big) \wedge \omega^{n-1} \\ &+ \int_X \operatorname{tr}\big(D_0''\dot{\varphi}_{0,t} \cdot I_1\big) \wedge \omega^{n-1}\Big).\end{aligned}$$

To analyze the sign of the above integration, we use the splitting $E = E_1 \oplus E_2$ to express

$$\dot{A}_0 = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

Because

$$H_{0,t} = \begin{pmatrix} \exp(\frac{t}{r_2}) \cdot I_1 & 0 \\ 0 & \exp(\frac{-t}{r_1}) \cdot I_2 \end{pmatrix},$$

the second term

$$-2\sqrt{-1}\int_X \operatorname{tr}\big([\dot{A}_0, [\dot{A}_0^*, H_{0,t}]H_{0,t}^{-1}] \cdot I_1\big) \wedge \omega^{n-1}$$

in $\ddot{r}(0,t)$ is, for $\alpha = \frac{1}{n_1} + \frac{1}{n_2}$,

$$\begin{aligned}&- 2\sqrt{-1}(1 - e^{-\alpha t})\int_X \operatorname{tr}\big(C_{12} \wedge C_{12}^*\big) \wedge \omega^{n-1} \\ &\qquad\qquad - 2\sqrt{-1}(1 - e^{\alpha t})\int_X \operatorname{tr}\big(C_{21}^* \wedge C_{21}\big) \wedge \omega^{n-1}.\end{aligned}$$

Similarly, because of (2.4) and $F_s^{2,0} = F_s^{0,2} = 0$,

$$\begin{aligned}\sqrt{-1}\int_X \operatorname{tr}\big(\ddot{F}_0 \cdot I_1\big) \wedge \omega^{n-1} &= 2\sqrt{-1}\int_X \operatorname{tr}\big(C_{12} \wedge C_{12}^*\big) \wedge \omega^{n-1} \\ &\quad + 2\sqrt{-1}\int_X \operatorname{tr}\big(C_{21}^* \wedge C_{21}\big) \wedge \omega^{n-1}.\end{aligned}$$

The last term in $\ddot{r}(0,t)$ is zero because of Lemma 2.3; the next-to-last term is

$$\begin{aligned}&- 2\sqrt{-1}\int_X \operatorname{tr}\big(\dot{A}_0 \cdot D'\dot{H}_{0,t} \cdot H_{0,t}^{-1} \cdot I_1\big) \wedge \omega^{n-1} \\ &\qquad\qquad + 2\sqrt{-1}\int_X \operatorname{tr}\big(D_0'\dot{H}_{0,t} \cdot H_{0,t}^{-1} \cdot \dot{A}_0 \cdot I_1\big) \wedge \omega^{n-1},\end{aligned}$$

which vanishes because $D_0''^* \dot{A}_0 = 0$ and Lemma 2.3. Therefore,

$$\ddot{r}(0,t) = \sqrt{-1}e^{-\alpha t}\int_X \operatorname{tr}(C_{12}\wedge C_{12}^*)\wedge\omega^{n-1} + \sqrt{-1}e^{\alpha t}\int_X \operatorname{tr}(C_{21}^*\wedge C_{21})\wedge\omega^{n-1}.$$

Because the associated cohomology class $[C_{ij}] = \kappa_{ij}$ and κ_{21} and κ_{12} are both non-zero,

$$A = \sqrt{-1}\int_X \operatorname{tr}(C_{12}\wedge C_{12}^*)\wedge\omega^{n-1} \qquad \text{and}$$
$$B = -\sqrt{-1}\int_X \operatorname{tr}(C_{21}^*\wedge C_{21})\wedge\omega^{n-1}$$

are positive. Hence, for sufficiently small s, the value $r(s,t) > 0$ for $t < \frac{1}{2\alpha}\ln\frac{A}{B}$ and $r(s,t) > 0$ for $t > \frac{1}{2\alpha}\ln\frac{A}{B}$. Hence, there is a function $t = \rho(s)$ so that $r(s,\rho(s)) = 0$. This proves that the system $L_s(H) = 0$ is solvable for small s. Here, the function $\rho(s)$ is not necessarily continuous, but $\lim_{s\to 0}\rho(s) = \frac{1}{2\alpha}\ln\frac{A}{B}$.

3. Linearization of Strominger's system

In this section, we will study the linearization of Strominger's system. Before we do this, we will first rephrase the system (1.1)–(1.4) in the form that is easier to handle.

We fix a Calabi–Yau threefold (X,ω_0) and a $(3,0)$-holomorphic form Ω so that $\Omega\wedge\bar{\Omega} = \omega_0^3$. We let (E, D'') be a rank r holomorphic bundle over X such that $c_1(E) = 0$ and $c_2(E) = c_2(X)$. We then choose a hermitian metric H on E and let $D_H = D'_H \oplus D''$ be the hermitian connection of (E, D'', H); its curvature $F_H = D_H\circ D_H$ satisfies

$$F_H^{2,0} = F_H^{0,2} = 0.$$

Thus, the second equation of the Strominger's system follows automatically.

The fourth equation of the system is a non-linear equation of a hermitian form ω involving the adjoint d_ω^* of ω. It turns out that this equation is equivalent to

$$d(\|\Omega\|_\omega\omega^2) = 0.$$

We now prove this equivalence. We let $\mathcal{H}(X)$ and $\mathcal{K}(X)$ be the cones of positive definite hermitian forms and Kahler forms on X respectively. Given an $\omega\in\mathcal{H}(X)$, we let $*_\omega$ be the (hermitian) star operator of ω; and let d_ω^* be the adjoint of d with respect to the metric ω.

The hermitian star operator has an explicit local expression. Given a hermitian form ω on X, it induces canonical hermitian metrics on $T_{X,\mathbb{C}}$ and on $\wedge^k T^\vee_{X,\mathbb{C}}$. Let $(\cdot,\cdot)_\omega$ be the hermitian metric on $\wedge^k T^\vee_{X,\mathbb{C}}$ and $\frac{1}{3!}\omega^3$ its associated volume form on X. The star operator $*_\omega$ is the $\mathbb{C}$-linear operator defined via

$$(\varphi,\psi)_\omega \cdot \frac{\omega^3}{3!} = \varphi \wedge *_\omega \bar{\psi}.$$

Let $p \in X$ be any point and let $\varphi_1, \varphi_2, \varphi_2$ be an $(\cdot,\cdot)_\omega$-orthonormal basis (a moving frame) of the $(1,0)$-forms near p obeying $(\varphi_i, \varphi_j)_\omega = 2\delta_{ij}$. Then, the hermitian form

$$\omega = \frac{\sqrt{-1}}{2}\sum_{i=1}^{3} \varphi_i \wedge \bar{\varphi}_i.$$

For any subset $I = \{i_1, \cdots, i_k\} \subset \{1,2,3\}$, we denote by $\varphi_I = \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$, and denote by I° the complement $\{1,2,3\} - I$. Following this convention,

$$*_\omega\,(c\,\bar{\varphi}_I \wedge \varphi_J) = \epsilon_{IJ}\sqrt{-1}\;\, 2^{|I|+|J|-3} c\,\varphi_{I^\circ} \wedge \bar{\varphi}_{J^\circ}, \quad c \in \mathbb{C}, \tag{3.1}$$

where ϵ_{IJ} is the parity of permuting $(I, J; I^\circ, J^\circ) \mapsto (1,2,3;1',2',3')$.

We now re-state and prove the mentioned equivalence.

Lemma 3.1. *Let ω_0 be the reference Kahler form as before. Then, the equation (1.4) is equivalent to*

$$*_{\omega_0}\, d\,(\|\Omega\|_\omega \omega^2) = 0. \tag{3.2}$$

Proof. Let f be a positive real valued function, then

$$d(f\omega^2) = f\,d\,\omega^2 + df \wedge \omega^2 = 2fd *_\omega \omega + df \wedge \omega^2.$$

Thus,

$$*_\omega d(f\omega^2) = 2f *_\omega d *_\omega \omega + *_\omega(df \wedge \omega^2) = -2fd^*_\omega\omega + 2d_c f.$$

Here, we have used the identity

$$*_\omega(df \wedge \omega^2) = 2d_c f,$$

which holds for all hermitian form ω. Replacing f by $\|\Omega\|$, we obtain

$$*_\omega d\big(\|\Omega\|_\omega\omega^2\big) = 2\|\Omega\|_\omega\big(-d^*_\omega\omega + d_c \log\|\Omega\|_\omega\big),$$

which vanishes if and only if

$$d^*_\omega\omega = d_c \log\|\Omega\|_\omega.$$

Finally, since $*_\omega$ and $*_{\omega_0}$ are both isomorphisms, $*_\omega d\big(\|\Omega\|^{-1}_\omega\omega^2\big) = 0$ if and only if

$$*_{\omega_0} d\big(\|\Omega\|_\omega\omega^2\big) = 0.$$

This proves the lemma. q.e.d.

To apply the implicit function theorem, we need to specify the range of the operators associated to Strominger's system. For that, noting that $2dd_c = \sqrt{-1}\partial\bar\partial$, we let $R(dd_c) \subset \Omega^{2,2}_{\mathbb{R}}(X)$ and $R(d^*_{\omega_0}) \subset \Omega^1_{\mathbb{R}}(X)$ be the range of

$$dd_c : \Omega^{1,1}_{\mathbb{R}}(X) \to \Omega^{2,2}_{\mathbb{R}}(X) \quad \text{and} \quad d^*_{\omega_0} : \Omega^{1,1}_{\mathbb{R}}(X) \to \Omega^1_{\mathbb{R}}(X).$$

Because (X, ω_0) is a Kahler manifold, by $\partial\bar\partial$-lemma, a real form $\alpha \in R(dd_c)$ if and only if $d\alpha = 0$. Hence, after picking a usual Banach norm on $\Omega^{2,2}_{\mathbb{R}}(X)$, $R(dd_c)$ is closed in it. As to $R(d^*_{\omega_0})$, since $d^*_{\omega_0}$ is part of an elliptic complex, it is also closed. This way, after replacing (1.4) by (3.2) and omitting the equation (1.2), the Strominger's system is equivalent to the vanishing of the operator

$$\mathbf{L} = \mathbf{L}_1 \oplus \mathbf{L}_2 \oplus \mathbf{L}_3 : \mathcal{H}(E)_1 \times \mathcal{H}(X) \longrightarrow \Omega^6_{\mathbb{R}}(\mathfrak{su}E) \oplus R(dd_c) \oplus R(d^*_{\omega_0}), \tag{3.3}$$

defined by

$$\mathbf{L}_1(H, \omega) = H^{-1/2} F_H H^{1/2} \wedge \omega^2 \in \Omega^6_{\mathbb{R}}(\mathfrak{su}E); \tag{3.4}$$

$$\mathbf{L}_2(H, \omega) = \frac{1}{2} dd_c\omega + \big(\mathrm{tr}(F_H \wedge F_H) - \mathrm{tr}(R_\omega \wedge R_\omega)\big) \in \Omega^{2,2}_{\mathbb{R}}(X); \tag{3.5}$$

$$\mathbf{L}_3(H, \omega) = *_{\omega_0} d\,(\|\Omega\|_\omega \omega^2) \in \Omega^1_{\mathbb{R}}(X). \tag{3.6}$$

Because $c_2(E) = c_2(TX)$ and X is a Kahler manifold, by $\partial\bar\partial$-lemma the image of $\mathbf{L}_2$ lies in $R(P)$. As to $\mathbf{L}_3$, because

$$*_{\omega_0} d = \pm *_{\omega_0} d *_{\omega_0} *_{\omega_0}^{-1} = \mp d^*_{\omega_0} *_{\omega_0}^{-1},$$

its image lies in the range of $d^*_{\omega_0}$ as well. Therefore, the operator $\mathbf{L}$ is well-defined.

Proposition 3.2. *Suppose $\mathbf{L}(H, \omega_0) = 0$. Then, the three summands of the linearization of $\mathbf{L}$ at (H, ω_0) are*

$$\begin{aligned}
\delta\mathbf{L}_1(H, \omega_0)(\delta h, \delta\omega) &= D'' D'_H \delta h \wedge \omega_0^2 + 2H^{-1/2} F_H H^{1/2} \wedge \omega_0 \wedge \delta\omega;\\
\delta\mathbf{L}_2(H, \omega_0)(\delta h, \delta\omega) &= \frac{1}{2} dd_c \delta\omega + 2\big(\mathrm{tr}(\delta F_H(\delta h) \wedge F_H)\\
&\qquad - \mathrm{tr}(\delta R_{\omega_0}(\delta\omega) \wedge R_{\omega_0})\big);\\
\delta\mathbf{L}_3(H, \omega_0)(\delta h, \delta\omega) &= 2d^*_{\omega_0}\delta\omega - d^*_{\omega_0}((\delta\omega, \omega_0)_{\omega_0}\omega_0).
\end{aligned}$$

Here, as before, we follow the convention $\delta H = H^{-1/2}\delta h H^{-1/2}$.

Proof. The formula for $\delta\mathbf{L}_1$ is well-known [**27**]; the formula for $\delta\mathbf{L}_2$ in the written form is a tautology; we stop short of finding an explicit form of δR since the current form is sufficient for our purposes.

We now prove the formula for $\delta\mathbf{L}_3$. Let ω_t be a smooth variation of the hermitian form ω_0; let $\varphi_1(t), \varphi_2(t), \varphi_3(t)$ be an orthonormal basis of $(1,0)$-forms, smooth in t, expressed in a holomorphic coordinate (z_1, z_2, z_3) near $p \in X$ by

$$\varphi_i(t) = \sum_j b_{ij}(t) dz_j, \quad b_{ij}(0)(p) = \delta_{ij} \quad \text{and} \quad (\varphi_i(t), \varphi_j(t))_{\omega_t} = 2\delta_{ij}.$$

We can compute explicitly $\frac{d}{dt}(\omega_t^2)|_{t=0}$. First,

$$\omega_t^2 = \frac{1}{2}\sum \varphi_{i^\circ}(t) \wedge \bar{\varphi}_{i^\circ}(t) = \frac{1}{2}\sum_{i,l,k} c_{ik}(t)\, \bar{c}_{il}(t)\, dz_{k^\circ} \wedge d\bar{z}_{l^\circ},$$

where $c_{ij}(t)$ is the ij-th minor of the matrix $(b_{ij}(t))_{3\times 3}$; namely

$$\big(c_{ij}(t)\big)^t = \det(b_{ij}(t)) \cdot \big(b_{ij}(t)\big)^{-1}. \tag{3.7}$$

Hence, at p,

$$\frac{d}{dt}\omega_t^2|_{t=0} = \frac{1}{2}\sum \big(\dot{c}_{lk}(0) + \dot{\bar{c}}_{kl}(0)\big) dz_{k^\circ} \wedge d\bar{z}_{l^\circ}.$$

Using the identity (3.7) above,

$$\dot{c}_{lk}(0) + \dot{\bar{c}}_{kl}(0) = -\dot{b}_{kl}(0) - \dot{\bar{b}}_{lk}(0) + c_{lk}(0)\sum \dot{b}_{ii}(0) + \bar{c}_{kl}(0)\sum_i \dot{\bar{b}}_{ii}(0).$$

Therefore, at p,

$$\begin{aligned}\frac{d}{dt}\omega_t^2|_{t=0} = \frac{-1}{2}&\sum_{l,k}(\dot{b}_{kl}(0) + \dot{\bar{b}}_{lk}(0)) dz_{k^\circ} \wedge d\bar{z}_{l^\circ} \\ &+ \frac{1}{2}\left(\sum_k dz_{k^\circ} \wedge d\bar{z}_{k^\circ}\right) \cdot \left(\sum \dot{b}_{ii}(0) + \dot{\bar{b}}_{ii}(0)\right).\end{aligned}$$

On the other hand, $\omega_0^2 = \frac{1}{2}\sum dz_{k^\circ} \wedge d\bar{z}_{k^\circ}$. Hence,

$$\frac{d}{dt}\omega_t^2|_{t=0} = \frac{-1}{2}\sum_{l,k}(\dot{b}_{kl}(0) + \dot{\bar{b}}_{lk}(0)) dz_{k^\circ} \wedge d\bar{z}_{l^\circ} + \omega_0^2\big(\sum \dot{b}_{ii}(0) + \dot{\bar{b}}_{ii}(0)\big). \tag{3.8}$$

Next, we compute

$$\frac{d}{dt}\log\|\Omega\|^2_{\omega_t}|_{t=0} = -\frac{d}{dt}\frac{\omega_t^3}{\omega_0^3}|_{t=0} = -\sum \dot{b}_{ii}(0) + \dot{\bar{b}}_{ii}(0).$$

Adding $\|\Omega\|_{\omega_0} \equiv 1$, we get

$$\frac{d}{dt}\big(\|\Omega\|_{\omega_t}\omega_t^2\big)|_{t=0} = \left(\frac{1}{2}\omega_0^2\frac{d}{dt}\log\|\Omega\|^2_{\omega_t} + \frac{d}{dt}\omega_t^2\right)|_{t=0}$$
$$= -\frac{1}{2}\sum_{l,k}(\dot b_{kl}(0) + \dot b_{lk}(0))dz_{k^\circ}\wedge d\bar z_{l^\circ}$$
$$+ \frac{1}{2}\big(\sum \dot b_{ii}(0) + \dot{\bar b}_{ii}(0)\big)\omega_0^2.$$

On the other hand, at p

$$\frac{d}{dt}\omega_t|_{t=0} = \frac{\sqrt{-1}}{2}\sum_i \dot\varphi_i\wedge\bar\varphi_i + \varphi_i\wedge\dot{\bar\varphi}_i = \frac{\sqrt{-1}}{2}\sum_{i,j}(\dot b_{ji}(0)+\dot{\bar b}_{ij}(0))dz_i\wedge d\bar z_j.$$

Hence,

$$*_{\omega_0}\dot\omega_0 = \frac{1}{4}\sum_{i,j}(\dot{\bar b}_{ji}(0) + \dot b_{ij}(0))dz_{i^\circ}\wedge d\bar z_{j^\circ}.$$

Combined, we obtain

$$\frac{d}{dt}\big(\|\Omega\|_{\omega_t}\omega_t^2\big)|_{t=0} = -2 *_{\omega_0}\dot\omega_0 + \big(\sum \dot b_{ii}(0) + \dot{\bar b}_{ii}(0)\big)\frac{\omega_0^2}{2}.$$

It remains to treat the term $\sum \dot b_{ii}(0) + \dot{\bar b}_{ii}(0)$. From

$$(\dot\omega_0, \omega_0)_{\omega_0}\frac{\omega_0^3}{3!} = \dot\omega_0\wedge *_{\omega_0}\omega_0 \quad \text{and} \quad *_{\omega_0}\omega_0 = \frac{1}{4}\sum dz_{k^\circ}\wedge d\overline{z}_{k^\circ},$$

we get

$$\begin{aligned}
&\dot\omega_0\wedge *_{\omega_0}\omega_0\\
&= \frac{\sqrt{-1}}{8}\sum(\dot b_{ij}(0) + \dot{\bar b}_{ji}(0))dz_i\wedge d\overline{z}_j\wedge dz_{k^\circ}\wedge d\overline{z}_{k^\circ}\\
&= -\frac{\sqrt{-1}}{8}\sum(\dot b_{ii}(0) + \dot{\bar b}_{ii}(0))dz_1\wedge d\overline{z}_1\wedge dz_2\wedge d\overline{z}_2\wedge dz_3\wedge d\overline{z}_3;
\end{aligned}$$

hence,

$$(\dot\omega_0, \omega_0)_{\omega_0} = \frac{\dot\omega_0\wedge *_{\omega_0}\omega_0}{\omega_0^3/3!} = \sum \dot b_{ii}(0) + \dot{\bar b}_{ii}(0).$$

This proves that

$$\frac{d}{dt}\big(\|\Omega\|_{\omega_t}\omega_t^2\big)|_{t=0} = -2 *_{\omega_0}\dot\omega_0 + \big(\sum \dot b_{ii}(0) + \dot{\bar b}_{ii}(0)\big)\omega_0^2$$
$$= -2 *_{\omega_0}\dot\omega_0 + *_{\omega_0}(\dot\omega_0, \omega_0)_{\omega_0}\omega_0.$$

Finally, Applying $*_{\omega_0}d$ to both sides of this identity, we obtain

$$\frac{d}{dt} *_{\omega_0} d\big(\|\Omega\|_{\omega_t}\omega_t^2\big)|_{t=0} = 2d^*_{\omega_0}\dot\omega_0 - d^*_{\omega_0}((\dot\omega_0, \omega_0)_{\omega_0}\omega_0).$$

This proves the Proposition. q.e.d.

Strominger's system admits a class of reducible solutions. Let

$$(E, D_0'') = \mathbb{C}_X^{\oplus r} \oplus TX$$

be the direct sum of the trivial holomorphic bundle $\mathbb{C}_X^{\oplus r}$ and the tangent bundle TX. We fix an isomorphism $\wedge^{r+3}E \cong \mathbb{C}_X$; we endow E with the hermitian metric $\langle\,,\rangle$ that is a direct sum of a constant metric on $\mathbb{C}_X^{\oplus r}$ and the Calabi–Yau metric ω_0 on TX. We normalize $\langle\,,\rangle$ so that its induced metric on $\wedge^{r+3}E \cong \mathbb{C}_X$ is the constant one metric. As before, the metric $\langle\,,\rangle$ is identified with the identity endomorphism $I: E \to E$.

Now, let $\mathcal{H}^+_{r\times r}$ be the space of positive definite hermitian symmetric $r \times r$ metrics; let I_1 and I_2 be the identity endomorphisms of $\mathbb{C}_X^{\oplus r}$ and TX respectively. By abuse of notation, for $T \in \mathcal{H}^+_{r\times r}$, we also view it as the constant endomorphism of $\mathbb{C}_X^{\oplus r}$ induced by T, viewed as an endomorphism of E. Then, the assignment

$$T \in \mathcal{H}^+_{r\times r} \longmapsto H_T = T \oplus |T|^{-1/3} I_2 \in \mathcal{H}(E)_1, \ |T| = \det T,$$

associates each $T \in \mathcal{H}^+_{r\times r}$ to a hermitian metric of E.

Obviously, the hermitian curvature F_{H_T} of $(E, \langle\,,\rangle_{H_T})$ is $0 \oplus R_{\omega_0}$; hence, $F_{H_T} \wedge F_{H_T} = R_{\omega_0} \wedge R_{\omega_0}$. Because ω_0 is d-closed,

$$\mathbf{L}_2(H_T, \omega_0) = \frac{1}{2} dd_c\omega_0 + \operatorname{tr}(F_{H_T} \wedge F_{H_T}) - \operatorname{tr}(R_{\omega_0} \wedge R_{\omega_0}) = 0.$$

Further, because $\langle\,,\rangle_{H_T}$ is Hermitian–Yang–Mills, and because $d^*_{\omega_0}\omega_0 = 0$ and $\Omega \wedge \bar{\Omega} = \omega_0^3$, $\mathbf{L}_1(H_T, \omega_0) = \mathbf{L}_3(H_T, \omega_0) = 0$. Therefore, (H_T, ω_0) is a solution to $\mathbf{L}(H, \omega) = 0$. Indeed, for any constant $c > 0$, the pair $(H_T, c\omega_0)$ is a solution to $\mathbf{L} = 0$. These solutions are reducible because the vector bundle E splits under the hermitian connection D_{H_T}. In this paper, we will call such solutions the *trivial* solutions to Strominger's system.

To construct irreducible solutions to Strominger's system, we will first form a family of holomorphic structures D_s'' on E that is a smooth deformation of D_0''; we then show that certain trivial solutions to Strominger's system on (E, D_0'') can be extended to (irreducible) solutions on (E, D_s''). We shall prove this by applying implicit function theorem to the operator $\mathbf{L}$ of (3.3).

To this end, we pick an integer k and a large p and endow the domain and the target of $\mathbf{L}$ the Banach space structures as indicated:

$$\mathcal{H}(E)_{1,L^p_k} \times \mathcal{H}(X)_{L^p_k} \longrightarrow \Omega^6_{\mathbb{R}}(\mathfrak{su}E)_{L^p_{k-2}} \oplus R(dd_c)_{L^p_{k-2}} \oplus R(d^*_{\omega_0})_{L^p_{k-1}}.$$

$\mathbf{L}$ becomes a smooth operator and its linearized operator $\delta\mathbf{L}$ at a solution (H,ω) becomes a linear map

$$\Omega^0(\mathfrak{Her}^0 E)_{L^p_k} \oplus \Omega^{1,1}_{\mathbb{R}}(X)_{L^p_k} \longrightarrow \Omega^6_{\mathbb{R}}(\mathfrak{su}E)_{L^p_{k-2}} \oplus R(dd_c)_{L^p_{k-2}} \oplus R(d^*_{\omega_0})_{L^p_{k-1}}.$$

Here, we used the canonical isomorphisms $T_H\mathcal{H}(E)_{1,L^p_k} \cong \Omega^0(\mathfrak{Her}^0 E)_{L^p_k}$ and $T_\omega\mathcal{H}(X)_{L^p_k} \cong \Omega^{1,1}_{\mathbb{R}}(X)_{L^p_k}$. For simplicity, in the following, we will abbreviate

$$\mathcal{W}_1 = \Omega^6_{\mathbb{R}}(\mathfrak{su}E)_{L^p_{k-2}} \quad \text{and} \quad \mathcal{W}_2 = R(dd_c)_{L^p_{k-2}} \oplus R(d^*_{\omega_0})_{L^p_{k-1}}.$$

Thus, $\delta\mathbf{L}(H,\omega)$ is a linear map

$$\delta\mathbf{L}(H,\omega): \Omega^0(\mathfrak{Her}^0 E)_{L^p_k} \oplus \Omega^{1,1}_{\mathbb{R}}(X)_{L^p_k} \longrightarrow \mathcal{W}_1 \oplus \mathcal{W}_2. \tag{3.9}$$

To study the kernel and the cokernel of $\delta\mathbf{L}$ at a trivial solution $(H_T, c\omega_0)$, we will first look at the linear map

$$\mathbf{F}(\delta h) = D''_0 D'_{0,H_T}(\delta h) \wedge \omega_0^2 : \Omega^0(\mathfrak{Her}^0 E)_{L^p_k} \longrightarrow \Omega^6_{\mathbb{R}}(\mathfrak{su}E)_{L^p_{k-2}}. \tag{3.10}$$

Here, according to our convention, $D_{H_T} = D'_{0,H_T} \oplus D''_0$ is the hermitian connection of (E, D''_0, H_T) for a $T \in \mathcal{H}^+_{r\times r}$. Since $(E, D''_0) = \mathbb{C}^{\oplus r}_X \oplus TX$, the above is a linear elliptic operator of index 0 whose kernel is

$$V_0 = \{M \oplus aI_2 \mid M \in \operatorname{End}(\mathbb{C}^{\oplus r}), M = M^*,\ \operatorname{tr} M + 3a = 0\}$$

and cokernel is

$$V_1 = \omega_0^3 \cdot V_0 \subset \mathcal{W}_1 = \Omega^6_{\mathbb{R}}(\mathfrak{su}E)_{L^p_{k-2}}. \tag{3.11}$$

We let $\mathbf{P}$ be the obvious projection

$$\mathbf{P}: \mathcal{W}_1 \longrightarrow \mathcal{W}_1/V_1.$$

Proposition 3.3. *Let (X,ω_0), Ω, $\langle\,,\rangle$ and $T \in \mathcal{H}^+_{r\times r}$ be as before. Then, there is a constant C so that for any $c > C$, the linear operator*

$$\mathbf{P} \circ \delta\mathcal{L}_1(H_T, c\omega_0) \oplus \delta\mathcal{L}_2(H_T, c\omega_0) \oplus \delta\mathcal{L}_3(H_T, c\omega_0)$$

from $\Omega^0(\mathfrak{Her}^0 E)_{L^p_k} \oplus \Omega^{1,1}_{\mathbb{R}}(X)_{L^p_k}$ to $\mathcal{W}_1/V_1 \oplus \mathcal{W}_2$ is surjective.

Proof. As we shall see, the proof of the Proposition relies on the understanding of the operator

$$T: \Omega^{1,1}_{\mathbb{R}}(X)_{L^p_k} \longrightarrow \mathcal{W}_2$$

defined by, after setting $P = \frac{1}{2}dd_c = \sqrt{-1}\partial\bar\partial$,

$$T\mu = \big(P\mu, 2d^*_{\omega_0}\mu - d^*_{\omega_0}\big((\mu,\omega_0)_{\omega_0}\omega_0\big)\big).$$

Before we go on, we remark that since in the proof of this Proposition we will solely work with the Kahler form ω_0, for convenience, we will abbreviate $*_{\omega_0}$ and $d^*_{\omega_0}$ to $*$ and d^*.

For the starter, we form the linear operator S:

$$S\mu = 2\mu - (\mu, \omega_0)_{\omega_0}\omega_0 : \Omega^{1,1}_{\mathbb{R}} \longrightarrow \Omega^{1,1}_{\mathbb{R}}$$

and its inverse

$$S^{-1}\phi = \frac{1}{2}(\phi - (\phi, \omega_0)_{\omega_0}\omega_0).$$

Then, by setting $\phi = S\mu$, $T\mu$ can be expressed as

$$T\mu = T \circ S^{-1}\phi = \left(P \circ S^{-1}\phi, d^*\phi\right).$$

Then, applying the Hodge decomposition to $\phi \in \Omega^{1,1}_{\mathbb{R}}(X)$,

$$\phi = dd^*\psi + d^*d\psi + h$$

for a real $(1,1)$-form ψ and harmonic h. By the $\partial\bar{\partial}$-lemma, we can rewrite $d^*d\psi = *P\alpha$ for a real form α.

As to the harmonic h, we check that the pairing $(h, \omega_0)_{\omega_0}$ is constant. Since (X, ω_0) is Kahler,

$$d_c * h = d^* * h \wedge \omega_0 - d^*(*h \wedge \omega_0);$$

and since $d^* * h = d_c * h = 0$, $d^*(*h \wedge \omega_0) = 0$. Hence, the defining identity

$$(h, \omega_0)*1 = *h \wedge \omega_0 \tag{3.12}$$

forces $(h, \omega_0)_{\omega_0}$ to be a constant. Therefore, the space of harmonic forms $\mathbb{H} \subset \Omega^{1,1}_{\mathbb{R}}(X)$ lies in the kernel of both T and $T \circ S^{-1}$.

With this said, to study the surjectivity of T, we only need to look at those ϕ that are orthogonal to $\mathbb{H}$ under the L^2-intersection pairing

$$\langle u, v\rangle = \int_X (u, v)_{\omega_0} * 1.$$

In particular, such ϕ has decomposition

$$\phi = *P\alpha + d^*d\psi,$$

and

$$T \circ S^{-1}\phi = \left(P \circ S^{-1}(*P\alpha) + P \circ S^{-1}(dd^*\psi), d^*d(d^*\psi)\right).$$

To proceed, we look at the operator U:

$$U\alpha = 2 * P \circ S^{-1}(*P\alpha) = *P\big(*P\alpha - (*P\alpha, \omega_0)_{\omega_0}\omega_0\big). \tag{3.13}$$

Because

$$P^* = \left(\sqrt{-1}\partial\bar{\partial}\right)^* = -\sqrt{-1}\bar{\partial}^*\partial^* = *\sqrt{-1}\partial\bar{\partial}* = *P*,$$

$U\alpha$ can be re-written as

$$U\alpha = P^*P\alpha - *P\big((*P(\alpha), \omega_0)_{\omega_0}\omega_0\big). \tag{3.14}$$

To proceed, we need to simplify the operator U. We first use the identities

(3.15)
$$\partial^*\mu\wedge\omega_0-\partial^*(\mu\wedge\omega_0)=\sqrt{-1}\bar\partial\mu \quad\text{and}\quad \bar\partial^*\mu\wedge\omega_0-\bar\partial^*(\mu\wedge\omega_0)=-\sqrt{-1}\partial\mu,$$

which hold for all Kahler manifolds, to derive

$$\partial^*(f{\omega_0}^2)=-2\sqrt{-1}\bar\partial f\wedge\omega_0.$$

Using $(*P\alpha,\omega_0)_{\omega_0}=*(P\alpha\wedge\omega_0)$, which follows from (3.12), we have

$$P\big((*P\alpha,\omega_0)_{\omega_0}\omega_0\big)=P\big(*(P\alpha\wedge\omega_0)\wedge\omega_0\big)=-\sqrt{-1}*\bar\partial^*\partial^*(P\alpha\wedge\omega_0)\wedge\omega_0.$$

Applying the identities (3.15) further, we obtain

$$\partial^*(P\alpha\wedge\omega_0)=\partial^*P\alpha\wedge\omega-\sqrt{-1}\bar\partial P\alpha=\partial^*P\alpha\wedge\omega$$

and

$$\bar\partial^*\partial^*\big(P\alpha\wedge\omega_0\big)=\bar\partial^*\big(\partial^*P\alpha\wedge\omega_0\big)=\bar\partial^*\partial^*P\alpha\wedge\omega_0+\sqrt{-1}\partial\partial^*P\alpha\wedge\omega_0.$$

Put together, we obtain

$$\begin{aligned}P\big((*P\alpha,\omega_0)_{\omega_0}\omega_0\big)&=-\sqrt{-1}*\bar\partial^*\partial^*\big(P\alpha\wedge\omega_0\big)\wedge\omega_0\\&=-\sqrt{-1}*\big(\bar\partial^*\partial^*P\alpha\wedge\omega_0+\sqrt{-1}\partial\partial^*P\alpha\wedge\omega_0\big)\wedge\omega_0\\&=*(P^*P\alpha\wedge\omega_0)\wedge\omega_0+*\big(\partial\partial^*P\alpha\wedge\omega_0\big)\wedge\omega_0.\end{aligned}$$

Because $\partial\partial^*P\alpha=\Box_\partial P\alpha$ since $\partial P\alpha=0$, the operator U becomes

$$U(\alpha)=P^*P\alpha-*\big(*(P^*P\alpha\wedge\omega)\wedge\omega_0\big)-*\left(*\big(\Box_\partial P\alpha\wedge\omega_0\big)\wedge\omega_0\right). \tag{3.16}$$

To continue, recall that for $\nu\in\Omega^{1,1}_{\mathbb{R}}(X)$ such that $(\nu,\omega_0)_{\omega_0}=0$, $*(\nu\wedge\omega_0)=-\nu$. Hence,

$$\begin{aligned}*(\nu\wedge\omega_0)&=*((\nu-\frac{1}{3}(\nu,\omega_0)_{\omega_0}\omega_0)\wedge\omega_0)+\frac{1}{3}*((\nu,\omega_0)_{\omega_0}{\omega_0}^2)\\&=-\nu+(\nu,\omega_0)_{\omega_0}\omega_0\end{aligned}$$

and

$$*(*(\nu\wedge\omega_0)\wedge\omega_0)=*\left(-\nu\wedge\omega_0+(\nu,\omega_0)_{\omega_0}*{\omega_0}^2\right)=\mu+(\nu,\omega_0)_{\omega_0}\omega_0.$$

Therefore, by (3.16),

$$\begin{aligned}U\alpha&=P^*P\alpha-\big(P^*P\alpha+(P^*P\alpha,\omega_0)_{\omega_0}\omega_0\big)-*\big(*\Box_\partial P\alpha\wedge\omega_0\big)\\&=-*\left(*\Box_\partial P\alpha\wedge\omega_0\right)-(P^*P\alpha,\omega_0)_{\omega_0}\omega_0.\end{aligned}$$

Now, we are ready to derive the estimate that for a universal constant C (in the sense that it only depends on (X,ω_0)),

(3.17)
$$C^{-1}\|T\circ S^{-1}\phi\|\le\|P\alpha\|_{L^p_k}+\|dd^*\psi\|_{L^p_k}\le C\|T\circ S^{-1}\phi\|,\quad\forall\phi\perp\mathbb{H}.$$

First, note that the first inequality holds because $T\circ S^{-1}$ is a bounded operator. As to the second, because $d^*d(d^*\psi)=\Box_\partial d^*\psi$ and that $d^*\psi$ is orthogonal to the harmonic forms, the elliptic estimate ensures that for a universal constant C_1,

$$\|d^*\psi\|_{L^p_{k+1}} \le C_1\|\Box_\partial d^*\psi\|_{L^p_{k-1}} \le C_1\|T\circ S^{-1}\phi\|.$$

Then, because

$$P\circ S^{-1}(dd^*\psi) = -\frac{1}{2}P(dd^*\psi,\omega_0)_{\omega_0}\omega_0$$

and because the right-hand side involves the third differentiation of $d^*\psi$,

$$\|P\circ S^{-1}(dd^*\psi)\|_{L^p_{k-2}} \le C_2\|d^*\psi\|_{L^p_{k+1}} \le C_1C_2\|T\circ S^{-1}\phi\|$$

holds for a universal constant C_2. On the other hand,

$$\frac{1}{2}*U\alpha = T\circ S^{-1}\phi + \frac{1}{2}P\circ S^{-1}(dd^*\psi,\omega_0) - d^*d(d^*\psi), \tag{3.18}$$

the previous estimates ensure that there is a universal constant C_3 so that

$$\|U\alpha\|_{L^p_{k-2}} \le C_3\|T\circ S^{-1}\phi\|. \tag{3.19}$$

Because

$$d*(*\Box_\partial P\alpha\wedge\omega_0)=0,$$

the formula of $U\alpha$ before (3.17) gives

$$d(P^*P\alpha,\omega_0)_{\omega_0}\wedge\omega_0 = d(U\alpha). \tag{3.20}$$

Combined with

$$\int_X (P^*P\alpha,\omega_0)_{\omega_0}*1 = \int_X (P\alpha,P\omega_0)_{\omega_0}*1 = 0,$$

and that wedging ω forms an isomorphism from $\Omega^{1,1}_{\mathbb{R}}(X)$ to $\Omega^{2,2}_{\mathbb{R}}(X)$ whose inverse has bounded norm, (3.20) and (3.19) implies that

$$\|(P^*P\alpha,\omega_0)_{\omega_0}\|_{L^p_{k-2}} \le C_4\|U\alpha\|_{L^p_{k-2}} \le C_3C_4\|T\circ S^{-1}\phi\|.$$

Thus, for a universal constant C_5,

$$\|\Box_\partial P\alpha\|_{L^p_{k-2}} \le C_5\|T\circ S^{-1}\phi\|.$$

Finally, because $\Box_\partial$ is elliptic,

$$\|P\alpha\|_{L^p_k} \le C_6\|T\circ S^{-1}\phi\|.$$

This proves that the second inequality in (3.17) holds for a universal constant C.

It remains to show that $T \circ S^{-1}$ is surjective. Because d^* surjects onto $R(d^*)$, we only need to verify that restricting to $\ker d^* \cap \Omega^{1,1}_{\mathbb{R}}(X)_{L^p_k}$ the operator $T \circ S^{-1}$ surjects onto $R(dd_c)_{L^p_{k-2}}$. Because

$$R(*dd_c)_{L^p_k} \subset \ker d^* \cap \Omega^{1,1}_{\mathbb{R}}(X)_{L^p_k},$$

it suffices to show that

$$* \, T \circ S^{-1}(*P(\cdot)) = \frac{1}{2} U(\cdot) : R(*dd_c)_{L^p_k} \longrightarrow R(*dd_c)_{L^p_{k-4}} \tag{3.21}$$

is surjective. For this, we note that the estimates derived so far show that (3.21) is injective and has closed range. Hence, if we can show that it is self-adjoint, it must be surjective as well. We now show that U is self-adjoint. Obviously, the first term P^*P appeared in U in (3.14) is self-adjoint. As to the second term, we observe that the L^2-intersection

$$\begin{aligned}\langle *P\big((*P\alpha, \omega_0)_{\omega_0}\omega_0\big), \beta\rangle &= \langle (*P\alpha, \omega_0)_{\omega_0} * \omega_0, P\beta\rangle \\ &= \int_X (*P\alpha, \omega_0)_{\omega_0}(*P\beta, \omega_0)_{\omega_0} *1.\end{aligned}$$

Because both α and β are real, the above expression is symmetrical in α and β. Therefore, the operator U is self-adjoint, and hence, is surjective.

We are ready to prove the Proposition now. By a change of trivialization of $\mathbb{C}^{\oplus r}_X$, we can assume without lose of generality that $T = I_{r\times r}$; thus, $H_T = I$. We next let $\mathfrak{Her}^0 E$ be the $\mathbb{R}$-sub-vector bundle of $\operatorname{End} E$ consisting of traceless pointwise $\langle\, ,\rangle$-hermitian symmetric endomorphisms of E. Clearly, $T_I\mathcal{H}(E)_{1,L^p_k} = \Omega^0(\mathfrak{Her}^0 E)_{L^p_k}$. We now define linear operators

$$\mathbf{T}_1,\ \mathbf{T}_2 : \Omega(\mathfrak{Her}^0 E)_{L^p_k} \oplus \Omega^{1,1}_{\mathbb{R}}(X)_{L^p_k} \longrightarrow \mathcal{W}_2$$

that are

$$\mathbf{T}_1(\delta h, \delta\omega) = \big(P\delta\omega, 2d^*_{\omega_0}\delta\omega - d^*_{\omega_0}\big((\delta\omega, \omega_0)_{\omega_0}\omega_0\big)\big)$$

and

$$\mathbf{T}_2(\delta h, \delta\omega) = 2\operatorname{tr}(\delta F_I(\delta h) \wedge F_I) - 2\operatorname{tr}(\delta R_{\omega_0}(\delta g) \wedge R_{\omega_0}).$$

Because

$$\begin{aligned}\delta\mathbf{L}_1(I, c\omega_0)(\delta h, c\delta\omega) &= c^2\delta\mathbf{L}_1(I, \omega_0)(\delta h, \delta\omega);\\ \delta\mathbf{L}_2(I, c\omega_0)(\delta h, c\delta\omega) &= \sqrt{-1}\,\partial\bar\partial c\delta\omega + 2\operatorname{tr}(\delta F_I(\delta h) \wedge F_I)\\ &\quad - 2\operatorname{tr}(\delta R_{c\omega_0}(c\delta\omega) \wedge R_{c\omega_0})\\ &= cP\delta\omega + 2\operatorname{tr}(\delta F_I(\delta h) \wedge F_I)\\ &\quad - 2\operatorname{tr}(\delta R_{\omega_0}(\delta\omega) \wedge R_{\omega_0}),\end{aligned}$$

and

$$\begin{aligned}
&\delta\mathbf{L}_3(I,c\omega_0)(\delta h,c\delta\omega)=2d^*c\delta\omega-d^*\big((c\delta\omega,\omega_0)_{\omega_0}\omega_0\big)\\
&\mathbf{P}\circ\delta\mathbf{L}_1(I,c\omega_0)\oplus\delta\mathbf{L}_2(I,c\omega_0)\oplus\delta\mathbf{L}_3(I,c\omega_0)\\
&\qquad=c^2\mathbf{P}\circ\delta\mathbf{L}_1(I,\omega_0)\oplus c(\mathbf{T}_1+c^{-1}\mathbf{T}_2).
\end{aligned}\tag{3.22}$$

Hence, to prove the Proposition, we need to show that the right-hand side is surjective. Based on the discussion before,

$$\mathbf{P}\circ\delta\mathbf{L}_1(I,\omega_0)(\delta h,0)=\mathbf{P}\circ\mathbf{F}(\delta h):\Omega^0(\mathfrak{Her}^0E)_{L^p_k}\longrightarrow\mathcal{W}_1/V_1$$

is surjective and its kernel is V_0. Also, we proved that

$$\mathbf{T}_1:\Omega^{1,1}_{\mathbb{R}}(X)_{L^p_k}\longrightarrow\mathcal{W}_2,$$

which is the operator T discussed before, is surjective with kernel $\mathbb{H}\subset\Omega^{1,1}_{\mathbb{R}}(X)$.

Now, let $\mathcal{V}\subset\Omega^0(\mathfrak{Her}^0E)_{L^p_k}\oplus\Omega^{1,1}_{\mathbb{R}}(X)_{L^p_k}$ be the orthogonal complement of $V_1\oplus\mathbb{H}$. For simplicity, we abbreviate $\mathbf{T}_0=\mathbf{P}\circ\delta\mathbf{L}_1(I,\omega_0)$. The discussion before shows that

$$(\mathbf{T}_0\oplus\mathbf{T}_1)|_{\mathcal{V}}:\mathcal{V}\longrightarrow\mathcal{W}_1/V_1\oplus\mathcal{W}_2$$

is surjective and that there is a constant C so that

$$\begin{aligned}
C^{-1}\|(u_1,u_2)\|&\le\big\|\big(\mathbf{T}_0(u_1,u_2),\mathbf{T}_1(u_1,u_2)\big)\big\|\\
&\le C\|(u_1,u_2)\|,\qquad(u_1,u_2)\in\mathcal{V}.
\end{aligned}\tag{3.23}$$

Because $\mathbf{T}_2$ is a bounded operator, for sufficiently large c,

$$\mathbf{T}_0\oplus(\mathbf{T}_1+c^{-1}\mathbf{T}_2):\Gamma(\mathrm{End}^0_{\mathrm{h}}E)_{L^p_k}\times\Omega^{1,1}_{\mathbb{R}}(X)\longrightarrow\mathcal{W}_1/V_1\oplus\mathcal{W}_2$$

is surjective. In particular, the left-hand side of (3.22) is surjective. This proves the Proposition. q.e.d.

4. Irreducible solutions to Strominger's system

In section two, assuming the existence of a non-degenerate deformation of holomorphic structures of the vector bundle $E_1\oplus E_2$, we showed how to use perturbation method to prove the existence of the Hermitian–Yang–Mills connections. In this section, we will construct solutions to Strominger's system using similar method. We will find an initial trivial solution to the Strominger's system and show that it can be extended to a family of irreducible solutions.

We continue to work with a Calabi–Yau threefold (X,ω_0) and the vector bundle

$$(E,D_0'')=\mathbb{C}_X^{\oplus r}\oplus TX;$$

we fix a smooth isomorphism $\wedge^{r+3} E \cong \mathbb{C}_X$ so that the D_0'' induces the standard holomorphic structure on $\mathbb{C}_X$; we let D_s'' be a smooth deformation of the holomorphic structure D_0''. As in section two, we write

$$D_s'' = D_0'' + A_s, \qquad A_s \in \Omega^{0,1}(\operatorname{End} E)$$

and write

$$\dot{A}_0 = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \in \Omega^{0,1}(\operatorname{End} E)$$

according to the decomposition $E = \mathbb{C}_X^{\oplus r} \oplus TX$. Because of Lemma 2.2, we can assume without lose of generality that C_{ij} are D_0''-harmonic. Since $E_1 = \mathbb{C}_x^{\oplus r}$ and $H^1_{\bar\partial}(X, \mathbb{C}_X) = 0$,

$$C_{11} = 0. \tag{4.1}$$

Because $\operatorname{Pic} X$ is discrete, we can assume further that $\operatorname{tr} A_s = 0$ for all s. This means that under given smooth isomorphism $\wedge^{r+3} E \cong \mathbb{C}_X$, the induced holomorphic structure on $\wedge^{r+3} E$ is the standard holomorphic structure on $\mathbb{C}_X$.

Next, we let H_1 be the standard constant metric on $\mathbb{C}_X^{\oplus r}$ and let H_2 be induced by the Calabi–Yau metric ω_0 normalized so that $\det(H_1 \oplus H_2)$ is the constant one metric on $\wedge^{r+3} E \cong \mathbb{C}_X$. The pair of $\langle\, , \rangle = H_1 \oplus H_2$ and ω_0 is a trivial solution of the Strominger's system on (E, D_0''). We fix such $\langle\, , \rangle$ as a reference hermitian metric on E. Following the convention in the previous section, all other determinant one hermitian metrics on E are of the forms $\langle \cdot, \cdot \rangle_H = \langle \cdot\, H, \cdot \rangle$ for some determinant one pointwise positive definite $\langle\, , \rangle$-hermitian symmetric endomorphisms of E.

Following this convention, the space of all trivial solutions to Strominger's system on (E, D_0'') with Kahler form ω_0 is isomorphic to the space of determinant one positive definite $r \times r$ hermitian symmetric matrices T with the correspondence

$$T \in \mathcal{H}^+_{r\times r} \longmapsto H_T = T \oplus |T|^{-1/3} I_2 \in \mathcal{H}(E)_1.$$

With the chosen Kahler form ω_0 and a hermitian metric H_T, the proposition 3.3 says that for V_1 the cokernel defined in (3.11) and for large enough c, the linearized operator $\delta \mathbf{L}$ at $(H_T, c\omega_0)$ surjects onto

$$\Omega^6_{\mathbb{R}}(\mathfrak{su} E)_{L^p_{k-2}} / V_1 \oplus R(dd_c)_{L^p_{k-2}} \oplus R(d^*_{\omega_0})_{L^p_{k-1}}. \tag{4.2}$$

With the connection forms A_s, the metric $\langle\, , \rangle$ and the Kahler form ω_0 so chosen, we can now define operators

$$\mathbf{L}_s = \mathbf{L}_{s,1} \oplus \mathbf{L}_{s,2} \oplus \mathbf{L}_{s,3}$$

between

$$\mathcal{H}(E)_{1, L^p_k} \times \mathcal{H}(X)_{L^p_k} \longrightarrow \Omega^6_{\mathbb{R}}(\mathfrak{su} E)_{L^p_{k-2}} \oplus R(dd_c)_{L^p_{k-2}} \oplus R(d^*_{\omega_0})_{L^p_{k-1}}$$

with $\mathbf{L}_{s,i}$ defined as in (3.4)–(3.6) of which the curvature form F_H is replaced by the hermitian curvature of (E, D''_s, H):

$$F_{s,H} = D_{s,H} \circ D_{s,H}.$$

Let $\mathbf{P}$ be the projection from

$$\Omega^6_{\mathbb{R}}(X)(\mathfrak{su}E)_{L^p_{k-2}} \oplus R(dd_c)_{L^p_{k-2}} \oplus R(d^*_{\omega_0})_{L^p_{k-1}}$$

to (4.2) and let $\mathcal{H}_\omega(X)_{L^p_k}$ be the space of those L^p_k-hermitian forms whose ω_0-harmonic parts are ω.

Lemma 4.1. *For any $T_0 \in \mathcal{H}^+_{r\times r}$, there are constants $a > 0$ and $C > 0$ such that for any $c > C$, there is a neighborhood $\mathcal{U}_c$ of $(H_{T_0}, c\omega_0) \in \mathcal{H}(E)_{1,L^p_k} \times \mathcal{H}_{c\omega_0}(X)_{L^p_k}$ such that for each $s \in [0,a)$, the set $\mathcal{S}_s = \big(\mathbf{P} \circ \mathbf{L}_s\big)^{-1}(0) \cap \mathcal{U}_c$ is a smooth r^2-dimensional manifold and that the union*

$$\mathcal{S} = \coprod_{s\in[0,a)} \mathcal{S}_s \times s \subset \mathcal{U}_c \times [0,a) \tag{4.3}$$

is a smooth (r^2+1)-dimensional manifold.

Proof. By proposition 3.3, there is a $C > 0$ such that the linearized operator of $\mathbf{P} \circ \mathbf{L}_0$ is surjective at $(H_{T_0}, c\omega_0)$. Hence, by the implicit theorem, for sufficiently small s, the solution set to $\mathbf{P} \circ \mathbf{L}_s = 0$ is smooth near $(H_{T_0}, c\omega_0)$ and has dimension equal to the index of the linear operator $\mathbf{P} \circ \delta\mathbf{L}_0$, which is $r^2 + \dim H^{1,1}(X, \mathbb{R})$. By restricting to the slice

$$\mathcal{H}(E)_{1,L^p_k} \times \mathcal{H}_{c\omega_0}(X)_{L^p_k} \subset \mathcal{H}(E)_{1,L^p_k} \times \mathcal{H}(X)_{L^p_k}$$

that is transversal to the kernel of $\mathbf{P} \circ \delta\mathbf{L}_0$, the solution set $\mathcal{S}_s$ will have the property as stated in the Lemma. This proves the Lemma. q.e.d.

Following our convention, $\mathcal{S}_0$ consists of all pairs

$$(H_{0,T}, \omega_{0,T}); \quad H_{0,T} = T \oplus |T|^{-1/3} I_2, \quad \omega_{0,T} = c\,\omega_0. \tag{4.4}$$

Since $\mathcal{S}_s$ and $\mathcal{S}$ are smooth, by shrinking $\mathcal{U}_c$ if necessary, we can parameterize $\mathcal{S}$ smoothly by (s,T) so that (s,T) parameterizes the set $\mathcal{S}$ that is consistent with the projection $\mathcal{S} \to [0,a)$ and the parameterization (4.4). By shrinking $\mathcal{U}_c$ if necessary, we can assume that under this parameterization, $\mathcal{S} \cong [0,a) \times B_\epsilon(T_0)$, where $B_\epsilon(T_0)$ is the ball of radius ϵ centered at T_0 in $\mathcal{H}^+_{r\times r}$. In the following, we denote by

$$(H_{s,T}, \omega_{s,T}) \in \mathcal{S}_s, \quad T \in B_\epsilon(T_0),$$

the solutions with parameters (s,T). For simplicity, we denote by $F_{s,T}$ the curvature of the hermitian vector bundle $(E, D''_s, H_{s,T})$. By our construction, it satisfies

$$\mathbf{L}_{s,1}(H_{s,T},\omega_{s,T})\equiv 0 \mod V_1,\quad \mathbf{L}_{s,2}(H_{s,T},\omega_{s,T})=0 \qquad \text{and}$$
$$\mathbf{L}_{s,3}(H_{s,T},\omega_{s,T})=0.$$

Hence, to find solutions to $\mathbf{L}_s=0$, it suffices to investigate the vanishing loci of the functional $\mathbf{r}(s,\cdot)$ from $B_\epsilon(T_0)$ to the Lie algebra $\mathfrak{u}(r)$ defined by

$$\mathbf{r}(s,T)=\int_X \big[\mathbf{L}_{s,1}(H_{s,T},\omega_{s,T})\big]_1, \tag{4.5}$$

where $[\cdot]_1$ is the projection from $\Omega^\bullet_{\mathbb{R}}\big(\mathfrak{s}uE\big)$ to $\Omega^\bullet_{\mathbb{R}}\big(\mathfrak{u}(\mathbb{C}_X^{\oplus r})\big)$. Here, $\mathfrak{u}(\mathbb{C}_X^{\oplus r})$ is the bundle of $\langle\,,\rangle$-hermitian antisymmetric endomorphisms of $\mathbb{C}_X^{\oplus r}$.

As in section two, we shall first prove $\dot{\mathbf{r}}(0,T)=0$ for all T. Indeed,

$$\begin{aligned}\dot{\mathbf{r}}(0,T)=&\int_X T^{-1/2}\big[\dot F_{0,T}\big]_1 T^{1/2}\wedge\omega_{0,T}^2\\ &+2\int_X T^{-1/2}\big[F_{0,T}\big]_1 T^{1/2}\wedge\omega_{0,T}\wedge\dot\omega_{0,T}.\end{aligned}\tag{4.6}$$

Because $H_{0,T}$ is a direct sum of a flat metric on $\mathbb{C}_X^{\oplus r}$ and a metric on TX, under the decomposition $E=\mathbb{C}_X^{\oplus 2}\oplus TX$,

$$F_{0,T}=\begin{pmatrix}0&0\\0&*\end{pmatrix}\in\Omega^{1,1}_{\mathbb{R}}(\mathfrak{s}uE).\tag{4.7}$$

What we will actually show is that

$$\dot F_{0,T}\wedge\omega_{0,T}^2=\begin{pmatrix}0&0\\0&*\end{pmatrix}\in\Omega^{6}_{\mathbb{R}}(\mathfrak{s}uE).$$

Since $(H_{s,T},\omega_{s,T})$ are solutions to $\mathbf{L}_s=0 \mod V_1$, there is a function $\mathbf{c}(s,T)$ taking values in V_1 with $\mathbf{c}(0,T)=0$ so that

$$F_{s,T}\wedge\omega_{s,T}^2=H_{s,T}^{1/2}\mathbf{c}(s,T)H_{s,T}^{-1/2}.$$

Taking derivative of s at $s=0$, and coupled with $\mathbf{c}(0,T)=0$, we obtain

$$\dot F_{0,T}\wedge\omega_{0,T}^2+2F_{0,T}\wedge\omega_{0,T}\wedge\dot\omega_{0,T}=H_{0,T}^{1/2}\dot{\mathbf{c}}(0,T)H_{0,T}^{-1/2},\tag{4.8}$$

which, after projecting to $\Omega^6_{\mathbb{R}}(\mathfrak{u}(\mathbb{C}_X^{\oplus r}))$, becomes

$$\big[\dot F_{0,T}\wedge\omega_{0,T}^2\big]_1=T^{1/2}\dot{\mathbf{c}}(0,T)T^{-1/2}.\tag{4.9}$$

Next, we let F_s as in (2.4) be the curvature of (E,D''_s,I). Because

$$F_{s,T}=F_s+D''_s(D'_sH_{s,T}\cdot H_{s,T}^{-1}),$$

because $D_0' H_{0,T} = 0$, and because D_s is a direct sum of a flat connection on $\mathbb{C}_X^{\oplus r}$ and a Hermitian Yang–Mills connection on TX,

$$[\dot F_{0,T}]_1 = [\dot F_0]_1 + D_0''[\dot D_0' H_{0,T}\cdot H_{0,T}^{-1}]_1 + D_0''[D_0'\dot H_{0,T}\cdot H_{0,T}^{-1}]_1.$$

Using the expression of F_s in (2.4), and that $C_{11}=0$ as stated in (4.1),

$$[\dot F_0]_1 = D_0' C_{11} - D_0'' C_{11}^* = 0. \tag{4.10}$$

Hence,

$$\big[\dot F_{0,T}\big]_1 = D_0'\varphi_1 + D_0''\varphi_2$$

for some sections φ_1 and φ_2. Therefore, by Lemma 2.3

$$\int_X T^{1/2}\dot{\mathbf{c}}(0,T)T^{-1/2} = \int_X \big[\dot F_{0,T}\big]_1\wedge\omega_{0,T}^2 = \int_X \big(D_0'\varphi_1 + D_0''\varphi_2\big)\wedge\omega_{0,T}^2 = 0.$$

Since $\dot{\mathbf{c}}(0,T)/\omega_0^3$ is a constant section of $\mathrm{End}(\mathbb{C}_X^{\oplus r})$, the above vanishing forces $\dot{\mathbf{c}}(0,T)=0$, which simplifies (4.8) to

$$\dot F_{0,T}\wedge\omega_{0,T}^2 + 2F_{0,T}\wedge\omega_{0,T}\wedge\dot\omega_{0,T} = 0.$$

Finally, because $F_{0,T}$ has vanishing entries as shown in (4.7),

$$\dot F_{0,T}\wedge\omega_{0,T}^2 = \begin{pmatrix} 0 & 0\\ 0 & * \end{pmatrix} \in \Omega^6_{\mathbb{R}}(\mathfrak{su}E). \tag{4.11}$$

The vanishing (4.6) follows from (4.7) and (4.11).

We next compute $\ddot{\mathbf{r}}(0,T)$. First, because $F_{0,T}=0$,

$$\big(H_{s,T}^{-1/2}F_{s,T}H_{s,T}^{1/2}\cdot\frac{d^2}{ds^2}\omega_{s,T}^2\big)|_{s=0} = 0$$

Because $\big[\dot F_{0,T}\big]_1 = 0$,

$$\big[\frac{d}{ds}\big(H_{s,T}^{-1/2}F_{s,T}H_{s,T}^{1/2}\big)\wedge\frac{d}{ds}\big(\omega_{s,T}^2\big)|_{s=0}\big]_1 = 0.$$

Hence,

$$\big[\frac{d^2}{ds^2}\big(H_{s,T}^{-1/2}F_{s,T}H_{s,T}^{1/2}\wedge\omega_{s,T}^2\big)|_{s=0}\big]_1 = \big[\frac{d^2}{ds^2}\big(H_{s,T}^{-1/2}F_{s,T}H_{s,T}^{1/2}\big)|_{s=0}\wedge\omega_{0,T}^2\big]_1.$$

Taking second order derivative of $H_{s,T}^{-1/2}F_{s,T}H_{s,T}^{1/2}$, we will encounter terms like

$$\frac{d^2}{ds^2}\big(H_{s,T}^{-1/2}\big)|_{s=0}\cdot F_{0,T}\cdot H_{0,T}^{1/2},$$

which are all zero because $F_{0,T}=0$. We will also encounter terms like

$$\frac{d}{ds}\big(H_{s,T}^{-1/2}\big)|_{s=0}\cdot \dot F_{0,T}\cdot H_{0,T}^{1/2};$$

after wedging it with $\omega_{0,T}^2$, because $H_{0,T}^{1/2}$ is diagonal and $\dot F_{0,T}\wedge\omega_{0,T}^2$ has vanishing shown in (4.11), their projections to $\Omega^6_{\mathbb{R}}(\mathfrak{u}(\mathbb{C}_X^{\oplus r}))$ are zero also. Hence, the only term left is

$$\Big[\frac{d^2}{ds^2}\big(H_{s,T}^{-1/2}F_{s,T}H_{s,T}^{1/2}\wedge\omega_{s,T}^2\big)|_{s=0}\Big]_1 = H_{s,T}^{-1/2}\ddot F_{0,T}H_{0,T}^{1/2}\wedge\omega_{0,T}^2.$$

As in the previous section, we compute

$$\begin{aligned}\int\big[H_{0,T}^{-1/2}\ddot F_{0,T}H_{0,T}^{1/2}\big]_1\wedge\omega_{0,T}^2\\ =\int_X T^{-1/2}[\ddot F_0]_1T^{1/2}\wedge\omega_{0,T}^2\\ -2\int_X T^{-1/2}\big[[\dot A_0,[\dot A_0^*,H_{0,T}]H_{0,T}^{-1}]\big]_1T^{1/2}\wedge\omega_{0,T}^2\\ -2\int_X T^{-1/2}\big[[\dot A_0,D_0'\dot H_{0,T}\cdot H_{0,T}^{-1}]\big]_1T^{1/2}\wedge\omega_{0,T}^2\\ +\int_X T^{-1/2}\big[D_0''\Phi_T\big]_1T^{1/2}\wedge\omega_{0,T}^2\end{aligned}$$

for some form $\Phi_{0,T}$. We now look at the four terms in the above identity: the last term vanishes because of Lemma 2.3; the next-to-last term is

$$\begin{aligned}-2\int_X T^{-1/2}\big[\dot A_0\cdot D_0'\dot H_{0,T}\cdot H_{0,T}^{-1}\big]_1T^{1/2}\wedge\omega_{0,T}^2\\ +2\int_X T^{-1/2}\big[D_0'\dot H_{0,T}\cdot H_{0,T}^{-1}\cdot\dot A_0\big]_1T^{1/2}\wedge\omega_{0,T}^2,\end{aligned}$$

which is zero because $D_0''^*\dot A_0=0$ and Lemma 2.3. Using

$$\dot A_0=\begin{pmatrix}0 & C_{12}\\ C_{21} & C_{22}\end{pmatrix}\quad\text{and}\quad H_{0,T}=\begin{pmatrix}T & 0\\ 0 & \alpha I_2\end{pmatrix},\qquad \alpha=|T|^{-1/3}$$

one computes

$$\big[[\dot A_0,D_0'\dot H_{0,T}\cdot H_{0,T}^{-1}]\big]_1=C_{12}\wedge C_{12}^*\big(I_1-\alpha T^{-1}\big)+\big(I_1-\alpha^{-1}T\big)C_{21}^*\wedge C_{21}.$$

For the same reason,

$$\begin{aligned}\big[\ddot F_0\big]_1\wedge\omega_{0,T}^2&=\big[2\dot A_0\wedge\dot A_0^*+2\dot A_0^*\wedge\dot A_0\big]_1\wedge\omega_{0,T}^2\\ &=2\big(C_{12}\wedge C_{12}^*+C_{21}^*\wedge C_{21}\big)\wedge\omega_{0,T}^2.\end{aligned}$$

Therefore,

$$\begin{aligned}&\ddot{\mathbf r}(0,T)\\ &\quad=2\int_X\big(\alpha T^{-1/2}C_{12}\wedge C_{12}^*T^{-1/2}+\alpha^{-1}T^{1/2}C_{21}^*\wedge C_{21}T^{1/2}\big)\wedge\omega_{0,T}^2.\end{aligned}$$

We now investigate the solvability of $(s, T) = 0$ for small s. For this, we need to make an assumption on the class C_{12} and C_{21}. Recall that C_{12} is a column vector $[\alpha_1, \cdots, \alpha_r]^t$ whose components are D_0''-harmonic

$$\alpha_i \in \Omega^{0,1}\big(TX^\vee \otimes \mathbb{C}_X\big);$$

C_{21} is a row vector $[\beta_1, \cdots, \beta_r]$ whose components are D_0''-harmonic

$$\beta_i \in \Omega^{0,1}\big(\mathbb{C}_X^\vee \otimes TX\big).$$

Since both α_i and β_i are (1,0)-forms,

$$\sqrt{-1}\, B = \sqrt{-1} \int_X C_{12} \wedge C_{12}^* \wedge \omega_{0,T}^2 \qquad \text{and} \tag{4.12}$$

$$\sqrt{-1}\, B' = -\sqrt{-1} \int_X C_{21}^* \wedge C_{21} \wedge \omega_{0,T}^2$$

are non-negative definite hermitian symmetric matrices. Because α_i are D_0''-harmonic, $\sqrt{-1}\, B$ is positive definite if and only if $[\alpha_1], \cdots, [\alpha_r]$ are linearly independent elements in $H^1_{\bar\partial}(TX^\vee)$. Similarly, $\sqrt{-1}\, B'$ is positive definite if $[\beta_1], \cdots, [\beta_r]$ are linearly independent in $H^1_{\bar\partial}(X, TX)$. Hence, the positivity of $\sqrt{-1}\, B$ and $\sqrt{-1}\, B'$ only depend on the Kodaira–Spencer class $\kappa \in H^1_{\bar\partial}(X, E^\vee \otimes E)$.

We now assume that *both matrices* $\sqrt{-1}\, B$ *and* $\sqrt{-1}\, B'$ *are positive definite*. By a GL(r,$\mathbb{C}$) change of basis of $\mathbb{C}_X^{\oplus r}$, we can assume that $\sqrt{-1}\, B' = I_{r\times r}$. Then,

$$\ddot{\mathbf{r}}(0, T) = 2|T|^{-1/3} T^{-1/2} B T^{-1/2} + 2\sqrt{-1}\, |T|^{1/3} T \in \mathfrak{u}(r).$$

Clearly, $\ddot{\mathbf{r}}(0, T) = 0$ if T is

$$T_0 \triangleq |\sqrt{-1}\, B_1|^{1/2(r+3)} \big(\sqrt{-1}\, B\big)^{1/2}.$$

Lemma 4.2. *The map* $\Phi : \partial B_\epsilon(T_0) \to S(1)$ *to the unit sphere* $S(1) \subset \mathfrak{u}(r)$ *defined by*

$$\Phi(T) = \frac{\ddot{\mathbf{r}}(0, T)}{\|\ddot{\mathbf{r}}(0, T)\|}$$

is a degree one map.

Proof. We define

$$\mathbf{u}_t(T) = |T|^{-1/3} \big(tT + (1-t)I\big)^{-1/2} \big(B + \sqrt{-1}\, |T|^{1/3} T^2\big) \big(tT + (1-t)I\big)^{-1/2}$$

and consider

$$\frac{\mathbf{u}_t(\cdot)}{\|\mathbf{u}_t(\cdot)\|} : \partial B_\epsilon(T_0) \to S(1).$$

It is well-defined since T and I are positive definite; it is Φ when $t = 1$. Hence, it provides a homotopy between Φ and

$$\Phi_1(\cdot) = \frac{\mathbf{u}_0(\cdot)}{\|\mathbf{u}_0(\cdot)\|} : \partial B_\epsilon(T_0) \to S(1).$$

Next, we consider

$$\mathbf{v}_t(T) = \mathbf{B} + \sqrt{-1}\,\big((1-t)|T|^{2/3} + t|T_0|^{2/3}\big)T^2.$$

We claim that $\mathbf{v}_t(T) \neq 0$ for all $t \in [0,1]$. Suppose for some $t_0 \in [0,1]$ and $T \in \partial B_\epsilon(T_0)$,

$$\mathbf{B} + \sqrt{-1}\,\big((1-t_0)|T|^{2/3} + t_0|T_0|^{2/3}\big)T^2 = 0,$$

then, $T = \eta(\sqrt{-1}\,B)^{1/2}$ for some $\eta \in \mathbb{R}^+$. Since $T \in \partial B_\epsilon(T_0)$, η satisfies

$$\|T - T_0\| = |\eta - |\sqrt{-1}\,B|^{-1/2(r+3)}|\,\|(\sqrt{-1}\,B)^{1/2}\| = \epsilon.$$

Hence, η can only take values

$$\eta_\pm = |\sqrt{-1}\,B|^{-1/2(r+3)} \pm \epsilon', \quad \epsilon' = \epsilon/\|(\sqrt{-1}\,B)^{1/2}\|.$$

But then $|\eta_+(\sqrt{-1}\,B)^{1/2}| > |\sqrt{-1}\,B|^{3/2(r+3)} = |T_0|$; and then,

$$\begin{aligned}&\big(t_0|\eta_+(\sqrt{-1}\,B)^{1/2}|^{2/3} + (1-t_0)|T_0|^{2/3}\big)\eta_+^2\\&\quad > \big(t_0|T_0|^{2/3} + (1-t_0)|T_0|^{2/3}\big)\big(|\sqrt{-1}\,B|^{-1/2(r+3)} + \epsilon'\big)^2 > 1.\end{aligned}$$

Hence, $\mathbf{v}_{t_0}(\eta_+(\sqrt{-1}\,B)^{1/2}) \neq 0$. Similarly, $\mathbf{v}_{t_0}(\eta_-(\sqrt{-1}\,B)^{1/2}) \neq 0$. This proves that

$$\frac{\mathbf{v}_t(\cdot)}{\|\mathbf{v}_t(\cdot)\|} : \partial B_\epsilon(T_0) \longrightarrow S(1)$$

are well-defined and is a homotopy between Φ_1 and

$$\Phi_2 : \partial B_\epsilon(T_0) \longrightarrow S(1); \quad \Phi_2(T) = \frac{\mathbf{B} + \sqrt{-1}\,|T_0|^{2/3}T^2}{\|\mathbf{B} + \sqrt{-1}\,|T_0|^{2/3}T^2\|}.$$

It remains to show that $\deg \Phi_2 = 1$. We write $T = T_0 + \epsilon\Delta T$ with ΔT varies in the unit sphere in the space of hermitian symmetric matrices $\mathcal{H}_{r\times r}$. Under this form the numerator of Φ_2 is

$$\begin{aligned}\mathbf{B} + \sqrt{-1}\,|T_0|^{2/3}(T_0 + \epsilon\Delta T)^2 &= \sqrt{-1}\,|T_0|^{2/3}\big(\Delta T T_0 + T_0\Delta T\big)\\&\quad + \epsilon^2|T_0|^{2/3}(\Delta T)^2.\end{aligned}$$

For ϵ small enough, the degree of Φ_2 is the same as the degree of

$$\Delta T \longmapsto \sqrt{-1}\,\frac{\Delta T T_0 + T_0\Delta T}{\|\Delta T T_0 + T_0\Delta T\|}, \tag{4.13}$$

which is the same as

$$\Delta T \longmapsto \sqrt{-1}\,\Delta T.$$

Because the map $\partial B_1(0) \subset \mathcal{H}_{r\times r} \to S(1) \subset \mathfrak{u}(r)$ by multiplying $\sqrt{-1}$ has degree one, the map Φ has degree one as well. This proves the Lemma. q.e.d.

We are now ready to prove the theorem

Theorem 4.3. *Let (X, ω_0) be a Calabi–Yau threefold; let D_s'' be a smooth deformation of the tautological holomorphic structure D_0'' on $E = \mathbb{C}_X^{\oplus r} \oplus TX$. Suppose the Kodaira–Spencer class $\kappa \in H^1_{\bar\partial}(X, E^\vee \otimes E)$ of the family D_s'' at $s = 0$ satisfies the non-degeneracy condition that both $\sqrt{-1}\,B$ and $\sqrt{-1}\,B'$ in (4.13) are positive definite. Then, for sufficiently large $c \in \mathbb{R}$ and small $a > 0$, there is a family of pairs of hermitian metrics and hermitian forms (H_s, ω_s), not necessarily continuous in $s \in [0, a)$, so that*

1. *the ω_0-harmonic part of ω_s is $c\omega_0$;*
2. *the pair (H_s, ω_s) is a solution to Strominger's system for the holomorphic vector bundle (E, D_s'');*
3. *$\lim_{s\to 0} \omega_s = c\omega_0$; $\lim_{s\to 0} H_s$ is a Hermitian Yang–Mills connection of E over (X, ω_0).*

Proof. First, we pick a basis of $\mathbb{C}_X^{\oplus r}$ so that the matrix $\sqrt{-1}\,B'$ in (4.13) is the identity matrix. We let B be the other matrix and let $T_0 = |\sqrt{-1}\,B|^{1/2(r+3)}(\sqrt{-1}\,B)^{1/2}$. By Lemma 4.1, we can choose C so that Lemma 4.1 holds for T_0 chosen. Then, for any $c > C$, we form solution set $\mathcal{S}_s$ of the system $\mathbf{P} \circ \mathbf{L}_s = 0$ and parameterize the solutions near $(H_{T_0}, c\omega_0)$ by $(s, T) \in [0, a) \times B_\epsilon(T_0)$. Based in this parameterization, we then form the functional $\mathbf{r}(s, T)$ in (4.5). Because $\dot{\mathbf{r}}(0, T) = 0$ and

$$\frac{\ddot{\mathbf{r}}(0, \cdot)}{\|\ddot{\mathbf{r}}(0, \cdot)\|} : \partial B_\epsilon(T_0) \longrightarrow S(1) \tag{4.14}$$

has degree one, for some small $0 < a' < a$ the maps

$$\mathbf{r}(s, \cdot) : \partial B_\epsilon(T_0) \longrightarrow \mathfrak{u}(r), \quad s \in (0, a')$$

does not take the value $0 \in \mathfrak{u}(r)$. Hence, the associated map

$$\frac{\mathbf{r}(s, \cdot)}{\|\mathbf{r}(s, \cdot)\|} : \partial B_\epsilon(T_0) \longrightarrow S(1) \subset \mathfrak{u}(r), \quad s \in (0, a'), \tag{4.15}$$

has the same degree as that of (4.14), which is one. Hence, the map

$$\mathbf{r}(s, \cdot) : B_\epsilon(T_0) \longrightarrow \mathfrak{u}(r), \quad s \in (0, a'),$$

attains value $0 \in \mathfrak{u}(r)$ for all $s \in (0, a')$ in $B_\epsilon(T_0)$. This proves the first two part of the theorem. The last part is true because we can choose ϵ arbitrarily small. q.e.d.

5. Irreducible Solutions on quintic threefolds

So far, we have derived a sufficient condition for the existence of irreducible solutions to Strominger's system. Our next step is to find examples that satisfy this condition. It is the purpose of this section to work out examples for SU(4) and SU(5).

We will first consider the Fermat quintic

$$X = \{z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0\} \subset \mathbf{P}^4;$$

we will find a deformation of the holomorphic structure of $\mathbb{C}_X \oplus TX$ and show that it satisfies the requirement of theorem 4.3. This will provide us SU(4) solutions to Strominger's system.

We begin with the Euler exact sequence of $T\mathbf{P}^4$ (the middle column), and the exact sequence relating TX and the restriction to X of the tangent bundle $T_X\mathbf{P}^4 = T\mathbf{P}^4|_X$ (the top row):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & TX & \xrightarrow{\varphi_1} & T_X\mathbf{P}^4 & \xrightarrow{\varphi_2} & \mathcal{O}_X(5) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & F & \longrightarrow & \mathcal{O}_X(1)^{\oplus 5} & \longrightarrow & \mathcal{O}_X(5) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{O}_X & = & \mathcal{O}_X & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{5.1}$$

We take F be the kernel of $\mathcal{O}_X(1)^{\oplus 5} \longrightarrow \mathcal{O}_X(5)$ and fill in the remainder entries to make up the exact diagram as shown above.

We claim that the left column in (5.1) is non-split. Assume not, say $F = TX \oplus \mathcal{O}_X$. Then, since F is a subsheaf of $\mathcal{O}_X(1)^{\oplus 5}$ with quotient sheaf $\mathcal{O}_X(5)$, $\mathcal{O}_X(1)^{\oplus 5}/TX$ must be locally free and an extension of $\mathcal{O}_X(5)$ by $\mathcal{O}_X$. Because $\mathrm{Ext}^1_X(\mathcal{O}_X(5), \mathcal{O}_X) = 0$, the only extension of $\mathcal{O}_X(5)$ by $\mathcal{O}_X$ is the direct sum $\mathcal{O}_X(5) \oplus \mathcal{O}_X$. Hence,

$$\mathcal{O}_X(1)^{\oplus 5}/TX \cong \mathcal{O}_X \oplus \mathcal{O}_X(5).$$

In particular, $\mathcal{O}_X$ becomes a quotient sheaf of $\mathcal{O}_X(1)^{\oplus 5}$ that is impossible. This proves that it does not split.

Next, we will construct a deformation of holomorphic structure of $\mathbb{C}_X \oplus TX$ so that its Kodaira–Spencer class is of the form

$$\kappa = \begin{pmatrix} 0 & 0 \\ \xi & 0 \end{pmatrix} \in \operatorname{Ext}^1_X(\mathcal{O}_X \oplus TX, \mathcal{O}_X \oplus TX) \tag{5.2}$$

whose only non-trivial entry is the extension class $\xi \in \operatorname{Ext}^1_X(TX, \mathcal{O}_X)$ of the left column exact sequence in (5.1); ξ is non-trivial because the exact sequence does not split. We let

$$\pi_1 : X \times \mathbf{A}^1 \longrightarrow X \quad \text{and} \quad \pi_2 : X \times \mathbf{A}^1 \longrightarrow \mathbf{A}^1$$

be the projections; we let t be the standard coordinate function on $\mathbf{A}^1$. The class

$$t \cdot \xi \in \Gamma(\mathcal{O}_{\mathbf{A}^1}) \otimes \operatorname{Ext}^1_X(TX, \mathcal{O}_X) = \operatorname{Ext}^1_{X\times \mathbf{A}^1}\big(\pi_1^* TX, \mathcal{O}_{X\times\mathbf{A}^1}\big)$$

defines an extension sheaf over $X \times \mathbf{A}^1$:

$$0 \longrightarrow \mathcal{O}_{X\times\mathbf{A}^1} \longrightarrow \mathcal{F} \longrightarrow \pi_1^* TX \longrightarrow 0. \tag{5.3}$$

The extension sheaf $\mathcal{F}$ is locally free; its restriction to $X \times t$, which we denote by F_t, form a one parameter family of holomorphic vector bundles whose special member $F_0 \cong \mathcal{O}_X \oplus TX$ and its general member $F_t \cong F$ for $t \neq 0$. Here, by abuse of notation, we use t to denote the point in $\mathbf{A}^1$ having coordinate t. It is a tautology that the Kodaira–Spencer class of this family at $t = 0$ is the κ in (5.2).

In terms of differential geometry, if we fix smooth isomorphisms $F_t \cong \mathbb{C}_X \oplus TX$ that also depend smoothly on t, then the holomorphic structure on F_t induces a family of holomorphic structures D''_t on $E = \mathbb{C}_X \oplus TX$ that is a deformation of the holomorphic structure D''_0 on $\mathbb{C}_X \oplus TX$. Following the convention of the first part of this paper, if we write $D''_t = D''_0 + A_t$ and use the splitting $E = \mathbb{C}_X \oplus TX$, then

$$\dot{D}''_0 = \dot{A}_0 = \begin{pmatrix} 0 & 0 \\ C_{21} & 0 \end{pmatrix}$$

and C_{21} represents the class ξ in $H^1(TX^\vee)$; thus $[C_{21}] \neq 0$.

What we aim at is to find a deformation of holomorphic structures D''_t of (E, D''_0) so that the first order deformation

$$\dot{D}''_0 = \begin{pmatrix} 0 & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

will have $[C_{12}] \neq 0$ and $[C_{21}] \neq 0$. To achieve this, we will construct a smooth family of holomorphic vector bundles (E, D''_u) parameterized by a smooth pointed domain $0 \in U$ so that

(1) D''_0 is the holomorphic structure on $\mathbb{C}_X \oplus TX$;

(2) there is a path $u = \rho_1(t)$ in U with $\rho_1(0) = 0$ so that $\dot{D}''_{\rho_1(0)} = \begin{pmatrix} 0 & * \\ C_{21} & * \end{pmatrix}$ and $[C_{21}] \neq 0$;

(3) there is another path $u = \rho_2(t)$ in U with $\rho_2(0) = 0$ so that $\dot{D}''_{\rho_2(0)} = \begin{pmatrix} 0 & C_{12} \\ * & * \end{pmatrix}$ and $[C_{12}] \neq 0$.

As we saw before, for the first path all, we need is to have it represent the family F_t constructed in (5.7). We now construct the second family that will represent the path ρ_2 that we need. We will work out the family over U after we have done this.

Using the top row exact sequence of the diagram (5.1), we can fit $\mathcal{O}_X \oplus TX$ into the exact sequence

$$0 \longrightarrow \mathcal{O}_X \oplus TX \longrightarrow \mathcal{O}_X \oplus T_X\mathbf{P}^4 \xrightarrow{\varphi} \mathcal{O}_X(5) \longrightarrow 0. \tag{5.4}$$

Here, $\varphi = (0, \varphi_2)^t$ is 0 when restricted to $\mathcal{O}_X$, and is the φ_2 in the diagram when restricted to $T_X\mathbf{P}^4$. We then pick a section $u \in H^0(\mathcal{O}_X(5))$, viewed as a homomorphism $\mathcal{O}_X \to \mathcal{O}_X(5)$, to form a new homomorphism of sheaves over $X \times \mathbf{A}^1$:

$$\Phi = (tu, \pi_1^*\varphi_2)^t : \mathcal{O}_{X\times\mathbf{A}^1} \oplus \pi_1^* T_X\mathbf{P}^4 \xrightarrow{\Phi} \pi_1^*\mathcal{O}_X(5)$$

whose restriction to $\mathcal{O}_{X\times\mathbf{A}^1}$ (resp. $\pi_1^*T_X\mathbf{P}^4$) is tu (resp. $\pi_1^*\varphi_2$). We let $\mathcal{F}'$ be the kernel of Φ. $\mathcal{F}'$ fits into the middle row exact sequence

(5.5)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \pi_1^*TX & = & \pi_1^*TX & & & & \\
 & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle (0,\pi_1^*\varphi_1)} & & & & \\
0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{O}_{X\times\mathbf{A}^1} \oplus \pi_1^*T_X\mathbf{P}^4 & \xrightarrow{\Phi} & \pi_1^*\mathcal{O}_X(5) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_{X\times\mathbf{A}^1} & \longrightarrow & \mathcal{O}_{X\times\mathbf{A}^1} \oplus \pi_1^*\mathcal{O}_X(5) & \longrightarrow & \pi_1^*\mathcal{O}_X(5) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Because the composite

$$\Phi \circ (0, \pi_1^*\varphi_1) = 0,$$

$(0, \pi_1^*\varphi_1)$ lifts to Ψ, shown in the diagram; its cokernel is $\mathcal{O}_{X\times\mathbf{A}^1}$.

We denote the restriction to $X \times \{t\}$ of $\mathcal{F}'$ by F'_t. Clearly, $F'_0 \cong \mathcal{O}_X \oplus TX$. The Kodaira–Spencer class of the first order deformation of the family $\mathcal{F}'$ at $t = 0$ is

$$\kappa' = \begin{pmatrix} 0 & \kappa'_{12} \\ 0 & 0 \end{pmatrix}.$$

To show that $\mathcal{F}'$ is the desired family, we need to show that $\kappa'_{12} \neq 0$. We now prove that this is true. We let $\mathbf{A}_2 = \operatorname{Spec} \mathbb{C}[t]/(t^2)$, which in plain language is the first order infinitesimal neighborhood of $0 \in \mathbf{A}^1$. Suppose $\kappa'_{12} = 0$, then based on deformation theory of vector bundles, the induced sheaf homomorphism

$\psi_2 : \mathcal{F}' \otimes_{\mathcal{O}_{X\times\mathbf{A}^1}} \mathcal{O}_{X\times\mathbf{A}_2} \longrightarrow \mathcal{O}_{X\times\mathbf{A}_2}$, or equivalently $\mathcal{F}'|_{X\times\mathbf{A}_2} \to \mathbb{C}_{X\times\mathbf{A}_2}$,

splits. Namely, there is a homomorphism

$$\tilde{\psi}_2 : \mathcal{O}_{X\times\mathbf{A}_2} \longrightarrow \mathcal{F}' \otimes_{\mathcal{O}_{X\times\mathbf{A}^1}} \mathcal{O}_{X\times\mathbf{A}_2} \tag{5.6}$$

so that

$$\psi_2 \circ \tilde{\psi}_2 = \mathrm{id}.$$

Let $p : X \times \mathbf{A}_2 \to X$ be the projection. Since $\mathcal{F}'$ is defined by the exact sequence (5.5), the homomorphism $\tilde{\psi}_2$ induces a homomorphism

$$\mathcal{O}_{X\times\mathbf{A}_2} \longrightarrow \mathcal{O}_{X\times\mathbf{A}_2} \oplus p^* T_X \mathbf{P}^4;$$

because $\operatorname{Ext}^1_X(\mathcal{O}_X, \mathcal{O}_X) = 0$ it lifts to a

$$\mu : \mathcal{O}_{X\times\mathbf{A}_2} \longrightarrow \mathcal{O}_{X\times\mathbf{A}_2} \oplus p^* \mathcal{O}_X(1)^{\oplus 5}.$$

Let

$$\lambda : \mathcal{O}_{X\times\mathbf{A}_2} \oplus p^* \mathcal{O}_X(1)^{\oplus 5} \longrightarrow p^* \mathcal{O}_X(5)$$

be the restriction of the composite of

$$\mathcal{O}_{X\times\mathbf{A}^1} \oplus p^* \mathcal{O}_X(1)^{\oplus 5} \longrightarrow \mathcal{O}_{X\times\mathbf{A}^1} \oplus p^* T_X \mathbf{P}^4$$

and Φ in (5.5) to $X \times \mathbf{A}_2$. Then, by definition

$$\lambda \circ \mu = 0.$$

To study this identity, we notice that the homomorphism μ must be of the form

$$\mu = [1 + at, bz_0 + t\alpha_0, \cdots, bz_4 + t\alpha_4]$$

with $[z_0, \cdots, z_4]$ the homogeneous coordinate of $\mathbf{P}^4$, $\alpha_i \in H^0(\mathcal{O}_X(1))$, $a \in \mathbb{C}$ and $b \in \mathbb{C}[t]/(t^2)$; the homomorphism λ is of the form

$$\begin{bmatrix} tu \\ z_0^4 \\ \vdots \\ z_4^4 \end{bmatrix}$$

Because $\lambda \circ \mu = 0$ holds over $X \times \mathbf{A}_2$, we have

$$[1 + a_1 t, bz_0 + t\alpha_0, \cdots, bz_4 + t\alpha_4] \begin{bmatrix} tu \\ z_0^4 \\ \vdots \\ z_4^4 \end{bmatrix} \equiv 0 \mod (t^2, z_0^5 + \cdots + z_4^5).$$

After simplification, the above identity reduces to

$$u + \alpha_0 z_0^4 + \cdots + \alpha_4 z_4^4 \equiv 0 \mod (z_0^5 + \cdots + z_4^5).$$

Now, we choose $u = z_0^2 z_1^3$. It is clear that there are no $\alpha_i \in H^0(\mathcal{O}_X(1))$ that make the above identity holds. Hence, with such choice of u, the lift $\tilde{\psi}_2$ does not exist. This proves $\kappa'_{12} \neq 0$.

It remains to find a family of holomorphic vector bundles that includes the two families $\mathcal{F}$ and $\mathcal{F}'$ as its subfamilies. We let $\eta \in \mathrm{Ext}^1_X(T_X\mathbf{P}^4, \mathcal{O}_X)$ be the extension class of the Euler exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(1)^{\oplus 5} \longrightarrow T_X\mathbf{P}^4 \longrightarrow 0. \tag{5.7}$$

Then, $t\eta$ is an extension class

$$t\eta \in \Gamma(\mathcal{O}_{\mathbf{A}^1}) \otimes \mathrm{Ext}^1(T_X\mathbf{P}^4, \mathcal{O}_X) = \mathrm{Ext}^1_{X\times\mathbf{A}^1}(\pi_1^* T_X\mathbf{P}^4, \mathcal{O}_{X\times\mathbf{A}^1})$$

that defines an exact sequence over $X \times \mathbf{A}^1$:

$$0 \longrightarrow \mathcal{O}_{X\times\mathbf{A}^1} \longrightarrow \mathcal{W} \longrightarrow \pi_1^* T_X\mathbf{P}^4 \longrightarrow 0.$$

Clearly, $\mathcal{W} \otimes_{\mathcal{O}_{X\times\mathbf{A}^1}} \mathcal{O}_{X\times\{0\}} = \mathcal{O}_X \oplus T_X\mathbf{P}^4$ while $\mathcal{W} \otimes_{\mathcal{O}_{X\times\mathbf{A}^1}} \mathcal{O}_{X\times\{t\}} = \mathcal{O}_X(1)^{\oplus 5}$ for $t \neq 0$. We claim that

$$\pi_{2*}\big(\mathcal{W}^\vee \otimes \pi_1^*\mathcal{O}_X(5)\big) \tag{5.8}$$

is a locally free sheaf of $\mathcal{O}_{\mathbf{A}^1}$-modules. By base change property, this is true if

$$H^1(X, \big(\mathcal{O}_X \oplus T_X\mathbf{P}^4)^\vee \otimes \mathcal{O}_X(5)\big) = 0$$

and

$$H^1\big(X, (\mathcal{O}_X(1)^{\oplus 5})^\vee \otimes \mathcal{O}_X(5)\big) = 0.$$

Since $X \subset \mathbf{P}^4$ is a smooth hypersurface, a standard long exact sequence chasing shows that $H^1(X, \mathcal{O}_X(a)) = 0$ for any integer a. To prove the above two identities, we only need to check that $H^1(X, T_X^\vee\mathbf{P}^4 \otimes \mathcal{O}_X(5)) = 0$. For this, we apply the long exact sequence of cohomologies to the dual of (5.7) tensored with $\mathcal{O}_X(5)$:

$$H^0(\mathcal{O}_X(4)^{\oplus 5}) \longrightarrow H^0(\mathcal{O}_X(5)) \longrightarrow H^1(T_X^\vee\mathbf{P}^4(5)) \longrightarrow H^1(\mathcal{O}_X(4)^{\oplus 5}).$$

Because the last term is zero, and because $H^0(\mathcal{O}_{\mathbf{P}^4}(a)) \to H^0(\mathcal{O}_X(a))$ is surjective, the first arrow is surjective. This shows that $H^1(T_X^\vee \mathbf{P}^4(5)) = 0$, and hence (5.8) is locally free.

We now let W be the total space of the vector bundle (5.8) and let

$$q\colon X \times W \to X \times \mathbf{A}^1$$

be the projection. Over $X \times W$, there is a tautological homomorphism

$$q^*\mathcal{W} \longrightarrow q^*\pi_1^*\mathcal{O}_X(5).$$

Let $\mathcal{E}$ be the kernel of the above sheaf homomorphism; for $w \in W$, we denote by E_w the restriction of $\mathcal{E}$ to $X \times w$.

It is now a matter of direct checking that there are two paths $\rho_1(t)$ and $\rho_2(t)$ in W so that $E_{\rho_1(t)}$ and $E_{\rho_2(t)}$ represent F_t and F'_t respectively. First of all, the homomorphism $\varphi\colon \mathcal{O}_X \oplus T_X\mathbf{P}^4 \to \mathcal{O}_X(5)$ in (5.4) represents a point in W; we designate this point to be the marked point $0 \in W$. The family $\mathcal{F}'$ is constructed as the kernel of Φ in (5.5) with Φ restricting to $X \times \{0\}$ being φ. Hence, Φ represents a path ρ_2 in W initiating from 0 and is contained in the fiber of $W \to \mathbf{A}^1$ over $0 \in \mathbf{A}^1$ that satisfies $E_{\rho_2(t)} \cong F'_t$.

As to the first family $\mathcal{F}$ constructed in (5.7), it fits into the exact diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{X\times\mathbf{A}^1} & \longrightarrow & \mathcal{F} & \longrightarrow & \pi_1^*TX & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{X\times\mathbf{A}^1} & \longrightarrow & \mathcal{W} & \longrightarrow & \pi_1^*T_X\mathbf{P}^4 & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \Psi} & & \downarrow & & \\
 & & & & \pi_1^*\mathcal{O}_X(5) & = & \pi_1^*\mathcal{O}_X(5) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since Ψ restricting to $X \times \{0\}$ is the φ in (5.4), it represents a path ρ_1 in W with $\rho_1(0) = 0$ so that $E_{\rho_1(t)}$ is the first family F_t constructed before.

From what we know of the families F_t and F'_t, their Kodaira–Spencer classes at $t = 0$ are of the form

$$\begin{pmatrix} 0 & 0 \\ \kappa_{21} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \kappa'_{12} \\ 0 & 0 \end{pmatrix}, \qquad \kappa_{21} \neq 0, \quad \kappa'_{12} \neq 0.$$

Since W is smooth, there is a path $\rho(t)$ with $\rho(0)=0$ so that $\dot\rho(0)=\dot\rho_1(0)+\dot\rho_2(0)$; hence, the family $E_{\rho(t)}$ has Kodaira–Spencer class at $t=0$

$$\begin{pmatrix} 0 & \kappa'_{12} \\ \kappa_{21} & 0 \end{pmatrix}, \qquad \kappa_{21}\neq 0, \quad \kappa'_{12}\neq 0.$$

It satisfies the requirement of theorem 4.3. This proves

Theorem 5.1. *Let $X\subset \mathbf{P}^4$ be a smooth quintic threefold and ω is a Calaby–Yau form (metric) on X. Then, there is a smooth deformation D''_s of $(E,D''_0)=\mathbb{C}_X\oplus TX$ so that for large $c>0$ and small s, there are irreducible regular solutions (H_s,ω_s) to Strominger's system on the vector bundle (E,D''_s) so that $\lim_{s\to 0}\omega_s=c\omega$ and $\lim_{s\to 0}H_s$ is a regular Hermitian Yang–Mills connection on $\mathbb{C}_X\oplus TX$.*

We next state the existence of solutions to SU(5)-strominger's system.

Theorem 5.2. *Let $X\subset \mathbf{P}^3\times\mathbf{P}^3$ be a smooth Calabi–Yau threefold cut out by three homogeneous polynomials of bi-degrees $(3,0)$, $(0,3)$ and $(1,1)$. Let ω be a Calabi–Yau form on X. Then, there is a smooth deformation D''_s of $(E,D''_0)=\mathbb{C}_X^{\oplus 2}\oplus TX$ so that for large $c>0$ and small s, there are irreducible regular solution (H_s,ω_s) to Strominger's system on (E,D''_s).*

Proof. We only need to produce a deformation of holomorphic structure of $\mathbb{C}_X^{\oplus 2}\oplus TX$. Let π_1 and $\pi_2 : X\to\mathbf{P}^3$ be the composite of the immersion $X\subset\mathbf{P}^3\times\mathbf{P}^3$ with the projections $\mathbf{P}^3\times\mathbf{P}^3\to\mathbf{P}^3$. Then, TX fits into the exact sequence

(5.9)
$$0\longrightarrow TX\longrightarrow \pi_1^*\mathbf{TP}^3\oplus\pi_2^*T\mathbf{P}^3\longrightarrow \mathcal{O}_X(3,0)\oplus\mathcal{O}_X(0,3)\oplus\mathcal{O}_X(1,1)\longrightarrow 0.$$

Here, $\mathcal{O}_X(i,j)$ is the restriction to X of $\pi_1^*\mathcal{O}_{\mathbf{P}^3}(i)\otimes\pi_2^*\mathcal{O}_{\mathbf{P}^3}(j)$. Composing the canonical

$$\mathcal{O}_X(1,0)^{\oplus 4}\oplus\mathcal{O}_X(0,1)^{\oplus 4}\longrightarrow \pi_1^*\mathbf{TP}^3\oplus\pi_2^*T\mathbf{P}^3$$

with the last arrow in (5.9), we obtain a surjective

$$\mathcal{O}_X(1,0)^{\oplus 4}\oplus\mathcal{O}_X(0,1)^{\oplus 4}\xrightarrow{\varphi_2}\mathcal{O}_X(3,0)\oplus\mathcal{O}_X(0,3)\oplus\mathcal{O}_X(1,1)$$

whose kernel, denoted by F_0, is an extension of TX by $\mathcal{O}_X^{\oplus 2}$. Next, we vary φ_2 to produce a variation of holomorphic structure of F_0. The bundle F_0 is a small deformation of $\mathbb{C}_X^{\oplus 2}\oplus TX$; varying φ_2 produces small deformation of F_0. We then mimic the argument in the proof of Theorem 5.1 to show that we can make this small deformation of small deformation into a single small deformation; it is our desired D''_s.

To complete the proof of the theorem, we need to check the non-degeneracy condition on the two matrices B and B' associated to the

Kodaira–Spencer class κ of this family. It is routine and shall be omitted. This completes the proof of the Theorem. q.e.d.

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