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NON-NEGATIVE PINCHING, MODULI SPACES AND BUNDLES WITH INFINITELY MANY SOULS

Vitali Kapovitch, Anton Petrunin & Wilderich Tuschmann

Abstract

We show that in each dimension $n \geq 10$, there exist infinite sequences of homotopy equivalent, but mutually non-homeomorphic closed simply connected Riemannian n-manifolds with $0 \leq \sec \leq 1$, positive Ricci curvature and uniformly bounded diameter. We also construct open manifolds of fixed diffeomorphism type which admit infinitely many complete non-negatively pinched metrics with souls of bounded diameter such that the souls are mutually non-homeomorphic. Finally, we construct examples of non-compact manifolds whose moduli spaces of complete metrics with $\sec \geq 0$ have infinitely many connected components.

1. Introduction

In this article, we discuss several infiniteness phenomena in non-negative sectional curvature.

Our first such result is motivated by the finiteness theorems in Riemannian geometry and a question of S.-T. Yau which asks whether there always exists only a finite number of diffeomorphism types of closed smooth manifolds of positive sectional curvature that are homotopy equivalent to a given positively curved manifold ([**37**], Problem 11).

If one relaxes the condition $\sec > 0$ to $\sec \geq 0$, then the answer to Yau's question is known to be false in all dimensions ≥ 7, even in the category of simply connected manifolds. Counterexamples can here be obtained in the following way: By a result of Grove and Ziller [**22**], the total space of any linear S^k-bundle over S^4 admits a Riemannian metric with non-negative curvature. However, for $k \geq 3$, the total spaces of such bundles fall into infinitely many homeomorphism, but only finitely many homotopy types ([**12**]; if $k = 3$, one has in addition to assume that the Euler class of the bundle be zero). On the other hand, when rescaled to have uniformly bounded diameter, by [**30**], these examples

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cannot satisfy any uniform upper curvature bound. More generally, it is natural to look at the following question:

Question 1.1. Given fixed $n \in \mathbb{N}, D > 0$ and c, $C \in \mathbb{R}$, are there at most finitely many diffeomorphism classes of pairwise homotopy equivalent closed Riemannian n-manifolds M^n with sectional curvature $c \leq \sec \leq C$ and diameter $\leq D$?

The diffeomorphism finiteness theorems in Riemannian geometry (see, e.g., [**1**, **10**, **28**, **14**, **16**, **21**, **30**, **34**]) leave this question in general dimensions completely open. However, the answer is known to be positive in some special situations.

This is, for example, the case if $c > 0$ and $n = 2m$ by [**24**, **10**], if M^n, $n \neq 4$, is simply connected and $C = 4c > 0$ by [**7**, **11**], if $D = D(C, c, n)$ is sufficiently small by Gromov's theorem on almost flat manifolds [**20**, **9**, **31**] and the rigidity of infranilmanifolds [**3**] (cf. [**17**]), if M is 2-connected by [**30**], or if $C \leq 0$ and $n \geq 5$ by results of Farrell and Jones [**18**, **19**]. Remarkably enough, in the latter case, one actually does not even need the lower curvature and the upper diameter bounds. In other words, for $n \geq 5$, the answer to the analogue of Yau's question for non-positive curvature (which in this case is a special case of the Borel conjecture) is yes.

As a preliminary result, we first show that, in general, the answer to Question 1.1 is actually negative in all dimensions ≥ 7:

Proposition 1.2. *There exists $D > 0$ such that for any $n \geq 7$, there exist an infinite sequence of homotopy equivalent, but mutually non-homeomorphic closed Riemannian n-manifolds M_k^n with*

$$|\sec(M_k^n)| \leq 1 \;\; \textit{and} \;\; \operatorname{diam}(M_k^n) \leq D.$$

If $n \neq 8$, all these manifolds can in addition be chosen to be simply-connected.

Notice that for simply connected manifolds, by [**14**, **34**], $n = 7$ is indeed the smallest dimension where such sequences can occur.

Our first main concern in this paper is, however, the analogue of Yau's question for non-negative pinching, i.e., the following special case of Question 1.1:

Question 1.3. Given fixed $n \in \mathbb{N}$ and $C, D > 0$, are there always at most finitely many diffeomorphism types of pairwise homotopy equivalent closed Riemannian n-manifolds with sectional curvature $0 \leq \sec \leq C$ and diameter $\leq D$?

Notice here that starting from dimension $n = 6$, from [**22**], one may infer the existence of infinite sequences of closed simply connected non-negatively curved n-manifolds of mutually distinct homotopy type, and in dimensions $n > 8$, $n \neq 10$ by [**33**], there even exist infinite sequences of closed simply connected non-negatively pinched Riemannian n-manifolds with pairwise non-isomorphic rational cohomology rings that also satisfy uniform upper diameter bounds. Totaro ([**33**]) also showed that there exist infinite sequences of closed simply connected non-negatively curved 6-manifolds with pairwise non-isomorphic rational cohomology rings, and (improving earlier work of [**15**] for manifolds of dimension $n \geq 22$) that for any $n \geq 7, n \neq 8$, there exist infinitely many closed simply connected Riemannian n-manifolds with $|sec| \leq 1$ and uniformly bounded diameter and pairwise non-isomorphic rational cohomology rings.

Our first main result shows that if $n \geq 10$, the answer to Question 1.3 is in general negative, even under the extra assumption of positive Ricci curvature.

Theorem A. *There exists $D > 0$ such that for each dimension $n \geq 10$, there exists an infinite sequence $(M_k^n)_{k\in\mathbb{N}}$ of pairwise homotopy equivalent, but mutually non-homeomorphic closed simply connected Riemannian n-manifolds satisfying*

$$0 \leq \sec(M_k^n) \leq 1,\ \mathrm{Ric}(M_k^n) > 0 \ \textit{and}\ \mathrm{diam}(M_k^n) \leq D.$$

Notice that this result relates Yau's problem to another long standing open question in Riemannian geometry:

Are there any obstructions to the existence of a Riemannian metric with positive sectional curvature on a closed simply connected manifold of non-negative sectional and positive Ricci curvature?

It is quite likely that the dimensional restriction $n \geq 10$ in Theorem A is not optimal. Again, by [**14**, **34**], this dimension must be at least 7.

We continue with a description of the further main results of this paper, which concern the souls and moduli spaces of metrics of open manifolds of non-negative sectional curvature.

Very recently, in [**5**], Belegradek constructed the first examples of manifolds admitting infinitely many non-negatively curved metrics with mutually non-homeomorphic souls. In our second main theorem, we sharpen this result by constructing such examples which in addition have uniform bounds on the curvature of the manifolds and the diameters of the souls:

Theorem B. *For any $k > 10$, the manifold $S^2 \times S^2 \times S^3 \times S^3 \times \mathbb{R}^k$ admits an infinite sequence of complete non-negatively curved metrics g_i with pairwise non-homeomorphic souls S_i such that*

$$0 \leq \sec(M, g_i) \leq 1 \ \textit{and}\ \mathrm{diam}(S_i) \leq D$$

where D is a positive constant independent of k and i.

Another interesting application of the construction that we employ in the proof of Theorem B concerns the moduli spaces of open manifolds with non-negative sectional curvature. Let $\mathfrak{R}_{\sec\geq 0}(M)$ denote the space of complete Riemannian metrics with non-negative sectional curvature on a given smooth manifold M. Then Diff(M), the group of diffeomorphisms of M, acts on this space by pulling back metrics, and the orbit space $\mathfrak{R}_{\sec\geq 0}(M)/\mathrm{Diff}(M)$ is called the moduli space of (complete) non-negatively curved Riemannian metrics on M. We show:

Theorem C. *There exists a manifold M^{22} which admits an infinite sequence of complete metrics g_i with pairwise non-homeomorphic souls S_i such that*

$$0 \leq \sec(M, g_i) \leq 1 \ \ \textit{and} \ \ \mathrm{diam}(S_i) \leq D$$

and such that the equivalence classes of the metrics g_i all lie in different connected components of the moduli space $\mathfrak{R}_{\sec\geq 0}(M)/\mathrm{Diff}(M)$ of complete metrics with $\sec \geq 0$ *on M.*

Moreover, for any closed non-negatively curved manifold (N, g), the product metrics $g_i \times g$ all lie in different connected components of the moduli space $\mathfrak{R}_{\sec\geq 0}(M \times N)/\mathrm{Diff}(M \times N)$.

To put Theorem C into further perspective, we note first that in [**25**], Kreck and Stoltz constructed a closed manifold M^7 such that the moduli space of metrics of positive Ricci curvature on M has infinitely many connected components. In fact, by choosing somewhat different metrics, their methods also show that $\mathfrak{R}_{\sec\geq 0}(M)/\mathrm{Diff}(M)$ also has infinitely many connected components! Since this was not observed in [**25**], let us briefly explain why that is true.

Kreck and Stoltz actually construct an invariant s which distinguishes components of $\mathfrak{R}_{\mathrm{scalar}>0}(M)/\mathrm{Diff}(M)$. They look at various S^1-bundles over $S^2 \times CP^2$ with indivisible Euler classes, which admit Einstein metrics of positive Ricci curvature constructed by Wang and Ziller ([**35**]). It is then shown [**25**, Theorem 3.11] that provided the metrics are S^1 invariant and have totally geodesic fibers (which is true for Wang–Ziller metrics), the invariant s depends only on the Euler class of the bundle. One can exhibit infinitely many bundles with distinct s invariants, but diffeomorphic total spaces [**25**, Theorems 3.2, 3.4]. Unfortunately, the Einstein metrics given by [**35**] do not have non-negative sectional curvature. However, we notice here that one can represent any S^1 bundle over $S^2 \times CP^2$ with an indivisible Euler class as a free isometric quotient $(S^3 \times S^5)/S^1$. The natural Riemannian submersion metric coming from the product metric on $S^3 \times S^5$ is easily seen to have $\sec \geq 0$ and $\mathrm{Ric} > 0$, it has totally geodesic fibers and it is S^1-invariant. Therefore, the same bundles as considered in [**25**], but taken with these metrics have distinct s-invariants and hence, lie in different components of

$\mathfrak{R}_{\text{scalar}>0}(M)/\text{Diff}(M)$. Since any metric of sec ≥ 0 on M has scalar ≥ 0 and scalar > 0 at a point, by [**2**] it can be deformed to a nearby metric of scalar > 0. Therefore, the above metrics also lie in different components of $\mathfrak{R}_{\sec\geq 0}(M)/\text{Diff}(M)$.

Observe, however, that all the different components of $\mathfrak{R}_{\text{scalar}>0}(M)/\text{Diff}(M)$ obviously become connected if we stabilize M by multiplying it by a closed manifold with non-negative sectional and positive scalar curvature, for example, by S^n with $n > 1$. Therefore, in contrast to Theorem C which yields non-connected moduli spaces of non-negative sectional curvature metrics in all dimensions ≥ 22, it is not clear if the components of $\mathfrak{R}_{\sec\geq 0}(M)/\text{Diff}(M)$ remain disconnected after such stabilization.

There are many other interesting results about the connectedness or disconnectedness of moduli spaces of metrics satisfying certain geometric bounds, for which we refer to, eg., [**26**, **27**, **29**, **30**].

We conclude the introduction with a short description of the ideas and outlines of the proofs.

To prove Proposition 1.2, we look at a 6-manifold X^6 which is homotopy equivalent to $S^2 \times S^2 \times S^2$, but has non-trivial first Pontrjagin class. By an easy topological argument, among the S^1-bundles over X^6, there are infinitely many spaces which are homotopy equivalent to $S^2 \times S^2 \times S^3$, but have distinct Pontrjagin classes. All S^1-bundles we consider can be represented as quotients of a fixed manifold Q by various subtori $T_i^2 \subset T^3$ where T^3 acts freely and isometrically on Q. This implies that the induced metrics on Q/T_i^2 have uniformly bounded curvatures and diameters.

To prove Theorem A, we fix a rank 2 bundle ξ over $S^2 \times S^2 \times S^2$ and look at the sphere bundle P of $\xi \oplus \epsilon^{k-1}$ with $k \geq 3$. We then look at various circle bundles $S^1 \to M_i \to P$. A topological argument shows that with an appropriate choice of ξ, infinitely many such bundles have total spaces homotopy equivalent to $S^2 \times S^2 \times S^3 \times S^k$, but distinct first Pontrjagin classes and thus are mutually non-homeomorphic.

We can represent all M_is as $S^3 \times S^3 \times S^3 \times S^k/T_i^2$ where $T_i^2 \subset T^3$ which acts freely and isometrically on $S^3 \times S^3 \times S^3 \times S^k$. This easily implies that the M_i satisfy all geometric constraints in Theorem A.

To prove Theorem B, we put $k = 3$, fix a rank 2 bundle ζ over P and look at the pullbacks of $\zeta \oplus \epsilon^{l-2}$ to M_i. By the same reasons as before, the total spaces of these pullbacks have metrics satisfying all geometric restrictions of Theorem B with souls isometric to M_i. Another topological argument then shows that with an appropriate choice of ζ, the total spaces of the pullbacks are diffeomorphic to $S^2 \times S^2 \times S^3 \times S^3 \times \mathbb{R}^l$ if $l > 10$.

To prove Theorem C, we modify the construction in the proof of Theorem B to produce a manifold with infinitely many non-diffeomorphic souls whose normal bundles have non-trivial rational Euler classes. We then show that for such a manifold, all elements of a connected component of $\mathfrak{R}_{\sec\geq 0}(M)/\mathrm{Diff}(M)$ have diffeomorphic souls.

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3. Proof of Proposition 1.2

Proof. Let α, β, γ be the standard basis of $H^2(S^2 \times S^2 \times S^2)$. By Lemma A.1 in the appendix, for some $m > 0$, there exists a closed manifold M^6 and a smooth homotopy equivalence $f\colon M \to S^2 \times S^2 \times S^2$ such that $p_1(M) = f^*(m\beta \wedge \gamma)$.

Consider the principal T^3 bundle $T^3 \xrightarrow{p} S^3 \times S^3 \times S^3 \to S^2 \times S^2 \times S^2$ and let $Q = f^*(p)$ be its pullback. Choose a Riemannian metric g on Q which is invariant under the T^3 action. For any subtorus $T^2 \subset T^3$, the quotient space Q/T^2 is naturally a principal S^1-bundle over $Q/T^3 = M^6$. Clearly, any principal S^1- bundle over M^6 with indivisible Euler class can be realized in this way. Let us denote the subtorus corresponding to the bundle with Euler class (a, b, c) by $T^2_{a,b,c}$. Here, the Euler class is written with respect to the natural product basis α, β, γ of $H^2(M^6) \cong H^2(S^2 \times S^2 \times S^2)$.

By Lemma A.2, all the quotients $Q/T^2_{a,b,c}$ with the induced submersion metrics satisfy $|\sec| \leq C, \mathrm{diam} \leq D$ for some $C, D > 0$.

Also from Lemma A.3, we see that all the spaces $N_{a,b} = Q/T^2_{a,b,0}$ with $(a, b) = 1$ are homotopy equivalent to $S^2 \times S^2 \times S^3$.

Now, for $\pi\colon N_{a,b} \to M$ with $(a, b) = 1$, we have $\pi^*(\alpha) = -b\omega, \pi^*(\beta) = a\omega, \pi^*(\gamma) = \gamma$ and thus $\pi^*(\beta \wedge \gamma) = a\omega \wedge \gamma$, where $\omega \wedge \gamma$ is the generator of $H^4(N_{a,b})$. Therefore, $p_1(N_{a,b}) = \pi^*(p_1(M)) = am\omega \wedge \gamma$. This means that all manifolds $N_{a,b}$ with distinct a and $(a, b) = 1$ have distinct Pontrjagin classes and thus are mutually non-homeomorphic.

Finally, observe that crossing the manifolds $N_{a,b}$ with round spheres produces examples satisfying the conclusion of Theorem 1.2 in all dimensions ≥ 8. q.e.d.

4. Proof of Theorem A

Proof. Fix $k \geq 3$.

Consider the standard free T^3 action on $S^3 \times S^3 \times S^3$ giving rise to the bundle $T^3 \to S^3 \times S^3 \times S^3 \to S^2 \times S^2 \times S^2$. For any subtorus $T^2 \subset T^3$, the homogeneous space $S^3 \times S^3 \times S^3/T^2$ is naturally a principal S^1-bundle over $S^3 \times S^3 \times S^3/T^3 = S^2 \times S^2 \times S^2$. Clearly, any principal S^1-bundle over $S^2 \times S^2 \times S^2$ with indivisible Euler class can be realized in this way. Let us denote the subtorus corresponding to the bundle with Euler class (a, b, c) by $T^2_{a,b,c}$. Here, the Euler class is written with respect to the natural product basis α, β, γ of $H^2(S^2 \times S^2 \times S^2)$. Let $N_{a,b,c}$ be the corresponding total space and $\pi\colon N_{a,b,c} \to S^2 \times S^2 \times S^2$ be the natural projection.

By Lemma A.3, $N_{a,b,0}$ is homotopy equivalent to $S^2 \times S^2 \times S^3$ if $(a, b) = 1$.

Let us fix a representation $\rho\colon T^3 \to SO(2)$ given by the weight (p, q, r). Look at the associated $\mathbb{R}^2$ bundle ξ over $S^2 \times S^2 \times S^2$ given by $S^3 \times S^3 \times S^3 \times_{T^3} \mathbb{R}^2$. Its Euler class is (p, q, r). Let $\eta = \xi \oplus \epsilon^{k-1}$ and let η^S be the corresponding *sphere* bundle $S^k \to P \xrightarrow{\eta^S} S^2 \times S^2 \times S^2$ (here and in what follows, ϵ^m denotes a trivial $\mathbb{R}^m$-bundle).

Next, look at the pullback of η to $N_{a,b,0}$. It can be written as $S^3 \times S^3 \times S^3 \times_{T^2_{a,b,0}} \mathbb{R}^2 \times \mathbb{R}^{k-1}$. We will denote this bundle by $\eta_{a,b}$. Let $S^k \to M_{a,b} \xrightarrow{\eta^S_{a,b}} N_{a,b,0}$ be the corresponding *sphere* bundle over $N_{a,b,0}$.

We claim that by choosing an appropriate $\rho\colon T^3 \to SO(2)$ and by varying a, b the manifolds $M_{a,b}$ provide examples satisfying the conclusion of Theorem A.

Let us first check the geometric conditions.

Observe that we can write η and η^S as $S^3 \times S^3 \times S^3 \times_{T^3} \mathbb{R}^{k+1}$, $S^3 \times S^3 \times S^3 \times_{T^3} S^k$ respectively. Here, T^3 acts on $S^3 \times S^3 \times S^3$ by the canonical homogeneous action and on $\mathbb{R}^{k+1}$ and S^k via ρ followed by the canonical inclusion $SO(2) \hookrightarrow SO(k + 1)$.

Hence, $M_{a,b} = S^3 \times S^3 \times S^3 \times_{T^2_{a,b,0}} S^k$.

Therefore, by Lemma A.2, when equipped with the induced quotient metrics, all total spaces have uniform curvature bounds $0 \le \sec \le C$ for some $C > 0$ and $\operatorname{diam} \le D$ for some $D > 0$.

Next, let us check that $\operatorname{Ric}(M_{a,b}) > 0$. Obviously, $\operatorname{Ric}(M_{a,b}) \ge 0$. Suppose there exists $x_0 \in T_p M_{a,b}$ such that $\operatorname{Ric}(x_0) = 0$. Let $\tilde{x}_0$ be its horizontal lift to $T_{\tilde{p}}(S^3 \times S^3 \times S^3 \times S^k)$. By O'Neill's formula, this means that $\sec(\tilde{x}_0, x) = 0$ for any horizontal vector $x \in T_{\tilde{p}}(S^3 \times S^3 \times S^3 \times S^k)$. Let $\mathfrak{h}$ and $\mathfrak{m}$, respectively, denote the horizontal and the vertical tangent space at $\tilde{p}$.

Then, $\tilde{x}_0$ contains a non-trivial component tangent to some sphere factor. By construction, the projection of $\mathfrak{m}$ to the tangent space to that sphere is at most one dimensional. Therefore, we can find a vector x tangent to that spherical factor and perpendicular to both $\mathfrak{m}$ and $\tilde{x}_0$. Then, $\sec(\tilde{x}_0, x) > 0$ which is a contradiction.

To finish the proof of Theorem A, it remains to show that among the spaces $M_{a,b}$, there are infinitely many homotopy equivalent, but mutually non-homeomorphic ones.

First, we claim that there exists an integer m such that for any ρ with weight (mp, mq, mr), *all* spaces $M_{a,b}$ are homotopy equivalent to $S^2 \times S^2 \times S^3 \times S^k$ if a and b are relatively prime.

Look at the sphere bundle $S^k \to P \xrightarrow{\eta} S^2 \times S^2 \times S^2$.

Up to fiberwise homotopy equivalences, such bundles are classified by the homotopy classes of maps in $[S^2 \times S^2 \times S^2, B\mathrm{Aut}(S^k)]$. Here, $B\mathrm{Aut}(S^k)$ is the classifying space for $\mathrm{Aut}(S^k)$ which is the identity component of the monoid of self-homotopy equivalences of S^k.

Moreover, in our case, by construction, the classifying map into $B\mathrm{Aut}(S^k)$ corresponding to the bundle η factors through $B\mathrm{Aut}_0(S^3)$ where $\mathrm{Aut}_0(S^k)$ is the subset of $\mathrm{Aut}(S^k)$ fixing a base point.

It is a well known fact that if k is odd, then $\pi_i(\mathrm{Aut}_0(S^k))$ is finite for any i. Indeed, it is easy to see that $\mathrm{Aut}_0(S^k)$ is the identity component of $\Omega^k(S^k)$, and therefore, for any $i > 0$, $\pi_i(\mathrm{Aut}_0(S^k)) \cong \pi_{k+i}(S^k)$, which is always finite if k is odd.

A standard obstruction theory argument now implies that $[S^2 \times S^2 \times S^2, B\mathrm{Aut}_0(S^3)]$ is finite.

Claim. There is an $m > 0$ such that if $e(\xi)$ is divisible by m, then the classifying map $f_\eta \colon S^2 \times S^2 \times S^2 \to B\mathrm{Aut}_0(S^3)$ is homotopic to a point.

For any $m > 0$, let $g_m \colon S^2 \to S^2$ be a map of degree m.

Let $F_m = g_m \times g_m \times g_m \colon S^2 \times S^2 \times S^2 \to S^2 \times S^2 \times S^2$. Clearly, $e(F_m^* \xi) = me(\xi)$ for any rank 2 bundle ξ over $S^2 \times S^2 \times S^2$. Hence, if $f_\eta \colon S^2 \times S^2 \times S^2 \to B\mathrm{Aut}_0(S^3)$ is the classifying map for the S^k bundle coming from ξ, then $f_\eta \circ F_m$ is the classifying map for the S^k bundle coming from the rank 2 bundle with Euler class equal to $me(\xi)$.

The claim now follows from a standard obstruction theory argument. Let us give a brief sketch. If $f_1, f_2 \colon S^2 \times S^2 \times S^2 \to B\mathrm{Aut}_0(S^3)$ are two maps which are homotopic on the $(i-1)$-skeleton, the obstruction to extending this homotopy to the i-skeleton lies in $\Gamma_i = H^i(S^2 \times S^2 \times S^2, \pi_i(B\mathrm{Aut}_0(S^3))$ which is finite by what has been said above. Let $m_i = |\Gamma_i|$. By naturality, the obstruction corresponding to the maps $f_1 \circ F_{m_i}$, $f_2 \circ F_{m_i}$ is zero. Repeating this process finitely many times, we see that for $m = m_1 \cdot \ldots \cdot m_6$, and any f_1, f_2, the maps $f_1 \circ F_m$, $f_2 \circ F_m$ are homotopic. By taking f_1 to be a constant map, we see that for any $f \colon S^2 \times S^2 \times S^2 \to B\mathrm{Aut}_0(S^3)$, the map $f \circ F_m$ is homotopic to a constant. This proves our claim.

From now on, we will assume that $e(\xi)$ is divisible by m and hence, the bundle $S^k \to P \xrightarrow{\eta^S} S^2 \times S^2 \times S^2$ is fiberwise homotopically trivial.

Of course, the same is true for any pullback of this bundle and hence, $\eta^S_{a,b}$ is fiberwise homotopically trivial for any a, b. Thus, its total space $M_{a,b}$ is homotopy equivalent to $N_{a,b,0} \times S^k$ which, by Lemma A.3, is homotopy equivalent to $S^2 \times S^2 \times S^3 \times S^k$ if $(a, b) = 1$.

Let us finally show that for appropriately chosen p, q, r, infinitely many of the spaces $M_{a,b}$ have distinct Pontrjagin classes and thus are mutually not diffeomorphic.

Consider the bundle $\pi\colon\ S^1 \to Q \to S^2 \times S^2$ with Euler class (a, b) with respect to the canonical generators α, β of $H^2(S^2 \times S^2)$. Let ω be the generator of $H^2(Q)$. Then, we see from the Gysin sequence that $\pi^*(\alpha) = -b\omega, \pi^*(\beta) = a\omega$.

Now, look at the bundle $\pi\colon\ M_{a,b} \to P$. Let ω, γ be the natural basis of $H^2(M_{a,b})$. Then, by the above, we have that $\pi^*(\alpha) = -b\omega, \pi^*(\beta) = a\omega, \pi^*(\gamma) = \gamma$. (We purposefully slightly abuse notations by denoting by γ elements of both $H^2(P)$ and $H^2(M_{a,b})$.)

We compute

$$\begin{aligned}
p_1(\zeta_{a,b}) &= p_1(\xi_{a,b}) \\
&= e(\xi_{a,b}) \cup e(\xi_{a,b}) \\
&= \pi^*(p\alpha + q\beta + r\gamma) \cup \pi^*(p\alpha + q\beta + r\gamma) \\
&= ((-pb + qa)\omega + r\gamma) \cup ((-pb + qa)\omega + r\gamma) \\
&= 2(-pb + qa)r\omega \wedge \gamma.
\end{aligned}$$

Notice that $\omega \wedge \gamma$ is the generator of $H^4(M_{a,b}) \cong Z$.

From the bundle $S^k \to M_{a,b} \stackrel{\eta^S_{a,b}}{\to} S^2 \times S^2 \times S^3$, using the Whitney formula, we see that $p_1(M_{a,b}) = p_1(\eta_{a,b}) + \eta^{S*}_{a,b}(p_1(S^2 \times S^2 \times S^3)) = 2(-pb + qa)r\omega \wedge \gamma$.

Clearly, for fixed p, q, r, infinitely many of these spaces have distinct p_1. For example, if $p = 0, q = mq_1 \neq 0, r = mr_1 \neq 0$, the spaces $M_{a,b}$ with distinct a will work, as in this case $p_1(M_{a,b}) = 2qar$.

By the above, all of these spaces are homotopy equivalent to $S^2 \times S^2 \times S^3 \times S^k$ and hence, they satisfy all conclusions of Theorem A. q.e.d.

Remark 4.1. Observe that unlike the examples constructed in [**5**], the manifolds constructed in the proof of Theorem A have infinite π_2. This is actually necessary by the π_2-finiteness theorem [**30**].

Also, all our examples are constructed as quotients of a fixed manifold E (in our case $E = S^3 \times S^3 \times S^3 \times S^k$) by free torus actions. This is also necessary by [**30**, Corollary 0.2].

5. Proof of Theorem B

We will use the notation and constructions employed in the proof of Theorem A.

We will make use of the following fact from algebraic topology which follows from a combination of results of Haefliger and Siebenmann [**23**, **32**].

Fact. Let $\mathbb{R}^l \to E_i \to M_i^n, (i = 1, 2)$ be two vector bundles over smooth closed manifolds. Suppose $f\colon E_1 \to E_2$ is a tangential homotopy equivalence and $l \geq 3, l > n$.

Then, f is homotopic to a diffeomorphism (cf. [**5**] for details).

In the proof of Theorem A, let now $k = 3$.

Recall that from the construction of P as $S^3 \times S^3 \times S^3 \times_{T^3} S^3$, we see that the map $\mathbb{Z}^3 \cong \pi_2(P) \to \pi_1(T^3) \cong \mathbb{Z}^3$ is an isomorphism.

Consider a representation $\phi\colon T^3 \to SO(2)$. It gives rise to a rank 2 bundle ζ over P. Its total space can be written as $E_\zeta = S^3 \times S^3 \times S^3 \times S^3 \times_{T^3} \mathbb{R}^2$, where T^3 acts on $S^3 \times S^3 \times S^3 \times S^3$ by the action described above and on $\mathbb{R}^2$ by ϕ. By the above, choosing appropriate ϕ, we can realize in this way *any* rank 2 bundle over P with Euler class (x, y, z) with respect to the basis α, β, γ of $H^2(P) \cong H^2(S^2 \times S^2 \times S^2)$.

Recall that by the proof of Theorem A, we can choose ξ so that $e(\xi) = (0, q, r)$.

Let us choose ζ so that $e(\zeta) = (\eta^S)^*(0, q, -r)$ where we recall that η^S is the sphere bundle $S^3 \to P \to S^2 \times S^2 \times S^2$.

Let $\zeta_{a,b}$ be the pullback of ζ to $M_{a,b}$ via the natural projection $\pi\colon M_{a,b} \to P$.

We will show that infinitely many of the stabilized bundles $\tilde{\zeta}_{a,b} = \zeta_{a,b} \oplus \epsilon^{l-2}$ satisfy the statement of Theorem B if $l > 10$.

Let us first check the geometric conditions. The total space of $\tilde{\zeta}_{a,b}$ can be written as $E(\tilde{\zeta}_{a,b}) = S^3 \times S^3 \times S^3 \times S^3 \times_{T^2_{a,b}} \mathbb{R}^l$. Here, $T^2_{a,b} \subset T^3$ is the subtorus which corresponds to the bundle $S^1 \to M_{a,b} \to P$. Therefore, by Lemma A.2, we have uniform curvature bounds. Note that while the manifolds in question are not compact, it is easy to see that curvature remains uniformly bounded at infinity, so that Lemma A.2 still applies. Alternatively, rather than taking $\mathbb{R}^l$ with a flat metric, we can take it with a rotationally symmetric non-negatively curved metric *isometric* to $S^{l-1} \times \mathbb{R}_+$ at infinity. Then, the uniform curvature bounds follow directly from Lemma A.2.

Of course, the soul of $E(\tilde{\zeta}_{a,b})$ is isometric to $M_{a,b}$, and thus, all the souls have bounded diameter and are not homeomorphic for different a.

Next, we will show that infinitely many of the bundles $\tilde{\zeta}_{a,b}$ have diffeomorphic total spaces.

First, by the same computation as in the proof of Theorem A, we find $p_1(\tilde{\zeta}_{a,b}) = -2qar$ and hence, $p_1(E((\tilde{\zeta}_{a,b})) = p_1(\tilde{\zeta}_{a,b}) + p_1(M_{a,b}) = -2qar + 2qar = 0$.

Since Pontrjagin classes determine a bundle up to finite ambiguity, infinitely many of the spaces $E(\tilde{\eta}_{a,1})$ are tangentially homotopy equivalent and hence diffeomorphic.

This observation is already sufficient to produce examples of manifolds with infinitely many non-negatively pinched metrics whose souls have bounded diameter and are mutually non-homeomorphic. Unfortunately, it does not give us the precise diffeomorphism type of these manifolds.

However, with a little more work, we can show that, in fact, *all* manifolds $E(\tilde{\zeta}_{a,b})$ with $(a,b)=1$ are diffeomorphic to $S^2\times S^2\times S^3\times S^3\times\mathbb{R}^l$.

Look at the following commutative diagram:

$$\begin{array}{ccccc} S^3 & \longrightarrow & M_{a,b} & \xrightarrow{\eta^S_{ab}} & N_{a,b,0} \\ & & \pi\big\downarrow & & \pi\big\downarrow \\ S^3 & \longrightarrow & P & \xrightarrow{\eta^S} & S^2\times S^2\times S^2 \end{array}$$

First, notice that the bundle ζ is the pullback via η^S of the bundle $\hat{\zeta}$ over $S^2\times S^2\times S^2$ with "the same" Euler class. Bundle $\hat{\zeta}$ can be written as $S^3\times S^3\times S^3\times_{T^3}\mathbb{R}^2$ where T^3 acts on $\mathbb{R}^2$ by the representation ϕ.

Similarly, $\zeta_{a,b}=\eta^{S*}_{a,b}(\hat{\zeta}_{a,b})$.

Next, observe that $TE(\tilde{\zeta}_{a,b})|_{M_{ab}}=TM_{a,b}\oplus\zeta_{a,b}\oplus\epsilon^{l-2}=TM_{a,b}\oplus\epsilon^1\oplus\zeta_{a,b}\oplus\epsilon^{l-3}=\eta^{S*}_{a,b}TN_{a,b,0}\oplus\eta^{S*}_{a,b}(\xi_{a,b}\oplus\epsilon^2)\oplus\zeta_{a,b}\oplus\epsilon^{l-3}=\eta^{S*}_{a,b}(TN_{a,b,0}\oplus\xi_{a,b}\oplus\hat{\zeta}_{a,b}\oplus\epsilon^{l-1})$.

Since $N_{a,b,0}$ is the total space of an S^1 bundle over $S^2\times S^2\times S^2$, it immediately follows that $TN_{a,b,0}\oplus\epsilon^{l-1}=e^{l+6}$. (Alternatively, this is also clear since $N_{a,b,0}$ is diffeomorphic to $S^2\times S^2\times S^3$ by [**4**].)

Thus, $TE(\tilde{\zeta}_{a,b})|_{M_{ab}}=\eta^{S*}_{a,b}(\xi_{a,b}\oplus\hat{\zeta}_{a,b})\oplus\epsilon^{l+6}=\eta^{S*}_{a,b}\pi^*(\xi\oplus\hat{\zeta})\oplus\epsilon^{l+6}=\eta^{S*}_{a,b}\pi^*(\epsilon^4)\oplus\epsilon^{l+6}=\epsilon^{10+l}$. Here, the next to last equality holds because by the choice of ζ, we have that $e(\hat{\zeta})=(0,q,-r)$ and $e(\xi)=(0,q,r)$ so that $\xi\oplus\hat{\zeta}=\epsilon^4$ by Lemma A.4 below.

Thus, $E(\tilde{\zeta}_{a,b})$ is tangentially equivalent and hence, diffeomorphic to $S^2\times S^2\times S^3\times S^3\times\mathbb{R}^l$. q.e.d.

Remark 5.1. Using the same procedure as in the proof of Theorem B, we can also construct manifolds with non-trivial p_1 which admit infinitely many non-negatively pinched metrics with non-homeomorphic souls.

6. Proof of Theorem C

We will use the same notations as in the proofs of Theorem A and Theorem B. Let us first construct the Riemannian manifolds in question. The construction is very similar to the one used in the proof of Theorem B. Therefore, we will skip some details.

Let m be as in the proof of Theorem A. Let us fix positive integers n and k and look at a rank 2 vector bundle ξ over $S^2 \times S^2 \times S^2$ with Euler class $m(0,1,k)$. Let $S^3 \to P \to S^2 \times S^2 \times S^2$ be the sphere bundle in $\xi \oplus \epsilon^2$. Look at the rank 2 bundles ζ_1, ζ_2 over P with Euler classes $m\pi^*(0,1,-k), m\pi^*(n+1,n,0)$, respectively. Now, look at the S^1 bundle $\pi_{ab}\colon N_{a,b} \to P$ over P with Euler class $\pi^*(a,b,0)$, where $(a,b) = 1$ and pull back $\zeta = \zeta_1 \oplus \zeta_2$ to $N_{a,b}$.

By the proof of Theorem A, $N_{a,b}$ is homotopy equivalent to $S^2 \times S^2 \times S^3 \times S^3$ for any pair of integers a, b with $(a,b) = 1$.

As before, we also see that the total space of the bundle $\pi^*_{a,b}(\zeta)$ admits a complete metric with $0 \le \sec \le C$ and the soul isometric to $N_{a,b}$ with $\mathrm{diam}(N_{a,b}) \le D$ where C, D are independent of a, b.

A computation similar to the one in the proof of Theorem B shows that for the first Pontrjagin and Euler classes of the bundles in question, we have

$$p_1(N_{a,b}) = 2m^2 ak\omega \wedge \gamma, \qquad p_1(\pi^*_{a,b}(\zeta)) = -2m^2 ak\omega \wedge \gamma,$$

and

$$e(\pi^*_{a,b}(\zeta)) = m^2(-b(n+1) + an)k\omega \wedge \gamma.$$

Set $a = 1 + r(n+1), b = 1 + rn$ where $r \in \mathbb{N}$ and let $N_r = N_{1+r(n+1),1+rn}$.

Then

$$p_1(N_r) = 2m^2 ak\omega \wedge \gamma, \qquad p_1(\pi^*_r(\zeta)) = -2m^2 ak\omega \wedge \gamma, \tag{1}$$

and

$$e(\pi^*_r(\zeta)) = -m^2 k\omega\gamma. \tag{2}$$

This means that the manifolds N_r have distinct Pontrjagin classes and hence are mutually non-homeomorphic.

Let E_r be the total space of the bundle $\pi^*_r(\zeta)$. From the above, we see that

$$p_1(E_r) = 2m^2 ak\omega \wedge \gamma - 2m^2 ak\omega \wedge \gamma = 0. \tag{3}$$

Look at the spaces $X_r = E_r \times TS^4$ with the product metric where we take the natural non-negatively curved metric on TS^4 given by the submersion metric on $TS^4 = SO(5) \times_{SO(4)} \mathbb{R}^4$.

We claim that the spaces X_r fall into finitely many diffeomorphism classes.

Indeed, let $f_r\colon N_1 \to E_r$ be the homotopy equivalence given by the homotopy equivalence of the souls followed by the embedding of the soul into E_r. Note that $H^4(N_r) \cong H^4(S^2 \times S^2 \times S^3 \times S^3) \cong \mathbb{Z}$. By (2), by possibly composing f_r with an orientation reversing self homotopy equivalence of N_1, we can assume that $f_r^*(e(\pi_r^*(\zeta))) = e(\pi_1^*(\zeta))$.

Observe that X_r is the total space of a rank $4+4=8$ vector bundle over $N_r \times S^4$ and $\dim N_r \times S^4 = 10 + 4 = 14$. Since $f_r \times \mathrm{Id}_{S^4}$ is a homotopy equivalence and $3\cdot 8 > 14+2$, we are in the metastable range and by Haefliger's Embedding Theorem [**23**], $f_r \times \mathrm{Id}_{S^4}$ is homotopic to an embedding g_r. Since the codimension of $N_r \times S^4$ in X_r is $= 8 > 3$, by [**32**], X_r is diffeomorphic to the total space of the normal bundle ν_{g_r}.

From (1) and (3), using the Whitney formula, we see that all ν_{g_r} have the same Pontrjagin classes. From (2), we see also that all ν_{g_r} have the same *non-trivial* Euler classes equal to $-2m^2k\omega \wedge \gamma \wedge [d_{vol}(S^4)]$. That is because the rational Euler class of ν_{g_r} is a homotopy invariant of g_r which can be defined homologically by the formula

$$\langle e(\nu_{g_r}), x\rangle = g_{r*}[X_1] \cdot g_{r*}(x)$$

for any $x \in H_8(X_1)$ where $\cdot$ is the algebraic intersection number. Thus,

$$e(\nu_{g_r}) = f_r^*(e(\pi_r^*(\zeta)) \cup e(S^4)) = -m^2k\omega\gamma \wedge 2[d_{vol}(S^4)]$$

by (2) and the fact that $\chi(S^4) = 2$. Here, we disregard the difference between rational and integer coefficients since all involved cohomology groups are torsion free. See also [**6**] for a more detailed discussion of invariants of maps.

Thus, we see that all the bundles ν_{g_r} have the same Euler and Pontrjagin classes. Since Euler and Pontrjagin classes determine a bundle up to a finite ambiguity, the bundles ν_{g_r} fall into finitely many isomorphism classes. Hence, the total spaces of ν_{g_r} fall into finitely many diffeomorphism classes. By the above, the total space of ν_{g_r} is diffeomorphic to X_r and hence all manifolds X_r also fall into finitely many diffeomorphism classes.

Thus, after passing to a subsequence, we can assume that all X_{r_i} are diffeomorphic to $M = X_{r_1}$.

We claim that M satisfies the conclusion of the Theorem. Observe that X_r carries by construction a natural metric of $0 \le sec \le C$ with soul isometric to $N_r \times S^4$ of diam $\le D$. Hence, all the souls have distinct Pontrjagin classes by (1) and thus are mutually not homeomorphic.

Since for any non-negatively curved metric g and any self-diffeomorphism of the underlying manifold ϕ, the souls of g and $\phi^*(g)$ are diffeomorphic, the statement of Theorem C will follow from the following

Lemma 6.1. *Let $(M, g_t), t \in [0,1]$ be a continuous family of non-negatively curved metrics such that the normal bundle to the soul of (M, g_0) has non-trivial rational Euler class.*

Then, all the souls of (M, g_t) are diffeomorphic.

Proof. Let S_t be the soul of (M, g_t). We claim that the family $(S_t, g_t|_{S_t})$ is continuous in Gromov–Hausdorff topology. Observe that since $S_t \hookrightarrow M$ is a homotopy equivalence, by the same argument as above, the rational Euler class of ν_{S_t} is non-zero for any t. Therefore, it is enough to show that $S_t \overset{G-H}{\to} S_0$ as $t \to 0$.

Let $\pi_t \colon M \to S_t$ be the Sharafutdinov retraction with respect to g_t.

Let d_t be the inner metric on M induced by g_t. Since $g_t \to g_0$ uniformly on compact sets, we clearly have that for any $x, y \in S_0$, $d_t(x, y) \leq d_0(x, y) + \epsilon_t$ where $\epsilon_t \to 0$ as $t \to 0$. Since π_t is distance non-increasing, we see that $d_t(\pi_t(x), \pi_t(y)) \leq d_0(x, y) + \epsilon_t$ for any $x, y \in S_0$. Since $\pi_t \colon S_0 \to S_t$ is a homotopy equivalence, it must be onto and hence $\mathrm{diam} S_t \leq \mathrm{diam} S_0 + \epsilon_t$.

From the assumption on the Euler class, we see that $S_t \cap S_0 \neq \emptyset$ for any t and since, by the above, all S_t have uniformly bounded diameters, they all must lie in some fixed closed ball $\bar{B}(p, D)$ where the ball is taken with respect to d_0. Again, using that g_t converges to g_0 uniformly on compact sets, we have that $d_0(x, y) \leq d_t(x, y) + \epsilon_t$ for any $x, y \in S_t$. Hence, $d_0(\pi_0(x), \pi_0(y)) \leq d_t(x, y) + \epsilon_t$ for any $x, y \in S_t$. Combining this with the above, we finally get that

$$d_0(\pi_0(\pi_t(x)), \pi_0(\pi_t(x))) \leq d_0(x, y) + 2\epsilon_t \quad \text{for any } x, y \in S_0.$$

By Lemma A.5, this implies that for some $\tilde{\epsilon}(t) \underset{t \to 0}{\to} 0$

$$\begin{aligned} d_0(x, y) - 2\tilde{\epsilon}_t &\leq d_0(\pi_0(\pi_t(x)), \pi_0(\pi_t(x))) \\ &\leq d_0(x, y) + 2\epsilon_t \quad \text{for any } x, y \in S_0. \end{aligned}$$

Hence, $\pi_0 \circ \pi_t \colon S_0 \to S_0$ is a $\max(\epsilon_t, \tilde{\epsilon}_t)$-Hausdorff approximation and the same is true for $\pi_0 \colon (S_t, d_t) \to (S_0, d_0)$ which proves that $S_t \overset{G-H}{\to} S_0$ as $t \to 0$.

Since S_t is a smooth manifold for any t and $\dim S_t = \dim S_0$, by Yamaguchi's Stability theorem [**36**], this implies that S_t is diffeomorphic to S_0 for all small t. q.e.d.

As observed before, Lemma 6.1 implies that all elements of a connected component of $\mathfrak{R}_{\mathrm{sec} \geq 0}(M)/\mathrm{Diff}(M)$ have diffeomorphic souls. This immediately implies the statement of Theorem C. q.e.d.

Remark 6.2. We suspect that Lemma 6.1 is true without any assumptions on the rational Euler class. If this holds true, then the examples constructed in the proof of Theorem B would directly yield Theorem C.

Appendix A.

We will need the following lemma which is an easy consequence of some well-known topological results:

Lemma A.1. *There exists an integer m such that for any element $p \in H^4(S^2 \times S^2 \times S^2)$, there exists a closed smooth manifold M^6 and a homotopy equivalence $f\colon\ M \to S^2 \times S^2 \times S^2$ such that $f^*(p) = mp_1(M)$.*

Proof.

By the Browder–Novikov Surgery Theorem [**8**, Thm. II.3.1, Cor. II.4.2], given a vector bundle ξ over a simply connected manifold X^6, there exists a manifold M^6 and a homotopy equivalence $M^6 \to X$, such that $f^*(\xi)$ is isomorphic to the stable normal bundle of M if and only if the stable spherical fibration coming from ξ is isomorphic to the Spivak normal spherical fibration $\nu(X)$.

If $X = S^2 \times S^2 \times S^2$, we obviously have that $\nu(X)$ is trivial. Recall that stable spherical fibrations are classified by the homotopy classes of maps into the classifying space BG and that all homotopy groups of BG are finite. The same obstruction theory argument as in the proof of Theorem A shows that there exists an m_1 such that for any $f\colon\ S^2 \times S^2 \times S^2 \to BG$, the map $f \circ F_{m_1}$ is homotopic to a point. Recall here that $F_m = g_m \times g_m \times g_m\colon\ S^2 \times S^2 \times S^2 \to S^2 \times S^2 \times S^2$ where $g_m\colon\ S^2 \to S^2$ has $\deg g_m = m$.

Next, observe that by looking at Whitney sums of rank 2 bundles, we can realize any *even* element of $H^4(S^2 \times S^2 \times S^2)$ as the first Pontrajagin class of a vector bundle.

Combining these two facts, we obtain the desired claim with $m = 2m_1$. q.e.d.

The geometric part of the proof of Theorem A is based on the following lemma, which is originally due to Eschenburg [**13**, Prop 22]. For convenience of the reader, we include a short outline of its proof.

Lemma A.2. *Let (M, g) be a closed Riemannian manifold on which a k-dimensional torus T^k acts freely and isometrically. Then, there exist $C, D > 0$ such that for any subtorus $T^m \subset T^k$, the quotient manifold M/T^m, when equipped with the induced quotient metric, satisfies*

$$|\sec(M/T^m)| \leq C \ \textit{and}\ \operatorname{diam}(M/T^m) \leq D.$$

Proof. The uniform diameter bound is obvious, and we need to find a uniform bound on the O'Neill term in the Gray–O'Neill curvature formula for Riemannian submersions. As the formula is local, it makes sense to look at the local quotients of M by $\mathbb{R}^m \subset \mathbb{R}^k$ where $\mathbb{R}^k$ is the universal cover of T^k. The compactness of the Grassmannian of m-planes in $\mathbb{R}^k$ now implies the result. q.e.d.

Lemma A.3. *Let $S^1 \to P \to S^2 \times S^2$ be a principal S^1 bundle such that P is simply connected. Then P is homotopy equivalent to $S^2 \times S^3$.*

In fact—though we will not need this fact in this paper—by a theorem of Barden [**4**], P is *diffeomorphic* to $S^2 \times S^3$.

Proof. It is easy to see that $H_2(P) \cong H_3(P) \cong \mathbb{Z}$.

We can write P as the homogeneous space $S^3 \times S^3/S^1$ for some $S^1 \subset S^3 \times S^3$. From the Gysin sequence, it is easy to see that the map $H_3(S^3 \times S^3) \to H_3(P)$ is onto. Since, by the Hurewicz theorem, $\pi_3(S^3 \times S^3) \cong H_3(S^3 \times S^3)$ and since, by the long exact homotopy sequence, $\pi_3(P) \cong \pi_3(S^3 \times S^3)$ we see that $\pi_3(P) \to H_3(P)$ is also surjective. Let $f\colon S^2 \to P$ be a map representing a generator of $\pi_2(P) \cong H_2(P) \cong \mathbb{Z}$. Let $g\colon S^3 \to P$ be a map representing a generator of $H_3(P) \cong Z$. Let $\hat{g}\colon S^3 \to S^3 \times S^3$ be a lift of g. Such a lift exists by the previous discussion. Consider the map $F\colon S^2 \times S^3 \to P$ given by $F(x, y) = \hat{g}(y) \cdot f(x)$, where $x \in S^2, y \in S^3$ and where the $\cdot$ represents the homogeneous space action of $S^3 \times S^3$ on P. It is straightforward to check that F induces an isomorphism on homology and thus is a homotopy equivalence. q.e.d.

Lemma A.4. *Let ξ_1, ξ_2 be rank 2 bundles over $S^2 \times S^2$ such that $e(\xi_1) = (q, r)$ and $e(\xi_2) = (q, -r)$ with respect to the canonical basis of $H^2(S^2 \times S^2)$.*

Then, $\xi_1 \oplus \xi_2$ is trivial.

Proof. A direct computation shows that $w_2(\xi_1 \oplus \xi_2) = p_1(\xi_1 \oplus \xi_2) = e(\xi_1 \oplus \xi_2) = 0$.

Since, $w_2(\xi_1 \oplus \xi_2) = 0$, its classifying map f into $BSO(4)$ factors as $f = g \circ c$. where $c\colon S^2 \times S^2 \to S^4$ is the collapsing map of degree 1. Thus, $\xi_1 \oplus \xi_2 = c^*(\eta)$ where η is a rank 4 bundle over S^4. Since c^* is an isomorphism on H^4, $p_1(\eta) = e(\eta) = 0$ and hence, η is trivial. q.e.d.

Lemma A.5. *Let S be a closed Riemannian manifold. There exists a function $\delta\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\delta(\epsilon) \to 0$ as $\epsilon \to 0$ and such that the following holds. If $f\colon S \to S$ is a homotopy equivalence satisfying*

$$d(f(x), f(y)) \le d(x, y) + \epsilon \qquad \text{for any } x, y \in S,$$

then

$$d(x, y) - \delta(\epsilon) \le d(f(x), f(y)) \le d(x, y) + \epsilon \qquad \text{for any } x, y \in S.$$

Proof. Suppose Lemma A.5 is false. Then, there exists a sequence $f_i\colon S \to S$ as well as a sequence $\epsilon_i \to 0$ satisfying

$$d(f(x), f(y)) \le d(x, y) + \epsilon_i \qquad \text{for any } x, y \in S$$

such that for some $\delta > 0$ there exist $x_i, y_i \in S$ such that $d(f(x_i), f(y_i)) \leq d(x_i, y_i) - \delta$. By Arzela–Ascoli and the compactness of S, we can assume that f_i uniformly converges to $f\colon S \to S$ and $x_i \to x_0, y_i \to y_0$. Then, f is 1-Lipschitz and $d(f(x_0), f(y_0)) \leq d(x_0, y_0) - \delta$. By uniform convergence f_i is homotopic to f for large i. Hence, f is onto. A surjective 1-Lipschitz self-map of a closed manifold has to preserve the volume which easily implies that it must be an isometry. Therefore, we must have $d(x_0, y_0) = d(f(x_0), f(y_0))$. This is a contradiction and hence Lemma A.5 is true. q.e.d.

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Department of Mathematics
University of Maryland
College Park, MD 20742
E-mail address: vtk@math.umd.edu

Department of Mathematics
Pennsylvania State University
University Park, State College, PA 16802
E-mail address: petrunin@math.psu.edu

Department of Mathematics
University of Kiel
Ludewig-Meyn–Strasse 4
D-24098 Kiel, Germany
E-mail address: tusch@math.uni-kiel.de