

## THE MOMENT MAP REVISITED

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### Abstract

In this paper, we show that the notion of moment map for the Hamiltonian action of a Lie group on a symplectic manifold is a special case of a much more general notion. In particular, we show that one can associate a moment map to a family of Hamiltonian symplectomorphisms, and we prove that its image is characterized, as in the classical case, by a generalized “energy-period” relation.

### 1. Introduction

In this paper, we will show that the classical moment map associated with a Hamiltonian action of a Lie group on a symplectic manifold,  $M$ , is a special example of a much more general moment map. More explicitly, in the set-up, we will consider below, the action

$$(1.1) \quad k \in K \longrightarrow \tau_k \in \text{Symplect}(M)$$

of a Lie group,  $K$ , on  $M$  gets replaced by a map

$$(1.2) \quad s \in S \longrightarrow f_s \in \text{Symplect}(M)$$

of a manifold,  $S$ , into the symplectomorphism group of  $M$ , or, more generally still, a map

$$(1.3) \quad s \longrightarrow \Gamma_s$$

of  $S$  into the space of canonical relations (see below) in  $M \times M$ .

We will show that, modulo some topological assumptions, one can associate to (1.2) a moment map

$$(1.4) \quad \Phi : M \times S \longrightarrow T^*S$$

which is compatible with the projection,  $M \times S \longrightarrow S$  and which makes the set

$$\Gamma = \{(m, f_s(m), \Phi(m, s)); (m, s) \in M \times S\}$$

into a Lagrangian submanifold of  $M \times M \times T^*S$ . In the group context, (1.1), the map  $\Phi$  gives, by restriction to  $M = M \times \{e\}$ , the usual moment map,  $\phi : M \longrightarrow \mathfrak{k}^*$ .

To prove that the map (1.4) exists and to describe some of its basic properties, it is useful to consider an even more general set-up than (1.2) and (1.3), namely, with  $M \times M^-$  replaced by  $M$ , a fibration,  $\pi : Z \rightarrow S$  and a map  $G : Z \rightarrow M$  which maps the fibers of  $\pi$  onto Lagrangian submanifolds of  $M$ . In this context, (1.4) gets replaced by a moment map

$$(1.5) \quad \Phi : Z \rightarrow T^*S.$$

(Moment maps of this type had been considered by us before in the context of integral geometry, and we will discuss the relation of the results of this paper to these earlier results of ours in Section 6 below.)

In the group context, the definition (1.4) of moment mapping is due to Alan Weinstein, [10], and in Section 2, we will discuss Weinstein's approach to moment geometry. Underlying this approach is the concept of the symplectic "category", a category in which the objects are symplectic manifolds and the morphisms are canonical relations and in Section 4 we will discuss some of the main features of this category which was introduced by Weinstein in [10] and by us in [3].

In Section 3, we formulate our results for families of symplectomorphisms as in equation (1.2) and in Section 5, we return to the general formulation and prove the main results.

In Section 5.2, we show that the derivative of the map (1.4) satisfies an identity similar to the "derivative identity" of the standard moment map.

In Section 7, we discuss the "image" of the moment map. As Weinstein shows in [10], for the usual moment map, it is useful to think of this "image" as a Lagrangian submanifold of  $T^*S$ . For instance, for a torus action, this image consists not only of the moment polytope, but of a labeling of its faces by isotropy groups. (What Sue Tolman calls the "x-ray" of the moment polytope.) More generally, for  $\mathbb{R}^n$  actions, this "image" is what is known in the theory of dynamical systems as the "period energy" relation, and the main result of Section 7.3 asserts that this is not only true of moment maps associated with  $\mathbb{R}^n$ -actions, but also of moment maps in general.

## 2. The classical moment map.

In this section, we review the classical notion of moment map from Weinstein's point of view.

Let  $(M, \omega)$  be a symplectic manifold,  $K$  a connected Lie group and  $\tau$  an action of  $K$  on  $M$  preserving the symplectic form. From  $\tau$ , one gets an infinitesimal action

$$(2.1) \quad \delta\tau : \mathfrak{k} \rightarrow \text{Vect}(M)$$

of the Lie algebra,  $\mathfrak{k}$ , of  $K$ , mapping  $\xi \in \mathfrak{k}$  to the vector field,  $\delta\tau(\xi) =: \xi_M$ . In particular, for  $p \in M$ , one gets from (2.1) a linear map,

$$(2.2) \quad d\tau_p : \mathfrak{k} \longrightarrow T_p M, \quad \xi \longrightarrow \xi_M(p);$$

and from  $\omega_p$ , a linear isomorphism,

$$(2.3) \quad T_p \longrightarrow T_p^* \quad v \longrightarrow i(v)\omega_p;$$

which can be composed with (2.2) to get a linear map

$$(2.4) \quad d\tilde{\tau}_p : \mathfrak{k} \longrightarrow T_p^* M.$$

**Definition.** A  $K$ -equivariant map

$$(2.5) \quad \phi : M \longrightarrow \mathfrak{k}^*$$

is a *moment map*, if for every  $p \in M$ :

$$(2.6) \quad d\phi_p : T_p M \longrightarrow \mathfrak{k}^*$$

is the transpose of the map (2.4).

The property (2.6) determines  $d\phi_p$  at all points  $p$ , and hence, determines  $\phi$  up to an additive constant,  $c \in (\mathfrak{k}^*)^K$  if  $M$  is connected. Thus, in particular, if  $K$  is semi-simple, the moment map, if it exists, is unique. As for the existence of  $\phi$ , the duality of (2.4) and (2.6) can be written in the form

$$(2.7) \quad i(\xi_M)\omega = d\langle\phi, \xi\rangle$$

for all  $\xi \in \mathfrak{k}$ ; and this shows that the vector field,  $\xi_M$ , has to be Hamiltonian. If  $K$  is compact, the converse is true. A sufficient condition for the existence of  $\phi$  is that each of the vector fields,  $\xi_M$ , be Hamiltonian. (See, for instance, [5], Section 26.) An equivalent formulation of this condition will be useful below:

**Definition.** A symplectomorphism,  $f : M \longrightarrow M$  is Hamiltonian if there exists a family of symplectomorphisms,  $f_t : M \longrightarrow M$ ,  $0 \leq t \leq 1$ , depending smoothly on  $t$  with  $f_0 = id_M$  and  $f_1 = f$ , such that the vector field

$$v_t = f_t^{-1} \frac{df_t}{dt}$$

is Hamiltonian for all  $t$ .

It is easy to see that  $\xi_M$  is Hamiltonian for all  $\xi \in \mathfrak{k}$  if and only if the symplectomorphism,  $\tau_g$ , is Hamiltonian for all  $g \in K$ .

Our goal in this article is to describe a generalized notion of moment mapping in which there are no group actions involved. First, however, we recall a very suggestive way of thinking about moment mappings and the “moment geometry” associated with moment mappings, due to

Alan Weinstein, [10]. From the left action of  $K$  on  $T^*K$ , one gets a trivialization

$$T^*K = K \times \mathfrak{k}^*$$

and via this trivialization, a Lagrangian submanifold

$$\Gamma_\tau = \{(m, \tau_g m, g, \phi(m)); m \in M, g \in K\},$$

of  $M \times M^- \times T^*K$ , which Weinstein calls *the moment Lagrangian*. He views this as a canonical relation between  $M \times M^-$  and  $T^*K$ , or as a “map”

$$\Gamma_\tau : M^- \times M \rightarrow T^*G,$$

and points out that many basic constructions in moment geometry can be formulated in a particularly succinct and illuminating way in terms of this “map”. For example, modulo clean intersection hypotheses, such a “map” maps Lagrangian submanifolds of  $M^- \times M$  onto Lagrangian submanifolds of  $T^*K$  and vice-versa. For instance, the diagonal in  $M^- \times M$  gets mapped by  $\Gamma_\tau$  into a disjoint union of Lagrangian submanifolds of  $T^*K$ , and these are just the pieces of the “character Lagrangian” of  $(M, \tau)$ . In the other direction, the zero-section of  $T^*K$  gets mapped onto the fiber product

$$(2.8) \quad \Pi_\tau = \Sigma \times_{M_{\text{red}}} \Sigma$$

where  $\Sigma = \phi^{-1}(0)$  and  $\Sigma \rightarrow M_{\text{red}}$  is the symplectic reduction of  $M$ . Since  $\Pi_\tau$  is a Lagrangian submanifold of  $M^- \times M$ , it can be viewed as a canonical relation or “map”

$$\Pi_\tau : M \rightarrow M$$

with the property,  $\Pi_\tau \circ \Pi_\tau = \Pi_\tau$ . Moreover, if  $Q(M)$  is a quantization of the action,  $\tau$ , the space of  $K$ -invariant elements in  $Q(M)$  is, by the principle of “quantization commutes with reduction” [4] and [2], the quantization of  $M_{\text{red}}$ ; so the orthogonal projection of  $Q(M)$  onto this subspace can be thought of as the quantization of  $\Pi_\tau$ .

This idea of looking at moment geometry with the view in mind of its “quantum” applications is useful in a much more general context: As Weinstein explains in [10] and as we explain in [7], one can think of symplectic manifolds and canonical relations between them as being the objects and morphisms (or “maps”) in a symplectic “category” and the “points” of a symplectic manifold as being its “categorical points”, i.e., its Lagrangian submanifolds. Unfortunately, the symplectic category is only euphemistically a “category” since one has to impose clean

intersection hypotheses on canonical relations in order to be able to compose them or to map “points” of the source manifold of a canonical relation onto “points” of the target manifold. We will review all this in Section 4 below. Nonetheless, this categorical way of thinking leads to interesting constructions in symplectic geometry that one would probably not have stumbled across otherwise (such as the character Lagrangian construction mentioned above.) In particular, it is what led us to our definition of the “image of the moment mapping” in Section 7 below.

### 3. Families of symplectomorphisms.

We now turn to the first stage of our generalization of the moment map, where the group action is replaced by a family of symplectomorphisms:

Let  $M, \omega$  be a symplectic manifold,  $S$  an arbitrary manifold and  $f_s, s \in S$ , a family of symplectomorphisms of  $M$  depending smoothly on  $s$ . For  $p \in M$  and  $s_0 \in S$ , let  $g_{s_0,p} : S \rightarrow M$  be the map,  $g_{s_0,p}(s) = f_s \circ f_{s_0}^{-1}(p)$ . Composing the derivative of  $g_{s_0,p}$  at  $s_0$

$$(3.1) \quad (dg_{s_0,p})_{s_0} : T_{s_0}S \rightarrow T_pM$$

with the map (2.3), one gets a linear map

$$(3.2) \quad (\widetilde{dg_{s_0,p}})_{s_0} : T_{s_0}S \rightarrow T_p^*M.$$

Now, let  $\Phi$  be a map of  $M \times S$  into  $T^*S$  which is compatible with the projection,  $M \times S \rightarrow S$  in the sense

$$\begin{array}{ccc} M \times S & \xrightarrow{\Phi} & T^*S \\ & \searrow & \downarrow \\ & & S \end{array}$$

commutes; and for  $s_0 \in S$  let

$$\Phi_{s_0} : M \rightarrow T_{s_0}^*S$$

be the restriction of  $\Phi$  to  $M \times \{s_0\}$ .

**Definition.**  $\Phi$  is a *moment map* if, for all  $s_0$  and  $p$ ,

$$(3.3) \quad (d\Phi_{s_0})_p : T_pM \rightarrow T_{s_0}^*S$$

is the transpose of the map (3.2).

We will prove below that a sufficient condition for the existence of  $\Phi$  is that the  $f_s$ 's be Hamiltonian; and, assuming that  $\Phi$  exists, we will consider the analogue for  $\Phi$  of Weinstein's moment Lagrangian,

$$(3.4) \quad \Gamma_\Phi = \{(m, f_s(m), \Phi(m, s)); m \in M, s \in S\},$$

and ask if the analogue of Weinstein's theorem is true. Is (3.4) a Lagrangian submanifold of  $M \times M^- \times T^*S$ ? Equivalently, consider the imbedding of  $M \times S$  into  $M \times M^- \times T^*S$  given by the map

$$G : M \times S \longrightarrow M \times M^- \times T^*S,$$

where  $G(m, s) = (m, f_s(m), \Phi(m, s))$ . Is this a Lagrangian imbedding? The answer is "no" in general, but we will prove:

**Theorem 1.** *The pull-back by  $G$  of the symplectic form on  $M \times M^- \times T^*S$  is the pull-back by the projection,  $M \times S \longrightarrow S$  of a closed two-form,  $\mu$ , on  $S$ .*

If  $\mu$  is exact, i.e., if  $\mu = d\nu$ , we can modify  $\Phi$  by setting

$$\Phi_{\text{new}}(m, s) = \Phi_{\text{old}}(m, s) - \nu_s,$$

and for this modified  $\Phi$ , the pull-back by  $G$  of the symplectic form on  $M \times M^- \times T^*S$  will be zero; so, we conclude:

**Theorem 2.** *If  $\mu$  is exact, there exists a moment map,  $\Phi : M \times S \longrightarrow T^*S$ , for which  $\Gamma_\Phi$  is Lagrangian.*

The following converse result is also true.

**Theorem 3.** *Let  $\Phi$  be a map of  $M \times S$  into  $T^*S$  which is compatible with the projection of  $M \times S$  onto  $S$ . Then, if  $\Gamma_\Phi$  is Lagrangian,  $\Phi$  is a moment map.*

#### Remarks.

- 1) A moment map with this property is still far from being unique; however, the ambiguity in the definition of  $\Phi$  is now a *closed* one-form,  $\nu \in \Omega^1(S)$ .
- 2) if  $[\mu] \neq 0$ , there is a simple expedient available for making  $\Gamma_\Phi$  Lagrangian. One can modify the symplectic structure of  $T^*S$  by adding to the standard symplectic form the pull-back of  $-\mu$  to  $T^*S$ .
- 3) Let  $\mathbb{G}_e$  be the group of Hamiltonian symplectomorphisms of  $M$ . Then, for every manifold,  $S$  and smooth map

$$F : S \longrightarrow \mathbb{G}_e$$

one obtains by the construction above a cohomology class  $[\mu]$  which is a homotopy invariant of the mapping  $F$ .

- 4) For a smooth map  $F : S \longrightarrow \mathbb{G}_e$ , there exists an analogue of the character Lagrangian. Think of  $\Gamma_\Phi$  as a canonical relation or "map"

$$\Gamma_\Phi : M^- \times M \rightrightarrows T^*S$$

and define the character Lagrangian of  $F$  to be the image with respect to  $\Gamma_\Phi$  of the diagonal in  $M^- \times M$ .

Our proof of the results above will be an illustration of the principle: the more general the statement of a theorem, the easier it is to prove. We will first generalize these results by assuming that the  $f_s$ 's are canonical relations rather than canonical transformations, i.e., are “maps” from  $M$  to  $M$  in Weinstein’s sense. Next, we will get rid of “maps” altogether and replace  $M \times M^-$  by  $M$  itself and canonical relations by Lagrangian submanifolds of  $M$ .

Before doing so, it will be useful to recall some ideas related to the symplectic “category”.

#### 4. The symplectic “category”.

This section is a summary of basic facts about the symplectic category. (Some of these are not easily accessible in the literature, so we have included them here for the convenience of the reader.)

**4.1. The linear symplectic category.** If  $V$  is a symplectic vector space we let  $V^-$  denote the same vector space but with the form  $\omega$  of  $V$  replaced by  $-\omega$ . If  $V_1$  and  $V_2$  are symplectic vector spaces, we let  $V_1^- \times V_2$  denote the symplectic vector space with the direct sum symplectic structure. A Lagrangian subspace  $\Gamma$  of  $V_1^- \times V_2$  is called a **linear canonical relation** from  $V_1$  to  $V_2$ . The purpose of this subsection is to define a category, **LinSymp** whose objects are symplectic vector spaces, whose morphisms are linear canonical relations and whose composition law is given by composition of relations. More explicitly, if  $V_3$  is a third symplectic vector space and

$$\Gamma_1 \text{ is a Lagrangian subspace of } V_1^- \oplus V_2$$

and

$$\Gamma_2 \text{ is a Lagrangian subspace of } V_2^- \oplus V_3.$$

then, as a *set* the composition

$$\Gamma_2 \circ \Gamma_1 \subset V_1 \times V_3$$

is defined by

$$(x, z) \in \Gamma_2 \circ \Gamma_1 \Leftrightarrow \exists y \in V_2 \text{ such that } (z, y) \in \Gamma_1 \text{ and } (y, x) \in \Gamma_2.$$

It is clear that the diagonal subspace of  $V^- \times V$  acts as the identity morphism and that the associative law holds. What must be checked is that the composition as defined above is a Lagrangian subspace of  $V_1^- \times V_3$ . It will be convenient to break up the proof of this into two steps:

**4.1.1. The space  $\Gamma_2 \star \Gamma_1$ .** Define

$$\Gamma_2 \star \Gamma_1 \subset \Gamma_1 \times \Gamma_2$$

to consist of all pairs  $((x, y), (y', z))$  such that  $y = y'$ . We will restate this definition in two convenient ways. Let

$$\pi : \Gamma_1 \rightarrow V_2, \quad \pi(v_1, v_2) = v_2$$

and

$$\rho : \Gamma_2 \rightarrow V_2, \quad \rho(v_2, v_3) = v_2.$$

Let

$$\tau : \Gamma_1 \times \Gamma_2 \rightarrow V_2$$

be defined by

$$\tau(\gamma_1, \gamma_2) := \pi(\gamma_1) - \rho(\gamma_2).$$

Then,  $\Gamma_2 \star \Gamma_1$  is determined by the exact sequence

$$(4.1) \quad 0 \rightarrow \Gamma_2 \star \Gamma_1 \rightarrow \Gamma_1 \times \Gamma_2 \xrightarrow{\tau} V_2 \rightarrow \text{Coker } \tau \rightarrow 0.$$

Another way of saying the same thing is to use the language of “fiber products” or “exact squares”. Let  $f : A \rightarrow C$  and  $g : B \rightarrow C$  be maps, say between sets. Then, we express the fact that  $F \subset A \times B$  consists of those pairs  $(a, b)$  such that  $f(a) = g(b)$  by saying that

$$\begin{array}{ccc} F & \longrightarrow & A \\ \downarrow & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

is an **exact square** or a **fiber product diagram**.

Thus, another way of expressing the definition of  $\Gamma_2 \star \Gamma_1$  is to say that

$$(4.2) \quad \begin{array}{ccc} \Gamma_2 \star \Gamma_1 & \longrightarrow & \Gamma_1 \\ \downarrow & & \downarrow \pi \\ \Gamma_1 & \xrightarrow{\rho} & V_2 \end{array}$$

is an exact square.

**4.1.2. The projection  $\alpha : \Gamma_2 \star \Gamma_1 \rightarrow \Gamma_2 \circ \Gamma_1$ .** Consider the map

$$(4.3) \quad \alpha : (x, y, y, z) \mapsto (x, z).$$

By definition,

$$\alpha : \Gamma_2 \star \Gamma_1 \rightarrow \Gamma_2 \circ \Gamma_1.$$



**4.1.3. The kernel and image of a linear canonical relation,** Let  $V_1$  and  $V_2$  be symplectic vector spaces and let  $\Gamma \subset V_1^- \times V_2$  be a linear canonical relation. Let

$$\pi : \Gamma \rightarrow V_2$$

be the projection onto the second factor. Define  $\ker \Gamma$  to be  $\{v \in V_1 \mid (v, 0) \in \Gamma\}$  and  $\text{Im } \Gamma$  to be  $\pi(\Gamma)$ . Let

$$(4.4) \quad \Gamma^\dagger := \{(v_2, v_1) \mid (v_1, v_2) \in \Gamma\}.$$

Thus,  $\Gamma^\dagger$  is a Lagrangian subspace of  $V_2^- \oplus V_1$ , and hence, both  $\ker \Gamma^\dagger$  and  $\text{Im } \Gamma$  are linear subspaces of the symplectic vector space  $V_2$ . We claim that

$$(4.5) \quad (\ker \Gamma^\dagger)^\perp = \text{Im } \Gamma.$$

Here,  $\perp$  means perpendicular relative to the symplectic structure on  $V_2$ .

*Proof.* Let  $\omega_1$  and  $\omega_2$  be the symplectic bilinear forms on  $V_1$  and  $V_2$  so that  $\tilde{\omega} = -\omega_1 \oplus (\omega_2)$  is the symplectic form on  $V_1^- \oplus V_2$ . So,  $v \in V_2$  is in  $\ker \Gamma^\dagger$  if and only if  $(0, v) \in \Gamma$ . Since  $\Gamma$  is Lagrangian,

$$(0, v) \in \Gamma^\perp \Leftrightarrow 0 = -\omega_1(0, v_1) + \omega_2(v, v_2) = -\omega_2(v, v_2) \quad \forall (v_1, v_2) \in \Gamma.$$

But this is precisely the condition that  $v \in (\text{Im } \Gamma)^\perp$ . q.e.d.

The kernel of  $\alpha$  consists of elements of the form,  $(0, v, v, 0)$ . We may thus identify

$$(4.6) \quad \ker \alpha = \ker \Gamma_1^\dagger \cap \ker \Gamma_2$$

as a subspace of  $V_2$ .

If we go back to the definition of the map  $\tau$ , we see that the image of  $\tau$  is given by

$$(4.7) \quad \text{Im } \tau = \text{Im } \Gamma_1 + \text{Im } \Gamma_2^\dagger,$$

a subspace of  $V_2$ . If we compare (4.6) with (4.7), we see that

$$(4.8) \quad \ker \alpha = (\text{Im } \tau)^\perp$$

as subspaces of  $V_2$  where  $\perp$  denotes orthocomplement relative to the symplectic form  $\omega_2$  of  $V_2$ .

**4.1.4. Proof that  $\Gamma_2 \circ \Gamma_1$  is Lagrangian.** Since  $\Gamma_2 \circ \Gamma_1 = \alpha(\Gamma_2 \star \Gamma_1)$  and  $\Gamma_2 \star \Gamma_1 = \ker \tau$ , it follows that  $\Gamma_2 \circ \Gamma_1$  is a linear subspace of  $V_1^- \oplus V_3$ .

It is equally easy to see that  $\Gamma_2 \circ \Gamma_1$  is an isotropic subspace of  $V_1^- \oplus V_2$ . Indeed, if  $(x, z)$  and  $(x', z')$  are elements of  $\Gamma_2 \circ \Gamma_1$ , then there are elements  $y$  and  $y'$  of  $V_2$  such that

$$(x, y) \in \Gamma_1, (y, z) \in \Gamma_2, (x', y') \in \Gamma_1, (y', z') \in \Gamma_2.$$

Then,

$$\omega_3(z, z') - \omega_1(x, x') = \omega_3(z, z') - \omega_2(y, y') + \omega_2(y, y') - \omega_1(x, x') = 0.$$

Hence it suffices to show that  $\dim \Gamma_2 \circ \Gamma_1 = \frac{1}{2} \dim V_1 + \frac{1}{2} \dim V_3$ . It follows from (4.8) that

$$\dim \ker \alpha = \dim V_2 - \dim \operatorname{Im} \tau$$

and from the fact that  $\Gamma_2 \circ \Gamma_1 = \alpha(\Gamma_2 \star \Gamma_1)$  that

$$\begin{aligned} \dim \Gamma_2 \circ \Gamma_1 &= \dim \Gamma_2 \star \Gamma_1 - \dim \ker \alpha = \\ &= \dim \Gamma_2 \star \Gamma_1 - \dim V_2 + \dim \operatorname{Im} \tau. \end{aligned}$$

Since  $\Gamma_2 \star \Gamma_1$  is the kernel of the map  $\tau : \Gamma_1 \times \Gamma_2 \rightarrow V_2$ , it follows that

$$\begin{aligned} \dim \Gamma_2 \star \Gamma_1 &= \dim \Gamma_1 \times \Gamma_2 - \dim \operatorname{Im} \tau = \\ &= \frac{1}{2} \dim V_1 + \frac{1}{2} \dim V_2 + \frac{1}{2} \dim V_2 + \frac{1}{2} \dim V_3 - \dim \operatorname{Im} \tau. \end{aligned}$$

Putting these two equations together, we see that

$$\dim \Gamma_2 \circ \Gamma_1 = \frac{1}{2} \dim V_1 + \frac{1}{2} \dim V_3$$

as desired. We have thus proved the following:

*The composite  $\Gamma_2 \circ \Gamma_1$  of two linear canonical relations is a linear canonical relation.*

We have already pointed out that the diagonal  $\Delta_V$  gives the identity morphism, so **LinSymp** is a category, as asserted.

**4.2. The symplectic “category”, Symp.** Let  $(M_i, \omega_i)$   $i = 1, 2$  be symplectic manifolds. A Lagrangian submanifold  $\Gamma$  of  $M_1^- \times M_2$  is called a **canonical relation**. For example, if  $f : M_1 \rightarrow M_2$  is a symplectomorphism, then  $\Gamma_f = \operatorname{graph} f$  is a canonical relation.

If  $\Gamma_1 \subset M_1 \times M_2$  and  $\Gamma_2 \subset M_2 \times M_3$ , we can form their composite

$$\Gamma_2 \circ \Gamma_1 \subset M_1 \times M_3$$

in the sense of the composite of relations. Hence,  $\Gamma_2 \circ \Gamma_1$  consists of all points  $(x, z)$  such that there exists a  $y \in M_2$  with  $(x, y) \in \Gamma_1$  and  $(y, z) \in \Gamma_2$ . Let us state this in the language of fiber products: Let

$$\pi : \Gamma_1 \rightarrow M_2$$

denote the restriction to  $\Gamma_1$  of the projection of  $M_1 \times M_2$  onto the second factor, and let

$$\rho : \Gamma_2 \rightarrow M_2$$

denote the restriction to  $\Gamma_2$  of the projection of  $M_2 \times M_3$  onto the first factor. Let

$$F \subset M_1 \times M_2 \times M_2 \times M_3$$

be defined by

$$F = (\pi \times \rho)^{-1} \Delta_{M_2}.$$

In other words,  $F$  is defined as the fiber product (or exact square)

$$(4.9) \quad \begin{array}{ccc} F & \xrightarrow{\iota_1} & \Gamma_1 \\ \iota_2 \downarrow & & \downarrow \pi \\ \Gamma_2 & \xrightarrow{\rho} & M_2 \end{array}$$

so

$$F \subset \Gamma_1 \times \Gamma_2 \subset M_1 \times M_2 \times M_2 \times M_3.$$

Let  $\text{pr}_{13}$  denote the projection of  $M_1 \times M_2 \times M_2 \times M_3$  onto  $M_1 \times M_3$  (projection onto the first and last components). Let  $\pi_{13}$  denote the restriction of  $\text{pr}_{13}$  to  $F$ . Then, as a *set*,

$$(4.10) \quad \Gamma_2 \circ \Gamma_1 = \pi_{13}(F).$$

The map  $\text{pr}_{13}$  is smooth, and hence its restriction to any submanifold is smooth. The problems are that  $F$  need not be a submanifold, and the restriction  $\pi_{13}$  of  $\text{pr}_{13}$  to  $F$  need not be an embedding. Hence, we need some additional hypotheses to ensure that  $\Gamma_2 \circ \Gamma_1$  is a submanifold of  $M_1 \times M_3$ . Once we impose these hypotheses we will find it easy to check that  $\Gamma_2 \circ \Gamma_1$  is a Lagrangian submanifold of  $M_1^- \times M_3$  and hence a canonical relation.

**4.2.1. Clean intersection.** Assume that the maps

$$\pi : \Gamma_1 \rightarrow M_2 \quad \text{and} \quad \rho : \Gamma_2 \rightarrow M_2$$

defined above intersect cleanly.

Notice that  $(m_1, m_2, m'_2, m_3) \in F$  if and only if

- $m_2 = m'_2$ ,
- $(m_1, m_2) \in \Gamma_1$ , and
- $(m'_2, m_3) \in \Gamma_2$ .

Therefore, we can think of  $F$  as the subset of  $M_1 \times M_2 \times M_3$  consisting of all points  $(m_1, m_2, m_3)$  with  $(m_1, m_2) \in \Gamma_1$  and  $(m_2, m_3) \in \Gamma_2$ . The clean intersection hypothesis involves two conditions. The first is that  $F$  be a manifold. The second is that the derived square be exact at

all points. Let us state this second condition more explicitly: Let  $m = (m_1, m_2, m_3) \in F$ . We have the following vector spaces:

$$\begin{aligned} V_1 &:= T_{m_1}M_1, \\ V_2 &:= T_{m_2}M_2, \\ V_3 &:= T_{m_3}M_3, \\ \Gamma_1^m &:= T_{(m_1, m_2)}\Gamma_1, \quad \text{and} \\ \Gamma_2^m &:= T_{(m_2, m_3)}\Gamma_2. \end{aligned}$$

So,

$$\Gamma_1^m \subset T_{(m_1, m_2)}(M_1 \times M_2) = V_1 \oplus V_2$$

is a linear Lagrangian subspace of  $V_1^- \oplus V_2$ . Similarly,  $\Gamma_2^m$  is a linear Lagrangian subspace of  $V_2^- \oplus V_3$ . The *clean intersection hypothesis* asserts that  $T_m F$  is given by the exact square

$$(4.11) \quad \begin{array}{ccc} T_m F & \xrightarrow{d(\iota_1)_m} & \Gamma_1^m \\ d(\iota_2)_m \downarrow & & \downarrow d\pi_{m_1, m_2} \\ \Gamma_2^m & \xrightarrow{d\rho_{(m_2, m_3)}} & T_{m_2}M_2 \end{array}$$

In other words,  $T_m F$  consists of all  $(v_1, v_2, v_3) \in V_1 \oplus V_2 \oplus V_3$  such that

$$(v_1, v_2) \in \Gamma_1^m \quad \text{and} \quad (v_2, v_3) \in \Gamma_2^m.$$

The exact square (4.11) is of the form (4.2) that we considered in Section 4.1. We know from Section 4.1 that  $\Gamma_2^m \circ \Gamma_1^m$  is a linear Lagrangian subspace of  $V_1^- \oplus V_3$ . In particular, its dimension is  $\frac{1}{2}(\dim M_1 + \dim M_3)$  which does not depend on the choice of  $m \in F$ . This implies the following: Let

$$\iota : F \rightarrow M_1 \times M_2 \times M_3$$

denote the inclusion map, and let

$$\kappa_{13} : M_1 \times M_2 \times M_3 \rightarrow M_1 \times M_3$$

denote the projection onto the first and third components. Then,

$$\kappa \circ \iota : F \rightarrow M_1 \times M_3$$

is a smooth map whose differential at any point  $m \in F$  maps  $T_m F$  onto  $\Gamma_2^m \circ \Gamma_1^m$  and so has locally constant rank. Furthermore, the image of  $T_m F$  is a Lagrangian subspace of  $T_{(m_1, m_3)}(M_1^- \times M_3)$ . We have proved:

**Theorem 4.** *If the canonical relations  $\Gamma_1 \subset M_1^- \times M_2$  and  $\Gamma_2 \subset M_2^- \times M_3$  intersect cleanly, then their composition  $\Gamma_2 \circ \Gamma_1$  is an immersed Lagrangian submanifold of  $M_1^- \times M_3$ .*

We must still impose conditions that will ensure that  $\Gamma_2 \circ \Gamma_1$  is a genuine submanifold of  $M_1 \times M_3$ . We will do this in the next section.

We will need a name for the manifold  $F$ , we created out of  $\Gamma_1$  and  $\Gamma_2$  above. We will call it  $\Gamma_2 \star \Gamma_1$ .

**4.2.2. Composable canonical relations.** We recall a theorem from differential topology:

**Theorem 5.** *Let  $X$  and  $Y$  be smooth manifolds and  $f : X \rightarrow Y$  is a smooth map of constant rank. Let  $W = f(X)$ . Suppose that  $f$  is proper and that for every  $w \in W$ ,  $f^{-1}(w)$  is connected and simply connected. Then,  $W$  is a smooth submanifold of  $Y$ .*

For a proof, see, for instance [8].

We apply this theorem to the map  $\kappa_{13} \circ \iota : F \rightarrow M_1 \times M_3$ . To shorten the notation, let us define

$$(4.12) \quad \kappa := \kappa_{13} \circ \iota.$$

**Theorem 6.** *Suppose that the canonical relations  $\Gamma_1$  and  $\Gamma_2$  intersect cleanly. Suppose, in addition, that the map  $\kappa$  is proper and that the inverse image of every  $\gamma \in \Gamma_2 \circ \Gamma_1 = \kappa(\Gamma_2 \star \Gamma_1)$  is connected and simply connected. Then,  $\Gamma_2 \circ \Gamma_1$  is a canonical relation. Furthermore,*

$$(4.13) \quad \kappa : \Gamma_2 \star \Gamma_1 \rightarrow \Gamma_2 \circ \Gamma_1$$

*is a smooth fibration with compact connected fibers.*

Thus to summarize, we cannot always compose the canonical relations  $\Gamma_2 \subset M_2^- \times M_3$  and  $\Gamma_1 \subset M_1^- \times M_2$  to obtain a canonical relation  $\Gamma_2 \circ \Gamma_1 \subset M_1^- \times M_3$ . We must impose some additional conditions (for example, those of the theorem). Following Weinstein, we put quotation marks around the word “category” to indicate this fact.

**4.3. Weinstein’s philosophy of “points” as Lagrangian submanifolds.** In a general category, where the objects are not necessarily sets, we cannot talk about the points of an object  $X$ . However, if we have a distinguished object  $\text{pt.}$ , then, we can *define* a “**point**” of  $X$  to be an element of  $\text{Morph}(\text{pt.}, X)$ . Then, a morphism  $\Gamma \in \text{Morph}(X, Y)$  yields a map from “points” of  $X$  to “points” of  $Y$ .

In the symplectic “category”, we will choose our point object to be the unique connected zero dimensional symplectic manifold and call it “pt.”. Then, a canonical relation between  $\text{pt.}$  and a symplectic manifold  $M$  is a Lagrangian submanifold of  $\text{pt.} \times M$  which may be identified with a Lagrangian submanifold of  $M$ . These are the “points” in our “category” **Symp**.

Suppose that  $\Lambda$  is a Lagrangian submanifold of  $M_1$  and  $\Gamma \in \text{Morph}(M_1, M_2)$  is a canonical relation. If we think of  $\Lambda$  as an element of  $\text{Morph}(\text{pt.}, M_1)$ , then if  $\Gamma$  and  $\Lambda$  are composable, we can form  $\Gamma \circ \Lambda \in \text{Morph}(\text{pt.}, M_2)$  which may be identified with a Lagrangian submanifold of  $M_2$ . (If we want to think of it this way, we will denote it by  $\Gamma(\Lambda)$  instead of  $\Gamma \circ \Lambda$ .)

### 5. The general set up.

Armed with this language, we now return to the situation described at the end of the introduction: Let  $(M, \omega)$  be a symplectic manifold. Let  $Z, X$  and  $S$  be manifolds and suppose that

$$\pi : Z \rightarrow S$$

is a fibration with fibers diffeomorphic to  $X$ . Let

$$G : Z \rightarrow M$$

be a smooth map and let

$$g_s : Z_s \rightarrow M, \quad Z_s := \pi^{-1}(s)$$

denote the restriction of  $G$  to  $Z_s$ . We assume that

$$(5.1) \quad g_s \text{ is a Lagrangian embedding}$$

and let

$$(5.2) \quad \Lambda_s := g_s(Z_s)$$

denote the image of  $g_s$ . Thus, for each  $s \in S$ ,  $G$  imbeds the fiber,  $Z_s = \pi^{-1}(s)$ , into  $M$  as the Lagrangian submanifold,  $\Lambda_s$ . Let  $s \in S$  and  $\xi \in T_s S$ . For  $z \in Z_s$  and  $w \in T_z Z_s$  tangent to the fiber  $Z_s$ ,

$$dG_z w = (dg_s)_z w \in T_{G(z)} \Lambda_s.$$

So,  $dG_z$  induces a map, which by abuse of language, we will continue to denote by  $dG_z$

$$(5.3) \quad dG_z : T_z Z / T_z Z_s \rightarrow T_m M / T_m \Lambda, \quad m = G(z).$$

But  $d\pi_z$  induces an identification

$$(5.4) \quad T_z Z / T_z(Z_s) = T_s S.$$

Furthermore, we have an identification

$$(5.5) \quad T_m M / T_m(\Lambda_s) = T_m^* \Lambda_s$$

given by

$$T_m M \ni u \mapsto i(u)\omega_m(\cdot) = \omega_m(u, \cdot).$$

Via these identifications, we can convert (5.3) into a map

$$(5.6) \quad T_s S \longrightarrow T_z^* Z_s.$$

Now, let  $\Phi : Z \longrightarrow T^* S$  be a lifting of  $\pi : Z \longrightarrow S$ , so that

$$\begin{array}{ccc} Z & \xrightarrow{\Phi} & T^* S \\ & \searrow \pi & \downarrow \\ & & S \end{array}$$

commutes; and for  $s \in S$ , let

$$\Phi_s : Z_s \longrightarrow T_s^* S$$

be the restriction of  $\Phi$  to  $Z_s$ .

**Definition.**  $\Phi$  is a *moment map* if, for all  $s$  and all  $z \in Z_s$ ,

$$(5.7) \quad (d\Phi_s)_z : T_z Z_s \longrightarrow T_s^* S$$

is the transpose of (5.6).

Note that this condition determines  $\Phi_s$  up to an additive constant  $\nu_s \in T_s^* S$  and hence, as in Section 3, determines  $\Phi$  up to a section,  $s \longrightarrow \nu_s$ , of  $T^* S$ .

When does a moment map exist? By (5.6) a vector,  $v \in T_s$ , defines, for every point,  $z \in Z_s$ , an element of  $T_z^* Z_s$  and hence, defines a one-form on  $Z_s$  which we will show to be closed. We will say that  $G$  is *exact* if for all  $s$  and all  $v \in T_s S$  this one-form is exact, and we will prove below that the exactness of  $G$  is a necessary and sufficient condition for the existence of  $\Phi$ .

Given a moment map,  $\Phi$ , one gets from it an imbedding

$$(5.8) \quad (G, \Phi) : Z \longrightarrow M \times T^* S$$

and we can ask how close this comes to being a Lagrangian imbedding. We will prove

**Theorem 7.** *The pull-back by (5.8) of the symplectic form on  $M \times T^* S$  is the pull-back by  $\pi$  of a closed two-form  $\mu$  on  $S$ .*

The cohomology class of this two-form is an intrinsic invariant of  $G$  (does not depend on the choice of  $\Phi$ ) and as in Section 3 one can show that this is the only obstruction to making (5.8) a Lagrangian imbedding.

**Theorem 8.** *If  $[\mu] = 0$ , there exists a moment map,  $\Phi$ , for which the imbedding (5.8) is Lagrangian.*

Conversely, we will prove:

**Theorem 9.** *Let  $\Phi$  be a map of  $Z$  into  $T^*S$  lifting the map,  $\pi$ , of  $Z$  into  $S$ . Then, if the imbedding (5.8) is Lagrangian,  $\Phi$  is a moment map.*

**5.1. Proofs.** Let us go back to the map (5.6). If we hold  $s$  fixed, but let  $z$  vary over  $Z_s$ , we see that each  $\xi \in T_s^*S$  gives rise to a one form on  $Z_s$ . To be explicit, let us choose a trivialization of our bundle around  $Z_s$  to give us an identification

$$H : Z_s \times U \rightarrow \pi^{-1}(U)$$

where  $U$  is a neighborhood of  $s$  in  $S$ . If  $t \mapsto s(t)$  is any curve on  $S$  with  $s(0) = s$ ,  $s'(0) = \xi$ , we get a curve of maps  $h_{s(t)}$  of  $Z_s \rightarrow M$  where

$$h_{s(t)} = g_{s(t)} \circ H.$$

We thus get a vector field  $v^\xi$  along the map  $h_s$

$$v^\xi : Z_s \rightarrow TM, \quad v^\xi(z) = \frac{d}{dt} h_{s(t)}(z)|_{t=0}.$$

Then, the one form in question is

$$\tau^\xi = h_s^*(i(v^\xi)\omega).$$

A direct check shows that this one form is exactly the one form described above (and hence is independent of all the choices). We claim that

$$(5.9) \quad d\tau^\xi = 0.$$

Indeed, the general form of the Weil formula (See[5] page 158 et. seq.) and the fact that  $d\omega = 0$  gives

$$\left( \frac{d}{dt} h_{s(t)}^* \omega \right)_{|t=0} = dh_s^* i(v^\xi) \omega$$

and the fact that  $\Lambda_s$  is Lagrangian for all  $s$  implies that the left-hand is zero. Let us now assume that  $G$  is *exact*, i.e., that for all  $s$  and  $\xi$  the one form  $\tau^\xi$  is exact. Then

$$\tau^\xi = d\phi^\xi$$

for some  $C^\infty$  function  $\phi^\xi$  on  $Z_s$ . The function  $\phi^\xi$  is uniquely determined up to an additive constant (if  $Z$  is connected) which we can fix (in various ways) so that it depends smoothly on  $s$  and linearly on  $\xi$ . For example, if we have a cross-section  $c : S \rightarrow Z$ , we can demand that  $\phi(c(s))^\xi \equiv 0$  for all  $s$  and  $\xi$ . Alternatively, we can equip each fiber  $Z_s$  with a compactly supported density  $dz_s$  which depends smoothly on  $s$  and whose integral over  $Z_s$  is one for each  $s$ . We can then demand that  $\int_{Z_s} \phi^\xi dz_s = 0$  for all  $\xi$  and  $s$ .



Suppose that we have made such choice. Then, for fixed  $z \in Z_s$ , the number  $\phi^\xi(z)$  depends linearly on  $\xi$ . Hence, we get a map

$$(5.10) \quad \Phi_0 : Z \rightarrow T^*S, \quad \Phi_0(z) = \lambda \Leftrightarrow \lambda(\xi) = \phi^\xi(z).$$

We shall see below that  $\Phi_0$  is a moment map by computing its derivative at  $z \in Z$  and checking that it is the transpose of (5.6).

If  $Z$  is connected, our choice determines  $\phi^\xi$  up to an additive constant  $\mu(s, \xi)$  which we can assume to be smooth in  $s$  and linear in  $\xi$ . Replacing  $\phi^\xi$  by  $\phi^\xi + \mu(s, \xi)$  has the effect of making the replacement

$$\Phi_0 \mapsto \Phi_0 + \mu \circ \pi$$

where  $\mu : S \rightarrow T^*S$  is the one form  $\langle \mu_s, \xi \rangle = \mu(s, \xi)$ .

Let  $\omega_S$  denote the canonical two form on  $T^*S$ .

**Theorem 10.** *There exists a closed two form  $\rho$  on  $S$  such that*

$$(5.11) \quad G^*\omega + \Phi^*\omega_S = \pi^*\rho.$$

If  $[\rho] = 0$ , then there is a one form  $\nu$  on  $S$  such that if we set

$$\Phi = \Phi_0 + \nu \circ \pi,$$

then

$$(5.12) \quad G^*\omega + \Phi^*\omega_S = 0.$$

As a consequence, the map

$$(5.13) \quad \tilde{G} : Z \rightarrow M \times T^*S, \quad z \mapsto (G(z), \Phi(z))$$

is a Lagrangian embedding.

*Proof.* We first prove a local version of the theorem. Locally, we may assume that  $Z = X \times S$ . This means that we have an identification of  $Z_s$  with  $X$  for all  $s$ . By the Weinstein tubular neighborhood theorem, we may assume (locally) that  $M = T^*X$  and that for a fixed  $s_0 \in S$  the Lagrangian submanifold  $\Lambda_{s_0}$  is the zero-section of  $T^*X$  and that the map

$$G : X \times S \rightarrow T^*X$$

is given by

$$G(x, s) = d_X\psi(x, s)$$

where  $\psi \in C^\infty(X \times S)$ . So, in terms of these choices, the maps  $h_{s(t)}$  used above are given by

$$h_{s(t)}(x) = d_X\psi(x, s(t))$$

and hence, the one form  $\tau^\xi$  is given by

$$d_S d_X \psi(x, \xi) = d_X \langle d_S \psi, \xi \rangle.$$

So, we may choose

$$\Phi(x, s) = d_S\psi(x, s).$$

Thus,

$$G^*\alpha_X = d_X\psi, \quad \Phi^*\alpha_S = d_S\psi$$

and hence,

$$G^*\omega_X + \Phi^*\omega_S = -dd\psi = 0.$$

This proves a local version of the theorem.

We now pass from the local to the global: By uniqueness, our global  $\Phi_0$  must agree with our local  $\Phi$  up to the replacement  $\Phi \mapsto \Phi + \mu \circ \pi$ . Therefore, we know that

$$G^*\omega + \Phi_0^*\omega_S = (\mu \circ \pi)^*\omega_S = \pi^*\mu^*\omega_S.$$

Here,  $\mu$  is a one form on  $S$  regarded as a map  $S \rightarrow T^*S$ . But

$$d\pi^*\mu^*\omega_S = \pi^*\mu^*d\omega_S = 0.$$

So, we know that  $G^*\omega + \Phi_0^*\omega_S$  is a closed two form which is locally and hence globally of the form  $\pi^*\rho$  where  $d\rho = 0$ . This proves (5.11).

Now, suppose that  $[\rho] = 0$  and hence  $\rho = d\nu$  for some one form  $\nu$  on  $S$ . Replacing  $\Phi_0$  by  $\Phi_0 + \nu$  replaces  $\rho$  by  $\rho + \nu^*\omega_S$ , but

$$\nu^*\omega_S = -\nu^*d\alpha_S = -d\nu = -\rho.$$

q.e.d.

**Remark.** If  $[\rho] \neq 0$ , we can modify the symplectic form on  $T^*S$  replacing  $\omega_S$  by  $\omega_S - \pi_S^*\rho$  where  $\pi_S$  denotes the projection  $T^*S \rightarrow S$ . Theorem 10 is then true for this modified form.

## 5.2. The derivative of $\Phi$ . Let

$$\Phi : Z \rightarrow T^*S.$$

be the map above and fix  $s \in S$ . The restriction of  $\Phi$  to the fiber  $Z_s$  maps  $Z_s \rightarrow T_s^*S$ . since  $T_s^*S$  is a vector space, we may identify its tangent space at any point with  $T_s^*S$  itself. Hence, for  $z \in Z_s$ , we may regard  $d\Phi_z$  as a linear map from  $T_zZ$  to  $T_s^*S$ . So, we write

$$(5.14) \quad d\Phi_z : T_zZ \rightarrow T_s^*S.$$

On the other hand, recall that using the identifications (5.4) and (5.5), we got a map

$$dG_z : T_sS \rightarrow T_m^*\Lambda, \quad m = G(z)$$

and hence, composing with  $d(g_s)_z^* : T_m^*\Lambda \rightarrow T_z^*Z_s$  a linear map

$$(5.15) \quad \chi_z := d(g_s)_z^* \circ dG_z : T_sS \rightarrow T_z^*Z.$$

**Theorem 11.** *The maps  $d\Phi_z$  given by (5.14) and  $\chi_z$  given by (5.15) are transposes of one another.*

*Proof.* Each  $\xi \in T_s^*S$  gives rise to a one form  $\tau^\xi$  on  $Z_s$  and by definition, the value of this one form at  $z \in Z_s$  is exactly  $\chi_z(\xi)$ . The function  $\phi^\xi$  was defined on  $Z_s$  so as to satisfy  $d\phi^\xi = \tau^\xi$ . In other words, for  $v \in T_zZ$

$$\langle \chi_z(\xi), v \rangle = \langle d\Phi_z(v), \xi \rangle.$$

q.e.d.

**Corollary 12.** *The kernel of  $\chi_z$  is the annihilator of the image of the map (5.14). In particular,  $z$  is a regular point of the map  $\Phi : Z_s \rightarrow T_s^*S$  if the map  $\chi_z$  is injective.*

**Corollary 13.** *The kernel of the map (5.14) is the annihilator of the image of  $\chi_z$ .*

**5.3. A converse.** The following is a converse to Theorem 10:

**Theorem 14.** *If  $\Phi : Z \rightarrow T^*S$  is a lifting of the map  $\pi : Z \rightarrow S$  to  $T^*S$  and  $(G, \Phi)$  is a Lagrangian imbedding, then  $\Phi$  is a moment map.*

*Proof.* It suffices to prove this in the local model described above, where  $Z = X \times S$ ,  $M = T^*X$  and  $G(x, s) = d_X\psi(x, s)$ . If  $\Phi : X \times S \rightarrow T^*S$  is a lifting of the projection  $X \times S \rightarrow X$ , then  $(G, \Phi)$  can be viewed as a section of  $T^*(X \times S)$ , i.e., as a one form  $\beta$  on  $X \times S$ . If  $(G, \Phi)$  is a Lagrangian imbedding, then  $\beta$  is closed. Moreover, the (1,0) component of  $\beta$  is  $d_X\psi$  so  $\beta - d\psi$  is a closed one form of type (0,1), and hence is of the form  $\mu \circ \pi$  for some closed one form on  $S$ . This shows that

$$\Phi = d_S\psi + \pi^*\mu$$

and hence, as verified above, is a moment map.

q.e.d.

**5.4. Families of symplectomorphisms.** Let us now specialize to the case of a parametrized family of symplectomorphisms. Let  $(M, \omega)$  be a symplectic manifold,  $S$  a manifold and

$$F : M \times S \rightarrow M$$

a smooth map such that

$$f_s : M \rightarrow M$$

is a symplectomorphism for each  $s$ , where  $f_s(m) = F(m, s)$ . We can apply the results of the preceding section where now  $\Lambda_s \subset M \times M^-$  is the

graph of  $f_s$  (and the  $M$  of the preceding section is replaced by  $M \times M^-$ ) and  $G$  is the map

$$(5.16) \quad G : M \times S \rightarrow M \times M^-, \quad G(m, s) = (m, F(m, s)).$$

Theorem 10 says that we get a map

$$\Phi : M \times S \rightarrow T^*S$$

and a moment Lagrangian

$$\Gamma_\Phi \subset M \times M^- \times T^*S$$

satisfying the conditions of Theorem 2.

**5.5. The equivariant situation.** Suppose that a compact Lie group  $K$  acts as fiber bundle automorphisms of  $\pi : Z \rightarrow S$  and acts as symplectomorphisms of  $M$ . Suppose further that the fibers of  $Z$  are compact and equipped with a density along the fiber which is invariant under the group action. (For example, we can put any density on  $Z_s$  varying smoothly on  $s$  and then replace this density by the one obtained by averaging over the group.) Finally, suppose that the map  $G$  is equivariant for the group actions of  $K$  on  $Z$  and on  $M$ . Then, the map  $\Phi$  is equivariant for the actions of  $K$  on  $Z$  and the induced action of  $K$  on  $M \times T^*S$ .

**5.5.1. Hamiltonian group actions.** Let us specialize further by assuming that  $S$  is a Lie group  $K$  and that  $F : M \times K \rightarrow M$  is a Hamiltonian group action. This gives us a map

$$G : M \times K \rightarrow M \times M^-, \quad (m, a) \mapsto (m, am).$$

Let  $K$  act on  $Z = M \times K$  via its left action on  $K$ . Thus  $a \in K$  acts on  $Z$  as

$$a : (m, b) \mapsto (m, ab).$$

To say that the action,  $F$ , is Hamiltonian with moment map  $\Psi : M \rightarrow \mathfrak{k}^*$  is to say that

$$i(\xi_M)\omega = -d\langle \Psi, \xi \rangle.$$

Thus, under the left invariant identification of  $T^*K$  with  $K \times \mathfrak{k}^*$   $\Psi$  determines a moment map in the sense of Theorem 2,

$$\Phi : M \times K \rightarrow T^*K, \quad \Phi(m, a) = (a, \Psi(m)).$$

So our  $\Phi$  of (5.10) is indeed a generalization of the moment map for Hamiltonian group actions.

## 6. Double fibrations

The set-up described in Section 3 has some legitimate applications of its own. For instance, suppose that the diagram

$$\begin{array}{ccc} & Z & \\ \pi \swarrow & & \searrow G \\ S & & M \end{array}$$

is a double fibration: i.e., both  $\pi$  and  $G$  are fiber mappings and the map

$$(G, \pi) : Z \longrightarrow M \times S$$

is an imbedding. In addition, suppose there exists a moment map  $\Phi : Z \longrightarrow T^*S$  such that

$$(6.1) \quad (G, \Phi) : Z \longrightarrow M \times T^*S$$

is a Lagrangian imbedding. We will prove

**Theorem 15.** *The moment map  $\Phi : Z \longrightarrow T^*S$  is a co-isotropic immersion.*

*Proof.* We leave as an exercise the following linear algebra result.

q.e.d.

**Lemma 1.** *Let  $V$  and  $W$  be symplectic vector spaces and  $\Gamma$  a Lagrangian subspace of  $V \times W$ . Suppose the projection of  $\Gamma$  into  $V$  is surjective. Then, the projection of  $\Gamma$  into  $W$  is injective and its image is a co-isotropic subspace of  $W$ .*

To prove the theorem, let  $\Gamma_\Phi$  be the image of the imbedding (6.1). Then, the projection,  $\Gamma_\Phi \longrightarrow M$ , is just the map,  $G$ ; so by assumption, it is a submersion. Hence by the lemma, the projection,  $\Gamma_\Phi \longrightarrow T^*S$ , which is just the map,  $\Phi$ , is a co-isotropic immersion.

The most interesting case of the theorem above is the case when  $\Phi$  is an imbedding. Then, its image,  $\Sigma$ , is a co-isotropic submanifold of  $T^*S$  and  $M$  is just the quotient of  $\Sigma$  by its null-foliation. This description of  $M$  gives one, in principle, a method for quantizing  $M$  as a Hilbert subspace of  $L_2(S)$ . (For examples of how this method works in practice, see [6].)

## 7. The moment image of a family of symplectomorphisms

As in Section 3, let  $M$  be a symplectic manifold and let  $\{f_s, s \in S\}$  be an exact family of symplectomorphisms. Let

$$\Phi : M \times S \longrightarrow T^*S$$

be the moment map associated with this family and let

$$(7.1) \quad \Gamma = \{(m, f_s(m)), \Phi(m, s); (m, s) \in M \times S\}$$

be its moment Lagrangian. From the perspective of Section 4,  $\Gamma$  is a morphism or “map”

$$\Gamma : M^- \times M \Rightarrow T^*S$$

mapping the categorical “points” (Lagrangian submanifolds) of  $M^- \times M$  into the categorical “points” (Lagrangian submanifolds) of  $T^*S$ . Let  $\Lambda_\Phi$  be the image with respect to this “map” of the diagonal,  $\Delta$ , in  $M \times M$ . In more prosaic terms, this image is just the image with respect to  $\Phi$  (in the usual sense) of the subset

$$(7.2) \quad X = \{(m, s) \in M \times S; f_s(m) = m\}$$

of  $M \times S$ . As we explained in Section 4, this image will be a Lagrangian submanifold of  $T^*S$  only if one imposes transversal or clean intersection hypotheses on  $\Gamma$  and  $\Delta$ . More explicitly, let

$$(7.3) \quad \rho : \Gamma \longrightarrow M \times M$$

be the projection of  $\Gamma$  into  $M \times M$ . Then, the pre-image in  $\Gamma$  of  $\Delta$  can be identified with the set (7.2), and if  $\rho$  intersects  $\Delta$  cleanly, the set (7.2) is a submanifold of  $M \times S$  and we know from Section 4 that:

**Theorem 16.** *The composition,*

$$(7.4) \quad \Phi \circ j : X \longrightarrow T^*S,$$

*of  $\Phi$  with the inclusion map,  $j$ , of  $X$  into  $M \times S$  is a mapping of constant rank and its image,  $\Delta_\Phi$ , is an immersed Lagrangian submanifold of  $T^*S$ .*

**Remark.**

- 1) If the projection (7.3) intersects  $\Delta$  transversely, one gets a stronger result. Namely, in this case, the map (7.4) is a Lagrangian immersion.
- 2) If the map (7.4) is proper and its level sets are simply connected, then  $\Lambda_\Phi$  is an imbedded Lagrangian submanifold of  $T^*S$ , and (7.4) is a fiber bundle mapping with  $X$  as fiber and  $\Lambda_\Phi$  as base.

Let us now describe what this “moment image”,  $\Lambda_\Phi$ , of the moment Lagrangian look like in some examples.

**7.1. The character Lagrangian.** Let  $K$  be the standard  $n$ -dimensional torus and  $\mathfrak{k}$  its Lie algebra. Given a Hamiltonian action,  $\tau$ , of  $K$  on a compact symplectic manifold,  $M$ , one has its usual moment mapping,  $\phi : M \longrightarrow \mathfrak{k}^*$ ; and if  $K$  acts faithfully, the image of  $\phi$  is a convex  $n$ -dimensional polytope,  $\mathbf{P}_\phi$ .

If we consider the moment map  $\Phi : M \rightarrow T^*K = K \times \mathfrak{k}^*$  in the sense of Section 3, the image of  $\Phi$  in the categorical sense can be viewed as a labeled polytope in which the open  $(n - k)$ -dimensional faces of  $\mathbf{P}_\Phi$  are labeled by  $k$ -dimensional subgroups of  $K$ . More explicitly, since  $M$  is compact, there are a finite number of subgroups of  $K$  occurring as stabilizer groups of points. Let

$$(7.5) \quad K_\alpha, \quad \alpha = 1, \dots, N$$

be a list of these subgroups and for each  $\alpha$ , let

$$(7.6) \quad M_{i,\alpha}, \quad i = 1, \dots, k_\alpha$$

be the connected components of the set of points whose stabilizer group is  $K_\alpha$ . Then, the sets

$$(7.7) \quad \phi(M_{i,\alpha}) = \mathbf{P}_{i,\alpha}$$

in  $\mathfrak{k}^*$  are the open faces of  $\mathbf{P}$  and the categorical image,  $\Lambda_\Phi$ , of the set of symplectomorphisms  $\{\tau_a, a \in K\}$  is the disjoint union of the Lagrangian manifolds

$$(7.8) \quad \Lambda_{i,\alpha} = K_\alpha \times \mathbf{P}_{i,\alpha}$$

**7.2. The period–energy relation.** If one replaces the group,  $K = T^n$  in this example by the non-compact group,  $K = \mathbb{R}^n$  one cannot expect  $\Lambda_\Phi$  to have this kind of polyhedral structure; however,  $\Lambda_\Phi$  does have some interesting properties from the dynamical systems perspective. If  $H : M \rightarrow (\mathbb{R}^n)^*$  is the moment map associated with the action of  $\mathbb{R}^n$  on  $M$ , the coordinates,  $H_i$ , of  $H$  can be viewed as Poisson-commuting Hamiltonians, and the  $\mathbb{R}^n$  action is generated by their Hamiltonian vector fields,  $\nu_{H_i}$ , i.e., by the map

$$(7.9) \quad s \in \mathbb{R}^n \rightarrow f_s = (\exp s_1 \nu_{H_1}) \dots (\exp s_n \nu_{H_n}).$$

Suppose now that  $H : M \rightarrow (\mathbb{R}^n)^*$  is a proper submersion. Then, each connected component,  $\Lambda$ , of  $\Lambda_\Phi$  in  $T^*\mathbb{R}^n = \mathbb{R}^n \times (\mathbb{R}^n)^*$  is the graph of a map

$$\xi \rightarrow \left( \frac{\partial \rho}{\partial \xi_1}, \dots, \frac{\partial \rho}{\partial \xi_n} \right)$$

over an open subset,  $U$ , of  $(\mathbb{R}^n)^*$  with  $\rho \in C^\infty(U)$ , and if  $\Lambda_\Phi$  is connected, the periodic trajectories of the system (7.9) on the level set,  $H_1 = c_1, \dots, H_n = c_n$ , all have the same period,  $T = \frac{\partial \rho}{\partial \xi}(c)$ . This result is known in the theory of dynamical systems as the *period–energy relation*.

**7.3. The period–energy relation for families of symplectomorphisms.** We will show that something similar to this period–energy relation is true for families of symplectomorphisms provided we impose some rather strong assumptions on  $M$  and  $\omega$ . Namely we will have to assume that  $\omega$  is exact and that  $H^1(M, \mathbb{R}) = 0$ . Modulo these assumptions, one can define, for a symplectomorphism,  $f : M \rightarrow M$ , and a fixed point,  $p$  of  $f$ , a natural notion of “the period of  $p$ ”.

The definition is the following. Choose a one-form,  $\alpha$ , with  $d\alpha = \omega$ . Then

$$d(\alpha - f^*\alpha) = \omega - f^*\omega = 0$$

so

$$(7.10) \quad \alpha - f^*\alpha = d\psi$$

for some  $\psi$  in  $C^\infty(M)$ . (Unfortunately,  $\psi$  is only defined up to an additive constant, and one needs some “intrinsic” way of normalizing this constant. For instance, if  $\psi$  is bounded and  $M$  has finite volume, one can require that the integral of  $\psi$  over  $M$  be zero, or if there is a natural base point,  $p_0$ , in  $M$  fixed by  $f$ , one can require that  $\psi(p_0) = 0$ .) Now, for every fixed point,  $p$ , set

$$(7.11) \quad T_p = \psi(p).$$

This definition depends on the normalization we have made of the additive constant in the definition of  $\psi$ , but we claim that it is independent of the choice of  $\alpha$ . In fact, if we replace  $\alpha$  by  $\alpha + dg$ ,  $g \in C^\infty(M)$ ,  $\psi$  gets changed to  $\psi + f^*g - g$  and at the fixed point,  $p$ ,

$$\psi(p) + (f^*g - g)(p) = \psi(p),$$

so, the definition (7.1) does not depend on  $\alpha$ .

There is also a dynamical systems method of defining these periods. By a variant of the mapping torus construction of Smale, one can construct a contact manifold,  $W$ , which is topologically identical with the usual mapping torus of  $f$ , and on this manifold, a contact flow having the following three properties.

- 1)  $M$  sits inside  $W$  and is a global cross-section of this flow.
- 2)  $f$  is the “first return” map.
- 3) If  $f(p) = p$ , the periodic trajectory of the flow through  $p$  has  $T_p$  as period.

Moreover, this contact manifold is unique up to contact isomorphism. (For details, see [1] or [7].) Let us apply these remarks to the set-up we are considering in this paper. As above, let  $F : M \times S \rightarrow M$  be a



smooth mapping such that for every  $s$  the map  $f_s : M \rightarrow M$ , mapping  $m$  to  $F(m, s)$ , is a symplectomorphism. Let us assume that

$$H^1(M \times S, \mathbb{R}) = 0.$$

Let  $\pi$  be the projection of  $M \times S$  onto  $M$ . Then, if  $\alpha$  is a one-form on  $M$  satisfying  $d\alpha = \omega$  and  $\alpha_S$  is the canonical one-form on  $T^*S$ , the moment map  $\Phi : M \times S \rightarrow M$  associated with  $F$  has the defining property

$$(7.12) \quad \pi^*\alpha - F^*\alpha + \Phi^*\alpha_S = d\psi$$

for some  $\psi$  in  $C^\infty(M \times S)$ . Let us now restrict both sides of (7.12) to  $M \times \{s\}$ . Since  $\Phi$  maps  $M \times \{s\}$  into  $T_s^*$ , and the restriction of  $\alpha_S$  to  $T_s^*$  is zero, we get:

$$(7.13) \quad \alpha - f_s^*\alpha = d\psi_s$$

where  $\psi_s = \psi|_{M \times \{s\}}$ .

Next, let  $X$  be the set, (7.2), i.e., the set:

$$\{(m, s) \in M \times S, \quad F(m, s) = m\}$$

and let us restrict (7.12) to  $X$ . If  $j$  is the inclusion map of  $X$  into  $M \times S$ , then  $F \circ j = \pi$ ; so

$$j^*(\pi^*\alpha - F^*\alpha) = 0$$

and we get from (7.12)

$$(7.14) \quad j^*(\Phi^*\alpha_S - d\psi) = 0.$$

The identities, (7.13) and (7.14) can be viewed as a generalization of the period–energy relation. For instance, suppose the map

$$\tilde{F} : M \times S \rightarrow M \times M$$

mapping  $(m, s)$  to  $(m, F(m, s))$  is proper and is transversal to  $\Delta$ . Then, by Theorem 16, the map  $\Phi \circ j : X \rightarrow T^*S$  is a Lagrangian immersion whose image is  $\Lambda_\Phi$ . Since  $\tilde{F}$  intersects  $\Delta$  transversely, the map

$$\tilde{f}_s : M \rightarrow M \times M, \quad \tilde{f}_s(m) = (m, f_s(m)),$$

intersects  $\Delta$  transversely for almost all  $s$ , and  $f_s$  is Lefschetz and has a countable number of fixed points,  $p_i(s)$ ,  $i = 1, 2, \dots$ . The functions,  $\psi_i(s) = \psi(p_i(s), s)$ , are, by (7.13), the periods of these fixed points and by (7.14) the Lagrangian manifolds

$$\Lambda_{\psi_i} = \{(s, \xi) \in T^*S \quad \xi = d\psi_i(s)\}$$

are the connected components of  $\Lambda_\Phi$ .

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