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OPTIMAL RIGIDITY ESTIMATES FOR NEARLY UMBILICAL SURFACES

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Dedicated to Hermann Karcher

Abstract

Let $\Sigma \subset \mathbf{R}^3$ be a smooth compact connected surface without boundary and denote by A its second fundamental form. We prove the existence of a universal constant C such that

$$\inf_{\lambda\in\mathbf{R}} \|A - \lambda \mathrm{Id}\|_{L^2(\Sigma)} \le C \left\|A - \tfrac{\operatorname{tr} A}{2}\mathrm{Id}\right\|_{L^2(\Sigma)} . \tag{1}$$

Building on this, we also show that, if the right-hand side of (1) is smaller than a geometric constant, Σ is $W^{2,2}$–close to a round sphere.

1. Introduction

Let $\Sigma \subset \mathbf{R}^3$ be a smooth surface. A point p of Σ is called umbilical if the principal curvatures of Σ at p are equal. A classical theorem in differential geometry states that if Σ is connected and all points of Σ are umbilical, then either Σ is a subset of a round sphere or it is a subset of a plane. Thus, if Σ is a compact surface without boundary, then Σ must be a round sphere and therefore, its second fundamental form is a constant multiple of the identity.

In the literature, some quantitative versions of this classical rigidity theorem are available. For instance, in [**11**], it is proved that if Σ is a closed convex surface and the ratio of its principal curvatures are uniformly close to 1, then Σ is close to a round sphere (see page 493). In [**16**], the author proves a similar result replacing the L^∞ condition by some integral versions of it. We refer to Chapter 6 of [**12**] for a survey of this and other results on convex surfaces which are almost umbilical. More recently, in their investigations on the gradient flow of the Willmore functional, in [**8**], the authors show that, without any convexity assumption, if $\left\|A - \frac{\operatorname{tr} A}{2}\mathrm{Id}\right\|_{L^2}$ is sufficiently small, then Σ flows toward

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a round sphere (as usual, we denote by A the second fundamental form of Σ).

The main theorem of this paper is the following. Here:

- Id denotes the identity $(1,1)$–tensor and the $(0,2)$–tensor naturally associated to it;
- $\mathring{A}$ denotes the traceless part of A, i.e., the tensor $A - \frac{\operatorname{tr} A}{2}\mathrm{Id}$;
- $\mathrm{id} : \mathbf{S}^2 \subset \mathbf{R}^3 \to \mathbf{R}^3$ is the standard isometric embedding of the round sphere.

Theorem 1.1. *Let $\Sigma \subset \mathbf{R}^3$ denote a smooth compact connected surface without boundary and for convenience normalize the area of Σ by $\mathrm{ar}(\Sigma) = 4\pi$. Then,*

$$\|A - \mathrm{Id}\|_{L^2(\Sigma)} \leq C \left\|\mathring{A}\right\|_{L^2(\Sigma)}, \tag{2}$$

where C is a universal constant. If in addition $\left\|\mathring{A}\right\|^2_{L^2(\Sigma)} \leq 8\pi$, then there exists a conformal parameterization $\psi : \mathbf{S}^2 \to \Sigma$ and a vector $c_\Sigma \in \mathbf{R}^3$ such that

$$\|\psi - (c_\Sigma + \mathrm{id})\|_{W^{2,2}(\mathbf{S}^2)} \leq C \left\|\mathring{A}\right\|_{L^2(\Sigma)}. \tag{3}$$

Note that (2) is a very natural estimate, since $\|\mathring{A}\|_{L^2(\Sigma)}$ is scaling invariant. Indeed (2) can be easily converted into the following scale–invariant estimate

$$\|A - r_\Sigma \mathrm{Id}\|_{L^2(\Sigma)} \leq C \left\|\mathring{A}\right\|_{L^2(\Sigma)} \qquad \text{where} \quad r_\Sigma = \sqrt{\frac{\mathrm{ar}(\Sigma)}{4\pi}}.$$

In order to have the second estimate of Theorem 1.1, it is sufficient to assume $\left\|\mathring{A}\right\|^2_{L^2} \leq 16\pi - \varepsilon$. In this case, C in (3) must be substituted by $C(\varepsilon)$, where $C(\varepsilon) \uparrow \infty$ as $\varepsilon \downarrow 0$.

Remark 1.2. Consider the conformal parameterization ψ of Theorem 1.1. Let us denote by g the metric of Σ, and by σ the standard metric on $\mathbf{S}^2$. For the conformal parameterization $\psi : \mathbf{S}^2 \to \Sigma$, we have $\psi_\# g = h^2\sigma$, where the positive smooth function h is the conformal factor of g in the coordinates induced by ψ. Then, suitably generalizing some arguments of [**10**], in [**3**] we prove that

$$\|h - 1\|_{C^0} \leq C \|A - \mathrm{Id}\|_{L^2(\Sigma)} \tag{4}$$

for some universal constant C.

In Section 7, we show that these estimates are optimal. More precisely, we construct a sequence of smooth connected compact surfaces Σ_n without boundary such that

- $\|\mathring{A}\|_{L^p} \to 0$ for every $p < 2$;
- Σ_n converges to the union of two spheres with different radii.

The starting point for proving Theorem 1.1 is the following observation. Let us fix an orthonormal frame e_1, e_2 on Σ and denote by A_{ij} the quantities $A(e_i, e_j)$ and by ∇A_{ijk} the quantities $[\nabla_{e_i} A](e_j, e_k)$. The Codazzi equations imply that $\nabla A_{ijk} = \nabla A_{jik}$. Hence, the symmetry of A gives that ∇A is a symmetric tensor. In view of this fact, straightforward algebraic computations give that $\nabla_{e_i}[A_{11} + A_{22}]$ can be written as a linear combination of $\nabla_{e_j}[A_{11} - A_{22}]$ and $\nabla_{e_j}[A_{12}]$ plus some error terms of type $A(\nabla_{e_j} e_k, e_l)$. Moreover, these error terms can be written as non-linear expressions involving $\mathring{A}$.

If $\mathring{A}$ were identically 0, then $\operatorname{tr} A$ would be constant. Roughly speaking, a control on $\mathring{A}$ gives some control on the oscillation of $\operatorname{tr} A = A_{11} + A_{22}$. Thus, if $\mathring{A}$ is small in a C^1 sense, then Σ would be close to a round sphere. This remark was used in [**7**] to give a definition of center of mass for isolated gravitating systems in General Relativity. In view of our result, one should be able to weaken the hypotheses under which Huisken–Yau's construction is possible.

1.1. Structure of the proof. In our case, the difficulties in getting the bound (2) are considerably increased by the weakness of the right-hand side of (2) and the non-linearity of the error terms of type $A(\nabla_{e_j} e_k, e_l)$. The outline of our proof is the following.

- First, we show that, when $\|\mathring{A}\|_{L^2}$ is sufficiently small, Σ is a sphere and there exists a good parameterization by a conformal map $\psi : \mathbf{S}^2 \to \Sigma$. By "good", we mean that, after a suitable rescaling, the conformal factor h satisfies uniform L^∞ and $W^{1,2}$ bounds (independent of Σ). In order to get these bounds, we derive Hardy space estimates on the Gauss curvature, using some ideas of [**10**]. This is accomplished in Section 3.
- We then perform the computations outlined above in the coordinate charts naturally induced by ψ. The control on ψ is sufficient to get an L^1 bound on the non-linear error terms. We use this bound and the regularity theory for the Laplacian to prove the existence of a universal constant C such that

$$\min_{\lambda \in \mathbf{R}} \|\operatorname{tr} A - \lambda\|_{L^{2,\infty}(\Sigma)} \le C \|\mathring{A}\|_{L^2(\Sigma)}, \tag{5}$$

  where $L^{2,\infty}$ is the weak Marcinkiewicz space (see Appendix B for the precise definition). This estimate is proved in Proposition 4.1.
- In Section 5, we show that the weak estimate (5) can be improved to the desired stronger estimate (2). This improvement heavily relies on some algebraic computations which exploit the special

structure of the tensor A. The proof uses Hardy space estimates for skew–symmetric quantities and the duality between the Hardy space $\mathcal{H}^1$ and BMO.

- In Section 6, we use (2) and the information derived in the previous sections to prove the existence of a conformal parameterization ψ which satisfies (3). The main difficulty here is due to the action of the conformal group of $\mathbf{S}^2$. The existence of ψ is proved into two steps: In the first one, we prove that there is a conformal parameterization with conformal factor $W^{1,2}$–close to 1; In the second step, we use the formalism of moving frames to show that this map is $W^{2,2}$–close to a smooth isometric embedding of the standard sphere.

2. Preliminaries

2.1. Notation. Throughout this paper, we will use the following notational conventions:

$\mathbf{S}^2$	standard sphere
Σ	compact connected smooth surface in $\mathbf{R}^3$ without boundary
$T_p\Sigma$, $T\Sigma$	tangent space in p, tangent bundle
$\mathrm{ar}(\Sigma)$, $\mathbf{g}(\Sigma)$	area of Σ, genus of Σ
$D_r(x)$, $\partial D_r(x)$	distance disk and distance circle of radius r and center x in 2d Riem. manifolds
$\mathcal{D}_1$, $\partial\mathcal{D}_1$	unit disk and unit circle in $\mathbf{R}^2$
g, σ	Riemannian metric on Σ, standard metric on $\mathbf{S}^2$
δ_{ij}, A, N	Kronecker symbol, second fundamental form, Gauss map
$\mathrm{tr}\, B$, $\det B$, $\lvert B\rvert$, Id	trace of B, determinant, Hilbert–Schmidt norm, identity matrix
κ_1, κ_2, K_G	principal curvatures, Gaussian curvature
$\mathrm{Deg}\,(\Gamma, \Sigma, u)$	topological degree of the map $u : \Gamma \to \Sigma$
L^p, $\mathcal{H}^1(\Omega)$	L^p spaces, Hardy space
Δ_Σ	Laplace operator on the Riemannian manifold Σ

Let $\psi : \Sigma \to \Gamma$ be an immersion and g a metric on Γ. Then, we denote by $\psi^* g$ the metric on Σ which is the pull back of g via ψ. That is

$$(\psi^* g)_p(v, w) := g_{\psi(p)}(d\psi(v), d\psi(w)) \qquad \text{for every } v, w \in T_p(\Sigma).$$

A system of coordinates on an open set $U \subset \Sigma$ can be regarded as a smooth diffeomorphism $\psi : \mathbf{R}^2 \supset \Omega \to U$. Hence, writing the metric in these coordinates is equivalent to calculating the pull–back metric $\psi^* g$.

In the rest of this paper, we assume that Σ is compact, connected, and without boundary. Moreover, we assume that $\mathrm{ar}(\Sigma) = 4\pi$ and we set

$$\delta^2 := \int_\Sigma \left|\mathring{A}\right|^2 . \tag{6}$$

We will make a frequent use of some elementary relations between differential geometric quantities, in particular, the identities

$$\left|\mathring{A}\right|^2 = \kappa_1^2 + \kappa_2^2 - 2\kappa_1\kappa_2 = |A|^2 - 2\det A = |A|^2 - 2K_G, \tag{7}$$

combined with Gauss–Bonnet Theorem:

$$\int_\Sigma |A|^2 = \int_\Sigma \left|\mathring{A}\right|^2 + 2\int_\Sigma K_G = \delta^2 + 2\int_\Sigma K_G = \delta^2 + 8\pi(1 - \mathbf{g}(\Sigma)). \tag{8}$$

Remark 2.1. Note that

$$\|A - \mathrm{Id}\|_{L^2}^2 \le 2\int_\Sigma |A|^2 + 2\mathrm{ar}(\Sigma).$$

Since $\mathbf{g}(\Sigma) \ge 0$, by (8) for every $c > 0$ there exists $C > 0$ such that

$$\|A - \mathrm{Id}\|_{L^2(\Sigma)} \le C \left\|\mathring{A}\right\|_{L^2(\Sigma)}^2 \qquad \text{for every } \Sigma \text{ with } \delta \ge c.$$

Thus, it suffices to show (2) for δ sufficiently small.

2.2. Σ is a sphere. In the following lemma, we show that, when δ is sufficiently small, Σ is a sphere. The proof uses well known elementary facts of differential geometry of surfaces. We report it for the reader's convenience.

Lemma 2.2. *If $\delta^2 < 16\pi$, then Σ is a sphere.*

Proof. Set $\eta := 16\pi - \delta^2$ and note that

$$\int_\Sigma |\det A| \le \frac{1}{2}\int_\Sigma |A|^2 \overset{(8)}{=} 8\pi - \frac{\eta}{2} + 4\pi(1 - \mathbf{g}(\Sigma)) < 4\pi(3 - \mathbf{g}(\Sigma)). \tag{9}$$

Hence, $\mathbf{g}(\Sigma)$ is either 0, 1, or 2. Let $N : \Sigma \to \mathbf{S}^2$ be the Gauss map, which to every point $x \in \Sigma$ associates the exterior unit normal to Σ in x. Since $A = dN$, the area formula gives

$$\int_\Sigma |\det A| = \int_{\mathbf{S}^2} \# N^{-1}(\{\xi\})\, d\xi. \tag{10}$$

Note that N is surjective. Indeed, let $\xi \in \mathbf{S}^2$ and consider the largest real number a such that the set $\mathrm{Ex} := \{x \in \Sigma : x \cdot \xi = a\}$ is not empty. For any $y \in \mathrm{Ex}$, we have $N(x) = \xi$.

This implies that $\#N^{-1}(\{\xi\}) \geq 1$ and hence gives $\int |\det A| \geq 4\pi$, which thanks to (9) rules out the possibility $\mathbf{g}(\Sigma) = 2$. Moreover, if $\mathbf{g}(\Sigma) = 1$ (i.e., if Σ were a torus), the degree $\mathrm{Deg}\,(\Sigma, \mathbf{S}^2, N)$ would necessarily be 0, which implies $\#N^{-1}(\{\xi\}) \geq 2$. Hence, (10) and (9) rule out the possibility $\mathbf{g}(\Sigma) = 1$. This gives $\mathbf{g}(\Sigma) = 0$ and completes the proof. q.e.d.

3. Existence of a good conformal parameterization

In this section, we show that, if δ is sufficiently small, then the surface Σ has a conformal parameterization which enjoys good bounds.

Definition 3.1. Denote by σ the metric on the standard sphere $\mathbf{S}^2$ and by g the standard metric on Σ as submanifold of $\mathbf{R}^3$. If $\psi : \mathbf{S}^2 \to \Sigma$ is conformal, then h denotes the unique function $h : \mathbf{S}^2 \to \mathbf{R}^+$ with $h^2\sigma = \psi^* g$.

Proposition 3.2. *Let $\delta^2 < 8\pi$ and set $\eta := 8\pi - \delta^2$. Then, there exists a constant $C(\eta)$ and a conformal parameterization $\psi : \mathbf{S}^2 \to \Sigma$ such that*

$$(C(\eta))^{-1} \leq h \leq C(\eta) \qquad \|dh\|_{L^2} \leq C(\eta). \tag{11}$$

A classical theorem (see for example [**9**]) implies the existence of conformal parameterizations $\psi : \mathbf{S}^2 \to \Sigma$. However, we cannot hope to have the bounds of Proposition 3.2 for *all* such ψ (due to the action of the conformal group). The choice of a good ψ is based on the following remark (cf. [**10**]). If $h = e^u$, then

$$\int_{\mathbf{S}^2} e^{2u} = 4\pi \qquad - \Delta_{\mathbf{S}^2} u = K e^{2u} - 1, \tag{12}$$

where $\Delta_{\mathbf{S}^2}$ is the Laplace operator on $\mathbf{S}^2$ and $K(x) = K_\Sigma(\psi(x))$. If we can bound the norm of the right-hand side of (12) in the Hardy space $\mathcal{H}^1$, then the proposition follows from the results of Fefferman and Stein [**6**] (for the definition of $\mathcal{H}^1$ and for a precise statement of the result of [**6**] needed here, see appendix A). Hence, it suffices to show the existence of a constant $C(\eta)$ and of a conformal ψ such that $\|Ke^{2u}\|_{\mathcal{H}^1(\mathbf{S}^2)} \leq C(\eta)$. To derive this estimate, we will use some ideas of [**10**] and the following result of [**2**]:

Theorem 3.3. *Let $u \in W^{1,n}(\mathbf{R}^n, \mathbf{R}^n)$. Then, there exists a constant c (depending only on n) such that*

$$\|\det du\|_{\mathcal{H}^1(\mathbf{R}^n)} \leq c\|du\|_{L^n}. \tag{13}$$

As already pointed out, in order to get the estimates (11), we have to mod out the action of the conformal group of the sphere. This is accomplished in the following

Lemma 3.4. *Assume that $\delta^2 < 8\pi$ and set $\eta := 8\pi - \delta^2$. Let x_1, x_2, and x_3 be standard coordinates on $\mathbf{R}^3$ and set $\mathbf{S}^{\pm}_i := \{\pm x_i > 0\} \cap \mathbf{S}^2$. Then, there exists a conformal $\psi : \mathbf{S}^2 \to \Sigma$ such that*

$$\int_{\psi(\mathbf{S}^j_i)} |A|^2 = 8\pi - \frac{\eta}{2} \quad \textit{for all } j \in \{+,-\} \textit{ and every } i \in \{1,2,3\}. \tag{14}$$

Proof. Thanks to Lemma 2.2, Σ is a sphere. Hence, equation (8) implies

$$\int_\Sigma |A|^2 = 16\pi - \eta.$$

Denote by e_i the vectors of the standard basis of $\mathbf{R}^3$ relative to the system of coordinates x_i. For each i, we denote by $\mathcal{S}^i : \mathbf{S}^2 \to \mathbb{C} \cup \{\infty\}$ the stereographic projection which maps e_i to the origin and the equator $\{x_i = 0\} \cap \mathbf{S}^2$ onto the unit circle $\{|z| = 1\}$. For each $r > 0$, we define $\mathcal{O}_r : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ by $\mathcal{O}_r(z) = rz$. For every $i \in \{1,2,3\}$ and $r > 0$, we denote by $F^i_r : \mathbf{S}^2 \to \mathbf{S}^2$ the conformal diffeomorphism $(\mathcal{S}^i)^{-1} \circ \mathcal{O}_r \circ \mathcal{S}^i$.

Choose a conformal parameterization $\varphi : \mathbf{S}^2 \to \Sigma$. Note that

$$\lim_{t\uparrow\infty} \int_{\varphi\left(F^1_t(\mathbf{S}^+_1)\right)} |A|^2 = \int_\Sigma |A|^2 \qquad \text{and} \qquad \lim_{t\downarrow 0} \int_{\varphi\left(F^1_t(\mathbf{S}^+_1)\right)} |A|^2 = 0.$$

By continuity, there exists a t such that

$$\int_{\varphi\left((F^1_t(\mathbf{S}^+_1)\right)} |A|^2 = \frac{1}{2}\int_\Sigma |A|^2 = 8\pi - \frac{\eta}{2}. \tag{15}$$

Define $\varphi_1 := \varphi \circ F^1_t$ and again note that for some τ, we have

$$\int_{\varphi_1\left(F^2_\tau(\mathbf{S}^+_2)\right)} |A|^2 = \frac{1}{2}\int_\Sigma |A|^2 = 8\pi - \frac{\eta}{2}. \tag{16}$$

Note that F^2_τ maps $\mathbf{S}^+_1$ onto itself. Thus, we have $\int_{\varphi_1\left(F^2_\tau(\mathbf{S}^+_1)\right)} |A|^2 = 8\pi - \eta/2$. A similar choice of F^3_σ shows that $\varphi \circ F^1_t \circ F^2_\tau \circ F^3_\sigma$ has the desired properties. q.e.d.

Below, we adopt the following convention. Let α be a 2–form on Σ (resp. on $\mathbf{S}^2$, on $\mathbf{R}^2$), let β be the standard volume form on Σ (resp. on $\mathbf{S}^2$, on $\mathbf{R}^2$), and denote by f the function such that $\alpha = f\beta$. If H is any function space, then we write $\|\alpha\|_H$ for $\|f\|_H$. When $H = \mathcal{H}^1$, i.e., the first Hardy space, the maximal function of f will be sometimes called "maximal function of α" (here and in what follows, we assume to have fixed a mollifier ζ and a finite atlas, see Appendix A).

Proof of Proposition 3.2. Fix ψ as in Lemma 3.4 and let $N : \Sigma \to \mathbf{S}^2$ be the Gauss map. Set $N' := N \circ \psi$ and note that $K' := Ke^{2u}$ is the Jacobian determinant of dN'.

The proof of the $\mathcal{H}^1$ estimate is based on some arguments of Section 3 of [**10**]. We first fix some notation. We denote by ω the standard volume form on $\mathbf{S}^2$. Then, $K'\omega$ is the pull–back of ω via the map N', that is $K'\omega = (N')^*\omega$. Moreover, any disk $D_\rho(x) \subset \mathbf{S}^2$ will be identified with a disk $\mathcal{D}_{\rho'} = \mathcal{D}_{\rho'}(0)$ in the complex plane via the standard stereographic projection which maps x onto 0.

We will show that there are constants r and $C(\eta)$ with the following property. For any $x \in \mathbf{S}^2$, there exists a map $M : \mathbb{C} \to \mathbf{S}^2$ such that

(i) $M = N'$ on $\mathcal{D}_{r'}$ ($\approx D_r(x)$);
(ii) M is constant on $\mathbb{C} \setminus \mathcal{D}_{(2r)'}$;
(iii) $\int_{\mathbb{C}} M^*\omega = 0$;
(iv) $\|M^*\omega\|_{W^{-1,2}} + \|dM\|_{L^2} \leq C(\eta)$.

Step 1. From (i)–(iv) to the $\mathcal{H}^1$ bound.

We first prove that the existence of M as the above gives an $\mathcal{H}^1$ bound for $(N')^*\omega$. We make the usual identification $\mathbf{S}^2 = P^1(\mathbb{C})$ and denote by $\pi : \mathbb{C}^2 \supset \mathbf{S}^3 \to P^1(\mathbb{C})$ the Hopf fibration. Then, Proposition 3.4.3 of [**10**] implies that M lifts to a map $F : \mathbb{C} \to \mathbf{S}^3 \subset \mathbb{C}^2$ (that is $M = \pi \circ F$) with

$$\|dF\|_{L^2} = \|dM\|_{L^2} + \|M^*\omega\|_{W^{-1,2}}. \tag{17}$$

Note that the existence of liftings is guaranteed by condition (iii) (see for example [**13**], Chapter 8). If F_1 and F_2 denote the components of F in a standard basis of $\mathbb{C}^2$, then $2M^*\omega = 2F^*\pi^*\omega = idF_1 \wedge d\overline{F}_1 + idF_2 \wedge d\overline{F}_2$. Writing F_j as $F_i^{re} + iF_i^{im}$, it is easy to see that $idF_1 \wedge d\overline{F}_1 + idF_2 \wedge d\overline{F}_2$ can be written as linear combination of forms of type $df_1 \wedge df_2$, where $df_1, df_2 \in L^2(\mathbb{C}) = L^2(\mathbf{R}^2)$. Clearly, $df_1 \wedge df_2 = (\det\, df) dx_1 \wedge dx_2$, where x_1, x_2 are standard coordinates in $\mathbf{R}^2$. Hence, we can apply Theorem 3.3 to derive

$$\|M^*\omega\|_{\mathcal{H}^1} \leq C\|dF\|_{L^2} \overset{(17)}{=} C\|dM\|_{L^2} + \|M^*\omega\|_{W^{-1,2}} \overset{\text{(iv)}}{\leq} C(\eta).$$

Let g be the maximal function of $M^*\omega$ (in the sense of equation (102)). Then

$$\|g\|_{L^1(D_{r/2}(x))} \leq \|g\|_{L^1(\mathbf{R}^2)} = \|M^*\omega\|_{\mathcal{H}^1} \leq C(\eta). \tag{18}$$

Let f be the maximal function of $(N')^*\omega$. Since $dN' \in L^2$, clearly, $\det dN' \in L^1$ and hence, $(N')^*\omega \in L^1$. By the definition of maximal functions, we have

$$\|f\|_{L^1(D_{r/2}(x))} \leq \|g\|_{L^1(D_{r/2}(x))} + C\|(N')^*\omega\|_{L^1},$$

where the constant C depends only on r. Since $\mathbf{S}^2$ can be covered with finitely many disks of radius $r/2$, we find that $\|(N')^*\omega\|_{\mathcal{H}^1(\mathbf{S}^2)}$ is bounded by a constant depending on η and r.

Step 2. Construction of M and $W^{-1,2}$ estimate.

We now come to the proof of the existence of constants r and $C(\eta)$ which satisfy (i)–(iv) above. We first construct an intermediate function $\zeta : \mathbb{C} \to \mathbf{S}^2$. The constant r is chosen so small that the disk $D_{2r}(x)$ is contained in one of the half spheres $\mathbf{S}_i^{\pm}$ of Lemma 3.4. Thus,

$$\int_{D_{2r}(x)} |\det dN'| \leq \frac{1}{2}\int_{\mathbf{S}_i^{\pm}} |dN'|^2 = 4\pi - \frac{\eta}{4}. \tag{19}$$

Using the Fubini–Tonelli Theorem, we can find a $\rho \in]r, 2r[$ such that

$$\int_{\partial D_\rho(x)} |dN'|^2 \leq \frac{4\pi}{r}. \tag{20}$$

We identify $D_\rho(x)$ with $\mathcal{D}_{\rho'} \subset \mathbb{C}$ (using the stereographic projection, see the discussion above) and we define $\zeta : \mathbb{C} \to \mathbf{S}^2$ by setting:

$$\zeta = N' \quad \text{on } \mathcal{D}_{\rho'} \qquad \text{and} \qquad \zeta(z) = N'\left(\rho' \tfrac{z}{|z|}\right) \quad \text{on } \mathbb{C} \setminus \mathcal{D}_{\rho'}.$$

Clearly, ζ satisfies (i). We now show that

(iv)′ $\|\zeta^*\omega\|_{W^{-1,2}(\mathcal{D}_{\rho'+1})}$ and $\|d\zeta\|_{L^2(\mathcal{D}_{\rho'+1})}$ are bounded by a constant $C(\eta)$.

The bound on $\|d\zeta\|_{L^2(\mathcal{D}_{\rho'+1})}$ is given by the fact that $\|dN'\|_{L^2(\mathbf{S}_i^{\pm})}$ is uniformly bounded and by the choice (20). We retain

$$\|d\zeta\|_{L^2(\mathcal{D}_{\rho'+1})} \leq C(\eta). \tag{21}$$

We now come to the $W^{-1,2}$ bound. Note that

$$\mathrm{ar}(\zeta(\mathbb{C})) \leq \int_{D_{2r}(x)} |\det dN'| \leq 4\pi - \frac{\eta}{4}.$$

Thus, $\mathbf{S}^2 \setminus \zeta(\mathbb{C})$ has area at least $\eta/4$. This means that we can find a closed set $E \subset \mathbf{S}^2 \setminus \zeta(\mathbb{C})$, with area $\eta/8$. Arguing as in the proof of Proposition 3.5.5 of [**10**], we can find a 1–form α_E on $\mathbf{S}^2 \setminus E$ such that

$$\|\alpha_E\|_{L^\infty(\mathbf{S}^2)} \leq \frac{C}{\mathrm{ar}(E)} \qquad \text{and} \qquad d\alpha_E = \omega \quad \text{on } \mathbf{S}^2 \setminus E, \tag{22}$$

where C is a universal constant. Using α_E, one finds $\zeta^*\omega = d(\zeta^*\alpha_E)$. Let $\varphi \in W^{1,2}(\mathcal{D}_{\rho'+1})$. Then, since ζ takes values in $\mathbf{S}^2 \setminus E$, we have

$$\int_{\mathcal{D}_{\rho'+1}} \varphi\, \zeta^*\omega = \int_{\partial\mathcal{D}_{\rho'+1}} \varphi\zeta^*\alpha - \int_{\mathcal{D}_{\rho'+1}} d\varphi \wedge \zeta^*\alpha.$$

Recall that $\zeta|_{\partial\mathcal{D}_{\rho'+1}} = N'|_{\partial D_{\rho'}}$. Thus, recalling that $\|\varphi\|_{L^2(\partial\mathcal{D}_{\rho'+1})} \leq C\|\varphi\|_{W^{1,2}(\mathcal{D}_{\rho'+1})}$, from (22), we get

$$\begin{aligned}\left|\int_{\partial\mathcal{D}_{\rho'+1}} \varphi\zeta^*\alpha\right| &\leq \frac{C}{\mathrm{ar}(E)}\|\varphi\|_{L^2(\partial\mathcal{D}_{\rho'+1})}\|d\zeta\|_{L^2(\partial\mathcal{D}_{\rho'+1})}\\ &\overset{(20)}{\leq} C(\eta)\|\varphi\|_{W^{1,2}(\mathcal{D}_{\rho'+1})}.\end{aligned}$$

Analogously,

$$\begin{aligned}\left|\int_{\mathcal{D}_{\rho'+1}} d\varphi \wedge \zeta^*\alpha\right| &\leq \frac{C}{\mathrm{ar}(E)}\|d\varphi\|_{L^2(\mathcal{D}_{\rho'+1})}\|d\zeta\|_{L^2(\mathcal{D}_{\rho'+1})}\\ &\overset{(21)}{\leq} C\|\varphi\|_{W^{1,2}(\mathcal{D}_{\rho'+1})}.\end{aligned}$$

This establishes the $W^{-1,2}$ bound of (iv)$'$.

Step 3. The existence of M.

In this step, we modify ζ so to reach (ii) and (iii), while keeping (i) and upgrading (iv)$'$ to (iv). Consider the restriction of ζ to $\mathcal{D}_{\rho'}$ and define for every regular value $x \in \mathbf{S}^2$ its degree $\deg(\zeta, x)$. Standard arguments give that $\deg(\zeta, x)$ is constant on the connected components of $\mathbf{S}^2 \setminus \zeta(\partial\mathcal{D}_{\rho'})$. Thus, by continuity, it can be extended to an integer valued piecewise constant function on $\mathbf{S}^2 \setminus \zeta(\partial\mathcal{D}_{\rho'})$. Define

$$U_0 := \left\{x \in \mathbf{S}^2 \middle| \deg(\zeta, x) = 0\right\}. \tag{23}$$

Then, U_0 is an open set contained in $\mathbf{S}^2 \setminus \zeta(\partial\mathcal{D}_{\rho'})$. The idea is to choose $y \in U_0$ and to take a retraction $R : [0,1] \times \mathbf{S}^2 \setminus \{y\} \to \mathbf{S}^2$ onto the antipodal of y. Then, we define $M = \zeta$ on $\mathcal{D}_{\rho'}$ and on $\mathcal{D}_{\rho'+1} \setminus \mathcal{D}_{\rho'}$ we put

$$M(z) = R\big(\rho' + 1 - |z|, \zeta(z)\big).$$

Since $\zeta(\mathcal{D}_{\rho'+1} \setminus \mathcal{D}_{\rho'}) = \zeta(\partial\mathcal{D}_{\rho'})$, we have $U_0 \cap \zeta(\mathcal{D}_{\rho'+1} \setminus \mathcal{D}_{\rho'}) = \emptyset$. Thus, M is well defined. From the definition of (23), we clearly have $\deg(\mathbb{C}, \mathbf{S}^2, M) \equiv 0$, and thus M satisfies (iii). Moreover, $M|_{\mathcal{D}_{\rho'}} = \zeta$ and $M|_{\mathbb{C}\setminus\mathcal{D}_{\rho'+1}}$ is constant; hence, M satisfies (i) and (ii). The only difficulty is to choose y and the retraction R so as to achieve the bound (iv).

Clearly, U_0 contains $\mathbf{S}^2\backslash\zeta(\mathbb{C})$ and thus $\mathrm{ar}(U_0) \geq \eta$. Moreover, U_0 is an open set bounded by a subset of the curve $\gamma = \zeta(\partial\mathcal{D}'_\rho) = N'(\partial D_\rho(x))$, which, in view of (20) has bounded length. Thanks to Lemma C.1, there exists a δ, depending on $\mathrm{ar}(U_0)$ and $\mathrm{length}(\gamma)$, such that U_0 contains a ball $D_\delta(y)$. Thus, δ can be chosen bigger than a constant which depends only on η.

Fix such a y and such a δ and define a C^1 map $R : [0,1] \times (\mathbf{S}^2 \setminus D_\delta(y)) \to \mathbf{S}^2$ which retracts on the antipode $\overline{y}$ of y. This can be done so that $\|R\|_{C^1}$ depends only on η. Thus,

$$\|M^*\omega\|_{W^{-1,2}(\mathbb{C})} \leq C_1(\eta)\big\|\zeta^*\omega\big\|_{W^{-1,2}\left(D_{\rho'+2}(0)\right)} \overset{\mathrm{(iv)}'}{\leq} C_2(\eta).$$

An analogous estimate holds for $\|dM\|_{L^2}$. This gives (iv) and completes the proof. q.e.d.

4. An $L^{2,\infty}$ estimate for $(A - \overline{H}\,\mathrm{Id})$

In this section, we prove the following.

Proposition 4.1. *There exists $C > 0$ such that, if*

$$\mathrm{ar}(\Sigma) = 4\pi, \qquad \textit{and} \qquad \int_\Sigma \big|\mathring{A}\big|^2 \leq \delta^2, \tag{24}$$

then

$$\left\|A - \left(\fint_\Sigma \frac{\mathrm{tr}\,A}{2}\right)\mathrm{Id}\right\|_{L^{2,\infty}(\Sigma)} \leq C\delta. \tag{25}$$

For the definition and properties of the Marcinkiewicz space $L^{2,\infty}$, we refer to Appendix B.

Proof. Below, we will prove the existence of a universal constant C such that, for every Σ with $\delta^2 \leq 4\pi$, there exist two conformal parameterizations $\varphi^+, \varphi^- : \mathcal{D}_1 \to \Sigma$ with the following properties:

(a) $\varphi^+(\mathcal{D}_1) \cup \varphi^-(\mathcal{D}_1) = \Sigma$;
(b) $\mathrm{ar}(\varphi^+(\mathcal{D}_1) \cap \varphi^-(\mathcal{D}_1)) \geq C^{-1}$;
(c) $\|\,\mathrm{tr}\ A - \lambda^\pm\|_{L^{2,\infty}(\varphi^\pm(\mathcal{D}_1))} \leq C\delta$ for some constants $\lambda^\pm$.

We first show how this would give (25). Note that

$$\begin{aligned} C^{-1}|\lambda^+ - \lambda^-| &\overset{\mathrm{(b)}}{\leq} \int_{\varphi^+(\mathcal{D}_1)\cap\varphi^-(\mathcal{D}_1)} |\lambda^+ - \lambda^-| \\ &\leq \int_{\varphi^+(\mathcal{D}_1)} |\,\mathrm{tr}\ A - \lambda^+| + \int_{\varphi^-(\mathcal{D}_1)} |\,\mathrm{tr}\ A - \lambda^-| \\ &\leq C_1\|\,\mathrm{tr}\ A - \lambda^+\|_{L^{2,\infty}(\varphi^+(\mathcal{D}_1))} \end{aligned}$$

$$+ C_1 \| \operatorname{tr} A - \lambda^- \|_{L^{2,\infty}(\varphi^-(\mathcal{D}_1))}$$
$$\stackrel{(c)}{\leq} 2C_1 C\delta.$$

Hence, $|\lambda^+ - \lambda^-| \leq 2C_1C^2\delta$. This means that $\| \operatorname{tr} A - \lambda^+ \|_{L^{2,\infty}(\Sigma)} \leq C_2\delta$, where C_2 is another universal constant. Let us set $2\overline{H} := \int_\Sigma \operatorname{tr} A$. Then,

$$4\pi |2\overline{H} - \lambda^+| \leq \int_\Sigma |\operatorname{tr} A - \lambda^+| \leq C_1 \| \operatorname{tr} A - \lambda^+ \|_{L^{2,\infty}(\Sigma)} \leq C_3\delta.$$

This gives $\| \operatorname{tr} A - 2\overline{H} \|_{L^{2,\infty}(\Omega)} \leq C_4\delta$. Then,

$$\|A - \overline{H}\mathrm{Id}\|_{L^{2,\infty}(\Omega)} \leq \left(\int_\Sigma |\mathring{A}|^2 \right)^{1/2} + \sqrt{2} \left\| \tfrac{\operatorname{tr} A}{2} - \overline{H} \right\|_{L^{2,\infty}(\Omega)} \leq C_6\delta.$$

Subsections 4.1 and 4.2 are devoted to prove the existence of $\varphi^\pm$ as above. To explain the underlying key idea, we have to set some notation. Let $\varphi : \mathcal{D}_1 \to \Sigma$ be a conformal parameterization of $\varphi(\mathcal{D}_1)$. We denote by x_1, x_2 a system of orthonormal coordinates in $\mathbf{R}^2$. Thus, in these conformal coordinates, the metric of Σ is given by $h^2\delta_{ij}$. We denote by $e_i \in T\Sigma$ the unit vectors $\frac{1}{h}\frac{\partial}{\partial x_i}$ and we set $A_{ij} := A(e_i, e_j)$.

Set $f := \operatorname{tr} A$, $f_d := A_{11} - A_{22}$, and $f_m := 2A_{12}$. In Subsection 4.1, we use the Codazzi–Mainardi equations to control ∇f in terms of f_m, f_d, ∇f_m, and ∇f_d (here, if $w : \Sigma \to \mathbf{R}$, then ∇w denotes the gradient of g in the Riemannian manifold Σ; that is, for any vector field $X : \Sigma \to T\Sigma$, we have $g(\nabla w, X) = dw(X)$).

Potentially, this control will depend in a rather subtle way on the conformal parameterization φ. This is not a surprise, since the functions f_d and f_m *depend* on φ (whereas $\operatorname{tr} A$ depends only on the immersion of Σ in $\mathbf{R}^3$). In Subsection 4.2, we use the results of Sections 2 and 3 in order to choose $\varphi^\pm$ which satisfy (a) and (b) and enjoy good bounds. We then show that these bounds and the relation derived in Subsection 4.1 are sufficient to prove (c).

4.1. Key calculation. Let φ, e_i, A_{ij}, f, f_d, and f_m be as above. When w is a function, $D_{e_i}w$ denotes the Lie derivative of w with respect to e_i, whereas we will use the notations $\partial_{x_i} w$ and w_i for $\frac{\partial}{\partial x_i}[w \circ \varphi] = D_{\frac{\partial}{\partial x_i}} w = hD_{e_i}w$.

If X is a vector field on Σ, then we denote by $\nabla_{e_i} X$ the covariant derivative of X with respect to e_i. For every $(2,0)$–tensor B on Σ, ∇B denotes the usual $(3,0)$–tensor given by

$$\nabla B(X, Y, Z) := D_X(B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z).$$

We set $\nabla B_{ijk} = \nabla B(e_i, e_j, e_k)$ and recall the Codazzi–Mainardi equations:

$$\nabla A_{ijk} = \nabla A_{jik}. \tag{26}$$

To compute ∇f, recall that $\nabla f = (D_{e_1} f)\, e_1 + (D_{e_2} f)\, e_2$. Straightforward calculations give

$$\begin{aligned}
D_{e_1} f &= D_{e_1}(A_{11} + A_{22}) \\
&= \nabla A_{111} + \nabla A_{122} + 2A(\nabla_{e_1} e_1, e_1) + 2A(\nabla_{e_1} e_2, e_2), \\
D_{e_1} f_d &= D_{e_1}(A_{11} - A_{22}) \\
&= \nabla A_{111} - \nabla A_{122} + 2A(\nabla_{e_1} e_1, e_1) - 2A(\nabla_{e_1} e_2, e_2), \\
D_{e_2} f_m &= 2D_{e_2} A_{12} \\
&= 2\nabla A_{212} + 2A(\nabla_{e_2} e_1, e_2) + 2A(e_1, \nabla_{e_2} e_2).
\end{aligned}$$

Thus, $D_{e_1} f = D_{e_1} f_d + D_{e_2} f_m + 2\tilde{R}_1$, where

$$\tilde{R}_1 = 2A(\nabla_{e_1} e_2, e_2) - A(\nabla_{e_2} e_1, e_2) - A(e_1, \nabla_{e_2} e_2). \tag{27}$$

We set $h_i := hD_{e_1} h = D_{\frac{\partial}{\partial x_i}} h$. Straightforward computations give:

$$\nabla_{e_1} e_1 = -\frac{h_2}{h^2} e_2 \qquad \nabla_{e_2} e_1 = \frac{h_1}{h^2} e_2 \tag{28}$$

$$\nabla_{e_1} e_2 = \frac{h_2}{h^2} e_1 \qquad \nabla_{e_2} e_2 = -\frac{h_1}{h^2} e_1. \tag{29}$$

Plugging these relations into (27), we get

$$\tilde{R}_1 := \frac{2h_2}{h^2} A_{12} + \frac{h_1}{h^2}(A_{11} - A_{22}) = \frac{h_2}{h^2} f_m + \frac{h_1}{h^2} f_d. \tag{30}$$

A similar computation for $D_{e_2} f$ yields $D_{e_2} f = -D_{e_2} f_d + D_{e_1} f_m + 2\tilde{R}_2$, where $\tilde{R}_2$ is given by an expression similar to the one of (30). Recall that $h_i = D_{\frac{\partial}{\partial x_i}} f = \partial_{x_i} f$. Hence,

$$\begin{cases} \partial_{x_1} f = \partial_{x_1} f_d + \partial_{x_2} f_m + 2h\tilde{R}_1 \\ \partial_{x_2} f = -\partial_{x_2} f_d + \partial_{x_1} f_m + 2h\tilde{R}_2. \end{cases} \tag{31}$$

Denote by R the vector

$$R := (R_1, R_2) := (2h\tilde{R}_1, 2h\tilde{R}_2), \tag{32}$$

by $\mathrm{div}_E R$ the "Euclidean" divergence $\partial_{x_1} R_1 + \partial_{x_2} R_2$ and by $\Delta_E f$ the "Euclidean laplacian" $\partial^2_{x_1} f + \partial^2_{x_2} f$. Then,

$$\Delta_E f = \partial^2_{x_1} f_d - \partial^2_{x_2} f_d + 2\partial_{x_1}\partial_{x_2} f_m + \mathrm{div}_E R. \tag{33}$$

4.2. Choice of $\varphi^{\pm}$. Thanks to Lemma 2.2 and Proposition 3.2, Σ is a sphere and there exist a universal constant C and a conformal parameterization $\psi : \mathbf{S}^2 \to \Sigma$ such that

$$\psi^* g = \overline{h}^2 \sigma \qquad C^{-1} \leq \overline{h} \leq C \qquad \|d\overline{h}\|_{L^2} \leq C. \tag{34}$$

Clearly, there exist a universal constant C_1 and two conformal parameterizations $\varphi_1, \varphi_2 : \mathbf{R}^2 \to \mathbf{S}^2$ such that

- (a′) $\varphi_1(\mathcal{D}_1) \cup \varphi_2(\mathcal{D}_1) = \mathbf{S}^2$;
- (b′) $\mathrm{ar}(\varphi_1(\mathcal{D}_1) \cap \varphi_2(\mathcal{D}_1)) \geq 1$;
- (c′) $\|\varphi_i\|_{C^0(K)} + \|\varphi_i\|_{C^1(K)} + \|\varphi_i\|_{C^2(K)} \leq C_1(K)$ for every compact set K.

Let us define $\varphi^+ := \psi \circ \varphi_1$ and $\varphi^- := \psi \circ \varphi_2$. Clearly, $\varphi^{\pm}$ are conformal and for some universal constant C, they satisfy (a) and (b). It remains to show (c). Without loss of generality, we show it for $\varphi = \varphi^+$. We fix a system of orthonormal coordinates x_1, x_2 in $\mathbf{R}^2 \supset \mathcal{D}_1$ and we adopt the notation of Subsection 4.1. Thus, in this system of conformal coordinates, the metric g on Σ is given by $h^2\delta_{ij}$. Set $f := \mathrm{tr}\, A$ as in Subsection 4.1.

Our goal is to bound $\|f - \lambda\|_{L^{2,\infty}(\varphi(\mathcal{D}_1))}$ for some $\lambda \in \mathbf{R}$. Since the conformal factor enjoys L^∞ estimates from above and from below, this is equivalent to show that $\|f - \lambda\|_{L^{2,\infty}(\mathcal{D}_1)} \leq C\delta$. Thus, from now on we work in the Euclidean disk $\mathcal{D}_1$: in order to achieve our estimate, we use equation (33).

First estimate. Let us denote by $\hat{w}$ the Fourier transform of w and by $\check{w}$ the inverse Fourier transform. Moreover, let ξ be the frequency variables. Recall that since $\varphi : \mathbf{R}^2 \to \mathbf{S}^2$, the functions f, f_m and f_d are defined everywhere on $\mathbf{R}^2$. Let ζ be a smooth cut–off function supported on $\mathcal{D}_{3/2}$ and such that $\varphi = 1$ on $\mathcal{D}_1$. Define f' as

$$f_1 := \frac{(\xi_1^2 - \xi_2^2)}{|\xi|^2}\widehat{\zeta f}_d + 2\frac{\xi_1\xi_2}{|\xi|^2}\widehat{\zeta f}_m \qquad f' := \check{f}_1.$$

By Plancherel theorem, there exists a constant C (which depends on the cut–off function φ) such that

$$\|f'\|_{L^2} \leq C\left(\|f_d\|_{L^2(\mathcal{D}_2)} + \|f_m\|_{L^2(\mathcal{D}_2)}\right) \leq C_1\delta.$$

Moreover, on the set $\mathcal{D}_{3/2}$, we have

$$\Delta_E f' = \partial^2_{x_1} f_d - \partial^2_{x_2} f_d + 2\partial_{x_1}\partial_{x_2} f_m. \tag{35}$$

Second estimate. Let $K(x) = \frac{1}{2\pi}\log(|x|)$ be the fundamental solution of the Laplacian in $\mathbf{R}^2$ and set $f'' = K * \mathrm{div}_{\mathrm{E}}\,\mathrm{R}$. Thus, $f'' = (\partial_{x_1}K) * R_1 + (\partial_{x_2}K) * R_2$. Recall the definition of R in (32). By (30), we have

$$R_1 = +\frac{\overline{h}_2}{\overline{\overline{h}}} f_m + \frac{\overline{h}_1}{\overline{\overline{h}}} f_d.$$

Hence, the estimate (34) gives that $\|R_1\|_{L^1} \leq C\delta$. An analogous estimate holds for R_2. The locality of convolution, Lemma B.1 and Lemma B.2 give that $\|f''\|_{L^{2,\infty}(\mathcal{D}_2)} \leq C\delta$. Moreover,

$$\Delta_E f'' = \mathrm{div} R. \tag{36}$$

Third estimate. Let $\alpha := f - f'' - f'$. Then, thanks to (33), (35), and (36), α is harmonic on $\mathcal{D}_{3/2}$. Moreover, the relations (31) give

$$\left\{ \begin{array}{rcl} \partial_{x_1}\alpha & = & \partial_{x_1} f_d + \partial_{x_2} f_m + R_1 - \partial_{x_1}(f' + f'') \\ \partial_{x_2}\alpha & = & -\partial_{x_2} f_d + \partial_{x_1} f_m + R_2 - \partial_{x_2}(f' + f'') \end{array} \right. .$$

Let $|||\cdot|||_{\mathcal{D}_{3/2}}$ be a norm which is controlled by both the $L^1(\mathcal{D}_{3/2})$ norm and the $W_0^{-1,2}(\mathcal{D}_{3/2})$ norm. Then, the various estimates give that $|||\nabla\alpha|||_{\mathcal{D}_{3/2}} \leq C\delta$. Since α is harmonic and $\mathcal{D}_1 \subset\subset \mathcal{D}_{3/2}$, there is a universal constant C_1 such that $\|\nabla\alpha\|_{L^\infty} \leq C_1\delta$. Thus, for some $\lambda > 0$ and for some universal constant C_2, we have $\|\alpha - \lambda\|_{L^\infty(\mathcal{D}_1)} \leq C_2\delta$. Since $f = f' + f'' + \alpha$, we get

$$\begin{aligned} \|f - \lambda\|_{L^{2,\infty}(\mathcal{D}_1)} \leq C_3\|f'\|_{L^2(\mathcal{D}_1)} + C_4\|f''\|_{L^{2,\infty}(\mathcal{D}_1)} \\ + C_5\|\alpha - \lambda\|_{L^\infty(\mathcal{D}_1)} \leq C_6\delta. \end{aligned} \tag{37}$$

5. Proof of the L^2 estimate for $A - \mathrm{Id}$

In the previous section, we have achieved the following: If we define $2\overline{H} := \int_\Sigma \mathrm{tr}\ A$, then $\|A - \overline{H}\mathrm{Id}\|_{L^{2,\infty}} \leq C\delta$. The goal of this section is to use this information to prove

$$\int_\Sigma |A - \mathrm{Id}|^2 \leq C\delta^2. \tag{38}$$

In order to do this, we will show that $|1 - \overline{H}^2| \leq C\delta^2$. This is sufficient to get (38). Indeed

$$\begin{aligned} |\,\mathrm{tr}\ A - 2\overline{H}|^2 &= \kappa_1^2 + \kappa_2^2 + 4\overline{H}^2 + 2\kappa_1\kappa_2 - 4\overline{H}\kappa_1 - 4\overline{H}\kappa_2 \\ &= |\kappa_1 - \kappa_2|^2 + 4\overline{H}^2 - 4\overline{H}\,\mathrm{tr}\ A + 4\det A. \end{aligned} \tag{39}$$

Integrating (39) and taking into account $\int_\Sigma \det A = 4\pi = \mathrm{ar}(\Sigma)$ and $\int_\Sigma \mathrm{tr}\, A = 2\overline{H}\mathrm{ar}(\Sigma)$, we have

$$\int_\Sigma |\,\mathrm{tr}\ A - 2\overline{H}\mathrm{Id}|^2 = \frac{1}{2}\int_\Sigma \left|\mathring{A}\right|^2 + 16\pi(1-\overline{H}^2).$$

Thus, $|1-\overline{H}^2| \le C\delta^2$ would imply $\int_\Sigma |A-\overline{H}\mathrm{Id}|^2 \le C\delta^2$. Moreover, for δ small enough, $|1-\overline{H}^2| \le C\delta^2$ implies $(1-\overline{H})^2 \le C\delta^2$. Since $|A-\mathrm{Id}|^2 \le 2|A-\overline{H}\mathrm{Id}|^2 + 2(1-\overline{H})^2$, this would give (38).

For later purposes, we collect the inequality

$$\|A-\overline{H}\mathrm{Id}\|^2_{L^2} \le C\delta^2 + C_1|1-\overline{H}^2|, \tag{40}$$

which is a direct consequence of the computations above. Moreover, we will make use of the following generalization of Wente's estimate:

Lemma 5.1. *Let $\alpha, \beta, \gamma \in C^\infty(\mathbf{S}^2)$. Then, there exists a universal constant C such that*

$$\int_{\mathbf{S}^2} \alpha\, d\beta \wedge d\gamma \le C\|d\alpha\|_{L^{2,\infty}}\|d\beta\|_{L^2}\|d\gamma\|_{L^2}. \tag{41}$$

Proof. In local charts, thanks to Theorem 3.3, we have the $\mathcal{H}^1$ estimate

$$\|d\beta\wedge d\gamma\|_{\mathcal{H}^1(\mathcal{D}_1)} \le C\|d\beta\|_{L^2(\mathcal{D}_1)}\|d\gamma\|_{L^2(\mathcal{D}_1)}$$

in the Euclidean disk $\mathcal{D}_1$. A finite covering of $\mathbf{S}^2$ with smooth coordinate patches yields

$$\|d\beta\wedge d\gamma\|_{\mathcal{H}^1(\mathbf{S}^2)} \le C\|d\beta\|_{L^2(\mathbf{S}^2)}\|d\gamma\|_{L^2(\mathbf{S}^2)}$$

Denote by $\overline{\alpha}$ the average of α on $\mathbf{S}^2$. Recalling that $\int d\beta\wedge d\gamma = 0$, we get

$$\int_{\mathbf{S}^2} \alpha\, d\beta\wedge d\gamma = \int_{\mathbf{S}^2} (\alpha - \overline{\alpha})\, d\beta\wedge d\gamma.$$

Thus, the duality between $\mathcal{H}^1$ and BMO (see Theorem A.6 and Corollary A.7) gives

$$\int_{\mathbf{S}^2} \alpha\, d\beta\wedge d\gamma \le C|\alpha|_{BMO}\|d\beta\|_{L^2}\|d\gamma\|_{L^2}. \tag{42}$$

Thanks to Lemma B.3, we have $|\alpha|_{BMO} \le C\|d\alpha\|_{L^{2,\infty}}$. q.e.d.

5.1. Setting. Using the Gauss–Bonnet formula and the identity $8\pi\overline{H} = \int_\Sigma \mathrm{tr}\ A$, we get that

$$4\pi(1-\overline{H}^2) = \int_\Sigma \det A - \overline{H}\int_\Sigma \mathrm{tr}\ A + \overline{H}^2\int_\Sigma 1. \tag{43}$$

We denote by $N : \Sigma \to \mathbf{S}^2 \subset \mathbf{R}^3$ the Gauss map. Fix a conformal map $\psi : \mathbf{S}^2 \to \Sigma \subset \mathbf{R}^3$ satisfying the requirements of Proposition 3.2 and a conformal map $\varphi : \mathbf{R}^2 \supset \mathcal{D}_1 \to \mathbf{S}^2$. Denote by

- $\Psi : \mathcal{D}_1 \to \Sigma \subset \mathbf{R}^3$ the conformal map $\psi\circ\varphi$;
- $\tilde{h}^2$ and h^2 the conformal factors of Ψ and ψ;
- M and N' the maps $N\circ\Psi$ and $N\circ\psi$.

Fix an orthonormal system of coordinates y_1, y_2, y_3 on $\mathbf{R}^3$ and an orthonormal system x_1, x_2 on $\mathcal{D}_1$. If a and b are two vectors of $\mathbf{R}^3$, then $a\times b$ denotes the vector of $\mathbf{R}^3$ which is the standard vector product of a and b.

5.2. Algebraic computations. As a first step, we give some formulae for $\tilde{h}^2$, $\tilde{h}^2(\det dN)\circ\Psi$ and $\tilde{h}^2(\mathrm{tr}\ dN)\circ\Psi$.

First Computation. Since Ψ is conformal, we have

$$\det d\Psi = |\Psi_{,x_1}\times\Psi_{,x_2}|\,, \tag{44}$$

where $\Psi_{,x_i}$ denotes the map $\frac{\partial\Psi}{\partial x_i} : \mathcal{D}_1 \to \mathbf{R}^3$. In equation (44), we make a slight abuse of notation. Indeed

- On the left-hand side, we consider Ψ as a map taking values on Σ. Thus, $\det d\Psi$ has the usual meaning, since $d\Psi_p$ is a linear map from $T_p\mathbf{R}^2 \to T_{\Psi(p)}\Sigma$.
- On the right-hand side, we consider Ψ as a map taking values on $\mathbf{R}^3$.

We now fix the convention on the wedge product of vectors of $\mathbf{R}^3$ in such a way that

$$M\cdot\Psi_{,x_1}\times\Psi_{,x_2} = |\Psi_{,x_1}\times\Psi_{,x_2}|\,. \tag{45}$$

Hence, we can write

$$\tilde{h}^2 = M\cdot\Psi_{,x_1}\times\Psi_{,x_2}. \tag{46}$$

Second Computation. The normal M is perpendicular to both $M_{,x_1}$ and $M_{,x_2}$. Moreover, the orientation convention which yields (45) gives

$$\det dM := M\cdot M_{,x_1}\times M_{,x_2}. \tag{47}$$

Similarly to (44), equation (47) must be understood in the following way:

- On the left-hand side, we consider M as a map taking values on $\mathbf{S}^2$. Thus, $\det dM$ has the usual meaning;
- On the right-hand side, we consider M as a map taking values on $\mathbf{R}^3$.

The discussion above gives the equality

$$\tilde{h}^2(\det dN)\circ\Psi = \det dM = M\cdot M_{,x_1}\times M_{,x_2}. \tag{48}$$

Third Computation. Note that $M_{,x_i} = [dN\circ\Psi](\Psi_{,x_i})$. Thus, thanks to the conformality of Ψ, we have

$$\begin{aligned}(\mathrm{tr}\ dN)\circ\Psi &= \left[dN\circ\Psi\left(\frac{\Psi_{,x_1}}{|\Psi_{,x_1}|}\right)\right]\cdot\frac{\Psi_{,x_1}}{|\Psi_{,x_1}|}\\ &\quad+\left[dN\circ\Psi\left(\frac{\Psi_{,x_2}}{|\Psi_{,x_2}|}\right)\right]\cdot\frac{\Psi_{,x_2}}{|\Psi_{,x_2}|}\\ &= \frac{1}{\tilde{h}^2}\left[M_{,x_1}\cdot\Psi_{,x_1}+M_{,x_2}\cdot\Psi_{,x_2}\right].\end{aligned}$$

Since Ψ is conformal, we have

$$M_{,x_1}\cdot\Psi_{,x_1} = M_{,x_1}\cdot(\Psi_{,x_2}\times M) = M\cdot M_{,x_1}\times\Psi_{,x_2}.$$

Thus, we get

$$\tilde{h}^2(\mathrm{tr}\ dN)\circ\Psi = \left(M\cdot M_{,x_1}\times\Psi_{,x_2}+M\cdot\Psi_{,x_1}\times M_{,x_2}\right). \tag{49}$$

Combining (46), (48), and (49), we get

$$\begin{aligned}&\int_{\Psi(\mathcal{D}_1)}\left(\det A-\overline{H}\,\mathrm{tr}\ A+\overline{H}^2\right)\zeta\\ &\quad=\int_{\mathcal{D}_1}\tilde{h}^2\left((\det dN)\circ\Psi-\overline{H}(\mathrm{tr}\ dN)\circ\Psi+\overline{H}^2\right)\zeta\circ\Psi\\ &\quad=\int_{\mathcal{D}_1}\left(M\cdot(M-\overline{H}\Psi)_{,x_1}\times(M-\overline{H}\Psi)_{,x_2}\right)\zeta\circ\Psi,\end{aligned} \tag{50}$$

for every $\zeta\in C_c^\infty(\Psi(\mathcal{D}_1))$.

5.3. Skew–symmetric quantities. Consider two smooth maps $\alpha,\beta:\mathcal{D}_1\to\mathbf{R}^3$. Denote by α_i,β_i, $i\in\{1,2,3\}$ the components of α and β in a system of orthonormal coordinates of $\mathbf{R}^3$. Then, straightforward computations give the following identity:

$$\left[\alpha\cdot\beta_{,x_1}\times\beta_{,x_2}\right]dx_1\wedge dx_2 = \sum_{i,j,k=1}^{3}\varepsilon_{ijk}\alpha_i\,d\beta_j\wedge d\beta_k. \tag{51}$$

where ε_{ijk} is the totally antisymmetric tensor given by

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, equations (50) and (51) give

$$\begin{aligned} (52)\quad & \int_{\Psi(\mathcal{D}_1)} \left(\det A - \overline{H} \operatorname{tr} A + \overline{H}^2\right) \zeta \\ & \quad = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \int_{\mathcal{D}_1} \left(M_i \, d\,[(M - \overline{H}\Psi)_j] \wedge d\,[(M - \overline{H}\Psi)_k]\right) \zeta \circ \Psi, \end{aligned}$$

for every $\zeta \in C_c^\infty(\Psi(D_1))$. Since $\varphi : \mathcal{D}_1 \to \varphi(\mathcal{D}_1) \subset \mathbf{S}^2$ is a diffeomorphism, we can use φ^{-1} to pull back the forms on the right-hand side of (52) on $\varphi(\mathcal{D}_1)$. Recalling that $N' = M \circ \varphi^{-1}$ and $\psi = \Psi \circ \varphi^{-1}$, we get

(53)

$$\begin{aligned} & \int_{\psi(\varphi(\mathcal{D}_1))} \left(\det A - \overline{H} \operatorname{tr} A + \overline{H}^2\right) \zeta \\ & \quad = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \int_{\varphi(\mathcal{D}_1)} \left(N'_i \, d\,[(N' - \overline{H}\psi)_j] \wedge d\,[(N' - \overline{H}\psi)_k]\right) \zeta \circ \psi. \end{aligned}$$

Hence, thanks to the arbitrariness of the conformal map φ, the previous equation gives that, for every $\zeta \in C^\infty(\mathbf{S}^2)$ which is supported in a set of diameter strictly less than 4π, we have

$$\begin{aligned} (54)\quad & \int_{\psi(\mathbf{S}^2)} \left(\det A - \overline{H} \operatorname{tr} A + \overline{H}^2\right) \zeta \circ \psi^{-1} \\ & \quad = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \int_{\mathbf{S}^2} \left(N'_i \, d\,[(N' - \overline{H}\psi)_j] \wedge d\,[(N' - \overline{H}\psi)_k]\right) \zeta. \end{aligned}$$

A partition of unity on $\mathbf{S}^2$ gives

$$\begin{aligned} (55)\quad & \int_{\Sigma} \left(\det A - \overline{H} \operatorname{tr} A + \overline{H}^2\right) \\ & \quad = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \int_{\mathbf{S}^2} N'_i \, d\,[(N' - \overline{H}\psi)_j] \wedge d\,[(N' - \overline{H}\psi)_k]. \end{aligned}$$

Integrating by parts, we can write

$$\int_{\mathbf{S}^2} N_i'\, d\,[(N'-\overline{H}\psi)_j] \wedge d\,[(N'-\overline{H}\psi)_k]$$
$$= \int_{\mathbf{S}^2} -(N'-\overline{H}\psi)_j\, dN_i' \wedge d\,[(N'-\overline{H}\psi)_k].$$

5.4. Final estimates. Thanks to Lemma 5.1, we have

$$\left|\int_{\mathbf{S}^2} [(N'-\overline{H}\psi)_j]\, dN_i' \wedge d\,[(N'-\overline{H}\psi)_k]\right| \leq \|d(N'-\overline{H}\psi)\|_{L^{2,\infty}} \|dN'\|_{L^2} \|d(N'-\overline{H}\psi)\|_{L^2}. \tag{56}$$

Thus, we conclude that

$$\left|\int_{\Sigma} (\det A - \overline{H}\,\mathrm{tr}\ A + \overline{H}^2)\right| \leq C\|dN'\|_{L^2(\mathbf{S}^2)} \|d(N'-\overline{H}\psi)\|_{L^2(\mathbf{S}^2)} \|d(N'-\overline{H}\psi)\|_{L^{2,\infty}(\mathbf{S}^2)}, \tag{57}$$

for some universal constant C. Since ψ is conformal and satisfies the bounds given by Proposition 3.2, we have that there exist universal constants C_1, C_2 such that

$$\|dN'\|_{L^2(\mathbf{S}^2)} \leq C_1 \|dN\|_{L^2(\Sigma)} \leq C_2$$

$$\|d(N'-\overline{H}\psi)\|_{L^2(\mathbf{S}^2)} \leq C_1 \|dN - \overline{H}\mathrm{Id}\|_{L^2(\Sigma)}$$

$$\|d(N'-\overline{H}\psi)\|_{L^{2,\infty}(\mathbf{S}^2)} \leq C_1 \|dN - \overline{H}\mathrm{Id}\|_{L^{2,\infty}(\Sigma)} \leq C_2\delta.$$

Thus, taking into account (43) and (57), we get

$$|1-\overline{H}^2| \leq C_3\delta \|A - \overline{H}\mathrm{Id}\|_{L^2(\Sigma)}. \tag{58}$$

Recalling (40), we conclude

$$\|A - \overline{H}\mathrm{Id}\|^2_{L^2(\Sigma)} \leq C\delta^2 + C_4\delta\|A - \overline{H}\mathrm{Id}\|_{L^2(\Sigma)},$$

which, by Young's inequality, yields

$$\|A - \overline{H}\mathrm{Id}\|^2_{L^2(\Sigma)} \leq C\delta^2 + \frac{C_4^2\delta^2}{2} + \frac{\|A - \overline{H}\mathrm{Id}\|^2_{L^2(\Sigma)}}{2}.$$

Hence,

$$\|A - \overline{H}\mathrm{Id}\|^2_{L^2(\Sigma)} \leq C_5\delta^2$$

and plugging this into (58), we get $|1-\overline{H}^2| \leq C_6\delta^2$, which completes the proof.

6. Σ is $W^{2,2}$ close to a round sphere

To complete the proof of Theorem 1.1, it remains to show the estimate (3), under the assumption that $\|\mathring{A}\|_{L^2}^2 \le 8\pi$. The difficulties in getting a conformal ψ satisfying (3) are considerably increased by the action of the conformal group of the sphere. In order to choose ψ, as a first step, we impose the normalization conditions of Lemma 3.4 and we show that these conditions imply that the conformal factor of ψ is $W^{1,2}$–close to 1 (see Subsection 6.1). In a second step, we prove that this, together with the bound on $\|A - \mathrm{Id}\|_{L^2(D_\rho)}$ implies that ψ is $W^{2,2}$–close to a smooth isometric embedding of $\mathbf{S}^2$ (see Subsections 6.2, and 6.3).

6.1. The conformal factor of ψ is close to 1. Fix ψ as in Lemma 3.4 and Proposition 3.2 and denote by $h = e^u$ its conformal factor. The goal of this subsection is to show the existence of a universal constant C such that

$$\|e^u - 1\|_{W^{1,2}} + \|u\|_{W^{1,2}} \le C\delta. \tag{59}$$

To do so, we first show that for $\delta \downarrow 0$, the map ψ must converge to a conformal map, in fact a rigid motion in view of the normalizations. Then, we use a linearization of the equation $-\Delta_{\mathbf{S}^2} u = K e^{2u} - 1$ to get the optimal estimate.

First, we gather all the information acquired in the previous sections (see (12) and Proposition 3.2):

$$u \text{ satisfies } -\Delta_{\mathbf{S}^2} u = K e^{2u} - 1 \text{ and } \int e^{2u} = 4\pi \tag{60}$$

$$\|u\|_{L^\infty} + \|u\|_{W^{1,2}} \le C \text{ for some universal constant } C \tag{61}$$

$$\text{Let } \mathbf{S}_i^\pm \text{ be as in Lemma 3.4. Then, } \int_{\mathbf{S}_i^\pm} |A|^2 e^{2u} = 4\pi + \delta^2/2. \tag{62}$$

$$\int_\Sigma |A - \mathrm{Id}|^2 \le C\delta^2 \tag{63}$$

Step 1. We begin by proving the following statement

$$\begin{aligned} &\text{Fix } p < \infty \text{ and } \eta > 0. \text{ If } \delta > 0 \text{ is} \\ &\text{sufficiently small, then } \|e^{2u} - 1\|_{L^p} + \|u\|_{L^p} \le \eta. \end{aligned} \tag{64}$$

Since e^{2v} is a locally Lipschitz function, thanks to (61), there exists a constant C, independent of u, such that

$$\left|e^{2u} - 1\right| \le C|u|. \tag{65}$$

Thus, we have $\|e^{2u} - 1\|_{L^p} \le C\|u\|_{L^p}$. Assume, by contradiction, that (64) were false. Then, there exist $\eta > 0$ and sequences $\delta_n \downarrow 0$, $\{u_n\} \subset C^\infty(\mathbf{S}^2)$ such that

- e^{u_n} are the conformal factors of the conformal diffeomorphisms $\psi_n : \mathbf{S}^2 \to \Sigma_n \subset \mathbf{R}^3$;
- (60), (61), (62), and (63) hold (with u_n, K_n, δ_n, Σ_n, and A^n in place of u, K, δ, Σ, and A);
- $\|u_n\|_{L^p} \geq \eta > 0$.

Thanks to these assumptions, $\Delta_{\mathbf{S}^2} u_n$ is a bounded sequence in L^1. Let $D(\Delta)$ be the set of functions $f \in L^1(\mathbf{S}^2)$ with zero average. Recall that $\Delta^{-1}_{\mathbf{S}^2} : D(\Delta) \to W^{1,q}$ is a compact operator for every $q < 2$. Thus, a subsequence of u_n, not relabeled, converges strongly in $W^{1,q}$ to some u_∞. Equations (63) and (62) give that $K_n - 1$ converges to 0 strongly in L^1. Since e^{2u_n} is bounded and converges strongly in L^q to e^{2u_∞}, by the dominated convergence Theorem, we conclude that $K_n e^{2u_n}$ converges strongly in L^1 to e^{2u_∞}. Passing to the limit in (60), (61), (62), and (63) we get

$$-\Delta_{\mathbf{S}^2} u_\infty = e^{2u_\infty} - 1, \tag{66}$$

$$\int_{\mathbf{S}^\pm_i} e^{2u_\infty} = 2\pi. \tag{67}$$

From [**1**], every solution of (66) is the logarithm of the conformal factor of a conformal diffemorphism $\tilde{\psi} : \mathbf{S}^2 \to \mathbf{S}^2$. Thus, the normalization condition (67) implies that $u_\infty = 0$.

Step 2. Consider the space of functions $\mathcal{S} := \{\|\zeta\|_\infty \leq C\}$. Then, we claim the existence of a universal constant C_1 such that

$$\|\zeta\|_{L^2} \leq C_1 \left(\|\Delta_{\mathbf{S}^2}\zeta + 2\zeta\|_{L^1} + \max_{i,j} \left| \int_{\mathbf{S}^j_i} e^{2\zeta} - 2\pi \right| \right) \qquad \forall \zeta \in \mathcal{S}. \tag{68}$$

Indeed, set

$$\eta := \|\Delta_{\mathbf{S}^2}\zeta + 2\zeta\|_{L^1} + \max_{i,j} \left| \int_{\mathbf{S}^j_i} e^{2\zeta} - 2\pi \right| \tag{69}$$

and consider the space

$$\mathcal{K} := \big\{\xi \big| -\Delta_{\mathbf{S}^2}\xi = 2\xi\big\}.$$

Note that, if we extend ξ to a 1–homogeneous function $\overline{\xi}$ on $\mathbf{R}^3$, we get that $\overline{\xi}$ is harmonic in $\mathbf{R}^3 \setminus \{0\}$. Since $\overline{\xi}$ is bounded in every ball, 0 is a removable singularity and $\overline{\xi}$ is an entire harmonic function with linear growth. By the Liouville Theorem, we conclude that $\overline{\xi}$ is a linear function. Thus, $\mathcal{K}$ is the three–dimensional space given by the restriction to $\mathbf{S}^2$ of linear functions of $\mathbf{R}^3$.

For $\zeta \in \mathcal{S}$, we denote by $P\zeta$ the L^2–projection of ζ on $\mathcal{K}$ and by $P^\perp\zeta$ the L^2–projection on the orthogonal complement of $\mathcal{K}$. Using Sobolev embeddings, it is easy to check that

$$\|P^\perp\zeta\|_{L^2} \le C_2\eta. \tag{70}$$

Since $\mathcal{K}$ has finite dimension, we have

$$\|P\zeta\|_\infty \le \|\zeta\|_\infty$$

and thus, for some universal constant C_3, we get

$$\|P\zeta\|_\infty + \|P^\perp\zeta\|_\infty \le C_3 \qquad \forall\zeta \in \mathcal{S}. \tag{71}$$

Clearly,

$$\left|e^{2\zeta} - e^{2P\zeta}\right| = \left|e^{2P\zeta}\right|\left|e^{2P^\perp\zeta} - 1\right| \overset{(71)}{\le} C_3\left|e^{2P^\perp\zeta} - 1\right|. \tag{72}$$

Moreover, since the exponential is a locally Lipschitz function, the bound (71) gives also

$$\int_{\mathbf{S}^2}\left|e^{2P^\perp\zeta} - 1\right| \le C_4\|P^\perp\zeta\|_{L^1} \le C_5\|P^\perp\zeta\|_{L^2} \overset{(70)}{\le} C_6\eta. \tag{73}$$

Thus, (69) and (73) give

$$\left|\int_{\mathbf{S}_i^\pm} e^{2P\zeta} - 2\pi\right| \le C_7\eta. \tag{74}$$

Since $P\zeta$ is the restriction of a linear function, it is straightforward to check that

$$\|P\zeta\|_{L^2} \le C_8\eta,$$

for some universal constant C_8. This completes the proof of (68).

Step 3. We rewrite the first identity of (60) as

$$-\Delta_{\mathbf{S}^2}u - 2u = \left(e^{2u} - 2u - 1\right) + (K-1)e^{2u}. \tag{75}$$

Since $\|u\|_\infty$ is bounded by a universal constant (see (61)), we have

$$\left\|e^{2u} - 2u - 1\right\|_{L^1} \le C_1\|u\|_{L^2}^2 \tag{76}$$

for some universal constant C_1. Moreover,

$$\begin{aligned} \|(K-1)e^{2u}\|_{L^1} &= \|\det A - 1\|_{L^1(\Sigma)} \\ &\le \|\det(A - \mathrm{Id})\|_{L^1(\Sigma)} + \|\operatorname{tr} A - 2\|_{L^1} \\ &\le \|A - \mathrm{Id}\|_{L^2(\Sigma)}^2 + C_2\|A - \mathrm{Id}\|_{L^2(\Sigma)} \overset{(63)}{\le} C\delta, \end{aligned} \tag{77}$$

and similarly, from (61), (62), and (63), we get

$$\left| \int_{\mathbf{S}_i^{\pm}} e^{2u} - 2\pi \right| \leq C\delta.$$

Hence, applying (68) and collecting all these inequalities, we get

$$\|u\|_{L^2} \leq C_3 \|u\|_{L^2}^2 + C_4 \delta. \tag{78}$$

Thanks to the first step, when δ is sufficiently small, we have $C_3\|u\|_{L^2} \leq 1/2$. Plugging this into (78), we get

$$\|u\|_{L^2} \leq 2C_4\delta. \tag{79}$$

Step 4. We multiply by u the equation

$$-\Delta_{\mathbf{S}^2} u = e^{2u} - 1 + (K-1)e^{2u}$$

and we integrate by parts to get

$$\left\|\nabla_{\mathbf{S}^2} u\right\|_{L^2}^2 \leq \int_{\mathbf{S}^2} |u| \left|e^{2u} - 1\right| + \int_{\mathbf{S}^2} |u||\mathrm{det} A - 1| e^{2u} \tag{80}$$

Notice that

$$\int |u| \left|e^{2u} - 1\right| \leq C_1 \|u\|_{L^2}^2.$$

Moreover, $|\mathrm{det} A - 1| \leq |\kappa_1 - 1||\kappa_2 - 1| + |\kappa_1 + \kappa_2 - 2|$, and recalling that $\|u\|_\infty$ is uniformly bounded, we get:

$$\int_{\mathbf{S}^2} |u||\mathrm{det} A - 1| e^{2u} \leq C_2 \|A - \mathrm{Id}\|_{L^2(\Sigma)}^2 + C_3 \|u\|_{L^2} \|A - \mathrm{Id}\|_{L^2(\Sigma)}. \tag{81}$$

Recalling (61), (63), and (79), we get

$$\left\|\nabla_{\mathbf{S}^2} u\right\|_{L^2}^2 \leq C_4 \delta^2, \tag{82}$$

which, together with (79), gives

$$\left\|u\right\|_{W^{1,2}} \leq C_5 \delta. \tag{83}$$

Since $\|u\|_\infty$ is bounded by a universal constant, the fact that the exponential map is locally Lipschitz gives (59).

6.2. Cartan formalism. Let D_ρ be a disk of $\mathbf{S}^2$ and let (e_1, e_2) be an orthonormal frame on D_ρ. We assume that this orthonormal frame is generated by a conformal map $\varphi : \mathcal{D}_r \to D_\rho$ via the relations $e_i = \partial_{x_i}\varphi / |\partial_{x_i}\varphi|$. Moreover, we assume that $\|\varphi\|_{C^1}$ is bounded by a universal constant (which is certainly possible if, for instance, $\rho \leq \pi$). We define two maps $\Phi, \Psi : D_\rho \to SO(3)$ in the following way

$$\Phi := (e_1, e_2, e_1 \times e_2). \tag{84}$$

$$\Psi := \Big(e^{-u}d\psi(e_1),\ e^{-u}d\psi(e_2),\ e^{-2u}d\psi(e_1) \times d\psi(e_2)\Big). \tag{85}$$

Note that $e^{-2u}d\psi(e_1) \times d\psi(e_2) = N \circ \psi$. Hereby, we fix a system of coordinates in $\mathbf{R}^3$ and we regard the elements of SO(3) as matrices: Thus, according to definition (84), for $x \in D_\rho$, $\Phi(x)$ is the matrix which has $e_1(x)$, $e_2(x)$, and $e_1(x) \times e_2(x)$ as row vectors. We endow SO(3) with the operator norm and we denote by $B \cdot F$ and by B^{-1} respectively the matrix product of B and F, and the inverse of B.

We want to show that there exist constants $\rho > 0$ and $C > 0$ such that

$$\min_{R \in \mathrm{SO}(3)} \|\Phi - R \cdot \Psi\|_{L^2(D_\rho)} \leq C\delta. \tag{86}$$

Note that the left-hand side of (86) is actually independent of the choice of the frame. Thus, though the estimate is derived for the particular frame of TD_ρ chosen above, we would conclude:

- Let (e_1, e_2) be *any orthonormal frame* and Φ, Ψ as in (84), (85). Then (86) holds.

An easy covering argument would yield a constant C' such that, for some $R \in SO(3)$:

$$\begin{aligned} &\text{For } \textit{every } V \text{ and for every frame } (e_1, e_2) \text{ on } TV, \\ &\text{we have } \|\Phi - R \cdot \Psi\|_{L^2(V)} \leq C'\delta. \end{aligned} \tag{87}$$

One basic property of moving frames (see for instance vol. 3 of [**14**]) is the existence of unique 1–forms with values in skew–symmetric matrices U and W such that

$$\begin{aligned} d\Phi &= \Phi \cdot U \\ d\Psi &= \Psi \cdot W. \end{aligned}$$

Alternatively, U and W can be regarded as matrices of 1–forms on $\mathbf{S}^2$. We define the norm of $|U_x|$ (for $x \in D_\rho$) as

$$|U_x| := \sup_{v \in T_x\mathbf{S}^2, |v|=1} |U_x(v)|,$$

where $|U_x(v)|$ is the operator norm of the matrix $U_x(v) \in \mathbb{M}^{3\times 3}$.

We now come to the proof of (86). Consider $\Lambda := \Phi \cdot \Psi^{-1}$ and compute

$$\begin{aligned} d\Lambda &= d\Phi \cdot \Psi^{-1} - \Phi \cdot \Psi^{-1} \cdot d\Psi \cdot \Psi^{-1} \\ &= \Phi \cdot U \cdot \Psi^{-1} - \Phi \cdot \Psi^{-1} \cdot \Psi \cdot W \cdot \Psi^{-1} = \Phi \cdot (U - W) \cdot \Psi^{-1}. \end{aligned}$$

The following Lemma is a standard Poincaré inequality (for the reader's convenience, we report its proof in Appendix D):

Lemma 6.1. *There exists a universal constant C such that for some $R \in SO(3)$, we have*

$$\|\Lambda - R\|_{L^2(D_\rho)} \le C\rho \|d\Lambda\|_{L^2(D_\rho)}.$$

Thus, since $\rho \le \pi$, there is a constant C such that

$$\|\Lambda - R\|_{L^2(D_\rho)} \le C\|U - W\|_{L^2(D_\rho)}.$$

To complete the proof of (86), it is sufficient to show that there is a universal constant C such that

$$\|U - W\|_{L^2(D_\rho)} \le C\delta. \tag{88}$$

Let θ_1, θ_2 be the basis of the cotangent space T^*M which is dual to (e_1, e_2). Moreover, recall that

- e^v is the conformal factor of $\varphi : \mathcal{D}_r \to D_\rho$;
- x_1, x_2 is an orthonormal basis for $\mathcal{D}_r$;
- $e_i = \partial_{x_i}\varphi / |\partial_{x_i}\varphi| = e^{-v}\partial_{x_i}\varphi$.

Since the second fundamental form of the sphere is the identity, we have (see e.g., p. 97 of Volume III of [**14**])

$$\begin{aligned}
-W_{31} = W_{13} &= A\big(e^{-u}d\psi(e_1), e^{-u}d\psi(e_1)\big)\theta_1 \\
&\quad + A\big(e^{-u}d\psi(e_1), e^{-u}d\psi(e_2)\big)\theta_2 \\
-W_{32} = W_{23} &= A\big(e^{-u}d\psi(e_1), e^{-u}d\psi(e_2)\big)\theta_1 \\
&\quad + A\big(e^{-u}d\psi(e_2), e^{-u}d\psi(e_2)\big)\theta_2 \\
-U_{31} = U_{13} &= \theta_1 \\
-U_{32} = U_{23} &= \theta_2.
\end{aligned}$$

Since $\|A - \mathrm{Id}\|_{L^2} \le C\delta$, the previous equations give $\|W_{i3} - U_{i3}\| \le C\delta$. Thus, it only remains to show that $\|U_{12} - W_{12}\| \le C\delta$. Recall that

$$\begin{aligned}
W_{12}(e_j) &= g\big(\nabla^{\Sigma}_{e^{-u}d\psi(e_j)}(e^{-u}d\psi(e_2)), e^{-u}d\psi(e_1)\big) \\
U_{12}(e_j) &= \theta^1\big(\nabla^{\mathbf{S}^2}_{e_j} e_2\big),
\end{aligned}$$

where g is the Riemannian metric on Σ. Thus

$$\begin{aligned}
U_{12} &= e^{-v}\Big\{\big[\partial_{x_2} v\big]\,\theta_1 - \big[\partial_{x_1} v\big]\,\theta_2\Big\} \\
W_{12} &= e^{-u\circ\varphi - v}\left\{\Big[\partial_{x_2}\big(v + u\circ\varphi\big)\Big]\,\theta_1 - \Big[\partial_{x_1}\big(v + u\circ\varphi\big)\Big]\,\theta_2\right\}.
\end{aligned}$$

Recall that $\|\varphi\|_{C^1}$ is bounded by a universal constant, that $\|e^{-u} - 1\|_{L^2} + \|u\|_{W^{1,2}} \le C\delta$ and $\|u\|_\infty \le C$. Hence, we conclude that

$$\|U_{12} - W_{12}\|_{L^2(D_\rho)} \le C\delta.$$

6.3. Conclusion. Let us compose ψ with the inverse of the rotation R appearing in (87). By abuse of notation, we denote this map by ψ as well. Then, the previous subsection shows the existence of constants C and ρ such that:

- For every disk D of radius ρ in $\mathbf{S}^2$ there exists a conformal map φ such that $\|\varphi\|_{C^2} \le C$ and, if we define $e_i := \partial_{x_i}\varphi/|\partial_{x_i}\varphi|$ and Φ, Ψ as in (84), (85), then:

$$\begin{aligned} &d\Psi = \Psi \cdot W \qquad d\Phi = \Phi \cdot U \\ &\|\Psi - \Phi\|_{L^2(D)} \le C\delta \qquad\qquad \|U - W\|_{L^2(D)} \le C\delta. \end{aligned} \tag{89}$$

Hence, we easily get that

$$\|d\Psi - d\Phi\|_{L^2(D)} \le \|\Psi \cdot (U - W)\|_{L^2(D)} + \|(\Phi - \Psi) \cdot U\|_{L^2(D)} \le C\delta, \tag{90}$$

where we have also used the fact that $\|U\|_{L^\infty}$ depends on $\|\varphi\|_{C^1}$, which is bounded by a uniform constant (recall the choice of φ). Denote by $\mathrm{id} : \mathbf{S}^2 \to \mathbf{R}^3$ the standard embedding of the round sphere in the Euclidean space. Note that (89) gives that $\|d\psi - d(\mathrm{id})\|_{L^2(D)} \le C\delta$. Thus, (since ρ is a fixed constant), by an easy covering argument, we get $\|d\psi - d(id)\|_{L^2(\mathbf{S}^2)} \le C_1\delta$ for some universal constant C_1. By the Poincaré inequality, there is a vector $c_\Sigma \in \mathbf{R}^3$ such that

$$\|\psi - (c_\Sigma + \mathrm{id})\|_{W^{1,2}(\mathbf{S}^2)} \le C_2\delta.$$

It is not difficult to see that (90) and (89) give an estimate on the second derivatives of $\psi - (c_\Sigma + \mathrm{id})$, yielding the desired bound

$$\|\psi - (c_\Sigma + \mathrm{id})\|_{W^{2,2}(\mathbf{S}^2)} \le C_3\delta.$$

Indeed fix a system coordinates on $\mathbf{R}^3$ and call ψ_k, id_k the components of ψ, id. Since $\|\varphi\|_{C^2}$ is bounded by a universal constant, it is sufficient to check

$$\left\|\partial^2_{x_i x_j}(\psi_k - \mathrm{id}_k)\right\|_{L^2(D)} \le C_4\delta. \tag{91}$$

Note that

$$\partial_{x_j}\psi_k = \left|\partial_{x_j}\varphi\right| \, \left[d\psi(e_j)\right]_k = h\,\Psi_{jk}$$

where Ψ_{jk} denotes the jk entry of the matrix Ψ and h is the conformal factor of φ. Thus,

$$\partial^2_{x_i x_j}\psi_k = \left(h\,\partial_{x_i} h\right)\Psi_{jk} + h^2\, d\Psi_{jk}(e_i).$$

Analogously

$$\partial^2_{x_i x_j}\mathrm{id}_k = \left(h\,\partial_{x_i} h\right)\Phi_{jk} + h^2\, d\Phi_{jk}(e_i).$$

Hence, thanks to the uniform bounds on $\|h\|_{L^\infty}$ and $\|\partial_{x_j} h\|_{L^\infty}$, the estimates (90) and (89) give (91).

7. Optimality

In this section, we prove the optimality of Theorem 1.1.

Proposition 7.1. *There exists a family of smooth connected compact surfaces $\Sigma_r \subset \mathbf{R}^3$ without boundary such that:*

$$C \geq \operatorname{ar}(\Sigma_r) \geq c > 0 \text{ for every } r \tag{92}$$

$$\lim_{r\downarrow 0} \int_{\Sigma_r} |\mathring{A}|^p = 0 \qquad \text{for every } p < 2 \tag{93}$$

$$\begin{aligned}&\Sigma_r \text{ converges, in the Hausdorff topology,}\\ &\text{to the union of two round spheres}\end{aligned} \tag{94}$$

$$\lim_{r\downarrow 0} \left(\inf_{\lambda} \int_{\Sigma_r} |A - \lambda \mathrm{Id}|^p \right) > 0. \tag{95}$$

Proof. The idea of the construction is the following. Let us take two round spheres Σ_1 and Σ_2 of radii 1 and 1/2. Then, we can glue them with a small hyperbolic neck Γ so that the integral $\int_\Gamma |A|^p$ is as small as we want. We now give the details of this construction. The estimate of the quantity $\int_\Gamma |A|^p$ will be simplified by using catenoid necks.

Detailed construction. Consider the family of curves $\{\gamma_r\}$ known as catenaries, i.e., the graphs of the functions $f_r : \mathbf{R} \to \mathbf{R}$ given by

$$f_r(x) := r \cosh\left(\frac{x}{r}\right).$$

The surface generated by a revolution of γ_r around the x–axis is called a catenoid and will be denoted by Γ_r. It is well known that catenoids are minimal surfaces (see for instance page 202 of [**4**]). Thus, $\operatorname{tr} A = \kappa_1 + \kappa_2 = 0$ everywhere on Γ_r.

Let x, y, z be a system of coordinates in $\mathbf{R}^3$ and assume that the catenoid Γ_r is given by $|(x,y)| = r\cosh\left(\frac{z}{r}\right)$. For every $r > 0$, we take:

- A round sphere of radius $\frac{1}{2}$ centered at a point $(0,0,z_1)$ with $z_1 > 0$ and tangent to Γ_r in a circle γ_r^1.
- A round sphere of radius 1 centered at a point $(0,0,z_2)$ with $z_2 < 0$ and tangent to Γ_r in a circle γ_r^2.

Consider the closed surface Σ_r which is made of:

- The part of the sphere Σ_1 lying above γ^1 (which we denote by S_r^2);
- The part of the sphere Σ_2 lying below γ^2 (which we denote by S_r^1);
- The portion of catenoid lying between γ^1 and γ^2 (which we denote by T_r).

See Fig. 1 below.

Step 1. Behavior of Σ_r for $r \downarrow 0$.

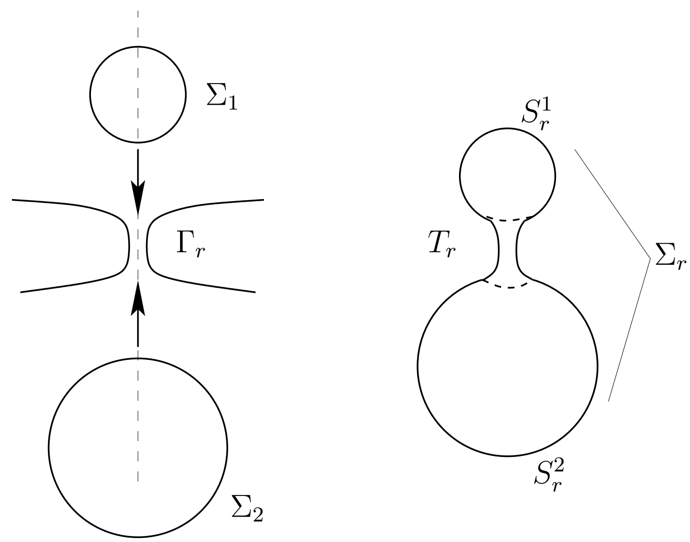

Figure 1. Construction of the surface Σ_r.

The circles γ_r^i are given by

$$\Gamma_r \cap \{z = z_i(r)\}$$

and straightforward computations give that

$$z_1(r) \text{ is the unique positive solution of } \cosh\left(\frac{z_1(r)}{r}\right) = \frac{1}{\sqrt{2r}}$$

$$z_2(r) \text{ is the unique negative solution of } \cosh\left(\frac{z_2(r)}{r}\right) = \frac{1}{\sqrt{r}}.$$

Hence, $z_i(r) \downarrow 0$ as $r \downarrow 0$. Moreover, the radius of γ_r^1 is $\sqrt{r/2}$, whereas the radius of γ_r^2 is $\sqrt{r}$. Hence, we conclude that

(96) The surfaces S_r^1 and S_r^2 converge, respectively, to a sphere S_∞^1 of radius $1/2$ and to a sphere S_∞^2 of radius 1, which are tangent at $(0,0,0)$.

(97) The area of the neck T_r converges to 0.

Step 2. Estimates.
We now prove that

$$\lim_{r\downarrow 0} \int_{T_r} |\mathring{A}|^p = 0. \tag{98}$$

Since T_r is a portion of a minimal surface, tr $A = 0$ on T_r. Thus, (98) is equivalent to

$$\lim_{r\downarrow 0} \int_{T_r} |A|^p = 0. \tag{99}$$

Again, because of the minimal surface equation, $2\det A = -|A|^2$ on T_r. Thus, by Gauss–Bonnet Theorem:

$$8\pi = \int_{\Sigma_r} 2\det A = \int_{S^1_r \cup S^2_r} 2\det A - \int_{T_r} |A|^2. \tag{100}$$

Since S^1_r and S^2_r are both portions of round spheres, we have

$$\int_{S^1_r \cup S^2_r} 2\det A \leq 16\pi.$$

Thus, $\int_{T_r} |A|^2 \leq 8\pi$ and, by Hölder inequality,

$$\int_{T_r} |A|^p \leq (\mathrm{ar}(T_r))^{\frac{2-p}{2}} \left(\int_{T_r} |A|^2\right)^{\frac{p}{2}} \leq (8\pi)^{\frac{p}{2}} (\mathrm{ar}(T_r))^{\frac{2-p}{2}}. \tag{101}$$

By (96), the inequality (101) yields (99). Thus:

- The bound (92) is trivially satisfied.
- Since S^1_r and S^2_r are subsets of round spheres, we have

$$\int_{\Sigma_r} |\mathring{A}|^p = \int_{T_r} |\mathring{A}|^p,$$

  and (93) follows from (98).
- Thanks to (97) and (99)

$$\begin{aligned}\lim_{r\downarrow 0}\left(\inf_\lambda \int_{\Sigma_r} |A-\lambda \mathrm{Id}|^p\right) &= \inf_\lambda \left(\int_{S^1_\infty} |A-\lambda\mathrm{Id}|^p + \int_{S^2_\infty} |A-\lambda\mathrm{Id}|^p\right)\\ &= \inf_\lambda \left[2\pi\left(\frac{1}{2}-\lambda\right)^2 + 8\pi(1-\lambda)^2\right] > 0,\end{aligned}$$

  which gives (95).

Note that the surfaces just constructed are C^1 and piecewise C^2. However, they are all surfaces of revolution: The curves which generate them are C^1 and piecewise C^∞, where the higher derivatives have four points of jump discontinuity. Hence, a standard smoothing argument yields a family of surfaces of revolution which are C^∞ and satisfy all the requirements of the Proposition. q.e.d.

Appendix A. Hardy and BMO spaces

We recall here the definitions of Hardy and BMO spaces (see for example [15], sections 1,2,3 and 4). We fix a $\zeta \in C_c^\infty(\mathbf{R}^n)$ with $\int \zeta = 1$ and we define ζ_ε as $\zeta_\varepsilon(x) = \varepsilon^{-n}\zeta\left(\frac{x}{\varepsilon}\right)$. Then, for every $f \in L^1_{loc}(\mathbf{R}^n)$, we define the maximal function $M_\zeta f$ as

$$M_\zeta f(x) := \sup_{r>0} |f * \zeta_r(x)|. \tag{102}$$

In a similar way, for $M > 0$, we define a local maximal function

$$M_\zeta^M f(x) := \sup_{M>r>0} |f * \zeta_r(x)|.$$

Definition A.1. The Hardy space $\mathcal{H}^1(\mathbf{R}^n)$ consists of the functions $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ such that $M_\zeta f \in L^1(\mathbf{R}^n)$ for some ζ. Similarly, if $\Omega \subset \mathbf{R}^n$, $\mathcal{H}^1_{\mathrm{loc}}(\Omega)$ is the subset of L^1_{loc} consisting of those functions f such that $M_\zeta^M f \in L^1_{\mathrm{loc}}$ for some ζ and some M.

Having fixed ζ, we can endow $\mathcal{H}^1(\mathbf{R}^n)$ with the norm $\|M_\zeta f\|_{L^1(\mathbf{R}^n)}$, thus getting a Banach space (see [15]). Different choices of ζ induce equivalent norms. Moreover, if $f \in \mathcal{H}^1_{\mathrm{loc}}(\Omega)$ and Φ is a diffeomorphism of Ω, then $f \circ \Phi \in \mathcal{H}^1_{\mathrm{loc}}(\Omega)$. Hence, using a finite atlas of coordinate patches, it is possible to define $\mathcal{H}^1(\Sigma)$ for any compact Riemannian manifold Σ. Similarly, after fixing a ζ, an $M > 0$, and a finite atlas, one can define a local maximal function Mf for $f \in \mathcal{H}^1(\Sigma)$ and a norm $\|f\|_{\mathcal{H}^1(\Sigma)} := \|Mf\|_{L^1}$. Different choices induce equivalent norms.

We recall the following celebrated result of [6]:

Theorem A.2. *Let $w \in \mathcal{H}^1(\mathbf{R}^2)$. Then, the equation $\Delta_{\mathbf{R}^2} u = w$ admits a continuous solution $u_0 : \mathbf{R}^2 \to \mathbf{R}$ which satisfies*

$$\|\nabla^2 u_0\|_{L^1} + \|du_0\|_{L^2} + \|u_0\|_{L^\infty} \le C\|w\|_{\mathcal{H}^1},$$

for some universal constant C.

Using a partition of unity and local coordinate patches, Theorem A.2 yields the following

Corollary A.3. *Let $w \in \mathcal{H}^1(\mathbf{S}^2)$. Then, the equation $\Delta_{\mathbf{S}^2} u = w$ admits a continuous solution u_0 which satisfies*

$$\|u_0\|_{W^{2,1}(\mathbf{S}^2)} + \|du_0\|_{L^2(\mathbf{S}^2)} + \|u_0\|_{L^\infty} \le C\|w\|_{\mathcal{H}^1(\mathbf{S}^2)}. \tag{103}$$

Remark A.4. Since harmonic functions on $\mathbf{S}^2$ are constant, the general solution of $\Delta_{\mathbf{S}^2} u = w$ can be written as $u = u_0 + c$. Thus, the normalization condition

$$\int_{\mathbf{S}^2} e^{2u} = 4\pi,$$

yields an estimate like (103) also for u.

In Section 5, we use the duality between BMO and $\mathcal{H}^1$, due to Fefferman.

Definition A.5. Let $f \in L^1_{loc}(\mathbf{R}^n)$. We say that $f \in BMO$ if

$$|f|_{BMO} := \sup_{x\in\mathbf{R}^n} \sup_{r>0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f - f^{x,r}| \qquad \text{is finite,}$$

where $f^{x,r}$ denotes the average of f on $B_r(x)$. We can extend the definition to compact surfaces by taking the second supremum among disks of radius smaller than the diameter of Σ.

Theorem A.6. *Let $f, w \in C^\infty_c(\mathbf{R}^n)$. Then,*

$$\left|\int fw\right| \leq C_\zeta \|f\|_{\mathcal{H}^1} |w|_{BMO},$$

where C_ζ depends only on the kernel $\zeta \in C^\infty_c(\mathbf{R}^n)$ which defines $\|f\|_{\mathcal{H}^1} = \|M_\zeta f\|_{L^1}$

Again, using local charts and a partition of unity, we get

Corollary A.7. *Let $f, w \in C^\infty(\mathbf{S}^2)$. Then, there exists a constant C (depending only on the choices involved in the definition of $\|f\|_{\mathcal{H}^1(\mathbf{S}^2)}$) such that*

$$\left|\int_{\mathbf{S}^2} fw\right| \leq C\|f\|_{\mathcal{H}^1(\mathbf{S}^2)}\left[|w|_{BMO(\mathbf{S}^2)} + \left|\fint_{\mathbf{S}^2} w\right|\right].$$

Appendix B. The space $L^{2,\infty}$

Given a measure space Ω with a σ–finite measure μ, the Marcinkiewicz space $L^{2,\infty}(\Omega,\mu)$ is defined as the set of functions

$$\left\{ f \,\middle|\, \text{there exists } C>0\text{:}\ \ \mu\big(\{f^2 \geq k\}\big) \leq \frac{C}{k} \ \text{ for every } k>0\right\}.$$

For every $f \in L^{2,\infty}$, it is natural to define

$$|f|_{L^{2,\infty}} := \inf\left\{C : \mu\big(\{f^2\geq k\}\big) \leq \frac{C}{k} \text{ for every } k>0\right\}. \tag{104}$$

$|\cdot|$ is not a norm. However, it is possible to define a norm $\|\cdot\|_{L^{2,\infty}}$ which endows $L^{2,\infty}$ of a Banach space structure and such that

$$\frac{1}{k}\|\cdot\|_{L^{2,\infty}} \leq |\cdot|_{L^{2,\infty}} \leq k\|\cdot\|_{L^{2,\infty}}, \tag{105}$$

see e.g. Section 1.8 of [**17**]. For the Proof of Proposition 4.1, we need the following two lemmas:

Lemma B.1. *If $f \in L^{2,\infty}(\mathbf{R}^n), w \in L^1(\mathbf{R}^n)$, then*

$$\|f * w\|_{L^{2,\infty}} \leq \|f\|_{L^{2,\infty}} \|w\|_{L^1}. \tag{106}$$

Lemma B.2. *Let K be the fundamental solution of the Laplacian in $\mathbf{R}^2$ given by $K(x) = \frac{1}{2\pi}\log(|x|)$. Then, $\nabla K \in L^{2,\infty}(U)$ for every bounded open set $U \subset \mathbf{R}^2$.*

Lemma B.1 follows easily from the fact that $\|\cdot\|_{L^{2,\infty}}$ is a norm, while Lemma B.2 is obtained directly from the definition of $|\cdot|_{L^{2,\infty}}$. Finally, in the proof of Theorem 1.1, we need the following

Lemma B.3. *Let $u \in C^\infty(\mathbf{S}^2, \mathbf{R})$. Then, there exists a universal constant C such that*

$$|u|_{BMO(\mathbf{S}^2)} \leq C\|du\|_{L^{2,\infty}(\mathbf{S}^2)}.$$

Proof. Lemma B.3 follows from the Sobolev embedding $W^{1,1}(\mathbf{S}^2) \hookrightarrow L^2(\mathbf{S}^2)$ and the fact that $|u|_{\mathbf{R}^2}$ and $|u|_{L^{2,\infty}(\mathbf{R}^2)}$ are both invariant under the rescalings $x \to rx$. We recall the argument for the reader's convenience.

Using local charts, it suffices to prove

$$|u|_{BMO(\mathcal{D}_1)} \leq C\|du\|_{L^{2,\infty}(\mathcal{D}_1)} \tag{107}$$

where $\mathcal{D}_1$ is the Euclidean unit disk. Recall that

$$|u|_{BMO(\mathcal{D}_1)} := \sup_{y\in\mathcal{D}_1}\left[\sup_{r<\operatorname{dist}(y,\partial\mathcal{D}_1)} \frac{1}{\operatorname{ar}(\mathcal{D}_r(y))}\int_{\mathcal{D}_r(y)} |u - u^{y,r}|\right], \tag{108}$$

In view of the definition of $|u|_{BMO(\mathcal{D}_1)}$, it would be sufficient to prove

$$\frac{1}{\operatorname{ar}(\mathcal{D}_r(y))}\int_{\mathcal{D}_r(y)} |u - u^{y,r}| \leq C\|du\|_{L^{2,\infty}(\mathcal{D}_r(y))} \qquad \text{for all } r < 1.$$

By invariance under translations, we can assume $y = 0$. Moreover, we can assume that $r = 1$. Indeed, define $u_r(x) := u(rx)$. Then,

$$\frac{1}{\operatorname{ar}(\mathcal{D}_r)}\int_{\mathcal{D}_r} \left|u - u^{0,r}\right| = \frac{1}{\operatorname{ar}(\mathcal{D}_1)}\int_{\mathcal{D}_1} \left|u_r - u_r^{0,1}\right|$$

and

$$\|u\|_{L^{2,\infty}(\mathcal{D}_r)} \leq k|u|_{L^{2,\infty}(\mathcal{D}_r)} = k|u_r|_{L^{2,\infty}(\mathcal{D}_1)} \leq k^2\|u_r\|_{L^{2,\infty}(\mathcal{D}_1)}.$$

Thus, the proof reduces to the inequality

$$\int_{\mathcal{D}_1} \left|u - u^{0,1}\right| \leq C\|du\|_{L^{2,\infty}(\mathcal{D}_1)}.$$

Clearly, for some universal constant C, we have

$$\|du\|_{L^1(\mathcal{D}_1)} \leq C\|du\|_{L^{2,\infty}(\mathcal{D}_1)}.$$

Moreover, the Poincaré and Schwartz inequalities give

$$\int_{\mathcal{D}_1} \left|u - u^{0,1}\right| \le \pi^{1/2}\|u - u^{0,1}\|_{L^2(\mathcal{D}_1)}$$
$$\le C_1\pi^{1/2}\|du\|_{L^1(\mathcal{D}_1)} \le C_1 C\pi^{1/2}\|du\|_{L^{2,\infty}(\mathcal{D}_1)}.$$

This completes the proof. q.e.d.

Appendix C. Lemma on open sets

Lemma C.1. *Let $U \subset \mathbf{S}^2$ be an open set and assume that $\partial U \subset \gamma$, where γ is a closed curve. Then, there exists a constant $\delta > 0$, depending only on $\mathrm{ar}(U)$ and $\mathrm{len}\,(\gamma)$ such that U contains an open disk of radius δ.*

Proof. We argue by contradiction. Then, there exist a sequence of open sets U_n and a sequence of closed curves γ_n such that:

1) $\lim_n \mathrm{len}\,(\gamma_n) = C_1 > 0$ and $\lim_n \mathrm{ar}(U_n) = C_2 > 0$;
2) For every $\delta > 0$, there exists N such that, for every $n > N$, U_n does not contain any disk of radius δ.

Let us parameterize γ_n by arc–length. Then, there is a subsequence, not relabeled, which converges uniformly to a Lipschitz curve γ_∞. Hence, up to subsequences, $\overline{U}_n$ converges, in the Hausdorff topology, to a closed set $\overline{U}_\infty$ whose boundary is contained in γ_∞. Due to 2., the set $\overline{U}_\infty$ has empty interior and thus $\mathrm{ar}(\overline{U}_\infty) = \mathrm{ar}(\partial\overline{U}_\infty) = 0$. But 1. implies that $\mathrm{ar}(\overline{U}_\infty) = C_2 > 0$. This is the desired contradiction. q.e.d.

Appendix D. Poincaré inequality for SO(3)–valued maps

Here, we give a proof of Lemma 6.1. We embed $SO(3) \subset \mathbb{M}^{3\times 3} = \mathbf{R}^9$ and we set

$$\overline{\Lambda} = \frac{1}{\mathrm{ar}(D_\rho)} \int_{D_\rho} \Lambda,$$

Since the operator norm on $\mathbb{M}^{3\times 3}$ is equivalent to the Euclidean norm on $\mathbf{R}^9$, the Poincaré inequality yields a constant C such that

$$\|\Lambda - \overline{\Lambda}\|_{L^2(D_\rho)} \le C\rho\|d\Lambda\|_{L^2(D\rho)}.$$

Note that

$$\begin{aligned}\operatorname{dist}(\overline{\Lambda}, \mathrm{SO}(3))^2 &= \frac{1}{\mathrm{ar}(D_\rho)} \int_{D_\rho} \operatorname{dist}(\overline{\Lambda}, SO(3))^2 \\ &\le \frac{1}{\mathrm{ar}(D_\rho)} \int_{D_\rho} \left(|\Lambda - \overline{\Lambda}| + \operatorname{dist}(\Lambda, SO(3))\right)^2 \\ &= \frac{1}{\mathrm{ar}(D_\rho)} \|\Lambda - \overline{\Lambda}\|^2_{L^2(D_\rho)}.\end{aligned}$$

Thus, there exists a map $R \in SO(3)$ such that

$$\|\Lambda - R\|_{L^2(D_\rho)} \le \sqrt{2} C \rho \|d\Lambda\|_{L^2(D\rho)}.$$

empty

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