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CANONICAL METRICS ON THE MODULI SPACE OF RIEMANN SURFACES I

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1. Introduction

One of the main purposes of this paper is to understand the geometry of the moduli and the Teichmüller spaces of Riemann surfaces. The most interesting results we have in this paper are the detailed understanding of two new complete Kähler metrics with nice properties and the Kähler–Einstein metric on the Teichmüller and the moduli spaces of Riemann surfaces. The two new metrics, the Ricci metric and the perturbed Ricci metric, are naturally defined as the negative Ricci curvature of the Weil–Petersson metric and a combination of it with the Weil–Petersson metric. We prove that these new metrics and the Kähler–Einstein metric on the Teichmüller and moduli spaces all have Poincaré type boundary behavior, and further, in [**7**], we prove that they all have bounded geometry. Note that the Kähler–Einstein metric is the key link between the differential geometric and algebraic geometric aspects of these spaces. So, it is most interesting and also most challenging to understand the Kähler–Einstein metric. In fact, by using our understanding of the Kähler–Einstein metric and the new metrics, we will derive in [**7**] the stability of the logarithmic cotangent bundle of the moduli space of Riemann surfaces. In this paper, we study in detail the asymptotic behaviors and the signs of the curvatures of these new metrics. In particular, we prove that the perturbed Ricci metric is a complete Kähler metric with bounded negative holomorphic sectional and Ricci curvature and bounded bisectional curvature. As a consequence, we show that, by using the new metrics as a bridge and some simple argument with Schwarz lemma, all of the classical complete metrics are equivalent to the Ricci and the perturbed Ricci metric on the Teichmüller and moduli spaces.

The study of the Teichmüller and moduli spaces of Riemann surfaces has a long history. It has been intensively studied by many mathematicians in complex analysis, differential geometry, topology and algebraic

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geometry for the past 60 years. They have also appeared in theoretical physics such as string theory. The moduli space can be viewed as the quotient of the corresponding Teichmüller space by the modular group. There are several classical metrics on these spaces: the Weil–Petersson metric, the Teichmüller metric, the Kobayashi metric, the Bergman metric, the Carathéodory metric and the Kähler–Einstein metric. These metrics have been studied over the years and have found many important applications in various areas of mathematics. Each of these metrics has its own advantages and disadvantages in studying different problems.

The Weil–Petersson metric is a Kähler metric as first proved by Ahlfors, both of its holomorphic sectional curvature and Ricci curvature have negative upper bounds as conjectured by Royden and proved by Wolpert. These properties have found many applications by Wolpert, and they were also used in solving problems from algebraic geometry by combining with the Schwarz lemma of Yau ([**6**], [**19**]). But as first proved by Wolpert and Chu, it is not a complete metric which prevents the understanding of some aspects of the geometry of the moduli spaces. Royden, Siu and Schumacher extended some results to higher dimensional cases. The works of Masur and Wolpert, Siu and Schumacher will play important roles in our study.

The Teichmüller metric, the Kobayashi metric and the Carathéodory metric are only Finsler metrics. They are very effective in studying the hyperbolic property of the moduli space. Royden proved that the Teichmüller metric is equal to the Kobayashi metric from which he deduced the important corollary that the isometry group of the Teichmüller space is exactly the modular group. Recently, McMullen introduced a new complete Kähler metric on the moduli space by perturbing the Weil–Petersson metric [**12**]. By using this metric, he was able to prove that the moduli space is Kähler hyperbolic, and also to derive several topological consequences. The McMullen metric has bounded geometry, but we lose control on the signs of its curvatures.

In the early 80s, Cheng and Yau [**2**] proved the existence of the Kähler–Einstein metric on the Teichmüller space. Since the Kähler–Einstein metric is canonical, it also descends to a complete Kähler metric on the moduli space. More than 20 years ago, Yau [**20**] conjectured the equivalence of the Kähler–Einstein metric to the Teichmüller metric. We will prove this conjecture in this paper. Since the McMullen metric is equivalent to the Teichmüller metric, so we have also proved the equivalence of the Kähler–Einstein metric and the McMullen metric. We will further show that the Bergman metric and the Carathéodory metric are also equivalent to the Kobayashi metric which was also first conjectured by Yau. Therefore, all of the classical metrics are equivalent to the Ricci and the perturbed Ricci metric.

The Ricci metric is induced by the negative Ricci curvature of the Weil–Petersson metric, see also [**15**], and the perturbed Ricci metric is a perturbation of the Ricci metric by the Weil–Petersson metric. We first study the asymptotic behaviors of the Ricci metric near the boundary of the moduli space, we prove that it is asymptotically equivalent to the Poincaré metric, and asymptotically, its holomorphic sectional curvature has negative upper and lower bound in the degeneration directions. But its curvatures in the non-degeneration directions near the boundary and in the interior of the moduli space cannot be controlled well. To solve this problem, we introduce another new complete Kähler metric which we call the perturbed Ricci metric, it is obtained by adding a multiple of the Weil–Petersson metric. We compute the holomorphic sectional curvature and the Ricci curvature of this new metric. We show in this paper and also in [**7**] that they are all bounded below and above, and the holomorphic sectional and Ricci curvature have negative upper and lower bounds. This is the first known complete Kähler metric on the moduli space with such good curvature property. Note that the curvatures of the Weil–Petersson metric do not have lower bound. By applying the Schwarz lemma of Yau, we can prove the equivalence of this new metric to the Kähler–Einstein metric. The equivalence of the perturbed Ricci metric to the McMullen metric is proved by an estimate of the asymptotic behavior of these two metrics. The equivalences of the Bergman metric, the Carathéodory metric and the Kobayashi metric are proved by simply using the Bers embedding and the Schwarz–Yau lemma.

Another important fact of the Ricci metric is that it is cohomologous to the Kähler–Einstein metric. By using the Ricci metric as the background metric, we can establish the Monge–Amperé equation and study the strongly bounded geometry of the Kähler–Einstein metric [**7**].

To state our main results in detail, let us introduce some definitions and notations. Here, for convenience, we will use the same notation for a Kähler metric and its Kähler form. First, two metrics ω_{τ_1} and ω_{τ_2} are called equivalent, if they are quasi-isometric to each other in the sense that

$$C^{-1}\omega_{\tau_2} \leq \omega_{\tau_1} \leq C\omega_{\tau_2}$$

for some positive constant C. We will write this as $\omega_{\tau_1} \sim \omega_{\tau_2}$.

Our first result is the following asymptotic behavior of the Ricci metric near the boundary divisor of the moduli space. Let $\mathcal{T}_g$ denote the Teichmüller space and $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus g where $g \geq 2$. $\mathcal{M}_g$ is a complex orbifold of dimension $3g-3$ as a quotient of $\mathcal{T}_g$ by the modular group. Let $n = 3g - 3$. Let ω_{WP} be the Weil–Petersson metric and $\omega_\tau = -Ric(\omega_{WP})$ be the Ricci metric. It is easy to show that there is an asymptotic Poincaré metric on $\mathcal{M}_g$. See Section 4 for the construction.

Theorem 1.1. *The Ricci metric is equivalent to the asymptotic Poincaré metric.*

This theorem is proved in Section 4. Our second result is the following estimates of the holomorphic sectional curvature of the Ricci metric. Note our convention of the sign of the curvature may be different from some literature.

Theorem 1.2. *Let $X_0 \in \overline{\mathcal{M}_g} \setminus \mathcal{M}_g$ be a codimension m point and let $(t_1, \cdots, t_m, s_{m+1}, \cdots, s_n)$ be the pinching coordinates at X_0 where $t_1, \cdots, t_m$ correspond to the degeneration directions. Then, the holomorphic sectional curvature of the Ricci metric is negative in the degeneration directions and is bounded in the non-degeneration directions. Precisely, there is a $\delta > 0$ such that if $|(t, s)| < \delta$, then*

$$\widetilde{R}_{i\bar{i}i\bar{i}} = \frac{3u_i^4}{8\pi^4 |t_i|^4}(1 + O(u_0)) > 0 \quad \text{if} \quad i \leq m$$

and

$$\widetilde{R}_{i\bar{i}i\bar{i}} = O(1) \quad \text{if} \quad i \geq m + 1.$$

Furthermore, on $\mathcal{M}_g$ the holomorphic sectional curvature, the bisectional curvature and the Ricci curvature of the Ricci metric are bounded from above and below.

This is Theorem 4.4 of Section 4 of this paper. One of the main purposes of our work was to find a natural complete metric whose holomorphic sectional curvature is negative. To do this, we introduce the perturbed Ricci metric. In Section 5, we will prove the following theorem:

Theorem 1.3. *For suitable choice of positive constant C, the perturbed Ricci metric*

$$\omega_{\widetilde{\tau}} = \omega_\tau + C\omega_{WP}$$

is complete and its holomorphic sectional curvatures are negative and bounded from above and below by negative constants. Furthermore, the Ricci curvature of the perturbed Ricci metric is bounded from above and below.

Note that the perturbed Ricci metric is equivalent to the Ricci metric, since its asymptotic behavior is dominated by the Ricci metric. Now we denote the Kähler–Einstein metric of Cheng–Yau by ω_{KE} which is another complete Kähler metric on the moduli space. By applying the Schwarz lemma of Yau, we derive our fourth result in Section 6:

Theorem 1.4. *We have the equivalence of the following three complete Kähler metrics on the moduli spaces of curves:*

$$\omega_{KE} \sim \omega_{\tau} \sim \omega_{\tilde{\tau}}.$$

Our fifth result in this paper proved in Section 6 is the equivalence of the Ricci metric and the perturbed Ricci metric to the McMullen metric. Let us denote the McMullen metric by ω_M.

Theorem 1.5. *We have the equivalence of the following metrics: the McMullen metric, the Ricci metric and the perturbed Ricci metric:*

$$\omega_M \sim \omega_{\tau} \sim \omega_{\tilde{\tau}}.$$

As a corollary, we know that these metrics are also equivalent to the Teichmüller metric, the Kobayashi metric, and the Kähler–Einstein metric. This proved the conjecture of Yau [**20**].

We denote by $\|\cdot\|_K$, $\|\cdot\|_B$ and $\|\cdot\|_C$ the norms defined by the Kobayashi, Bergman and Carathéodory metrics. In the last section, we showed that these metrics are equivalent.

Theorem 1.6. *On the Teichmüller space $\mathcal{T}_g$ with $g \geq 2$, the Kobayashi metric, the Bergman metric and the Carathéodory metrics are equivalent. Namely,*

$$\|\cdot\|_K \sim \|\cdot\|_B \sim \|\cdot\|_C.$$

These results imply that all the above complete metrics have Poincaré type growth on the moduli space.

In the second part of this work [**7**], we will prove that the Kähler–Einstein metric, Ricci and the perturbed Ricci metric all have (strongly) bounded geometry, and derive the stability of the logarithmic cotangent bundle of the moduli space of Riemann surfaces. It would be interesting to see how this result can be proved by algebraic geometric method. In the third part of our work [**8**], we will prove the goodness of the Weil–Petersson metric, the Ricci and the perturbed Ricci metric and other metrics in the sense of Mumford, derive some other nice properties of these metrics and find interesting applications.

This paper is organized as follows. In Section 2, we set up some notations and introduce the Weil–Petersson metric and its curvatures. In Section 3, we introduce various operators needed for our computations, we compute and simplify the curvature of the Ricci metric by using these operators and their various special properties. This section consists of long and complicated computations. Section 4 consists of several subtle estimates of the Ricci metric and its curvatures near the boundary of the moduli space. In Section 5, we introduce the perturbed Ricci metric, compute its curvature and study its asymptotic behavior near the boundary of the moduli space. These results are then used in Section 6 to prove the equivalence of the several well-known classical complete Kähler metrics as stated above. Finally, in Section 7, by using the Ber's embedding theorem and basic properties of the Kobayashi, Bergman and Carathéodory metric, we show that these metrics are equivalent. This finishes the proof of the equivalence of all of the known complete metrics. In the appendix, we add some details of the computations for the convenience of the readers.

For simplicity, we state all of our results for the moduli and Teichmüller spaces of closed Riemann surfaces. All of the theorems hold for moduli spaces $\mathcal{M}_{g,n}$ of hyperbolic Riemann surfaces with punctures.

Some history of this research can be found in our survey paper [**9**].

2. The Weil–Petersson metric

The purpose of this section is to set up notations for our computations. We will introduce the Weil–Petersson metric and recall some of its basic properties. Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus g where $g \geq 2$. $\mathcal{M}_g$ is a complex orbifold of dimension $3g-3$. Let $n = 3g-3$. Let $\mathfrak{X}$ be the total space and $\pi : \mathfrak{X} \to \mathcal{M}_g$ be the projection map. There is a natural metric, called the Weil–Petersson metric which is defined on the orbifold $\mathcal{M}_g$ as follows:

Let $s_1, \cdots, s_n$ be holomorphic local coordinates near a regular point $s \in \mathcal{M}_g$ and assume that z is a holomorphic local coordinate on the fiber $X_s = \pi^{-1}(s)$. For the local holomorphic vector fields $\frac{\partial}{\partial s_1}, \cdots, \frac{\partial}{\partial s_n}$, there are vector fields $v_1, \cdots, v_n$ on $\pi^{-1}(U) \subset \mathfrak{X}$ where U is a small neighborhood of s in $\mathcal{M}_g$ such that

1) $\pi_*(v_i) = \frac{\partial}{\partial s_i}$ for $i = 1, \cdots, n$;
2) $\overline{\partial} v_i$ are harmonic TX_s-valued $(0,1)$ forms for $i = 1, \cdots, n$. Here, $\overline{\partial}$ is the operator on the fiber X_s.

The vector fields $v_1, \cdots, v_n$ are called the harmonic lift of the vectors $\frac{\partial}{\partial s_1}, \cdots, \frac{\partial}{\partial s_n}$. The existence of such harmonic vector fields was pointed out by Siu [**14**]. In his work [**13**], Schumacher gave an explicit construction of such lift which we now describe.

Since $g \geq 2$, we can assume that each fiber is equipped with the Kähler–Einstein, or the Poincaré metric, $\lambda = \frac{\sqrt{-1}}{2}\lambda(z,s)dz \wedge d\overline{z}$. The

Kähler–Einstein condition gives the following equation:

$$\partial_z \partial_{\overline{z}} \log \lambda = \lambda. \tag{2.1}$$

For the rest of this paper, we denote $\frac{\partial}{\partial s_i}$ by ∂_i and $\frac{\partial}{\partial z}$ by ∂_z. Let

$$a_i = -\lambda^{-1} \partial_i \partial_{\overline{z}} \log \lambda \tag{2.2}$$

and let

$$A_i = \partial_{\overline{z}} a_i. \tag{2.3}$$

Then, we have the following

Lemma 2.1. *The harmonic horizontal lift of ∂_i is*

$$v_i = \partial_i + a_i \partial_z.$$

In particular

$$B_i = A_i \partial_z \otimes d\overline{z} \in H^1(X_s, T_{X_s})$$

is harmonic. Furthermore, the lift $\partial_i \mapsto B_i$ gives the Kodaira–Spencer map $T_s\mathcal{M}_g \to H^1(X_s, T_{X_s})$.

Remark 2.1. In the above lemma, the space $H^1(X_s, T_{X_s})$ is the space of harmonic forms with value in the holomorphic tangent sheaf of X_s. We used the Dolbeault isomorphism implicitly.

Now, we define the well-known Weil–Petersson metric:

Definition 2.1. *The Weil–Petersson metric on $\mathcal{M}_g$ is defined to be*

$$h_{i\overline{j}}(s) = \int_{X_s} B_i \cdot \overline{B_j}\ dv = \int_{X_s} A_i \overline{A_j}\ dv, \tag{2.4}$$

where $dv = \frac{\sqrt{-1}}{2} \lambda dz \wedge d\overline{z}$ is the volume form on the fiber X_s.

It is known that the curvature tensor of the Weil–Petersson metric can be represented by

$$R_{i\overline{j}k\overline{l}} = \int_{X_s} \big\{ (B_i \cdot \overline{B}_j)(\Box + 1)^{-1} (B_k \cdot \overline{B}_l) + (B_i \cdot \overline{B}_l)(\Box + 1)^{-1} (B_k \cdot \overline{B}_j) \big\}\, dv,$$

where $\Box$ is the complex Laplacian defined by

$$\Box = -\lambda^{-1} \frac{\partial^2}{\partial z \partial \overline{z}}.$$

By the expression of the curvature operator, we know that the curvature operator is non-positive. Furthermore, the Ricci curvature of the metric is negative.

However, the Weil–Petersson metric is incomplete. In [15], Trapani proved the negative Ricci curvature of the Weil–Petersson metric is a complete Kähler metric on the moduli space. We call this metric the Ricci metric. It is interesting to understand the curvature of the Ricci metric, at least asymptotically. To estimate it, we first derive an integral formula of its curvature.

3. Ricci metric and its curvature

In this section, we establish an integral formula (3.28) of the curvature of the Ricci metric. The importance of this formula is that the functions being integrated only involve derivatives in the fiber direction which we are able to control. Thus, we can use this formula to estimate the asymptotic of the curvature of the Ricci metric in next section.

The main tool we use is the harmonic lift of Siu and Schumacher described in the previous section. These lifts together with formula (3.2) enable us to transfer derivatives in the moduli direction into derivatives in the fiber direction.

We use the same notations as in the previous section. We first introduce several operators which will be used for the computations and simplifications of the curvatures of the Ricci metric.

Define an $(1,1)$ form on the total space $\mathfrak{X}$ by

$$g=\frac{\sqrt{-1}}{2}\partial\overline{\partial}\log\lambda=\frac{\sqrt{-1}}{2}(g_{i\overline{j}}ds_i\wedge d\overline{s}_j-\lambda a_i ds_i\wedge d\overline{z}-\lambda\overline{a}_i dz\wedge d\overline{s}_i+\lambda dz\wedge d\overline{z}).$$

The form g is not necessarily positive. Introduce

$$e_{i\overline{j}}=\frac{2}{\sqrt{-1}}g(v_i,\overline{v}_j)=g_{i\overline{j}}-\lambda a_i\overline{a_j}$$

be a global function. Let us write $f_{i\overline{j}}=A_i\overline{A_j}$. Schumacher proved the following result:

Lemma 3.1. *By using the same notations as above, we have*

$$(\Box+1)e_{i\overline{j}}=f_{i\overline{j}}. \tag{3.1}$$

Since $e_{i\overline{j}}$ and $f_{i\overline{j}}$ are the building blocks of the Ricci metric, it is interesting to study its property under the action of the vector fields v_i's.

Lemma 3.2. *With the same notations as above, we have*

$$v_k(e_{i\overline{j}})=v_i(e_{k\overline{j}}).$$

Proof. Since $dg=0$, we have the following

$$\begin{aligned}0=dg(v_i,v_k,\overline{v}_j)&=v_i(e_{k\overline{j}})-v_k(e_{i\overline{j}})+\overline{v}_j g(v_i,v_k)\\&-g(v_i,[v_k,\overline{v}_j])+g(v_k,[v_i,\overline{v}_j])-g(\overline{v}_j,[v_i,v_k]).\end{aligned}$$

The Lie bracket of v_j with $\overline{v}_j$ or v_k are vector fields tangential to X_s, which are perpendicular to the horizontal vector fields v_i with respect to the form g. Thus, the last three terms of the above equations are zero. On the other hand, $g(v_i, v_k) = 0$. The lemma thus follows from the above equation. q.e.d.

We also need to define the following operator

$$P: \ C^\infty(X_s) \to \Gamma(\Lambda^{1,0}(T^{0,1}X_s)), \ f \mapsto \partial_z(\lambda^{-1}\partial_z f).$$

The dual operator P^* can be written as follows

$$P^*: \ \Gamma(\Lambda^{0,1}(T^{1,0}X_s)) \to C^\infty(X_s), \ B \mapsto \lambda^{-1}\partial_z(\lambda^{-1}\partial_z(\lambda B)).$$

The operator P is actually a composition of the Maass operators. We recall the definitions from [**18**]. Let X be a Riemann surface and let κ be its canonical bundle. For any integer p, let $S(p)$ be the space of smooth sections of $(\kappa \otimes \overline{\kappa}^{-1})^{\frac{p}{2}}$. Fix a conformal metric $ds^2 = \rho^2(z)|dz|^2$.

Definition 3.1. *The Maass operators K_p and L_p are defined to be the metric derivatives $K_p: \ S(p) \to S(p+1)$ and $L_p: \ S(p) \to S(p-1)$ given by*

$$K_p(\sigma) = \rho^{p-1}\partial_z(\rho^{-p}\sigma)$$

and

$$L_p(\sigma) = \rho^{-p-1}\partial_{\overline{z}}(\rho^p\sigma)$$

where $\sigma \in S(p)$.

Clearly, we have $P = K_1 K_0$. Also, each element $\sigma \in S(p)$ has a well-defined absolute value $|\sigma|$ which is independent of the choice of the local coordinate. We define the C^k norm of σ as in [**18**]:

Definition 3.2. *Let Q be an operator which is a composition of operators K_* and L_*. Denote by $|Q|$ the number of such factors. For any $\sigma \in S(p)$, define*

$$\|\sigma\|_0 = \sup_X |\sigma|$$

and

$$\|\sigma\|_k = \sum_{|Q| \leq k} \|Q\sigma\|_0.$$

We can also localize the norm on a subset of X. Let $\Omega \subset X$ be a domain. We can define

$$\|\sigma\|_{0,\Omega} = \sup_\Omega |\sigma|$$

and

$$\|\sigma\|_{k,\Omega} = \sum_{|Q| \leq k} \|Q\sigma\|_{0,\Omega}.$$

Both of the above definitions depend on the choice of conformal metric on X. In the following, we always use the Kähler–Einstein metric on the surface unless otherwise stated.

Since the Weil–Petersson metric is defined by using the integral along the fibers, the following formula is very useful:

$$\partial_i \int_{X_s} \eta = \int_{X_s} L_{v_i} \eta \tag{3.2}$$

where η is a relative $(1,1)$ form on $\mathfrak{X}$.

The Lie derivative defined here is slightly different from the ordinary definition. Let φ_t be the one parameter group generated by the vector field v_i. Then, φ_t can be viewed as a diffeomorphism between two fibers $X_s \to X_{s'}$. Then, we define

$$L_{v_i}\eta = \lim_{t\to 0} \frac{1}{t}(\varphi_t^*(\sigma) - \sigma)$$

for any one form σ. On the other hand, let ξ be a vector field on the fiber X_s. Then, we define

$$L_{v_i}\xi = \lim_{t\to 0} \frac{1}{t}((\varphi_{-t})_* \xi - \xi).$$

We have the following

Proposition 3.1. *By using the above notations, we have*

$$L_{v_i}\sigma = i(v_i)d_1\sigma + d_1 i(v_i)\sigma,$$

where d_1 is the differential operator along the fiber, and

$$L_{v_i}\xi = [v_i, \xi].$$

In the following, we denote L_{v_i} by L_i.

Lemma 3.3. *By using the above notations, we have*

1) $L_i dv = 0$;
2) $L_{\bar{l}}(B_i) = -\overline{P}(e_{i\bar{l}}) - f_{i\bar{l}}\partial_{\bar{z}} \otimes d\bar{z} + f_{i\bar{l}}\partial_z \otimes dz$;
3) $L_k(\overline{B_j}) = -P(e_{k\bar{j}}) - f_{k\bar{j}}\partial_z \otimes dz + f_{k\bar{j}}\partial_{\bar{z}} \otimes d\bar{z}$;
4) $L_k(B_i) = (v_k(A_i) - A_i\partial_z a_k)\partial_z \otimes d\bar{z}$;
5) $L_{\bar{l}}(\overline{B_j}) = (\overline{v_l}(\overline{A_l}) - \overline{A_l}\partial_{\bar{z}}\overline{a_l})\partial_{\bar{z}} \otimes dz$.

Proof. The first formula was proved by Schumacher in [**13**]. To check the other formulae, we note that the third and fifth formulae follow from the second and fourth, which we will prove, by taking conjugation. We first have

$$\begin{aligned}\partial_z a_k &= \partial_z(-\lambda^{-1}\partial_k\partial_{\bar{z}}\log\lambda) = \lambda^{-2}\partial_z\lambda\partial_k\partial_{\bar{z}}\log\lambda - \lambda^{-1}\partial_z\partial_k\partial_{\bar{z}}\log\lambda \\ &= -\lambda^{-1}\partial_z\lambda a_k - \lambda^{-1}\partial_k\partial_z\partial_{\bar{z}}\log\lambda = -\lambda^{-1}\partial_z\lambda a_k - \lambda^{-1}\partial_k\lambda.\end{aligned}$$

We also have

$$\begin{aligned}\partial_{\bar{l}} a_i &= \partial_{\bar{l}}(-\lambda^{-1}\partial_i\partial_{\bar{z}}\log\lambda)) = \lambda^{-2}\partial_{\bar{l}}\lambda\partial_i\partial_{\bar{z}}\log\lambda - \lambda^{-1}\partial_{\bar{z}}\partial_i\partial_{\bar{l}}\log\lambda\\ &= -\lambda^{-1}\partial_{\bar{l}}\lambda a_i - \lambda^{-1}\partial_{\bar{z}} g_{i\bar{l}} = -\lambda^{-1}\partial_{\bar{l}}\lambda a_i - \lambda^{-1}\partial_{\bar{z}}(e_{i\bar{l}} + \lambda a_i\overline{a_l})\\ &= -\lambda^{-1}\partial_{\bar{l}}\lambda a_i - \lambda^{-1}\partial_{\bar{z}} e_{i\bar{l}} - \lambda^{-1}\partial_{\bar{z}}\lambda a_i\overline{a_l} - A_i\overline{a_l} - a_i\partial_{\bar{z}}\overline{a_l}\\ &= -(\lambda^{-1}\partial_{\bar{l}}\lambda + \lambda^{-1}\partial_{\bar{z}}\lambda\overline{a_l} + \partial_{\bar{z}}\overline{a_l})a_i - \lambda^{-1}\partial_{\bar{z}} e_{i\bar{l}} - A_i\overline{a_l}\\ &= -\lambda^{-1}\partial_{\bar{z}} e_{i\bar{l}} - A_i\overline{a_l}.\end{aligned}$$

For the second formula, we have

$$\begin{aligned}L_{\bar{l}}(B_i) &= \overline{v_l}(A_i)\partial_z\otimes d\bar{z} + A_i(-\partial_z\overline{a_l}\partial_{\bar{z}})\otimes d\bar{z} + A_i\partial_z\otimes(\partial_z\overline{a_l}dz + \partial_{\bar{z}}\overline{a_l}d\bar{z})\\ &= (\overline{v_l}(A_i) + A_i\partial_{\bar{z}}\overline{a_l})\partial_z\otimes d\bar{z} - f_{i\bar{l}}\partial_{\bar{z}}\otimes d\bar{z} + f_{i\bar{l}}\partial_z\otimes dz.\end{aligned}$$

So, we only need to check that $\overline{v_l}(A_i) + A_i\partial_{\bar{z}}\overline{a_l} = -\partial_{\bar{z}}(\lambda^{-1}\partial_{\bar{z}} e_{i\bar{l}})$. To prove this, we have

$$\begin{aligned}\overline{v_l}(A_i) + A_i\partial_{\bar{z}}\overline{a_l} &= \overline{a_l}\partial_{\bar{z}} A_i + \partial_{\bar{l}} A_i + A_i\partial_{\bar{z}}\overline{a_l} = \partial_{\bar{z}}(A_i\overline{a_l}) + \partial_{\bar{z}}\partial_{\bar{l}} a_i\\ &= \partial_{\bar{z}}(A_i\overline{a_l}) - \partial_{\bar{z}}(\lambda^{-1}\partial_{\bar{z}} e_{i\bar{l}}) - \partial_{\bar{z}}(A_i\overline{a_l}) = -\partial_{\bar{z}}(\lambda^{-1}\partial_{\bar{z}} e_{i\bar{l}}).\end{aligned}$$

This proves the second formula. For the fourth one, we have

$$\begin{aligned}L_k(B_i) &= v_k(A_i)\partial_z\otimes d\bar{z} + A_i(-\partial_z a_k\partial_z)\otimes d\bar{z}\\ &= (v_k(A_i) - A_i\partial_z a_k)\partial_z\otimes d\bar{z}.\end{aligned}$$

This finishes the proof. q.e.d.

An interesting and useful fact is that the Lie derivative of B_i in the direction of v_k is still harmonic. This result is true only for the moduli space of Riemann surfaces. In the general case of moduli space of Kähler–Einstein manifolds, we only have $\overline{\partial}^* L_k B_i = 0$.

Lemma 3.4. *$L_k(B_i) \in H^1(X_s, TX_s)$ is harmonic.*

Proof. From Lemma 3.3, we know that $L_k(B_i) = (v_k(A_i) - A_i\partial_z a_k)\partial_z\otimes d\bar{z} \in A^{0,1}(X_s, T_{X_s})$. So, it is clear that $\overline{\partial}(L_k(B_i)) = 0$ for the dimensional consideration. The fact that $\overline{\partial}^* L_k B_i = 0$ was proved in [**14**]. q.e.d.

The above lemma is very helpful in computing the curvature when we use normal coordinates of the Weil–Petersson metric. We have

Corollary 3.1. *Let $s_1, \cdots, s_n$ be normal coordinates at $s \in \mathcal{M}_g$ with respect to the Weil–Petersson metric. Then, at s, we have, for all i, k,*

$$L_k B_i = 0.$$

Proof. From Lemma 3.4, we know that L_kB_i is harmonic. Since $B_1, \cdots, B_n$ is a basis of $T_s\mathcal{M}_g$, we have

$$L_kB_i = h^{p\overline{q}}(\int_{X_s} L_kB_i \cdot \overline{B_q}\ dv)B_p = h^{p\overline{q}}\partial_k h_{i\overline{q}} B_p = 0.$$

q.e.d.

The commutator of v_k and $\overline{v_l}$ will be used later. We give a formula here which is essentially due to Schumacher.

Lemma 3.5. $[\overline{v_l}, v_k] = -\lambda^{-1}\partial_{\overline{z}} e_{k\overline{l}}\partial_z + \lambda^{-1}\partial_z e_{k\overline{l}}\partial_{\overline{z}}$.

Proof. From a direct computation, we have

$$[\overline{v_l}, v_k] = \overline{v_l}(a_k)\partial_z - v_k(\overline{a_l})\partial_{\overline{z}}.$$

By using the proof of Lemma 3.3, we have

$$\overline{v_l}(a_k) = \overline{a_l}\partial_{\overline{z}} a_k + \partial_{\overline{l}} a_k = -\lambda^{-1}\partial_{\overline{z}} e_{k\overline{l}}$$

and

$$v_k(\overline{a_l}) = a_k\partial_z\overline{a_l} + \partial_k\overline{a_l} = -\lambda^{-1}\partial_z e_{k\overline{l}}.$$

These finish the proof. q.e.d.

Remark 3.1. In the rest of this paper, we will use the following notation for curvature:
Let (M, g) be a Kähler manifold. Then, the curvature tensor is given by

$$R_{i\overline{j}k\overline{l}} = \frac{\partial^2 g_{i\overline{j}}}{\partial z_k \partial \overline{z}_l} - g^{p\overline{q}}\frac{\partial g_{i\overline{q}}}{\partial z_k}\frac{\partial g_{p\overline{j}}}{\partial \overline{z}_l}. \tag{3.3}$$

By using this convention, the Ricci curvature is given by

$$R_{i\overline{j}} = -g^{k\overline{l}} R_{i\overline{j}k\overline{l}},$$

and the holomorphic sectional curvature of g is negative means

$$R(v, \overline{v}, v, \overline{v}) > 0$$

for any holomorphic tangent vector v at any point.

In [**14**] and [**13**], Siu and Schumacher proved the following curvature formula for the Weil–Petersson metric. This formula was also proved by Wolpert in [**16**]. We give a short proof here since we need to use the techniques.

Theorem 3.1. *The curvature of Weil–Petersson metric is given by*

$$R_{i\overline{j}k\overline{l}} = \int_{X_s} (e_{i\overline{j}} f_{k\overline{l}} + e_{i\overline{l}} f_{k\overline{j}})\ dv. \tag{3.4}$$

Proof. We have

(3.5)
$$\begin{aligned}
R_{i\bar{j}k\bar{l}} =&\partial_{\bar{l}}\partial_k h_{i\bar{j}} - h^{p\bar{q}}\partial_k h_{i\bar{q}}\partial_{\bar{l}} h_{p\bar{j}} \\
=&\partial_{\bar{l}}\int_{X_s} L_k B_i \cdot \overline{B_j}\ dv - h^{p\bar{q}}\int_{X_s} L_k B_i \cdot \overline{B_q}\ dv \int_{X_s} B_p \cdot L_{\bar{l}}\overline{B_j}\ dv \\
=&\int_{X_s} (L_{\bar{l}} L_k B_i \cdot \overline{B_j} + L_k B_i \cdot L_{\bar{l}}\overline{B_j})\ dv \\
&\quad - h^{p\bar{q}}\int_{X_s} L_k B_i \cdot \overline{B_q}\ dv \int_{X_s} B_p \cdot L_{\bar{l}}\overline{B_j}\ dv.
\end{aligned}$$

Since $B_1, \cdots, B_n$ is a basis of $T_s M_g$, we have

$$h^{p\bar{q}}\int_{X_s} L_k B_i \cdot \overline{B_q}\ dv \int_{X_s} B_p \cdot L_{\bar{l}}\overline{B_j}\ dv = \int_{X_s} L_k B_i \cdot L_{\bar{l}}\overline{B_j}\ dv.$$

By combining this formula with (3.5), we have

(3.6)
$$\begin{aligned}
R_{i\bar{j}k\bar{l}} =& \int_{X_s} L_{\bar{l}} L_k B_i \cdot \overline{B_j}\ dv = \int_{X_s} L_k L_{\bar{l}} B_i \cdot \overline{B_j}\ dv + \int_{X_s} L_{[\overline{v_l}, v_k]} B_i \cdot \overline{B_j}\ dv \\
=&\partial_k \int_{X_s} L_{\bar{l}} B_i \cdot \overline{B_j}\ dv - \int_{X_s} L_{\bar{l}} B_i \cdot L_k\overline{B_j}\ dv + \int_{X_s} L_{[\overline{v_l}, v_k]} B_i \cdot \overline{B_j}\ dv \\
=& - \int_{X_s} L_{\bar{l}} B_i \cdot L_k\overline{B_j}\ dv + \int_{X_s} L_{[\overline{v_l}, v_k]} B_i \cdot \overline{B_j}\ dv
\end{aligned}$$

since $\int_{X_s} L_{\bar{l}} B_i \cdot \overline{B_j}\ dv = 0$. Now, we compute $\int_{X_s} L_{[\overline{v_l}, v_k]} B_i \cdot \overline{B_j}\ dv$. Let $\pi_1^1(L_{[\overline{v_l}, v_k]} B_i)$ be the projection of $L_{[\overline{v_l}, v_k]} B_i$ onto $H^{0,1}(X_s, T_{X_s})$ which gives the $\partial_z \otimes d\bar{z}$ part of $L_{[\overline{v_l}, v_k]} B_i$. Since B_i is harmonic, we know $\partial_z(\lambda A_i) = 0$ which implies $\partial_z A_i = -\lambda^{-1}\partial_z \lambda A_i$. By Lemma 3.5, we have

$$\begin{aligned}
\pi_1^1(L_{[\overline{v_l}, v_k]} B_i) =&(-\lambda^{-1}\partial_{\bar{z}} e_{k\bar{l}}\partial_z A_i + A_i\partial_z(\lambda^{-1}\partial_{\bar{z}} e_{k\bar{l}}) \\
&+ \partial_{\bar{z}}(\lambda^{-1} A_i \partial_z e_{k\bar{l}}))\partial_z \otimes d\bar{z} \\
=&(\lambda^{-2}\partial_z \lambda A_i \partial_{\bar{z}} e_{k\bar{l}} - \lambda^{-2}\partial_z \lambda A_i \partial_{\bar{z}} e_{k\bar{l}} - A_i \Box e_{k\bar{l}} \\
&+ \partial_{\bar{z}}(\lambda^{-1} A_i \partial_z e_{k\bar{l}}))\partial_z \otimes d\bar{z} \\
=&(-A_i \Box e_{k\bar{l}} + \partial_{\bar{z}}(\lambda^{-1} A_i \partial_z e_{k\bar{l}}))\partial_z \otimes d\bar{z}.
\end{aligned} \tag{3.7}$$

This implies

(3.8)
$$\begin{aligned}\int_{X_s} L_{[\overline{v_l},v_k]}B_i\cdot\overline{B_j}\,dv &= \int_{X_s}\pi_1^1(L_{[\overline{v_l},v_k]}B_i)\cdot\overline{B_j}\,dv\\ &= \int_{X_s}(-A_i\Box e_{k\bar l}+\partial_{\bar z}(\lambda^{-1}A_i\partial_z e_{k\bar l}))\overline{A_j}\,dv\\ &= -\int_{X_s} f_{i\bar j}\Box e_{k\bar l}\,dv+\int_{X_s}\partial_{\bar z}(\lambda^{-1}A_i\partial_z e_{k\bar l})\overline{A_j}\,dv\\ &= -\int_{X_s} f_{i\bar j}\Box e_{k\bar l}\,dv-\int_{X_s}\lambda^{-2}A_i\partial_z e_{k\bar l}\partial_{\bar z}(\lambda\overline{A_j})\,dv\\ &= -\int_{X_s} f_{i\bar j}\Box e_{k\bar l}\,dv.\end{aligned}$$

To compute $\int_{X_s} L_{\bar l}B_i\cdot L_k\overline{B_j}\,dv$, by using Lemma 3.3, we obtain

(3.9)
$$\begin{aligned}\int_{X_s} L_{\bar l}B_i\cdot L_k\overline{B_j}\,dv &= \int_{X_s}(\partial_{\bar z}(\lambda^{-1}\partial_{\bar z}e_{i\bar l})\partial_z(\lambda^{-1}\partial_z e_{k\bar j})-2f_{k\bar j}f_{i\bar l})\,dv\\ &= \int_{X_s}(\lambda^{-2}\partial_z e_{k\bar j}\partial_z(\lambda\partial_{\bar z}(\lambda^{-1}\partial_{\bar z}e_{i\bar l}))\,dv-2\int_{X_s}f_{k\bar j}f_{i\bar l}\,dv\\ &= -\int_{X_s}(\lambda^{-2}\partial_z\lambda\partial_z e_{k\bar j}\partial_{\bar z}(\lambda^{-1}\partial_{\bar z}e_{i\bar l})\\ &\qquad +\lambda^{-1}\partial_z e_{k\bar j}\partial_z\partial_{\bar z}(\lambda^{-1}\partial_{\bar z}e_{i\bar l}))\,dv-2\int_{X_s}f_{k\bar j}f_{i\bar l}\,dv\\ &= \int_{X_s}(\lambda^{-2}\partial_{\bar z}e_{i\bar l}\partial_{\bar z}(\lambda^{-1}\partial_z\lambda\partial_z e_{k\bar j})\\ &\qquad +\lambda^{-1}\partial_z\partial_{\bar z}e_{k\bar j}\partial_z(\lambda^{-1}\partial_{\bar z}e_{i\bar l}))\,dv-2\int_{X_s}f_{k\bar j}f_{i\bar l}\,dv\\ &= \int_{X_s}(\lambda^{-2}\partial_{\bar z}e_{i\bar l}(\lambda\partial_z e_{k\bar j}-\partial_z\lambda\Box e_{k\bar j})\\ &\qquad -\Box e_{k\bar j}(-\lambda^{-2}\partial_z\lambda\partial_{\bar z}e_{i\bar l}-\Box e_{i\bar l}))\,dv-2\int_{X_s}f_{k\bar j}f_{i\bar l}\,dv\\ &= \int_{X_s}(\lambda^{-1}\partial_{\bar z}e_{i\bar l}\partial_z e_{k\bar j})+\Box e_{k\bar j}\Box e_{i\bar l})\,dv-2\int_{X_s}f_{k\bar j}f_{i\bar l}\,dv\\ &= \int_{X_s}(\Box e_{k\bar j}e_{i\bar l}+\Box e_{k\bar j}\Box e_{i\bar l})\,dv-2\int_{X_s}f_{k\bar j}f_{i\bar l}\,dv\\ &= \int_{X_s}(\Box e_{k\bar j}f_{i\bar l}-2f_{k\bar j}f_{i\bar l})\,dv=-\int_{X_s}(f_{k\bar j}f_{i\bar l}+e_{k\bar j}f_{i\bar l})\,dv.\end{aligned}$$

By combining (3.6), (3.8) and (3.9) with the identity $f_{k\bar{j}}f_{i\bar{l}} = A_i\overline{A_j}A_k\overline{A_l} = f_{i\bar{j}}f_{k\bar{l}}$, we have

(3.10)
$$R_{i\bar{j}k\bar{l}} = \int_{X_s} (f_{k\bar{j}}f_{i\bar{l}} + e_{k\bar{j}}f_{i\bar{l}} - f_{i\bar{j}}\Box e_{k\bar{l}})\, dv = \int_{X_s} (f_{i\bar{j}}e_{k\bar{l}} + f_{i\bar{l}}e_{k\bar{j}})\, dv$$
$$= \int_{X_s} (e_{i\bar{j}}f_{k\bar{l}} + e_{i\bar{l}}f_{k\bar{j}})\, dv.$$

Here, we have used the fact the $(\Box + 1)$ is a self-adjoint operator. This finishes the proof. q.e.d.

It is well-known that the Ricci curvature of the Weil–Petersson metric is negative which implies that the negative Ricci curvature of the Weil–Petersson metric defines a Kähler metric on the moduli space $\mathcal{M}_g$.

Definition 3.3. *The Ricci metric $\tau_{i\bar{j}}$ on the moduli space $\mathcal{M}_g$ is the negative Ricci curvature of the Weil–Petersson metric. That is*

$$\tau_{i\bar{j}} = -R_{i\bar{j}} = h^{\alpha\overline{\beta}}R_{i\bar{j}\alpha\overline{\beta}}. \tag{3.11}$$

Now, we define a new operator which acts on functions over the fibers.

Definition 3.4. *For each $1 \leq k \leq n$ and for any smooth function f on the fibers, we define the commutator operator ξ_k which acts on a function f by*

$$\xi_k(f) = \overline{\partial}^*(i(B_k)\partial f) = -\lambda^{-1}\partial_z(A_k\partial_z f). \tag{3.12}$$

The reason we call ξ_k the commutator operator is that ξ_k is the commutator of $(\Box + 1)$ and v_k and the following lemma.

Lemma 3.6. *As operators acting on functions, we have*
(1) $(\Box + 1)v_k - v_k(\Box + 1) = \Box v_k - v_k\Box = \xi_k$;
(2) $(\Box + 1)\overline{v_l} - \overline{v_l}(\Box + 1) = \Box\overline{v_l} - \overline{v_l}\Box = \overline{\xi_l}$;
(3) $\xi_k(f) = -A_k\partial_z(\lambda^{-1}\partial_z f) = -A_kP(f) = -A_kK_1K_0(f)$.
Furthermore, we have

$$(\Box + 1)v_k(e_{i\bar{j}}) = \xi_k(e_{i\bar{j}}) + \xi_i(e_{k\bar{j}}) + L_kB_i \cdot \overline{B_j}. \tag{3.13}$$

Proof. To prove (1), we have

$$\begin{aligned}
&(\square+1)v_k - v_k(\square+1)\\
&\quad= \square v_k + v_k - v_k\square - v_k = \square v_k - v_k\square\\
&\quad= -\lambda^{-1}\partial_z\partial_{\overline{z}}(a_k\partial_z+\partial_k) - (a_k\partial_z+\partial_k)(-\lambda^{-1}\partial_z\partial_{\overline{z}})\\
&\quad= -\lambda^{-1}\partial_z(A_k\partial_z + a_k\partial_z\partial_{\overline{z}} + \partial_k\partial_{\overline{z}})\\
&\qquad + a_k\partial_z(\lambda^{-1})\partial_z\partial_{\overline{z}} + \lambda^{-1}a_k\partial_z\partial_z\partial_{\overline{z}} + \partial_k(\lambda^{-1})\partial_z\partial_{\overline{z}} + \lambda^{-1}\partial_k\partial_z\partial_{\overline{z}}\\
&\quad= -\lambda^{-1}\partial_z(A_k\partial_z) - \lambda^{-1}\partial_z a_k\partial_z\partial_{\overline{z}} - \lambda^{-1}a_k\partial_z\partial_z\partial_{\overline{z}} - \lambda^{-1}\partial_k\partial_z\partial_{\overline{z}}\\
&\qquad - \lambda^{-2}\partial_z\lambda a_k\partial_z\partial_{\overline{z}} + \lambda^{-1}a_k\partial_z\partial_z\partial_{\overline{z}} - \lambda^{-2}\partial_k\lambda\partial_z\partial_{\overline{z}} + \lambda^{-1}\partial_k\partial_z\partial_{\overline{z}}\\
&\quad= \xi_k - \lambda^{-1}(\partial_z a_k + \lambda^{-1}\partial_z\lambda a_k + \lambda^{-1}\partial_k\lambda)\partial_z\partial_{\overline{z}} = \xi_k
\end{aligned}$$

where we have used Lemma 3.3 in the last equality of the above formula. By taking conjugation, we can prove (2) by using (1). To prove (3), we use the harmonicity of B_k. Since $\overline{\partial}^* B_k = 0$, we have $\partial_z(\lambda A_k) = 0$. So

$$\begin{aligned}
\xi_k(f) &= -\lambda^{-1}\partial_z(A_k\partial_z f) = -\lambda^{-1}\partial_z(\lambda A_k\lambda^{-1}\partial_z f)\\
&= -\lambda^{-1}\lambda A_k\partial_z(\lambda^{-1}\partial_z f) = -A_k\partial_z(\lambda^{-1}\partial_z f).
\end{aligned}$$

To prove the last part, by using part 1 of this lemma, we have

$$\begin{aligned}
(\square+1)v_k(e_{i\overline{j}}) =& v_k((\square+1)(e_{i\overline{j}})) + \xi_k(e_{i\overline{j}}) = v_k(f_{i\overline{j}}) + \xi_k(e_{i\overline{j}})\\
=& L_kB_i\cdot\overline{B_j} + B_i\cdot L_k\overline{B_j} + \xi_k(e_{i\overline{j}})\\
=& L_kB_i\cdot\overline{B_j} - A_i\partial_z(\lambda^{-1}\partial_z e_{k\overline{j}}) + \xi_k(e_{i\overline{j}})\\
=& L_kB_i\cdot\overline{B_j} + \xi_i(e_{k\overline{j}}) + \xi_k(e_{i\overline{j}}).
\end{aligned}$$

This finishes the proof. q.e.d.

Remark 3.2. From Corollary 3.1 and the above lemma, when we use the normal coordinates on the moduli space with respect to the Weil–Petersson metric, we have the clean formula $(\square+1)v_k(e_{i\overline{j}}) = \xi_i(e_{k\overline{j}}) + \xi_k(e_{i\overline{j}})$.

The main result in this section is the curvature formula of the Ricci metric. The terms produced here are very symmetric with respect to indices. For convenience, we introduce the symmetrization operator.

Definition 3.5. *Let U be any quantity which depends on indices $i, k, \alpha, \overline{j}, \overline{l}, \overline{\beta}$. The symmetrization operator σ_1 is defined by taking the* *summation of all orders of the triple* (i,k,α)*. That is*

$$\begin{aligned}\sigma_1(U(i,k,\alpha,\overline{j},\overline{l},\overline{\beta})) = {} & U(i,k,\alpha,\overline{j},\overline{l},\overline{\beta}) + U(i,\alpha,k,\overline{j},\overline{l},\overline{\beta}) \\ & + U(k,i,\alpha,\overline{j},\overline{l},\overline{\beta}) + U(k,\alpha,i,\overline{j},\overline{l},\overline{\beta}) \\ & + U(\alpha,i,k,\overline{j},\overline{l},\overline{\beta}) + U(\alpha,k,i,\overline{j},\overline{l},\overline{\beta}).\end{aligned}$$

Similarly, σ_2 *is the symmetrization operator of* $\overline{j}$ *and* $\overline{\beta}$ *and* $\widetilde{\sigma_1}$ *is the symmetrization operator of* $\overline{j}$*,* $\overline{l}$ *and* $\overline{\beta}$*.*

Now, we are ready to compute the curvature of the Ricci metric. For the first order derivative, we have

Theorem 3.2.

$$\partial_k \tau_{i\overline{j}} = h^{\alpha\overline{\beta}} \left\{ \sigma_1 \int_{X_s} (\xi_k(e_{i\overline{j}}) e_{\alpha\overline{\beta}})\, dv \right\} + \tau_{p\overline{j}} \Gamma_{ik}^p \tag{3.14}$$

where Γ_{ik}^p *is the Christoffell symbol of the Weil–Petersson metric.*

Proof. From Lemma 3.1, we know that $(\Box + 1)e_{i\overline{j}} = f_{i\overline{j}}$. By using Lemma 3.6 and Theorem 3.1, we have

(3.15)

$$\begin{aligned}\partial_k R_{i\overline{j}\alpha\overline{\beta}} = {} & \partial_k \int_{X_s} (e_{i\overline{j}} f_{\alpha\overline{\beta}} + e_{i\overline{\beta}} f_{\alpha\overline{j}})\, dv \\ = {} & \int_{X_s} (v_k(e_{i\overline{j}}) f_{\alpha\overline{\beta}} + e_{i\overline{j}} v_k(f_{\alpha\overline{\beta}}) + v_k(e_{i\overline{\beta}}) f_{\alpha\overline{j}} + e_{i\overline{\beta}} v_k(f_{\alpha\overline{j}}))\, dv \\ = {} & \int_{X_s} ((\Box+1) v_k(e_{i\overline{j}}) e_{\alpha\overline{\beta}} + e_{i\overline{j}} v_k(f_{\alpha\overline{\beta}}) + (\Box+1) v_k(e_{i\overline{\beta}}) e_{\alpha\overline{j}} \\ & \quad + e_{i\overline{\beta}} v_k(f_{\alpha\overline{j}}))\, dv \\ = {} & \int_{X_s} (v_k(f_{i\overline{j}}) e_{\alpha\overline{\beta}} + e_{i\overline{j}} v_k(f_{\alpha\overline{\beta}}) + v_k(f_{i\overline{\beta}}) e_{\alpha\overline{j}} \\ & \quad + e_{i\overline{\beta}} v_k(f_{\alpha\overline{j}}))\, dv \\ & \quad + \int_{X_s} (\xi_k(e_{i\overline{j}}) e_{\alpha\overline{\beta}} + \xi_k(e_{i\overline{\beta}}) e_{\alpha\overline{j}})\, dv \\ = {} & \int_{X_s} ((L_k B_i \cdot \overline{B_j}) e_{\alpha\overline{\beta}} + (L_k B_\alpha \cdot \overline{B_\beta}) e_{i\overline{j}} \\ & + (L_k B_i \cdot \overline{B_\beta}) e_{\alpha\overline{j}} + (L_k B_\alpha \cdot \overline{B_j}) e_{i\overline{\beta}})\, dv \\ & + \int_{X_s} ((B_i \cdot L_k \overline{B_j}) e_{\alpha\overline{\beta}} + (B_\alpha \cdot L_k \overline{B_\beta}) e_{i\overline{j}} + (B_i \cdot L_k \overline{B_\beta}) e_{\alpha\overline{j}} \\ & + (B_\alpha \cdot L_k \overline{B_j}) e_{i\overline{\beta}})\, dv + \int_{X_s} (\xi_k(e_{i\overline{j}}) e_{\alpha\overline{\beta}} + \xi_k(e_{i\overline{\beta}}) e_{\alpha\overline{j}})\, dv.\end{aligned}$$

Now, we simplify the right-hand side of (3.15). Since $B_1, \cdots, B_n$ is a basis of $T_s\mathcal{M}_g$, we know that the first line of the right-hand side of (3.15) is

(3.16)
$$\begin{aligned}
\int_{X_s} &((L_kB_i \cdot \overline{B_j})e_{\alpha\overline{\beta}} + (L_kB_\alpha \cdot \overline{B_\beta})e_{i\overline{j}} + (L_kB_i \cdot \overline{B_\beta})e_{\alpha\overline{j}} \\
&\qquad + (L_kB_\alpha \cdot \overline{B_j})e_{i\overline{\beta}})\ dv \\
&= \int_{X_s} (L_kB_i \cdot (\overline{B_j}e_{\alpha\overline{\beta}} + \overline{B_\beta}e_{\alpha\overline{j}}) + L_kB_\alpha \cdot (\overline{B_j}e_{i\overline{\beta}} + \overline{B_\beta}e_{i\overline{j}}))\ dv \\
&= h^{p\overline{q}} \int_{X_s} (L_kB_i \cdot \overline{B_q})\ dv \int_{X_s} (B_p \cdot (\overline{B_j}e_{\alpha\overline{\beta}} + \overline{B_\beta}e_{\alpha\overline{j}})\ dv \\
&\quad + h^{p\overline{q}} \int_{X_s} (L_kB_\alpha \cdot \overline{B_q})\ dv \int_{X_s} (B_p \cdot (\overline{B_j}e_{i\overline{\beta}} + \overline{B_\beta}e_{i\overline{j}})\ dv \\
&= h^{p\overline{q}}\partial_k h_{i\overline{q}} R_{p\overline{j}\alpha\overline{\beta}} + h^{p\overline{q}}\partial_k h_{\alpha\overline{q}} R_{i\overline{j}p\overline{\beta}} = \Gamma^p_{ik} R_{p\overline{j}\alpha\overline{\beta}} + \Gamma^p_{\alpha k} R_{i\overline{j}p\overline{\beta}}.
\end{aligned}$$

We deal with the second line of the right-hand side of (3.15) by using Lemmas 3.3 and 3.6 to get

$$B_i \cdot L_k\overline{B_j} = -A_i\partial_z(\lambda^{-1}\partial_z e_{k\overline{j}}) = \xi_i(e_{k\overline{j}}). \tag{3.17}$$

This implies

$$\begin{aligned}
\int_{X_s} &((B_i \cdot L_k\overline{B_j})e_{\alpha\overline{\beta}} + (B_\alpha \cdot L_k\overline{B_\beta})e_{i\overline{j}} \\
&\qquad + (B_i \cdot L_k\overline{B_\beta})e_{\alpha\overline{j}} + (B_\alpha \cdot L_k\overline{B_j})e_{i\overline{\beta}})\ dv \\
&= \int_{X_s} (\xi_i(e_{k\overline{j}})e_{\alpha\overline{\beta}} + \xi_\alpha(e_{k\overline{\beta}})e_{i\overline{j}} + \xi_i(e_{k\overline{\beta}})e_{\alpha\overline{j}} + \xi_\alpha(e_{k\overline{j}})e_{i\overline{\beta}})\ dv.
\end{aligned} \tag{3.18}$$

We also have

(3.19)
$$\partial_k\tau_{i\overline{j}} = h^{\alpha\overline{\beta}}\partial_k R_{i\overline{j}\alpha\overline{\beta}} + \partial_k h^{\alpha\overline{\beta}} R_{i\overline{j}\alpha\overline{\beta}} = h^{\alpha\overline{\beta}}(\partial_k R_{i\overline{j}\alpha\overline{\beta}} - R_{i\overline{j}p\overline{\beta}}\Gamma^p_{k\alpha}).$$

By combining (3.15), (3.16), (3.18) and (3.19), together with the fact that ξ_i is a real symmetric operator and the definition of $\tau_{i\overline{j}}$, we have proved this theorem. q.e.d.

To compute the second order derivative, we need to compute the commutator of ξ_k and $\overline{v_l}$. We have

Lemma 3.7. *For any smooth function $f \in C^\infty(X_s)$,*

$$\overline{v_l}(\xi_k f) - \xi_k(\overline{v_l} f) = \overline{P}(e_{k\overline{l}})P(f) - 2f_{k\overline{l}}\Box f + \lambda^{-1}\partial_z f_{k\overline{l}}\partial_{\overline{z}} f. \tag{3.20}$$

Proof. We will fix local holomorphic coordinates and compute locally. First, we know that the commutator of $\overline{v_l}$ and ∂_z is

$$\overline{v_l}\partial_z - \partial_z\overline{v_l} = -\partial_z\overline{a_l}\partial_{\overline{z}} = -\overline{A_l}\partial_{\overline{z}}. \tag{3.21}$$

Similarly, the commutator of $\overline{v_l}$ and $\lambda^{-1}\partial_z$ is

$$\begin{aligned}\overline{v_l}(\lambda^{-1}\partial_z) - \lambda^{-1}\partial_z\overline{v_l} &= \overline{v_l}(\lambda^{-1})\partial_z + \lambda^{-1}(\overline{v_l}\partial_z - \partial_z\overline{v_l})\\ &= \lambda^{-1}\partial_{\overline{z}}\overline{a_l}\partial_z - \lambda^{-1}\overline{A_l}\partial_{\overline{z}}.\end{aligned} \tag{3.22}$$

The above two formulae imply

$$\begin{aligned}-(\overline{v_l}P - P\overline{v_l}) &= -\overline{v_l}(\partial_z(\lambda^{-1}\partial_z)) + \partial_z(\lambda^{-1}\partial_z)\overline{v_l}\\ &= (\overline{A_l}\partial_{\overline{z}} - \partial_z\overline{v_l})(\lambda^{-1}\partial_z) + \partial_z(\overline{v_l}(\lambda^{-1}\partial_z) - \lambda^{-1}\partial_{\overline{z}}\overline{a_l}\partial_z\\ &\quad + \lambda^{-1}\overline{A_l}\partial_{\overline{z}})\\ &= \overline{A_l}\partial_{\overline{z}}(\lambda^{-1}\partial_z) - \partial_z(\lambda^{-1}\partial_{\overline{z}}\overline{a_l}\partial_z) + \partial_z(\lambda^{-1}\overline{A_l}\partial_{\overline{z}})\\ &= -\lambda^{-2}\partial_{\overline{z}}\lambda\overline{A_l}\partial_z + \lambda^{-1}\overline{A_l}\partial_z\partial_{\overline{z}} + \lambda^{-2}\partial_z\lambda\partial_{\overline{z}}\overline{a_l}\partial_z\\ &\quad - \lambda^{-1}\partial_{\overline{z}}\overline{A_l}\partial_z - \lambda^{-1}\partial_{\overline{z}}\overline{a_l}\partial_z\partial_z\\ &\quad - \lambda^{-2}\partial_z\lambda\overline{A_l}\partial_{\overline{z}} + \lambda^{-1}\partial_z\overline{A_l}\partial_{\overline{z}} + \lambda^{-1}\overline{A_l}\partial_z\partial_{\overline{z}}.\end{aligned} \tag{3.23}$$

By using the harmonicity, we have $\partial_{\overline{z}}(\lambda\overline{A_l}) = 0$ which implies $\partial_{\overline{z}}\overline{A_l} = -\lambda^{-1}\partial_{\overline{z}}\lambda\overline{A_l}$. By plugging this into formula (3.23), we have

$$\begin{aligned}-(\overline{v_l}P - P\overline{v_l}) =& -2\overline{A_l}\Box + \lambda^{-2}\partial_z\lambda\partial_{\overline{z}}\overline{a_l}\partial_z - \lambda^{-1}\partial_{\overline{z}}\overline{a_l}\partial_z\partial_z\\ & - \lambda^{-2}\partial_z\lambda\overline{A_l}\partial_{\overline{z}} + \lambda^{-1}\partial_z\overline{A_l}\partial_{\overline{z}}\\ =& -2\overline{A_l}\Box - \partial_{\overline{z}}\overline{a_l}P - \lambda^{-2}\partial_z\lambda\overline{A_l}\partial_{\overline{z}} + \lambda^{-1}\partial_z\overline{A_l}\partial_{\overline{z}}.\end{aligned} \tag{3.24}$$

Now, since $\xi_k = -A_kP$, we have

$$\begin{aligned}\overline{v_l}(\xi_k f) - \xi_k(\overline{v_l}f) =& -\overline{v_l}(A_k)P(f) - A_k(\overline{v_l}P(f) - P\overline{v_l}(f))\\ =& -(\overline{v_l}(A_k) + A_k\partial_{\overline{z}}\overline{a_l})P(f) - 2f_{k\bar{l}}\Box f\\ & - \lambda^{-2}\partial_z\lambda A_k\overline{A_l}\partial_{\overline{z}}f + \lambda^{-1}A_k\partial_z\overline{A_l}\partial_{\overline{z}}f.\end{aligned} \tag{3.25}$$

From the proof of lemma 3.3, we know $\overline{v_l}(A_k) + A_k\partial_{\overline{z}}\overline{a_l} = -\overline{P}(e_{k\bar{l}})$. By using the harmonicity, we have $-\lambda^{-1}\partial_z\lambda A_k = \partial_z A_k$. So, from (3.25), we have

$$\begin{aligned}\overline{v_l}(\xi_k f) - \xi_k(\overline{v_l}f) =& \overline{P}(e_{k\bar{l}})P(f) - 2f_{k\bar{l}}\Box f + \lambda^{-1}\partial_z A_k\overline{A_l}\partial_{\overline{z}}f\\ & + \lambda^{-1}A_k\partial_z\overline{A_l}\partial_{\overline{z}}f\\ =& \overline{P}(e_{k\bar{l}})P(f) - 2f_{k\bar{l}}\Box f + \lambda^{-1}\partial_z f_{k\bar{l}}\partial_{\overline{z}}f.\end{aligned} \tag{3.26}$$

This finishes the proof. q.e.d.

From the above lemma, it is convenient to define the commutator of ξ_k and $\overline{v_l}$ as an operator.

Definition 3.6. *For each k,l, we define the operator $Q_{k\bar{l}}$ which acts on a function to produce another function by*

$$Q_{k\bar{l}}(f) = \overline{P}(e_{k\bar{l}})P(f) - 2f_{k\bar{l}}\Box f + \lambda^{-1}\partial_z f_{k\bar{l}}\partial_{\bar{z}} f. \tag{3.27}$$

Now, we are ready to compute the curvature tensor of the Ricci metric. The formula consists of four types of terms.

Theorem 3.3. *Let $s_1, \cdots, s_n$ be local holomorphic coordinates at $s \in \mathcal{M}_g$. Then at s, we have*

$$\begin{aligned}\widetilde{R}_{i\bar{j}k\bar{l}} =& h^{\alpha\overline{\beta}}\left\{\sigma_1\sigma_2\int_{X_s}\Big\{(\Box+1)^{-1}(\xi_k(e_{i\bar{j}}))\overline{\xi}_l(e_{\alpha\overline{\beta}})\right.\\ &\left.\qquad + (\Box+1)^{-1}(\xi_k(e_{i\bar{j}}))\overline{\xi}_\beta(e_{\alpha\bar{l}})\Big\}\ dv\right\}\\ &+ h^{\alpha\overline{\beta}}\left\{\sigma_1\int_{X_s} Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\overline{\beta}}\ dv\right\}\\ &- \tau^{p\bar{q}}h^{\alpha\overline{\beta}}h^{\gamma\bar{\delta}}\left\{\sigma_1\int_{X_s}\xi_k(e_{i\bar{q}})e_{\alpha\overline{\beta}}\ dv\right\}\left\{\widetilde{\sigma}_1\int_{X_s}\overline{\xi}_l(e_{p\bar{j}})e_{\gamma\bar{\delta}})\ dv\right\}\\ &+ \tau_{p\bar{j}}h^{p\bar{q}}R_{i\bar{q}k\bar{l}}.\end{aligned} \tag{3.28}$$

Proof. By Lemma 3.4, we know that L_kB_i is harmonic. Since $B_1, \cdots, B_n$ is a basis of harmonic Beltrami differentials, from the proof of Theorem 3.1, we have

$$L_kB_i = \Gamma_{ik}^s B_s. \tag{3.29}$$

We first compute $\partial_{\bar{l}}\int_{X_s}\xi_k(e_{i\bar{j}})e_{\alpha\overline{\beta}}\ dv$. By Lemmas 3.6 and 3.7, we have

$$\begin{aligned}\partial_{\bar{l}}\int_{X_s}\xi_k(e_{i\bar{j}})e_{\alpha\overline{\beta}}\ dv &= \int_{X_s}(\overline{v}_l(\xi_k(e_{i\bar{j}}))e_{\alpha\overline{\beta}} + \xi_k(e_{i\bar{j}})\overline{v}_l(e_{\alpha\overline{\beta}}))\ dv\\ &= \int_{X_s}(\xi_k(\overline{v}_l(e_{i\bar{j}}))e_{\alpha\overline{\beta}} + \xi_k(e_{i\bar{j}})\overline{v}_l(e_{\alpha\overline{\beta}}) + Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\overline{\beta}})\ dv\\ &= \int_{X_s}(\xi_k(e_{\alpha\overline{\beta}})\overline{v}_l(e_{i\bar{j}}) + \xi_k(e_{i\bar{j}})\overline{v}_l(e_{\alpha\overline{\beta}}) + Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\overline{\beta}})\ dv\\ &= \int_{X_s}(\Box+1)^{-1}(\xi_k(e_{\alpha\overline{\beta}}))(\Box+1)(\overline{v}_l(e_{i\bar{j}}))\ dv\end{aligned} \tag{3.30}$$

$$
\begin{aligned}
&+\int_{X_s}(\Box+1)^{-1}(\xi_k(e_{i\overline{j}}))(\Box+1)(\overline{v}_l(e_{\alpha\overline{\beta}}))\ dv \\
&\qquad+\int_{X_s}Q_{k\overline{l}}(e_{i\overline{j}})e_{\alpha\overline{\beta}}\ dv \\
=&\int_{X_s}(\Box+1)^{-1}(\xi_k(e_{\alpha\overline{\beta}}))(\overline{\xi}_l(e_{i\overline{j}})+\overline{v}_l(f_{i\overline{j}}))\ dv \\
&+\int_{X_s}(\Box+1)^{-1}(\xi_k(e_{i\overline{j}}))(\overline{\xi}_l(e_{\alpha\overline{\beta}})+\overline{v}_l(f_{\alpha\overline{\beta}}))\ dv \\
&+\int_{X_s}Q_{k\overline{l}}(e_{i\overline{j}})e_{\alpha\overline{\beta}}\ dv \\
=&\int_{X_s}((\Box+1)^{-1}(\xi_k(e_{\alpha\overline{\beta}}))\overline{\xi}_l(e_{i\overline{j}}) \\
&\qquad+(\Box+1)^{-1}(\xi_k(e_{i\overline{j}}))\overline{\xi}_l(e_{\alpha\overline{\beta}}))\ dv \\
&+\int_{X_s}(\Box+1)^{-1}(\xi_k(e_{\alpha\overline{\beta}}))(\overline{\xi}_j(e_{i\overline{l}})+A_i\cdot L_{\overline{l}}\overline{A_j})\ dv \\
&+\int_{X_s}(\Box+1)^{-1}(\xi_k(e_{i\overline{j}}))(\overline{\xi}_\beta(e_{\alpha\overline{l}})+A_\alpha\cdot L_{\overline{l}}\overline{A_\beta})\ dv \\
&+\int_{X_s}Q_{k\overline{l}}(e_{i\overline{j}})e_{\alpha\overline{\beta}}\ dv.
\end{aligned}
$$

Now, by using (3.29), we have

(3.31)

$$
\begin{aligned}
&\int_{X_s}((\Box+1)^{-1}(\xi_k(e_{\alpha\overline{\beta}}))(A_i\cdot L_{\overline{l}}\overline{A_j})+(\Box+1)^{-1}(\xi_k(e_{i\overline{j}}))(A_\alpha\cdot L_{\overline{l}}\overline{A_\beta}))\ dv \\
&\quad=\int_{X_s}((\Box+1)^{-1}(\xi_k(e_{\alpha\overline{\beta}}))(\overline{\Gamma^t_{jl}}A_i\cdot\overline{A_t}) \\
&\qquad\quad+(\Box+1)^{-1}(\xi_k(e_{i\overline{j}}))(\overline{\Gamma^t_{\beta l}}A_\alpha\cdot\overline{A_t}))\ dv \\
&\quad=\overline{\Gamma^t_{jl}}\int_{X_s}\xi_k(e_{\alpha\overline{\beta}})(\Box+1)^{-1}(A_i\cdot\overline{A_t})\ dv \\
&\qquad\quad+\overline{\Gamma^t_{\beta l}}\int_{X_s}\xi_k(e_{i\overline{j}})(\Box+1)^{-1}(A_\alpha\cdot\overline{A_t})\ dv \\
&\quad=\overline{\Gamma^t_{jl}}\int_{X_s}\xi_k(e_{\alpha\overline{\beta}})e_{i\overline{t}}\ dv+\overline{\Gamma^t_{\beta l}}\int_{X_s}\xi_k(e_{i\overline{j}})e_{\alpha\overline{t}}\ dv.
\end{aligned}
$$

By combining (3.30) and (3.31), we have

(3.32)

$$\begin{aligned}\partial_{\bar l}\int_{X_s}\xi_k(e_{i\bar j})e_{\alpha\bar\beta}\,dv =&\int_{X_s}(\Box+1)^{-1}(\xi_k(e_{i\bar j}))(\overline{\xi}_l(e_{\alpha\bar\beta})+\overline{\xi}_\beta(e_{\alpha\bar l}))\,dv\\&+\int_{X_s}(\Box+1)^{-1}(\xi_k(e_{\alpha\bar\beta}))(\overline{\xi}_l(e_{i\bar j})+\overline{\xi}_j(e_{i\bar l}))\,dv\\&+\overline{\Gamma^t_{jl}}\int_{X_s}\xi_k(e_{\alpha\bar\beta})e_{i\bar t}\,dv+\overline{\Gamma^t_{\beta l}}\int_{X_s}\xi_k(e_{i\bar j})e_{\alpha\bar t}\,dv\\&+\int_{X_s}Q_{k\bar l}(e_{i\bar j})e_{\alpha\bar\beta}\,dv.\end{aligned}$$

We also have

$$\begin{aligned}\partial_{\bar l}\Gamma^p_{ik}=&\partial_{\bar l}(h^{p\bar q}\partial_k h_{i\bar q})=-h^{p\bar\beta}h^{\alpha\bar q}\partial_{\bar l}h_{\alpha\bar\beta}\partial_k h_{i\bar q}+h^{p\bar q}\partial_{\bar l}\partial_k h_{i\bar q}\\=&h^{p\bar q}(\partial_{\bar l}\partial_k h_{i\bar q}-h^{\alpha\bar\beta}\partial_{\bar l}h_{\alpha\bar q}\partial_k h_{i\bar\beta})=h^{p\bar q}R_{i\bar q k\bar l}.\end{aligned}\tag{3.33}$$

From Theorem 3.2, formula (3.32) and (3.33), we derive

(3.34)

$$\begin{aligned}\partial_{\bar l}\partial_k\tau_{i\bar j}=&(\partial_{\bar l}h^{\alpha\bar\beta})\left\{\sigma_1\int_{X_s}\xi_k(e_{i\bar j})e_{\alpha\bar\beta}\,dv\right\}+h^{\alpha\bar\beta}\left\{\sigma_1\partial_{\bar l}\int_{X_s}\xi_k(e_{i\bar j})e_{\alpha\bar\beta}\,dv\right\}\\&+h^{\gamma\bar\delta}\left\{\widetilde{\sigma}_1\int_{X_s}\overline{\xi}_l(e_{p\bar j})e_{\gamma\bar\delta}\,dv\right\}\Gamma^p_{ik}+\tau_{p\bar q}\Gamma^p_{ik}\overline{\Gamma^q_{jl}}+\tau_{p\bar j}h^{p\bar q}R_{i\bar q k\bar l}\\=&-h^{\alpha\bar t}\overline{\Gamma^\beta_{lt}}\left\{\sigma_1\int_{X_s}\xi_k(e_{i\bar j})e_{\alpha\bar\beta}\,dv\right\}\\&+h^{\alpha\bar\beta}\left\{\sigma_1\sigma_2\int_{X_s}(\Box+1)^{-1}(\xi_k(e_{i\bar j}))(\overline{\xi}_l(e_{\alpha\bar\beta})+\overline{\xi}_\beta(e_{\alpha\bar l}))\,dv\right\}\\&+h^{\alpha\bar\beta}\left\{\sigma_1\int_{X_s}Q_{k\bar l}(e_{i\bar j})e_{\alpha\bar\beta}\,dv\right\}+h^{\alpha\bar\beta}\overline{\Gamma^t_{jl}}\left\{\sigma_1\int_{X_s}\xi_k(e_{i\bar t})e_{\alpha\bar\beta}\,dv\right\}\\&+h^{\alpha\bar\beta}\overline{\Gamma^t_{\beta l}}\left\{\sigma_1\int_{X_s}\xi_k(e_{i\bar j})e_{\alpha\bar t}\,dv\right\}+h^{\gamma\bar\delta}\left\{\widetilde{\sigma}_1\int_{X_s}\overline{\xi}_l(e_{p\bar j})e_{\gamma\bar\delta}\,dv\right\}\Gamma^p_{ik}\\&+\tau_{p\bar q}\Gamma^p_{ik}\overline{\Gamma^q_{jl}}+\tau_{p\bar j}h^{p\bar q}R_{i\bar q k\bar l}.\end{aligned}$$

Now, from the above formula, by using Theorem 3.2, we can easily check the formula (3.28). q.e.d.

The curvature formula of the Ricci metric would be simpler if we have used the normal coordinates. However, when we estimate the asymptotic behavior of the curvature, it is hard to describe the normal coordinates near the boundary points. Thus, we will use this general formula directly in our computations. The estimates are quite subtle.

4. The asymptotic of the Ricci metric and its curvatures

From formula (3.4), we can easily see the sign of the curvature of the Weil–Petersson metric directly. However, the sign of the curvature of the Ricci metric cannot be derived from formula (3.28). In this section, we estimate the asymptotic of the Ricci metric and its curvatures. We first describe the local pinching coordinates near the boundary of the moduli space due to the plumbing construction of Wolpert. Then, we use Masur's construction of the holomorphic quadratic differentials to estimate the harmonic Beltrami differentials. Finally, we construct $\widetilde{e}_{i\bar{j}}$ which is an approximation of $e_{i\bar{j}}$. By doing this, we avoid the estimates of the Green function of $\Box + 1$ on the Riemann surfaces.

Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus $g \geq 2$ and let $\overline{\mathcal{M}}_g$ be its Deligne–Mumford compactification [**3**]. Each point $y \in \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ corresponds to a stable nodal surface X_y. A point $p \in X_y$ is a node if there is a neighborhood of p which is isometric to the germ $\{(u,v) \mid uv = 0,\ |u|, |v| < 1\} \subset \mathbb{C}^2$.

We first recall the rs-coordinate on a Riemann surface defined by Wolpert in [**18**]. There are two cases: the puncture case and the short geodesic case. For the puncture case, we have a nodal surface X and a node $p \in X$. Let a, b be two punctures which are glued together to form p.

Definition 4.1. *A local coordinate chart (U, u) near a is called rs-coordinate if $u(a) = 0$ where u maps U to the punctured disc $0 < |u| < c$ with $c > 0$, and the restriction to U of the Kähler–Einstein metric on X can be written as $\frac{1}{2|u|^2(\log|u|)^2}|du|^2$. The rs-coordinate (V, v) near b is defined in a similar way.*

For the short geodesic case, we have a closed surface X, a closed geodesic $\gamma \subset X$ with length $l < c_*$ where c_* is the collar constant.

Definition 4.2. *A local coordinate chart (U, z) is called rs-coordinate at γ if $\gamma \subset U$ where z maps U to the annulus $c^{-1}|t|^{\frac{1}{2}} < |z| < c|t|^{\frac{1}{2}}$, and the Kähler–Einstein metric on X can be written as $\frac{1}{2}(\frac{\pi}{\log|t|}\frac{1}{|z|}\csc\frac{\pi\log|z|}{\log|t|})^2$ $|dz|^2$.*

Remark 4.1. We put the factor $\frac{1}{2}$ in the above two definitions to normalize metrics such that (2.1) hold.

By Keen's collar theorem [**4**], we have the following lemma:

Lemma 4.1. *Let X be a closed surface and let γ be a closed geodesic on X such that the length l of γ satisfies $l < c_*$. Then, there is a collar Ω on X with holomorphic coordinate z defined on Ω such that*

1) *z maps Ω to the annulus $\frac{1}{c}e^{-\frac{2\pi^2}{l}} < |z| < c$ for $c > 0$;*
2) *the Kähler–Einstein metric on X restricted to Ω is given by*

$$\left(\frac{1}{2}u^2 r^{-2}\csc^2\tau\right)|dz|^2 \tag{4.1}$$

where $u = \frac{l}{2\pi}$, $r = |z|$ and $\tau = u\log r$;

3) *the geodesic γ is given by the equation $|z| = e^{-\frac{\pi^2}{l}}$.*

We call such a collar Ω a genuine collar.

We notice that the constant c in the above lemma has a lower bound such that the area of Ω is bounded from below. Also, the coordinate z in the above lemma is rs-coordinate. In the following, we will keep using the above notations u, r and τ.

Now, we describe the local manifold cover of $\overline{\mathcal{M}}_g$ near the boundary. We take the construction of Wolpert [**18**]. Let $X_{0,0}$ be a nodal surface corresponding to a codimension m boundary point. $X_{0,0}$ have m nodes $p_1, \cdots, p_m$. $X_0 = X_{0,0} \setminus \{p_1, \cdots, p_m\}$ is a union of punctured Riemann surfaces. Fix the rs-coordinate charts (U_i, η_i) and (V_i, ζ_i) at p_i for $i = 1, \cdots, m$ such that all the U_i and V_i are mutually disjoint. Now, pick an open set $U_0 \subset X_0$ such that the intersection of each connected component of X_0 and U_0 is a non-empty relatively compact set and the intersection $U_0 \cap (U_i \cup V_i)$ is empty for all i. Now, pick Beltrami differentials $\nu_{m+1}, \cdots, \nu_n$ which are supported in U_0 and span the tangent space at X_0 of the deformation space of X_0. For $s = (s_{m+1}, \cdots, s_n)$, let $\nu(s) = \sum_{i=m+1}^n s_i\nu_i$. We assume $|s| = (\sum |s_i|^2)^{\frac{1}{2}}$ small enough such that $|\nu(s)| < 1$. The nodal surface $X_{0,s}$ is obtained by solving the Beltrami equation $\overline{\partial} w = \nu(s)\partial w$. Since $\nu(s)$ is supported in U_0, (U_i, η_i) and (V_i, ζ_i) are still holomorphic coordinates on $X_{0,s}$. Note that they are no longer rs-coordinates. By the theory of Alhfors and Bers [**1**] and Wolpert [**18**], we can assume that there are constants $\delta, c > 0$ such that when $|s| < \delta$, η_i and ζ_i are holomorphic coordinates on $X_{0,s}$ with $0 < |\eta_i| < c$ and $0 < |\zeta_i| < c$. Now, we assume $t = (t_1, \cdots, t_m)$ has small norm. We do the plumbing construction on $X_{0,s}$ to obtain $X_{t,s}$. We remove from $X_{0,s}$ the discs $0 < |\eta_i| \le \frac{|t_i|}{c}$ and $0 < |\zeta_i| \le \frac{|t_i|}{c}$ for each $i = 1, \cdots, m$, and identify $\frac{|t_i|}{c} < |\eta_i| < c$ with $\frac{|t_i|}{c} < |\zeta_i| < c$ by the rule $\eta_i\zeta_i = t_i$. This defines the surface $X_{t,s}$. The tuple $(t_1, \cdots, t_m, s_{m+1}, \cdots, s_n)$ are the local pinching coordinates for the manifold cover of $\overline{\mathcal{M}}_g$. We call the coordinates η_i (or ζ_i) the plumbing coordinates on $X_{t,s}$ and the collar defined by $\frac{|t_i|}{c} < |\eta_i| < c$ the plumbing collar.

REMARK 4.2. From the estimate of Wolpert [**17**], [**18**] on the length of short geodesic, we have $u_i = \frac{l_i}{2\pi} \sim -\frac{\pi}{\log |t_i|}$.

We also need the following version of the Schauder estimate proved by Wolpert [**18**].

Theorem 4.1. *Let X be a closed Riemann surface equipped with the unique Kähler–Einstein metric. Let f and g be smooth functions on X such that $(\Box + 1)g = f$. Then, for any integer $k \geq 0$, there is a constant c_k such that $\|g\|_{k+1} \leq c_k \|f\|_k$ where the norm is defined by* (3.2).

Now, we estimate the asymptotic of the Ricci metric in the pinching coordinates. We will use the following notations. Let $(t,s) = (t_1, \cdots, t_m, s_{m+1}, \cdots, s_n)$ be the pinching coordinates near $X_{0,0}$. For $|(t,s)| < \delta$, let Ω_c^j be the j-th genuine collar on $X_{t,s}$ which contains a short geodesic γ_j with length l_j. Let $u_j = \frac{l_j}{2\pi}$, $u_0 = \sum_{j=1}^m u_j + \sum_{j=m+1}^n |s_j|$, $r_j = |z_j|$ and $\tau_j = u_j \log r_j$ where z_j is the properly normalized rs-coordinate on Ω_c^j such that

$$\Omega_c^j = \{z_j \mid c^{-1} e^{-\frac{2\pi^2}{l_j}} < |z_j| < c\}.$$

From the above argument, we know that the Kähler–Einstein metric λ on $X_{t,s}$ restrict to the collar Ω_c^j is given by

$$\lambda = \frac{1}{2} u_j^2 r_j^{-2} \csc^2 \tau_j. \tag{4.2}$$

For convenience, we let $\Omega_c = \cup_{j=1}^m \Omega_c^j$ and $R_c = X_{t,s} \setminus \Omega_c$. In the following, we may change the constant c finitely many times. Clearly, this will not affect the estimates.

To estimate the curvature of the Ricci metric, the first step is to find all the harmonic Beltrami differentials $B_1, \cdots, B_n$ which correspond to the tangent vectors $\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial s_n}$. In [**11**], Masur constructed $3g-3$ regular holomorphic quadratic differentials $\psi_1, \cdots, \psi_n$ on the plumbing collars by using the plumbing coordinate η_j. These quadratic differentials correspond to the cotangent vectors $dt_1, \cdots, ds_n$.

However, it is more convenient to estimate the curvature if we use the rs-coordinate on $X_{t,s}$ since we have the accurate form of the Kähler–Einstein metric λ in this coordinate. In [**15**], Trapani used the graft metric constructed by Wolpert [**18**] to estimate the difference between the plumbing coordinate and rs-coordinate and gave the holomorphic quadratic differentials constructed by Masur in the rs-coordinate. We collect Trapani's results (Lemma 6.2–6.5, [**15**]) in the following theorem:

Theorem 4.2. *Let (t,s) be the pinching coordinates on $\overline{\mathcal{M}}_g$ near $X_{0,0}$ which corresponds to a codimension m boundary point of $\overline{\mathcal{M}}_g$. Then, there exist constants $M,\delta>0$ and $1>c>0$ such that if $|(t,s)|<\delta$, then the j-th plumbing collar on $X_{t,s}$ contains the genuine collar Ω_c^j. Furthermore, one can choose rs-coordinate z_j on the collar Ω_c^j properly such that the holomorphic quadratic differentials $\psi_1,\cdots,\psi_n$ corresponding to the cotangent vectors $dt_1,\cdots,ds_n$ have the form $\psi_i=\varphi_i(z_j)dz_j^2$ on the genuine collar Ω_c^j for $1\le j\le m$, where*

1) $\varphi_i(z_j)=\frac{1}{z_j^2}(q_i^j(z_j)+\beta_i^j)$ *if* $i\ge m+1$;
2) $\varphi_i(z_j)=(-\frac{t_j}{\pi})\frac{1}{z_j^2}(q_j(z_j)+\beta_j)$ *if* $i=j$;
3) $\varphi_i(z_j)=(-\frac{t_i}{\pi})\frac{1}{z_j^2}(q_i^j(z_j)+\beta_i^j)$ *if* $1\le i\le m$ *and* $i\ne j$.

Here, β_i^j and β_j are functions of (t,s), q_i^j and q_j are functions of (t,s,z_j) given by

$$q_i^j(z_j)=\sum_{k<0}\alpha_{ik}^j(t,s)t_j^{-k}z_j^k+\sum_{k>0}\alpha_{ik}^j(t,s)z_j^k$$

and

$$q_j(z_j)=\sum_{k<0}\alpha_{jk}(t,s)t_j^{-k}z_j^k+\sum_{k>0}\alpha_{jk}(t,s)z_j^k$$

such that

1) $\sum_{k<0}|\alpha_{ik}^j|c^{-k}\le M$ *and* $\sum_{k>0}|\alpha_{ik}^j|c^k\le M$ *if* $i\ne j$;
2) $\sum_{k<0}|\alpha_{jk}|c^{-k}\le M$ *and* $\sum_{k>0}|\alpha_{jk}|c^k\le M$;
3) $|\beta_i^j|=O(|t_j|^{\frac{1}{2}-\epsilon})$ *with* $\epsilon<\frac{1}{2}$ *if* $i\ne j$;
4) $|\beta_j|=(1+O(u_0))$.

An immediate consequence of the above theorem is the following refined version of Masur's estimates of the Weil–Petersson metric. In the following, we will fix (t,s) with small norm and let $X=X_{t,s}$.

Corollary 4.1. *Let (t,s) be the pinching coordinates. Then*

1) $h^{i\bar i}=2u_i^{-3}|t_i|^2(1+O(u_0))$ *and* $h_{i\bar i}=\frac{1}{2}\frac{u_i^3}{|t_i|^2}(1+O(u_0))$ *for* $1\le i\le m$;
2) $h^{i\bar j}=O(|t_it_j|)$ *and* $h_{i\bar j}=O(\frac{u_i^3u_j^3}{|t_it_j|})$, *if* $1\le i,j\le m$ *and* $i\ne j$;
3) $h^{i\bar j}=O(1)$ *and* $h_{i\bar j}=O(1)$, *if* $m+1\le i,j\le n$;
4) $h^{i\bar j}=O(|t_i|)$ *and* $h_{i\bar j}=O(\frac{u_i^3}{|t_i|})$ *if* $i\le m<j$;
5) $h^{i\bar j}=O(|t_j|)$ *and* $h_{i\bar j}=O(\frac{u_j^3}{|t_j|})$ *if* $j\le m<i$.

Proof. We need the following simple calculus results:

$$\int_{c^{-1}e^{-\frac{2\pi^2}{l_j}}}^{c} \frac{1}{r_j}\sin^2\tau_j \ dr_j = u_j^{-1}(\frac{\pi}{2}+O(u_j)). \tag{4.3}$$

For any $k \geq 1$,

$$\int_{c^{-1}e^{-\frac{2\pi^2}{l_j}}}^{c} r_j^{k-1}\sin^2\tau_j \ dr_j = O(u_j^2)c^k \tag{4.4}$$

and for $k \leq -1$,

$$\int_{c^{-1}e^{-\frac{2\pi^2}{l_j}}}^{c} r_j^{k-1}\sin^2\tau_j \ dr_j = O(u_j^2)c^{-k}\left(e^{-\frac{2\pi^2}{l_j}}\right)^k. \tag{4.5}$$

On the collar Ω_c^j, the metric λ is given by (4.2). $h^{i\bar{j}}$ is given by the formula

$$h^{i\bar{j}} = \int_X \psi_i\overline{\psi_j}\lambda^{-2}dv.$$

By using the above calculus facts, we can compute the above integral on the collars. The bounds on R_c was calculated in [**11**]. A simple computation shows that the first parts of all of the above claims hold. The second parts of these claims can be obtained by inverting the matrix $(h^{i\bar{j}})$ together with Masur's result on the non-degenerate extension of the submatrix $(h^{i\bar{j}})_{i,j>m}$. This finishes the proof. q.e.d.

Now, we are ready to compute the harmonic Beltrami differentials $B_i = A_i\partial_z \otimes d\bar{z}$.

Lemma 4.2. *For c small, on the genuine collar Ω_c^j, the coefficient functions A_i of the harmonic Beltrami differentials have the form:*

1) $A_i = \frac{z_j}{\bar{z}_j}\sin^2\tau_j\left(\overline{p_i^j(z_j)}+\overline{b_i^j}\right)$ *if $i \neq j$;*
2) $A_j = \frac{z_j}{\bar{z}_j}\sin^2\tau_j\left(\overline{p_j(z_j)}+\overline{b_j}\right)$

where

1) $p_i^j(z_j) = \sum_{k\leq -1} a_{ik}^j\rho_j^{-k}z_j^k + \sum_{k\geq 1} a_{ik}^j z_j^k$ *if $i \neq j$;*
2) $p_j(z_j) = \sum_{k\leq -1} a_{jk}\rho_j^{-k}z_j^k + \sum_{k\geq 1} a_{jk}z_j^k$.

In the above expressions, $\rho_j = e^{-\frac{2\pi^2}{l_j}}$ and the coefficients satisfy the following conditions:

1) $\sum_{k\leq -1}|a_{ik}^j|c^{-k} = O(u_j^{-2})$ *and* $\sum_{k\geq 1}|a_{ik}^j|c^k = O(u_j^{-2})$ *if $i \geq m+1$;*
2) $\sum_{k\leq -1}|a_{ik}^j|c^{-k} = O(u_j^{-2})O\left(\frac{u_i^3}{|t_i|}\right)$ *and* $\sum_{k\geq 1}|a_{ik}^j|c^k = O(u_j^{-2})O\left(\frac{u_i^3}{|t_i|}\right)$ *if $i \leq m$ and $i \neq j$;*

3) $\sum_{k\le -1}|a_{jk}|c^{-k} = O(\frac{u_j}{|t_j|})$ *and* $\sum_{k\ge 1}|a_{jk}|c^{k} = O(\frac{u_j}{|t_j|})$;
4) $|b_i^j| = O(u_j)$ *if* $i \ge m+1$;
5) $|b_i^j| = O(u_j)O\big(\frac{u_i^3}{|t_i|}\big)$ *if* $i \le m$ *and* $i \ne j$;
6) $b_j = -\frac{u_j}{\pi \bar{t}_j}(1+O(u_0))$.

Proof. The duality between the harmonic Beltrami differentials and the holomorphic quadratic differentials is given by

$$B_i = \lambda^{-1}\sum_{l=1}^{n} h_{i\bar{l}}\overline{\psi_l} \tag{4.6}$$

which implies $A_i = \lambda^{-1}\sum_{l=1}^{n} h_{i\bar{l}}\overline{\varphi_l}$. Now, by Wolpert's estimate on the length of the short geodesic γ_j in [18], we have $l_j = -\frac{2\pi^2}{\log|t_j|}(1+O(u_j))$. This implies there is a constant $0<\mu<1$ such that $\mu|t_j| < \rho_j < \mu^{-1}|t_j|$. The lemma follows from equation (4.6) by replacing c by μc, a simple computation together with Theorem 4.2 and Corollary 4.1. q.e.d.

To estimate the curvature of the Ricci metric, we need to estimate the asymptotic of the Ricci metric by using Theorem 3.1. So, we need the following estimates on the norms of the harmonic Beltrami differentials.

Lemma 4.3. *Let* $\|\cdot\|_k$ *be the norm as defined in Definition 3.2. We have*

1) $\|A_i\|_{0,\Omega_c^i} = O\big(\frac{u_i}{|t_i|}\big)$ *and* $\|A_i\|_{0,X\setminus\Omega_c^i} = O\big(\frac{u_i^3}{|t_i|}\big)$, *if* $i\le m$;
2) $\|A_i\|_0 = O(1)$, *if* $i \ge m+1$;
3) $\|f_{i\bar{i}}\|_{0,\Omega_c^i} = O\big(\frac{u_i^2}{|t_i|^2}\big)$ *and* $\|f_{i\bar{i}}\|_{0,X\setminus\Omega_c^i} = O\big(\frac{u_i^6}{|t_i|^2}\big)$, *if* $i\le m$;
4) $\|f_{i\bar{j}}\|_0 = O(1)$, *if* $i,j \ge m+1$;
5) $\|f_{i\bar{j}}\|_{0,\Omega_c^i} = O\big(\frac{u_iu_j^3}{|t_it_j|}\big)$ *and* $\|f_{i\bar{j}}\|_{0,\Omega_c^j} = O\big(\frac{u_i^3u_j}{|t_it_j|}\big)$ *and* $\|f_{i\bar{j}}\|_{0,X\setminus(\Omega_c^i\cup\Omega_c^j)} = O\big(\frac{u_i^3u_j^3}{|t_it_j|}\big)$ *if* $i,j\le m$ *and* $i\ne j$;
6) $\|f_{i\bar{j}}\|_{0,\Omega_c^i} = O\big(\frac{u_i}{|t_i|}\big)$ *and* $\|f_{i\bar{j}}\|_{0,X\setminus\Omega_c^i} = O\big(\frac{u_i^3}{|t_i|}\big)$, *if* $i\le m$ *and* $j \ge m+1$;
7) $|f_{i\bar{j}}|_{L^1} = O(1)$, *if* $i,j\ge m+1$;
8) $|f_{i\bar{j}}|_{L^1} = O(\frac{u_i^3}{|t_i|})$, *if* $i\le m$ *and* $j\ge m+1$;
9) $|f_{i\bar{j}}|_{L^1} = O(\frac{u_i^3u_j^3}{|t_it_j|})$, *if* $i,j\le m$ *and* $i\ne j$.

Proof. We choose c small enough such that for each $1\le j\le m$,

$$\tan(u_j\log c) < -10u_j$$

when $|(t,s)| < \delta$. A simple computation shows that, when $1 \le p \le 10$, on the collar Ω_c^j, we have

$$|r_j^k \sin^p \tau_j| \le c^k |\log c|^p u_j^p$$

if $k \ge 1$, and

$$|r_j^k \sin^p \tau_j| \le c^{-k} |\log c|^p \rho_j^k u_j^p$$

if $k \le -1$.

To prove the first claim, note that on Ω_c^i, we have

$$\begin{aligned}
|A_i| = \left|\frac{z_i}{\overline{z_i}}\right| |\sin^2 \tau_i(\overline{p_i} + \overline{b_i})| \le & \sum_{k\le -1} |a_{ik}| \rho_i^{-k} r_i^k \sin^2 \tau_i \\
& + \sum_{k \ge 1} |a_{ik}| r_i^k \sin^2 \tau_i + |b_i| \\
\le & (\log c)^2 u_i^2 \left(\sum_{k \le -1} |a_{ik}| c^{-k} + \sum_{k\ge 1} |a_{ik}| c^k \right) + |b_i| \\
= & O(u_i^2) O\left(\frac{u_i}{|t_i|}\right) + O(u_i^2) O\left(\frac{u_i}{|t_i|}\right) + O\left(\frac{u_i}{|t_i|}\right) = O\left(\frac{u_i}{|t_i|}\right).
\end{aligned}$$

Similarly, on Ω_c^j with $j \ne i$, we have $|A_i| = O\big(\frac{u_i^3}{|t_i|}\big)$. Also, on R_c we have $|A_i| = O\big(\frac{u_i^3}{|t_i|}\big)$ by the work of Masur [**11**], equation (4.6) together with Theorem 4.2 and Corollary 4.1. This finishes the proof of the first claim.

The second claim can be proved in a similar way. Claim (3)–(6) follow from the first and second claims by using the fact that $f_{i\bar{j}} = A_i \overline{A_j}$. Claim (7) follows from claim (4) and the fact that the area of X is a fixed positive constant using the Gauss–Bonnet theorem.

Now, we prove claim (9). On Ω_c^i, by using a similar estimate as above, we have

$$\begin{aligned}
|f_{i\bar{j}}| = & |\sin^4 \tau_i (\overline{p_i} + \overline{b_i})(p_j^i + b_j^i)| \le |\sin^4 \tau_i \overline{p_i} p_j^i| + |\sin^4 \tau_i \overline{b_i} p_j^i| \\
& + |\sin^4 \tau_i \overline{p_i} b_j^i| + |\sin^4 \tau_i \overline{b_i} b_j^i| \\
\le & O\big(\frac{u_i^3 u_j^3}{|t_i t_j|}\big) + |\sin^4 \tau_i \overline{b_i} b_j^i| = O\big(\frac{u_i^3 u_j^3}{|t_i t_j|}\big) + O\big(\frac{u_i^2 u_j^3}{|t_i t_j|}\big) \sin^4 \tau_i.
\end{aligned}$$

So

$$|f_{i\bar{j}}|_{L^1(\Omega_c^i)} \le \int_{\Omega_c^i} \left(O\left(\frac{u_i^3 u_j^3}{|t_i t_j|}\right) + O\left(\frac{u_i^2 u_j^3}{|t_i t_j|}\right) \sin^4 \tau_i \right) dv = O\left(\frac{u_i^3 u_j^3}{|t_i t_j|}\right).$$

Similarly, $|f_{i\bar{j}}|_{L^1(\Omega_c^j)} \leq O\big(\frac{u_i^3 u_j^3}{|t_i t_j|}\big)$. The estimate $|f_{i\bar{j}}|_{L^1(X\setminus(\Omega_c^i \cup \Omega_c^j))} = O\big(\frac{u_i^3 u_j^3}{|t_i t_j|}\big)$ follows from claim (5). This proves claim (9). Similarly, we can prove claim (8). q.e.d.

In the following, we will denote the operator $(\square + 1)^{-1}$ by T. We then have the following estimates about L^2 norms:

Lemma 4.4. *Let $f \in C^\infty(X, \mathbb{C})$. Then we have*

$$\int_X |Tf|^2 \, dv \leq \int_X Tf \cdot \overline{f} \, dv \leq \int_X |f|^2 \, dv. \tag{4.7}$$

Proof. This lemma is a simple application of the spectral decomposition of the operator $(\square + 1)$ and the fact that all eigenvalues of this operator are greater than or equal to 1. One can also prove it directly by using integration by part. q.e.d.

To estimate the Ricci metric, we also need to estimate the functions $e_{i\bar{j}}$. We localize these functions on the collars by constructing the following approximation functions.

Pick a positive constant $c_1 < c$ and define the cut-off function $\eta \in C^\infty(\mathbb{R}, [0,1])$ by

$$\begin{cases} \eta(x) = 1, & x \leq \log c_1; \\ \eta(x) = 0, & x \geq \log c; \\ 0 < \eta(x) < 1, & \log c_1 < x < \log c. \end{cases} \tag{4.8}$$

It is clear that the derivatives of η are bounded by constants which only depend on c and c_1. Let $\widetilde{e_{i\bar{j}}}(z)$ be the function on X defined in the following way where z is taken to be z_i on the collar Ω_c^i:

1) if $i \leq m$ and $j \geq m+1$, then

$$\widetilde{e_{i\bar{j}}}(z) = \begin{cases} \frac{1}{2}\sin^2 \tau_i \overline{b_i} b_j^i, & z \in \Omega_{c_1}^i; \\ (\frac{1}{2}\sin^2 \tau_i \overline{b_i} b_j^i)\eta(\log r_i), & z \in \Omega_c^i \text{ and } c_1 < r_i < c; \\ (\frac{1}{2}\sin^2 \tau_i \overline{b_i} b_j^i)\eta(\log \rho_i - \log r_i), & z \in \Omega_c^i \text{ and } c^{-1}\rho_i < r_i < c_1^{-1}\rho_i; \\ 0, & z \in X \setminus \Omega_c^i; \end{cases}$$

2) if $i,j\le m$ and $i\ne j$, then

$$\widetilde{e_{i\bar{j}}}(z)=\begin{cases}\frac{1}{2}\sin^2\tau_i\overline{b_i}b^i_j, & z\in\Omega^i_{c_1};\\ (\frac{1}{2}\sin^2\tau_i\overline{b_i}b^i_j)\eta(\log r_i), & z\in\Omega^i_c \text{ and } c_1<r_i<c;\\ (\frac{1}{2}\sin^2\tau_i\overline{b_i}b^i_j)\eta(\log\rho_i-\log r_i), & z\in\Omega^i_c \text{ and } c^{-1}\rho_i<r_i<c_1^{-1}\rho_i;\\ \frac{1}{2}\sin^2\tau_j\overline{b^j_i}b_j, & z\in\Omega^j_{c_1};\\ (\frac{1}{2}\sin^2\tau_i\overline{b^j_i}b_j)\eta(\log r_j), & z\in\Omega^j_c \text{ and } c_1<r_j<c;\\ (\frac{1}{2}\sin^2\tau_i\overline{b^j_i}b_j)\eta(\log\rho_j-\log r_j), & z\in\Omega^j_c \text{ and } c^{-1}\rho_j<r_j<c_1^{-1}\rho_j;\\ 0, & z\in X\setminus(\Omega^i_c\cup\Omega^j_c);\end{cases}$$

3) if $i\le m$, then

$$\widetilde{e_{i\bar{i}}}(z)=\begin{cases}\frac{1}{2}\sin^2\tau_i|b_i|^2, & z\in\Omega^i_{c_1};\\ (\frac{1}{2}\sin^2\tau_i|b_i|^2)\eta(\log r_i), & z\in\Omega^i_c \text{ and } c_1<r_i<c;\\ (\frac{1}{2}\sin^2\tau_i|b_i|^2)\eta(\log\rho_i-\log r_i), & z\in\Omega^i_c \text{ and } c^{-1}\rho_i<r_i<c_1^{-1}\rho_i;\\ 0, & z\in X\setminus\Omega^i_c.\end{cases}$$

Also, let $\widetilde{f_{i\bar{j}}}=(\Box+1)\widetilde{e_{i\bar{j}}}$. It is clear that the supports of these approximation functions are contained in the corresponding collars. We have the following estimates:

Lemma 4.5. *Let $\widetilde{e_{i\bar{j}}}$ be the functions constructed above. Then*

1) $e_{i\bar{i}}=\widetilde{e_{i\bar{i}}}+O\big(\frac{u_i^4}{|t_i|^2}\big)$, *if* $i\le m$;
2) $e_{i\bar{j}}=\widetilde{e_{i\bar{j}}}+O\big(\frac{u_i^3u_j^3}{|t_it_j|}\big)$, *if* $i,j\le m$ *and* $i\ne j$;
3) $e_{i\bar{j}}=\widetilde{e_{i\bar{j}}}+O\big(\frac{u_i^3}{|t_i|}\big)$, *if* $i\le m$ *and* $j\ge m+1$;
4) $\|e_{i\bar{j}}\|_0=O(1)$, *if* $i,j\ge m+1$.

Proof. The last claim follows from the maximum principle and Lemma 4.3. To prove the first claim, we note that the maximum principle implies

$$\|e_{i\bar{i}}-\widetilde{e_{i\bar{i}}}\|_0\le\|f_{i\bar{i}}-\widetilde{f_{i\bar{i}}}\|_0.$$

Now, we compute the right-hand side of the above inequality. Since $\widetilde{f_{i\bar{i}}}\mid_{X\setminus\Omega^i_c}=0$, by Lemma 4.3, we know that $\|f_{i\bar{i}}-\widetilde{f_{i\bar{i}}}\|_{0,X\setminus\Omega^i_c}=O\big(\frac{u_i^6}{|t_i|^2}\big)$. On $\Omega^i_{c_1}$, we have

$$|f_{i\bar{i}}-\widetilde{f_{i\bar{i}}}|\le|\sin^4\tau_i\overline{p_i}b_i|+|\sin^4\tau_i\overline{b_i}p_i|+|\sin^4\tau_i\overline{p_i}p_i|=O\left(\frac{u_i^6}{|t_i|^2}\right)$$

which implies $\|f_{i\bar{i}} - \widetilde{f_{i\bar{i}}}\|_{0,\Omega^i_{c_1}} = O\big(\frac{u_i^6}{|t_i|^2}\big)$. On $\Omega^i_c \setminus \Omega^i_{c_1}$ with $c_1 \le r_i \le c$, we have

$$\begin{aligned}|f_{i\bar{i}} - \widetilde{f_{i\bar{i}}}| \le &(1-\eta)|b_i|^2 \sin^4\tau_i + |\sin^4\tau_i \overline{p_i} b_i| + |\sin^4\tau_i \overline{b_i} p_i| + |\sin^4\tau_i \overline{p_i} p_i| \\ &+ \frac{|b_i|^2 u_i^{-2}|\eta''|}{4}\sin^4\tau_i + \frac{|b_i|^2 u_i^{-1}|\eta'|}{2}\sin^2\tau_i|\sin 2\tau_i| \\ =& O\left(\frac{u_i^4}{|t_i|^2}\right).\end{aligned}$$

Similarly, on $\Omega^i_c \setminus \Omega^i_{c_1}$ with $c^{-1}\rho_i \le r_i \le c_1^{-1}\rho_i$, we have $|f_{i\bar{i}} - \widetilde{f_{i\bar{i}}}| \le O\big(\frac{u_i^4}{|t_i|^2}\big)$. By combining the above estimate, we have $\|f_{i\bar{i}} - \widetilde{f_{i\bar{i}}}\|_0 = O\big(\frac{u_i^4}{|t_i|^2}\big)$ which implies the first claim. The second and the third claims can be proved in a similar way. q.e.d.

As a corollary, we prove the following estimates which are more refined than those of Trapani's on the Ricci metric [15]. The precise constants of the leading terms will be used later to compute the curvature of the Ricci metric.

Corollary 4.2. *Let (t,s) be the pinching coordinates. Then, we have*

1) $\tau_{i\bar{i}} = \frac{3}{4\pi^2}\frac{u_i^2}{|t_i|^2}(1+O(u_0))$ *and* $\tau^{i\bar{i}} = \frac{4\pi^2}{3}\frac{|t_i|^2}{u_i^2}(1+O(u_0))$, *if* $i \le m$;
2) $\tau_{i\bar{j}} = O\left(\frac{u_i^2 u_j^2}{|t_i t_j|}(u_i+u_j)\right)$ *and* $\tau^{i\bar{j}} = O(|t_i t_j|)$, *if* $i,j \le m$ *and* $i \ne j$;
3) $\tau_{i\bar{j}} = O\big(\frac{u_i^2}{|t_i|}\big)$ *and* $\tau^{i\bar{j}} = O(|t_i|)$, *if* $i \le m$ *and* $j \ge m+1$;
4) $\tau_{i\bar{j}} = O(1)$, *if* $i,j \ge m+1$.

Remark 4.3. The second part of the above corollary can be made sharper. However, it will not be useful for our later estimates.

Proof. The second part of the corollary is obtained by inverting the matrix $(\tau_{i\bar{j}})$ in the first part together with the fact that the matrix $(h_{i\bar{j}})_{i,j\ge m+1}$ is non-degenerate which was proved by Masur and the fact that the matrix $(\tau_{i\bar{j}})_{i,j\ge m+1}$ is bounded from below by a constant multiple of the matrix $(h_{i\bar{j}})_{i,j\ge m+1}$ which was proved by Wolpert.

Now, we prove the first part. In the following, we use C_0 to denote all universal constants which may change. Recall that

$$\tau_{i\bar{j}} = h^{\alpha\overline{\beta}} R_{i\bar{j}\alpha\overline{\beta}}. \tag{4.9}$$

To prove the last claim, let $i,j \ge m+1$. We first notice that if $\alpha \ne \beta$ or $\alpha = \beta \ge m+1$, then $|h^{\alpha\overline{\beta}}|\|A_\alpha\|_0\|A_\beta\|_0 = O(1)$ by Lemma 4.3 and Corollary 4.1. In this case, we have

$$
\begin{aligned}
|R_{i\bar{j}\alpha\bar{\beta}}| \leq & \left|\int_X e_{i\bar{j}} f_{\alpha\bar{\beta}}\ dv\right| + \left|\int_X e_{i\bar{\beta}} f_{\alpha\bar{j}}\ dv\right| \\
\leq & C_0(\|e_{i\bar{j}}\|_0 \|f_{\alpha\bar{\beta}}\|_0 + \|e_{i\bar{\beta}}\|_0 \|f_{\alpha\bar{j}}\|_0) \\
\leq & C_0(\|f_{i\bar{j}}\|_0 \|f_{\alpha\bar{\beta}}\|_0 + \|f_{i\bar{\beta}}\|_0 \|f_{\alpha\bar{j}}\|_0) = O(1)\|A_\alpha\|_0 \|A_\beta\|_0
\end{aligned}
$$

which implies $|h^{\alpha\bar{\beta}} R_{i\bar{j}\alpha\bar{\beta}}| = O(1)$. If $\alpha = \beta \leq m$, we have

$$
\begin{aligned}
|R_{i\bar{j}\alpha\bar{\alpha}}| \leq & \left|\int_X e_{i\bar{j}} f_{\alpha\bar{\alpha}}\ dv\right| + \left|\int_X e_{i\bar{\alpha}} f_{\alpha\bar{j}}\ dv\right| \leq \|e_{i\bar{j}}\|_0 |f_{\alpha\bar{\alpha}}|_{L^1} \\
& + \left(\int_X |e_{i\bar{\alpha}}|^2\ dv \int_X |f_{\alpha\bar{j}}|^2\ dv\right)^{\frac{1}{2}} \\
\leq & O(1) O\left(\frac{u_i^3}{|t_i|^2}\right) + \left(\int_X |f_{i\bar{\alpha}}|^2\ dv \int_X |f_{\alpha\bar{j}}|^2\ dv\right)^{\frac{1}{2}} \\
= & O\left(\frac{u_i^3}{|t_i|^2}\right) + \left(\int_X f_{i\bar{i}} f_{\alpha\bar{\alpha}}\ dv \int_X f_{\alpha\bar{\alpha}} f_{j\bar{j}}\ dv\right)^{\frac{1}{2}} \leq O\left(\frac{u_i^3}{|t_i|^2}\right) \\
& + \|A_i\|_0 \|A_j\|_0 |f_{\alpha\bar{\alpha}}|_{L^1} = O\left(\frac{u_i^3}{|t_i|^2}\right)
\end{aligned}
$$

which implies $|h^{\alpha\bar{\alpha}} R_{i\bar{j}\alpha\bar{\alpha}}| = O(1)$. So, we have proved that last claim.

To prove the third claim, let $i \leq m$ and $j \geq m+1$. If $\alpha \neq \beta$ or $\alpha = \beta \geq m+1$ in formula (4.9), by using integration by part, we have

$$
\begin{aligned}
|R_{i\bar{j}\alpha\bar{\beta}}| \leq & \left|\int_X f_{i\bar{j}} e_{\alpha\bar{\beta}}\ dv\right| + \left|\int_X f_{i\bar{\beta}} e_{\alpha\bar{j}}\ dv\right| \\
\leq & C_0(\|e_{\alpha\bar{\beta}}\|_0 |f_{i\bar{j}}|_{L^1} + \|e_{\alpha\bar{j}}\|_0 |f_{i\bar{\beta}}|_{L^1}) \\
\leq & C_0(\|f_{\alpha\bar{\beta}}\|_0 |f_{i\bar{j}}|_{L^1} + \|f_{\alpha\bar{j}}\|_0 |f_{i\bar{\beta}}|_{L^1}) \\
= & O\left(\frac{u_i^3}{|t_i|}\right) \|A_\alpha\|_0 \|A_\beta\|_0 + O(1)\|A_\alpha\|_0 |f_{i\bar{\beta}}|_{L^1}.
\end{aligned}
$$

By the above argument, we have $|h^{\alpha\bar{\beta}} O(\frac{u_i^3}{|t_i|}) \|A_\alpha\|_0 \|A_\beta\|_0| = O(\frac{u_i^3}{|t_i|})$ and by Lemma 4.3, we have $|h^{\alpha\bar{\beta}} \|A_\alpha\|_0 |f_{i\bar{\beta}}|_{L^1}| = O(\frac{u_i^3}{|t_i|})$. So, the claim is true in this case.

If $\alpha = \beta \leq m$ and $\alpha \neq i$, we have

$$
|R_{i\bar{j}\alpha\bar{\alpha}}| \leq \left|\int_X f_{i\bar{j}} e_{\alpha\bar{\alpha}}\ dv\right| + \left|\int_X f_{i\bar{\alpha}} e_{\alpha\bar{j}}\ dv\right|.
$$

To estimate the second term in the above formula, we have

$$\begin{aligned}\left|\int_X f_{i\overline{\alpha}} e_{\alpha\overline{j}}\ dv\right| &\leq \|e_{\alpha\overline{j}}\|_0 |f_{i\overline{\alpha}}|_{L^1} \leq \|f_{\alpha\overline{j}}\|_0 |f_{i\overline{\alpha}}|_{L^1}\\ &= O\left(\frac{u_\alpha}{|t_\alpha|}\right) O\left(\frac{u_i^3 u_\alpha^3}{|t_i t_\alpha|}\right) = O\left(\frac{u_i^3 u_\alpha^4}{|t_i||t_\alpha|^2}\right).\end{aligned}$$

To estimate the first term, we have

$$\begin{aligned}\left|\int_X f_{i\overline{j}} e_{\alpha\overline{\alpha}}\ dv\right| &\leq \left|\int_X f_{i\overline{j}} \widetilde{e}_{\alpha\overline{\alpha}}\ dv\right| + \left|\int_X f_{i\overline{j}} (e_{\alpha\overline{\alpha}} - \widetilde{e}_{\alpha\overline{\alpha}})\ dv\right|\\ &\leq \left|\int_{\Omega_c^\alpha} f_{i\overline{j}} \widetilde{e}_{\alpha\overline{\alpha}}\ dv\right| + \|e_{\alpha\overline{\alpha}} - \widetilde{e}_{\alpha\overline{\alpha}}\|_0 |f_{i\overline{j}}|_{L^1}\\ &\leq \|f_{i\overline{j}}\|_{0,\Omega_c^\alpha} |\widetilde{e}_{\alpha\overline{\alpha}}|_{L^1} + O\left(\frac{u_\alpha^4}{|t_\alpha|^2}\right) O\left(\frac{u_i^3}{|t_i|}\right) = O\left(\frac{u_i^3 u_\alpha^3}{|t_i||t_\alpha|^2}\right)\end{aligned}$$

which implies $|h^{\alpha\overline{\alpha}} R_{i\overline{j}\alpha\overline{\alpha}}| = O\left(\frac{u_i^3}{|t_i|}\right)$.

Finally, if $\alpha = \beta = i$, we have

$$\begin{aligned}|R_{i\overline{j}i\overline{i}}| &= 2\left|\int_X f_{i\overline{j}} e_{i\overline{i}}\ dv\right| \leq 2\|e_{i\overline{i}}\|_0 |f_{i\overline{j}}|_{L^1} \leq 2\|f_{i\overline{i}}\|_0 |f_{i\overline{j}}|_{L^1}\\ &= O\left(\frac{u_i^2}{|t_i|^2}\right) O\left(\frac{u_i^3}{|t_i|}\right) = O\left(\frac{u_i^5}{|t_i|^3}\right)\end{aligned}$$

which implies $|h^{i\overline{i}} R_{i\overline{j}i\overline{i}}| = O\left(\frac{u_i^2}{|t_i|}\right)$. This proves the third claim.

The second claim can be proved in a similar way. Now, we prove the first claim. If $\alpha \neq \beta$ or $\alpha = \beta \geq m+1$ in formula (4.9), we have

$$\begin{aligned}|R_{i\overline{i}\alpha\overline{\beta}}| &\leq \left|\int_X f_{i\overline{i}} e_{\alpha\overline{\beta}}\ dv\right| + \left|\int_X f_{i\overline{\beta}} e_{\alpha\overline{i}}\ dv\right| \leq \|e_{\alpha\overline{\beta}}\|_0 |f_{i\overline{i}}|_{L^1}\\ &\quad + \left(\int_X |e_{\alpha\overline{i}}|^2\ dv \int_X |f_{i\overline{\beta}}|^2\ dv\right)^{\frac{1}{2}}\\ &\leq \|f_{\alpha\overline{\beta}}\|_0 |f_{i\overline{i}}|_{L^1} + \left(\int_X |f_{\alpha\overline{i}}|^2\ dv \int_X |f_{i\overline{\beta}}|^2\ dv\right)^{\frac{1}{2}}\\ &\leq (\|f_{\alpha\overline{\beta}}\|_0 + \|A_\alpha\|_0 \|A_\beta\|_0) |f_{i\overline{i}}|_{L^1}\end{aligned}$$

which implies $|h^{\alpha\overline{\beta}} R_{i\overline{i}\alpha\overline{\beta}}| = O\left(\frac{u_i^3}{|t_i|^2}\right)$.

If $\alpha = \beta \leq m$ and $\alpha \neq i$, we have

$$|R_{i\overline{i}\alpha\overline{\alpha}}| \leq \left|\int_X e_{i\overline{i}} f_{\alpha\overline{\alpha}}\ dv\right| + \left|\int_X e_{i\overline{\alpha}} f_{\alpha\overline{i}}\ dv\right|.$$

To estimate the second term in the above inequality, we have

$$\left|\int_X e_{i\overline{\alpha}} f_{\alpha\overline{i}}\, dv\right| \le \|e_{i\overline{\alpha}}\|_0 |f_{\alpha\overline{i}}|_{L^1} \le \|f_{i\overline{\alpha}}\|_0 |f_{\alpha\overline{i}}|_{L^1}$$
$$= O\left(\frac{u_i u_\alpha}{|t_i t_\alpha|}\right) O\left(\frac{u_i^3 u_\alpha^3}{|t_i t_\alpha|}\right) = O\left(\frac{u_i^4 u_\alpha^4}{|t_i t_\alpha|^2}\right).$$

To estimate the first term in the above inequality, we have

$$\begin{aligned}\left|\int_X e_{i\overline{i}} f_{\alpha\overline{\alpha}}\, dv\right| &\le \left|\int_X \widetilde{e}_{i\overline{i}} f_{\alpha\overline{\alpha}}\, dv\right| + \left|\int_X (e_{i\overline{i}} - \widetilde{e}_{i\overline{i}}) f_{\alpha\overline{\alpha}}\, dv\right| \\ &\le \left|\int_{\Omega_c^i} \widetilde{e}_{i\overline{i}} f_{\alpha\overline{\alpha}}\, dv\right| + \|e_{i\overline{i}} - \widetilde{e}_{i\overline{i}}\|_0 |f_{\alpha\overline{\alpha}}|_{L^1} \\ &\le \|f_{\alpha\overline{\alpha}}\|_{0,\Omega_c^i} |\widetilde{e}_{i\overline{i}}|_{L^1} + \|e_{i\overline{i}} - \widetilde{e}_{i\overline{i}}\|_0 |f_{\alpha\overline{\alpha}}|_{L^1} \\ &= O\left(\frac{u_\alpha^6}{|t_\alpha|^2}\right) O\left(\frac{u_i^3}{|t_i|^2}\right) \\ &\quad + O\left(\frac{u_\alpha^3}{|t_\alpha|^2}\right) O\left(\frac{u_i^4}{|t_i|^2}\right) = O\left(\frac{u_i^3 u_\alpha^3}{|t_i t_\alpha|^2}\right).\end{aligned}$$

These imply $|h^{\alpha\overline{\alpha}} R_{i\overline{i}\alpha\overline{\alpha}}| = O\big(\frac{u_i^3}{|t_i|^2}\big)$.

Finally, we compute $h^{i\overline{i}} R_{i\overline{i}i\overline{i}}$. Clearly, $R_{i\overline{i}i\overline{i}} = 2\int_X e_{i\overline{i}} f_{i\overline{i}}\, dv$ and

$$\int_X e_{i\overline{i}} f_{i\overline{i}}\, dv = \int_X \widetilde{e}_{i\overline{i}} \widetilde{f}_{i\overline{i}}\, dv + \int_X \widetilde{e}_{i\overline{i}} (f_{i\overline{i}} - \widetilde{f}_{i\overline{i}})\, dv + \int_X (e_{i\overline{i}} - \widetilde{e}_{i\overline{i}}) f_{i\overline{i}}\, dv.$$

We also have

$$\left|\int_X \widetilde{e}_{i\overline{i}} (f_{i\overline{i}} - \widetilde{f}_{i\overline{i}})\, dv\right| \le \|f_{i\overline{i}} - \widetilde{f}_{i\overline{i}}\|_0 |\widetilde{e}_{i\overline{i}}|_{L^1} = O\left(\frac{u_i^7}{|t_i|^4}\right)$$

and

$$\left|\int_X f_{i\overline{i}} (e_{i\overline{i}} - \widetilde{e}_{i\overline{i}})\, dv\right| \le \|e_{i\overline{i}} - \widetilde{e}_{i\overline{i}}\|_0 |f_{i\overline{i}}|_{L^1} = O\left(\frac{u_i^7}{|t_i|^4}\right).$$

Also, we have $\|\widetilde{e}_{i\overline{i}}\|_{0,\Omega_c^i \setminus \Omega_{c_1}^i} = O\big(\frac{u_i^4}{|t_i|^2}\big)$ and $\|\widetilde{f}_{i\overline{i}}\|_{0,\Omega_c^i \setminus \Omega_{c_1}^i} = O\big(\frac{u_i^4}{|t_i|^2}\big)$. So

$$\begin{aligned}\int_X \widetilde{e}_{i\overline{i}} \widetilde{f}_{i\overline{i}}\, dv &= \int_{\Omega_{c_1}^i} \widetilde{e}_{i\overline{i}} \widetilde{f}_{i\overline{i}}\, dv + \int_{\Omega_c^i \setminus \Omega_{c_1}^i} \widetilde{e}_{i\overline{i}} \widetilde{f}_{i\overline{i}}\, dv \\ &= \frac{3\pi^2}{16} |b_i|^4 u_i (1 + O(u_0)) + O\left(\frac{u_i^8}{|t_i|^4}\right).\end{aligned}$$

By using Corollary 4.1, we have $h^{i\overline{i}} R_{i\overline{i}i\overline{i}} = \frac{3}{4\pi^2} \frac{u_i^2}{|t_i|^2} (1 + O(u_0))$. By combining the above results, we have proved this corollary. q.e.d.

It is well known that there is a complete asymptotic Poincaré metric ω_p on $\mathcal{M}_g$. We briefly describe it here. Please see [**10**] for more details.

Let $\overline{M}$ be a compact Kähler manifold of dimension m. Let $Y \subset \overline{M}$ be a divisor of normal crossings and let $M = \overline{M} \backslash Y$. Cover $\overline{M}$ by coordinate charts $U_1, \cdots, U_p, \cdots, U_q$ such that $(\overline{U}_{p+1} \cup \cdots \cup \overline{U}_q) \cap Y = \Phi$. We also assume that, for each $1 \leq \alpha \leq p$, there is a constant n_α such that $U_\alpha \setminus Y = (\Delta^*)^{n_\alpha} \times \Delta^{m-n_\alpha}$ and on U_α, Y is given by $z_1^\alpha \cdots z_{n_\alpha}^\alpha = 0$. Here, Δ is the disk of radius $\frac{1}{2}$ and Δ^* is the punctured disk of radius $\frac{1}{2}$. Let $\{\eta_i\}_{1 \leq i \leq q}$ be the partition of unity subordinate to the cover $\{U_i\}_{1 \leq i \leq q}$. Let ω be a Kähler metric on $\overline{M}$ and let C be a positive constant. Then, for C large, the Kähler form

$$\omega_p = C\omega + \sum_{i=1}^{p} \sqrt{-1}\partial\overline{\partial}\left(\eta_i \sum_{j=1}^{n_i} \log\log \frac{1}{|z_j^i|}\right)$$

defines a complete metric on M with finite volume since on each U_i with $1 \leq i \leq p$, ω_p is bounded from above and below by the local Poincaré metric on U_i. We call this metric the asymptotic Poincaré metric.

As a direct application of the above corollary, we have

Theorem 4.3. *The Ricci metric is equivalent to the asymptotic Poincaré metric. More precisely, there is a positive constant C such that*

$$C^{-1}\omega_p \leq \omega_\tau \leq C\omega_p.$$

Now, we estimate the holomorphic sectional curvature of the Ricci metric. We will show that the holomorphic sectional curvature is negative in the degeneration directions and is bounded in other directions. We will need the following estimates on the norms to estimate the error terms.

Lemma 4.6. *Let $f, g \in C^\infty(X, \mathbb{C})$ be smooth functions such that $(\Box + 1)f = g$. Then, there is a constant C_0 such that*

1) $|K_0 f|_{L^2} \leq C_0 |K_0 g|_{L^2}$;
2) $|K_1 K_0 f|_{L^2} \leq C_0 |K_0 g|_{L^2}$;

Proof. Let $h = |K_0 f|^2$. By using Schwarz inequality, we easily see that the lemma follows from the Bochner formula:

$$\Box h + h + |K_1 K_0 f|^2 = K_0 f \overline{K_0 g} + \overline{K_0 f} K_0 g - |f - g|^2.$$

q.e.d.

We also need the estimates on the sections $K_0 f_{i\overline{j}}$. We have:

Lemma 4.7. *Let K_0 and K_1 be the Maass operators defined in Section 3. Then*

1) $\|K_0 f_{i\bar{i}}\|_{0,\Omega_c^i} = O\big(\frac{u_i^2}{|t_i|^2}\big)$ *and* $\|K_0 f_{i\bar{i}}\|_{0,X\setminus\Omega_c^i} = O\big(\frac{u_i^6}{|t_i|^2}\big)$, *if* $i \le m$*;*
2) $\|K_0 f_{i\bar{j}}\|_0 = O(1)$, *if* $i, j \ge m+1$*;*
3) $\|K_0 f_{i\bar{j}}\|_{0,\Omega_c^i} = O\big(\frac{u_i u_j^3}{|t_i t_j|}\big)$ *and* $\|K_0 f_{i\bar{j}}\|_{0,\Omega_c^j} = O\big(\frac{u_i^3 u_j}{|t_i t_j|}\big)$ *and* $\|K_0 f_{i\bar{j}}\|_{0,X\setminus(\Omega_c^i\cup\Omega_c^j)} = O\big(\frac{u_i^3 u_j^3}{|t_i t_j|}\big)$, *if* $i, j \le m$ *and* $i \ne j$*;*
4) $\|K_0 f_{i\bar{j}}\|_{0,\Omega_c^i} = O\big(\frac{u_i}{|t_i|}\big)$ *and* $\|K_0 f_{i\bar{j}}\|_{0,X\setminus\Omega_c^i} = O\big(\frac{u_i^3}{|t_i|}\big)$, *if* $i \le m$ *and* $j \ge m+1$*;*
5) $\|f_{i\bar{i}} - \widetilde{f}_{i\bar{i}}\|_1 = O\big(\frac{u_i^4}{|t_i|^2}\big)$, *if* $i \le m$.

This lemma can be proved by using similar methods as we used in the proof of Lemma 4.3 together with direct computations. So are the following L^1 and L^2 estimates:

Lemma 4.8. *Let* $P = K_1 K_0$ *be the operator defined Section 3. We have*

1) $|f_{i\bar{i}}|_{L^2}^2 = O\big(\frac{u_i^5}{|t_i|^4}\big)$, *if* $i \le m$*;*
2) $|K_0 f_{i\bar{i}}|_{L^2}^2 = O\big(\frac{u_i^5}{|t_i|^4}\big)$, *if* $i \le m$*;*
3) $|K_0 f_{i\bar{j}}|_{L^2}^2 = O\big(\frac{u_i^3 u_j^3}{|t_i t_j|^2}\big)$, *if* $i, j \le m$ *and* $i \ne j$*;*
4) $|K_0 f_{i\bar{j}}|_{L^2}^2 = O\big(\frac{u_i^3}{|t_i|^2}\big)$, *if* $i \le m$ *and* $j \ge m+1$*;*
5) $|K_0 f_{i\bar{j}}|_{L^2}^2 = O(1)$, *if* $i, j \ge m+1$*;*
6) $|P(\widetilde{e}_{i\bar{i}})|_{L^1} = O\big(\frac{u_i^3}{|t_i|^2}\big)$, *if* $i \le m$.

To estimate the curvature of the Ricci metric by using formula (3.28), we first expand the term $\int_X Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\overline{\beta}}\, dv$. A simple computation shows that

Lemma 4.9. *We have*

$$\int_X Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\overline{\beta}}\, dv = -\int_X f_{k\bar{l}}(K_0 e_{i\bar{j}}\overline{K}_0 e_{\alpha\overline{\beta}} + \overline{K}_0 e_{i\bar{j}} K_0 e_{\alpha\overline{\beta}})\, dv - \int_X (\Box e_{i\bar{j}} K_0 e_{\alpha\overline{\beta}}\overline{K}_0 e_{k\bar{l}} + \Box e_{\alpha\overline{\beta}} K_0 e_{i\bar{j}}\overline{K}_0 e_{k\bar{l}})\, dv.$$

To estimate the holomorphic sectional curvature, in formula (3.28), we let $i = j = k = l$. We decompose $\widetilde{R}_{i\bar{i}i\bar{i}}$ into two parts:

$$\widetilde{R}_{i\bar{i}i\bar{i}} = G_1 + G_2$$

where G_1 consists of those terms in the right-hand side of (3.28) with all indices α, β, γ, δ, p and q equal to i and $G_2 = \widetilde{R}_{i\bar{i}i\bar{i}} - G_1$ consists of those terms in (3.28) where, in each term, at least one of the indices α,

β, γ, δ, p or q is not i. If $i \leq m$, the leading term is G_1 which is given by

$$
\begin{aligned}
G_1 =& 24h^{i\bar{i}} \int_X (\Box + 1)^{-1}(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{i\bar{i}})\, dv \\
&+ 6h^{i\bar{i}} \int_X Q_{i\bar{i}}(e_{i\bar{i}})e_{i\bar{i}}\, dv \\
&- 36\tau^{i\bar{i}}(h^{i\bar{i}})^2 \left| \int_X \xi_i(e_{i\bar{i}})e_{i\bar{i}}\, dv \right|^2 \\
&+ \tau_{i\bar{i}} h^{i\bar{i}} R_{i\bar{i}i\bar{i}}.
\end{aligned} \tag{4.10}
$$

The main theorem of this section is the following estimate of the holomorphic sectional curvature of the Ricci metric.

Theorem 4.4. *Let $X_0 \in \overline{\mathcal{M}_g} \setminus \mathcal{M}_g$ be a codimension m point and let $(t_1, \cdots, t_m, s_{m+1}, \cdots, s_n)$ be the pinching coordinates at X_0 where $t_1, \cdots, t_m$ correspond to the degeneration directions. Then, the holomorphic sectional curvature is negative in the degeneration directions and is bounded in the non-degeneration directions. More precisely, there is a $\delta > 0$ such that, if $|(t, s)| < \delta$, then*

$$
-\widetilde{R}_{i\bar{i}i\bar{i}} = -\frac{3u_i^4}{8\pi^4|t_i|^4}(1 + O(u_0)) < 0 \tag{4.11}
$$

if $i \leq m$ and

$$
\widetilde{R}_{i\bar{i}i\bar{i}} = O(1) \tag{4.12}
$$

if $i \geq m+1$.

Furthermore, on $\mathcal{M}_g$, the holomorphic sectional curvature, the bisectional curvature and the Ricci curvature of the Ricci metric are bounded from above and below.

Proof. We first compute the asymptotic of the holomorphic sectional curvature. By Lemma 4.9, we know that

$$
\int_X Q_{i\bar{i}}(e_{i\bar{i}})e_{i\bar{i}}\, dv = \int_X |K_0 e_{i\bar{i}}|^2 (2e_{i\bar{i}} - 4f_{i\bar{i}})\, dv.
$$

By (4.10), we have

(4.13)

$$
\begin{aligned}
G_1 =& 24h^{i\bar{i}} \int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{i\bar{i}})\, dv + 6h^{i\bar{i}} \int_X |K_0 e_{i\bar{i}}|^2 (2e_{i\bar{i}} - 4f_{i\bar{i}})\, dv \\
&- 36\tau^{i\bar{i}}(h^{i\bar{i}})^2 \left| \int_X \xi_i(e_{i\bar{i}})e_{i\bar{i}}\, dv \right|^2 + \tau_{i\bar{i}} h^{i\bar{i}} R_{i\bar{i}i\bar{i}}.
\end{aligned}
$$

We first consider the degeneration directions. Assume $i \leq m$. In this case, G_1 is the leading term. We have the following lemma.

Lemma 4.10. *If $i \leq m$, then $|G_2| = O\big(\frac{u_i^5}{|t_i|^4}\big)$.*

Proof. The lemma follows from a case by case check. We will prove it in the appendix. q.e.d.

Now, we go back to the proof of Theorem 4.4. We compute each term of G_1. By the proof of Corollary 4.2, we know that $h^{i\bar{i}}R_{i\bar{i}i\bar{i}} = \frac{3}{4\pi^2}\frac{u_i^2}{|t_i|^2}(1+O(u_0))$. So, we have

$$\tau_{i\bar{i}}h^{i\bar{i}}R_{i\bar{i}i\bar{i}} = \left(\frac{3u_i^2}{4\pi^2|t_i|^2}\right)^2(1+O(u_0)) = \frac{9u_i^4}{16\pi^4|t_i|^4}(1+O(u_0)). \tag{4.14}$$

Now, we compute the second term. We have

$$\begin{aligned}&\int_X |K_0 e_{i\bar{i}}|^2(2e_{i\bar{i}} - 4f_{i\bar{i}})\,dv \\ &= \int_X |K_0\widetilde{e}_{i\bar{i}}|^2(2\widetilde{e}_{i\bar{i}} - 4\widetilde{f}_{i\bar{i}})\,dv + \int_X (|K_0 e_{i\bar{i}}|^2 - |K_0\widetilde{e}_{i\bar{i}}|^2)(2\widetilde{e}_{i\bar{i}} - 4\widetilde{f}_{i\bar{i}})\,dv \\ &\quad + \int_X |K_0 e_{i\bar{i}}|^2(2(e_{i\bar{i}} - \widetilde{e}_{i\bar{i}}) - 4(f_{i\bar{i}} - \widetilde{f}_{i\bar{i}}))\,dv.\end{aligned} \tag{4.15}$$

For the second term in the above equation, we have

$$\begin{aligned}&\left|\int_X (|K_0 e_{i\bar{i}}|^2 - |K_0\widetilde{e}_{i\bar{i}}|^2)(2\widetilde{e}_{i\bar{i}} - 4\widetilde{f}_{i\bar{i}})\,dv\right| \\ &\quad\leq \||K_0 e_{i\bar{i}}|^2 - |K_0\widetilde{e}_{i\bar{i}}|^2\|_0 \int_X (2|\widetilde{e}_{i\bar{i}}| + 4|\widetilde{f}_{i\bar{i}}|)\,dv \\ &\quad\leq \||K_0 e_{i\bar{i}}| + |K_0\widetilde{e}_{i\bar{i}}|\|_0 \|K_0(e_{i\bar{i}} - \widetilde{e}_{i\bar{i}})\|_0 \int_X (2|\widetilde{e}_{i\bar{i}}| + 4|\widetilde{f}_{i\bar{i}}|)\,dv \\ &\quad= O\left(\frac{u_i^2}{|t_i|^2}\right)O\left(\frac{u_i^4}{|t_i|^2}\right)O\left(\frac{u_i^3}{|t_i|^2}\right) = O\left(\frac{u_i^9}{|t_i|^6}\right).\end{aligned}$$

For the second term in the above equation, we have

$$\begin{aligned}&\left|\int_X |K_0 e_{i\bar{i}}|^2(2(e_{i\bar{i}} - \widetilde{e}_{i\bar{i}}) - 4(f_{i\bar{i}} - \widetilde{f}_{i\bar{i}}))\,dv\right| \\ &\quad\leq C_0\|K_0 e_{i\bar{i}}\|_0^2(2\|e_{i\bar{i}} - \widetilde{e}_{i\bar{i}}\|_0 + 4\|f_{i\bar{i}} - \widetilde{f}_{i\bar{i}}\|_0) \\ &\quad= O\left(\frac{u_i^4}{|t_i|^4}\right)O\left(\frac{u_i^4}{|t_i|^2}\right) = O\left(\frac{u_i^8}{|t_i|^6}\right).\end{aligned}$$

So, we get

$$
\begin{aligned}
&\int_X |K_0 e_{i\bar{i}}|^2 (2e_{i\bar{i}} - 4f_{i\bar{i}})\, dv \\
&\quad = \int_X |K_0 \widetilde{e}_{i\bar{i}}|^2 (2\widetilde{e}_{i\bar{i}} - 4\widetilde{f}_{i\bar{i}})\, dv + O\left(\frac{u_i^8}{|t_i|^6}\right) \\
&\quad = \int_{\Omega^i_{c_1}} |K_0 \widetilde{e}_{i\bar{i}}|^2 (2\widetilde{e}_{i\bar{i}} - 4\widetilde{f}_{i\bar{i}})\, dv \\
&\qquad\quad + \int_{\Omega^i_c \setminus \Omega^i_{c_1}} |K_0 \widetilde{e}_{i\bar{i}}|^2 (2\widetilde{e}_{i\bar{i}} - 4\widetilde{f}_{i\bar{i}})\, dv + O\left(\frac{u_i^8}{|t_i|^6}\right).
\end{aligned} \tag{4.16}
$$

We also have the estimate

$$
\begin{aligned}
&\left| \int_{\Omega^i_c \setminus \Omega^i_{c_1}} |K_0 \widetilde{e}_{i\bar{i}}|^2 (2\widetilde{e}_{i\bar{i}} - 4\widetilde{f}_{i\bar{i}})\, dv \right| \\
&\qquad \le C_0 \|K_0 \widetilde{e}_{i\bar{i}}\|_0^2 (\|\widetilde{e}_{i\bar{i}}\|_{0,\Omega^i_c \setminus \Omega^i_{c_1}} + \|\widetilde{f}_{i\bar{i}}\|_{0,\Omega^i_c \setminus \Omega^i_{c_1}}) = O\left(\frac{u_i^8}{|t_i|^6}\right).
\end{aligned}
$$

A direct computation shows that

$$
\int_{\Omega^i_{c_1}} |K_0 \widetilde{e}_{i\bar{i}}|^2 (2\widetilde{e}_{i\bar{i}} - 4\widetilde{f}_{i\bar{i}})\, dv = -\frac{3u_i^7}{64\pi^4 |t_i|^6}(1 + O(u_0)).
$$

So

$$
6h^{i\bar{i}} \int_X |K_0 e_{i\bar{i}}|^2 (2e_{i\bar{i}} - 4f_{i\bar{i}})\, dv = -\frac{9u_i^4}{16\pi^4 |t_i|^4}(1 + O(u_0)). \tag{4.17}
$$

Now, we compute the third term. We have

$$
\begin{aligned}
\int_X \xi_i(e_{i\bar{i}}) e_{i\bar{i}}\, dv = &\int_X \xi_i(\widetilde{e}_{i\bar{i}}) \widetilde{e}_{i\bar{i}}\, dv + \int_X \xi_i(\widetilde{e}_{i\bar{i}})(e_{i\bar{i}} - \widetilde{e}_{i\bar{i}})\, dv \\
&+ \int_X \xi_i(e_{i\bar{i}} - \widetilde{e}_{i\bar{i}}) e_{i\bar{i}}\, dv.
\end{aligned} \tag{4.18}
$$

By using the same method as above, we obtain

$$
\begin{aligned}
\left| \int_X \xi_i(\widetilde{e}_{i\bar{i}})(e_{i\bar{i}} - \widetilde{e}_{i\bar{i}})\, dv \right| &\le C_0 \|\xi_i(\widetilde{e}_{i\bar{i}})\|_0 \|e_{i\bar{i}} - \widetilde{e}_{i\bar{i}}\|_0 \\
&\le C_0 \|A_i\|_0 \|K_1 K_0(\widetilde{e}_{i\bar{i}})\|_0 \|e_{i\bar{i}} - \widetilde{e}_{i\bar{i}}\|_0 \\
&\le C_0 \|A_i\|_0 \|\widetilde{e}_{i\bar{i}}\|_2 \|e_{i\bar{i}} - \widetilde{e}_{i\bar{i}}\|_0 \\
&= O\left(\frac{u_i}{|t_i|}\right) O\left(\frac{u_i^2}{|t_i|^2}\right) O\left(\frac{u_i^4}{|t_i|^2}\right) \\
&= O\left(\frac{u_i^7}{|t_i|^5}\right)
\end{aligned}
$$

and

$$\begin{aligned}\left|\int_X \xi_i(e_{i\bar{i}}-\widetilde{e}_{i\bar{i}})e_{i\bar{i}}\ dv\right| &\le \|\xi_i(e_{i\bar{i}}-\widetilde{e}_{i\bar{i}})\|_0\int_X e_{i\bar{i}}\ dv\\ &\le \|A_i\|_0\|e_{i\bar{i}}-\widetilde{e}_{i\bar{i}}\|_2 h_{i\bar{i}}\\ &\le \|A_i\|_0\|f_{i\bar{i}}-\widetilde{f}_{i\bar{i}}\|_1 h_{i\bar{i}}\\ &= O\left(\frac{u_i}{|t_i|}\right)O\left(\frac{u_i^4}{|t_i|^2}\right)O\left(\frac{u_i^3}{|t_i|^2}\right)\\ &= O\left(\frac{u_i^8}{|t_i|^5}\right)\end{aligned}$$

and

$$\left|\int_{\Omega_c^i\setminus\Omega_{c_1}^i}\xi_i(\widetilde{e}_{i\bar{i}})\widetilde{e}_{i\bar{i}}\ dv\right| \le C_0\|\xi_i(\widetilde{e}_{i\bar{i}})\|_0\|\widetilde{e}_{i\bar{i}}\|_{0,\Omega_c^i\setminus\Omega_{c_1}^i} = O\left(\frac{u_i^7}{|t_i|^5}\right).$$

By putting the above results together, we get

$$\int_X \xi_i(e_{i\bar{i}})e_{i\bar{i}}\ dv = \int_{\Omega_{c_1}^i}\xi_i(\widetilde{e}_{i\bar{i}})\widetilde{e}_{i\bar{i}}\ dv + O\left(\frac{u_i^7}{|t_i|^5}\right).$$

On $\Omega_{c_1}^i$, we have

$$\xi_i(\widetilde{e}_{i\bar{i}}) = -\frac{z_i}{\overline{z_i}}\sin^2\tau_i\overline{b_i}P(\widetilde{e}_{i\bar{i}}) - \frac{z_i}{\overline{z_i}}\sin^2\tau_i\overline{p_i}P(\widetilde{e}_{i\bar{i}}).$$

However, we have $\|\frac{z_i}{\overline{z_i}}\sin^2\tau_i\overline{p_i}P(\widetilde{e}_{i\bar{i}})\|_{0,\Omega_{c_1}^i} = O\big(\frac{u_i^5}{|t_i|^3}\big)$ which implies

$$\left|\int_{\Omega_{c_1}^i}\frac{z_i}{\overline{z_i}}\sin^2\tau_i\overline{p_i}P(\widetilde{e}_{i\bar{i}})\widetilde{e}_{i\bar{i}}\ dv\right| = O\left(\frac{u_i^8}{|t_i|^5}\right).$$

A direct computation shows that

$$\int_{\Omega_{c_1}^i} -\frac{z_i}{\overline{z_i}}\sin^2\tau_i\overline{b_i}P(\widetilde{e}_{i\bar{i}}))\widetilde{e}_{i\bar{i}}\ dv = -\frac{u_i^6}{32\pi^3|t_i|^4 t_i}(1+O(u_0))$$

which implies

$$\int_X \xi_i(e_{i\bar{i}})e_{i\bar{i}}\ dv = -\frac{u_i^6}{32\pi^3|t_i|^4 t_i}(1+O(u_0)).$$

So, we obtain

$$(4.19)\quad 36\tau^{i\bar{i}}(h^{i\bar{i}})^2\left|\int_X \xi_i(e_{i\bar{i}})e_{i\bar{i}}\ dv\right|^2 = \frac{3u_i^4}{16\pi^4|t_i|^4}(1+O(u_0)).$$

Now, we estimate the first term. We have

$$\begin{aligned}\int_X T\xi_i(e_{i\bar{i}})\overline{\xi_i}(e_{i\bar{i}})\, dv = &\int_X T\xi_i(\widetilde{e}_{i\bar{i}})\overline{\xi_i}(\widetilde{e}_{i\bar{i}})\, dv \\ &+ \int_X T\xi_i(e_{i\bar{i}} - \widetilde{e}_{i\bar{i}})\overline{\xi_i}(\widetilde{e}_{i\bar{i}})\, dv \\ &+ \int_X T\xi_i(e_{i\bar{i}})\overline{\xi_i}(e_{i\bar{i}} - \widetilde{e}_{i\bar{i}})\, dv.\end{aligned}$$

By using the same method, we can get

$$\begin{aligned}\left|\int_X T\xi_i(e_{i\bar{i}} - \widetilde{e}_{i\bar{i}})\overline{\xi_i}(\widetilde{e}_{i\bar{i}})\, dv\right| \leq & C_0\|T\xi_i(e_{i\bar{i}} - \widetilde{e}_{i\bar{i}})\|_0\|\overline{\xi_i}(\widetilde{e}_{i\bar{i}})\|_0 \\ \leq & C_0\|\xi_i(e_{i\bar{i}} - \widetilde{e}_{i\bar{i}})\|_0\|\overline{\xi_i}(\widetilde{e}_{i\bar{i}})\|_0 \\ = & O\left(\frac{u_i^5}{|t_i|^3}\right) O\left(\frac{u_i^3}{|t_i|^3}\right) = O\left(\frac{u_i^8}{|t_i|^6}\right).\end{aligned}$$

Similarly,

$$\left|\int_X T\xi_i(e_{i\bar{i}})\overline{\xi_i}(e_{i\bar{i}} - \widetilde{e}_{i\bar{i}})\, dv\right| = O\left(\frac{u_i^8}{|t_i|^6}\right).$$

So, we have

$$\int_X T\xi_i(e_{i\bar{i}})\overline{\xi_i}(e_{i\bar{i}})\, dv = \int_X T\xi_i(\widetilde{e}_{i\bar{i}})\overline{\xi_i}(\widetilde{e}_{i\bar{i}})\, dv + O\left(\frac{u_i^8}{|t_i|^6}\right).$$

To estimate $T\xi_i(\widetilde{e}_{i\bar{i}})$, we introduce another approximation function. Pick $c_2 < c_1$ and let $\eta_1 \in C^\infty(\mathbb{R}, [0,1])$ be the cut-off function defined by

$$\eta_1 = \begin{cases} \eta_1(x) = 1, & x \leq \log c_2; \\ \eta_1(x) = 0, & x \geq \log c_1; \\ 0 < \eta_1(x) < 1, & \log c_2 < x < \log c_1. \end{cases} \tag{4.20}$$

For $i \leq m$, define the function d_i by

$$d_i(z) = \begin{cases} -\frac{1}{8}\sin^2\tau_i\cos 2\tau_i|b_i|^2\overline{b_i}, & z \in \Omega_{c_2}^i; \\ (-\frac{1}{8}\sin^2\tau_i\cos 2\tau_i|b_i|^2\overline{b_i})\eta_1(\log r_i), & z \in \Omega_{c_1}^i \text{ and } c_2 < r_i < c_1; \\ (-\frac{1}{8}\sin^2\tau_i\cos 2\tau_i|b_i|^2\overline{b_i})\eta_1(\log\rho_i - \log r_i), & z \in \Omega_{c_1}^i \text{ and } c_1^{-1}\rho_i < r_i < c_2^{-1}\rho_i; \\ 0, & z \in X \setminus \Omega_{c_1}^i. \end{cases}$$

A simple computation shows that

$$\|\xi_i(\widetilde{e}_{i\bar{i}}) - (\Box + 1)d_i\|_0 = O\left(\frac{u_i^5}{|t_i|^3}\right)$$

which implies

$$\|T\xi_i(\widetilde{e}_{i\bar{i}}) - d_i\|_0 = O\left(\frac{u_i^5}{|t_i|^3}\right).$$

So

$$\int_X T\xi_i(\widetilde{e}_{i\bar{i}})\overline{\xi_i}(\widetilde{e}_{i\bar{i}})\ dv = \int_X d_i\overline{\xi_i}(\widetilde{e}_{i\bar{i}})\ dv + \int_X (T\xi_i(\widetilde{e}_{i\bar{i}}) - d_i)\overline{\xi_i}(\widetilde{e}_{i\bar{i}})\ dv.$$

We have the estimate

$$\left|\int_X (T\xi_i(\widetilde{e}_{i\bar{i}}) - d_i)\overline{\xi_i}(\widetilde{e}_{i\bar{i}})\ dv\right| \le C_0\|T\xi_i(\widetilde{e}_{i\bar{i}}) - d_i\|_0\|\overline{\xi_i}(\widetilde{e}_{i\bar{i}})\|_0 = O\left(\frac{u_i^8}{|t_i|^6}\right)$$

which implies

$$\int_X T\xi_i(e_{i\bar{i}})\overline{\xi_i}(e_{i\bar{i}})\ dv = \int_X d_i\overline{\xi_i}(\widetilde{e}_{i\bar{i}})\ dv + O\left(\frac{u_i^8}{|t_i|^6}\right).$$

We also have

$$d_i\overline{\xi_i}(\widetilde{e}_{i\bar{i}}) = -d_i\frac{\overline{z_i}}{z_i}\sin^2\tau_i b_i\overline{P}(\widetilde{e}_{i\bar{i}}) - d_i\frac{\overline{z_i}}{z_i}\sin^2\tau_i p_i\overline{P}(\widetilde{e}_{i\bar{i}}).$$

Since $\|d_i\frac{\overline{z_i}}{z_i}\sin^2\tau_i p_i\overline{P}(\widetilde{e}_{i\bar{i}})\|_0 = O\big(\frac{u_i^8}{|t_i|^6}\big)$ and $\|d_i\frac{\overline{z_i}}{z_i}\sin^2\tau_i b_i\overline{P}(\widetilde{e}_{i\bar{i}})\|_{0,\Omega_{c_1}^i\setminus\Omega_{c_2}^i}$ $= O\big(\frac{u_i^8}{|t_i|^6}\big)$, we get

$$\int_X T\xi_i(e_{i\bar{i}})\overline{\xi_i}(e_{i\bar{i}})\ dv = \int_{\Omega_{c_2}^i} d_i\overline{\xi_i}(\widetilde{e}_{i\bar{i}})\ dv + O\left(\frac{u_i^8}{|t_i|^6}\right).$$

A direct computation shows that

$$\int_X T\xi_i(e_{i\bar{i}})\overline{\xi_i}(e_{i\bar{i}})\ dv = \frac{3u_i^7}{256\pi^4|t_i|^6}(1 + O(u_0))$$

which implies

$$24h^{i\bar{i}}\int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{i\bar{i}})\ dv = \frac{9u_i^4}{16\pi^4|t_i|^4}(1 + O(u_0)). \tag{4.21}$$

By combining formulas (4.21), (4.17), (4.19) and (4.14), we obtain

$$G_1 = \frac{3u_i^4}{8\pi^4|t_i|^4}(1 + O(u_0)).$$

Together with Lemma 4.10, we proved formula (4.11). The formula (4.12) can be proved using similar method with a case by case like the proof of Lemma 4.10.

Now, we give a weak estimate on the full curvature of the Ricci metric. Let

1) $\Lambda_i = \frac{u_i}{|t_i|}$ if $i \le m$;
2) $\Lambda_i = 1$ if $i \ge m+1$.

We can check the following estimates by using the methods in the proof of Lemma 4.10. We have

$$\widetilde{R}_{i\bar{j}k\bar{l}} = O(1) \tag{4.22}$$

if $i, j, k, l \ge m+1$ and

$$\widetilde{R}_{i\bar{j}k\bar{l}} = O(\Lambda_i\Lambda_j\Lambda_k\Lambda_l)O(u_0) \tag{4.23}$$

if at least one of these indices i, j, k, l is less than or equal to m and they are not all equal to each other.

Now, we prove the boundedness of the curvatures. We first consider the holomorphic sectional curvature. We need to show that there is a positive constant c such that for each point $p \in \mathcal{M}_g$ and each tangent vector $v \in T_p\mathcal{M}_g$, $|R(v, \overline{v}, v, \overline{v})| \leq c\|v\|_\tau^4$. We first check on a pinching coordinate chart near a codimension m boundary point. Assume the coordinates are $(t_1, \cdots, s_n)$. We assume $v = \sum_{i=1}^m a_i \frac{\partial}{\partial t_i} + \sum_{j=m+1}^n a_j \frac{\partial}{\partial s_j}$. By Corollary 4.2, we know that there is a constant $c_0 > 0$ such that

$$\|v\|_\tau^2 \geq c_0 \sum_{i=1}^n |a_i|^2 \Lambda_i^2.$$

Now, we have

$$|R(v, \overline{v}, v, \overline{v})| \leq \sum_{i,j,k,l} |a_i a_j a_k a_l||R_{i\overline{j}k\overline{l}}|.$$

The conclusion follows from Theorem 4.4, formulas (4.11), (4.12) and Schwarz inequality.

We cover the divisor $Y = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ by such open coordinate charts. Since Y is compact, we can pick finitely many such coordinate charts $\Xi_1, \cdots, \Xi_q$ such that $Y \subset \bigcup_{s=1}^q \Xi_s$. Clearly, there is an open neighborhood N of Y such that $\overline{N} \subset \bigcup_{s=1}^q \Xi_s$. From formulae (4.22), (4.23) and the above argument, we know that the holomorphic sectional curvature of τ is bounded from above and below on N. However, $\mathcal{M}_g \setminus N$ is a compact set of $\mathcal{M}_g$, so the holomorphic sectional curvature is also bounded on $\mathcal{M}_g \setminus N$ which implies the holomorphic sectional curvature is bounded on $\mathcal{M}_g$.

The boundedness of the bisectional curvature and the Ricci curvature of the Ricci metric can be proved by using (4.22), (4.23) and a similar argument as above, together with the covering and compactness argument. This finishes the proof. q.e.d.

Remark 4.4. The estimates of the bisectional curvature and the Ricci curvature are not optimal. A sharper estimate will be given in our next paper [**7**].

5. The perturbed Ricci metric and its curvatures

In this section, we introduce another new metric, the perturbed Ricci metric. This metric is obtained by adding a constant multiple of the Weil–Petersson metric to the Ricci metric. By doing this, we construct a natural complete metric whose holomorphic sectional curvature is negatively bounded. We will see that the holomorphic sectional curvature of the perturbed Ricci metric near an interior point of the moduli space is dominated by the curvature of the large constant multiple of the Weil–Petersson metric. Similar argument holds for the holomorphic sectional curvature of the perturbed Ricci metric in the non-degenerate directions near a boundary point.

Definition 5.1. *For any constant $C > 0$, we call the metric*

$$\widetilde{\tau}_{i\bar{j}} = \tau_{i\bar{j}} + Ch_{i\bar{j}}$$

the perturbed Ricci metric with constant C.

We first give the curvature formula of the perturbed Ricci metric. We use $P_{i\bar{j}k\bar{l}}$ to denote the curvature tensor of the perturbed Ricci metric.

Theorem 5.1. *Let $s_1, \cdots, s_n$ be local holomorphic coordinates at $s \in M_g$. Then at s, we have*

(5.1)

$$\begin{aligned}
P_{i\bar{j}k\bar{l}} =& h^{\alpha\overline{\beta}} \left\{ \sigma_1\sigma_2 \int_{X_s} \left\{ (\Box+1)^{-1}(\xi_k(e_{i\bar{j}}))\overline{\xi}_l(e_{\alpha\overline{\beta}}) \right.\right. \\
& \left.\left. +(\Box+1)^{-1}(\xi_k(e_{i\bar{j}}))\overline{\xi}_\beta(e_{\alpha\bar{l}}) \right\} \, dv \right\} \\
& + h^{\alpha\overline{\beta}} \left\{ \sigma_1 \int_{X_s} Q_{k\bar{l}}(e_{i\bar{j}}) e_{\alpha\overline{\beta}} \, dv \right\} \\
& - \widetilde{\tau}^{p\overline{q}} h^{\alpha\overline{\beta}} h^{\gamma\overline{\delta}} \left\{ \sigma_1 \int_{X_s} \xi_k(e_{i\overline{q}}) e_{\alpha\overline{\beta}} \, dv \right\} \left\{ \widetilde{\sigma}_1 \int_{X_s} \overline{\xi}_l(e_{p\bar{j}}) e_{\gamma\overline{\delta}}) \, dv \right\} \\
& + \tau_{p\bar{j}} h^{p\overline{q}} R_{i\overline{q}k\bar{l}} + CR_{i\bar{j}k\bar{l}}.
\end{aligned}$$

Proof. Let $s_1, \cdots, s_n$ be normal coordinates at a point $s \in \mathcal{M}_g$ with respect to the Weil–Petersson metric. By formula (3.14), at the point s, we have

$$\begin{aligned}
\partial_k \widetilde{\tau}_{i\bar{j}} &= \partial_k \tau_{i\bar{j}} + C\partial_k h_{i\bar{j}} \\
&= h^{\alpha\overline{\beta}} \left\{ \sigma_1 \int_{X_s} (\xi_k(e_{i\bar{j}}) e_{\alpha\overline{\beta}}) \, dv \right\} + \tau_{p\bar{j}} \Gamma^p_{ik} + C\partial_k h_{i\bar{j}} \\
&= h^{\alpha\overline{\beta}} \left\{ \sigma_1 \int_{X_s} (\xi_k(e_{i\bar{j}}) e_{\alpha\overline{\beta}}) \, dv \right\}
\end{aligned} \tag{5.2}$$

since $\Gamma^p_{ik} = \partial_k h_{i\bar{j}} = 0$ at this point. Now, at s the curvature of the Weil–Petersson metric is

$$R_{i\bar{j}k\bar{l}} = \partial_{\bar{l}} \partial_k h_{i\bar{j}}.$$

The theorem follows from formulas (3.3), (5.2) and (3.34). q.e.d.

Now, we estimate the curvature of the perturbed Ricci metric using formula (5.1). The following two linear algebra lemmas will be used to handle the inverse matrix $\widetilde{\tau}^{i\bar{j}}$ near an interior point and a boundary point.

Lemma 5.1. *Let D be a neighborhood of 0 in $\mathbb{C}^n$ and let A and B be two positive definite $n \times n$ Hermitian matrix functions on D such that they are bounded from above and below on D and each entry of them are bounded. Then, each entry of the inverse matrix $(A+CB)^{-1} = O(C^{-1})$ when C is very large.*

Proof. Consider the determinant $\det(A + CB)$. It is a polynomial of C of degree n and the coefficient of the leading term is $\det(B)$ which is bounded from below. All other coefficients are bounded since they only depend on the entries of A and B. So, we can pick C large such that $\det(A+CB) \geq \frac{1}{2}\det(B)C^n$. Now, the determinant of the (i,j)-minor of $A+CB$ is a polynomial of C of degree at most $n-1$ and the coefficients are bounded since they only depend on the entries of A and B. From the fact that the (i,j)-entry is the quotient of the determinant of the (i,j)-minor and the determinant of the matrix $A + CB$, the lemma follows directly. q.e.d.

Lemma 5.2. *Let $X_0 \in \overline{\mathcal{M}}_g$ be a codimension m boundary point and let $(t_1, \cdots, s_n)$ be the pinching coordinates near X_0. Then, for $|(t,s)| < \delta$ with δ small, we have that, for any $C > 0$,*

1) $0 < \widetilde{\tau}^{i\bar{i}} < \tau^{i\bar{i}}$ *for all* i*;*
2) $\widetilde{\tau}^{i\bar{j}} = O(|t_i t_j|)$*, if* $i,j \leq m$ *and* $i \neq j$*;*
3) $\widetilde{\tau}^{i\bar{j}} = O(|t_i|)$*, if* $i \leq m$ *and* $j \geq m+1$*;*
4) $\widetilde{\tau}^{i\bar{j}} = O(1)$*, if* $i,j \geq m+1$*.*

Furthermore, the bounds in the last three claims are independent of the choice of C*.*

Proof. The first claim is a general fact of linear algebra. To prove the last three claims, we denote the submatrices $(\tau_{i\bar{j}})_{i,j\geq m+1}$ and $(h_{i\bar{j}})_{i,j\geq m+1}$ by A and B. These two matrices represent the non-degenerate directions of the Ricci metric and the Weil–Petersson metric respectively. By the work of Masur, we know that the matrix B can be extended to the boundary non-degenerately. This implies that B has a positive lower bound. By Corollary (4.1), we know that B is bounded from above. Now, by the work of Wolpert, since $\omega_\tau \geq \widetilde{C}\omega_{WP}$ where $\widetilde{C}$ only depend on the genus of the Riemann surface, we know that A has a positive lower bound. By Corollary 4.2, we know that A is bounded from above. So, both matrices A and B are bounded from above and below and all their entries are bounded as long as $|(t,s)| \leq \delta$.

By Corollarys 4.1 and 4.2, we know that

$$(\widetilde{\tau}_{i\bar{j}}) = \begin{pmatrix} \Upsilon & \Psi \\ \overline{\Psi}^T & A + CB \end{pmatrix}$$

where Υ is an $m \times m$ matrix given by

$$\Upsilon = \begin{pmatrix} \frac{u_1^2}{|t_1|^2}(\frac{3}{4\pi^2} + \frac{Cu_1}{2})(1 + O(u_0)) & \dots & \frac{u_1^2 u_m^2}{|t_1 t_m|}(O(u_0) + CO(u_1 u_m)) \\ \vdots & \vdots & \vdots \\ \frac{u_1^2 u_m^2}{|t_1 t_m|}(O(u_0) + CO(u_1 u_m)) & \dots & \frac{u_m^2}{|t_m|^2}(\frac{3}{4\pi^2} + \frac{Cu_m}{2})(1 + O(u_0)) \end{pmatrix}$$

which represent the degenerate directions of the perturbed Ricci metric and Ψ is an $m \times (n-m)$ matrix given by

$$\Psi = \begin{pmatrix} \frac{u_1^2}{|t_1|}(O(1) + CO(u_1)) & \dots & \frac{u_1^2}{|t_1|}(O(1) + CO(u_1)) \\ \vdots & \vdots & \vdots \\ \frac{u_m^2}{|t_m|}(O(1) + CO(u_m)) & \dots & \frac{u_m^2}{|t_m|}(O(1) + CO(u_m)) \end{pmatrix}$$

which represents the mixed directions of the perturbed Ricci metric.

A direct computation shows that

$$\det \widetilde{\tau} = \left\{ \prod_{i=1}^m \frac{u_i^2}{|t_i|^2} \left(\frac{3}{4\pi^2} + \frac{Cu_i}{2} \right) \right\} \det(A + CB)(1 + O(u_0))$$

where the $O(u_0)$ term is independent of C. Let Φ_{ij} be the (i,j)-minor of $(\widetilde{\tau}_{i\bar{j}})$ obtained by deleting the i-th row and j-th column of $(\widetilde{\tau}_{i\bar{j}})$. By using the fact that

$$|\widetilde{\tau}^{i\bar{j}}| = \left| \frac{\det \Phi_{ij}}{\det \widetilde{\tau}} \right|,$$

the lemma follows from a direct computation of the determinant of Φ_{ij}. q.e.d.

Now, we prove the main theorem of this section.

Theorem 5.2. *For a suitable choice of positive constant C, the perturbed Ricci metric $\widetilde{\tau}_{i\bar{j}} = \tau_{i\bar{j}} + Ch_{i\bar{j}}$ is complete and its holomorphic sectional curvatures are negative and bounded from above and below by negative constants. Furthermore, the Ricci curvature of the perturbed Ricci metric is bounded from above and below.*

Proof. It is clear that the metric $\widetilde{\tau}_{i\bar{j}}$ is complete as long as $C \geq 0$ since it is greater than the Ricci metric which is complete.

Now, we estimate the holomorphic sectional curvature. We first show that, for any codimension m point $X_0 \in \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$, there are constants $C_0, \delta > 0$ such that, if $(t,s) = (t_1, \cdots, t_m, s_{m+1}, \cdots, s_n)$ is the pinching coordinates at X_0 with $|(t,s)| < \delta$ and $C \geq C_0$, then the holomorphic sectional curvature of the metric $\widetilde{\tau}$ is negative. We first consider the degeneration directions. Let $i = j = k = l \le m$. As in the proof of Theorem 4.4, we let

(5.3)

$$\begin{aligned}\widetilde{G}_1 =& 24h^{i\bar{i}} \int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{i\bar{i}})\ dv + 6h^{i\bar{i}} \int_X |K_0 e_{i\bar{i}}|^2 (2e_{i\bar{i}} - 4f_{i\bar{i}})\ dv \\ & - 36\widetilde{\tau}^{i\bar{i}}(h^{i\bar{i}})^2 \left| \int_X \xi_i(e_{i\bar{i}}) e_{i\bar{i}}\ dv \right|^2 + \tau_{i\bar{i}} h^{i\bar{i}} R_{i\bar{i}i\bar{i}}\end{aligned}$$

and $\widetilde{G}_2$ be the summation of those terms in (5.1) in which at least one of the indices $p, q, \alpha, \beta, \gamma, \delta$ is not i. We have $P_{i\bar{i}i\bar{i}} = \widetilde{G}_1 + \widetilde{G}_2 + CR_{i\bar{i}i\bar{i}}$. We notice here that we can use Lemma 5.2 instead of Corollary 4.2 in the proof of Lemma 4.10 without changing any estimate. This implies that $|\widetilde{G}_2| = O\big(\frac{u_i^5}{|t_i|^4}\big)$. By the proof of Theorem 4.4, we have

$$(5.4)\ \widetilde{G}_1 = \left(\frac{9}{16\pi^4} - \frac{3}{16\pi^4} \left(1 + \frac{2\pi^2 C u_i}{3} \right)^{-1} \right) \frac{u_i^4}{|t_i|^4} (1 + O(u_0))$$

which implies

$$(5.5)\qquad \begin{aligned} P_{i\bar{i}i\bar{i}} = & \left(\left(\frac{9}{16\pi^4} - \frac{3}{16\pi^4} \left(1 + \frac{2\pi^2 C u_i}{3} \right)^{-1} \right) \frac{u_i^4}{|t_i|^4} \right. \\ & \left. + \frac{3C}{8\pi^2} \frac{u_i^5}{|t_i|^4} \right) (1 + O(u_0)) > 0 \end{aligned}$$

as long as δ is small enough. Furthermore, $P_{i\bar{i}i\bar{i}}$ is bounded above and below by constant multiple of $\widetilde{\tau}_{i\bar{i}}^2$ where the constants may depend on C. However, when C is fixed, the constants are universal if δ is small enough.

Now, we let $i = j = k = l \ge m + 1$. By the proof of Theorem 4.4 and Lemma 5.2, we know that $P_{i\bar{i}i\bar{i}} = O(1) + CR_{i\bar{i}i\bar{i}}$. We also know that $R_{i\bar{i}i\bar{i}} > 0$ has a positive lower bound. Again, by using the extension theorem of Masur, we can choose C_0 large enough such that, when $C \ge C_0$, we have $P_{i\bar{i}i\bar{i}} > 0$. Furthermore, $P_{i\bar{i}i\bar{i}}$ is bounded from above and below by constant multiples of $\widetilde{\tau}_{i\bar{i}}^2$ where the constants may depend on C,m,n, X_0 and the choice of $\nu_{m+1}, \cdots, \nu_n$ if δ is small enough. We also have estimates similar to (4.22) and (4.23):

$$(5.6)\qquad P_{i\bar{j}k\bar{l}} = O(1) + CR_{i\bar{j}k\bar{l}}$$

if $i, j, k, l \ge m + 1$ and

$$(5.7)\qquad P_{i\bar{j}k\bar{l}} = O(\Lambda_i \Lambda_j \Lambda_k \Lambda_l) O(u_0) + CR_{i\bar{j}k\bar{l}}$$

if at least one of these indices i, j, k, l is less than or equal to m and they are not all equal to each other. So, we can choose δ small such that, if $|(t, s)| \leq \delta$, then the holomorphic sectional curvature is bounded from above and below by negative constants which may depend on C.

Now, we consider the interior points. Fix a point $p \in \mathcal{M}_g$ and a small neighborhood D of p such that $\overline{D} \subset \mathcal{M}_g$. Since the Ricci metric and Weil–Petersson metric are uniformly bounded in $\overline{D}$, we have $P_{i\bar{i}i\bar{i}} = O(1) + CR_{i\bar{i}i\bar{i}}$. Using a similar argument as above, we can choose a C_0 such that, when $C > C_0$, the holomorphic sectional curvature is bounded from above and below by negative constants which may depend on C.

Since the divisor $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ is compact, we can find finitely many boundary charts of $\mathcal{M}_g$ described above such that the holomorphic sectional curvature of $\widetilde{\tau}$ is pinched by two negative constants which depend on C on these charts. Furthermore, there is a neighborhood N of $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ in $\overline{\mathcal{M}}_g$ such that $\overline{N}$ is contained in the union of these charts. It is clear that we can find a constant C_1 such that on N, the holomorphic sectional curvature of $\widetilde{\tau}$ is pinched by negative constants when $C \geq C_1$.

Also, since the set $\mathcal{M}_g \setminus N$ is compact, by the above argument, we can find finitely many interior charts described above such that their union covers $\mathcal{M}_g \setminus N$ and a constant C_2, such that the holomorphic sectional curvature of $\widetilde{\tau}$ is pinched by negative constants when $C > C_2$. Again, the bounds may depend on C. By taking a constant $C > \max\{C_1, C_2\}$, we have proved the first part of the theorem. The Ricci curvature can be estimated in a similar way as we did in the proof of Theorem 4.4 together with Lemmas 5.1 and 5.2. q.e.d.

Remark 5.1. By using the negativity of the Ricci curvature of the Weil–Petersson metric and estimates (5.5), (5.6) and (5.7), we can actually show that the Ricci curvature of the perturbed Ricci metric is pinched between two negative constants. The detail will be given in our next paper.

6. Equivalent metrics on the moduli space

In this section, we prove the equivalence among the Ricci metric, perturbed Ricci metric, Kähler–Einstein metric and the McMullen metric. These equivalences imply that the Teichmüller metric is equivalent to the Kähler–Einstein metric which gives a positive answer to Yau's Conjecture. The main tool we use is the Schwarz–Yau Lemma. Also, to control the McMullen metric, we give a simple formula of the first derivative of the geodesic length functions.

Lemma 6.1. *The Weil–Petersson metric is bounded above by a constant multiple of the Ricci metric. Namely, there is a constant $\alpha > 0$ such that $\omega_{WP} \le \alpha\omega_\tau$.*

Proof. This lemma follows from Corollarys 4.1 and 4.2. It also follows directly from Schwarz–Yau Lemma. q.e.d.

By using this simple result, we have

Theorem 6.1. *The Ricci metric and the perturbed Ricci metric are equivalent.*

Proof. Since $\widetilde{\tau}_{i\bar{j}} = \tau_{i\bar{j}} + Ch_{i\bar{j}}$ and $C > 0$, we know that the Ricci metric is bounded above by the perturbed Ricci metric. By using the above lemma, we also have the bound of the other side. q.e.d.

By the work of Cheng and Yau [**2**], there is a unique complete Kähler–Einstein metric on the moduli space whose Ricci curvature is -1. One of the main results of this section is the equivalence of the Kähler–Einstein metric and the Ricci metric. To prove this result, we need the following simple fact of linear algebra.

Lemma 6.2. *Let A and B be positive definite $n \times n$ Hermitian matrices and let α, β be positive constants such that $B \ge \alpha A$ and $\det(B) \le \beta \det(A)$. Then, there is a constant $\gamma > 0$ depending on α, β and n such that $B \le \gamma A$.*

Theorem 6.2. *The Ricci metric is equivalent to the Kähler–Einstein metric g_{KE}.*

Proof. Consider the identity map $i : (\mathcal{M}_g, g_{KE}) \to (\mathcal{M}_g, \widetilde{\tau})$. We know that the Kähler–Einstein metric is complete and its Ricci curvature is -1. By Theorem 5.2, we know that the holomorphic sectional curvatures of the perturbed Ricci metric is bounded above by a negative constant. From the Schwarz–Yau Lemma, there is a constant $c_0 > 0$ such that

$$g_{KE} \ge c_0 \widetilde{\tau}.$$

From Theorem 6.1, we know that the Kähler–Einstein metric is bounded below by a constant multiple of the Ricci metric

$$g_{KE} \ge \tilde{c}_0 \tau. \tag{6.1}$$

Now, we consider the identity map $j : (\mathcal{M}_g, \tau) \to (\mathcal{M}_g, g_{KE})$. By Theorem 4.4, we know that the Ricci curvature of the Ricci metric is bounded from below. Also, the Ricci curvature of the Kähler–Einstein metric is -1. From the Schwarz–Yau Lemma for volume forms, there is a constant $c_1 > 0$ such that

$$\det(g_{KE}) \leq c_1 \det(\tau). \tag{6.2}$$

By combining formula (6.1), (6.2) and Lemma 6.2, we have proved the theorem. q.e.d.

Now, we consider the McMullen metric. In [**12**], McMullen constructed a new metric $g_{1/l}$ on $\mathcal{M}_g$ which is equivalent to the Teichmüller metric and is Kähler hyperbolic. More precisely, let $Log: \mathbb{R}_+ \to [0,\infty)$ be a smooth function such that

1) $Log(x) = \log x$ if $x \geq 2$;
2) $Log(x) = 0$ if $x \leq 1$.

For suitable choices of small constants $\delta, \epsilon > 0$, the Kähler form of the McMullen metric $g_{1/l}$ is

$$\omega_{1/l} = \omega_{WP} - i\delta \sum_{l_\gamma(X)<\epsilon} \partial\overline{\partial} Log \frac{\epsilon}{l_\gamma}$$

where the sum is taken over primitive short geodesics γ on X. We will also write this as ω_M.

To compare the Ricci metric and the McMullen metric, we compute the first order derivative of the short geodesics.

Lemma 6.3. *Let $X_0 \in \overline{\mathcal{M}}_g$ be a codimension m boundary point and let $(t_1, \cdots, s_n)$ be the pinching coordinates near X_0. Let l_j be the length of the geodesic on the collar Ω_c^j. Then*

$$\partial_i l_j = -\pi u_j \overline{b_i^j}$$

if $i \neq j$ and

$$\partial_i l_j = -\pi u_j \overline{b_i}$$

if $i = j$. Here, b_i^j and b_i are defined in Lemma 4.2.

Proof. It is clear that on the genuine collar Ω_c^j, λA_i is an anti-holomorphic quadratic differential. By using the rs-coordinate z on Ω_c^j, we can denote λA_i by $\kappa_i(\overline{z})d\overline{z}^2$. We consider the coefficient of the term $\overline{z}^{-2}$ in the expansion of κ_i and denote it by $C_{-2}(\kappa_i)$. From formula (4.2) and Lemma 4.2, we know that

$$C_{-2}(\kappa_i) = \frac{1}{2} u_j^2 \overline{b_i^j}. \tag{6.3}$$

Now, we use a different way to compute $C_{-2}(\kappa_i)$. Fix (t_0, s_0) with small norm and let $X = X_{t_0,s_0}$. Let w be the rs-coordinates on the j-th collar of $X_{t,s}$ and let z be the rs-coordinate on the j-th collar of X. Clearly, $w = w(z,t,s)$ is holomorphic with respect to z and when $(t,s) = (t_0,s_0)$, we have $w = z$. We pull-back the metric on the j-th collar of $X_{t,s}$ to X. We have

$$\Lambda = \frac{1}{2}u_j^2|w|^{-2}\csc^2(u_j \log|w|)\left|\frac{\partial w}{\partial z}\right|^2$$

is the Kähler–Einstein metric on the j-th collar of $X_{t,s}$. Now, from formulae (2.2) and (2.3), at point (t_0, s_0), a simple computation shows that

$$\begin{aligned}(6.4)\qquad \kappa_i(\overline{z}) = &-\frac{u_j\partial_i u_j}{\overline{z}^2} + \frac{u_j^2+1}{\overline{z}^3}\partial_i\overline{w}\mid_{(t_0,s_0)} - \frac{u_j^2+1}{\overline{z}^2}\partial_i\partial_{\overline{z}}\overline{w}\mid_{(t_0,s_0)}\\ &- \partial_i\partial_{\overline{z}}\partial_{\overline{z}}\partial_{\overline{z}}\overline{w}\mid_{(t_0,s_0)}.\end{aligned}$$

From the above formula, we can see that $C_{-2}(\kappa_i) = -u_j\partial_i u_j$ since the contribution of the last three terms in the above formula to $C_{-2}(\kappa_i)$ is 0. By comparing equations (6.3) and (6.4), we have

$$\partial_i u_j = -\frac{1}{2}u_j\overline{b_i^j}.$$

The lemma follows from the fact that $l_j = 2\pi u_j$. Again, the above argument also works when $i = j$. In this case, we replace b_i^j by b_i. q.e.d.

Now, we can prove another main theorem of this section.

Theorem 6.3. *The Ricci metric is equivalent to the McMullen metric, the Teichmüller metric and the Kobayashi metric.*

Proof. Royden proved that the Teichmüller metric is the same as the Kobayashi metric. Also, the equivalence of the McMullen metric and the Teichmüller metric was proved by McMullen [**12**]. We only need to show the equivalence between the Ricci metric and the McMullen $g_{1/l}$ metric.

Since the Ricci curvature of the $g_{1/l}$ metric is bounded from below and it is complete, by the Schwarz–Yau lemma, we know that

$$\tau < \widetilde{\tau} \leq C_0 g_{1/l}$$

for some constant C_0. Now, we prove the other bound. Fix a boundary point X_0 and the pinching coordinates near X_0. By Theorems 1.1 and 1.7 of [**12**], we know that there are constants c_1, c_2 such that, when

$i \le m$,

(6.5)
$$(g_{1/l})_{i\bar{i}} = \left\| \frac{\partial}{\partial t_i} \right\|_{g_{1/l}}^2 < c_1 \left\| \frac{\partial}{\partial t_i} \right\|_T^2 \le c_2 \left(\left\| \frac{\partial}{\partial t_i} \right\|_{WP}^2 + \sum_{l_\gamma < \epsilon} \left| (\partial \log l_\gamma) \frac{\partial}{\partial t_i} \right|^2 \right)$$
$$= c_2 \left(\left\| \frac{\partial}{\partial t_i} \right\|_{WP}^2 + \sum_{j=1}^m |\partial_i \log l_j|^2 \right).$$

By Lemma 6.3, we know that

$$|\partial_i \log l_j|^2 = \left| \frac{-\pi u_j \overline{b_i^j}}{l_j} \right|^2 = \frac{1}{4} \left| b_i^j \right|^2.$$

From Lemma 4.2, we have

$$\sum_{j=1}^m |\partial_i \log l_j|^2 = \frac{1}{4} \frac{u_i^2}{\pi^2 |t_i|^2} (1 + O(u_0)).$$

From the above formulae and Corollarys 4.1 and 4.2, we know that there is a constant c_3 such that

$$\left\| \frac{\partial}{\partial t_i} \right\|_{WP}^2 + \sum_{j=1}^m |\partial_i \log l_j|^2 \le c_3 \tau_{i\bar{i}}$$

which implies

$$(g_{1/l})_{i\bar{i}} \le c_4 \tau_{i\bar{i}} \tag{6.6}$$

where c_4 is another constant. The same argument works when $i \ge m+1$. So formula (6.6) holds for all i. Since the McMullen metric is bounded from below by a constant multiple of the Ricci metric and the diagonal terms of its metric matrix is bounded from above by a constant multiple of the diagonal terms of matrix of the Ricci metric, a simple linear algebra fact shows that there is a constant c_5 such that

$$\tau \ge c_5 g_{1/l}.$$

The theorem follows from a compactness argument as we have used in previous sections. q.e.d.

7. The Carathéodory Metric and the Bergman Metric

In this section, we prove that the Carathéodory metric and the Bergman metric on the Teichmüller space are equivalent to the Kobayashi metric by using the Bers' embedding theorem. This achieves one of our initial goals on the equivalence of all known complete metrics on the Teichmüller space. The proof of these equivalences can be applied to holomorphic homogeneous regular manifolds.

We first describe the idea. By the Bers' embedding theorem, we know that for each point p in the Teichmüller space $\mathcal{T}_g$, we can find an embedding map of the Teichmüller space into $\mathbb{C}^n$ with $n = 3g-3$ such that p is mapped to the origin and the image of the Teichmüller space contains the ball of radius 2 and is contained inside the ball of radius 6. The Kobayashi metric and the Carathéodory metric of these balls coincide and can be computed directly. Also, both of these metrics have restriction property. Roughly speaking, the metrics on a submanifold are larger than those on the ambient manifold. We use explicit form of these metrics on the balls together with this property to estimate the Kobayashi and the Carathéodory metric on the Teichmüller space and compare them on a smaller ball. On the other hand, the norm defined by the Bergman metric at each point can be estimated by using the quotient of peak sections at this point. We use upper and lower bounds of these peak sections to compare the Bergman metric, the Kobayashi metric and the Euclidean metric on a small ball in the image under the Bers' embedding of the Teichmüller space.

At first, we briefly recall the definitions of the Carathéodory, Bergman and Kobayashi metric on a complex manifold. Please see [5] for details.

Let X be a complex manifold of dimension n. let Δ_R be the disk in $\mathbb{C}$ with radius R. Let $\Delta = \Delta_1$ and let ρ be the Poincaré metric on Δ. Let $p \in X$ be a point and let $v \in T_pX$ be a holomorphic tangent vector. Let $\mathrm{Hol}(X, \Delta_R)$ and $\mathrm{Hol}(\Delta_R, X)$ be the spaces of holomorphic maps from X to Δ_R and from Δ_R to X respectively. The Carathéodory norm of the vector v is defined to be

$$\|v\|_C = \sup_{f \in \mathrm{Hol}(X,\Delta)} \|f_* v\|_{\Delta,\rho}$$

and the Kobayashi norm of v is defined to be

$$\|v\|_K = \inf_{f \in \mathrm{Hol}(\Delta_R, X),\ f(0)=p,\ f'(0)=v} \frac{2}{R}.$$

Now, we define the Bergman metric on X. Let K_X be the canonical bundle of X and let W be the space of L^2 holomorphic sections of K_X

in the sense that if $\sigma \in W$, then

$$\|\sigma\|_{L^2}^2 = \int_X (\sqrt{-1})^{n^2} \sigma \wedge \overline{\sigma} < \infty.$$

The inner product on W is defined to be

$$(\sigma, \rho) = \int_X (\sqrt{-1})^{n^2} \sigma \wedge \overline{\rho}$$

for all $\sigma, \rho \in W$. Let $\sigma_1, \sigma_2, \cdots$ be an orthonormal basis of W. The Bergman kernel form is the non-negative (n,n)-form

$$B_X = \sum_{j=1}^{\infty} (\sqrt{-1})^{n^2} \sigma_j \wedge \overline{\sigma}_j.$$

With a choice of local coordinates $z_i, \cdots, z_n$, we have

$$B_X = BE_X(z, \overline{z})(\sqrt{-1})^{n^2} dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n$$

where $BE_X(z, \overline{z})$ is called the Bergman kernel function. If the Bergman kernel B_X is positive, one can define the Bergman metric

$$B_{i\overline{j}} = \frac{\partial^2 \log BE_X(z, \overline{z})}{\partial z_i \partial \overline{z}_j}.$$

The Bergman metric is well-defined and is non-degenerate if the elements in W separate points and the first jet of X.

We will use the following notations:

Definition 7.1. *Let X be a complex space. For each point $p \in X$ and each holomorphic tangent vector $v \in T_pX$, we denote by $\|v\|_{B,X,p}$, $\|v\|_{C,X,p}$ and $\|v\|_{K,X,p}$ the norms of v measured in the Bergman metric, the Carathéodory metric and the Kobayashi metric of the space X respectively.*

Now, we fix an integer $g \geq 2$ and denote by $\mathcal{T} = \mathcal{T}_g$ the Teichmüller space of closed Riemann surface of genus g. Our main theorem of this section is the following:

Theorem 7.1. *Let $\mathcal{T}$ be the Teichmüller space of closed Riemann surfaces of genus g with $g \geq 2$. Then, there is a positive constant C only depending on g such that for each point $p \in \mathcal{T}$ and each vector $v \in T_p\mathcal{T}$, we have*

$$C^{-1}\|v\|_{K,\mathcal{T},p} \leq \|v\|_{B,\mathcal{T},p} \leq C\|v\|_{K,\mathcal{T},p}$$

and

$$C^{-1}\|v\|_{K,\mathcal{T},p} \leq \|v\|_{C,\mathcal{T},p} \leq C\|v\|_{K,\mathcal{T},p}.$$

Proof. We will show that the norms defined by these metrics are uniformly equivalent at each point of $\mathcal{T}$. We first collect some known results in the following lemma. q.e.d.

Lemma 7.1. *Let X be a complex space. Then*

1) $\|\cdot\|_{C,X} \leq \|\cdot\|_{K,X}$;
2) *Let Y be another complex space and $f : X \to Y$ be a holomorphic map. Let $p \in X$ and $v \in T_pX$. Then, $\|f_*(v)\|_{C,Y,f(p)} \leq \|v\|_{C,X,p}$ and $\|f_*(v)\|_{K,Y,f(p)} \leq \|v\|_{K,X,p}$;*
3) *If $X \subset Y$ is a connected open subset and $z \in X$ is a point. Then, with any local coordinates, we have $BE_Y(z) \leq BE_X(z)$;*
4) *If the Bergman kernel is positive, then at each point $z \in X$, a peak section σ at z exists. Such a peak section is unique up to a constant factor c with norm 1. Furthermore, with any choice of local coordinates, we have $BE_X(z) = |\sigma(z)|^2$;*
5) *If the Bergman kernel of X is positive, then $\|\cdot\|_{C,X} \leq 2\|\cdot\|_{B,X}$;*
6) *If X is a bounded convex domain in $\mathbb{C}^n$, then $\|\cdot\|_{C,X} = \|\cdot\|_{K,X}$;*
7) *Let B_r be the open ball with center 0 and radius r in $\mathbb{C}^n$. Then, for any holomorphic tangent vector v at 0,*

$$\|v\|_{C,B_r,0} = \|v\|_{K,B_r,0} = \frac{2}{r}|v|$$

where $|v|$ is the Euclidean norm of v.

Proof. The first six claims are Propositions 4.2.4, 4.2.3, 3.5.18, 4.10.4 and 4.10.3, Theorems 4.10.18 and 4.8.13 of [5].

The last claim follows from the second claim easily. By rotation, we can assume that $v = b\frac{\partial}{\partial z_1}$. Let Δ_r be the disk with radius r in $\mathbb{C}$ with standard coordinate z and let $\widetilde{v} = b\frac{\partial}{\partial z}$ be the corresponding tangent vector of Δ_r at 0. Now, consider the maps $i : \Delta_r \to B_r$ and $j : B_r \to \Delta_r$ given by $i(z) = (z, 0, \cdots, 0)$ and $j(z_1, \cdots, z_n) = z_1$. We have $i_*(\widetilde{v}) = v$ and $j_*(v) = \widetilde{v}$. By the Schwarz lemma, it is easy to see that $\|\widetilde{v}\|_{C,\Delta_r,0} = \frac{2}{r}|\widetilde{v}|$. So, we have

$$\|v\|_{C,B_r,0} \geq \|j_*(v)\|_{C,\Delta_r,0} = \|\widetilde{v}\|_{C,\Delta_r,0} = \frac{2}{r}|\widetilde{v}| = \frac{2}{r}|v|$$

and

$$\|v\|_{C,B_r,0} = \|i_*(\widetilde{v})\|_{C,B_r,0} \leq \|\widetilde{v}\|_{C,\Delta_r,0} = \frac{2}{r}|v|.$$

This shows that the last claim holds for the Carathéodory metric. By the sixth claim, we know that the last claim also holds for the Kobayashi metric. This finishes the proof. q.e.d.

Now, we prove the theorem. We first compare the Carathéodory metric and the Kobayashi metric. By the above lemma, it is easy to see that if $X \subset Y$ is a subspace, then $\|\cdot\|_{C,Y} \leq \|\cdot\|_{C,X}$ and $\|\cdot\|_{K,Y} \leq \|\cdot\|_{K,X}$. Let $p \in \mathcal{T}$ be an arbitrary point and let $n = 3g-3 = \dim_{\mathbb{C}} \mathcal{T}$. Let $f_p : \mathcal{T} \to \mathbb{C}^n$ be the Bers' embedding map with $f_p(p) = 0$. In the following, we will identify $\mathcal{T}$ with $f_p(\mathcal{T})$ and $T_p\mathcal{T}$ with $T_0\mathbb{C}^n$. We know that

$$B_2 \subset \mathcal{T} \subset B_6. \tag{7.1}$$

Let $v \in T_0\mathbb{C}^n$ be a holomorphic tangent vector. By using the above lemma, we have

$$\|v\|_{C,\mathcal{T},0} \leq \|v\|_{K,\mathcal{T},0} \tag{7.2}$$

and

$$\begin{aligned} \|v\|_{C,\mathcal{T},0} \geq \|v\|_{C,B_6,0} = \frac{1}{3}|v| &= \frac{1}{3}\|v\|_{C,B_2,0} \\ &= \frac{1}{3}\|v\|_{K,B_2,0} \geq \frac{1}{3}\|v\|_{K,\mathcal{T},0}. \end{aligned} \tag{7.3}$$

By combining the above two inequalities, we have

$$\frac{1}{3}\|v\|_{K,\mathcal{T},0} \leq \|v\|_{C,\mathcal{T},0} \leq \|v\|_{K,\mathcal{T},0}.$$

Since the above constants are independent of the choice of p, we proved the second claim of the theorem.

Now, we compare the Bergman metric and the Kobayashi metric. By the above lemma, we know that the Bergman norm is bounded from below by half of the Carathéodory norm provided the Bergman kernel is non-zero. For each point $p \in \mathcal{T}_g$, let f_p be the Bers' embedding map with $f_p(p) = 0$. Since $f_p(\mathcal{T}_g) \subset B_6$, by the above lemma, we know that $BE_{f_p(\mathcal{T}_g)}(0) \geq BE_{B_6}(0)$. However, we know that the Bergman kernel on B_6 is positive. This implies that the Bergman kernel is non-zero at every point of the Teichmüller space.

By the above lemma and the equivalence of the Carathéodory metric and the Kobayashi metric, we know that the Bergman metric is bounded from below by a constant multiple of the Kobayashi metric.

When we fix a point p and the Bers' embedding map f_p, from inequality (7.3), we know that

$$|v| \leq 3\|v\|_{C,\mathcal{T},0} \leq 3\|v\|_{K,\mathcal{T},0}. \tag{7.4}$$

Let $z_1, \cdots, z_n$ be the standard coordinates on $\mathbb{C}^n$ with $r_i = |z_i|$ and let $dV = \left(\sqrt{-1}\right)^n dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n$ be the volume form. Let

$\sigma = \alpha(z)dz_1 \wedge \cdots \wedge dz_n$ be a peak section over $\mathcal{T}$ at 0 such that

$$\int_{\mathcal{T}} |\alpha|^2 \, dV = 1.$$

Then, we have $BE_{\mathcal{T}}(0) = |\alpha(0)|^2$. Now, we consider a peak section $\sigma_1 = \alpha_1(z)dz_1 \wedge \cdots \wedge dz_n$ over B_6 at 0 with $\int_{B_6} |\alpha_1|^2 \, dV = 1$. Similarly, we have that $BE_{B_6}(0) = |\alpha_1(0)|^2$. By the above lemma and (7.1), we have

$$|\alpha(0)| = (BE_{\mathcal{T}}(0))^{\frac{1}{2}} \geq (BE_{B_6}(0))^{\frac{1}{2}} = |\alpha_1(0)|. \tag{7.5}$$

Let $v_n = \int_{B_1} dV$ be the volume of the unit ball in $\mathbb{C}^n$ and let

$$w_n = \frac{1}{n} \int_{x_1^2 + \cdots + x_n^2 \leq 4, \ x_i \geq 0} (x_1^2 + \cdots + x_n^2) x_1 \cdots x_n \, dx_1 \cdots dx_n$$

where $x_1, \cdots, x_n$ are real variables. We see that both v_n and w_n are positive constants only depending on $n = 3g - 3$.

Now, we consider the constant section $\sigma_2 = a \, dz_1 \wedge \cdots \wedge dz_n$ over B_6 where $a = 6^{-\frac{n}{2}} v_n^{-\frac{1}{2}}$. we have $\int_{B_6} a^2 \, dV = 1$. Since σ_1 is a peak section at 0, we know that $|\alpha_1(0)| \geq a$. By using inequality (7.5), we have

$$|\alpha(0)| \geq 6^{-\frac{n}{2}} v_n^{-\frac{1}{2}}. \tag{7.6}$$

To estimate the Bergman norm of v, by rotation, we may assume $v = b \frac{\partial}{\partial z_1}$. So $|v| = |b|$. Let $\tau = f(z)dz_1 \wedge \cdots \wedge dz_n$ be an arbitrary section over $\mathcal{T}$ with $f(0) = 0$ and $\int_{\mathcal{T}} |f|^2 \, dV = 1$. We have $\int_{B_2} |f|^2 \, dV \leq 1$.

Let I be the index set $I = \{(i_1, \cdots, i_n) \mid i_k \geq 0, \ \sum i_k \geq 1\}$. Since $f(0) = 0$ and f is holomorphic, we can expand f as a power series on B_2 as

$$f(z) = \sum_{(i_1, \cdots, i_n) \in I} a_{i_1 \cdots i_n} z_1^{i_1} \cdots z_n^{i_n}.$$

This implies $df(v) = a_{10\cdots0} b$. Since $\int_{B_2} |f|^2 \, dV \leq \int_{\mathcal{T}} |f|^2 \, dV = 1$, we have

$$\begin{aligned} 1 \geq \int_{B_2} |f|^2 \, dV &= \int_{B_2} \sum_{(i_1, \cdots, i_n) \in I} |a_{i_1 \cdots i_n}|^2 r_1^{2i_1} \cdots r_n^{2i_n} \, dV \\ &\geq \int_{B_2} |a_{10\cdots0}|^2 r_1^2 \, dV \\ &= |a_{10\cdots0}|^2 (4\pi)^n \int_{r_1^2 + \cdots + r_n^2 \leq 4} r_1^3 r_2 \cdots r_n \, dr_1 \cdots dr_n \\ &= |a_{10\cdots0}|^2 (4\pi)^n w_n \end{aligned}$$

which implies

$$|a_{10\cdots 0}| \leq (4\pi)^{-\frac{n}{2}} w_n^{-\frac{1}{2}}. \tag{7.7}$$

So, we have

$$|df(v)| = |a_{10\cdots 0}||b| \leq (4\pi)^{-\frac{n}{2}} w_n^{-\frac{1}{2}} |v|. \tag{7.8}$$

Let W' be the set of sections over $\mathcal{T}$ such that

$$W' = \{\tau = f(z)dz_1 \wedge \cdots \wedge dz_n \mid f(0) = 0, \int_{\mathcal{T}} |f|^2 \, dV = 1\}.$$

By combining (7.4), (7.6) and (7.8), we have

$$\begin{aligned} \|v\|_{B,\mathcal{T},0} = \sup_{\tau \in W'} \frac{|df(v)|}{|\alpha(0)|} &\leq \frac{(4\pi)^{-\frac{n}{2}} w_n^{-\frac{1}{2}} |v|}{6^{-\frac{n}{2}} v_n^{-\frac{1}{2}}} = \left(\frac{3}{2\pi}\right)^{\frac{n}{2}} \left(\frac{v_n}{w_n}\right)^{\frac{1}{2}} |v| \\ &\leq 3 \left(\frac{3}{2\pi}\right)^{\frac{n}{2}} \left(\frac{v_n}{w_n}\right)^{\frac{1}{2}} \|v\|_{K,\mathcal{T},0}. \end{aligned} \tag{7.9}$$

Since the constant in the above inequality only depends on the dimension n, we know that the Bergman metric is uniformly equivalent to the Kobayashi metric. This finished the proof. q.e.d.

Remark 7.1. After we proved this theorem, the second author was informed by McMullen that the equivalence of the Carathéodory metric and the Kobayashi metric may be already known. A more interesting question is whether these two metrics coincide or not. We would like to study this problem in the future.

Finally, we introduce the notion of holomorphic homogeneous regular manifolds. This generalizes the idea of Morrey.

Definition 7.2. *A complex manifold X of dimension n is called holomorphic homogeneous regular if there are positive constants $r < R$ such that for each point $p \in X$, there is a holomorphic map $f_p : X \to \mathbb{C}^n$ which satisfies*

1) $f_p(p) = 0$;
2) $f_p : X \to f_p(X)$ *is a biholomorphism;*
3) $B_r \subset f_p(X) \subset B_R$ *where B_r and B_R are Euclidean balls with center 0 in $\mathbb{C}^n$.*

The following theorem follows from the proof of Theorem 7.1 directly.

Theorem 7.2. *Let X be a holomorphic homogeneous regular manifold. Then, the Kobayashi metric, the Bergman metric and the Carathéodory metric on X are equivalent.*

We will study in detail the holomorphic homogeneous regular manifolds and the possible complete Kähler–Einstein metric on them in our future paper [8].

8. Appendix: the proof of Lemma 4.10

We will prove Lemma 4.10 in this appendix which consists of some computational details. We fix a nodal surface X_0 which corresponding to a codimension m boundary point in $\mathcal{M}_g$. Let (t,s) be the pinching coordinates near X_0 such that $X_{0,0}=X_0$. Fix (t,s) with small norm, we denote $X_{t,s}$ by X. In the curvature formula (3.28), we let $i=j=k=l\le m$. The term G_2 is a summation of the following four types of terms:

1) $I=h^{\alpha\overline{\beta}}\left\{\sigma_1\sigma_2\int_X\left\{T(\xi_k(e_{i\overline{j}}))\overline{\xi}_l(e_{\alpha\overline{\beta}})+T(\xi_k(e_{i\overline{j}}))\overline{\xi}_\beta(e_{\alpha\overline{l}})\right\}\ dv\right\}$ with $(\alpha,\beta)\neq(i,i)$;
2) $II=h^{\alpha\overline{\beta}}\left\{\sigma_1\int_{X_s}Q_{k\overline{l}}(e_{i\overline{j}})e_{\alpha\overline{\beta}}\ dv\right\}$ with $(\alpha,\beta)\neq(i,i)$;
3) $III=\tau^{p\overline{q}}h^{\alpha\overline{\beta}}h^{\gamma\overline{\delta}}\left\{\sigma_1\int_{X_s}\xi_k(e_{i\overline{q}})e_{\alpha\overline{\beta}}\ dv\right\}\left\{\widetilde{\sigma}_1\int_{X_s}\overline{\xi}_l(e_{p\overline{j}})e_{\gamma\overline{\delta}})\ dv\right\}$ with $(p,q,\alpha,\beta,\gamma,\delta)\neq(i,i,i,i,i,i)$;
4) $IV=\tau_{p\overline{j}}h^{p\overline{q}}R_{i\overline{q}k\overline{l}}$ with $(p,q)\neq(i,i)$

where $T=(\square+1)^{-1}$. Now, we check that the norm of each type is bounded by $O\big(\frac{u_i^5}{|t_i|^4}\big)$. In the following, C_0 will be a universal constant which may change, but is independent of the Riemann surface as long as (t,s) has small norm.

Case 1. We check that each term in the sum IV has the desired bound. By Corollary 4.2 and its proof, we have

$$R_{i\overline{q}i\overline{i}}=\begin{cases}O\big(\frac{u_i^5}{|t_i|^3}\big),\ \text{if } q\ge m+1;\\ O\big(\frac{u_i^5u_q^3}{|t_i|^3|t_q|}\big),\ \text{if } q\le m,\ \text{and } q\neq i;\\ O\big(\frac{u_i^5}{|t_i|^4}\big),\ \text{if } q=i.\end{cases}$$

By using the above formula and Corollarys 4.1 and 4.2, and by a case by case check, we have

$$|\tau_{p\overline{i}}h^{p\overline{q}}R_{i\overline{q}i\overline{i}}|=O\big(\frac{u_i^7}{|t_i|^4}\big).$$

This proves that the norm of the last term is bounded by $=O\big(\frac{u_i^5}{|t_i|^4}\big)$.

Case 2. We check that each term in the sum I has the desired bound. Firstly, when $i = j = k = l$, we have

$$
\begin{aligned}
(8.1)\quad & \sigma_1\sigma_2 \left\{ T(\xi_k(e_{i\bar{j}}))\overline{\xi}_l(e_{\alpha\overline{\beta}}) + T(\xi_k(e_{i\bar{j}}))\overline{\xi}_\beta(e_{\alpha\bar{l}}) \right\} \\
&= 2\left\{ T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{\alpha\overline{\beta}}) + 2T(\xi_i(e_{i\overline{\beta}}))\overline{\xi}_i(e_{\alpha\bar{i}}) + T(\xi_i(e_{i\bar{i}}))\overline{\xi}_\beta(e_{\alpha\bar{i}}) \right\} \\
&\quad + 2\left\{ T(\xi_i(e_{\alpha\bar{i}}))\overline{\xi}_i(e_{i\overline{\beta}}) + 2T(\xi_i(e_{\alpha\overline{\beta}}))\overline{\xi}_i(e_{i\bar{i}}) + T(\xi_i(e_{\alpha\bar{i}}))\overline{\xi}_\beta(e_{i\bar{i}}) \right\} \\
&\quad + 2\left\{ T(\xi_\alpha(e_{i\bar{i}}))\overline{\xi}_i(e_{i\overline{\beta}}) + T(\xi_\alpha(e_{i\overline{\beta}}))\overline{\xi}_i(e_{i\bar{i}}) + T(\xi_\alpha(e_{i\bar{i}}))\overline{\xi}_\beta(e_{i\bar{i}}) \right\} \\
&\quad + 2T(\xi_\alpha(e_{i\overline{\beta}}))\overline{\xi}_i(e_{i\bar{i}}).
\end{aligned}
$$

We estimate the integration of each term in the above summation. To estimate these terms, we note that, if $\alpha \neq \beta$ or $\alpha = \beta \geq m+1$, then

$$\left| h^{\alpha\overline{\beta}} \|f_{\alpha\overline{\beta}}\|_1 \right| = O(1). \tag{8.2}$$

Also, we have

$$\|P(e_{\alpha\overline{\beta}})\|_0 \leq \|e_{\alpha\overline{\beta}}\|_2 \leq C_0 \|f_{\alpha\overline{\beta}}\|_1. \tag{8.3}$$

These formulae can be checked easily by using Theorem 4.1, Corollary 4.1, Lemmas 4.3 and 4.7.

Now, we estimate $\left| h^{\alpha\overline{\beta}} \int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{\alpha\overline{\beta}})\, dv \right|$. If $\alpha \neq \beta$ or $\alpha = \beta \geq m+1$, we have

$$
\begin{aligned}
\left| \int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{\alpha\overline{\beta}})\, dv \right| &\leq \left(\int_X |T(\xi_i(e_{i\bar{i}}))|^2\, dv \int_X |\overline{\xi}_i(e_{\alpha\overline{\beta}})|^2\, dv \right)^{\frac{1}{2}} \\
&\leq \left(\int_X |\xi_i(e_{i\bar{i}})|^2\, dv \int_X |\overline{\xi}_i(e_{\alpha\overline{\beta}})|^2\, dv \right)^{\frac{1}{2}} \\
&= \left(\int_X f_{i\bar{i}} |P(e_{i\bar{i}})|^2\, dv \int_X f_{i\bar{i}} |P(e_{\alpha\overline{\beta}})|^2\, dv \right)^{\frac{1}{2}} \\
&\leq \|P(e_{i\bar{i}})\|_0 \|P(e_{\alpha\overline{\beta}})\|_0 h_{i\bar{i}} \leq C_0 \|f_{i\bar{i}}\|_1 \|f_{\alpha\overline{\beta}}\|_1 h_{i\bar{i}} = O\left(\frac{u_i^5}{|t_i|^4} \right) \|f_{\alpha\overline{\beta}}\|_1
\end{aligned}
$$

since $\|f_{i\bar{i}}\|_1 = O\big(\frac{u_i^2}{|t_i|^2}\big)$. Together with formula (8.2), we have

$$\left| h^{\alpha\overline{\beta}} \int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{\alpha\overline{\beta}})\, dv \right| = O\left(\frac{u_i^5}{|t_i|^4} \right).$$

If $\alpha = \beta \le m$ and $\alpha \ne i$, we have

$$\left|\int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{\alpha\overline{\alpha}})\,dv\right| \le \left|\int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(\widetilde{e_{\alpha\overline{\alpha}}})\,dv\right| + \left|\int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{\alpha\overline{\alpha}} - \widetilde{e_{\alpha\overline{\alpha}}})\,dv\right|. \tag{8.4}$$

From Lemma 4.7, we have

$$\|P(e_{\alpha\overline{\alpha}} - \widetilde{e_{\alpha\overline{\alpha}}})\|_0 \le \|e_{\alpha\overline{\alpha}} - \widetilde{e_{\alpha\overline{\alpha}}}\|_2 \le \|f_{\alpha\overline{\alpha}} - \widetilde{e_{\alpha\overline{\alpha}}}\|_1 = O\left(\frac{u_\alpha^4}{|t_\alpha|^2}\right).$$

So

$$\begin{aligned}
&\left|\int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{\alpha\overline{\alpha}} - \widetilde{e_{\alpha\overline{\alpha}}})\,dv\right| \\
&\le \|P(e_{\alpha\overline{\alpha}} - \widetilde{e_{\alpha\overline{\alpha}}})\|_0 \left|\int_X |T(\xi_i(e_{i\bar{i}}))||A_i|\,dv\right| \\
&\le \|P(e_{\alpha\overline{\alpha}} - \widetilde{e_{\alpha\overline{\alpha}}})\|_0 \left(\int_X |T(\xi_i(e_{i\bar{i}}))|^2\,dv \int_X f_{i\bar{i}}\,dv\right)^{\frac{1}{2}} \\
&\le \|P(e_{\alpha\overline{\alpha}} - \widetilde{e_{\alpha\overline{\alpha}}})\|_0 \left(\int_X |\xi_i(e_{i\bar{i}})|^2\,dv \int_X f_{i\bar{i}}\,dv\right)^{\frac{1}{2}} \\
&= \|P(e_{\alpha\overline{\alpha}} - \widetilde{e_{\alpha\overline{\alpha}}})\|_0 \left(\int_X f_{i\bar{i}}|P(e_{i\bar{i}})|^2\,dv \int_X f_{i\bar{i}}\,dv\right)^{\frac{1}{2}} \\
&\le \|P(e_{\alpha\overline{\alpha}} - \widetilde{e_{\alpha\overline{\alpha}}})\|_0 \|e_{i\bar{i}}\|_2 h_{i\bar{i}} = O\left(\frac{u_\alpha^4}{|t_\alpha|^2}\right) O\left(\frac{u_i^5}{|t_i|^4}\right).
\end{aligned} \tag{8.5}$$

Since the support of $\widetilde{e_{\alpha\overline{\alpha}}}$ is inside Ω_c^α, we know the support of $P(\widetilde{e_{\alpha\overline{\alpha}}})$ is inside Ω_c^α. From Lemma 4.8, we have

$$\begin{aligned}
&\left|\int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(\widetilde{e_{\alpha\overline{\alpha}}})\,dv\right| = \left|\int_{\Omega_c^\alpha} T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(\widetilde{e_{\alpha\overline{\alpha}}})\,dv\right| \\
&\le \|A_i\|_{0,\Omega_c^\alpha} \|T(\xi_i(e_{i\bar{i}}))\|_0 |P(\widetilde{e_{\alpha\overline{\alpha}}})|_{L^1} \le \|A_i\|_{0,\Omega_c^\alpha} \|\xi_i(e_{i\bar{i}})\|_0 |P(\widetilde{e_{\alpha\overline{\alpha}}})|_{L^1} \\
&= \|A_i\|_{0,\Omega_c^\alpha} \|A_i\|_0 \|P(e_{i\bar{i}})\|_0 |P(\widetilde{e_{\alpha\overline{\alpha}}})|_{L^1} \\
&= O\left(\frac{u_i^3}{|t_i|}\right) O\left(\frac{u_i}{|t_i|}\right) O\left(\frac{u_i^2}{|t_i|^2}\right) O\left(\frac{u_\alpha^3}{|t_\alpha|^2}\right) \\
&= O\left(\frac{u_i^6}{|t_i|^4}\right) O\left(\frac{u_\alpha^3}{|t_\alpha|^2}\right).
\end{aligned} \tag{8.6}$$

By combining the inequalities (8.5) and (8.6), we know that

$$\left|\int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{\alpha\overline{\alpha}})\ dv\right| = O\big(\frac{u_i^5}{|t_i|^4}\big)O\big(\frac{u_\alpha^3}{|t_\alpha|^2}\big).$$

From Lemma 4.1, we have

$$\left|h^{\alpha\overline{\alpha}}\int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{\alpha\overline{\alpha}})\ dv\right| = O\big(\frac{u_i^5}{|t_i|^4}\big).$$

We finish the estimate of the first term in the sum (8.1). The integration of other terms in this sum can be estimated in a similar way.

Case 3. We check that each term in the sum III has the desired bound. By Lemma 4.2, we first prove that when $q \neq i$ and $k = i$,

$$\left|h^{\alpha\overline{\beta}}\left\{\sigma_1\int_X \xi_k(e_{i\overline{q}})e_{\alpha\overline{\beta}}\ dv\right\}\right| = \begin{cases} O\big(\frac{u_i^{\frac{5}{2}}}{|t_i|^2}\big)O\big(\frac{u_q}{|t_q|}\big) \text{ if } q \leq m \\ O\big(\frac{u_i^{\frac{5}{2}}}{|t_i|^2}\big) \text{ if } q \geq m+1 \end{cases} \tag{8.7}$$

Again, we do a case by case check. First, we estimate $\left|h^{\alpha\overline{\beta}}\int_X \xi_i(e_{i\overline{q}})e_{\alpha\overline{\beta}}\ dv\right|$. If $\alpha \neq \beta$ or $\alpha = \beta \geq m+1$, we have

$$\begin{aligned}\left|\int_X \xi_i(e_{i\overline{q}})e_{\alpha\overline{\beta}}\ dv\right| &= \left|\int_X e_{i\overline{q}}\xi_i(e_{\alpha\overline{\beta}})\ dv\right| \\ &\leq \left(\int_X |\xi_i(e_{\alpha\overline{\beta}})|^2\ dv \int_X |e_{i\overline{q}}|^2\ dv\right)^{\frac{1}{2}} \\ &\leq \left(\int_X f_{i\bar{i}}|P(e_{\alpha\overline{\beta}})|^2\ dv \int_X |f_{i\overline{q}}|^2\ dv\right)^{\frac{1}{2}} \\ &\leq \|P(e_{\alpha\overline{\beta}})\|_0\left(\int_X f_{i\bar{i}}\ dv\int_X f_{i\bar{i}}f_{q\overline{q}}\ dv\right)^{\frac{1}{2}} \\ &\leq \|P(e_{\alpha\overline{\beta}})\|_0\|A_q\|_0 h_{i\bar{i}} = O\big(\frac{u_i^3}{|t_i|^2}\big)\|f_{\alpha\overline{\beta}}\|_1\|A_q\|_0.\end{aligned} \tag{8.8}$$

This implies

$$\left|h^{\alpha\overline{\beta}}\int_X \xi_i(e_{i\overline{q}})e_{\alpha\overline{\beta}}\ dv\right| = O\big(\frac{u_i^3}{|t_i|^2}\big)\|A_q\|_0.$$

If $\alpha = \beta \leq m$ and $\alpha \neq i$, we have

$$\left|\int_X \xi_i(e_{i\overline{q}})e_{\alpha\overline{\alpha}}\ dv\right| \leq \left|\int_X \xi_i(e_{i\overline{q}})\widetilde{e_{\alpha\overline{\alpha}}}\ dv\right| + \left|\int_X \xi_i(e_{i\overline{q}})(e_{\alpha\overline{\alpha}} - \widetilde{e_{\alpha\overline{\alpha}}})\ dv\right|.$$

For the second term in the above formula, we have

$$\begin{aligned}
&\left|\int_X \xi_i(e_{i\overline{q}})(e_{\alpha\overline{\alpha}} - \widetilde{e_{\alpha\overline{\alpha}}})\ dv\right| \\
&= \left|\int_X e_{i\overline{q}}\xi_i(e_{\alpha\overline{\alpha}} - \widetilde{e_{\alpha\overline{\alpha}}})\ dv\right| \leq \left(\int_X |e_{i\overline{q}}|^2\ dv \int_X |\xi_i(e_{\alpha\overline{\alpha}} - \widetilde{e_{\alpha\overline{\alpha}}})|^2\ dv\right)^{\frac{1}{2}} \\
&\leq \left(\int_X |f_{i\overline{q}}|^2\ dv \int_X f_{i\overline{i}}|P(e_{\alpha\overline{\alpha}} - \widetilde{e_{\alpha\overline{\alpha}}})|^2\ dv\right)^{\frac{1}{2}} \\
&\leq \|P(e_{\alpha\overline{\alpha}} - \widetilde{e_{\alpha\overline{\alpha}}})\|_0 \left(\int_X f_{i\overline{i}}f_{q\overline{q}}\ dv \int_X f_{i\overline{i}}\ dv\right)^{\frac{1}{2}} \leq \|e_{\alpha\overline{\alpha}} - \widetilde{e_{\alpha\overline{\alpha}}}\|_2\|A_q\|_0 h_{i\overline{i}} \\
&\leq \|f_{\alpha\overline{\alpha}} - \widetilde{e_{\alpha\overline{\alpha}}}\|_2\|A_q\|_0 h_{i\overline{i}} = O\left(\frac{u_\alpha^4}{|t_\alpha|^2}\right) O\left(\frac{u_i^3}{|t_i|^2}\right)\|A_q\|_0.
\end{aligned}$$

For the first term in the above formula, we have

$$\begin{aligned}
&\left|\int_X \xi_i(e_{i\overline{q}})\widetilde{e_{\alpha\overline{\alpha}}}\ dv\right| = \left|\int_{\Omega_c^\alpha} \xi_i(e_{i\overline{q}})\widetilde{e_{\alpha\overline{\alpha}}}\ dv\right| \leq \|A_i\|_{0,\Omega_c^\alpha}\|P(e_{i\overline{q}})\|_0 \int_{\Omega_c^\alpha} \widetilde{e_{\alpha\overline{\alpha}}}\ dv \\
&\quad \leq \|A_i\|_{0,\Omega_c^\alpha}\|e_{i\overline{q}}\|_2 \int_{\Omega_c^\alpha} \widetilde{e_{\alpha\overline{\alpha}}}\ dv \leq O\big(\frac{u_\alpha^3}{|t_\alpha|^2}\big)O\big(\frac{u_i^3}{|t_i|}\big)\|f_{i\overline{q}}\|_1.
\end{aligned}$$

By combining the above two formulae, we have the desired bound for $\left|h^{\alpha\overline{\alpha}}\int_X \xi_i(e_{i\overline{q}})e_{\alpha\overline{\alpha}}\ dv\right|$.

When $\alpha = \beta = i$, by using a similar method, we can show that $\left|h^{i\overline{i}}\int_X \xi_i(e_{i\overline{q}})e_{i\overline{i}}\ dv\right| = O\big(\frac{u_i^3}{|t_i|^2}\big)\|A_q\|_0$. From the above estimates, we have proved that the term $\left|h^{\alpha\overline{\beta}}\int_X \xi_i(e_{i\overline{q}})e_{\alpha\overline{\beta}}\ dv\right|$ in formula (8.7) has the desired estimate. By using similar method, we can show that the other terms in (8.7) have the desired estimate. This proves formula (8.7).

In a similar way, in the case $q = i$, we can prove that, when $k = i$,

$$(8.9)\qquad \left|h^{\alpha\overline{\beta}}\left\{\sigma_1\int_X \xi_k(e_{i\overline{q}})e_{\alpha\overline{\beta}}\ dv\right\}\right| = \begin{cases} O\big(\frac{u_i^3}{|t_i|^3}\big), & \text{if } \alpha = \beta = i; \\ O\big(\frac{u_i^4}{|t_i|^3}\big), & \text{if } \alpha \neq i \text{ or } \beta \neq i. \end{cases}$$

By combining formulas (8.8) and (8.9), we conclude that each term in the sum III is of order $O\big(\frac{u_i^5}{|t_i|^4}\big)$.

Case 4. We need to show that each term in the sum II is of order $O\big(\frac{u_i^5}{|t_i|^4}\big)$. This case can be proved by a case by case check by using the similar estimates as in the third case together with Lemma 4.9. This finishes the proof. q.e.d.

Remark 8.1. The method we estimate these terms can be directly applied to the computations of the full curvature tensor and we can get certain bounds for the bisectional curvature and the Ricci curvature of the Ricci metric as well as the perturbed Ricci metric.

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References

[1] L. Ahlfors & L. Bers. Riemann's mapping theorem for variable metrics. *Ann. of Math. (2)*, 72:385–404, 1960, MR0115006 (22 #5813), Zbl 0104.29902.

[2] S. Y. Cheng & S. T. Yau. On the existence of a complete Kähler metric on non-compact complex manifolds and the regularity of Fefferman's equation. *Comm. Pure Appl. Math.*, 33(4):507–544, 1980, MR0575736 (82f:53074), Zbl 0506.53031.

[3] P. Deligne & D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969, MR0262240 (41 #6850), Zbl 0181.48803.

[4] L. Keen. Collars on Riemann surfaces. In *Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973)*, pages 263–268. Ann. of Math. Studies, No. 79. Princeton Univ. Press, Princeton, N.J., 1974, MR0379833 (52 #738), Zbl 0304.30014.

[5] S. Kobayashi. *Hyperbolic complex spaces*, volume 318 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1998, MR1635983 (99m:32026), Zbl 0917.32019.

[6] K. Liu. Remarks on the geometry of moduli spaces. *Proc. Amer. Math. Soc.*, 124(3):689–695, 1996, MR1291785 (96f:32032), Zbl 0873.14031.

[7] K. Liu, X. Sun, & S.-T. Yau. Canonical metrics on the moduli space of Riemann surface II. to appear in J. Differential Geometry, 2004.

[8] K. Liu, X. Sun, & S.-T. Yau. Good metrics on the moduli space of riemann surface. Preprint, 2004.

[9] K. Liu, X. Sun, & S.-T. Yau. Geometric aspects of the moduli space of Riemann surfaces. math.DG/0411247

[10] Z. Lu & X. Sun. On the Weil–Petersson volume and the first chern class o of the Moduli Spaces of Calabi–Yau Manifolds. to appear in Communications in Mathematical Physics, 2003.

[11] H. Masur. Extension of the Weil–Petersson metric to the boundary of Teichmuller space. *Duke Math. J.*, 43(3):623–635, 1976, MR0417456 (54 #5506), Zbl 0358.32017.

[12] C. T. McMullen. The moduli space of Riemann surfaces is Kähler hyperbolic. *Ann. of Math. (2)*, 151(1):327–357, 2000, MR1745010 (2001m:32032), Zbl 0988.32012.

[13] G. Schumacher. The curvature of the Petersson–Weil metric on the moduli space of Kähler–Einstein manifolds. In *Complex analysis and geometry*, Univ. Ser. Math., pages 339–354. Plenum, New York, 1993, MR1211890 (94g:32038), Zbl 0804.53094.

[14] Y. T. Siu. Curvature of the Weil–Petersson metric in the moduli space of compact Kähler–Einstein manifolds of negative first Chern class. In *Contributions to several complex variables*, Aspects Math., E9, pages 261–298. Vieweg, Braunschweig, 1986, MR0859202 (87k:32042), Zbl 0598.32020.

[15] S. Trapani. On the determinant of the bundle of meromorphic quadratic differentials on the Deligne–Mumford compactification of the moduli space of Riemann surfaces. *Math. Ann.*, 293(4):681–705, 1992, MR1176026 (93g:58157), Zbl 0735.32017.

[16] S. A. Wolpert. Chern forms and the Riemann tensor for the moduli space of curves. *Invent. Math.*, 85(1):119–145, 1986, MR0842050 (87j:32070), Zbl 0595.32031.

[17] S. A. Wolpert. Asymptotics of the spectrum and the Selberg zeta function on the space of Riemann surfaces. *Comm. Math. Phys.*, 112(2):283–315, 1987, MR0905169 (89c:58136), Zbl 0629.58029.

[18] S. A. Wolpert. The hyperbolic metric and the geometry of the universal curve. *J. Differential Geometry*, 31(2):417–472, 1990, MR1037410 (91a:32030), Zbl 0698.53002.

[19] S. T. Yau. A general Schwarz lemma for Kähler manifolds. *Amer. J. Math.*, 100(1):197–203, 1978, MR0486659 (58 #6370), Zbl 0424.53040.

[20] S.-T. Yau. *Nonlinear analysis in geometry*, volume 33 of *Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique]*. L'Enseignement Mathématique, Geneva, 1986. Série des Conférences de l'Union Mathématique Internationale [Lecture Series of the International Mathematics Union], 8, MR0865650 (88e:53001), Zbl 0631.53003.

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