J. DIFFERENTIAL GEOMETRY
68 (2004) 433-514

A RIGIDITY THEOREM FOR STABLE MINIMAL HYPERCONES

Neshan Wickramasekera

Abstract

We prove that any cone having vertex density strictly between 1 and 3, occurring as the weak limit of a sequence of oriented, immersed, stable minimal hypersurfaces and lying near a pair of hyperplanes must itself be a pair of hyperplanes.

1. Introduction

Our goal in this paper is to prove a rigidity theorem (Theorem 1.1) for certain cones that arise as weak (i.e., varifold) limits of oriented, smooth, immersed, stable, minimal hypersurfaces in $B_2^{n+1}(0)$, the open ball in the $(n+1)$-dimensional Euclidean space $\mathbf{R}^{n+1}$ $(n \geq 2)$ with radius 2 and center the origin. In order to state the theorem precisely, we introduce the following notation and terminology.

Let $\mathcal{I}$ denote the family of oriented, smooth, immersed, stable, minimal hypersurfaces in $B_2^{n+1}(0)$ of finite volume. Let $\overline{\mathcal{I}}$ be the closure of $\mathcal{I}$ in varifold topology. (See Section 2 for a brief discussion of varifolds.) By a *pair of hyperplanes*, we mean either the sum (as varifolds) of two transversely intersecting hyperplanes of $\mathbf{R}^{n+1}$ or a single hyperplane of $\mathbf{R}^{n+1}$ with multiplicity 2.

The theorem we shall prove is the following.

Theorem 1.1. *Let $\alpha \in (0,2)$. There exists a number $\epsilon_0 = \epsilon_0(n,\alpha) > 0$, depending only on n and α, such that the following is true. Suppose $\mathbf{C} \in \overline{\mathcal{I}}$ is a cone with $1 < \Theta(\|\mathbf{C}\|, 0) \leq 3 - \alpha$ and*

$$\mathrm{dist}_{\mathcal{H}}\,(\mathrm{spt}\,\|\mathbf{C}\| \cap \overline{B}_2^{n+1}(0), \mathrm{spt}\,\|\mathbf{P}\| \cap \overline{B}_2^{n+1}(0)) \;\leq\; \epsilon_0$$

for some pair of hyperplanes $\mathbf{P}$; namely, $\mathbf{P}$ is either

(a) *a multiplicity two hyperplane or*
(b) *the sum of two transverse hyperplanes.*

Received 06/09/2003.

Then $\mathbf{C}$ *is equal to a pair of hyperplanes. Here* $\Theta(\|\mathbf{C}\|, 0)$ *denotes the density of* $\mathbf{C}$ *at the origin and* $\mathrm{dist}_{\mathcal{H}}(S, T)$ *denotes the Hausdorff distance between the sets* S *and* T*.*

It is well known by Allard's regularity theorem ([**1**], Section 8; [**10**], Theorem 23.1; [**3**], Theorem 2.1) that if a stationary n-varifold (of arbitrary codimension) is a cone having vertex density greater than or equal to 1 and less than or equal to $2 - \alpha$, where $\alpha \in (0, 1]$ is a given number, and if it is weakly sufficiently close (depending on α) to an n-dimensional plane, then it must be equal to an n-dimensional plane. Thus, for cones in $\overline{\mathcal{I}}$, our theorem is an extension of this result.

Theorem 1.1 can be applied to understanding the local behavior of certain stable varifolds. It gives results concerning the types of tangent cones at low density singularities (i.e., those with density not much greater than 2) of an arbitrary varifold $V \in \overline{\mathcal{I}}$. These and other applications of the theorem will appear in [**13**].

Remark. We shall first prove case (a) of the theorem; i.e., case when $\mathbf{P}$ is a multiplicity 2 hyperplane. Case (b), in which $\mathbf{P}$ is a transverse pair of hyperplanes, will follow from case (a) and (slightly modified) results of Section 6. Section 7 will provide a detailed account of the proof of case (b).

In view of the preceding remark, until the end of Section 6 of the paper, we assume that $\mathbf{P}$ is a multiplicity two hyperplane, and without loss of generality, we take $\mathbf{P} = \mathbf{v}(\mathbf{R}^n, 2)$, multiplicity 2 varifold associated with $\mathbf{R}^n$, the hyperplane $\{x^{n+1} = 0\}$ of $\mathbf{R}^{n+1}$.

The proof of Theorem 1.1 is based on two separate "blow-up" processes. The idea is as follows. Suppose the theorem is false. Then, we can satisfy the following.

HYPOTHESES $(\star)$

(1) $\alpha \in (0, 2)$.
(2) $\mathbf{C}_k \in \overline{\mathcal{I}}$ is a sequence of cones with $1 < \Theta(\|\mathbf{C}_k\|, 0) \leq 3 - \alpha$.
(3) $\mathbf{C}_k \neq$ a pair of hyperplanes for each k.
(4) $\mathrm{spt}\, \|\mathbf{C}_k\| \cap \overline{B}_2^{n+1}(0)$ converges to the ball $\overline{B}_2^n(0) \subseteq \mathbf{R}^n$ in the sense of Hausdorff distance.
(5) For each k, there exists a smooth, immersed, stable, hypersurface M_k in $B_2^{n+1}(0)$ approximating $\mathbf{C}_k$ sufficiently closely. (This means that $M_k = M_k^{(j)}$ for any $j \geq j_0(k)$, where for each k, $\{M_k^{(j)}\}$ is a sequence of immersed, stable hypersurfaces converging as varifolds to $\mathbf{C}_k$ and $j_0(k)$ is chosen sufficiently large depending on the requirements to be specified during the course of the proof. In particular, we require that $d_{\mathcal{H}}\left(\mathrm{spt}\, \|\mathbf{C}_k\| \cap \overline{B}_2^{n+1}(0), M_k^{(j)} \cap \overline{B}_2^{n+1}(0)\right) \leq$

$\delta(k)$ for all $j \geq j_0(k)$ where $\delta(k) \searrow 0$ as $k \to \infty$ sufficiently rapidly. By (4) above, this of course implies that $M_k \cap \overline{B}_2^{n+1}(0)$ converge to $\overline{B}_2^n(0) \subseteq \mathbf{R}^n$ in Hausdorff distance. Such M_k exist by the definition of $\overline{\mathcal{I}}$.)

We shall prove that under HYPOTHESES ($\star$), for infinitely many k, M_k must be equal to a pair of transverse hyperplanes. This means that for infinitely many k, $\mathbf{C}_k$ is the varifold limit of a sequence of pairs of hyperplanes, and hence, $\mathbf{C}_k$ must itself be a pair of hyperplanes, giving the contradiction necessary to prove the theorem.

The first step of the proof involves obtaining a two-valued approximate graphical decomposition of M_k over the unit ball $B_1^n(0)$ of $\mathbf{R}^n$. We do this in Section 3, following exactly the methods used by Schoen and Simon in [**8**], and often just quoting the relevant results from [**8**].

We then obtain, in Section 4, a (two-valued) blow-up of these graphs by blowing them up by the tilt-excess of $M_k \cap (B_1^n(0) \times \mathbf{R})$ relative to the hyperplane $\mathbf{R}^n$. In Section 4, we also establish several key properties satisfied by the blow-up.

In Section 5, we prove that this blow-up is equal to the union of two transverse hyperplanes of $\mathbf{R}^{n+1}$. Our analysis uses a dimension reducing argument based on the monotonicity of a frequency function, a technique first used by F. J. Almgren, Jr. in [**2**] in his work on multi-valued functions minimizing the Dirichlet integral. Our setting, however, differs from that of [**2**] in two important aspects. On the one hand, we are in codimension 1, which is much simpler than the arbitrary codimension dealt with in [**2**]. On the other hand, we do not have the minimizing hypothesis. Our arguments demonstrate that a frequency function can nevertheless be used under the weaker hypothesis of stability.

In Section 6, we blow up the M_k's again, but this time by their (finer) excess measured relative to the (appropriately vertically scaled) hyperplanes of the first blow up. This is done following very closely the work of Simon in [**11**], and we show that this second blow-up is also equal to a pair of hyperplanes. This leads to a contradiction unless the M_k are pairs of hyperplanes for infinitely many k, completing the proof of the theorem in case (a).

Finally, in Section 7, we indicate how case (b) of the theorem follows.

2. Notation and varifold preliminaries

We shall adopt the following notation and conventions throughout the paper.

$\mathbf{R}^{n+1}$ denotes the $(n+1)$-dimensional Euclidean space and $(x^1, \ldots, x^{n+1})$ denotes a general point in $\mathbf{R}^{n+1}$. We shall identify $\mathbf{R}^n$ with the hyperplane $\{x^{n+1} = 0\}$ of $\mathbf{R}^{n+1}$ (except in Section 6, where we choose notation to be consistent with that of [11].)

$B^{n+1}_\rho(x)$ denotes the open ball in $\mathbf{R}^{n+1}$ with radius ρ and center x. We shall often use the abbreviation $B_\rho(x)$ for $B^n_\rho(x)$.

ω_n denotes the volume of a ball in $\mathbf{R}^n$ with radius 1.

For compact sets $S, T \subseteq \mathbf{R}^{n+1}$, $\mathrm{dist}_{\mathcal{H}}(S,T)$ denotes the Hausdorff distance between S and T.

$\mathcal{H}^n(S)$ denotes the n-dimensional Hausdorff measure of the set S and $\mathcal{L}^n(S)$ the n-dimensional Lebesgue measure of S.

$\mathcal{I}$ denotes the collection of smooth, immersed, stable, minimal hypersurfaces $M \subset B^{n+1}_2(0)$ with $\mathcal{H}^n(M) < \infty$.

$\overline{\mathcal{I}}$ denotes the closure of $\mathcal{I}$ taken in the varifold topology. (See the discussion on varifolds below.)

$\Theta(V,x)$ denotes the n-dimensional density of the n-varifold V in $\mathbf{R}^{n+1}$ at the point $x \in \mathbf{R}^{n+1}$ and $\|V\|$ denotes the weight measure associated with V.

$\mathrm{spt}\,\mu$ denotes the support of the measure μ.

$\mathbf{v}(M,\theta)$ denotes the rectifiable n-varifold associated to the smooth n-manifold taken with multiplicity θ. We shall abbreviate $\mathbf{v}(M,1)$ as $\mathbf{v}(M)$.

D, ∇^M denote the gradient operators on $\mathbf{R}^n$ and the smooth manifold M respectively. Δ_M denotes the Laplacian on M.

For $y \in \mathbf{R}^{n+1}$ and $\rho > 0$, the map $\eta_{y,\rho}\colon \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is defined by $\eta_{y,\rho}(x) = \frac{x-y}{\rho}$.

$\mathbf{p} : \mathbf{R}^{n+1} \to \mathbf{R}^n$ denotes the orthogonal projection. For a subspace S, $\mathbf{p}_S$ denotes the orthogonal projection of $\mathbf{R}^{n+1}$ onto S.

All constants c depend only on n and α unless stated otherwise.

Next we briefly record, following [10], the notions and theorems in the theory of varifolds relevant for the purposes of this paper. We refer the reader to [10], Chapter 8 or [1] for a detailed exposition of the theory.

Let $G(n)$ denote the set of hyperplanes of $\mathbf{R}^{n+1}$ equipped with the metric given by

$$\rho(S_1,S_2) = |\mathbf{p}_{S_1} - \mathbf{p}_{S_2}| \equiv \left(\sum_{i,j=1}^{n+1} (\mathbf{p}^{ij}_{S_1} - \mathbf{p}^{ij}_{S_2})^2\right)^{\frac{1}{2}},$$

where $\mathbf{p}_S$ denotes the orthogonal projection of $\mathbf{R}^{n+1}$ onto S and $\mathbf{p}^{ij}_S = e_i \cdot \mathbf{p}_S(e_j)$, $1 \le i, j \le n+1$ gives the matrix of $\mathbf{p}_S$ with respect to the standard orthonormal basis $\{e_k\}_{k=1}^{n+1}$ of $\mathbf{R}^{n+1}$.

For $U \subseteq \mathbf{R}^{n+1}$ open, let $G_n(U) = U \times G(n)$, equipped with the product metric. Thus, $G_n(U)$ is a locally compact metric space—indeed it is the countable union of the compact subsets $G_n(K_j)$, where K_j are compact subsets of U with $U = \cup_{j=1}^{\infty} K_j$.

Let $\mathbf{V}_n(U)$ denote the set of n-varifolds on U; i.e. the collection of (non-negative) Radon measures on $G_n(U)$ topologized by the weak* topology. Thus, if $V,\ V_j \in \mathbf{V}_n(U)$, $j = 1, 2, \ldots$, then $V_j \to V$ if and only if

$$\int_{G_n(U)} f(x,S)\, dV_j(x,S) \to \int_{G_n(U)} f(x,S)\, dV(x,S) \tag{2.1}$$

for each fixed continuous function $f : G_n(U) \to \mathbf{R}$ with compact support.

Each given $V \in \mathbf{V}_n(U)$ defines a Radon measure $\|V\|$ on U, called the weight measure associated with V, via

$$\|V\|(A) = V(\{(x,S) \in G_n(U) : x \in A\}).$$

The mass $\mathbf{M}(V)$ of V is defined by $\mathbf{M}(V) = \|V\|(U)$.

If U, $\widetilde{U}$ are open subsets of $\mathbf{R}^{n+1}$ and $f : U \to \widetilde{U}$ is a proper C^1 function, then, given $V \in \mathbf{V}_n(U)$, we define the image varifold $f_\sharp V \in \mathbf{V}_n(\widetilde{U})$ by the mapping formula

$$f_\sharp V(E) = \int_{F^{-1}(E)} J_S\, f(x)\, dV(x,S)$$

for each Borel set $E \subseteq G_n(\widetilde{U})$. Here, $F : G_n^+(U) \to G_n(\widetilde{U})$ is defined by $F(x,S) = (f(x), df_x(S))$, $J_S\, f(x) = (\det((df_x|_S)^* \circ (df_x|_S)))^{1/2}$ where $(df_x|_S)^*$ denotes the adjoint of $df_x|_S$ and $G_n^+(U) = \{(x,S) \in G_n(U) : J_S\, f(x) \neq 0\}$.

The first and the second variations of $V \in \mathbf{V}_n(U)$, denoted δV and $\delta^2 V$ respectively, are the linear functionals on the set of C^1 vector fields $X : U \to \mathbf{R}^{n+1}$ with compact support defined by

$$\delta V(X) = \left.\frac{d}{dt}\right|_{t=0} \mathbf{M}(\varphi_{t\,\sharp} V \llcorner G_n(K)) \tag{2.2}$$

and

$$\delta^2 V(X) = \left.\frac{d^2}{dt^2}\right|_{t=0} \mathbf{M}(\varphi_{t\,\sharp} V \llcorner G_n(K)), \tag{2.3}$$

where K is a compact subset of U and φ_t, $t \in (-1,1)$ is a family of diffeomorphisms of U onto U such that

(i) $\varphi_t(x) = \varphi(t,x)$ where $\varphi : (-1,1) \times U \to U$ is C^2,

(ii) $\varphi_0(x) \equiv x$,

(iii) $\varphi_t(x) \equiv x$ for each $t \in (-1, 1)$ whenever $x \in U \setminus K$ and
(iv) $\frac{\partial}{\partial t}\big|_{t=0} \varphi(t, x) = X(x)$.

A computation involving the above mapping formula for varifolds and differentiation under the integral sign then yields

$$\delta V(X) = \int_{G_n(U)} \operatorname{div}_S X(x)\, dV(x, S), \tag{2.4}$$

where for $S \in G(n)$ and $X = (X^1, \ldots, X^{n+1})$, $\operatorname{div}_S X = \sum_{j=1}^{n+1} e_j \cdot \nabla^S X^j$ with $\nabla^S f(x) = \mathbf{p}_S\, D\, f(x)$.

A varifold $V \in \mathbf{V}_n(U)$ is said to be stationary in U if

$$\delta V(X) = 0 \tag{2.5}$$

for every C^1 vector field $X : U \to \mathbf{R}^{n+1}$ with compact support. Stationary varifolds satisfy the following monotonicity identity:

$$\begin{aligned} &\frac{\|V\|(B^{n+1}_\rho(y))}{\rho^n} - \frac{\|V\|(B^{n+1}_\sigma(y))}{\sigma^n} \\ &\qquad = \int_{G_n(B^{n+1}_\rho(y) \setminus B^{n+1}_\sigma(y))} \frac{|\mathbf{p}_{S^\perp}\,(x-y)|^2}{|x-y|^{n+2}}\, dV(x, S) \end{aligned} \tag{2.6}$$

for $y \in U$ and $0 < \sigma \leq \rho \leq \operatorname{dist}(y, \partial U)$. In particular, this means that the quantity $\frac{\|V\|(B^{n+1}_\rho(y))}{\omega_n\, \rho^n}$ is monotonically non-decreasing in ρ and therefore, the density

$$\Theta(\|V\|, y) = \lim_{\rho \downarrow 0} \frac{\|V\|(B^{n+1}_\rho(y))}{\omega_n\, \rho^n}$$

exists and is finite.

A varifold $\mathbf{C} \in \mathbf{V}_n(\mathbf{R}^{n+1})$ is a *cone* if $\eta_{0,\lambda\,\sharp}\, \mathbf{C} = \mathbf{C}$ for all $\lambda > 0$. If $\mathbf{C}$ is a cone, then $\Theta(\|\mathbf{C}\|, 0) = \frac{\|\mathbf{C}\|(B^{n+1}_\rho(0))}{\omega_n \rho^n}$ for every $\rho > 0$.

Given a countably n-rectifiable $\mathcal{H}^n$-measurable subset M of U and a non-negative, locally $\mathcal{H}^n$-integrable function θ on M, we can define an n-varifold $\mathbf{v}\,(M, \theta) \in \mathbf{V}_n(U)$ (called a countably n-rectifiable varifold or rectifiable n-varifold) by setting

$$\mathbf{v}\,(M, \theta)(E) = \int_{\pi(TM \cap E)} \theta\, d\mathcal{H}^n \tag{2.7}$$

for each $E \subseteq G_n(U)$, where $TM = \{\, (x, T_x M) \,:\, x \in M_* \,\}$ and M_* is the set of points $x \in M$ where M has an approximate tangent space $T_x M$ with respect to θ and $\pi : G_n(U) \to U$ is the projection map. When $\theta \equiv 1$, we simply write $\mathbf{v}(M)$ for $\mathbf{v}(M, 1)$. The function θ is referred to as the multiplicity of the varifold (M, θ).

Thus, we may regard each smooth minimal hypersurface $M \subseteq U$ (in particular, each $M \in \mathcal{I}$) as a rectifiable n-varifold $\mathbf{v}(M)$ on U with multiplicity 1 via

$$\mathbf{v}(M)(E) = \mathcal{H}^n\left(\pi(TM \cap E)\right)$$

for $E \subseteq G_n(U)$, where $TM = \{(x, T_xM) : x \in M\}$ and T_xM is the (classical) tangent space to M at the point $x \in M$. Notice that then, $\|\mathbf{v}(M)\|(A) = \mathcal{H}^n(M \cap A)$ for $A \subseteq \mathbf{R}^{n+1}$. Then, $\overline{\mathcal{I}}$ is the closure of $\mathcal{I} \subseteq \mathbf{V}_n(B_2^{n+1}(0))$ taken with respect to the weak* topology on $\mathbf{V}_n(B_2^{n+1}(0))$. Note that by the well known compactness theorem of Allard ([**10**], Theorem 42.7), every sequence $M_j \in \mathcal{I}$ with $\sup_{j\geq 1} \mathcal{H}^n(M_j) < \infty$ has a subsequence converging as varifolds to an integer multiplicity, stationary, rectifiable n-varifold.

For a smooth minimal hypersurface $M \subseteq U$, the first variation identity (2.5) says that

$$\int_M \operatorname{div}_M X \, d\mathcal{H}^n = 0 \tag{2.8}$$

for every C^1 vector field $X : U \to \mathbf{R}^{n+1}$ with compact support in U, and the monotonicity formula (2.6) says that

$$\begin{aligned} &\frac{\mathcal{H}^n\left(M \cap B_\rho^{n+1}(Y)\right)}{\rho^n} - \frac{\mathcal{H}^n\left(M \cap B_\sigma^{n+1}(Y)\right)}{\sigma^n} \\ &\qquad = \int_{M \cap (B_\rho^{n+1}(Y) \setminus B_\sigma^{n+1}(Y))} \frac{|\mathbf{p}_{T_X M^\perp}(X - Y)|^2}{|X - Y|^{n+2}} \, d\mathcal{H}^n(X) \end{aligned} \tag{2.9}$$

for $Y \in M$ and $0 < \sigma \leq \rho$ with $\overline{B}_\rho^{n+1}(Y) \subseteq U$. The minimal hypersurface M is stable if the second variation in identity (2.3) (with $V = \mathbf{v}(M)$) is non-negative for every C^1 vector field X with compact support in U. A standard computation ([**10**], Remark 9.8) shows that stability of M implies that

$$\int_M |A|^2 \varphi^2 \leq \int_M |\nabla^M \varphi|^2 \tag{2.10}$$

for every C^1 function φ with compact support in M, where A is the second fundamental form of M and $|A|$ the length of A.

3. Approximate decomposition into graphs

Suppose HYPOTHESES $(\star)$ of Section 1 hold, and fix a k, chosen sufficiently large. In this section we use the argument in Section 3 of [**8**] to show that a "large" part G_k of $M_k \cap (B_{1/2}(0) \times \mathbf{R})$ can be written as the union of graphs (not necessarily disjoint) of two Lipschitz functions over an open, connected sub-domain $\Omega_k \subseteq B_{1/2}(0)$ in such a way that the gradients of the graphs are controlled $\mathcal{L}^n$ almost everywhere in their domain. G_k is large in the sense that the n-dimensional Hausdorff measure of $(M_k \setminus G_k) \cap (B_{1/2}(0) \times \mathbf{R})$ is of lower order than the square of the "tilt-excess" ϵ_k of $M_k \cap (B_1(0) \times \mathbf{R})$ relative to $\mathbf{R}^n$. The definition of ϵ_k appears below.

We shall begin by giving the statement of an integral curvature estimate proved in [8]. This result was first proved by Schoen in his thesis [7]. It is a key ingredient in obtaining the approximate graphical decomposition as well as in both our subsequent blow-up procedures.

Lemma 3.2 ([8, Lemma 1])**.** *For every bounded, locally Lipschitz function φ vanishing in a neighborhood of $M_k \cap (\partial B_1(0) \times \mathbf{R})$, we have*

$$\int_{M_k} |A_k|^2 \varphi^2 \leq c \int_{M_k} \left(1 - (\nu_k(X).e_{n+1})^2\right) |\nabla^{M_k} \varphi|^2, \tag{3.1}$$

where c is a constant depending only on n, A_k is the second fundamental form of M_k and ν_k is the unit normal to M_k.

Proof. See Lemma 1 of [8]. q.e.d.

Definitions. Let $g_k(X) = 1 - (\nu_k(X).e_{n+1})^2$ for $X \in M_k$. Define a tilt-excess ϵ_k by

$$\epsilon_k = \left(\int_{M_k \cap (B_1(0) \times \mathbf{R})} \left(1 - (\nu_k(X).e_{n+1})^2\right) \right)^{\frac{1}{2}}.$$

Remark. Since $M_k \cap B_2^{n+1}(0)$ converge to $B_2(0)$ in Hausdorff distance, it follows that

$$\sup_{X \in M_k \cap (B_{3/2}(0) \times \mathbf{R})} |x^{n+1}| \to 0$$

as $k \to \infty$, where $X = (x, x^{n+1})$. By the first variation formula, this implies that $\epsilon_k \to 0$. To see this implication, repeat the argument leading to inequality (4.18) of Section 4 with an appropriate choice of the cut off function ζ.

Lemma 3.3 ([8])**.** *For each k sufficiently large such that $\epsilon_k \leq \delta$, where $\delta = \delta(n, \alpha) \in (0, 1)$ is a fixed constant depending only on n and α, there exist a regular value $\theta_k \in (1/4, 1/2)$ of g_k and a unique open, connected component Ω_k of $B_{1/2}(0) \setminus \Gamma_k$ with $\partial\Omega_k \cap B_{1/2}(0) \subseteq \Gamma_k$, where $\Gamma_k = \mathbf{p}\{ X \in M_k \cap (B_{1/2}(0) \times \mathbf{R}) : g_k(X) = \theta_k \}$, such that the following is true:*

(a) $\mathcal{H}^{n-1}(\{ X \in M_k \cap (B_{1/2}(0) \times \mathbf{R}) : g_k(X) = \theta_k\}) \leq c\, \epsilon_k^2$.

(b) $\mathcal{H}^n(B_{1/2}(0) \setminus \Omega_k) \leq c\,\epsilon_k^2$.

(c) *For every $x \in \Omega_k$, there exists a ball $B_r(x) \subseteq \Omega_k$, $r = r(x)$, and two $C^2(B_r(x))$ functions $u_k^{x,(1)}$, $u_k^{x,(2)}$ (depending on x) such that*

$$|Du_k^{x,(1)}|^2,\ |Du_k^{x,(2)}|^2 \leq \frac{\theta_k}{1-\theta_k} \quad \textit{in } B_r(x) \textit{ and}$$

$$\{\, X \in M_k \,:\, g_k(X) < \theta_k \,\} \cap (B_r(x) \times \mathbf{R}) = \operatorname{graph} u_k^{x,(1)} \cup \operatorname{graph} u_k^{x,(2)}.$$

(We emphasize here our hypothesis that M_k is smooth with no branch points.)

(d) $\mathcal{H}^n\big(M_k \cap (B_{1/2}(0) \times \mathbf{R}) \setminus (\{X \in M_k : g_k(X) < \theta_k\} \cap (\Omega_k \times \mathbf{R}))\big) \leq c\,\epsilon_k^2$.

Here, $c = c(n)$.

Remark. $\operatorname{graph} u_k^{x,(1)}$ and $\operatorname{graph} u_k^{x,(2)}$ are not necessarily disjoint.

Remark. For notational convenience, we shall often suppress the dependence of $u_k^{x,(i)}$ on x and simply write $u_k^{(i)}$, $i = 1, 2$ instead.

Proof. The proofs of parts (a), (b) and (c) are essentially the same as the proof of Lemma 2 of [8] and (d) follows exactly by the corresponding argument on pp. 755–756 of [8]. We do not repeat the entire argument here, except to justify that in part (c), the set $\{X \in M_k : g_k(X) < \theta_k\} \cap (B_r(x) \times \mathbf{R})$ is the union of exactly two graphs.

For each sufficiently large k, the argument of [8] gives an integer $m(k) \geq 0$ and locally defined C^2 functions $u_k^{x,(1)}, \ldots, u_k^{x,(m(k))}$ for each $x \in \Omega_k$ with the properties as in part (c). i.e., $|Du^{x,(i)}| \leq \theta_k/(1-\theta_k)$ for $i = 1, \ldots, m(k)$ and

$$\{X \in M_k : g_k(X) < \theta_k\} \cap (B_r(x) \times \mathbf{R}) = \cup_{i=1}^{m(k)} \operatorname{graph} u_k^{x,(i)}.$$

To show that $m(k) = 2$, we proceed as follows.

First, notice that it is clear from part (d) that $m(k) \geq 1$.

For each k and each $x \in \Omega_k$, we may order the numbers $u_k^{x,(i)}(x)$, $i = 1, \ldots, m(k)$, in a non-decreasing sequence. Define functions $u_{k,i} : \Omega_k \to \mathbf{R}$ by setting $u_{k,i}(x) =$ the ith term of this sequence, where $u_{k,1}(x)$ is the smallest. Let $G_k^i = \operatorname{graph} u_{k,i}$.

By part (b),

$$\omega_n \left(\frac{1}{2}\right)^n - c\,\epsilon_k^2 \leq \mathcal{L}^n(\Omega_k) \leq \mathcal{H}^n(G_k^i) \tag{3.2}$$

for each i. Summing over i in inequalities (3.2) gives

$$(3.3) \qquad \left(\omega_n \left(\frac{1}{2}\right)^n - c\,\epsilon_k^2\right) m(k) \leq \mathcal{H}^n(M_k \cap (B_{1/2}(0) \times \mathbf{R})).$$

Now, $M_k \cap (B_{1/2}(0) \times \mathbf{R}) \subseteq M_k \cap B^{n+1}_{R_k}(0)$, where $R_k = \left((\frac{1}{2})^2 + \delta_k^2\right)^{\frac{1}{2}}$, $\delta_k = \sup_{(x,x^{n+1}) \in M_k \cap (B_{3/2}(0) \times \mathbf{R})} |x^{n+1}|$. By the definition of $\overline{\mathcal{I}}$, we may choose M_k such that

$$(3.4) \qquad \begin{aligned} \mathcal{H}^n(M_k \cap B^{n+1}_{R_k}(0)) &\leq \left(1 + \frac{1}{k}\right) \|\mathbf{C}_k\|(B^{n+1}_{R_k}(0)) \\ &\leq \left(1 + \frac{1}{k}\right)(3 - \alpha)\omega_n R_k^n. \end{aligned}$$

Hence

$$(3.5) \qquad \begin{aligned} m(k) &\leq \frac{\left(1 + \frac{1}{k}\right)(3 - \alpha)\omega_n R_k^n}{\omega_n \left(\frac{1}{2}\right)^n - c\,\epsilon_k^2} \\ &< 3 \end{aligned}$$

for sufficiently large k depending on n, α.

On the other hand, if $m(k) = 1$ for infinitely many k, then $\{X \in M_k : g_k(X) < \theta_k\} \cap (\Omega_k \times \mathbf{R})$ is the graph of a single $C^2(\overline{\Omega_k})$ function u_k for infinitely many k. A standard single valued blow up argument (e.g. first extending u_k to all of $B_{1/2}(0)$ by multiplying u_k by the cut-off function $\overline{\psi_k}$ of Section 4, and then following the argument leading to the identity (4.30), making use of part (d) of Lemma 3.3) shows then that M_k, and hence $\mathbf{C}_k$, must be equal to a (multiplicity 1) hyperplane for infinitely many k. But this contradicts the hypothesis $\Theta(\|\mathbf{C}_k\|, 0) > 1$.

We thus conclude that

$$(3.6) \qquad m(k) = 2$$

for all sufficiently large k. q.e.d.

Definitions. Let $G_k = \{\, X \in M_k \,:\, g_k(X) < \theta_k \,\} \cap (\Omega_k \times \mathbf{R})$.

For $x \in \Omega_k$, define

$$u_k^+(x) = \max\{u_k^{x,(1)}(x), u_k^{x,(2)}(x)\}$$

and

$$u_k^-(x) = \min\{u_k^{x,(1)}(x), u_k^{x,(2)}(x)\},$$

where $u_k^{x,(1)}$ and $u_k^{x,(2)}$ are as in Lemma 3.3, part (c). The functions u_k^+ and u_k^- are Lipschitz with $u_k^+ \geq u_k^-$ and $|Du_k^+|$, $|Du_k^-| \leq \theta_k/(1 - \theta_k)$ $\mathcal{L}^n$-a.e. in Ω_k. Furthermore, $G_k = \operatorname{graph} u_k^+ \cup \operatorname{graph} u_k^-$.

Let $G_k^+ = \operatorname{graph} u_k^+$ and $G_k^- = \operatorname{graph} u_k^-$.

We will need the following result later in the paper.

Lemma 3.4. *For every locally Lipschitz function φ with compact support in M_k,*

$$\int_{M_k} |A_k|^4 \varphi^2 \le c \int_{M_k} |A_k|^2 |\nabla^{M_k} \varphi|^2,$$

where $c = c(n)$.

Proof. The lemma follows from (i) the stability inequality (2.10) and (ii) the following pointwise estimate proved in [**9**]:

If A is the second fundamental form of a smooth, stable, minimal hypersurface M immersed in $\mathbf{R}^{n+1}$ then,

$$|A|\,\Delta_M\,|A| + |A|^4 \ge c\,|\nabla^M\,|A||^2, \tag{3.7}$$

where c depends only on n. The proof of the lemma is as follows.

Let φ be a locally Lipschitz function with compact support in M_k. Using the stability inequality (2.10) with M_k in place of M and $|A_k|\varphi$ in place of φ (a valid choice by approximation), we obtain that

$$\begin{aligned} &\int_{M_k} |A_k|^4\,\varphi^2 \\ &\quad \le \int_{M_k} |\nabla^{M_k}\,|A_k||^2\,\varphi^2 + 2\varphi|A_k|\,\nabla^{M_k}\,\varphi\cdot\nabla^{M_k}\,|A_k| + |A_k|^2|\nabla^{M_k}\varphi|^2. \end{aligned} \tag{3.8}$$

Multiplying inequality (3.7) by φ^2 and integrating the resulting inequality over M_k, we have

$$\begin{aligned} c\int_{M_k} |\nabla^{M_k}\,|A_k||^2\varphi^2 \le \int_{M_k} |A_k|^4\varphi^2 &- \int_{M_k} |\nabla^{M_k}\,|A_k||^2\varphi^2 \\ &- 2\int_{M_k} \varphi|A_k|\,\nabla^{M_k}\,\varphi\cdot\nabla^{M_k}\,|A_k|. \end{aligned} \tag{3.9}$$

The inequalities (3.8) and (3.9) imply that

$$\int_{M_k} |\nabla^{M_k}\,|A_k||^2\varphi^2 \le c\int_{M_k} |A_k|^2|\nabla^{M_k}\varphi|^2 \tag{3.10}$$

and the lemma follows from this, the inequality (3.8) and the fact that $2\,\varphi|A_k|\,\nabla^{M_k}\,\varphi\cdot\nabla^{M_k}\,|A_k| \le |\nabla^{M_k}\,|A_k||^2\varphi^2 + |A_k|^2|\nabla^{M_k}\,\varphi|^2$. q.e.d.

Next, we reproduce from [8] the construction of two important cut-off functions, which will play a crucial role in the blow up argument of Section 4.

For the first of these, we begin with a C^2 function $\gamma\colon[0,1]\to\mathbf{R}$ with the properties that $\gamma(t)=t$ if $0\le t\le 2/3$, $\gamma(t)=0$ if $t\ge 3/4$, $\gamma(t)\ge 0$ and $|\gamma'(t)|\le 12$ for all t, and set $g_k^0(X)=\gamma((g_k(X))^{\frac{1}{2}})$ for $X\in M_k$. Then $g_k^0(X)=g_k^{\frac{1}{2}}(X)$ if $g_k^{\frac{1}{2}}(X)\le 1/2$ and $|\nabla^{M_k} g_k^0|\le 12|\nabla^{M_k} g_k^{\frac{1}{2}}|\le 12|A_k|$ on M_k.

Define $\varphi_k^0\colon B_1(0)\to\mathbf{R}$ by

$$\varphi_k^0(x)=\begin{cases}0 & \text{if } M_k\cap\mathbf{p}^{-1}(x)=\emptyset,\\ \sup\{g_k^0(X): X\in M_k\cap\mathbf{p}^{-1}(x)\} & \text{otherwise.}\end{cases}$$

φ_k^0 is locally Lipschitz in $B_1(0)$. The proof of this is the same as in [8], p. 758, except $\{\operatorname{graph} w_j\}$ are not disjoint (notation as in [8]),but this does not cause a problem.

We also have the following key estimate for all $x\in B_1(0)$. (See [8], p. 758.)

$$|D\varphi_k^0(x)|\le 24\max\{|A_k|(X): X\in M_k\cap\mathbf{p}^{-1}(x)\}. \tag{3.11}$$

Extend φ_k^0 to $B_1(0)\times\mathbf{R}$ by setting $\varphi_k^0(x,x_{n+1})=\varphi_k^0(x)$.

Our second cut off function is a locally Lipschitz function on M_k that separates $M_k\setminus G_k$ from most of G_k. To construct this, first let $\eta\in(0,1/4]$, and $\beta_\eta\colon[0,\infty)\to\mathbf{R}$ be a C^2 function such that

$$\beta_\eta(t)=\begin{cases}0 & \text{if } t\le\eta/2,\\ 1 & \text{if } t\ge\eta\end{cases}$$

and $0\le\beta_\eta'(t)\le 4/\eta$ for all t. Then, define the required cut-off function $\psi_k^{(\eta)}\colon M_k\to\mathbf{R}$ by setting

$$\psi_k^{(\eta)}(X)=\begin{cases}\beta_\eta(\varphi_k^0(X)) & \text{if } X\in G_k,\\ 1 & \text{if } X\in M_k\setminus G_k.\end{cases}$$

$\psi_k^{(\eta)}$ is locally Lipschitz and

$$|D\psi_k^{(\eta)}(x,u_k^{\pm}(x))|\le 96\eta^{-1}\max\{|A_k|(X): X\in M_k\cap\mathbf{p}^{-1}(x)\} \tag{3.12}$$

for all $x\in\Omega_k$. (See [8], p. 759.)

We use these cut-off functions just as they were used in [8] to show that the non-graphical part $(M_k\setminus G_k)\cap(B_{1/2}(0)\times\mathbf{R})$ as well as the part of G_k, where the unit normal to M_k deviates from the e_{n+1}-direction by a "large" amount (i.e., the part where φ_k^0 assumes "large" values) has lower order measure.

For $\sigma \in (0,1/2]$ and $\lambda \in (0,1/4]$, let

$$L_k^{\sigma,\lambda} = (M_k \cap (B_\sigma(0) \times \mathbf{R}) \setminus G_k) \cup \left(M_k \cap (B_\sigma(0) \times \mathbf{R}) \cap G_k \cap \{X : \varphi_k^0(X) \geq \lambda\}\right). \tag{3.13}$$

Lemma 3.5 ([8]). *If $\sigma \in (0,1/2]$ and $\lambda \in (0,1/4]$, then*

$$\mathcal{H}^n(L_k^{\sigma/2,\lambda}) \leq c\,\sigma^{-(2+\mu)}\,\lambda^{-(2+\mu)}\epsilon_k^{2+\mu}, \tag{3.14}$$

where $\mu = \frac{4}{(n-2)}$ and $c = c(n)$.

Proof. This is a coarser version of Lemma 3 of [8], sufficient for our purposes. All of the steps of its proof are contained in [8], but we include here a complete proof since the lemma does not appear in a quotable form in [8].

If $x \in \Omega_k \cap B_\sigma(0) \cap \{\varphi_k^0 \geq \lambda\}$, we have that

$$\begin{aligned} \lambda^2 &\leq \left(\varphi_k^0(x)\right)^2 \\ &= \sup\left\{ \left(g_k^0(X)\right)^2 : X \in M_k \cap \mathbf{p}^{-1}(x) \right\} \\ &\leq g_k(x, u_k^+(x)) + g_k(x, u_k^-(x)) + \chi_{\mathbf{p}(M_k \cap B_\sigma(0) \times \mathbf{R}) \setminus G_k)}(x) \\ &= \frac{|Du_k^+(x)|^2}{1+|Du_k^+(x)|^2} + \frac{|Du_k^-(x)|^2}{1+|Du_k^-(x)|^2} + \chi_{\mathbf{p}(M_k \cap B_\sigma(0) \times \mathbf{R}) \setminus G_k)}(x). \end{aligned} \tag{3.15}$$

Integrating both sides of the above inequality, we obtain

$$\begin{aligned} \lambda^2 \mathcal{L}^n(\Omega_k \cap B_\sigma(0) \cap \{\varphi_k^0 \geq \lambda\}) &\leq \int_{\Omega_k \cap B_\sigma(0) \cap \{\varphi_k^0 \geq \lambda\}} \frac{|Du_k^+(x)|^2}{1+|Du_k^+(x)|^2} \\ &\quad + \int_{\Omega_k \cap B_\sigma(0) \cap \{\varphi_k^0 \geq \lambda\}} \frac{|Du_k^-(x)|^2}{1+|Du_k^-(x)|^2} \\ &\quad + \mathcal{H}^n(M_k \cap (B_\sigma(0) \times \mathbf{R}) \setminus G_k) \\ &\leq \int_{M_k} \left(1 - (\nu_k . e_{n+1})^2\right) + \mathcal{H}^n(M_k \cap (B_\sigma(0) \times \mathbf{R}) \setminus G_k) \\ &\leq \epsilon_k^2 + c\,\epsilon_k^2 \leq c\,\epsilon_k^2. \end{aligned} \tag{3.16}$$

Thus,

$$\mathcal{L}^n(\Omega_k \cap B_\sigma(0) \cap \{\varphi_k^0 \geq \lambda\}) \leq c\,\lambda^{-2}\epsilon_k^2 \tag{3.17}$$

and therefore, since $u_k^\pm$ have bounded gradient, we get

$$\mathcal{H}^n(M_k \cap (B_\sigma(0) \times \mathbf{R}) \cap G_k \cap \{\varphi_k^0 \geq \lambda\}) \leq c\,\lambda^{-2}\epsilon_k^2. \tag{3.18}$$

In view of Lemma 3.3 (d), it follows from inequality (3.18) that

$$\mathcal{H}^n(L_k^{\sigma,\lambda}) \leq c\lambda^{-2}\epsilon_k^2. \tag{3.19}$$

Next, we use the Sobolev inequality of [**6**] to show that the n-dimensional Hausdorff measure of $(L_k^{\sigma,\lambda})$ is in fact of lower order than ϵ_k^2.

The Sobolev inequality on M_k says that

$$\left(\int_{M_k} |f|^{2\kappa}\, d\mathcal{H}^n\right)^{\frac{1}{\kappa}} \leq c\int_{M_k} |\nabla^{M_k} f|^2\, d\mathcal{H}^n \tag{3.20}$$

for every locally Lipschitz function f with compact support in M_k, where $\kappa = \frac{n}{(n-2)}$.

Let $\zeta : [0,\infty) \to \mathbf{R}$ be a Lipschitz function satisfying $0 \leq \zeta \leq 1$, $\zeta(r) \equiv 1$ for $0 \leq r \leq \sigma/2$, $\zeta \equiv 0$ for $r \geq \sigma$ and $|\zeta'| \leq 3/\sigma$. Set $f(X) = \zeta(r(X))\psi_k^{(\lambda)}(X)$ in inequality (3.20), where $r(X) = |\mathbf{p}(X)|$ for $X \in M_k$. This yields

$$\begin{aligned}\left(\mathcal{H}^n(L_k^{\sigma/2,\lambda})\right)^{\frac{1}{\kappa}} \leq c\int_{M_k} \zeta^2(r(X))|\nabla^{M_k}\psi_k^{(\lambda)}(X)|^2 \\ + c\,\sigma^{-2}\int_{M_k\cap(B_\sigma(0)\times\mathbf{R})} (\psi_k^{(\lambda)}(X))^2.\end{aligned} \tag{3.21}$$

Since $\nabla^{M_k}\psi_k^{(\lambda)} = 0$ on $M_k \setminus G_k$, this implies that

(3.22)
$$\begin{aligned}\left(\mathcal{H}^n(L_k^{\sigma/2,\lambda})\right)^{\frac{1}{\kappa}} &\leq c\,\lambda^{-2}\int_{M_k\cap(B_\sigma(0)\times\mathbf{R})} |A_k|^2(X)\zeta^2(r(X)) \\ &\quad + c\,\sigma^{-2}\,\mathcal{H}^n((M_k \cap (B_\sigma(0) \times \mathbf{R})) \cap \{\varphi_k^0 \geq \lambda/2\}) \\ &\leq c\,\lambda^{-2}\int_{M_k} \left(1 - (\nu_k(X).e_{n+1})^2\right)|\nabla^{M_k}\zeta^2(r(X))| \\ &\quad + c\,\sigma^{-2}\mathcal{H}^n(L_k^{\sigma,\lambda/2}).\end{aligned}$$

The last inequality of the above follows from Lemma 3.2. Observing that $|\nabla^{M_k} r(X)| \leq 1$ and $|\zeta'| \leq 3/\sigma$, we conclude from inequalities (3.22), (3.19) and the definition of ϵ_k that

$$\mathcal{H}^n\left(L_k^{\sigma/2,\lambda}\right) \leq c\,\sigma^{-2\kappa}\lambda^{-2\kappa}\epsilon_k^{2\kappa}. \tag{3.23}$$

This completes the proof of the lemma. q.e.d.

Remark. In particular, choosing $\sigma = 1/2$ and $\lambda = 1/4$, we have

$$\mathcal{H}^n\left(M_k \cap (B_{1/4}(0) \times \mathbf{R}) \setminus G_k\right) \leq c\,\epsilon_k^{2+\mu} \tag{3.24}$$

where $\mu = \frac{4}{n-2}$.

Next, we define a third cut off function $\overline{\psi}_k$ on $B_{1/2}(0)$ by setting

$$\overline{\psi}_k(x) = \begin{cases} 1 - \overline{\beta}(\varphi_k^0(x)) & \text{if } \mathbf{p}^{-1}(x) \cap M_k \neq \emptyset \text{ and } \mathbf{p}^{-1}(x) \cap M_k \subseteq G_k \\ 0 & \text{otherwise,} \end{cases}$$

where $\overline{\beta}\colon [0,\infty) \to \mathbf{R}$ is a C^2 function such that

$$\overline{\beta}(t) = \begin{cases} 0 & \text{if } \; t \leq 1/8, \\ 1 & \text{if } \; t \geq 1/4 \end{cases}$$

and $0 \leq \overline{\beta}'(t) \leq 16$ for all t.

Extend $\overline{\psi}_k$ to $B_{1/2}(0) \times \mathbf{R}$ by setting $\overline{\psi}_k(x, x_{n+1}) = \overline{\psi}_k(x)$. Since $\theta_k \in [1/4, 1/2]$, it follows from the definition of φ_k^0 that $\overline{\psi}_k(X) = 1 - \psi_k^{(1/4)}(X)$ for $X \in M_k$.

Also, $\overline{\psi}_k$ is locally Lipschitz in $B_{1/2}(0)$ (since φ_k^0 is locally Lipschitz) and, by inequality (3.11), we have that

$$|D\overline{\psi}_k(x)| \leq 384 \max\{|A_k|(X) : X \in M_k \cap \mathbf{p}^{-1}(x)\}. \tag{3.25}$$

Furthermore, $\overline{\psi}_k(X) = 0$ for $X \in M_k \setminus G_k$ and $\overline{\psi}_k(X) = 1$ for $X \in G_k \cap \{X : \varphi_k^0(X) \leq 1/8\}$. By Lemma 3.5, $\mathcal{H}^n(\{X \in M_k : \varphi_k^0(X) \geq 1/8\}) \leq c\,\epsilon_k^{2+\mu}$ and therefore,

$$\mathcal{H}^n\left(B_{1/2}(0) \setminus \{x : \overline{\psi}_k(x) \equiv 1\}\right) \leq c\,\epsilon_k^{2+\mu}. \tag{3.26}$$

So, in particular, $\overline{\psi}_k(x) \to 1$ for $\mathcal{L}^n$-a.e. $x \in B_{1/2}(0)$.

Two more important properties of $\overline{\psi}_k$ are that

$$\int_{B_{1/2}(0)} |D\overline{\psi}_k| \leq c\,\epsilon_k^{2+\mu/2} \tag{3.27}$$

and

$$\int_{B_{1/2}(0)} |D\overline{\psi}_k|^2 \leq c\,\epsilon_k^{2+\mu/2}. \tag{3.28}$$

To see estimate (3.27), first observe that $S_k \equiv \operatorname{spt}(D\overline{\psi}_k) \subseteq \{x \in B_{1/2}(0) : \varphi_k^0(x) \geq 1/8\}$. Therefore, writing $M_k^\sigma = M_k \cap (B_\sigma(0) \times \mathbf{R})$, we see that

(3.29)

$$\begin{aligned}\int_{B_{1/2}(0)} |D\overline{\psi}_k| &= \int_{S_k} |D\overline{\psi}_k| \\ &\leq c \int_{\{X\in M_k^{1/2}:\varphi_k^0(X)\geq 1/8\}} |A_k| \\ &\leq c\left(\mathcal{H}^n(\{X\in M_k^{1/2} : \varphi_k^0(X)\geq 1/8\})\right)^{1/2}\left(\int_{M_k^{1/2}} |A_k|^2\right)^{1/2} \\ &\leq c\,\epsilon_k^{1+\mu/2}\,\epsilon_k = c\,\epsilon_k^{2+\mu/2}.\end{aligned}$$

Here, we have used Lemma (3.2), the definition of ϵ_k and the fact that $\mathcal{H}^n(\{X \in M_k^{1/2} : \varphi_k^0(X) \geq 1/8\}) \leq c\,\epsilon_k^{2+\mu}$. This last assertion follows from Lemma 3.5 with $\sigma = 1/2$ and $\lambda = 1/8$.

To see estimate (3.28), we proceed similarly.

(3.30)

$$\begin{aligned}\int_{B_{1/2}(0)} |D\overline{\psi}_k|^2 &= \int_{S_k} |D\overline{\psi}_k|^2 \\ &\leq c \int_{\{X\in M_k^{1/2}:\varphi_k^0(X)\geq 1/8\}} |A_k|^2 \\ &\leq c\left(\mathcal{H}^n(\{X\in M_k^{1/2} : \varphi_k^0(X)\geq 1/8\})\right)^{1/2}\left(\int_{M_k^{1/2}} |A_k|^4\right)^{1/2} \\ &\leq c\left(\mathcal{H}^n(\{X\in M_k^{1/2} : \varphi_k^0(X)\geq 1/8\})\right)^{1/2}\left(\int_{M_k^{3/4}} |A_k|^2\right)^{1/2} \\ &\leq c\,\epsilon_k^{1+\mu/2}\,\epsilon_k = c\,\epsilon_k^{2+\mu/2}.\end{aligned}$$

Note that here we have used Lemma 3.4 with a choice of φ that satisfies e.g., $\varphi \equiv 1$ on $M_k \cap (B_{1/2}(0) \times \mathbf{R})$, $\varphi \equiv 0$ outside $M_k \cap (B_{3/4}(0) \times \mathbf{R})$ and $|\nabla^{M_k} \varphi| \leq 8$.

4. First blow-up

Let $\tilde{\zeta}\colon B_2^{n+1}(0) \to \mathbf{R}$ be a C^1 function with compact support. Taking $X = \tilde{\zeta}\, e_{n+1}$ in the identity (2.8) gives

$$\int_{M_k} \nabla^{M_k} x^{n+1} \cdot \nabla^{M_k} \tilde{\zeta} = 0. \tag{4.1}$$

Our goal in this section is to use the identity (4.1) and the results of Section 3 to obtain convergence in a suitable sense as $k \to \infty$ of the sequences of functions obtained by blowing up (i.e., dividing) $\overline{\psi}_k u_k^+$ and $\overline{\psi}_k u_k^-$ by the tilt excess ϵ_k. Here, $\overline{\psi}_k$ is the cut-off function defined in Section 3. (Notice that we have $\epsilon_k > 0$ for all k because if $\epsilon_k = 0$, then M_k must obviously be a (multiplicity 1) hyperplane which is impossible by (3.6).) We shall also establish a number of important properties of the limit functions, among which is a symmetry property between them. (See part (e) of Lemma 4.6.) This symmetry will be very useful for our later analysis because it will enable us to avoid having to address the complexities that would otherwise arise from the two-valued nature of the problem.

Extend u_k^+ and u_k^- to be zero in $B_{1/2}(0) \setminus \Omega_k$, and set $v_k^+ = u_k^+/\epsilon_k$, $v_k^- = u_k^-/\epsilon_k$. Also, set $v_k^{(1)} = u_k^{(1)}/\epsilon_k$ and $v_k^{(2)} = u_k^{(2)}/\epsilon_k$. (Recall that $u_k^{(1)}$ and $u_k^{(2)}$ are our respective abbreviations for the C^2 functions $u_k^{x,(1)}$ and $u_k^{x,(2)}$ of Lemma 3.3 defined locally in Ω_k. Thus, $v_k^{(1)}$ and $v_k^{(2)}$ are C^2 functions defined locally in Ω_k.)

We shall also use the following notation:

$$|Du_k| = |Du_k^+| + |Du_k^-| \quad \text{and} \quad |Dv_k| = |Dv_k^+| + |Dv_k^-|. \tag{4.2}$$

Lemma 4.6. *There exist functions $v^+, v^- \in W^{1,2}(B_{1/4}(0))$ with $v^+ \geq v^-$ such that, after passing to a subsequence,*

(a) $\overline{\psi}_k\, v_k^+ \to v^+$ *strongly in* $L^2(B_{1/4}(0))$ *and weakly in* $W^{1,2}(B_{1/4}(0))$.
(b) $\overline{\psi}_k\, v_k^- \to v^-$ *strongly in* $L^2(B_{1/4}(0))$ *and weakly in* $W^{1,2}(B_{1/4}(0))$.
(c) v^+ *and* v^- *are not both identically equal to zero on* $B_{1/4}(0)$.
(d) $v^+ \not\equiv v^-$.
(e) $v^+ + v^-$ *is a linear function. Hence, by rotation, we assume without loss of generality that* $v^+ + v^- \equiv 0$.
(f) $\overline{\psi}_k\, |Dv_k^+| \to |Dv^+|$ *strongly in* $L^2(B_{1/4}(0))$.
(g) $\overline{\psi}_k\, |Dv_k^-| \to |Dv^-|$ *strongly in* $L^2(B_{1/4}(0))$.
(h) $|Dv^+|,\ |Dv^-| \in W^{1,2}(B_{1/4}(0))$ *and* $\overline{\psi}_k |Dv_k| \to |Dv^+| + |Dv^-|$ *weakly in* $W^{1,2}(B_{1/4}(0))$.

Terminology. The functions v^+ and v^- (and their graphs) will be referred to as the (first) blow-up of the sequence of hypersurfaces $M_k \cap (B_{1/2}(0) \times \mathbf{R})$ (or of the sequences of functions $\overline{\psi}_k u_k^+$ and $\overline{\psi}_k u_k^-$.)

Before giving the proof of Lemma 4.6, we derive an estimate (inequality (4.6)) for the energy of $\overline{\psi}_k\, v_k^\pm$ in the region where $|\overline{\psi}_k\, v_k^\pm|$ is small. We will need this estimate in the proof of Lemma 4.6. To obtain this estimate, we proceed as follows.

Let $\delta > 0$ be arbitrary and $\zeta \in C^1_c(B_1(0))$. There exists a $C^1_c(B_1(0)\times \mathbf{R})$ cut-off function $\tilde\zeta$ that agrees with ζ_1 in a neighborhood of M_k, where $\zeta_1(x, x^{n+1}) = \zeta(x)$. Replace $\tilde\zeta$ in identity (4.1) by $F_\delta(x^{n+1})\tilde\zeta^2$, where F_δ is defined by

$$F_\delta(t) = \begin{cases} -\delta & \text{if } \ t < -\delta, \\ t & \text{if } \ |t| \leq \delta, \\ \delta & \text{if } \ t > \delta. \end{cases}$$

This yields

$$\begin{aligned} \int_{M_k \cap \{|x^{n+1}| \leq \delta\}} &|\nabla^{M_k} x^{n+1}|^2 \tilde\zeta^2 \\ &= -2 \int_{M_k} F_\delta(x^{n+1}) \tilde\zeta \nabla^{M_k} \tilde\zeta \cdot \nabla^{M_k} x^{n+1} \\ &\leq 2\,\delta \int_{M_k} |\tilde\zeta||\nabla^{M_k}\tilde\zeta||\nabla^{M_k} x^{n+1}| \\ &\leq c\,\delta \left(\int_{M_k} \tilde\zeta^2\, |\nabla^{M_k}\tilde\zeta|^2\, |\nabla^{M_k} x^{n+1}|^2\right)^{1/2}. \end{aligned} \tag{4.3}$$

Now, choose ζ such that $\zeta \equiv 1$ on $B_{1/4}(0)$, $\zeta \equiv 0$ outside $B_{1/2}(0)$ and $|D\zeta| \leq 4$ everywhere. Writing the integral on the left-hand side of the above as the sum of integrals over $G_k^+ \cap \{|x^{n+1}| \leq \delta\}$, $G_k^- \cap \{|x^{n+1}| \leq \delta\}$ and $(M_k \setminus G_k) \cap \{|x^{n+1}| \leq \delta\}$ and using the fact that $|\nabla^{M_k} x^{n+1}|^2 = 1 - (\nu_k \cdot e_{n+1})^2$ on the right hand side, we obtain from inequality (4.3) that

$$\begin{aligned} \int_{\Omega_k \cap B_{1/4}(0) \cap \{|u_k^+| \leq \delta\}} &\frac{|Du_k^+|^2}{\sqrt{1 + |Du_k^+|^2}} \\ &+ \int_{\Omega_k \cap B_{1/4}(0) \cap \{|u_k^-| \leq \delta\}} \frac{|Du_k^-|^2}{\sqrt{1 + |Du_k^-|^2}} \\ &\leq c\,\delta\epsilon_k. \end{aligned} \tag{4.4}$$

Setting $\delta = \epsilon\epsilon_k$ in the above inequality, we deduce that for arbitrary $\epsilon > 0$,

$$\int_{\Omega_k \cap B_{1/4}(0) \cap \{|v_k^+| \leq \epsilon\}} |Dv_k^+|^2 + \int_{\Omega_k \cap B_{1/4}(0) \cap \{|v_k^-| \leq \epsilon\}} |Dv_k^-|^2 \leq c\,\epsilon. \tag{4.5}$$

In view of inequality (3.28), estimate (4.5) implies that

$$\int_{B_{1/4}(0)\cap\{|\overline{\psi}_k v_k^+|\le\epsilon\}} |D(\overline{\psi}_k v_k^+)|^2 + \int_{B_{1/4}(0)\cap\{|\overline{\psi}_k v_k^-|\le\epsilon\}} |D(\overline{\psi}_k v_k^-)|^2 \le c\,\epsilon \tag{4.6}$$

for sufficiently large k depending on ϵ. This is the desired energy estimate.

Proof of Lemma 4.6. To prove parts (a) and (b) of the lemma, we first show that $\{\overline{\psi}_k\, v_k^+\}$ and $\{\overline{\psi}_k\, v_k^-\}$ are bounded in $L^2(B_{1/4}(0))$. Note that this is equivalent to proving that the "height-excess" of M_k relative to $\mathbf{R}^n$ ($\equiv \sqrt{\int_{M_k\cap B^{n+1}_{1/2}(0)} |x^{n+1}|^2}$) is bounded from above by a constant times the tilt-excess ϵ_k. We first establish this bound (in a varifold setting) for the cones $\mathbf{C}_k$. The statement for M_k (chosen sufficiently close to $\mathbf{C}_k$) then follows directly from the definition of varifold convergence.

The monotonicity formula (2.6) applied to the cone $\mathbf{C}_k$ says that

$$\int_{B_{1/2}(0)\times G(n)} \frac{|\mathbf{p}_{S^\perp} X|^2}{|X|^{n+2}}\, d\mathbf{C}_k(X,S) = 0, \tag{4.7}$$

where $G(n)$ denotes the set of hyperplanes in $\mathbf{R}^{n+1}$.

On the other hand, writing $X = (x', x^{n+1})$ and letting $\nu_S = (\nu'_S, \nu_S^{n+1})$ denote the unit normal to the hyperplane S and noting that $\mathbf{p}_{S^\perp} X = X\cdot\nu_S = x'\cdot\nu'_S + x^{n+1}\nu_S^{n+1}$, we obtain using $(a+b)^2 \ge \frac{1}{2}a^2 - b^2$ with $a = x^{n+1}\,\nu_S^{n+1}$ and $b = (\nu'_S\cdot x')$ that

$$\begin{aligned}
&\int_{B^{n+1}_{1/2}(0)\times G(n)} \frac{|\mathbf{p}_{S^\perp} X|^2}{|X|^{n+2}}\, d\mathbf{C}_k(X,S) \\
&\quad\ge 2^{n+2}\int_{B^{n+1}_{1/2}(0)\times G(n)} |\mathbf{p}_{S^\perp} X|^2\, d\mathbf{C}_k(X,S) \\
&\quad\ge 2^{n+1}\int_{B^{n+1}_{1/2}(0)\times G(n)} (x^{n+1})^2(\nu_S^{n+1})^2\, d\mathbf{C}_k(X,S) \\
&\qquad - 2^{n+2}\int_{B^{n+1}_{1/2}(0)\times G(n)} (\nu'_S\cdot x')^2\, d\mathbf{C}_k(X,S) \\
&\quad\ge 2^{n+1}\int_{B^{n+1}_{1/2}(0)\times G(n)} (x^{n+1})^2(\nu_S^{n+1})^2\, d\mathbf{C}_k(X,S) \\
&\qquad - 2^{n}\int_{B^{n+1}_{1/2}(0)\times G(n)} |\nu'_S|^2\, d\mathbf{C}_k(X,S) \\
&\quad= 2^{n+1}\int_{B^{n+1}_{1/2}(0)\times G(n)} (x^{n+1})^2(\nu_S^{n+1})^2\, d\mathbf{C}_k(X,S)
\end{aligned} \tag{4.8}$$

$$
\begin{aligned}
&- 2^n \int_{B^{n+1}_{1/2}(0)\times G(n)} (1-(\nu_S^{n+1})^2)\, d\mathbf{C}_k(X,S) \\
\geq\ & 2^{n+1} \int_{B^{n+1}_{1/2}(0)\times G(n)} (x^{n+1})^2\, d\mathbf{C}_k(X,S) \\
&- 3.2^{n-1} \int_{B^{n+1}_{1/2}(0)\times G(n)} (1-(\nu_S^{n+1})^2)\, d\mathbf{C}_k(X,S).
\end{aligned}
$$

Combining equation (4.7) and inequality (4.8), we obtain that

$$
\begin{aligned}
0 &< \int_{B^{n+1}_{1/2}(0)\times G(n)} (x^{n+1})^2\, d\mathbf{C}_k(X,S) \\
&\leq \frac{3}{4} \int_{B^{n+1}_{1/2}(0)\times G(n)} (1-(\nu_S^{n+1})^2)\, d\mathbf{C}_k(X,S).
\end{aligned} \tag{4.9}
$$

The first inequality of the above follows from item (3) of HYPOTHESIS ($\star$), Section 1. By the definition of varifold convergence (2.1), we then have that for M_k chosen sufficiently close to $\mathbf{C}_k$ (depending on k),

$$
\int_{M_k\cap B^{n+1}_{1/2}(0)} (x^{n+1})^2 \leq \int_{M_k\cap B^{n+1}_{1/2}(0)} 1-(\nu_k^{n+1})^2. \tag{4.10}
$$

This implies in particular that for sufficiently large k,

$$
\int_{B_{1/4}(0)} (\overline{\psi}_k u_k^+)^2 + (\overline{\psi}_k u_k^-)^2 \leq \epsilon_k^2 \tag{4.11}
$$

or, equivalently, that

$$
\int_{B_{1/4}(0)} (\overline{\psi}_k v_k^+)^2 + (\overline{\psi}_k v_k^-)^2 \leq 1 \tag{4.12}
$$

as required.

Next, we show that $D(\overline{\psi}_k v_k^{\pm})$ are bounded in $L^2(B_{1/4}(0))$, by estimating as follows:

$$\begin{aligned}
(4.13)\qquad & \int_{B_{1/4}(0)} |D(\overline{\psi}_k\, v_k^{+})|^2 + |D(\overline{\psi}_k v_k^{-})|^2 \\
&\leq \frac{2}{\epsilon_k^2} \int_{B_{1/4}(0)} |D\overline{\psi}_k|^2 \left(|u_k^{+}|^2 + |u_k^{-}|^2\right) \\
&\quad + \frac{2}{\epsilon_k^2} \int_{B_{1/4}(0)} \overline{\psi}_k^2 (|Du_k^{+}|^2 + |Du_k^{-}|^2) \\
&\leq c\,\epsilon_k^{\mu/2} + \frac{4}{\epsilon_k^2} \int_{B_{1/4}(0)} \frac{\overline{\psi}_k^2 |Du_k^{+}|^2}{\sqrt{1+|Du_k^{+}|^2}} + \frac{\overline{\psi}_k^2 |Du_k^{-}|^2}{\sqrt{1+|Du_k^{-}|^2}} \\
&\leq c\,\epsilon_k^{\mu/2} + \frac{4}{\epsilon_k^2} \int_{G_k} 1 - (\nu_k \cdot e_{n+1})^2 \\
&\leq c\,\epsilon_k^{\mu/2} + 4,
\end{aligned}$$

where we have used inequality (3.28) and the fact that $|Du_k^{\pm}| \leq 1$.

In view of inequalities (4.12) and (4.13), Rellich's compactness lemma implies that there exist functions v^{+}, $v^{-} \in W^{1,2}(B_{1/4}(0))$ such that, after passing to a subsequence,

$$(4.14)\qquad \overline{\psi}_k\, v_k^{+} \to v^{+} \text{ strongly in } L^2(B_{1/4}(0)) \text{ and weakly in } W^{1,2}(B_{1/4}(0))$$

and

$$(4.15)\qquad \overline{\psi}_k\, v_k^{-} \to v^{-} \text{ strongly in } L^2(B_{1/4}(0)) \text{ and weakly in } W^{1,2}(B_{1/4}(0)).$$

This proves parts (a) and (b) of the Lemma 4.6.

To prove part (c), we first show that the tilt-excess ϵ_k is bounded from above by a constant times the height-excess. To do this, first notice that replacing $\tilde{\zeta}$ in identity (4.1) with $x^{n+1}\tilde{\zeta}^2$, where $\tilde{\zeta} \in C_c^1(B_2^{n+1}(0))$, and using the Cauchy–Schwarz inequality, we obtain that

$$\begin{aligned}
(4.16)\qquad & \int_{M_k} |\nabla^{M_k} x^{n+1}|^2 \tilde{\zeta}^2 \\
&\leq 2 \int_{M_k} |\tilde{\zeta}||x^{n+1}||\nabla^{M_k}\tilde{\zeta}||\nabla^{M_k} x^{n+1}| \\
&\leq 2 \left(\int_{M_k} |\nabla^{M_k} x^{n+1}|^2 \tilde{\zeta}^2\right)^{1/2} \left(\int_{M_k} |x^{n+1}|^2 |\nabla^{M_k}\tilde{\zeta}|^2\right)^{1/2}.
\end{aligned}$$

Hence,

$$\int_{M_k} |\nabla^{M_k} x^{n+1}|^2 \tilde{\zeta}^2 \leq 4 \int_{M_k} |x^{n+1}|^2 |\nabla^{M_k} \tilde{\zeta}|^2. \tag{4.17}$$

Now, let $\zeta \in C^1_c(\mathbf{R}^n)$ be such that $\zeta = 1$ on $B_{1/8}(0)$, $\zeta = 0$ outside $B_{1/4}(0)$ and $|D\zeta| \leq 16$, and let ζ_1 be the extension of ζ to $\mathbf{R}^{n+1}$ such that ζ_1 is constant in e_{n+1} direction. i.e., $\zeta_1(x', x^{n+1}) = \zeta(x')$. Choosing $\tilde{\zeta}$ in inequality (4.17) to be a $C^1_c(B_1(0) \times \mathbf{R})$ function such that $\tilde{\zeta} \equiv \zeta_1$ in a neighborhood of $M_k \cap (B_1(0) \times \mathbf{R})$, we obtain that

$$\int_{M_k \cap (B_{1/8}(0) \times \mathbf{R})} |\nabla^{M_k} x^{n+1}|^2 \leq 128 \int_{M_k \cap (B_{1/4}(0) \times \mathbf{R})} |x^{n+1}|^2. \tag{4.18}$$

Notice that since $\mathbf{C}_k$ is a cone (i.e., since $\eta_{0,\lambda\,\sharp}\, \mathbf{C}_k = \mathbf{C}_k$ for every $\lambda > 0$), we have that

$$\begin{aligned} &\int_{B^{n+1}_{1/8}(0) \times G(n)} |\nabla^S\, x^{n+1}|^2 \, d\mathbf{C}_k(X, S) \\ &\qquad = c \int_{B^{n+1}_{3/2}(0) \times G(n)} |\nabla^S\, x^{n+1}|^2 \, d\mathbf{C}_k(X, S). \end{aligned} \tag{4.19}$$

Therefore, for M_k chosen to approximate $\mathbf{C}_k$ sufficiently closely (depending on k) we have, by the definition of varifold convergence and the fact that

$$\int_{B^{n+1}_1(0) \times G(n)} |\nabla^S\, x^{n+1}|^2 \, d\mathbf{C}_k(X, S) > 0$$

(which follows from HYPOTHESES $(\star)$, item (3)), that

$$\int_{M_k \cap (B_{1/8}(0) \times \mathbf{R})} |\nabla^{M_k} x^{n+1}|^2 \geq c \int_{M_k \cap (B_1(0) \times \mathbf{R})} |\nabla^{M_k} x^{n+1}|^2. \tag{4.20}$$

Since $|\nabla^{M_k} x^{n+1}|^2 = 1 - (\nu_k \cdot e_{n+1})^2$, we obtain from inequalities (4.18) and (4.20) that

$$\epsilon_k^2 \leq c \int_{M_k \cap (B_{1/4}(0) \times \mathbf{R})} |x^{n+1}|^2 \tag{4.21}$$

as required. Here, $c = c(n)$.

Now, expressing the integral on the right-hand side of inequality (4.21) as the sum of integrals over $G_k \cap (B_{1/4}(0) \times \mathbf{R})$ and $(M_k \setminus G_k) \cap$

$(B_{1/4}(0) \times \mathbf{R})$, we have that

(4.22)

$$\epsilon_k^2 \le c \left(\int_{\Omega_k \cap B_{1/4}(0)} (u_k^+)^2 + (u_k^-)^2 + \int_{(M_k \setminus G_k) \cap (B_{1/4}(0) \times \mathbf{R})} |x^{n+1}|^2 \right).$$

Using inequalities (3.24) and (3.26), we then have that

$$\epsilon_k^2 \le c\,(\epsilon_k^{2+\mu}) + c \int_{B_{1/4}(0)} (\overline{\psi}_k u_k^+)^2 + (\overline{\psi}_k u_k^-)^2. \tag{4.23}$$

For sufficiently large k (depending only on n), we may absorb the first term on the right-hand side of the above into the left-hand side, and the resulting inequality yields

$$\int_{B_{1/4}(0)} (\overline{\psi}_k v_k^+)^2 + (\overline{\psi}_k v_k^-)^2 \ge c > 0, \tag{4.24}$$

where $c = c(n)$. Letting $k \to \infty$ in this gives

$$\int_{B_{1/4}(0)} (v^+)^2 + (v^-)^2 \ge c > 0 \tag{4.25}$$

proving part (c) of Lemma 4.6.

To see part (d), notice that if $v^+ \equiv v^-$, then geometrically, we have a single hyperplane as the blow-up, and it is standard, then, that for sufficiently large k, each M_k must itself be a hyperplane. (To see why, repeat the blow-up argument with the tilt-excess of each of $M_k \cap (B_1(0) \times \mathbf{R})$ measured relative to the "optimal" reference hyperplane; i.e., the hyperplane with respect to which the tilt-excess is the minimum among all tilt-excesses with respect to hyperplanes.) By (3.6), however, it is not possible that M_k is a hyperplane, proving part (d).

For part (e), by identity (4.1), we have that

$$\int_{G_k} \nabla^{M_k} x^{n+1} \cdot \nabla^{M_k} \tilde{\zeta} = - \int_{M_k \setminus G_k} \nabla^{M_k} x^{n+1} \cdot \nabla^{M_k} \tilde{\zeta}. \tag{4.26}$$

Let ζ be an arbitrary function in $C_c^1(B_{1/4}(0))$ and set $\zeta_1(x', x^{n+1}) = \zeta(x')$. Taking $\tilde{\zeta}$ in identity (4.26) to be a $C_c^1(B_{1/4}(0) \times \mathbf{R})$ function that agrees with ζ_1 in a neighborhood of $M_k \cap (B_{1/4}(0) \times \mathbf{R})$ and using also that $|\nabla^{M_k} x^{n+1}| \le 1$, we obtain that

$$\begin{aligned}(4.27)\qquad &\left|\int_{G_k\cap(B_{1/4}(0)\times\mathbf{R})}\nabla^{M_k}x^{n+1}\cdot\nabla^{M_k}\zeta_1\right|\\ &\quad\le\sup|D\zeta|\,\mathcal{H}^n\left((M_k\setminus G_k)\cap(B_{1/4}(0)\times\mathbf{R})\right)\\ &\quad\le c\sup|D\zeta|\,\epsilon_k^{2+\mu},\end{aligned}$$

where we have used estimate (3.24).

Now, observe that for any two C^1 functions φ, ψ on $B_1(0)$, if we let $\widetilde{\varphi}(x,x_{n+1})=\varphi(x)$ and $\widetilde{\psi}(x,x_{n+1})=\psi(x)$, then

$$\begin{aligned}\nabla^{M_k}\widetilde{\phi}&=\mathbf{p}_{T_xM_k}D\widetilde{\phi}\\ &=D\widetilde{\phi}-(\nu_k\cdot D\widetilde{\phi})\nu_k\\ &=D\widetilde{\phi}-((\nu_k-(\nu_k\cdot e_{n+1})e_{n+1})\cdot D\widetilde{\phi})\nu_k\end{aligned}$$

and hence,

$$(4.28)\qquad |\nabla^{M_k}\widetilde{\psi}\cdot\nabla^{M_k}\widetilde{\phi}-D\widetilde{\psi}\cdot D\widetilde{\phi}|\le(1-(\nu^k\cdot e_{n+1})^2)|D\widetilde{\psi}||D\widetilde{\phi}|.$$

Letting $\widetilde{u}_k^+(x,x^{n+1})=u_k^+(x)$, $\widetilde{u}_k^-(x,x^{n+1})=u_k^-(x)$ and using inequalities (4.27) and (4.28), we then estimate as follows:

$$\begin{aligned}(4.29)\qquad &\left|\int_{G_k^+}D\widetilde{u}_k^+\cdot D\zeta_1+\int_{G_k^-}D\widetilde{u}_k^-\cdot D\zeta_1\right|\\ &\le\left|\int_{G_k^+}\nabla^{M_k}\widetilde{u}_k^+\cdot\nabla^{M_k}\zeta_1+\int_{G_k^-}\nabla^{M_k}\widetilde{u}_k^-\cdot\nabla^{M_k}\zeta_1\right|\\ &\quad+\int_{G_k^+}\left(1-(\nu_k\cdot e_{n+1})^2)\right)|D\widetilde{u}_k^+||D\zeta_1|+\\ &\quad+\int_{G_k^-}\left(1-(\nu_k\cdot e_{n+1})^2)\right)|D\widetilde{u}_k^-||D\zeta_1|\\ &\le\left|\int_{G_k}\nabla^{M_k}x^{n+1}\cdot\nabla^{M_k}\zeta_1\right|+\sup|D\zeta|\int_{G_k}\left(1-(\nu_k\cdot e_{n+1})^2\right)\\ &\le 2\,c\sup|D\zeta|\,\epsilon_k^2.\end{aligned}$$

Dividing both sides of the above inequality by ϵ_k and using the area formula to express each of the integrals on the left-hand side as an integral over Ω_k, and noting that $\sqrt{1+|Du_k^+|^2}$, $\sqrt{1+|Du_k^-|^2}\le 2$, we obtain, by letting $k \to \infty$ (and using inequalities (3.26) and (3.28)) that

$$\int_{B_{1/4}(0)} (D(v^+ + v^-)) \cdot D\zeta = 0. \tag{4.30}$$

Since ζ is an arbitrary C^1 function with compact support in $B_{1/4}(0)$, it follows from Weyl's lemma that $v^+ + v^-$ is smooth and harmonic in $B_{1/4}(0)$. Also, v^+ and v^- are homogeneous of degree one (and hence defined everywhere in $\mathbf{R}^n$), because, by item (5) of HYPOTHESES ($\star$), Section 1, $\operatorname{graph} u_k^+ \cup \operatorname{graph} u_k^-$ approximates a cone. Applying the maximum principle in $B_{1/4}(0)$ to each of the partial derivatives $D_j(v^+ + v^-)$ (which are homogeneous of degree zero), we conclude that $v^+ + v^-$ is linear. By rotation, we assume without loss of generality that $v^+ + v^- \equiv 0$. This completes the proof of parts (e).

To prove parts (f), (g) and (h) of the lemma, we first show that the sequence $\{\overline{\psi}_k\, |Dv_k|\}$ is bounded in $W^{1,2}(B_{1/4}(0))$. Note that $\overline{\psi}_k\, |Dv_k|$ is locally Lipschitz, and hence is in $W^{1,\infty}_{\mathrm{loc}}(B_{1/4}(0))$. (Indeed, $|Dv_k^+| + |Dv_k^-| = |Dv_k^{(1)}| + |Dv_k^{(2)}|$ is a C^2 function in Ω_k; even though individually $Dv_k^{(1)}$ and $Dv_k^{(2)}$ are only locally defined, their sum has an unambiguous meaning everywhere in Ω_k and is in $C^2(\Omega_k)$.) Note also that $\operatorname{spt} \overline{\psi}_k \subseteq \Omega_k$. We estimate as follows:

$$\begin{aligned}
&\int_{B_{1/4}(0)} \left(\overline{\psi}_k\, |Dv_k|\right)^2 \\
&\quad \le \frac{2}{\epsilon_k^2} \int_{B_{1/4}(0)} \overline{\psi}_k^2\, (|Du_k^+|^2 + |Du_k^-|^2) \\
&\quad \le \frac{4}{\epsilon_k^2} \int_{B_{1/4}(0)} \overline{\psi}_k^2 \left(\frac{|Du_k^+|^2}{\sqrt{1 + |Du_k^+|^2}} + \frac{|Du_k^-|^2}{\sqrt{1 + |Du_k^-|^2}} \right) \\
&\quad = \frac{4}{\epsilon_k^2} \int_{G_k} \left(1 - (\nu_k . e_{n+1})^2\right) \\
&\quad \le 4.
\end{aligned} \tag{4.31}$$

$$\begin{aligned}
&\int_{B_{1/4}(0)} |D(\overline{\psi}_k\, |Dv_k|)|^2 \\
&\quad \le \frac{2}{\epsilon_k^2} \int_{B_{1/4}(0)} |D\overline{\psi}_k|^2\, |Du_k|^2 + \overline{\psi}_k^2\, |D(|Du_k|)|^2 \\
&\quad \le \sup |Du_k|^2\, \frac{2}{\epsilon_k^2} \int_{B_{1/4}(0)} |D\overline{\psi}_k|^2
\end{aligned} \tag{4.32}$$

$$
\begin{aligned}
&+ \frac{2}{\epsilon_k^2} \int_{B_{1/4}(0)} \overline{\psi}_k^2 \, |D(|Du_k|)|^2 \\
&\leq c \left(\epsilon_k^{\mu/2} + \frac{2}{\epsilon_k^2} \int_{M_k \cap (B_{1/4}(0) \times \mathbf{R})} |A_k|^2 \right) \\
&\leq c.
\end{aligned}
$$

We have used inequality (3.28) and Lemma 3.1.

Thus, by Rellich's compactness lemma, there exists $w \in W^{1,2}(B_{1/4}(0))$ such that, after passing to a subsequence,

$$
\overline{\psi}_k \, |Dv_k| \to w \tag{4.33}
$$

weakly in $W^{1,2}(B_{1/4}(0))$, strongly in $L^2(B_{1/4}(0))$ and pointwise a.e. in $B_{1/4}(0)$.

For parts (f) and (g), we want to establish the L^2 convergence of $\overline{\psi}_k \, Dv_k^+$ and $\overline{\psi}_k \, Dv_k^-$ separately; thus, we would like to get L^2 bounds on the gradients of these functions. However, $\bar{\psi}_k \, Dv_k^+$ and $\bar{\psi}_k \, Dv_k^-$ do not belong to the Sobolev space $W^{1,2}(B_{1/4}(0))$ because the functions v_k^+ and v_k^- have "corners." To get around this difficulty, we work with a suitable symmetric combination of v_k^+ and v_k^-. Then, corners would not cause a problem because our symmetric expression in v_k^+ and v_k^- will always be smooth (in fact, real analytic) as can be seen by rewriting it in terms of the real analytic functions $v_k^{(1)}$ and $v_k^{(2)}$. Notice that even though the functions $v_k^{(1)}$ and $v_k^{(2)}$ are only defined locally, the unordered pair $\{v_k^+, v_k^-\}$ is the same as the unordered pair $\{v_k^{(1)}, v_k^{(2)}\}$, and hence any symmetric expression in v_k^+ and v_k^- will continue to have an unambiguous meaning when v_k^+ and v_k^- are replaced by $v_k^{(1)}$ and $v_k^{(2)}$ respectively. The explicit technique we adopt is as follows.

Take $\epsilon > 0$ and let $\gamma_\epsilon \colon \mathbf{R} \to \mathbf{R}$ be a C^1 cut-off function with $\gamma_\epsilon(t) = 0$ if $t \leq \epsilon$, $\gamma_\epsilon(t) = 1$ if $t > 2\epsilon$, $\gamma_\epsilon(t) \geq 0$ and $\gamma_\epsilon'(t) \leq 2/\epsilon$ for all t. Let $V_k^\epsilon = \overline{\psi}_k(\gamma_\epsilon(v_k^+)Dv_k^+ + \gamma_\epsilon(v_k^-)Dv_k^-)$. Notice that V_k^ϵ is locally Lipschitz, and hence is in $W^{1,\infty}_{\mathrm{loc}}(B_{1/4}(0))$. (This is because, as indicated in the preceding paragraph, the unordered pair $\{v_k^+, v_k^-\}$ is the same as the unordered pair $\{v_k^{(1)}, v_k^{(2)}\}$ and therefore, $V_k^\epsilon = \overline{\psi}_k \gamma_\epsilon(v_k^{(1)})Dv_k^{(1)} + \gamma_\epsilon(v_k^{(2)})Dv_k^{(2)}$.) It follows from inequality (4.31) that:

$$
\int_{B_{1/4}(0)} |V_k^\epsilon|^2 \leq c
$$

and, by inequalities (3.27), (4.31), (3.1) and the fact that $|DDu_k^+| + |DDu_k^-| \leq |A_k|$ pointwise, that

$$\int_{B_{1/4}(0)} |DV_k^\epsilon| \leq \frac{c}{\epsilon^2}.$$

Therefore, by the compactness of the embedding $W^{1,1}(B_{1/4}(0)) \to L^1(B_{1/4}(0))$, there exists $V^\epsilon \in L^1(B_{1/4}(0))$ such that, after passing to a subsequence, $V_k^\epsilon \to V^\epsilon$ in $L^1(B_{1/4}(0))$ and pointwise a.e. in $B_{1/4}(0)$. Now, notice that since we may write $V_k^\epsilon = \overline{\psi}_k\, D(\Gamma_\epsilon(v_k^+)+\Gamma_\epsilon(v_k^-))$, where $\Gamma'(t) = \gamma(t)$, we have, for an arbitrary vector-valued C_c^1 function ζ on $B_{1/4}(0)$, that

$$\begin{aligned}
\int_{B_{1/4}(0)} \zeta \cdot V_k^\epsilon &= \int_{B_{1/4}(0)} \zeta \cdot \overline{\psi}_k D(\Gamma_\epsilon(v_k^+) + \Gamma_\epsilon(v_k^-)) \\
&= -\int_{B_{1/4}(0)} \zeta \cdot D\overline{\psi}_k(\Gamma_\epsilon(v_k^+) + \Gamma_\epsilon(v_k^-)) \\
&\quad - \int_{B_{1/4}(0)} \operatorname{div}\zeta\, \overline{\psi}_k(\Gamma_\epsilon(v_k^+) + \Gamma_\epsilon(v_k^-)).
\end{aligned} \tag{4.34}$$

Taking the limit as $k \to \infty$ on both sides of the above, and using inequality (3.28) to conclude that the first of the integrals on the right-hand side converges to zero, we obtain that

$$\begin{aligned}
\int_{B_{1/4}(0)} \zeta \cdot V^\epsilon &= -\int_{B_{1/4}(0)} \operatorname{div}\zeta(\Gamma_\epsilon(v^+) + \Gamma_\epsilon(v^-)) \\
&= \int_{B_{1/4}(0)} \zeta \cdot D(\Gamma_\epsilon(v^+) + \Gamma_\epsilon(v^-))
\end{aligned} \tag{4.35}$$

for every $C_c^1(B_{1/4}(0))$ vector field ζ. Therefore, $V^\epsilon = D(\Gamma_\epsilon(v^+) + \Gamma_\epsilon(v^-)) = \gamma_\epsilon(v^+)Dv^+ + \gamma_\epsilon(v^-)Dv^- = \gamma_\epsilon(v^+)Dv^+$. Thus, we have shown that, after passing to a subsequence,

$$\overline{\psi}_k(\gamma_\epsilon(v_k^+)Dv_k^+ + \gamma_\epsilon(v_k^-)Dv_k^-) \to \gamma_\epsilon(v^+)Dv^+ \tag{4.36}$$

in $L^1(B_{1/4}(0))$ and pointwise a.e. in $B_{1/4}(0)$.

Similarly, we obtain that

$$\overline{\psi}_k(\gamma_\epsilon(-v_k^+)Dv_k^+ + \gamma_\epsilon(-v_k^-)Dv_k^-) \to \gamma_\epsilon(-v^-)Dv^- \tag{4.37}$$

in $L^1(B_{1/4}(0))$ and pointwise a.e. in $B_{1/4}(0)$.

However, $\overline{\psi}_k\gamma_\epsilon(v_k^-)Dv_k^-$, $\overline{\psi}_k\gamma_\epsilon(-v_k^+)Dv_k^+ \to 0$ in L^2; in fact

$$\int_{\{v_k^- \geq \epsilon\}} (\overline{\psi}_k\,|Dv_k^-|)^2, \quad \int_{\{v_k^+ \leq \epsilon\}} (\overline{\psi}_k\,|Dv_k^+|)^2 \to 0. \tag{4.38}$$

This follows from (the generalized) Lebesgue Dominated Convergence Theorem because $|\overline{\psi}_k\gamma_\epsilon(v_k^-)Dv_k^-|^2$ and $|\overline{\psi}_k\gamma_\epsilon(-v_k^+)Dv_k^+|^2$ are bounded by $(\overline{\psi}_k|Dv_k|)^2$ and, by (4.33), the latter converges pointwise a.e. and in L^1 to w^2. Also, the pointwise a.e. limits of $\chi_{\{v_k^- \geq \epsilon\}}\,\overline{\psi}_k\,Dv_k^-$ and $\chi_{\{v_k^+ \leq \epsilon\}}\,\overline{\psi}_k\,Dv_k^+$ are both zero because $|\overline{\psi}_k Dv_k^{\pm}|$ are bounded by $2|w|$, $\bar{\psi}_k\,v_k^- \to v^- \leq 0$, $\bar{\psi}_k\,v_k^+ \to v^+ \geq 0$ and $\bar{\psi}_k \to 1$ a.e. in $B_{1/4}(0)$.

We, therefore, have that

$$\overline{\psi}_k\gamma_\epsilon(v_k^+)Dv_k^+ \to \gamma_\epsilon(v^+)Dv^+ \tag{4.39}$$

and

$$\overline{\psi}_k\gamma_\epsilon(-v_k^-)Dv_k^- \to \gamma_\epsilon(-v^-)Dv^- \tag{4.40}$$

in $L^1(B_{1/4}(0))$ and pointwise a.e. in $B_{1/4}(0)$.

Now, it follows by letting $k \to \infty$ in the estimate (4.6) and using the lower semi-continuity of energy that

$$\int_{B_{1/4}(0)\cap\{|v^+|\leq\epsilon\}} |Dv^+|^2 + \int_{B_{1/4}(0)\cap\{|v^-|\leq\epsilon\}} |Dv^-|^2 \leq c\,\epsilon. \tag{4.41}$$

In view of inequalities (4.6), (4.41), (4.38) (all with $2\,\epsilon$ in place of ϵ) and the Cauchy–Schwarz inequality, we may let $\epsilon \to 0$ in (4.39) and (4.40) to conclude that

$$\overline{\psi}_k Dv_k^+ \to Dv^+ \tag{4.42}$$

and

$$\overline{\psi}_k Dv_k^- \to Dv^- \tag{4.43}$$

in $L^1(B_{1/4}(0))$ and pointwise a.e. in $B_{1/4}(0)$.

This in particular shows that the function w in (4.33) must be equal to $|Dv^+| + |Dv^-|$, and therefore that

$$\bar{\psi}_k\,|Dv_k| \to |Dv^+| + |Dv^-| \tag{4.44}$$

strongly in $L^2(B_{1/4}(0))$, weakly in $W^{1,2}(B_{1/4}(0))$ and pointwise a.e. in $B_{1/4}(0)$. Since $w \in W^{1,2}(B_{1/4}(0))$, part (h) of the lemma follows.

To complete the proof of parts (f) and (g), we show that the convergence in (4.36) and (4.37) is indeed in $L^2(B_{1/4}(0))$. This would follow immediately if we could prove that DV_k^ϵ are uniformly bounded in

$L^2(B_{1/4}(0))$, but the terms resulting from differentiating the γ_ϵ factor of V_k^ϵ are of degree 4 in the partial derivatives of v_k and hence are not in L^2 uniformly. Therefore, we modify V_k^ϵ further by cutting off regions where $|Dv_k|$ is large. Thus, we let $V_k^{\epsilon,K} = \varphi_K(|Dv_k|)V_k^\epsilon$, where $K > 0$ is a (large) fixed number and $\varphi_K \in C^1(\mathbf{R})$ with $\varphi_K(t) \equiv 1$ for $t \leq K/2$, $\varphi_K(t) \equiv 0$ for $t \geq K$ and $|D\varphi_K(t)| \leq 4/K$ for all t. It is easy to see then, by inequalities (4.31), (3.28), (3.1) and the fact that $|DDu_k^+| + |DDu_k^-| \leq c\,|A_k|$ pointwise, that

$$\int_{B_{1/4}(0)} |V_k^{\epsilon,K}|^2 \leq c$$

and

$$\int_{B_{1/4}(0)} |DV_k^{\epsilon,K}|^2 \leq \frac{c\,K^4}{\epsilon^2}.$$

Therefore, there exists a function $V^{\epsilon,K} \in W^{1,2}(B_{1/4}(0))$ such that

$$V_k^{\epsilon,K} \to V^{\epsilon,K}$$

in $L^2(B_{1/4}(0))$, weakly in $W^{1,2}(B_{1/4}(0))$ and pointwise a.e. in $B_{1/4}(0)$ as $k \to \infty$. However, by (4.36) and (4.44), we know that

$$V_k^{\epsilon,K} \to \varphi_K(|Dv^+| + |Dv^-|)\,\gamma_\epsilon(v^+)Dv^+$$

pointwise a.e., and therefore, we have that

$$V_k^{\epsilon,K} \to \varphi_K(|Dv^+| + |Dv^-|)\,\gamma_\epsilon(v^+)Dv^+ \tag{4.45}$$

in $L^2(B_{1/4}(0))$, weakly in $W^{1,2}(B_{1/4}(0))$ and pointwise a.e. in $B_{1/4}(0)$ as $k \to \infty$.

Now, $\mathcal{L}^n\{x \in B_{1/4}(0) : |Dv_k(x)| \geq K/2\} \to 0$ uniformly in k as $K \to \infty$ because by (4.44), $|Dv_k|$ converge in $L^1(B_{1/4}(0))$. Therefore, in view of the fact that V_k^ϵ converge in $L^1(B_{1/4}(0))$ (by (4.36)), we have by Lebesgue Dominated Convergence theorem that

$$\int_{B_{1/4}(0)\cap\{x\,:\,|Dv_k|\geq K/2\}} |V_k^\epsilon|^2 \to 0 \tag{4.46}$$

uniformly in k as $K \to \infty$. In view of (4.44) and (4.45), letting $k \to \infty$ in (4.46), we also have that

$$\int_{B_{1/4}(0)\cap\{x:|Dv^+|+|Dv^-|\geq K/2\}} |\gamma_\epsilon(v^+)Dv^+|^2 \to 0 \tag{4.47}$$

as $K \to \infty$.

By (4.46) and (4.47), we may let $K \to \infty$ in (4.45) to conclude that the convergence in (4.36) is in $L^2(B_{1/4}(0))$. A similar argument shows that the convergence in (4.37) is in $L^2(B_{1/4}(0))$. In view of (4.38), this implies that the convergence in (4.39) and (4.40) is in $L^2(B_{1/4}(0))$. Finally, in view of inequalities (4.6) and (4.41), we may let $\epsilon \to 0$ in (4.39) and (4.40) to conclude that the convergence in (4.42) and (4.43) is in $L^2(B_{1/4}(0))$. This completes the proof of parts (f) and (g). q.e.d.

Remark. We have that

$$v^+ \not\equiv 0. \tag{4.48}$$

This follows from parts (d) and (e) of Lemma 4.6.

In the following lemma, we prove several integral identities and estimates involving v^+ which we will use in the next section. Notice that since $v^- = -v^+$, these identities and estimates also hold with v^- in place of v^+.

Lemma 4.7. *Let $\epsilon \in (0, 1/2)$, $y \in B_{1/8}(0)$ and $\sigma \in (0, 1/8]$ be arbitrary. Then*

(i) $\int_{B_{1/4}(0)} |Dv^+|^2\zeta = -\int_{B_{1/4}(0)} v^+ Dv^+ \cdot D\zeta$ *for every* $\zeta \in C^1_c(B_{1/4}(0))$.

(ii) $\int_{B_\sigma(y)} |Dv^+|^2 = \int_{\partial B_\sigma(y)} v^+ \frac{\partial v^+}{\partial R}$ *where* $\frac{\partial v^+}{\partial R}(x) = Dv^+(x) \cdot \frac{x-y}{\sigma}$.

(iii) $\int_{B_{\sigma/2}(y)} |Dv^+|^2 \leq 8\sigma^{-2} \int_{B_\sigma(y)} (v^+)^2$.

(iv) $\int_{B_{\sigma/2}(y) \cap \{x : v^+(x) \leq \epsilon\sigma^{-n/2}\|v^+\|_{L^2(B_\sigma(y))}\}} |Dv^+|^2 \leq c\epsilon\sigma^{-2} \int_{B_\sigma(y)} (v^+)^2$.

Here, $c = c(n)$.

Proof. We begin by replacing $\tilde{\zeta}$ with $x^{n+1}\tilde{\zeta}$ in identity (4.1) to deduce that

$$\int_{M_k} |\nabla^{M_k} x^{n+1}|^2 \tilde{\zeta} = -\int_{M_k} x^{n+1} \nabla^{M_k} x^{n+1} \cdot \nabla^{M_k} \tilde{\zeta} \tag{4.49}$$

for every $\tilde{\zeta} \in C^1_c(\mathbf{R}^{n+1})$.

Let $\zeta \in C^1_c(B_{1/4}(0))$ be arbitrary. There exists a $\tilde{\zeta} \in C^1_c(B_{1/4}(0) \times \mathbf{R})$ such that $\tilde{\zeta} \equiv (\overline{\psi}_k)^2\zeta_1$ in a neighborhood of M_k where $\zeta_1(x, x^{n+1}) = \zeta(x)$. (Here again, $\overline{\psi}_k$ is the cut-off function defined in Section 3.) Therefore, identity (4.49) holds with $(\overline{\psi}_k)^2\zeta_1$ in place of $\tilde{\zeta}$, and we obtain that

(4.50)

$$\begin{aligned}\int_{G_k} &|\nabla^{M_k} x^{n+1}|^2(\overline{\psi}_k)^2\zeta_1\\ &= -\int_{G_k} x^{n+1}\nabla^{M_k} x^{n+1}\cdot\nabla^{M_k}((\overline{\psi}_k)^2\zeta_1)\\ &\quad -\int_{M_k\setminus G_k} |\nabla^{M_k} x^{n+1}|^2(\overline{\psi}_k)^2\zeta_1 + x^{n+1}\nabla^{M_k} x^{n+1}\cdot\nabla^{M_k}((\overline{\psi}_k)^2\zeta_1).\end{aligned}$$

Since $\nabla^{M_k}((\overline{\psi}_k)^2\zeta_1) = \mathbf{p}_{T_xM_k}D((\overline{\psi}_k)^2\zeta_1) = D((\overline{\psi}_k)^2\zeta_1) - (D((\overline{\psi}_k)^2\zeta_1)\cdot\nu_k)\nu_k$ and $\nabla^{M_k}x^{n+1} = e_{n+1} - (\nu_k.e_{n+1})\nu_k$, we have that (since ζ_1 is independent of x^{n+1}), $\nabla^{M_k}((\overline{\psi}_k)^2\zeta_1)\cdot\nabla^{M_k}x^{n+1} = -(D((\overline{\psi}_k)^2\zeta_1)\cdot\nu_k)(\nu_k\cdot e_{n+1})$ and $|\nabla^{M_k}x^{n+1}|^2 = 1-(\nu_k\cdot e_{n+1})^2$. Substituting these expressions into the identity (4.50), and using also the fact that $\nu_k = (-Du_k^+,1)/(1+|Du_k^+|^2)^{1/2}$ on G_k^+ and $\nu_k = (-Du_k^-,1)/(1+|Du_k^-|^2)^{1/2}$ on G_k^-, we deduce, after dividing both sides of (4.50) by ϵ_k^2, that

(4.51)

$$\begin{aligned}\int_{B_{1/4}(0)} &\left(\frac{(\overline{\psi}_k|Dv_k^+|)^2}{\sqrt{1+|Du_k^+|^2}} + \frac{(\overline{\psi}_k|Dv_k^-|)^2}{\sqrt{1+|Du_k^-|^2}}\right)\zeta\\ &= -\int_{B_{1/4}(0)} \frac{\overline{\psi}_k v_k^+\overline{\psi}_k Dv_k^+\cdot D\zeta}{\sqrt{1+|Du_k^+|^2}} - \int_{B_{1/4}(0)} \frac{\overline{\psi}_k v_k^-\overline{\psi}_k Dv_k^-\cdot D\zeta}{\sqrt{1+|Du_k^-|^2}} -\\ &\quad -\frac{1}{\epsilon_k^2}\int_{B_{1/4}(0)} 2\left(\frac{\overline{\psi}_k\zeta u_k^+ Du_k^+}{\sqrt{1+|Du_k^+|^2}} + \frac{\overline{\psi}_k\zeta u_k^- Du_k^-}{\sqrt{1+|Du_k^-|^2}}\right)\cdot D\overline{\psi}_k\\ &\quad -\frac{1}{\epsilon_k^2}\int_{M_k\setminus G_k} |\nabla^{M_k} x^{n+1}|^2(\overline{\psi}_k)^2\zeta_1 + x^{n+1}\nabla^{M_k} x^{n+1}\cdot\nabla^{M_k}(\overline{\psi}_k)^2\zeta_1.\end{aligned}$$

Since $|Du_k^\pm|\to 0$ a.e. (because e.g., $\overline{\psi}_k Du_k^\pm = \epsilon_k\overline{\psi}_k Dv_k^\pm$ and $\overline{\psi}_k Dv_k^\pm \to Dv^\pm$, $\overline{\psi}_k\to 1$ a.e.), the integral on the left-hand side of the above converges, by Lemma 4.6 (e) and (f), to $\int_{B_{1/4}(0)}(|Dv^+|^2+|Dv^-|^2)\zeta$. On the right-hand side, the first and the second integrals converge to $\int_{B_{1/4}(0)} v^+Dv^+\cdot D\zeta$ and $\int_{B_{1/4}(0)} v^-Dv^-\cdot D\zeta$, respectively, because $\sqrt{1+|Du_k^\pm|^2}\to 1$ a.e., $\overline{\psi}_k v_k^\pm\to v^\pm$ in $L^2(B_{1/4}(0))$ and $\overline{\psi}_k Dv_k^\pm\to Dv^\pm$ in $L^2(B_{1/4}(0))$ by Lemma 4.6. The third integral on the right-hand side converges to zero by estimate (3.27) because $u_k^\pm$, $Du_k^\pm$, $\overline{\psi}_k$ and ζ are bounded. And by estimate (3.24), the last integral on the right-hand side converges to zero also because the integrand is bounded. Thus, taking the limit as $k \to \infty$ on both sides and using $v^- \equiv -v^+$, we obtain from (4.51) that

$$\int_{B_{1/4}(0)} |Dv^+|^2 \zeta = -\int_{B_{1/4}(0)} v^+ Dv^+ \cdot D\zeta \tag{4.52}$$

for every $\zeta \in C^1_c(B_{1/4}(0))$. This is part (i) of the lemma.

Parts (ii) and (iii) of the lemma are direct consequences of part (i). To see (ii), first observe that by approximation, we may take ζ in the identity in (i) to be Lipschitz. For $0 < \sigma,\, \tau \leq 1/8$, let $\gamma\colon [0,1] \to \mathbf{R}$ be defined by $\gamma(s) = 1$ for $0 \leq s \leq \sigma$, $\gamma(s) = 0$ for $\sigma + \tau \leq s \leq 1$ and linear elsewhere. Taking $\zeta(x) = \gamma(|x-y|)$ in (i) and letting $\tau \to 0$, we obtain that

$$\int_{B_\sigma(y)} |Dv^+|^2 = \int_{\partial B_\sigma(y)} v^+ \frac{\partial v^+}{\partial R} \tag{4.53}$$

for all $y \in B_{1/8}(0)$ and all $\sigma \in (0, 1/8]$, where $\frac{\partial v^+}{\partial R}(x) = Dv^+(x).\frac{x-y}{\sigma}$, proving part (ii).

To see (iii), first replace ζ in the identity in part (i) by ζ^2 to get

$$\int_{B_{1/4}(0)} |Dv^+|^2 \zeta^2 = -2\int_{B_{1/4}(0)} v^+ \zeta Dv^+ \cdot D\zeta. \tag{4.54}$$

Using Cauchy–Schwarz inequality on the right-hand side of (4.54) yields

$$\int_{B_{1/4}(0)} |Dv^+|^2 \zeta^2 \leq 4\int_{B_{1/4}(0)} \left(v^+\right)^2 |D\zeta|^2. \tag{4.55}$$

Now, choose the cut-off function ζ such that $\zeta \equiv 1$ in $B_{\sigma/2}(y)$, $\zeta \equiv 0$ in $B_{1/4}(0) \setminus B_\sigma(y)$ and $|D\zeta| \leq 2/\sigma$. This gives the required estimate that

$$\int_{B_{\sigma/2}(y)} |Dv^+|^2 \leq 8\sigma^{-2} \int_{B_\sigma(y)} \left(v^+\right)^2 \tag{4.56}$$

for all $y \in B_{1/8}(0)$ and $\sigma \in (0, 1/8]$.

Finally, to prove part (iv), we recall the inequality (4.3). Choosing $\zeta \in C^1_c(B_1(0))$ to be a standard cut-off function satisfying $\zeta(t) \equiv 1$ on $B_{\sigma/2}(y)$, $\zeta \equiv 0$ outside $B_{3\sigma/4}(y)$ and $|D\zeta| \leq 8/\sigma$, and taking $\tilde{\zeta}$ in (4.3) to be the extension of ζ to $\mathbf{R}^{n+1}$ that is constant in the x^{n+1} direction, we obtain from (4.3) that

$$
\begin{aligned}
(4.57)\qquad &\int_{M_k\cap(B_{\sigma/2}(y)\times\mathbf{R})\cap\{|x^{n+1}|\le\delta\}} |\nabla^{M_k}x^{n+1}|^2 \\
&\quad\le c\,\delta\,\sigma^{-1}\int_{M_k\cap(B_{3\sigma/4}(y)\times\mathbf{R})}|\nabla^{M_k}x^{n+1}| \\
&\quad\le c\,\delta\,\sigma^{-1}\left(\int_{M_k\cap G_k\cap(B_{3\sigma/4}(y)\times\mathbf{R})}|\nabla^{M_k}x^{n+1}|+\epsilon_k^{2+\mu}\right),
\end{aligned}
$$

where we have used inequality (3.24). Since $|Du_k^{\pm}|\le 1$, it follows that $M_k\cap G_k^+\cap(B_{3\sigma/4}(y)\times\mathbf{R})\subseteq B^{n+1}_{3\sigma/2}(y^+)$ and $M_k\cap G_k^-\cap(B_{3\sigma/4}(y)\times\mathbf{R})\subseteq B^{n+1}_{3\sigma/2}(y^-)$ for suitable $y^+,\ y^-\in\mathbf{R}^{n+1}$, and therefore by the monotonicity inequality for M_k and the Cauchy–Schwarz inequality, it follows from (4.57) that

(4.58)

$$
\begin{aligned}
&\int_{M_k\cap(B_{\sigma/2}(y)\times\mathbf{R})\cap\{|x^{n+1}|\le\delta\}} |\nabla^{M_k}x^{n+1}|^2 \\
&\quad\le c\,\delta\,\sigma^{-1}\left(\sigma^{n/2}\left(\int_{M_k\cap G_k\cap(B_{3\sigma/4}(y)\times\mathbf{R})}|\nabla^{M_k}x^{n+1}|^2\right)^{1/2}+\epsilon_k^{1+\mu}\sigma^n\right)
\end{aligned}
$$

for sufficiently large k depending on σ.

By making an appropriate choice of ζ in inequality (4.17), we have on the other hand that

$$
(4.59)\qquad \int_{M_k\cap B_{3\sigma/4}(y)\times\mathbf{R}}|\nabla^{M_k}x^{n+1}|^2\le c\,\sigma^{-2}\int_{M_k\cap B_{\sigma/2}(y)\times\mathbf{R}}(x^{n+1})^2.
$$

Combining inequalities (4.58) and (4.59), we have that

$$
\begin{aligned}
(4.60)\qquad &\int_{M_k\cap(B_{\sigma/2}(y)\times\mathbf{R})\cap\{|x^{n+1}|\le\delta\}} |\nabla^{M_k}x^{n+1}|^2 \\
&\quad\le c\,\delta\,\sigma^{-2}\left(\sigma^{n/2}\left(\int_{M_k\cap B_\sigma(y)\times\mathbf{R}}(x^{n+1})^2\right)^{1/2}+\epsilon_k^{1+\mu}\sigma^n\right).
\end{aligned}
$$

It follows from this, the boundedness of $|Du_k^{\pm}|$ and inequality (3.26) that

(4.61)

$$\int_{B_{\sigma/2}(y)\cap\{\overline{\psi}_k|u_k^+|\leq\delta\}}\overline{\psi}_k|Du_k^+|^2+\int_{B_{\sigma/2}(y)\cap\{\overline{\psi}_k|u_k^-|\leq\delta\}}\overline{\psi}_k|Du_k^-|^2$$
$$\leq c\,\delta\,\sigma^{n-2}\left(\sigma^{-n/2}\left(\int_{B_\sigma(y)}(\overline{\psi}_k\,u_k^+)^2+(\overline{\psi}_k\,u_k^-)^2\right)^{1/2}+\epsilon_k^{1+\mu/2}\right).$$

Choosing $\delta=\delta_k\equiv\frac{1}{2}\epsilon\left(\sigma^{-n/2}\left(\int_{B_\sigma(y)}(\overline{\psi}_k u_k^+)^2+(\overline{\psi}_k u_k^-)^2\right)^{1/2}+\epsilon_k^{1+\mu/2}\right)$ in (4.61) and dividing both sides by ϵ_k^2, and using also inequalities (3.28) and (3.26), we obtain from inequality (4.61) that

$$\int_{B_{\sigma/2}(y)\cap\{|\overline{\psi}_k v_k^+|\leq\frac{\delta_k}{\epsilon_k}\}}\overline{\psi}_k|Dv_k^+|^2+\int_{B_{\sigma/2}(y)\cap\{|\overline{\psi}_k v_k^-|\leq\frac{\delta_k}{\epsilon_k}\}}\overline{\psi}_k|Dv_k^-|^2$$
$$\leq c\,\epsilon\,\sigma^{-2}\left(\int_{B_\sigma(y)}(\overline{\psi}_k v_k^+)^2+(\overline{\psi}_k v_k^-)^2+\epsilon_k^\mu\right). \tag{4.62}$$

The estimate in part (iv) now follows by letting $k\to\infty$ in this and observing that $\frac{\delta_k}{\epsilon_k}\to\epsilon\,\sigma^{-n/2}\|v^+\|_{L^2(B_\sigma(y))}$. This completes the proof of the lemma. q.e.d.

We also have the following important lemma. We will use this lemma to prove monotonicity of the "frequency function" of Section 5, a key ingredient in the proof of Theorem 1.1.

Lemma 4.8. *v^+ satisfies the following harmonic identity:*

$$\sum_{i,j=1}^n\int_{B_\sigma(y)}(\delta_{ij}|Dv^+|^2-2D_iv^+D_jv^+)D_i\zeta^j=0 \tag{4.63}$$

for all $y\in B_{1/8}(0)$, $\sigma\in(0,1/8]$ and $\zeta^j\in C_c^1(B_\sigma(y))$, $\quad j=1,\ldots,n$.

Proof. Let $y\in B_{1/8}(0)$, $\sigma\in(0,1/8]$ be arbitrary and suppose $\zeta^j\in C_c^1(B_\sigma(y))$, $j=1,2\ldots,n$. First, extend each ζ^j to $B_\sigma(y)\times\mathbf{R}$ by setting $\tilde{\zeta}^j(x,x^{n+1})=\zeta^j(x)$ and then let ζ_1^j be a $C_c^1(B_\sigma(y)\times\mathbf{R})$ function that agrees with $\tilde{\zeta}^j$ in a neighborhood of $M_k\cap(B_\sigma(y)\times\mathbf{R})$. Let Z_k be the locally Lipschitz vector field defined by $Z_k=\overline{\psi}_k^2(\zeta_1^1,\zeta_1^2,\ldots,\zeta_1^n,0)$, where $\overline{\psi}_k$ is the cut-off function defined in Section 3. Notice that $\operatorname{spt}Z_k\subset\subset B_\sigma(y)\cap\Omega_k$. We shall use identity (2.8) with Z_k in place of X. First, we compute $\operatorname{div}_{M_k}Z_k$.

Let $\nu_k = (\nu_k^1, \nu_k^2, \ldots, \nu_k^{n+1})$ denote the unit normal on M_k. Then,

$$
\begin{aligned}
\operatorname{div}_{M_k} Z_k &= \sum_{i=1}^{n+1} e_i \cdot \nabla^{M_k}(\overline{\psi}_k^2 \zeta_1^i) \\
&= \sum_{i=1}^{n+1} e_i \cdot \mathbf{p}_{T_x M_k}\, D(\overline{\psi}_k^2\, \zeta_1^i) \\
&= \sum_{i=1}^{n+1} e_i \cdot \Big(D(\overline{\psi}_k^2\, \zeta_1^i) - (D(\overline{\psi}_k^2\, \zeta_1^i) \cdot \nu_k)\nu_k \Big) \\
&= \sum_{i=1}^{n+1} D_i(\overline{\psi}_k^2\, \zeta_1^i) - \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} D_j(\overline{\psi}_k^2\, \zeta_1^i)\nu_k^j \nu_k^i \\
&= \sum_{i,j=1}^{n} \Big(\delta_{ij} - \nu_k^i \nu_k^j \Big) D_j(\overline{\psi}_k^2\, \zeta_1^i).
\end{aligned}
\tag{4.64}
$$

Using (4.64) in (2.8), we then obtain

$$
\sum_{i,j=1}^{n} \int_{M_k} \Big(\delta_{ij} - \nu_k^i \nu_k^j \Big) D_j(\overline{\psi}_k^2\, \zeta_1^i) = 0. \tag{4.65}
$$

This can be written as

$$
\sum_{i,j=1}^{n} \int_{G_k} \Big(\delta_{ij} - \nu_k^i \nu_k^j \Big) D_j(\overline{\psi}_k^2\, \zeta_1^i) = F_k, \tag{4.66}
$$

where $F_k = \sum_{i,j=1}^{n} \int_{M_k \setminus G_k} \Big(\delta_{ij} - \nu_k^i \nu_k^j \Big) D_j(\overline{\psi}_k^2\, \zeta_1^i)$. Now

$$
\begin{aligned}
\int_{M_k \setminus G_k} & |(\delta_{ij} - \nu_k^i\, \nu_k^j) D_j(\overline{\psi}_k^2\, \zeta_1^i)| \\
&\leq \int_{M_k \setminus G_k} |\delta_{ij} - \nu_k^i\, \nu_k^j| |\overline{\psi}_k|^2 |D_j \zeta_1^i| \\
&\quad + 2 \int_{M_k \setminus G_k} |\delta_{ij} - \nu_k^i\, \nu_k^j| \overline{\psi}_k | D_j \overline{\psi}_k| |\zeta_1^i|.
\end{aligned}
\tag{4.67}
$$

The integrand of the first integral on the right-hand side of inequality (4.67) is bounded, and by inequality (3.24),

$$
\mathcal{H}^n \left((M_k \setminus G_k) \cap (B_{1/2}(0) \times \mathbf{R}) \right) \leq c\, \epsilon_k^{2+\mu}.
$$

Since $|\delta_{ij} - \nu_k^i\,\nu_k^j||\zeta_1^i|\overline{\psi}_k$ is bounded, we also have by inequality (3.27) that the second integral on the right-hand side of inequality (4.67) is bounded by $c\,\epsilon_k^{2+\mu/2}$. Therefore, $F_k/\epsilon_k^2 \to 0$ as $k \to \infty$.

Since $\nu_k^i = \frac{-D_i u_k^+}{\sqrt{1+|Du_k^+|^2}}$ on G_k^+ and $\nu_k^i = \frac{-D_i u_k^-}{\sqrt{1+|Du_k^-|^2}}$ on G_k^-, we obtain from Equation (4.66) and the area formula that

(4.68)
$$\sum_{i,j=1}^n \int_{B_\sigma(y)} \left(\sqrt{1+|Du_k^+|^2} + \sqrt{1+|Du_k^-|^2}\right) \delta_{ij}\, D_j(\overline{\psi}_k^2\,\zeta^i)$$
$$- \sum_{i,j=1}^n \int_{B_\sigma(y)} \left(\frac{D_i u_k^+\, D_j u_k^+}{\sqrt{1+|Du_k^+|^2}} + \frac{D_i u_k^-\, D_j u_k^-}{\sqrt{1+|Du_k^-|^2}}\right) D_j(\overline{\psi}_k^2\,\zeta^i) = F_k.$$

By the divergence theorem, we also trivially have that

$$\sum_{i,j=1}^n \int_{B_\sigma(y)} 2\,\delta_{ij}\, D_j(\overline{\psi}_k^2 \zeta^i) = 0. \tag{4.69}$$

Now, subtract (4.69) from (4.68) and divide both sides by ϵ_k^2. The left-hand side of the resulting identity is then the sum over i, j of the integrals

$$(I_1) \quad \frac{1}{\epsilon_k^2}\int_{B_\sigma(y)} \frac{2|Du_k^+|^2}{1+\sqrt{1+|Du_k^+|^2}}\delta_{ij}\zeta^i\overline{\psi}_k D_j\overline{\psi}_k,$$

$$(I_2) \quad \int_{B_\sigma(y)} \frac{|Dv_k^+|^2}{1+\sqrt{1+|Du_k^+|^2}}\delta_{ij}\overline{\psi}_k^2\, D_j\zeta^i,$$

$$(I_3) \quad \frac{1}{\epsilon_k^2}\int_{B_\sigma(y)} \left(\frac{2\,D_i u_k^+\, D_j u_k^+}{\sqrt{1+|Du_k^+|^2}}\right)\zeta^i\overline{\psi}_k D_j\overline{\psi}_k,$$

$$(I_4) \quad \int_{B_\sigma(y)} \left(\frac{D_i v_k^+\, D_j v_k^+}{\sqrt{1+|Du_k^+|^2}}\right)\overline{\psi}_k^2 D_j\zeta^i$$

and similar integrals with u_k^- in place of u_k^+. Since $D_i v_k^+ \to D_i v^+$, $|Du_k^+| \to 0$ and $\overline{\psi}_k \to 1$ pointwise a.e. in $B_{1/4}(0)$, we see by the generalized Lebesgue Dominated Convergence Theorem that the integrals in (I_2) and (I_4) converge to $\frac{1}{2}\int_{B_\sigma(y)} |Dv^+|^2\delta_{ij}D_j\zeta^i$ and $\int_{B_\sigma(y)} D_i v^+\, D_j v^+\, D_j\zeta^i$, respectively. (The integrands in (I_2) and (I_4)

are bounded by $c\,\overline{\psi}_k^2 |Dv_k^+|^2$ which converges a.e. and in L^1 to $c\,|Dv^+|$.) The integrals in (I_1) and (I_3) converge to zero by estimate (3.27) because $\frac{|Du_k^+|^2}{1+\sqrt{1+|Du_k^+|^2}}$ and $\frac{D_i u_k^+ \, D_j u_k^+}{\sqrt{1+|Du_k^+|^2}}$ are bounded. Thus, the lemma follows by subtracting (4.69) from (4.68) and letting $k \to \infty$ in the resulting identity after dividing it through by ϵ_k^2. q.e.d.

Next, we establish boundedness of the gradient of v^+, a result that will play a very important role in Section 5, where we uncover the geometry of the blow-up.

Lemma 4.9. *$|Dv^+|^2$ is sub-harmonic in $B_{1/4}(0)$, and hence, by the mean value property, $|Dv^+|$ is bounded in $B_{1/8}(0)$.*

Proof. We use the identity

$$\Delta_{M_k} (1 - \nu_k \cdot e_{n+1}) = (\nu_k \cdot e_{n+1}) |A_k|^2. \tag{4.70}$$

Let φ be a smooth, non-negative function with compact support in $B_{1/4}(0)$. Extend φ to $B_{1/4}(0) \times \mathbf{R}$ by $\widetilde{\varphi}(x, x_{n+1}) \equiv \varphi(x)$ and let φ_1 be any smooth function with compact support in $B_{1/4}(0) \times \mathbf{R}$ that agrees with $\widetilde{\varphi}$ in a neighborhood of $M_k \cap (B_{1/4}(0) \times \mathbf{R})$.

Multiplying both sides of identity (4.70) by $\overline{\psi}_k \varphi_1$ and integrating over M_k gives

$$\int_{M_k} \overline{\psi}_k \varphi_1 \, \Delta_{M_k} (1 - (\nu_k \cdot e_{n+1})) = \int_{M_k} \overline{\psi}_k \varphi_1 \, (\nu_k \cdot e_{n+1}) \, |A_k|^2. \tag{4.71}$$

Integrating by parts on the left-hand side of Equation (4.71) and decomposing the integral on the right-hand side as integrals over G_k and $M_k \setminus G_k$, we obtain that

$$\begin{aligned} &\int_{M_k} \nabla^{M_k} (\overline{\psi}_k \varphi_1) \cdot \nabla^{M_k} (1 - (\nu_k \cdot e_{n+1})) \\ &\quad = - \int_{G_k} \overline{\psi}_k \varphi_1 \, (\nu_k \cdot e_{n+1}) \, |A_k|^2 - \int_{M_k \setminus G_k} \overline{\psi}_k \varphi_1 \, (\nu_k \cdot e_{n+1}) \, |A_k|^2. \end{aligned} \tag{4.72}$$

The first integral on the right-hand side of (4.72) is non-negative because the integrand is non-negative on G_k. Therefore, we have that

$$\int_{M_k} \nabla^{M_k} (\overline{\psi}_k \, \varphi_1) \cdot \nabla^{M_k} \, (1 - (\nu_k \cdot e_{n+1})) \leq - \int_{M_k \setminus G_k} \overline{\psi}_k \, \varphi_1 (\nu_k \cdot e_{n+1}) |A_k|^2. \tag{4.73}$$

We estimate the integral on the right-hand side of inequality (4.73) as follows:

$$
\begin{aligned}
(4.74)\qquad & \left|\int_{M_k\setminus G_k} \overline{\psi}_k\varphi_1\,(\nu_k\cdot e_{n+1})\,|A_k|^2\right| \\
&\leq \sup|\varphi_1| \int_{(M_k\setminus G_k)\cap(B_{1/4}(0)\times\mathbf{R})} |A_k|^2 \\
&\leq \sup|\varphi_1| \left(\mathcal{H}^n((M_k\setminus G_k)\cap(B_{1/4}(0)\times\mathbf{R}))\right)^{\frac{1}{2}} \\
&\qquad\cdot\left(\int_{(M_k\setminus G_k)\cap(B_{1/4}(0)\times\mathbf{R})} |A_k|^4\right)^{\frac{1}{2}} \\
&\leq c\sup|\varphi_1|\,\epsilon_k^{1+\mu/2}\left(\int_{M_k\cap(B_1(0)\times\mathbf{R})} |A_k|^2\right)^{\frac{1}{2}} \\
&\leq c\sup|\varphi_1|\,\epsilon_k^{2+\mu/2}.
\end{aligned}
$$

Note that we have used Lemmas 3.4 and 3.1 here. Inequality (4.74) implies that

$$
(4.75)\qquad \frac{1}{\epsilon_k^2}\int_{M_k\setminus G_k} \overline{\psi}_k\varphi_1\,(\nu_k\cdot e_{n+1})\,|A_k|^2 \to 0
$$

as $k\to\infty$.

Now, we split the integral on the left-hand side of inequality (4.73) in the usual way as follows:

$$
\begin{aligned}
(4.76)\qquad & \int_{M_k} \nabla^{M_k}(\overline{\psi}_k\varphi_1)\cdot\nabla^{M_k}(1-(\nu_k\cdot e_{n+1})) \\
&= \int_{G_k} \nabla^{M_k}(\overline{\psi}_k\varphi_1)\cdot\nabla^{M_k}(1-(\nu_k\cdot e_{n+1})) + \\
&\quad + \int_{M_k\setminus G_k} \nabla^{M_k}(\overline{\psi}_k\varphi_1)\cdot\nabla^{M_k}(1-(\nu_k\cdot e_{n+1})).
\end{aligned}
$$

Since $\nabla^{M_k}\varphi_1$ is bounded and $|\nabla^{M_k}(1-(\nu_k\cdot e_{n+1}))|\leq|A_k|$, Cauchy–Schwarz inequality, Lemma 3.1, inequalities (3.24) and (3.28) imply that the second integral on the right-hand side of Equation (4.76) is bounded from above by $c\,\epsilon_k^{2+\mu/2}$. Thus,

$$
(4.77)\qquad \frac{1}{\epsilon_k^2}\int_{M_k\setminus G_k} \nabla^{M_k}(\overline{\psi}_k\varphi_1)\cdot\nabla^{M_k}(1-(\nu_k\cdot e_{n+1})) \to 0
$$

as $k\to\infty$.

Using the area formula and the fact that

$$\nabla^{M_k}\zeta_1 \cdot \nabla^{M_k}\zeta_2 = D\zeta_1 \cdot D\zeta_2 - (D\zeta_1 \cdot \nu_k)(D\zeta_2 \cdot \nu_k)$$

the first integral on the right-hand side of Equation (4.76) can be expressed as follows:

$$\begin{aligned}(4.78) \quad & \int_{G_k} \nabla^{M_k}(\overline{\psi}_k \varphi_1) \cdot \nabla^{M_k}(1 - \nu_k \cdot e_{n+1}) \\ &= \frac{1}{2}\int_{B_{1/4}(0)} \frac{D(\overline{\psi}_k\varphi) \cdot D(|Du_k^+|^2)}{1+|Du_k^+|^2} + \frac{D(\overline{\psi}_k\varphi) \cdot D(|Du_k^-|^2)}{1+|Du_k^-|^2} \\ &\quad - \frac{1}{2}\int_{B_{1/4}(0)} \frac{(D(\overline{\psi}_k\varphi) \cdot Du_k^+)(D(|Du_k^+|^2) \cdot Du_k^+)}{(1+|Du_k^+|^2)^{5/2}} \\ &\quad + \frac{(D(\overline{\psi}_k\varphi) \cdot Du_k^-)(D(|Du_k^-|^2) \cdot Du_k^-)}{(1+|Du_k^-|^2)^{5/2}}.\end{aligned}$$

We remark here that even though the individual terms $D|Du_k^+|$ and $D|Du_k^-|$ may not belong to the Sobolev space $W^{1,2}(B_{1/4}(0))$, the integrands of both integrals on the right hand side of the above are in $W^{1,2}(B_{1/4}(0))$, because they are symmetric expressions in u_k^+ and u_k^- and hence can be expressed unambiguously in terms of the smooth functions $u_k^{(1)}$, $u_k^{(2)}$ and the derivatives of $u_k^{(1)}$ and $u_k^{(2)}$ (since the unordered pair $\{u_k^+, u_k^-\}$ is the same as the unordered pair $\{u_k^{(1)}, u_k^{(2)}\}$).

Since $|Du_k^+| \to 0$ a.e., after dividing by ϵ_k^2, the second integral on the right-hand side of Equation (4.78) converges to zero. Using inequality (3.27) and integration by parts, we also see that, after dividing by ϵ_k^2, the first integral on the right-hand side of Equation (4.78) converges to

$$\frac{1}{2}\int_{B_{1/4}(0)} D\varphi \cdot \left(D(|Dv^+|^2) + D(|Dv^-|^2)\right).$$

Since $v^- = -v^+$, it follows that $\int_{B_{1/4}(0)} D\varphi \cdot D(|Dv^+|^2) \le 0$. q.e.d.

Corollary 4.10. *v^+ is smooth and harmonic everywhere in $B_{1/8}(0) \setminus Z_{v^+}$, where $Z_{v^+} \equiv \{z \in B_{1/8}(0) : v^+(z) = 0\}$.*

Proof. The identity of Lemma 4.7, part (i) says that v^+ satisfies $\Delta(v^+)^2 = 2\,|Dv^+|^2$ weakly in $B_{1/4}(0)$. By Lemma 4.9, $|Dv^+|$ is bounded in $B_{1/8}(0)$. Therefore, by elliptic regularity theory [**4**], $(v^+)^2$ is in $C^{1,\alpha}(B_{1/8}(0))$ for every $\alpha < 1$. Hence, v^+ is $C^{1,\alpha}$ everywhere in $(B_{1/8}(0) \setminus Z_{v^+})$. Therefore, $|Dv^+|$ is $C^{0,\alpha}$ everywhere in $(B_{1/8}(0) \setminus Z_{v^+})$ and by elliptic regularity again, v^+ is in $C^{2,\alpha}$ everywhere in $(B_{1/8}(0) \setminus Z_{v^+})$. By

Lemma 4.7, part (i) again, we then have that $v\Delta v = 0$, and hence, $\Delta v(x) = 0$, at every point $x \in B_{1/8}(0) \setminus Z_{v^+}$. q.e.d.

Preparatory to the analysis of Section 5, we next want to describe how we can blow up v^+ iteratively. In Section 5, we shall prove that the graph of v^+ is equal to the union of two n-dimensional half spaces of $\mathbf{R}^{n+1}$ intersecting along an $(n-1)$-dimensional subspace of $\mathbf{R}^n$. The key step there will be a dimension reducing process which will be used to prove that the zero set Z_{v^+} of v^+ is an $(n-1)$-dimensional subspace of $\mathbf{R}^{n+1}$. This involves blowing up v^+ at a point $z \in Z_{v^+}$; by that, we mean choosing an arbitrary sequence of radii $\rho_i \downarrow 0$ and taking a subsequential limit (w_1, say) of the sequence of functions obtained by rescaling v^+ at z, where the domain variables of v^+ (i.e., the $\mathbf{R}^n$ variables) are translated and scaled so that the ball $B_{\rho_i}(z)$ becomes the unit ball centered at the origin, and the dependent variable is scaled by dividing by the (scale invariant) L^2 norm of v^+ over $B_{\rho_i}(z)$ (provided of course this quantity is non-zero, a fact that will be proved in Section 5). See the Definition (4.79). The dimension reducing argument of Section 5 relies on being able to repeat this process—i.e., being able to blow up the blow-up w_1 at a zero of w_1 and then blow up the second blow-up (w_2, say) at a zero of w_2 and so on—preserving each time all of the regularity properties of v^+, we have established so far in Lemmas 4.7–4.9. This can indeed be done, and we conclude the present section with a detailed discussion of this iterative blow-up procedure including a proof of the fact that each blow-up does indeed inherit all of the properties of v^+ established in Lemmas 4.7–4.9.

Since it is convenient to prove this claim as an abstract result usable iteratively with each blow-up, we switch notation at this point and introduce an abstract function v in place of v^+.

So, suppose v is a Lipschitz function in $B_{1/4}(0)$, v does not identically vanish in any ball $B_\rho(z) \subseteq B_{1/4}(0)$ and that Lemmas 4.7–4.9 hold with v in place of v^+.

Let $z \in B_{1/8}(0)$ and $\rho \in (0, 1/8]$ be arbitrary. (In Section 5, where we take v to be, in turn, v^+ and its subsequent blow-ups, we shall choose z to be a zero of v, but here it is an arbitrary point.) For $x \in B_1(0)$, define

$$\widetilde{v}_{z,\rho}(x) = \frac{v(z + \rho x)}{\rho^{-n/2}\|v\|_{L^2(B_\rho(z))}}. \tag{4.79}$$

$\widetilde{v}_{z,\rho}$ is well defined by hypothesis. (In Section 5, once we have more tools at our disposal, we shall prove that v^+ as well as its subsequent blow-ups do not identically vanish on any ball $B_\rho(z) \subseteq B_{1/4}(0)$. See Lemma 5.15.)

It follows directly from the definition (4.79) that

$$\int_{B_{1/2}(0)} \widetilde{v}_{z,\rho}^2 \leq 1 \tag{4.80}$$

and part (iii) of Lemma 4.7 with z, ρ in place of y, σ implies that

$$\int_{B_{1/2}(0)} |D\widetilde{v}_{z,\rho}|^2 \leq 8. \tag{4.81}$$

Now let $\{\rho_i\}$ be a sequence of positive numbers with $\rho_i \downarrow 0$ as $i \to \infty$ and write $\widetilde{v}_i = \widetilde{v}_{z,\rho_i}$. The inequalities (4.80) and (4.81) imply that there exists a function $w \in W^{1,2}(B_{1/2}(0))$ such that, for a subsequence of $\{i\}$ (which we also denote $\{i\}$), $\widetilde{v}_i \to w$ strongly in $L^2(B_{1/2}(0))$, weakly in $W^{1,2}(B_{1/2}(0))$ and pointwise a.e. in $B_{1/2}(0)$. This convergence is in fact strong in $W^{1,2}(B_{1/4}(0))$. Equivalently, the energy convergence $\int_{B_{1/4}(0)} |D\widetilde{v}_i|^2 \to \int_{B_{1/4}(0)} |Dw|^2$ holds. To prove this, we shall first use the technique employed in the proof of parts (e) and (f) of Lemma 4.6 to obtain L^1 convergence of the gradients.

Take $\epsilon > 0$ and let γ_ϵ be a non-negative $C^1(\mathbf{R})$ cut-off function with $\gamma_\epsilon(t) \equiv 0$ for $t \leq \epsilon$, $\gamma_\epsilon(t) \equiv 1$ for $t \geq 2\,\epsilon$ and $|\gamma_\epsilon'| \leq 4/\epsilon$. The sequence $\gamma_\epsilon(\widetilde{v}_i)D\widetilde{v}_i$ is uniformly bounded in $W^{1,1}(B_{1/2}(0))$, and hence converges to some vector valued function $\widetilde{V}^\epsilon$ in $L^1(B_{1/2}(0))$ and pointwise a.e. in $B_{1/2}(0)$. Since we may write $\gamma_\epsilon(\widetilde{v}_i)D\widetilde{v}_i = D\Gamma_\epsilon(\widetilde{v}_i)$, where $\Gamma_\epsilon'(t) = \gamma_\epsilon(t)$, we can integrate by parts to get $\int \zeta \cdot \gamma_\epsilon(\widetilde{v}_i)D\widetilde{v}_i = -\int \operatorname{div}\zeta\, \Gamma_\epsilon(\widetilde{v}_i)$. Letting $i \to \infty$ in this shows that $\widetilde{V}^\epsilon = \gamma_\epsilon(w)Dw$. Thus,

$$\gamma_\epsilon(\widetilde{v}_i)D\widetilde{v}_i \to \gamma_\epsilon(w)Dw \tag{4.82}$$

in $L^1(B_{1/2}(0))$ and pointwise a.e. in $B_{1/2}(0)$.

By the definition of $\widetilde{v}_i$ and part (iv) of Lemma 4.7, we know that

$$\int_{B_{1/2}(0)\cap\{x:\widetilde{v}_i(x)\leq\epsilon\}} |D\widetilde{v}_i| \leq c\,\epsilon. \tag{4.83}$$

Letting $i \to \infty$ in (4.83) and using Fatou's Lemma, we also have that

$$\int_{B_{1/2}(0)\cap\{x\,:\,w(x)\leq\epsilon\}} |Dw| \leq c\,\epsilon. \tag{4.84}$$

Hence, we can let $\epsilon \to 0$ in (4.82) to conclude that

$$D\widetilde{v}_i \to Dw \tag{4.85}$$

in $L^1(B_{1/2}(0))$ and pointwise a.e. in $B_{1/2}(0)$.

Next, observe that $|D\widetilde{v}_i|^2$ are weakly subharmonic in $B_1(0)$ because by Lemma 4.9, $|Dv|^2$ is weakly subharmonic in $B_{1/4}(0)$. By the mean value property and inequality (4.81), we then have that $|D\widetilde{v}_i|$ are uniformly (i.e., independently of i) bounded in $B_{1/4}(0)$. This and (4.85) imply that

$$\int_{B_{1/4}(0)} |D\widetilde{v}_i|^2 \to \int_{B_{1/4}(0)} |Dw|^2 \tag{4.86}$$

giving the required energy convergence. We have thus shown that

$$\widetilde{v}_i \to w \text{ strongly in } W^{1,2}(B_{1/4}(0)). \tag{4.87}$$

It also follows from (4.87) and the fact that $|D\widetilde{v}_i|$ are uniformly bounded that $|Dw|$ is bounded in $B_{1/4}(0)$ and therefore, that w is Lipschitz and the convergence in (4.87) is uniform on compact subsets of $B_{1/4}(0)$.

We also have that Lemmas 4.7–4.9 all hold with w in place of v. This is easily seen in view of (4.87) and the fact that these lemmas hold with $\widetilde{v}_i$ in place of v.

We have thus established the following abstract implication.

Lemma 4.11. *Suppose v is Lipschitz in $B_{1/4}(0)$, not identical to 0 on any ball $B_\rho(z) \subseteq B_{1/4}(0)$ and that Lemmas 4.7–4.9 hold with v in place of v^+ and with arbitrary $y \in B_{1/8}(0)$ and $\sigma \in (0, 1/8]$. Let $z \in B_{1/8}(0)$, $\rho_i \in (0, 1/8]$ be arbitrary with $\rho_i \downarrow 0$. Then,*

(a) *after passing to a subsequence, $\{\widetilde{v}_{z,\rho_i}\}$ converge in $W^{1,2}(B_{1/4}(0))$ to a function w. Here, $\widetilde{v}_{z,\rho}$ is as in (4.79). w is Lipschitz in $B_{1/4}(0)$ and the convergence is uniform on compact subsets of $B_{1/4}(0)$*
and
(b) *Lemmas 4.7–4.9 hold with w in place of v^+ and with arbitrary $y \in B_{1/8}(0)$ and $\sigma \in (0, 1/8]$.*

5. Geometric picture of the first blow-up

In this section, we complete the analysis of the blow-up v^+ by proving that the graph of v^+ is equal to the union of two n-dimensional half spaces of $\mathbf{R}^{n+1}$ intersecting along an $(n-1)$-dimensional subspace of $\mathbf{R}^n$. Central to the argument here is a dimension reducing procedure using a frequency function (defined below), a method first used by Almgren, Jr. [2] to study multi-valued Dirichlet energy minimizing functions.

Definition. Let f be a Lipschitz function on a domain $\Omega \subseteq \mathbf{R}^n$. For $z \in \Omega$ and $0 < \rho < \operatorname{dist}(z, \partial\,\Omega)$ for which $\int_{\partial B_\rho(z)} f^2 \neq 0$, we define the frequency function $N_{f,z}(.)$ of f at z by

$$N_{f,z}(\rho) = \frac{\rho \int_{B_\rho(z)} |Df|^2}{\int_{\partial B_\rho(z)} f^2}. \tag{5.1}$$

We shall use the frequency function of v^+ at points of its zero set $Z_{v^+} \equiv \{z \in \mathbf{R}^n : v^+(z) = 0\}$ to show that Z_{v^+} is an $(n-1)$-dimensional subspace of $\mathbf{R}^n$ and that v^+ is invariant under translation by any element of Z_{v^+}; from this, it will follow that the the graph of v^+ consists of two half spaces. Proposition 5.18 at the end of this section gives the proof of these claims.

In order to prove Proposition 5.18, we need to establish several key properties of the frequency functions associated with v^+ and its successive blow-ups and some consequences of these properties. We obtain these results in Lemmas 5.12–5.17. The proof of Proposition 5.18 will be based on these lemmas, and will require us to use the lemmas iteratively with successive blow-ups of v^+. In view of this, we will establish Lemmas 5.13–5.17 for an abstract function v, which we can replace v^+ and its blow-ups in the applications of the lemmas. So, suppose

(5.2) (i) v is a Lipschitz function with domain $B_{1/4} \equiv B^n_{1/4}(0)$.

(ii) Lemmas 4.7–4.9 hold with v in place of v^+.

Lemma 5.12. *Suppose hypotheses (5.2) hold. Then,*

$$N_{v,z}(\rho) = \frac{\rho \frac{d}{d\rho} \int_{S^{n-1}} v^2_{z,\rho}}{2 \int_{S^{n-1}} v^2_{z,\rho}}$$

for $z \in B_{1/8}(0)$ and $\rho \in (0, 1/8)$ with $\int_{S^{n-1}} v^2_{z,\rho} \neq 0$.

Proof. This is a direct consequence of Lemma 4.7, part (ii). q.e.d.

Lemma 5.13. *Suppose hypotheses (5.2) hold. Let $z \in B_{1/8}(0)$, $\rho_1,\, \rho_2 \in (0, 1/8)$ with $0 < \rho_1 < \rho_2$. Suppose $\int_{\partial B_\rho(z)} v^2 \neq 0$ for every $\rho \in (\rho_1,\, \rho_2)$. Then, $N_{v,z}(\rho)$ is a monotonically non-decreasing function of ρ for $\rho \in (\rho_1,\, \rho_2)$.*

Proof. The harmonic identity (4.63) implies that

$$\frac{d}{d\rho}\left(\rho^{2-n} \int_{B_\rho(z)} |Dv|^2\right) = 2\rho^{2-n} \int_{\partial B_\rho(z)} \left|\frac{\partial v}{\partial R}\right|^2 \tag{5.3}$$

for almost all $\rho \in (0, 1/8)$, where $\frac{\partial v}{\partial R}(x) = Dv(x) \cdot \frac{x-z}{|x-z|}$ is the radial derivative. This follows by taking $(x^j - z^j)\,\zeta_l$ in place of ζ^j in identity (4.63) and letting $l \to \infty$, where ζ_l is a sequence of $C_c^\infty(B_\rho(z))$ functions converging to the characteristic function of the ball $B_\rho(z)$. (We omit the details here. This is exactly the argument used to derive the standard monotonicity formula for stationary harmonic maps, and can be found e.g., in [**12**], Chapter 2.)

Now, by a change of variables in the denominator of (5.1), we have that $N_{v,z}(\rho) = \frac{\rho^{2-n}\int_{B_\rho(z)} |Dv|^2}{\int_{\mathbf{S}^{n-1}} v_{z,\rho}^2}$, where $v_{z,\rho}(x) = v(z + \rho x)$. Using this, Equations (5.3), (4.53) and the fact that $\left(\frac{\partial v_{z,\rho}}{\partial R}\right)(\omega) = \rho\frac{\partial}{\partial\rho}v(z + \rho\,\omega)$, we compute as follows:

(5.4)

$$\begin{aligned}
&\frac{d}{d\rho}\, N_{v,z}(\rho)\\
&\quad= \frac{\frac{d}{d\rho}\left(\rho^{2-n}\int_{B_\rho(z)} |Dv|^2\right)}{\int_{\mathbf{S}^{n-1}} v_{z,\rho}^2} - \frac{2\rho^{1-n}\int_{B_\rho(z)} |Dv|^2\, \int_{\mathbf{S}^{n-1}} v_{z,\rho}\frac{\partial v_{z,\rho}}{\partial R}}{\left(\int_{\mathbf{S}^{n-1}} v_{z,\rho}^2\right)^2}\\
&\quad= \frac{2\rho^{1-n}\left(\rho\int_{\mathbf{S}^{n-1}} v_{z,\rho}^2\, \int_{\partial B_\rho(z)}\left|\frac{\partial v}{\partial R}\right|^2 - \int_{\partial B_\rho(z)} v\frac{\partial v}{\partial R}\, \int_{\mathbf{S}^{n-1}} v_{z,\rho}\frac{\partial v_{z,\rho}}{\partial R}\right)}{\left(\int_{\mathbf{S}^{n-1}} v_{z,\rho}^2\right)^2}\\
&\quad= \frac{2\rho^{-1}\left(\int_{\mathbf{S}^{n-1}} v_{z,\rho}^2\, \int_{\mathbf{S}^{n-1}}\left|\frac{\partial v_{z,\rho}}{\partial R}\right|^2 - \left(\int_{\mathbf{S}^{n-1}} v_{z,\rho}\frac{\partial v_{z,\rho}}{\partial R}\right)^2\right)}{\left(\int_{\mathbf{S}^{n-1}} v_{z,\rho}^2\right)^2}.
\end{aligned}$$

The lemma follows from the above and the Cauchy–Schwarz inequality. q.e.d.

Lemma 5.14. *Suppose hypotheses (5.2) hold. Let $z \in B_{1/8}(0)$, $\rho_1,\ \rho_2 \in (0, 1/8)$ with $0 < \rho_1 < \rho_2$. Suppose that $\int_{\partial B_\rho(z)} v^2 \neq 0$ for every $\rho \in (\rho_1,\ \rho_2]$. Then, for $\tau,\ \sigma$ with $\rho_1 < \tau \leq \sigma \leq \rho_2$, we have that*

$$\frac{\int_{S^{n-1}} v_{z,\tau}^2}{\tau^{N_2}} \geq \frac{\int_{S^{n-1}} v_{z,\sigma}^2}{\sigma^{N_2}}, \tag{5.5}$$

where $N_2 = 2N_{v,z}(\rho_2)$.

Proof. By monotonicity of $N_{v,z}(\rho)$ and Lemma 5.12, we have that

$$\rho \frac{d}{d\rho} \int_{S^{n-1}} v_{z,\rho}^2 \over \int_{S^{n-1}} v_{z,\rho}^2 \leq N_2 \tag{5.6}$$

for $\rho \in (\rho_1, \rho_2)$. The lemma follows by integrating this differential inequality over $[\tau, \sigma]$. q.e.d.

Lemma 5.15. *Suppose hypotheses* (5.2) *hold and that v is homogeneous* (*of any degree*) *from the origin.* (*Thus, v is defined everywhere in $\mathbf{R}^n$ and $v(rx) = r^\alpha v(x)$ for some α and all $r > 0$, $x \in \mathbf{R}^n$.*) *Then, either $v \equiv 0$ in $\mathbf{R}^n$ or $\int_{\partial B_\rho(z)} v^2 \neq 0$ for every $z \in \mathbf{R}^n$ and every $\rho > 0$.*

Proof. Let $z \in \mathbf{R}^n$ be arbitrary. If v is not identically equal to zero, there exists $\rho_0 > 0$ such that $\int_{\partial B_{\rho_0}(z)} v^2 \neq 0$. By continuity, there exist ρ_1, ρ_2 with $0 < \rho_1 < \rho_0 < \rho_2$ such that $\int_{\partial B_\rho(z)} v^2 \neq 0$ for every $\rho \in (\rho_1, \rho_2]$. Applying Lemma 5.14 with $\sigma = \rho_2$, $\tau = \tau_j$, where $\{\tau_j\}$ is a sequence with $\rho_1 < \tau_j < \rho_2$ and $\tau_j \downarrow \rho_1$, and taking the limit as $j \to \infty$, we then have that $\int_{\partial B_{\rho_1}(z)} v^2 \neq 0$. (Note that since the domain of v is all of $\mathbf{R}^n$, the restrictions $z \in B_{1/8}(0)$ and ρ_1, $\rho_2 \in (0, 1/8)$ in Lemma 5.14 are unnecessary.) This argument then shows that $\int_{\partial B_\rho(z)} v^2 \neq 0$ for all $\rho \in (0, \rho_0]$. Homogeneity of v obviously implies that $\int_{\partial B_\rho(z)} v^2 \neq 0$ for all $\rho > \rho_0$. q.e.d.

Definition. Write $\mathcal{N}_v(z) = \lim_{\rho \to 0} N_{v,z}(\rho)$ whenever the limit exists. Note that by Lemmas 5.15 and 5.13, if hypotheses (5.2) hold and v is homogeneous (of any degree) from the origin, this limit exists for every point $z \in \mathbf{R}^n$.

Lemma 5.16. *Suppose hypotheses* (5.2) *hold. Then, v is homogeneous of degree α from a point $z \in B_{1/8}(0)$* (*that is, $v(z + r_1\omega) = \left(\frac{r_1}{r_2}\right)^\alpha v(z + r_2\omega)$ for all $r_1, r_2 \in (0, 1/8)$ and all $\omega \in \mathbf{S}^{n-1}$*) *if and only if $N_{v,z}(\rho)$ is constant for $\rho \in (0, 1/8)$ and $\alpha = \mathcal{N}_v(z) = N_{v,z}(\rho)$.*

Proof. Suppose v is homogeneous of degree α from $z \in B_{1/8}(0)$ and let $\rho_0 \in (0, 1/8)$ be fixed. By Lemma 5.12, we have that for $\rho \in (0, 1/8)$,

$$\begin{aligned} N_{v,z}(\rho) &= \frac{\rho \frac{d}{d\rho} \int_{S^{n-1}} v^2(z + \rho\,\omega) d\mathcal{H}^{n-1}(\omega)}{2 \int_{S^{n-1}} v^2(z + \rho\,\omega) d\mathcal{H}^{n-1}(\omega)} \\ &= \frac{\rho \frac{d}{d\rho} \left(\frac{\rho}{\rho_0}\right)^{2\,\alpha} \int_{S^{n-1}} v^2(z + \rho_0\,\omega) d\mathcal{H}^{n-1}(\omega)}{2 \left(\frac{\rho}{\rho_0}\right)^{2\,\alpha} \int_{S^{n-1}} v^2(z + \rho_0\,\omega) d\mathcal{H}^{n-1}(\omega)} \end{aligned} \tag{5.7}$$

$$= \frac{\rho\, 2\alpha\, \rho^{2\alpha-1} \int_{S^{n-1}} v^2(z+\rho_0\omega) d\mathcal{H}^{n-1}(\omega)}{2\, \rho^{2\alpha} \int_{S^{n-1}} v^2(z+\rho_0\omega) d\mathcal{H}^{n-1}(\omega)}$$
$$= \alpha.$$

Conversely, suppose $N_{v,z}(\rho)$ is independent of ρ for $\rho \in (0, 1/8)$. Then, $\frac{d}{d\rho} N_{v,z}(\rho) = 0$. By Equation (5.4), this means that

$$\frac{\partial v_{z,\rho}}{\partial \rho}(\omega) = \beta\, v_{z,\rho}(\omega) \tag{5.8}$$

for some constant β, all $\rho \in (0, 1/8)$ and all $\omega \in \mathbf{S}^{n-1}$. (This is precisely the condition under which equality holds in the Cauchy–Schwarz inequality.) Using $\left(\frac{\partial v_{z,\rho}}{\partial \rho}\right)(\omega) = \rho \frac{\partial}{\partial \rho} v(z+\rho\,\omega)$ in Equation (5.8) and integrating the resulting differential identity from r_1 to r_2, where $r_1, r_2 \in (0, 1/8)$, we obtain that

$$v(z + r_1\,\omega) = \left(\frac{r_1}{r_2}\right)^{\beta} v(z + r_2\,\omega). \tag{5.9}$$

Applying the first part of the lemma, we conclude that $\beta = N_{v,z}(\rho) = \mathcal{N}_v(z)$. q.e.d.

Remark. Since v^+ is homogeneous of degree 1 from the origin, Lemma 5.16 implies that $\mathcal{N}_{v^+}(0) = 1$.

Lemma 5.17. *Suppose hypotheses* (5.2) *hold and that v is homogeneous from the origin.* (*Note that the degree of homogeneity then is $\mathcal{N}_v(0)$ by Lemma* 5.16.) *Then, $\mathcal{N}_v(z) \le \mathcal{N}_v(0)$ for every $z \in \mathbf{R}^n$. The equality holds if and only if v is homogeneous of degree $\mathcal{N}_v(0)$ from z, which holds if and only if v is cylindrical in the direction of z.* (*Cylindrical in the direction of z means $v(x) = v(x + tz)$ for all $x \in \mathbf{R}^n$ and all $t \in \mathbf{R}$.*)

Proof. For arbitrary σ, ρ with $0 < \sigma < \rho$, $N_{v,z}(\rho) - N_{v,z}(\sigma) = \int_\sigma^\rho \frac{dN_{v,z}}{ds}\, ds$. Letting $\sigma \downarrow 0$ in this, we get

$$N_{v,z}(\rho) = \mathcal{N}_v(z) + \int_0^\rho \frac{dN_{v,z}}{ds}\, ds. \tag{5.10}$$

On the other hand, using homogeneity of v from the origin, we have that

(5.11)

$$
\begin{aligned}
N_{v,z}(\rho) &= \frac{\rho \int_{B_\rho(z)} |Dv|^2}{\int_{\partial B_\rho(z)} v^2} \\
&\le \left(\frac{\rho}{\rho+|z|}\right)\left(\frac{\int_{\partial B_{\rho+|z|}(0)} v^2}{\int_{\partial B_\rho(z)} v^2}\right)\left(\frac{(\rho+|z|)\int_{B_{\rho+|z|}(0)} |Dv|^2}{\int_{\partial B_{\rho+|z|}(0)} v^2}\right) \\
&= \left(\frac{\rho}{\rho+|z|}\right)\left(\frac{\int_{\partial B_{\rho+|z|}(0)} v^2}{\int_{\partial B_\rho(z)} v^2}\right)\mathcal{N}_v(0).
\end{aligned}
$$

Observe that as $\rho \uparrow \infty$, $\frac{\rho}{\rho+|z|} \to 1$ and, since v is homogeneous of degree $\mathcal{N}_v(0)$ from the origin, $\frac{\int_{\partial B_{\rho+|z|}(0)} v^2}{\int_{\partial B_\rho(z)} v^2} = \frac{(\rho+|z|)^{n-1+2\mathcal{N}_v(0)} \int_{S^{n-1}} v^2(x)}{\rho^{n-1+2\mathcal{N}_v(0)} \int_{S^{n-1}} v^2(\frac{z}{\rho}+x)} \to 1$, so combining Equations (5.10) and (5.11) and letting $\rho \uparrow \infty$, we obtain that

$$
\mathcal{N}_v(z) + \int_0^\infty \frac{dN_{v,z}}{ds}\, ds \le \mathcal{N}_v(0). \tag{5.12}
$$

Thus, $\mathcal{N}_v(0) \ge \mathcal{N}_v(z)$. If the equality holds, then $\frac{dN_{v,z}}{ds} \equiv 0$. By Lemma 5.16, this means that

$$
v(z+rx) = r^{\mathcal{N}_v(0)} v(z+x) \quad \text{for all} \quad x \quad \text{and} \quad r>0. \tag{5.13}
$$

Conversely, if (5.13) holds, then by Lemma 5.16, $\mathcal{N}_v(z) = \mathcal{N}_v(0)$.

That v is cylindrical in z-direction when $\mathcal{N}_v(z) = \mathcal{N}_v(0)$ follows readily from the fact that v is homogeneous of the same degree from the origin and from z. To see this, let $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$ be arbitrary, and choose $\lambda > 0$ such that $t = \lambda - \lambda^{-1}$. Then, writing $\mathcal{N}_v(0) = \mathcal{N}_v(z) = \alpha$, we have that

$$
\begin{aligned}
v(x) &= \lambda^{-\alpha} v(\lambda x) = \lambda^{-\alpha} v(z + (\lambda x - z)) \\
&= \lambda^{\alpha} v(z + \lambda^{-2}(\lambda x - z)) = v(\lambda(z + \lambda^{-2}(\lambda x - z))) \\
&= v(x + t z), \quad \text{as required.}
\end{aligned} \tag{5.14}
$$

Finally, if (5.14) holds for all $x \in \mathbf{R}^n$ and all $t \in \mathbf{R}$, then for $r > 0$, $v(z + r x) = v(r x) = r^{\mathcal{N}_v(0)} v(x) = r^{\mathcal{N}_v(0)} v(z+x)$, and therefore by Lemma 5.16, $\mathcal{N}_v(z) = \mathcal{N}_v(0)$. This completes the proof of the lemma.

q.e.d.

Using the results of the preceding lemmas, we are now ready to prove the following.

Proposition 5.18.

(1) *The zero set* $Z_{v^+} \equiv \{x \in \mathbf{R}^n : v^+(x) = 0\}$ *of* v^+ *is an* $(n-1)$-*dimensional subspace of* $\mathbf{R}^n$.

(2) $\operatorname{graph} v^+ = \mathbf{H}^{(1)} \cup \mathbf{H}^{(2)}$ *where* $\mathbf{H}^{(1)}$ *and* $\mathbf{H}^{(2)}$ *are* n-*dimensional half-spaces of* $\mathbf{R}^{n+1}$ *with* $\mathbf{H}^{(1)} \cap \mathbf{R}^n = \mathbf{H}^{(2)} \cap \mathbf{R}^n = \mathbf{H}^{(1)} \cap \mathbf{H}^{(2)} = Z_v^+$.

Remark. Since $v^- = -v^+$, Proposition 5.18 implies that $\operatorname{graph} v^+ \cup \operatorname{graph} v^-$ is equal to the union of four n-dimensional half spaces of $\mathbf{R}^{n+1}$ meeting along an $(n-1)$-dimensional subspace of $\mathbf{R}^n$. In fact, since $|Dv^+| \in W^{1,2}(B_{1/4}(0))$ (by Lemma 4.6 (h)), $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ must make equal angles with $\mathbf{R}^n$, and therefore, $\operatorname{graph} v^+ \cup \operatorname{graph} v^-$ must be equal to a pair of hyperplanes intersecting transversely along an $(n-1)$-dimensional subspace of $\mathbf{R}^n$. $\square$

Proof of Proposition 5.18. By continuity of v^+, Z_{v^+} is closed. First, we claim that

$$\mathcal{H}^{n-1}(Z_{v^+} \cap B_{1/16}(0)) > 0. \tag{5.15}$$

If this is not true, then for any $\epsilon > 0$, there exists a finite collection of balls $B_{r_i}(y_i) \subseteq B_{1/16}(0)$, $i = 1, \ldots, N$, such that $Z_{v^+} \cap B_{1/16}(0) \subseteq \cup_{i=1}^N B_{r_i}(y_i)$ and $\sum_{i=1}^N r_i^{n-1} \leq \epsilon$. For $1 \leq i \leq N$, let $\zeta_i \colon B_{1/16}(0) \to \mathbf{R}$ be a C^1 cut off function satisfying $\zeta_i(x) = 0$ if $x \in B_{r_i}(y_i)$, $\zeta_i(x) = 1$ if $x \in B_{1/16}(0) \setminus B_{2r_i}(y_i)$, $0 \leq \zeta_i \leq 1$ and $|D\zeta_i| \leq 2/r_i$ everywhere in $B_{1/16}(0)$. Define $\zeta_\epsilon : B_{1/16}(0) \to \mathbf{R}$ by $\zeta_\epsilon(x) = \prod_{i=1}^N \zeta_i(x)$. It is clear from this definition that ζ_ϵ is a C^1 function with compact support in $B_{1/16}(0) \setminus Z_{v^+}$.

Let ζ be an arbitrary C^1 function with compact support in $B_{1/16}(0)$. By Corollary 4.10, v^+ is smooth and harmonic in $B_{1/16}(0) \setminus Z_{v^+}$ and therefore, we have that

$$\int_{B_{1/16}(0)} \Delta v^+ \zeta_\epsilon \zeta = 0. \tag{5.16}$$

Integrating by parts in the above equation, we obtain that

$$\int_{B_{1/16}(0)} \zeta_\epsilon Dv^+ \cdot D\zeta = -\int_{B_{1/16}(0)} \zeta Dv^+ \cdot D\zeta_\epsilon. \tag{5.17}$$

On the other hand, since $|Dv^+|$ is bounded (by Lemma 4.9), we may estimate as follows:

$$\left|\int_{B_{1/16}(0)} \zeta Dv^+ \cdot D\zeta_\epsilon\right| \le \sup|\zeta||Dv^+| \int_{B_{1/16}(0)} |D\zeta_\epsilon| \tag{5.18}$$
$$\le c \sup|\zeta||Dv^+| \sum_{i=1}^{N} \frac{\mathcal{H}^n(B_{2r_i}(y_i))}{r_i}$$
$$\le c \sup|\zeta||Dv^+| \sum_{i=1}^{N} r_i^{n-1}$$
$$\le c \sup|\zeta||Dv^+|\, \epsilon.$$

In view of inequality (5.18), letting $\epsilon \to 0$ in Equation (5.17) and noting that $\zeta_\epsilon \to 1$, we conclude that v^+ is harmonic in $B_{1/16}(0)$. Since v^+ is homogeneous of degree one, v^+ must be linear (by the maximum principle applied to each $D_i v^+$), and therefore, since v^+ is non-negative with $v^+(0) = 0$, v^+ must be identically equal to zero, which is a contradiction. This proves (5.15).

We shall prove shortly that for every $z \in Z_{v^+} \cap (B_{1/16}(0) \setminus \{0\})$, v^+ is cylindrical in the direction of z. Since, by (5.15), there exist $(n-1)$ points $z_1, \ldots, z_{n-1} \in Z_{v^+} \cap (B_{1/16}(0) \setminus \{0\})$ that are linearly independent as vectors in $\mathbf{R}^n$, we shall then have that $v^+(x) = 0$ for all $x \in \operatorname{span}\{z_i\}$. Because v^+ is not identically equal to zero, this must mean that $Z_{v^+} = \operatorname{span}\{z_i\}$, establishing the first part of the proposition.

So to complete the proof of the first part, consider an arbitrary point $z \in Z_{v^+} \cap (B_{1/16}(0) \setminus \{0\})$. Let $\{\rho_i\}$ be a sequence of positive numbers with $\rho_i \downarrow 0$ as $i \to \infty$. Let $\widetilde{v^+}_{z,\rho_i}$ be the blow-up sequence as in (4.79) with v^+ in place of v and ρ_i in place of ρ. By Lemma 5.15, $\widetilde{v^+_{z,\rho_i}}$ are well defined. By Lemma 4.11, $\widetilde{v^+}_{z,\rho_i}$ converge (after passing to a subsequence) to a Lipschitz function w_1 strongly in $W^{1,2}(B_{1/4}(0))$ and uniformly on compact subsets of $B_{1/4}(0)$, and Lemmas 4.7–4.9 hold with w_1 in place of v^+.

We next establish the following key properties of w_1:

- (a$_1$) w_1 is not identically equal to zero on any ball $B_\rho(0)$, $0 < \rho < 1/4$.
- (b$_1$) w_1 is homogeneous of degree $\mathcal{N}_{v^+}(z)$ from the origin. Hence w_1 extends to all of $\mathbf{R}^n$ as a homogeneous function.
- (c$_1$) $\int_{\partial B_\sigma(q)} w_1^2 > 0$ for all $q \in \mathbf{R}^n$ and all $\sigma > 0$.
- (d$_1$) w_1 is cylindrical in the direction of z.
- (e$_1$) $\{tz : t \in \mathbf{R}\} \subseteq Z_{w_1}$, where Z_{w_1} is the zero set of w_1.
- (f$_1$) w_1 is harmonic where it is non-zero.
- (g$_1$) $\mathcal{H}^{n-1}(Z_{w_1}) > 0$.

Proof of (a$_1$). Let $\rho_0 \in (0, \mathrm{dist}\,(z, \partial\, B_{1/4}(0)))$ be fixed. Taking $\tau = \sigma\rho$, $\rho_2 = \rho_0$ and $v = v^+$ in Lemma 5.14, we have that

$$\rho^{-N_0} \int_{S^{n-1}} {v^+}^2_{z,\sigma\rho} \geq \int_{S^{n-1}} {v^+}^2_{z,\sigma} \tag{5.19}$$

for all $\sigma \in (0, \rho_0]$, where $N_0 = 2N_{v^+,z}(\rho_0)$. Now, for sufficiently large i, $\rho_i \leq \rho_0$, so for each such i, multiply both sides of the above inequality by σ^{n-1} and integrate over $[0, \rho_i]$ to obtain

$$\begin{aligned}\rho^{-(N_0+n-1)} \int_0^{\rho_i} (\sigma\rho)^{n-1} \int_{S^{n-1}} {v^+}^2_{z,\sigma\rho}(x)\, d\mathcal{H}^{n-1}(x)\, d\sigma \\ \geq \int_0^{\rho_i} \sigma^{n-1} \int_{S^{n-1}} {v^+}^2_{z,\sigma}(x)\, d\mathcal{H}^{n-1}(x)\, d\sigma.\end{aligned} \tag{5.20}$$

This is the same as

$$\rho^{-(N_0+n)} \int_{B_\rho(0)} {v^+}^2_{z,\rho_i} \geq \int_{B_1(0)} {v^+}^2_{z,\rho_i} \tag{5.21}$$

or

$$\int_{B_\rho(0)} \widetilde{v^+}^2_{z,\rho_i} \geq \rho^{N_0+n}. \tag{5.22}$$

Passing to the limit as $i \to \infty$, we conclude that

$$\int_{B_\rho(0)} w_1^2 \geq \rho^{N_0+n}. \tag{5.23}$$

Proof of (b$_1$). For arbitrary $\rho \in (0, 1/8]$, we have that

$$\frac{\rho \int_{B_\rho(0)} |D\widetilde{v^+}_{z,\rho_i}|^2}{\int_{\partial B_\rho(0)} \widetilde{v^+}^2_{z,\rho_i}} = \frac{\rho\,\rho_i \int_{B_{\rho\,\rho_i}(z)} |Dv^+|^2}{\int_{\partial B_{\rho\,\rho_i}(z)} {v^+}^2}. \tag{5.24}$$

Letting $i \to \infty$ in this, we obtain that

$$N_{w_1,0}(\rho) = \frac{\rho \int_{B_\rho(0)} |Dw_1|^2}{\int_{\partial B_\rho(0)} w_1^2} = \mathcal{N}_{v^+}(z). \tag{5.25}$$

In view of (a_1), this shows that $\int_{\partial\, B_\rho(0)} w_1^2 > 0$ for all $\rho \in (0, 1/8]$, and that $N_{w_1,0}(\rho)$ is independent of ρ for $\rho \in (0, 1/8]$. Therefore, we have that $\mathcal{N}_{w_1}(0) = \mathcal{N}_{v^+}(z)$. By Lemma 4.11, the identities (4.53) and (4.63) hold with w_1 in place of v^+, and therefore, the identity (5.4) holds with w_1 in place of v, $z = 0$ and $\rho \in (0, 1/8]$. (We do not however have this for general z yet because we do not know if $N_{w_1,z}(\rho)$ is well defined at a general point z.) Constancy of $N_{w_1,0}(\rho)$ then implies, by the argument of the "if" direction of Lemma 5.16 with w_1 in place of v^+ and $z=0$, that w_1 is homogeneous from the origin. Notice that since w_1 is technically not defined on all of $\mathbf{R}^n$ at this point, homogeneity means that $w_1(r_1\,\omega) = \left(\frac{r_1}{r_2}\right)^{\alpha} w_1(r_2\,\omega)$ for some $\alpha \geq 0$ and all $r_1, r_2 \in (0, 1/8]$ and all $\omega \in \mathbf{S}^{n-1}$. We then extend w_1 to all of $\mathbf{R}^n$ so that w_1 is homogeneous of degree α from the origin, and thus $w_1(r\,\omega) = r^{\alpha} w_1(\omega)$ for all $r > 0$ and all $\omega \in \mathbf{S}^{n-1}$. By Lemma 5.16 with w_1 in place of v and $z = 0$, it follows that the degree of homogeneity α is then equal to $\mathcal{N}_{v^+}(z)$.

Proof of (c_1). By Lemma 4.11, the hypotheses (5.2) hold with w_1 in place of v. By (a_1) and (b_1) above, $\int_{\partial B_\rho(0)} w_1^2 > 0$ for every $\rho > 0$. Hence, Lemma 5.15 applies with w_1 in place of v. In view of (a_1), this proves (c_1).

Proof of (d_1). Let $x \in B_{1/16}(0)$ and $t \in [-1, 1]$. Using the homogeneity of v^+ from the origin, we have

$$
\begin{aligned}
(5.26)\qquad w_1(x + t\,z) &= \lim_{i\to\infty} \widetilde{v^+}_{z,\rho_i}(x + t\,z) = \lim_{i\to\infty} \frac{v^+(z + \rho_i(x + tz))}{\rho_i^{-n/2}\|v^+\|_{L^2(B_{\rho_i}(z))}} \\
&= \lim_{i\to\infty} (1 + t\,\rho_i)\frac{v^+(z + \rho_i(1 + t\,\rho_i)^{-1}x)}{\rho_i^{-n/2}\|v^+\|_{L^2(B_{\rho_i}(z))}} \\
&= \lim_{i\to\infty} \widetilde{v^+}_{z,\rho_i}((1 + t\,\rho_i)^{-1}x) = w_1(x),
\end{aligned}
$$

where we have used the uniform convergence of $\widetilde{v^+}_{z,\rho_i}$ to w_1 in $B_{1/8}(0)$. The result for arbitrary $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$ follows from this and the homogeneity of w_1 from the origin.

Proof of (e_1). Since $v^+(z) = 0$, it follows that $w_1(0) = 0$ because w_1 is the pointwise limit of $\widetilde{v^+}_{z,\rho_i}$ in $B_{1/8}(0)$. The result follows immediately from this by taking $x = 0$ in Equation (5.26).

Proof of (f_1). First, observe that since $|Dv^+|^2$ is subharmonic in $B_{1/2}(0)$ and hence bounded in $B_{1/4}(0)$, so is every $|D\widetilde{v^+}_{z,\rho_i}|^2$. Passing to the limit, we obtain that $|Dw_1|^2$ is weakly sub-harmonic in $B_{1/4}(0)$ and hence bounded in $B_{1/8}(0)$. Also, since v^+ satisfies $v^+\Delta v^+ = 0$ weakly (i.e. $\Delta({v^+}^2) = 2|Dv^+|^2$ weakly), so do all $\widetilde{v^+}_{z,\rho_i}$, and again passing to the limit, we see that $\Delta(w_1^2) = 2|Dw_1|^2$ weakly. By elliptic regularity [**4**], boundedness of $|Dw_1|^2$ implies that $w_1^2 \in C^{1,\gamma}(B_{1/8}(0))$ so that w_1 is $C^{1,\gamma}$, where it is non-zero. Thus, $|Dw_1|$ is $C^{0,\gamma}$, where w_1

is non-zero, and this in turn implies w_1^2 is $C^{2,\gamma}$, where w_1 is non-zero. Thus, we have that w_1 is smooth and harmonic where it is non-zero.

Proof of (e$_1$). If $\mathcal{H}^{n-1}(Z_{w_1}) = 0$, then the argument used to establish (5.15) implies w_1 is harmonic in $B_{1/8}(0)$. Since $w_1 \geq 0$ with $w_1(0) = 0$, by the maximum principle, w_1 must be identically equal to zero, contradicting (a_1).

Now, we repeat the above blow up procedure; first with w_1 in place of v^+ and a point $q_1 \in Z_{w_1} \cap B_{1/16}(0) \setminus \{tz : t \in \mathbf{R}\}$ in place of z to obtain a limit w_2 in place of w_1, and then with w_2 in place of v^+ and a point $q_2 \in Z_{w_2} \cap B_{1/16}(0) \setminus \operatorname{span}\{z, q_1\}$ in place of z to obtain a limit w_3 in place of w_1 and so on. Proceeding this way, we obtain a sequence of functions $\{w_i\}_{i=1}^{n-1}$, corresponding to a sequence $\{q_i\}_{i=0}^{n-2}$ of linearly independent blow up points in $B_{1/16}(0)$. Here, $q_0 = z$. Applying Lemma 4.11 with w_{i-1}, w_i, q_{i-1} in place of v^+, w and z, we see inductively that Lemmas 4.7–4.9 hold with w_i in place of v^+ for each i, $1 \leq i \leq (n-2)$. Here, $w_0 = v^+$. This in turn enables us to use Lemmas 5.13–5.17 with w_i in place of v^+, and consequently, prove that w_i satisfies the following properties:

- (a$_i$) w_i is not identical to zero.
- (b$_i$) w_i is homogeneous of degree $\mathcal{N}_{w_{i-1}}(q_{i-1})$ from the origin.
- (c$_i$) $\int_{\partial B_\sigma(q)} w_i^2 > 0$ for all $q \in \mathbf{R}^n$ and all $\sigma > 0$.
- (d$_i$) w_i is cylindrical in the directions of $q_0,\ q_1,\ q_2, \ldots, q_{i-1}$. Hence, w_i is invariant under translations by the elements of $\operatorname{span}\{q_0,\ q_1,\ q_2, \ldots, q_{i-1}\}$.
- (e$_i$) $\operatorname{span}\{q_0,\ q_1,\ q_2,\ \ldots, q_{i-1}\} \subseteq Z_{w_i}$.
- (f$_i$) w_i is harmonic where it is non-zero.
- (g$_i$) $\mathcal{H}^{n-1}(Z_{w_i}) > 0$.

Furthermore, repeated application of Lemma 5.17 yields that

$$
\begin{aligned}
(5.27) \quad 1 = \mathcal{N}_{v^+}(0) \geq \mathcal{N}_{v^+}(z) &= \mathcal{N}_{w_1}(0) \\
&\geq \mathcal{N}_{w_1}(q_1) = \mathcal{N}_{w_2}(0) \geq \cdots = \mathcal{N}_{w_{n-1}}(0).
\end{aligned}
$$

Since by the property $(d_{(n-1)})$, w_{n-1} is invariant under translations by the elements of an $(n-1)$-dimensional subspace L ($\equiv$ the subspace spanned by $\{q_0,\ q_1,\ \ldots,\ q_{n-2}\}$) contained in its zero set and since w_{n-1} is not identically equal to zero and harmonic where it is non-zero, it follows that w_{n-1} is linear in each of the two components of $\mathbf{R}^n \setminus L$. This then readily implies that w_{n-1} is homogeneous of degree one from the origin and therefore, by Lemma 5.16, we have that

$$
(5.28) \qquad \mathcal{N}_{w_{n-1}}(0) = 1.
$$

The resulting equality in (5.27) yields

$$\mathcal{N}_{v^+}(0) = \mathcal{N}_{v^+}(z). \tag{5.29}$$

Lemma 5.17 then readily implies that v^+ is cylindrical in z-direction, as was required.

The second assertion of the proposition follows readily. Since v^+ is non-negative and invariant under translation by the elements of its zero set Z_{v^+} which is an $(n-1)$-dimensional subspace, $\widetilde{v^+} \equiv v^+|_{Z^\perp_{v^+}}$ is a non-negative function of a single variable whose zero set consists of the origin. Furthermore, $\widetilde{v^+}$ is harmonic away from the origin since v^+ is harmonic away from Z_{v^+}. Thus the graph of $\widetilde{v^+}$ must consists of two rays. (Alternatively, we may use the homogeneity of $\widetilde{v^+}$ here.) This means that the graph of v^+ must consist of two n-dimensional half-spaces meeting along Z_{v^+}, completing the proof of the proposition. q.e.d.

6. Second Blow-up

In this section, we complete the proof of case (a) of Theorem 1.1 by using a second blow-up argument. Since the first blow-up of the hypersurfaces M_k off hyperplanes (taken to be $\mathbf{R}^n$) is equal to the union of two hyperplanes intersecting along an $(n-1)$-dimensional subspace (see Proposition 5.18 and the remark thereafter), for sufficiently large k, M_k is closer to the union $\mathbf{H}_k$ of two hyperplanes ($\mathbf{H}_k \equiv \operatorname{graph} \epsilon_k\, v^+ \cup \operatorname{graph} \epsilon_k\, v^-$, see notation below) than it is to $\mathbf{R}^n$. More precisely, the L^2 height excess β_k (see definition (6.9) below) of M_k relative to $\mathbf{H}_k$ (in a suitable ball) is of lower order than its excess ϵ_k relative to $\mathbf{R}^n$. The idea now is to blow up M_k by β_k. Following the work of L. Simon in [**11**], we will show in the present section that this second blow-up too is the union of four half-spaces meeting along a common $(n-1)$-dimensional axis, which gives a contradiction implying that M_k are pairs of transverse hyperplanes for infinitely many k. (See paragraph entitled **Completion of the Proof of Theorem 1.1 in Case** (a) at the end of the present section.) We note here that in [**11**], Simon shows how to blow up a sequence of multiplicity one minimal submanifolds off a sequence of multiplicity one cones converging to a *multiplicity one* cone. Our setting differs from his in that the sequence $\mathbf{H}_k$ converges to a multiplicity *two* hyperplane, and hence modification and replacement of some of the arguments of Simon is necessary here.

For notational convenience, we assume in this section that convergence in Lemma 4.6 of the sequences $\overline{\psi}_k v_k^{\pm}$ is in $W^{1,2}(B_1^n(0))$ (rather than in $W^{1,2}(B^n_{1/4}(0))$).

We now introduce some further notation.

Let $\mathbf{H} = \operatorname{graph} v^+ \cup \operatorname{graph} v^-$, where v^+ and v^- are the first blow-up as in Sections 4 and 5. By Proposition 5.18, $\mathbf{H}$ is equal to the union of two hyperplanes intersecting along an $(n-1)$-dimensional subspace $\mathbf{T}$ of $\mathbf{R}^n$. Without loss of generality, we take $\mathbf{T}$ to be $\{0\} \times \mathbf{R}^{n-1}$.

In the present section, our notation for points in various Euclidean spaces is chosen to be consistent with that of [**11**]. Thus, $(x^1, x^2, y^1, \ldots, y^{n-1})$ will denote a general point in $\mathbf{R}^{n+1}$ and $e_1, e_2, \ldots, e_{n+1}$ the standard orthonormal basis vectors in $\mathbf{R}^{n+1}$. We identify $\mathbf{R}^n$ with the hyperplane $\{x^1 = 0\}$ of $\mathbf{R}^{n+1}$ and $\mathbf{R}^{n-1}$, $\mathbf{R}^2$ with the subspaces $\{x^1 = x^2 = 0\}$, $\{y^1 = \ldots = y^{n-1} = 0\}$ respectively. If we write (x, y) to denote a point in $\mathbf{R}^n$, we are thinking of $\mathbf{R}^n$ as $\mathbf{R} \times \mathbf{R}^{n-1}$, with $x \in \mathbf{R}$ and $y \in \mathbf{R}^{n-1}$. If (x, y) denotes a point in $\mathbf{R}^{n+1}$, we are taking $\mathbf{R}^{n+1} \equiv \mathbf{R}^2 \times \mathbf{R}^{n-1}$ with $x \in \mathbf{R}^2$ and $y \in \mathbf{R}^{n-1}$.

Let

$$\mathbf{R}^{n\,+} \equiv \{(x, y) \in \mathbf{R}^n \,:\, x > 0\} \text{ and } \mathbf{R}^{n\,-} \equiv \{(x, y) \in \mathbf{R}^n \,:\, x < 0\}.$$

Let

$$\begin{aligned}
\mathbf{H}_k^{(1)} &= \operatorname{graph} \epsilon_k\, v^+|_{\mathbf{R}^{n\,+}}, \\
\mathbf{H}_k^{(2)} &= \operatorname{graph} \epsilon_k\, v^+|_{\mathbf{R}^{n\,-}}, \\
\mathbf{H}_k^{(3)} &= \operatorname{graph} \epsilon_k\, v^-|_{\mathbf{R}^{n\,-}}, \\
\mathbf{H}_k^{(4)} &= \operatorname{graph} \epsilon_k\, v^-|_{\mathbf{R}^{n\,+}},
\end{aligned}$$

where ϵ_k is the tilt-excess defined in Section 3, $\mathbf{L}_k^{(i)} = \mathbf{H}_k^{(i)} \cap \mathbf{R}^2$ for $i = 1, \ldots, 4$, $\mathbf{H}_k = \cup_{i=1}^4 \mathbf{H}_k^{(i)}$ and $\mathbf{L}_k = \cup_{i=1}^4 \mathbf{L}_k^{(i)}$.

The fact that the first blow-up $\mathbf{H}$, away from $\mathbf{R}^{n-1}$, consists of smooth, disjoint pieces suggests that for sufficiently large k, the two "sheets" of M_k are well separated (i.e. their union is embedded) at least in the graphical region G_k away from a small tubular neighborhood of $\mathbf{R}^{n-1}$. This is indeed the case. Precisely, we claim that for every $\tau \in (0, 1/8]$,

$$(G_k^+ \cap G_k^-) \setminus \widetilde{T}_k = \emptyset \tag{6.1}$$

for sufficiently large k (depending on τ), where

$$\widetilde{T}_k = \{(\xi^1, \xi^2) \in \mathbf{R}^2 \,:\, (\xi^1)^2 + \epsilon_k^2 (\xi^2)^2 \leq \tau\, \epsilon_k^2\,\} \times \mathbf{R}^{n-1}.$$

To see (6.1), we use an argument of R. Hardt and L. Simon [**5**], based on the monotonicity formula (2.9). If the claim is false, then for every k in a subsequence of the indices $\{k\}$ (which we also denote $\{k\}$), there exists a point $y_k \in (G_k^+ \cap G_k^-) \setminus \widetilde{T}_k$. Since this implies that $\Theta_{M_k}(y_k) \geq 2$, we have by the monotonicity formula (2.9) with $\sigma \downarrow 0$ and $\rho \leq \tau/2$ that

$$\omega_n \rho^n \int_{M_k \cap B_\rho^{n+1}(y_k)} \frac{((x - y_k)\cdot \nu)^2}{|x - y_k|^{n+2}} = \mathcal{H}^n(M_k \cap B_\rho^{n+1}(y_k)) - 2\omega_n \rho^n \leq c\, \epsilon_k^2 \rho^n \tag{6.2}$$

where $c = c(n)$. (The last inequality in (6.2) is easily seen by writing $\mathcal{H}^n(M_k \cap B_\rho^{n+1}(y_k)) = \mathcal{H}^n(G_k \cap B_\rho^{n+1}(y_k)) + \mathcal{H}^n((M_k \setminus G_k) \cap B_\rho^{n+1}(y_k))$ and expressing $\mathcal{H}^n(G_k \cap B_\rho^{n+1}(y_k))$ as the sum of integrals of $\sqrt{1 + |D(\overline{\psi}_k u_k^\pm)|^2}$ plus a term which is of lower order (by (3.26)) and noting that by inequality (3.24), $\mathcal{H}^n((M_k \setminus G_k) \cap B_\rho^{n+1}(y_k)) \leq c\, \epsilon_k^{2+\mu}$.)

The integral on the left hand side of (6.2) can be estimated from below as follows.

$$\begin{aligned}\int_{M_k \cap B_\rho^{n+1}(y_k)} &\frac{((x - y_k)\cdot \nu)^2}{|x - y_k|^{n+2}} \\ &\geq \int_{B_\rho^n(y_k')} \frac{((x' - y_k')\cdot D(\overline{\psi}_k u_k^+) - (\overline{\psi}_k u_k^+ - y_k^1))^2}{(|x' - y_k'|^2 + |\overline{\psi}_k u_k^+ - y_k^1|^2)^{\frac{n+2}{2}} \sqrt{1 + |D(\overline{\psi}_k u_k^+)|^2}} \\ &\quad + \int_{B_\rho^n(y_k')} \frac{((x' - y_k')\cdot D(\overline{\psi}_k u_k^-) - (\overline{\psi}_k u_k^- - y_k^1))^2}{(|x' - y_k'|^2 + |\overline{\psi}_k u_k^- - y_k^1|^2)^{\frac{n+2}{2}} \sqrt{1 + |D(\overline{\psi}_k u_k^-)|^2}}\end{aligned} \tag{6.3}$$

where we used the notation $x' = (x^2, \ldots, x^{n+1})$ and $y_k' = (y_k^2, \ldots, y_k^{n+1})$. Since $u_k^\pm, Du_k^\pm \to 0$ a.e. and $y_k^1 \to 0$, Fatou's lemma and (6.3) imply that

$$\begin{aligned}\liminf_{k\to\infty} \frac{1}{\epsilon_k^2} &\int_{M_k \cap B_\rho^{n+1}(y_k)} \frac{((x - y_k)\cdot \nu)^2}{|x - y_k|^{n+2}} \\ &\geq \int_{B_\rho^n(y')} R^{2-n} \left(\frac{\partial (u^+/R)}{\partial R}\right)^2 + R^{2-n} \left(\frac{\partial (u^-/R)}{\partial R}\right)^2 dx'\end{aligned} \tag{6.4}$$

where $\frac{\partial}{\partial R}$ denotes the radial derivative, $R = |x' - y'|$, $y' = \lim y'_k$ is such that $\operatorname{dist}(y', \mathbf{R}^{n-1}) \geq \tau > 0$ and $u^{\pm} = \lim \epsilon_k^{-1}(u_k^{\pm} - y_k^1)$ (i.e. $u^{\pm} = v^{\pm} - y^1$ where $y^1 = \lim \epsilon_k^{-1} y_k^1$) having *distinct* values at y'. Thus we finally conclude from (6.2) that

$$(6.5) \qquad \int_{B^n_\rho(y')} R^{2-n}\left(\frac{\partial(u^+/R)}{\partial R}\right)^2 + R^{2-n}\left(\frac{\partial(u^-/R)}{\partial R}\right)^2 dx' \leq c < \infty$$

which of course cannot be true, because at least one of the two smooth functions $u^{\pm}$ has value $\neq 0$ at y'. This proves (6.1).

Now let $\tau_k \downarrow 0$ be a given sequence of positive numbers. In view of (6.1), we have that for an appropriately chosen subsequence of the hypersurfaces M_k (which we denote M_k again),

$$(6.6) \qquad (G_k^+ \cap G_k^-) \setminus T_k = \emptyset$$

for all k, where

$$(6.7) \qquad T_k = \{(\xi^1, \xi^2) \in \mathbf{R}^2 \, : \, (\xi^1)^2 + \epsilon_k^2 (\xi^2)^2 \leq \tau_k \, \epsilon_k^2\} \times \mathbf{R}^{n-1}.$$

We may now use Schoen-Simon regularity theorem (Theorem 1 of [8], applied in fixed sized balls in $B_1^n(0)$ away from $\mathbf{R}^{n-1}$) to conclude that for all sufficiently large k, $M_k \cap (B_1^{n+1}(0) \setminus T_k)$ is the union of four disjoint smooth hypersurfaces. (Notice that even though Schoen-Simon regularity theorem as it is stated in [8] assumes embeddedness of the hypersurfaces everywhere, the proof of it only requires embeddedness of the graphical part, which we now have by (6.6).) Applying Allard's regularity theorem ([1], [10]) to each of these hypersurfaces, we obtain that for sufficiently large k, $M_k \cap (B_1^{n+1}(0) \setminus T_k) = \cup_{i=1}^4 \operatorname{graph} g_k^{(i)}$ with $g_k^{(i)} \in C^2(U_k^{(i)}, \mathbf{H}_k^{(i)\,\perp})$ satisfying

$$(6.8) \qquad \sup_{U_k^{(i)}} |g_k^{(i)}| + |\nabla^{\mathbf{H}_k^{(i)}} g_k^{(i)}| \leq c \, \beta_k,$$

where $U_k^{(i)} = \mathbf{H}_k^{(i)} \cap (B_1^{n+1}(0) \setminus T_k)$ for $i = 1, \ldots, 4$ and

$$(6.9) \qquad \beta_k = \left(\int_{M_k \cap B_1^{n+1}(0)} \operatorname{dist}^2(x, \mathbf{H}_k) \, d\mathcal{H}^n(x)\right)^{1/2}.$$

Observe that by writing $M_k \cap B_{1/4}^{n+1}(0)$ as the union of $G_k \cap B_{1/4}^{n+1}(0)$ and $(M_k \setminus G_k) \cap B_{1/4}^{n+1}(0)$ and using (6.9), the definition of $\mathbf{H}_k$, parts

(a), (b) of Lemma 4.6 and inequalities (3.24) and (3.26), it is easy to see that

$$\beta_k/\epsilon_k \to 0. \tag{6.10}$$

Now, for $i = 1, 4$, $\mathbf{H}_k^{(i)} = \operatorname{graph} l_k^{(i)}|_{\mathbf{R}^{n\,+}}$ and for $i = 2, 3$, $\mathbf{H}_k^{(i)} = \operatorname{graph} l_k^{(i)}|_{\mathbf{R}^{n\,-}}$ where $l_k^{(i)}$, $1 \le i \le 4$, are linear functions over $\mathbf{R}^n$ with $|Dl_k^{(i)}| \le c\,\epsilon_k$. Thus, letting $\tilde{g}_k^{(i)}(X) = g_k^{(i)}(l_k^{(i)}(X), X)$, where $X \in B_1^n(0) \cap (\mathbf{R}^{n\,+} \setminus \mathbf{p}T_k)$ if $i = 1, 4$ and $X \in B_1^n(0) \cap (\mathbf{R}^{n\,-} \setminus \mathbf{p}T_k)$ if $i = 2, 3$ (where $\mathbf{p}$ is the orthogonal projection of $\mathbf{R}^{n+1}$ onto $\mathbf{R}^n$), we obtain that

$$M_k \cap (B_1^{n+1}(0) \setminus T_k) = \cup_{i=1}^4 \operatorname{graph} \tilde{g}_k^{(i)} \tag{6.11}$$

with $\tilde{g}_k^{(i)}$ satisfying

$$\sup_{\mathbf{p}U_k^{(i)}} |\tilde{g}_k^{(i)}| + |D\tilde{g}_k^{(i)}| \le c\,\beta_k. \tag{6.12}$$

Let $w_k^{(i)} = \beta_k^{-1}\tilde{g}_k^{(i)}$. By elliptic estimates for $g_k^{(i)}$, we have that for $j \le 3$ and for every compact $K \subset B_1(0) \setminus (\{0\} \times \mathbf{R}^{n-1})$,

$$\sup_{K \cap \mathbf{R}^{n\,+}} |D^j w_k^{(i)}| \le c \tag{6.13}$$

for $i = 1, 4$, and

$$\sup_{K \cap \mathbf{R}^{n\,-}} |D^j w_k^{(i)}| \le c \tag{6.14}$$

for $i = 2, 3$, where $c = c(K)$. This implies that there exist functions $w^{(i)}$, $1 \le i \le 4$, with $w^{(1)}, w^{(4)} \in C^2(\mathbf{R}^{n\,+} \cap B_1^n(0))$ and $w^{(2)}, w^{(3)} \in C^2(\mathbf{R}^{n\,-} \cap B_1^n(0))$ such that

$$w^{(i)} \text{ is homogeneous of degree one,} \tag{6.15}$$

$$\Delta\, w^{(i)} = 0 \tag{6.16}$$

and

$$w_k^{(i)} \to w^{(i)} \tag{6.17}$$

in the C^2-norm on each compact subset of the domain of $w^{(i)}$. Note that (6.15) is a consequence of item (5) of HYPOTHESES ($\star$) of Section 1. The functions $w^{(i)}$ collectively are the "second blow-up". We intend to prove, following [**11**], that for each i, the graph of $w^{(i)}$ is an n-dimensional half-space of $\mathbf{R}^{n+1}$, and that these four half spaces meet along a common $(n-1)$-dimensional axis.

Except in two dimensions, (i.e. when $n = 2$), just the conditions (6.15) and (6.16) need not imply that each $w^{(i)}$ is linear in its domain. However, as in [**11**], we can show that the $w^{(i)}$ satisfy certain additional properties (given below by the L^2 estimates of (6.53), (6.54), (6.55) and (6.56)) which, together with (6.15) and (6.16), do indeed guarantee that the $w^{(i)}$ are linear, and furthermore, that the graphs of $w^{(i)}$ form a pair of hyperplanes. Lemma 6.23 below will establish this assertion.

Before discussing the aforementioned additional properties of the $w^{(i)}$, we want to make one more important point here; namely, that in [**11**], the L^2 estimates analogous to our estimates (6.53)– (6.56) are proved under a certain "no-large-gaps" assumption (hypothesis ($**$) of Remark 1.14 therein) regarding $\operatorname{sing} M_k$; (See Lemma 6.19 below for a precise statement of this. $\operatorname{sing} M_k$ in our case means the set of points of self-intersection of the immersion M_k. These are the points z where $\Theta_{M_k}(z) \geq 2$.) In our setting where we assume co-dimension one stability (unlike in [**11**]), M_k do indeed satisfy this hypothesis for sufficiently large k. This is a consequence of Lemma 3.2. We precisely state and prove this assertion in the following lemma.

Lemma 6.19. *If k is sufficiently large, there are no "large gaps" in the set $\{ z \in M_k \cap B_1^{n+1}(0) : \Theta_{M_k}(z) \geq 2 \}$. Precisely,*

$$\{0\} \times \{y \in \mathbf{R}^{n-1} : |y| \leq 1/2\} \subset \big(\{z \in M_k \cap B_1^{n+1}(0) : \Theta_{M_k}(z) \geq 2\}\big)_{\delta_k}$$

for sufficiently large k, where $(S)_\delta$ means the δ-neighborhood of the set S, $\delta_k = c\sqrt{\tau_k}$ (τ_k as in (6.7)) and $c = c(n)$.

Proof. It suffices to show that

$$\mathcal{H}^{n-1}(E_k) \leq c\sqrt{\tau_k}, \tag{6.18}$$

where $c = c(n)$ and

$$E_k = \{y \in B_{1/2}^{n-1}(0) : \Theta_{M_k}(z) < 2 \,\forall\, z \in M_k \cap \mathbf{p}^{-1}(y) \cap B_1^{n+1}(0)\}.$$

Since M_k is smooth, we have that

$$E_k = \{y \in B^{n-1}_{1/2}(0) \,:\, \Theta_{M_k}(z) = 1 \,\forall\, z \in M_k \cap \mathbf{p}^{-1}(y) \cap B^{n+1}_1(0)\}.$$

By Sard's theorem,

$$\begin{aligned}\mathcal{H}^{n-1}(\{y \in B^{n-1}_{1/2}(0) \,:\, \Theta_{M_k}(z) = 1 \,\forall\, z \in M_k \cap \mathbf{p}^{-1}(y) \cap B^{n+1}_1(0)\})& \\ = \mathcal{H}^{n-1}(\widetilde{E}_k),&\end{aligned}$$

where

$$\begin{aligned}\widetilde{E}_k = \Big\{y \in B^{n-1}_{1/2}(0) \,:\, M_k \cap \mathbf{p}^{-1}(y) \cap B^{n+1}_1(0)& \\ \text{is a smooth, 1-dim manifold}\}.&\end{aligned}$$

For $y \in \widetilde{E}_k$, let $\Sigma^k_y \equiv M_k \cap \mathbf{p}^{-1}(y) \cap (B^2_{2\tau_k}(0) \times \{(0,y)\})$. By the definition of $\widetilde{E}_k$, Σ^k_y is the union of two disjoint, smooth curves of finite length. Suppose $\gamma^k_y(s)$, $0 \le s \le L^k_y$ is the arc length parameterization of one of them. Here L^k_y is the length of this curve.

Since the two curves that make up Σ^k_y are disjoint, the two end points $\gamma^k_y(0)$ and $\gamma^k_y(L)$ cannot lie in the 1^{st} and the 3^{rd} quadrants, or in the 2^{nd} and the 4^{th} quadrants. This means that $\operatorname{dist}(\gamma^k_y(0), \mathbf{H}^{(i)}_k) \le c\,\beta_k$ and $\operatorname{dist}(\gamma^k_y(L), \mathbf{H}^{(j)}_k) \le c\,\beta_k$, where $\{i,j\} = \{1,2\}$, $\{2,3\}$, $\{3,4\}$ or $\{4,1\}$. Therefore, since the angle between $\mathbf{H}^{(i)}_k$ and $\mathbf{H}^{(j)}_k$ is $\ge c\,\epsilon_k$ for such $\{i,j\}$, we have that

$$|\nu_k(\gamma^k_y(L)) - \nu_k(\gamma^k_y(0))| \ge c\,\epsilon_k. \tag{6.19}$$

We also have that

$$\begin{aligned}\nu_k(\gamma^k_y(L)) - \nu_k(\gamma^k_y(0)) &= \int_0^L \frac{d}{ds}\nu_k(\gamma^k_y(s))\,ds \\ &= \int_0^L D_{\dot{\gamma}^k_y(s)}\nu_k(\gamma^k_y(s)) \cdot \dot{\gamma}^k_y(s)\,ds \\ &= -\int_0^L (A_k)_{\gamma^k_y(s)}\,ds\end{aligned} \tag{6.20}$$

where $(A_k)_{\gamma^k_y(s)} = A_k(\dot{\gamma}^k_y(s), \dot{\gamma}^k_y(s))$.

Combining inequality (6.19) and equation (6.20), we get that

$$\int_0^L |(A_k)_{\gamma_y^k(s)}|\,ds \geq c\,\epsilon_k, \tag{6.21}$$

which gives

$$c\,\epsilon_k \mathcal{H}^{n-1}(\widetilde{E}_k) \leq \int_{\widetilde{E}_k} \int_0^L |(A_k)_{\gamma_y^k(s)}|\,ds\,d\mathcal{H}^{n-1}(y). \tag{6.22}$$

Using the co-area formula and the Cauchy-Schwarz inequality (and observing that the Jacobian $J\mathbf{p} \leq 1$), we obtain from inequality (6.22) that

$$\begin{aligned} c\,\epsilon_k \mathcal{H}^{n-1}(\widetilde{E}_k) &\leq \int_{M_k \cap (B^2_{2\tau_k}(0) \times B^{n-1}_{1/2}(0))} |A_k|\,d\mathcal{H}^n \\ &\leq \Big(\mathcal{H}^n(M_k \cap (B^2_{2\tau_k}(0) \times B^{n-1}_{1/2}(0))\Big)^{1/2} \\ &\qquad \times \left(\int_{M_k \cap (B^2_{2\tau_k}(0) \times B^{n-1}_{1/2}(0))} |A_k|^2\right)^{1/2}. \end{aligned} \tag{6.23}$$

Now, by the monotonicity formula (2.9), we have that, for every $\rho \in (0,\, 1/2]$ and every $x \in M_k \cap B^{n+1}_{1/2}(0)$,

$$\begin{aligned} \frac{\mathcal{H}^n(M_k \cap B^{n+1}_\rho(x))}{\omega_n \rho^n} \leq& \frac{\mathcal{H}^n(M_k \cap B^{n+1}_{1/2}(x))}{\omega_n (1/2)^n} \\ &\leq 2^n \frac{\mathcal{H}^n(M_k \cap B^{n+1}_1(0))}{\omega_n} \leq 2^n.3 \end{aligned} \tag{6.24}$$

for sufficiently large k.

Using the above inequality in each of the balls of a (finite) collection of balls of radius ϵ_k covering $B^2_{2\tau_k}(0) \times B^{n-1}_1(0)$, and noting that the number of such balls required is $\leq \frac{\mathrm{vol}\,(B^2_{2\tau_k}(0) \times B^{n-1}_1(0))}{\mathrm{vol}\,(B^{n+1}_{\tau_k}(0))} = \frac{C(n)}{\tau_k^{n-1}}$, we obtain that

$$\mathcal{H}^n(M_k \cap (B^2_{2\tau_k}(0) \times B^{n-1}_{1/2}(0)) \leq c\,\tau_k. \tag{6.25}$$

Since $B^2_{2\tau_k}(0) \times B^{n-1}_{1/2}(0)) \subset B^{n+1}_{3/4}(0)$, using Lemma 3.2 with a choice of φ that satisfies $\varphi \equiv 1$ in $B_{3/4}(0)$, $\varphi \equiv 0$ outside $B_{7/8}(0)$ and $|D\varphi| \leq 16$, we have that

(6.26) $$\int_{M_k\cap(B^2_{2\tau_k}(0)\times B^{n-1}_{1/2}(0))}|A_k|^2\le c\,\epsilon_k^2.$$

It follows from inequalities (6.23), (6.25) and (6.26) that $\mathcal{H}^{n-1}(\widetilde{E}_k)\le c\,\tau_k^{\frac{1}{2}}$. This completes the proof of the lemma. q.e.d.

Lemma 6.20. [[**11**], **Lemma 3.4**] *For sufficiently large k, if $Z\in M_k\cap B^{n+1}_{1/2}(0)$ is such that $\Theta_{M_k}(Z)\ge 2$, then,*

(i) $\int_{M_k\cap B^{n+1}_{1/4}(Z)}\frac{(\nu_k\cdot((x,y)-Z))^2}{|(x,y)-Z|^{n+2}}\le c\int_{M_k\cap B^{n+1}_1(0)}\operatorname{dist}^2((x,y),\tau_Z\mathbf{H}_k)$,

(ii) $\int_{M_k\cap B^{n+1}_{1/4}(Z)}\sum_{j=1}^{n-1}(\nu_k\cdot e_{2+j})^2\le c\int_{M_k\cap B^{n+1}_1(0)}\operatorname{dist}^2((x,y),\tau_Z\mathbf{H}_k)$

and

(iii) $\int_{M_k\cap B^{n+1}_{1/4}(Z)}\frac{\operatorname{dist}^2((x,y),\tau_Z\mathbf{H}_k)}{|(x,y)-Z|^{n+3/2}}\le c\int_{M_k\cap B^{n+1}_1(0)}\operatorname{dist}^2((x,y),\tau_Z\mathbf{H}_k)$

where c depends only on n and τ_Z is the translation $X\mapsto X-Z$.

Proof. By translation and scaling, we may assume $Z=0$. The proof is essentially as in the proof of Lemma 3.4 of [**11**]. The only change is that the definition of the function u_k and the graphical part G_k ($=\operatorname{graph}u_k$) (which correspond to u and G respectively in the proof of Lemma 3.4 of [**11**]) have to be different here. In [**11**], u_k, G_k together with the estimate

(6.27) $$\int_{M_k\cap(B^{n+1}_{1/2}(0)\setminus G_k)}r^2+\int_{U_k\cap B^{n+1}_{1/2}(0)}\left(|u_k|^2+r^2|Du_k|^2\right)\le C\beta_k^2$$

are obtained via the regularity result (1.8) there. (c.f. Lemma 2.6 of [**11**].) Since (1.8) of [**11**] depends on the assumption that M belongs to a multiplicity 1 class (the crucial point being 1.3(b) of [**11**] holds) we cannot use the same approach here. Instead, we proceed differently to argue that (6.27) holds for a suitable *two valued* graph G_k over a domain U_k in the plane $\{0\}\times\mathbf{R}^n$. (Thus u_k will be a two valued function over a domain $U_k\subset B^n_1(0)$ which satisfies (6.27).)

To see this we use the notation that, for $\rho<1/2$ and $|y|<1$, $A_\rho(y)=\{(r\omega,z):(r-\rho/2)^2+|z-y|^2<\rho^2/16,\ \omega\in S^1\}$, $\widetilde{A}_\rho(y)=\{(r\omega,z):(r-\rho/2)^2+|z-y|^2<\rho^2/32,\ \omega\in S^1\}$.

Take any $\rho\in(0,1/2)$ and for $\delta=\delta(n)$ and $c=c(n,\alpha)$ small (to be chosen) consider the alternatives:

(a) $\int_{M_k\cap A_\rho(y)}h_k^2<c\rho^{n+2}$ and $\int_{M_k\cap\widetilde{A}_\rho(y)}d_k^2>\delta\int_{M_k\cap A_\rho(y)}h_k^2$,

(b) $\int_{M_k\cap A_\rho(y)}h_k^2\ge c\rho^{n+2}$ and $\int_{M_k\cap\widetilde{A}_\rho(y)}d_k^2>\delta\int_{M_k\cap A_\rho(y)}h_k^2$,

(c) $\int_{M_k \cap A_\rho(y)} d_k^2 < c\rho^{n+2}$ and $\int_{M_k \cap \widetilde{A}_\rho(y)} d_k^2 \leq \delta \int_{M_k \cap A_\rho(y)} h_k^2$, and
(d)$\int_{M_k \cap A_\rho(y)} d_k^2 \geq c\rho^{n+2}$ and $\int_{M_k \cap \widetilde{A}_\rho(y)} d_k^2 \leq \delta \int_{M_k \cap A_\rho(y)} h_k^2$
where h_k is the distance to the plane $x_1 = 0$ and d_k is the distance to the pair of planes $\mathbf{H}_k$.

In case of alternative (a), provided $c = c(n, \alpha)$ is sufficiently small, we can use the Schoen-Simon approximate graphical decomposition (as in Lemma 3.3 above) relative to the single plane $x_1 = 0$ to get a 2-valued graph $G_\rho^{(k)}(y) = \operatorname{graph} u_{\rho,y}^{(k)}$ of small gradient over a domain $U_\rho^{(k)}(y)$ in the $x_1 = 0$ plane such that (see (3.24))

(6.28)
$$\mathcal{H}^n(M_k \cap \widetilde{A}_\rho(y) \setminus G_\rho^{(k)}(y)) \leq C\rho^{-2} \int_{M_k \cap A_\rho(y)} h_k^2 \leq C\delta^{-1}\rho^{-2} \int_{M_k \cap \widetilde{A}_\rho(y)} d_k^2$$

and

$$\int_{U_\rho^{(k)}(y)} \left(|u_{\rho,y}^{(k)}|^2 + r^2 |Du_{\rho,y}^k|^2 \right) \leq C \int_{M_k \cap \widetilde{A}_\rho(y)} d_k^2. \tag{6.29}$$

Note that (6.28) implies

$$\int_{M_k \cap \widetilde{A}_\rho(y) \setminus G_\rho^{(k)}(y)} r^2 \leq C \int_{M_k \cap A_\rho(y)} d_k^2. \tag{6.30}$$

In case of alternative (c) note that we have $\int_{M_k \cap A_\rho(y)} h_k^2 \geq C\rho^{n+2}\epsilon_k^2$ (since by definition of alternative (c), it follows that there is a subset of $M_k \cap A_\rho(y)$ of measure $\geq C\rho^n$, $C = C(n)$, on which $h_k \geq \frac{1}{16}\epsilon_k^2\rho^2$) and so, for small enough δ depending only on n, by the argument leading to the estiamte (6.5), (which shows that we can't have two sheets joining up when the L^2 distance from $x_1 = 0$ is much bigger than the L^2 distance to a pair of almost parallel planes) we conclude that either (i) $M_k \cap \widetilde{A}_\rho(y)$ decomposes into two regular graphs over the respective planes $\mathbf{H}_{k,1}$, $\mathbf{H}_{k,2}$ of $\mathbf{H}_k$ or else (ii) the whole $M_k \cap \widetilde{A}_\rho(y)$ is contained in an $\delta\epsilon_k\rho/2$ neighborhood of one of the planes, and we can apply an argument similar to that for alternative (a), except that it is done relative to the plane $\mathbf{H}_{k,1}$ or $\mathbf{H}_{k,2}$ instead of the plane $x_1 = 0$. In either of the cases (i) and (ii), by composing the defining functions of the graphs thus obtained with the functions that express the planes $\mathbf{H}_{k,1}$, $\mathbf{H}_{k,2}$ as graphs over $x_1 = 0$, we get a 2 valued function $u_{\rho,y}^{(k)}$ with small gradient over a domain $U_\rho^{(k)}(y)$ in $x_1 = 0$ satisfying (6.29) and (6.30).

The alternatives (b) and (d) give trivially that $\int_{M_k\cap\tilde{A}_\rho(y)} r^2 \le C \int_{M_k\cap A_\rho(y)} d_k^2$, so we take $G_\rho^{(k)}(y) = \emptyset$ in these cases.

Thus all alternatives lead to the conclusions (6.29) and (6.30). Defining $U_k = \cup_{\rho\in(0,1/2),\, y\in B_1^{n-1}(0)} U_\rho^{(k)}(y)$ and the (2-valued) function u_k over U_k by setting $u_k|_{U_\rho^{(k)}(y)} \equiv u_{\rho,y}^{(k)}$, an elementary covering argument completes the proof of (6.27). Note that u_k is well defined by unique continuation of solutions to the minimal surface equation. With the estimate (6.27), the proof of the lemma can be completed as in Lemma 3.4 of [**11**]. q.e.d.

Lemma 6.21. **[[11], Theorem 3.1 with minor modification]** *For sufficiently large k, if $Z = (\xi,\eta) \in M_k \cap B_{1/2}^{n+1}(0)$ is such that $\Theta_{M_k}(Z) \ge 2$, then,*

(i) $|\xi^{\perp_0}|^2 + \epsilon_k^2\,|\xi|^2 \le c\,\beta_k^2$,

(ii) $\int_{M_k\cap B_{1/4}^{n+1}(Z)} \sum_{j=1}^{n-1} (\nu_k\cdot e_{2+j})^2 + \int_{M_k\cap B_{1/4}^{n+1}(Z)} \frac{d_k^2}{|(x,y)-Z|^{n-1/4}} \le c\,\beta_k^2$ *and*

(iii) $\sum_{i=1}^4 \int_{U_k^{(i)}\cap B_{1/4}^{n+1}(Z)} \frac{|g_k^{(i)}(x,y)-\xi^{\perp_k}(x,y)|^2}{|(x,y)-Z|^{n+3/2}} \le c\,\beta_k^2$

where $\xi^{\perp_0}$ means the orthogonal projection of $(\xi,0)$ onto $(\mathbf{R}^n)^\perp$, $\xi^{\perp_k}(x,y)$ means the orthogonal projection of $(\xi,0)$ onto $(T_{(x',y)}\mathbf{H}_k)^\perp$ where x' is the nearest point projection of x onto $\mathbf{L}_k$, ($T_{(x',y)}\mathbf{H}_k \equiv$ subspace containing $\mathbf{H}_k^{(i)}$ if $(x',y)\in\mathbf{H}_k^{(i)}$), $d_k(x,y) = \operatorname{dist}((x,y),\mathbf{H}_k)$ and c depends only on n.

Proof. To prove (i), notice that since $Z\in T_k$, there exist $\theta_k > 0$ with $\theta_k \searrow 0$ such that for

$$X = (x,y) \in W_k \equiv M_k \cap \left(B_1^{n+1}(0) \setminus (B_{\theta_k}^2(0)\times\mathbf{R}^{n-1})\right),$$

$$\operatorname{dist}(X,\tau_Z\mathbf{H}_k) = |(x,y)-(x',y)-\xi^{\perp_k}| \tag{6.31}$$

where τ_Z is the translation $X\mapsto X-Z$. (This is because for $X\in W_k$, the nearest of the four half spaces of $\mathbf{H}_k$ to X and $X+Z$ are the same.)

Since

$$|(x,y)-(x',y)| = d_k(X), \tag{6.32}$$

it follows that

$$|\xi^{\perp_k}| \le \operatorname{dist}(X,\tau_Z\mathbf{H}_k) + d_k(X) \tag{6.33}$$

for $X \in W_k$. We also have that for each $\rho_0 \in (0, 1/8]$ and all sufficiently large k (depending on ρ_0),

$$|\xi^{\perp_0}|^2 + \epsilon_k^2 \, |\xi|^2 \leq c \, \rho_0^{-n} \int_{M_k \cap W_k \cap B^{n+1}_{\rho_0}(Z)} |\xi^{\perp_k}|^2 \tag{6.34}$$

and that

$$|\xi^{\perp_k}| \leq c \, (|\xi^{\perp_0}| + \epsilon_k \, |\xi|). \tag{6.35}$$

The inequality (6.35) holds because for each i, the angle between $\mathbf{H}_k^{(i)}$ and $\mathbf{R}^n$ is $\leq c \, \epsilon_k$, and hence

$$|\xi^{\perp_k} - \xi^{\perp_0}| \leq c \, \epsilon_k |\xi|. \tag{6.36}$$

To see estimate (6.34), first notice that for sufficiently large k,

$$\epsilon_k^2 \, |\xi|^2 \leq c \, \rho_0^{-n} \int_{M_k \cap W_k \cap B^{n+1}_{\rho_0}(Z)} |\xi^{\perp_k}|^2.$$

This is true because for sufficiently large k, $Z \in T_k$ and therefore, for any given ρ_0, there exists $k_0 = k_0(\rho_0)$ such that for each $k \geq k_0$ there exists a subset S_k (depending on Z) of $M_k \cap W_k \cap B_{\rho_0}(Z)$ with $\mathcal{H}^n(S_k) \geq \frac{1}{2} \omega_n \rho_0^n$ such that for every $X = (x, y) \in S_k$, the angle between $(\xi, 0)$ and the nearest of $\{\mathbf{L}_k^{(i)}\}_{i=1}^4$ to $(x, 0)$ is $\geq c \, \epsilon_k$, or, equivalently, $|\xi^{\perp_k}(X)| \geq c \, \epsilon_k |\xi|$. Integrating this last inequality over S_k yields that $\epsilon_k^2 \, |\xi|^2 \leq c \, \rho_0^{-n} \int_{M_k \cap W_k \cap B^{n+1}_{\rho_0}(Z)} |\xi^{\perp_k}|^2$. Inequality (6.34) then follows from this and inequality (6.36) because $|\xi^{\perp_0}|^2 \leq 2 \, |\xi^{\perp_0} - \xi^{\perp_k}|^2 + 2 \, |\xi^{\perp_k}|^2$.

By inequalities (6.34) and (6.33), we have that

$$|\xi^{\perp_0}|^2 + \epsilon_k^2 \, |\xi|^2 \leq c \, \rho_0^{-n} \left(\int_{M_k \cap B^{n+1}_{\rho_0}(Z)} \mathrm{dist}^2 \, (X, \tau_Z \mathbf{H}_k) + \int_{M_k \cap B^{n+1}_1(0)} d_k^2 \right) \tag{6.37}$$

where c is independent of k and ρ_0.

By Lemma 6.20, we have that

$$\rho_0^{-n-3/2} \int_{M_k \cap B^{n+1}_{\rho_0}(Z)} \operatorname{dist}^2(X, \tau_Z \mathbf{H}_k) \leq c \int_{M_k \cap B^{n+1}_1(0)} \operatorname{dist}^2(X, \tau_Z \mathbf{H}_k) \leq c \int_{M_k \cap B^{n+1}_1(0)} d_k^2 + c \int_{M_k \cap B^{n+1}_1(0)} |\xi^{\perp_k}|^2 \tag{6.38}$$

and in view of inequality (6.35), this implies that

$$\rho_0^{-n} \int_{M_k \cap B^{n+1}_{\rho_0}(Z)} \operatorname{dist}^2(X, \tau_Z \mathbf{H}_k) \leq c\, \rho_0^{3/2} \int_{M_k \cap B^{n+1}_1(0)} d_k^2 + c\, \rho_0^{3/2} \left(|\xi^{\perp_0}|^2 + \epsilon_k^2 \, |\xi|^2 \right). \tag{6.39}$$

Combining inequalities (6.37) and (6.39) and choosing $\rho_0 = \rho_0(n)$ sufficiently small, we obtain that

$$|\xi^{\perp_0}|^2 + \epsilon_k^2 |\xi|^2 \leq c \int_{M_k \cap B^{n+1}_1(0)} d_k^2. \tag{6.40}$$

This is part (i) of the lemma.

To prove part (ii), first notice that we have by the triangle inequality that

$$|\operatorname{dist}((x,y), \tau_Z \mathbf{H}_k) - \operatorname{dist}((x,y), \mathbf{H}_k)| \leq |\xi^{\perp_k^i}| \tag{6.41}$$

for some i, $1 \leq i \leq 4$, where $\xi^{\perp_k^i}$ denotes the orthogonal projection of $(\xi, 0)$ onto the direction normal to $\mathbf{H}_k^{(i)}$. Since $|\xi^{\perp_k^i}|^2 \leq c\,(|\xi^{\perp_0}|^2 + \epsilon_k^2 |\xi|^2)$ for each i, it follows from part (i) that

$$|\operatorname{dist}((x,y), \tau_Z \mathbf{H}_k) - \operatorname{dist}((x,y), \mathbf{H}_k)| \leq c\, \beta_k. \tag{6.42}$$

By Lemma 6.20 (together with the obvious fact that $|(x,y) - Z|^{n+3/2} \leq |(x,y) - Z|^{n-1/4}$ for $(x,y) \in B_{1/4}(Z)$) we have that

$$\int_{M_k \cap B^{n+1}_{1/4}(Z)} \frac{\operatorname{dist}^2((x,y), \tau_Z \mathbf{H}_k)}{|(x,y) - Z|^{n-1/4}} \leq c \int_{M_k \cap B^{n+1}_1(0)} \operatorname{dist}^2((x,y), \tau_Z \mathbf{H}_k). \tag{6.43}$$

By inequalities (6.42), (6.43) and the fact that $\int_{M_k \cap B^{n+1}_{1/4}(Z)} |X - Z|^{-n+1/4} \leq c(n)$ (which is a consequence of the volume growth estimate $\mathcal{H}^n(M_k \cap B^{n+1}_r(Z)) \leq c\, r^n$), we conclude that

$$\int_{M_k \cap B^{n+1}_{1/4}(Z)} \frac{d_k^2}{|(x,y) - Z|^{n-1/4}} \leq c\, \beta_k^2 \tag{6.44}$$

which, together with Lemma 6.20, gives part (ii).

Finally, for part (iii), observe that by part (i), equation (6.31) holds for $(x,y) \in U_k^{(i)}$, so we have by (6.31) that

$$\text{dist}\,((x,y) + g_k^{(i)}(x,y), \tau_Z \mathbf{H}_k) = |g_k^{(i)}(x,y) - \xi^{\perp_k}(x,y)|. \tag{6.45}$$

The estimate of part (iii) follows from Lemma 6.20, the area formula, inequality (6.42), equation (6.45) and the estimate of part (i). This completes the proof of the lemma. q.e.d.

Lemma 6.22. **[[11], Corollary 3.2]** *For sufficiently large k,*

$$(i) \quad \sum_{i=1}^{4} \int_{U_k^{(i)} \cap B^{n+1}_{1/8}(0)} \frac{|g_k^{(i)}(x,y) - (\kappa_k^1\, e_1^{\perp_k} + \kappa_k^2\, e_2^{\perp_k})(x,y)|^2}{r_k^{3/2}} \leq c\, \beta_k^2$$

and

$$(ii) \quad \int_{M_k \cap B^{n+1}_{1/8}(0)} \frac{d_k^2}{r_k^{1/4}} \leq c\, \beta_k^2.$$

Here c depends only on n, $r_k(x,y) = \max\{r, \delta_k\}$ $(r = |x|)$ with δ_k as in Lemma 6.19, $e_j^{\perp_k}(x,y)$, $j = 1, 2$, means the orthogonal projection of e_j onto $(T_{(x,y)}\mathbf{H}_k)^\perp$ where e_1, e_2 are the standard basis vectors of $\mathbf{R}^2$ and $\kappa_k^j(x,y) = \widetilde{\kappa}_k^j(r,y)$ with

$$\widetilde{\kappa}_k^j : D_k \equiv \{(r,y) \,:\, r > 0,\, y \in \mathbf{R}^{n-1},\, (r\omega_k^i, y) \in U_k^i \text{ for } i = 1, \ldots, 4\} \to \mathbf{R}$$

(where $\omega_k^i = \mathbf{L}_k^i \cap \mathbf{S}^1$) satisfying

$$\sup\nolimits_{(r,y) \in D_k} \left((\widetilde{\kappa}_k^1)^2 + \epsilon_k^2\left((\widetilde{\kappa}_k^1)^2 + (\widetilde{\kappa}_k^2)^2\right)\right) \leq c\, \beta_k^2.$$

Proof. To prove (ii), let $z \in B^{n-1}_{1/2}(0)$ and $\rho \in (\delta_k, 1/8)$ be arbitrary. Here δ_k is as in Lemma 6.19. By Lemma 6.19, there exists $Z \in M_k$ with $\Theta_{M_k}(Z) \geq 2$ such that $|Z - (0, z)| \leq \delta_k$. Thus, if $X \in B^{n+1}_\rho(0, z)$ then $|X - Z| \leq 2\rho$ and therefore, using part (ii) of Lemma 6.21, we obtain that

$$\rho^{-n+1/4} \int_{M_k \cap B^{n+1}_\rho(0,z)} d_k^2 \leq 2^{n-1/4} \int_{M_k \cap B^{n+1}_\rho(0,z)} \frac{d_k^2}{|X-Z|^{n-1/4}} \leq 2^{n-1/4} \int_{M_k \cap B^{n+1}_{1/4}(Z)} \frac{d_k^2}{|X-Z|^{n-1/4}} \leq c\,\beta_k^2. \tag{6.46}$$

We can cover $B^{n+1}_{1/2}(0) \cap (B^2_{\rho/2}(0) \times \mathbf{R}^{n-1})$ by a collection of balls $B^{n+1}_\rho(0, z_j)$ with $z_j \in B^{n-1}_{1/2}(0)$ such that the number of balls in the collection is $\leq \frac{c(n)}{\rho^{n-1}}$. Using such a covering, we obtain from inequality (6.46) that

$$\rho^{-3/4} \int_{M_k \cap B^{n+1}_{1/2}(0) \cap (B^2_{\rho/2}(0) \times \mathbf{R}^{n-1})} d_k^2 \leq c\,\beta_k^2 \tag{6.47}$$

for $\rho \in (\delta_k, 1/8)$.

The required inequality now follows from Fubini's theorem by multiplying both sides of (6.47) by $\rho^{-1/2}$ and integrating with respect to ρ over $(\delta_k, 1/8)$.

To prove (i), for $\rho \in (\delta_k, 1/8)$, let $\{B^{n+1}_\rho(0, z_j)\}$ be the same covering as the one used in the proof of (ii) above. As before, for each j, there exists by Lemma 6.19 $Z_j = (\xi_j, \eta_j) \in M_k$ with $\Theta_{M_k}(Z_j) \geq 2$ and $|Z_j - (0, z_j)| \leq \delta_k$. Furthermore, writing $\xi_j = (\xi_j^1, \xi_j^2) (\in \mathbf{R}^2)$ and noting that $\xi_j^{\perp_0} = (\xi_j^1, 0)$, we have by part (i) of Lemma 6.21 that

$$\left(|\xi_j^1|^2 + \epsilon_k^2\,\left(|\xi_j^1|^2 + |\xi_j^2|^2\right)\right) \leq c\,\beta_k^2 \tag{6.48}$$

and by part (iii) of Lemma 6.21, that

$$\sum_{i=1}^{4} \rho^{-n-3/2} \int_{U_k^{(i)} \cap B^{n+1}_\rho(0,z_j)} |g_k^{(i)}(x,y) - (\xi_j^1\, e_1^{\perp_k} + \xi_j^2\, e_2^{\perp_k})(x,y)|^2 \leq c\,\beta_k^2. \tag{6.49}$$

For each k and each $(r, y) \in (\delta_k, 1/8] \times B^{n-1}_{1/4}(0)$, choose $\widetilde{\kappa}^1_k(r, y)$ and $\widetilde{\kappa}^2_k(r, y)$ such that

$$|\widetilde{\kappa}^1_k(r, y)|^2 + \epsilon_k^2 \left(|\widetilde{\kappa}^1_k(r, y)|^2 + |\widetilde{\kappa}^2_k(r, y)|^2\right) \leq c\, \beta_k^2 \tag{6.50}$$

and

$$\begin{aligned}(6.51)\quad \sum_{i=1}^4 &|g_k^{(i)}(r\, \omega_k^i, y) - (\widetilde{\kappa}^1_k(r, y)\, e_1^{\perp_k}(r\, \omega_k^i, y) + \widetilde{\kappa}^2_k(r, y)\, e_2^{\perp_k}(r\, \omega_k^i, y))|^2 \\ &= \inf \sum_{i=1}^4 |g_k^{(i)}(r\, \omega_k^i, y) - (\lambda_1\, e_1^{\perp_k}(r\, \omega_k^i, y) + \lambda_2\, e_2^{\perp_k}(r\, \omega_k^i, y))|^2\end{aligned}$$

where $\omega_k^i = \mathbf{L}_k^{(i)} \cap \mathbf{S}^1$ and the inf is taken over all $\lambda = (\lambda_1, \lambda_2) \in \mathbf{R}^2$ such that $\lambda_1^2 + \epsilon_k^2\, (\lambda_1^2 + \lambda_2^2) \leq c\, \beta_k^2$, with $c = c(n)$ as in inequality (6.48).

It follows from inequality (6.49) then that

$$\sum_{i=1}^4 \rho^{-n-3/2} \int_{U_k^{(i)} \cap B_\rho^{n+1}(0, z_j)} |g_k^{(i)}(x, y) - (\kappa_k^1\, e_1^{\perp_k} + \kappa_k^2\, e_2^{\perp_k})(x, y)|^2 \leq c\, \beta_k^2.$$

Summing over j, we obtain from this that

$$\sum_{i=1}^4 \rho^{-5/2} \int_{U_k^{(i)} \cap (B^2_{\rho/2}(0) \times \mathbf{R}^{n-1})} |g_k^{(i)}(x, y) - (\kappa_k^1\, e_1^{\perp_k} + \kappa_k^2\, e_2^{\perp_k})(x, y)|^2 \leq c\, \beta_k^2$$

and the inequality of part (i) of the lemma follows from this by integrating it with respect to ρ over $(\delta_k, 1/8)$. This completes the proof of the lemma. q.e.d.

We now return to our second blow-up $w^{(i)}$, $i = 1, \ldots 4$, as in (6.15), (6.16) and (6.17). The estimate of Lemma 6.22 (ii) in particular implies that for $i = 1, \ldots, 4$,

$$\int_{U_k^{(i)} \cap B^{n+1}_{1/8}(0) \cap \left(B^2_{\delta_k}(0) \times \mathbf{R}^{n-1}\right)} |g_k^{(i)}|^2 \leq c\, \delta_k^{1/4}\, \beta_k^2. \tag{6.52}$$

This means that the convergence in (6.17) is strong in $L^2(\mathbf{R}^{n\,+} \cap B_{1/16}(0))$ for $i = 1$, 4 and in $L^2(\mathbf{R}^{n\,-} \cap B_{1/16}(0))$ for $i = 2$, 3. Therefore, by part (i) of Lemma 6.22, we have that

$$\int_0^{1/16} \int_{B^{n-1}_{1/16}(0)} \frac{|w^{(1)}(x^2, y) - (\kappa_1(r, y) - \kappa_2(r, y))|^2}{r^{3/2}} < \infty, \tag{6.53}$$

$$\int_0^{1/16} \int_{B^{n-1}_{1/16}(0)} \frac{|w^{(2)}(-x^2, y) - (\kappa_1(r, y) + \kappa_2(r, y))|^2}{r^{3/2}} < \infty, \tag{6.54}$$

$$\int_0^{1/16} \int_{B^{n-1}_{1/16}(0)} \frac{|w^{(3)}(-x^2, y) - (\kappa_1(r, y) - \kappa_2(r, y))|^2}{r^{3/2}} < \infty \tag{6.55}$$

and

$$\int_0^{1/16} \int_{B^{n-1}_{1/16}(0)} \frac{|w^{(4)}(x^2, y) - (\kappa_1(r, y) + \kappa_2(r, y))|^2}{r^{3/2}} < \infty \tag{6.56}$$

for bounded functions $\kappa_1\,, \kappa_2 : (0, 1/16] \times B^{n-1}_{1/16}(0) \to \mathbf{R}$.

Lemma 6.23. *The part of the union of the closures of the graphs of functions $w^{(1)}$ and $w^{(3)}$ in $B_{1/16}(0) \times \mathbf{R}$ is equal to the graph of a single harmonic function over $B_{1/16}(0)$. Similarly, the part of the union of the closures of the graphs of $w^{(2)}$ and $w^{(4)}$ in $B_{1/16}(0) \times \mathbf{R}$ is equal to the graph of a single harmonic function over $B_{1/16}(0)$.*

Hence in particular, since $w^{(i)}$ are homogeneous of degree 1, the second blow-up consists of two intersecting hyperplanes.

Proof. Here we shall assume that $\mathbf{H} = \text{graph} v^+ \cup \text{graph}\, v^-$ is the union of the two hyperplanes given by $x^1 = \pm x^2$. Thus $\mathbf{H}_k$ is given by $x^1 = \pm \epsilon_k x^2$. (We may arrange this by replacing ϵ_k by $c\epsilon_k$ for a suitable fixed constant c independent of k.) Let $\psi : \mathbf{R} \to \mathbf{R}$ be a C^2 cut-off function with $\psi(t) \equiv 1$ if $t \leq 1/8$, $\psi(t) \equiv 0$ if $t \geq 1/4$, $|\psi'(t)| \leq 16$ and $|\psi''(t)| \leq 128$ for all t. For each k, let $\psi_k : \mathbf{R} \to \mathbf{R}$ be a C^2 cut-off function with $\psi_k(t) \equiv 1$ if $t \in (-\infty, -\frac{\epsilon_k}{2}]$, $\psi_k(t) \equiv 0$ if $t \in [\frac{\epsilon_k}{2}, \infty)$, $|\psi'(t)| \leq 2\epsilon_k^{-1}$ and $|\psi''(t)| \leq 4\epsilon_k^{-2}$ for all t.

Let $\tau \in (0, 1/16^3)$ be arbitrary and $\zeta \in C^2_c(B_{1/16}(0))$ with $D_2\, \zeta \equiv 0$ in $B_{1/16}(0) \cap \{|x^2| \leq \tau\}$. By the first variation formula, we have that

$$\int_{M_k} \nabla\,(x^1 - \epsilon_k x^2)\cdot\nabla(\widetilde{\zeta}\psi(|\nu_k'|^2)\psi_k(\nu_k^2)) = 0 \tag{6.57}$$

where $\widetilde{\zeta}$ is a $C_c^1(B_{1/16}(0)\times\mathbf{R})$ function which agrees with $\zeta_1(x^1, x^2, \ldots, x^{n+1}) \equiv \zeta(x^2, \ldots, x^{n+1})$ in a neighborhood of $M_k \cap (B_{1/16}(0)\times\mathbf{R})$, $\nu_k = (\nu_k^1, \nu_k^2, \ldots, \nu_k^{n+1})$ is the unit normal vector field to M_k and we use the notation $\nu_k' = (\nu_k^2, \ldots, \nu_k^{n+1})$.

Now

$$\int_{M_k} \nabla\,(x^1 - \epsilon_k x^2)\cdot\nabla(\widetilde{\zeta}\psi(|\nu_k'|^2)\psi_k(\nu_k^2)) = I_1^{(k)} + I_2^{(k)} + I_3^{(k)} + I_4^{(k)} \tag{6.58}$$

where

$$I_1^{(k)} = \int_{M_k\setminus T} \psi(|\nu_k'|^2)\psi_k(\nu_k^2)\nabla\,(x^1 - \epsilon_k x^2)\cdot\nabla\,\widetilde{\zeta}$$

$$I_2^{(k)} = \int_{M_k\cap T} \psi(|\nu_k'|^2)\psi_k(\nu_k^2)\nabla\,(x^1 - \epsilon_k x^2)\cdot\nabla\,\widetilde{\zeta}$$

$$I_3^{(k)} = 2\sum_{j=2}^{n+1}\int_{M_k} \widetilde{\zeta}\psi_k(\nu_k^2)\psi'(|\nu_k'|^2)\nu_k^j(e^1 - \epsilon_k e^2)\cdot\nabla\,\nu_k^j$$

and

$$I_4^{(k)} = \int_{M_k} \widetilde{\zeta}\psi(|\nu_k'|^2)\psi_k'(\nu_k^2)(e^1 - \epsilon_k e^2)\cdot\nabla\nu_k^2.$$

Here $T = B_{\tau/2}^2(0)\times\mathbf{R}^{n-1}$. We estimate each of $I_1^{(k)}, \ldots, I_4^{(k)}$ as follows:

For $I_3^{(k)}$ and $I_4^{(k)}$, first note that the support of the integrand is contained in the set $M_k \cap \{|\nu_k'|^2 \leq 1/4\}$, and hence, ν_k can locally be written as a function of just the variables x_2 and y. Hence, by direct computation, the integrand $F_3^{(k)}$ of $I_3^{(k)}$ can locally be written as

$$\begin{aligned} F_3^{(k)} &= -2\sum_{j=2}^{n+1}\widetilde{\zeta}\psi_k(\nu_k^2)\psi'(|\nu_k'|^2)\nu_k^j\left(\epsilon_k D_2\nu_k^j + (\nu_k^1 - \epsilon_k\nu_k^2)\sum_{i=2}^{n+1}\nu_k^i D_i\nu_k^j\right)\\ &= -\widetilde{\zeta}\psi_k(\nu_k^2)\psi'(|\nu_k'|^2)\nu_k^2\left(\epsilon_k + (\nu_k^1 - \epsilon_k\nu_k^2)\nu_k^2\right)D_2\nu_k^2 + S_3^{(k)} \qquad (6.59)\end{aligned}$$

where $S_3^{(k)}$ is the sum of all the terms of $F_3^{(k)}$ having a factor ν_k^j, $j = 3, \ldots, (n+1)$. Thus

$$|S_3^{(k)}| \leq C|\zeta||A_k|\sqrt{\sum_{j=3}^{n+1}(\nu_k^j)^2}. \tag{6.60}$$

Now

$$D_2\nu_k^2 = -\sum_{i=3}^{n+1} D_i\nu_k^i \tag{6.61}$$

which is the minimal surface equation for minimal graphs over the x_2, y plane. Using this in (6.59) and integrating by parts (which can be justified using a suitable partition of unity), we conclude that, in view of the fact that for all sufficiently large k, the supports of $\psi'(|\nu_k'|^2)$ and $\psi''(|\nu_k'|^2)$ are contained in $M_k \cap T$,

$$\begin{aligned}
|I_3^{(k)}| &\leq C\sup|\zeta| \int_{M_k\cap T\cap B_{1/8}^{n+1}(0)} |A_k|\sqrt{\sum_{j=3}^{n+1}(\nu_k^j)^2} \\
&\quad + C\sup|D\zeta| \int_{M_k\cap T\cap B_{1/8}^{n+1}(0)} \sqrt{\sum_{j=3}^{n+2}(\nu_k^j)^2} \\
&\leq C\sup(|\zeta| + |D\zeta|)\sqrt{\tau}\beta_k. \qquad (6.62)
\end{aligned}$$

Here we have used the fact that $|\psi_k'| \leq C\epsilon_k^{-1}$, that $|\nu_k^2| \leq C\epsilon_k$ at every point in the support of $\psi_k'(\nu_k^2)$, that $\mathcal{H}^n(M_k \cap T \cap B_{1/8}^{n+1}(0)) \leq C\tau$ and the estimate $\int_{M_k\cap B_{1/8}^{n+1}(0)} \sum_{j=3}^{n+1}(\nu_k^j)^2 \leq C\beta_k^2$. This last estimate follows from Lemma 6.21, part (ii).

To estimate $I_4^{(k)}$, note that by (6.61) and direct computation as before, the integrand $F_4^{(k)}$ of $I_4^{(k)}$ can locally be written as

$$F_4^{(k)} = \widetilde{\zeta}\psi(|\nu_k'|^2)\psi_k'(\nu_k^2)\left(\epsilon_k + (\nu_k^1 - \epsilon_k\nu_k^2)\nu_k^2\right)\sum_{j=3}^{n+1} D_j\nu_k^j - S_4^{(k)} \tag{6.63}$$

where

$$S_4^{(k)} = \widetilde{\zeta}\psi(|\nu_k'|^2)\psi_k'(\nu_k^2)(\nu_k^1 - \epsilon_k\nu_k^2)\sum_{j=3}^{n+1}\nu_k^j D_j\nu_k^2. \tag{6.64}$$

Thus

$$|S_4^{(k)}| \leq C|\zeta||\psi_k'(\nu_k^2)||A_k|\sqrt{\sum_{j=3}^{n+1}(\nu_k^j)^2}. \tag{6.65}$$

Using this and integrating the terms involving $D_j\nu_k^j$, $j = 3, \ldots, (n+1)$, by parts (which again can be justified using a partition of unity) we see that since for all sufficiently large k, the support of $\psi_k'(\nu_k^2)$ is contained in $M_k \cap T \cap \{|\nu_k^2| \leq \frac{1}{2}\epsilon_k\}$,

$$\begin{aligned} |I_4^{(k)}| \quad &\leq \quad C\sup\left(|\zeta| + |D\zeta|\right)\left(\int_{M_k\cap T\cap\{|\nu_k^2|\leq\frac{1}{2}\epsilon_k\}\cap B_{1/8}^{n+1}(0)}\sqrt{\sum_{j=3}^{n+1}(\nu_k^j)^2}\, + \right. \\ &\left. + \,\epsilon_k^{-1}\int_{M_k\cap T\cap\{|\nu_k^2|\leq\frac{1}{2}\epsilon_k\}\cap B_{1/8}^{n+1}(0)}|A_k|\sqrt{\sum_{j=3}^{n+1}(\nu_k^j)^2}\right). \end{aligned} \tag{6.66}$$

Note that here we have used the fact that $|\psi_k'| \leq C\epsilon_k^{-1}$, $|\psi_k''| \leq C\epsilon_k^{-2}$ and that $|\nu_k^2| \leq C\epsilon_k$ at every point of the support of $\psi_k'(\nu_k^2)$.

Using Cauchy-Schwarz inequality and Lemma 6.24 below, we conclude from this that

$$|I_4^{(k)}| \leq C\sup\left(|\zeta| + |D\zeta|\right)\tau^{1/6}\beta_k. \tag{6.67}$$

To estimate $I_2^{(k)}$, note that since $\widetilde{\zeta}$ is independent of x_1 everywhere and independent of x_2 in T, we have that on $M_k \cap T$, the integrand $F_2^{(k)}$ of $I_2^{(k)}$ is given by

$$F_2^{(k)} = -\psi(|\nu_k'|^2)\psi_k(\nu_k^2)(\nu_k^1 - \epsilon_k\nu_k^2)\left(\sum_{j=3}^{n+1}D_j\zeta\nu_k^j\right) \tag{6.68}$$

and hence

$$|I_2^{(k)}| \leq C\sup|D\zeta|\sqrt{\tau}\beta_k. \tag{6.69}$$

Finally, note that for all sufficiently large k, $\psi(|\nu_k'|^2) \equiv 1$ on $M_k \setminus T$, $\psi_k(\nu_k^2) \equiv 0$ on $(\operatorname{graph} g_k^{(2)} \cup \operatorname{graph} g_k^{(4)}) \setminus T$ and $\psi_k(\nu_k^2) \equiv 1$ on $(\operatorname{graph} g_k^{(1)} \cup \operatorname{graph} g_k^{(3)}) \setminus T$. Thus, since $x^1 = \epsilon_k x^2 + \tilde{g}_k^{(i)} \cdot e^1$ on $\operatorname{graph} g_k^{(i)}$, $i = 1, 3$, we have that

(6.70)
$$\lim_{k\to\infty}\beta_k^{-1}I_1^{(k)}=\int_{\mathbf{R}^{n\,+}\setminus\{|x^2|\le\tau/2\}}Dw^{(1)}\cdot D\zeta+\int_{\mathbf{R}^{n\,-}\setminus\{|x^2|\le\tau/2\}}Dw^{(3)}\cdot D\zeta.$$

Integrating by parts in this keeping in mind that $D_2\zeta\equiv 0$ in $\{|x^2|<\tau\}$, we see that

$$\lim_{k\to\infty}\beta_k^{-1}I_1^{(k)}=\int_{\mathbf{R}^{n\,+}\setminus\{|x^2|\le\tau/2\}}(w^{(1)}(x^2,y)+w^{(3)}(-x^2,y))\Delta\,\zeta \tag{6.71}$$

for any $C_c^2(B_{1/16}(0))$ function with $D_2\zeta=0$ in a neighborhood of $x^2=0$ and which is even in the x^2 variable. Hence, in view of the estimates (6.62), (6.67) and (6.69) together with the fact that the function $w^{(1)}(x^2,y)+w^{(3)}(-x^2,y)$ is in $L^2(\mathbf{R}^{n\,+}\cap B_{1/4}(0))$, we conclude by dividing (6.57) by β_k and first letting $k\to\infty$ and then letting $\tau\to 0$ that

$$\int_{\mathbf{R}^{n\,+}\cap B_{1/16}(0)}(w^{(1)}(x^2,y)+w^{(3)}(-x^2,y))\Delta\zeta=0 \tag{6.72}$$

for any $\zeta\in C_c^2(B_{1/16}(0))$ with $D_2\zeta\equiv 0$ in some neighborhood of $x^2=0$ which is even in the x^2 variable. By approximation, (6.72) holds for *every* $C_c^2(B_{1/16}(0))$ function which is even in the x^2 variable. Now let $w_e^{(1\,3)}$ be the even reflection in the x^2 variable of the function $w^{(1)}(x^2,y)+w^{(3)}(-x^2,y)$. Then by (6.72),

$$\int_{B_{1/16}(0)}w_e^{(1\,3)}\Delta\zeta=0 \tag{6.73}$$

for every $\zeta\in C_c^2(B_{1/16}(0))$ even in the x^2 variable. Also, since $w_e^{(1\,3)}$ is even in the x^2 variable, (6.73) trivially holds for any $C^2(B_{1/16}(0))$ function which is odd in the x^2 variable. Hence we conclude that (6.73) holds for arbitrary $\zeta\in C_c^2(B_{1/16}(0))$ and therefore that $w_e^{(1\,3)}$ is harmonic in $B_{1/16}(0)$.

Next note that the conditions (6.53) and (6.55) imply that

$$\lim_{\rho\to 0^+}\rho^{-1}\int_0^\rho\int_{B^{n-1}_{1/16}(0)}|w^{(1)}(x^2,y)-w^{(3)}(-x^2,y)|^2=0. \tag{6.74}$$

Thus, if $w_o^{(1\,3)}$ is the odd reflection in the x^2 variable of the function $w^{(1)}(x^2,y)-w^{(3)}(-x^2,y)$ (which is harmonic in $B_{1/16}(0)\cap\{x_2>$

$0\}$), then (6.74) implies that $w_o^{(1\,3)}$ extends to a harmonic function in $B_{1/16}(0)$. Now define a function $w^{(1\,3)}$ on $B_{1/16}(0) \setminus \{x_2 = 0\}$ by setting $w^{(1\,3)}(x^2, y) = w^{(1)}(x^2, y)$ if $x^2 > 0$ and $w^{(1\,3)}(x^2, y) = w^{(3)}(x^2, y)$ if $x^2 < 0$. Then $w^{(1\,3)} = \frac{1}{2}(w_e^{(1\,3)} + w_o^{(1\,3)})$ everywhere in $B_{1/16}(0) \setminus \{x_2 = 0\}$ and hence we have shown that $w^{(1\,3)}$ extends to all of $B_{1/16}(0)$ as a harmonic function. i.e. that the union of the closures of the graphs of $w^{(1)}$ and $w^{(3)}$ in $B_{1/16}(0) \times \mathbf{R}$ is equal to the graph of a single harmonic function over $B_{1/16}(0)$.

Repeating the entire argument with $(1 - \psi_k)$ in place of ψ_k and using (6.54), (6.56) in place of (6.53), (6.55), we also conclude that the union of the closures of the graphs of $w^{(2)}$ and $w^{(4)}$ in $B_{1/16}(0) \times \mathbf{R}$ is equal to the graph of a single harmonic function over $B_{1/16}(0)$.

Finally, since $w^{(i)}$ are homogeneous of degree 1, we conclude that the second blow-up consists of 2 intersecting hyperplanes. q.e.d.

Lemma 6.24. *For each $\tau \in (0, 1/16^3)$, if $T_\tau = \{(x, y) \in \mathbf{R}^2 \times \mathbf{R}^{n-1} : |x| < \tau\}$, then we have*

$$\int_{M_k \cap \{|\nu_k^2| \leq \frac{1}{2}\epsilon_k\} \cap T_\tau \cap B^{n+1}_{1/8}(0)} \sum_{j=3}^{n+1} (\nu_k^j)^2 \leq C\tau^{1/3}\beta_k^2,$$

for all sufficiently larg k depending on τ. Here $C = C(n)$.

Proof. Let $a^2 = \frac{1}{8^2} - \frac{1}{16^6}$. For each $(0, \eta) \in B_a(0) \cap \{0\} \times \mathbf{R}^{n-1}$ such that there is at least one $\xi \in \mathbf{R}^2$ with $\Theta_{M_k}(\xi, \eta) \geq 2$, let $\xi(\eta)$ be any one of the points $\xi \in \mathbf{R}^2$ with $\Theta_{M_k}(\xi, \eta) \geq 2$. By (6.18), for large enough k the set S_k of such η has $(n-1)$-dimensional measure close to full measure on $B_a(0) \cap \{0\} \times \mathbf{R}^{n-1}$ and by Lemma 6.20 and inequality (6.42), for each $\eta \in S_k$,

(6.75)
$$\int_{M_k \cap B_{1/8}} \frac{\big((x - \xi(\eta)) \cdot \nu_k^x(x, y) + (y - \eta) \cdot \nu_k^y(x, y)\big)^2}{|(x - \xi(\eta), y - \eta)|^{n+2}} d\mathcal{H}^n(x, y) \leq C\beta_k^2,$$

where $\nu_k^x(x, y) = (\nu_k^1(x, y), \nu_k^2(x, y))$ and $\nu_k^y(x, y) = (\nu_k^3(x, y), \ldots, \nu_k^{n+1}$ $(x, y))$. S_k is a closed set, and we can integrate with respect to $\eta \in S_k$ and use Fubini's theorem, whence

$$\int_{M_k\cap B_{1/8}}\int_{S_k}\frac{\big((x-\xi(\eta))\cdot\nu_k^x(x,y)+(y-\eta)\cdot\nu_k^y(x,y)\big)^2}{|(x-\xi(\eta),y-\eta)|^{n+2}}\,d\eta\,d\mathcal{H}^n(x,y)\le C\beta_k^2. \tag{6.76}$$

Using the change of variable $\psi=y-\eta\in y-S_k$ in the inner integral, we have

$$\int_{M_k\cap B_{1/8}}\int_{y-S_k}\frac{\big((x-\xi(y-\psi))\cdot\nu_k^x(x,y)+\psi\cdot\nu_k^y(x,y)\big)^2}{|(x-\xi(y-\psi),\psi)|^{n+2}}\,d\psi\,d\mathcal{H}^n(x,y)\le C\beta_k^2, \tag{6.77}$$

and so, making the change of variable $\psi\mapsto 2\psi$ in (6.77)

$$\int_{M_k\cap B_{1/8}}\int_{\frac12(y-S_k)}\frac{\big((x-\xi(y-2\psi))\cdot\nu_k^x(x,y)+2\psi\cdot\nu_k^y(x,y)\big)^2}{|(x-\xi(y-2\psi),2\psi)|^{n+2}}\,d\psi\,d\mathcal{H}^n(x,y)\le C\beta_k^2, \tag{6.78}$$

which evidently implies

$$\int_{M_k\cap B_{1/8}}\int_{\frac12(y-S_k)}\frac{\big((x-\xi(y-2\psi))\cdot\nu_k^x(x,y)+2\psi\cdot\nu_k^y(x,y)\big)^2}{|(x-\xi(y-2\psi),\psi)|^{n+2}}\,d\psi\,d\mathcal{H}^n(x,y)\le C\beta_k^2. \tag{6.79}$$

For any $\tau\in(0,1/16)$ we know that $|\xi(\eta)|<\tau/2$ for all η such that $\xi(\eta)$ is defined, so (6.77), (6.79) evidently imply

$$\int_{M_k\cap B_{1/8}}\int_{S_k(y)}\frac{\big((x-\xi(y-\psi))\cdot\nu_k^x(x,y)+\psi\cdot\nu_k^y(x,y)\big)^2}{|(x_\tau,\psi)|^{n+2}}\,d\psi\,d\mathcal{H}^n(x,y)\le C\beta_k^2, \tag{6.80}$$

and

$$\int_{M_k\cap B_{1/8}}\int_{S_k(y)}\frac{\big((x-\xi(y-2\psi))\cdot\nu_k^x(x,y)+2\psi\cdot\nu_k^y(x,y)\big)^2}{|(x_\tau,\psi)|^{n+2}}\,d\psi\,d\mathcal{H}^n(x,y)\le C\beta_k^2, \tag{6.81}$$

respectively, for all sufficiently large k (depending on τ), where $S_k(y) = (y - S_k) \cap \frac{1}{2}(y - S_k)$ and x_τ is defined by $x_\tau = x$ if $|x| > \tau$ and $x_\tau = \tau|x|^{-1}x$ if $|x| < \tau$.

Taking differences we conclude

(6.82)
$$\int_{M_k \cap B_{1/8}} \int_{S_k(y)} \frac{\Big(\big(\xi(y-2\psi) - \xi(y-\psi)\big) \cdot \nu_k^x(x,y) + \psi \cdot \nu_k^y(x,y)\Big)^2}{|(x_\tau, \psi)|^{n+2}} \\ d\psi d\mathcal{H}^n(x,y) \leq C\beta_k^2.$$

Now by Lemma 6.21, $\sup|\xi_1(\eta)| + \epsilon_k|\xi_2(\eta)| \leq \gamma\beta_k$ for some $\gamma = \gamma(n) > 1$. Letting $M_{k,+} = \{(x,y) \in M_k \cap B_{1/8} : |\nu_k^y| \geq 4\tau^{-1}\gamma\beta_k,\ |\nu_k^2| \leq \frac{1}{2}\epsilon_k\}$, we have $|\big(\xi(y-2\psi) - \xi(y-\psi)\big) \cdot \nu_k^x(x,y) + \psi \cdot \nu_k^y(x,y)| \geq \frac{1}{4}|\psi||\nu_k^y(x,y)|$ for all $(x,y) \in M_{k,+}$ and all $\psi \in (y - S_k)$ with $\psi \cdot \nu_k^y(x,y) \geq \frac{1}{2}|\psi|\,|\nu_k^y(x,y)|$ and $|\psi| \in (\tau, 1/8)$, and hence

(6.83)
$$\int_{M_{k,+}} \int_{\Omega(x,y)} |(x_\tau, s\omega)|^{-n-2} s^2 |\nu_k^y(x,y)|^2\, s^{n-2}\, ds\, d\omega\, d\mathcal{H}^n(x,y) \leq C\beta_k^2,$$

where $\Omega(x,y)$ is the region in $\{0\} \times \mathbf{R}^{n-1}$ given by $\Omega(x,y) = (y - S_k) \cap \frac{1}{2}(y - S_k) \cap \{s\omega : s \in (\tau, 1/8),\ \omega \in \Sigma(x,y)\}$, with $\Sigma(x,y) = \{\omega \in S^{n-2} : \omega \cdot \nu_k^y(x,y) \geq \frac{1}{2}|\nu_k^y(x,y)|\}$. Thus

(6.84)
$$\int_{M_{k,+}} |\nu_k^y(x,y)|^2 \Big(\int_{\Omega(x,y)} (r_\tau^2 + s^2)^{-(n+2)/2} s^n\, ds\, d\omega\Big)\, d\mathcal{H}^n(x,y) \leq C\beta_k^2,$$

where $r_\tau = |x_\tau|$.

Now since S_k has almost full measure for sufficiently large k, we deduce that there is a fixed constant $C > 0$, independent of $\tau \in (0, 1/16)$ and $(x,y) \in M_{k,+}$, such that, for k large enough, $\int_{\Omega(x,y)} (r_\tau^2 + s^2)^{-(n+2)/2} s^n\, ds\, d\omega \geq C r_\tau^{-1}$ for each $(x,y) \in M_k \cap B_{1/8}$, and hence (6.84) implies

$$\int_{M_{k,+}} |\nu_k^y(x,y)|^2 r_\tau^{-1}\, d\mathcal{H}^n(x,y) \leq C\beta_k^2, \tag{6.85}$$

so

$$\tau^{-1}\int_{M_{k,+}\cap T_\tau} |\nu_k^y(x,y)|^2\, d\mathcal{H}^n(x,y) \le C\beta_k^2, \tag{6.86}$$

and hence in particular

$$\tau^{-1}\int_{M_{k,+}\cap T_{\tau^3}} |\nu_k^y(x,y)|^2\, d\mathcal{H}^n(x,y) \le C\beta_k^2. \tag{6.87}$$

Of course since we trivially have

$$\tau^{-1}\int_{(M_k\cap\{|\nu_k^2|\le\frac{1}{2}\epsilon_k\}\cap B_{1/8}\setminus M_{k,+})\cap T_{\tau^3}} |\nu_k^y(x,y)|^2\, d\mathcal{H}^n(x,y) \le C\beta_k^2, \tag{6.88}$$

$C = C(n)$, (by the definition of $M_{k,+}$ and the fact that $\mathcal{H}^n\,(M_k \cap T_\sigma \cap B_{1/8}) \le C\sigma$ for any $\sigma \in (0, 1/16)$), we then deduce that

$$\tau^{-1}\int_{M_k\cap\{|\nu_k^2|\le\frac{1}{2}\epsilon_k\}\cap T_{\tau^3}\cap B_{1/8}} |\nu_k^y(x,y)|^2\, d\mathcal{H}^n(x,y) \le C\beta_k^2. \tag{6.89}$$

Replacing τ with $\tau^{1/3}$, we deduce that for any $\tau \in (0, 1/16^3)$ and k suffciently large

$$\int_{M_k\cap\{|\nu_k^2|\le\frac{1}{2}\epsilon_k\}\cap T_\tau\cap B_{1/8}} |\nu_k^y(x,y)|^2\, d\mathcal{H}^n(x,y) \le C\tau^{1/3}\beta_k^2 \tag{6.90}$$

with $C = C(n)$ as claimed. q.e.d.

Completion of the Proof of Theorem 1.1 in Case (a)**:** First observe that the entire analysis in the present section can be carried out with the sequence $\{\mathbf{H}_k\}_{k=1}^\infty$ replaced by an arbitrary sequence $\{\widetilde{\mathbf{H}}_k\}_{k=1}^\infty$ with

$$d_{\mathcal{H}}\left(\mathbf{H}_k \cap B_1^{n+1}(0),\, \widetilde{\mathbf{H}}_k \cap B_1^{n+1}(0)\right) \le c\,\beta_k,$$

where, for each k, $\widetilde{\mathbf{H}}_k$ is a pair of hyperplanes and c is any fixed positive constant independent of k. Now let $\beta_k^{(m)}$ be the infimum of the L^2 height-excesses of $M_k \cap B_1^{n+1}(0)$ over all pairs of hyperplanes. To prove Theorem 1.1, we want to show that under HYPOTHESES $(\star)$, $\beta_k^{(m)} = 0$ for infinitely many k. So suppose $\beta_k^{(m)} > 0$ for all sufficiently large k. For each such k, choose $\widetilde{\mathbf{H}}_k$, such that the L^2 height-excess $\widetilde{\beta}_k$ of $M_k \cap B_1^{n+1}(0)$ relative to $\widetilde{\mathbf{H}}_k$ satisfies $\beta_k^{(m)} \le \widetilde{\beta}_k \le 2\,\beta_k^{(m)}$. For each k, choose an orthogonal transformation q_k so that $q_k\,\widetilde{\mathbf{H}}_k$ has axis coinciding with $\{0\} \times \mathbf{R}^{n-1}$, is symmetric about $\{0\} \times \mathbf{R}^n$ and $q_k\,\widetilde{\mathbf{H}}_k \to \{0\} \times \mathbf{R}^n$.

Repeating the analysis of the present section with $q_k\,\widetilde{\mathbf{H}}_k$, $q_k\,M_k$, $\widetilde{\beta}_k$ in place of $\mathbf{H}_k$, M_k, β_k, we obtain blow-ups $\widetilde{w}^{(i)}$ in place of $w^{(i)}$ ($w^{(i)}$ as in 6.17). By Lemma 6.23, the closures of graph $\widetilde{w}^{(i)}$, $i = 1, \ldots, 4$ form a pair of hyperplanes. In view of the estimate (6.52) (with $\widetilde{\beta}_k$, $\widetilde{d}_k$ in place of β_k, d_k where $\widetilde{d}_k(x, y) = \text{dist}\,((x, y), q_k\,\widetilde{\mathbf{H}}_k))$, this implies that the excess of $M_k \cap B_1^{n+1}(0)$ relative to a new pair of hyperplanes (given by $q_k^{-1} \cup_{i=1}^4 \text{graph}\,(\widetilde{\beta}_k \widetilde{w}^{(i)} + h_k^{(i)})$ where $h_k^{(i)}$ are the functions whose graphs are the four half-spaces of $q_k\,\widetilde{\mathbf{H}}_k$) is of lower order than $\widetilde{\beta}_k$. For sufficiently large k, this contradicts the definition of $\beta_k^{(m)}$ since we chose $\widetilde{\mathbf{H}}_k^{(i)}$ such that $\widetilde{\beta}_k \leq 2\beta_k^{(m)}$.

This concludes the proof of case (a) of Theorem 1.1. q.e.d.

7. Transverse Case

In this section, we indicate how case (b) of Theorem 1.1 follows from case (a) and the work in Section 6.

Suppose the theorem is false in case (b). Then we would have a sequence $\{\mathbf{C}_k\}$ of cones in $\overline{\mathcal{I}}$ and a sequence $\{\mathbf{P}_k\}$ of transverse pairs of hyperplanes for $k = 1, 2, \ldots$, such that $d_{\mathcal{H}}\,(\text{spt}\,\|\mathbf{C}_k\| \cap B_2^{n+1}(0), \text{spt}\,\|\mathbf{P}_k\| \cap B_2^{n+1}(0)) \to 0$ as $k \to \infty$ and $\mathbf{C}_k \neq$ a pair of hyperplanes for every k. After passing to a subsequence, $\mathbf{P}_k \to \mathbf{P}_0$ for some pair of hyperplanes $\mathbf{P}_0$ and so $d_{\mathcal{H}}\,(\text{spt}\,\|\mathbf{C}_k\| \cap B_2^{n+1}(0), \text{spt}\,\|\mathbf{P}_0\| \cap B_2^{n+1}(0)) \to 0$. If $\mathbf{P}_0$ is a multiplicity 2 hyperplane, case (a) of the theorem implies that $\mathbf{C}_k$ is equal to a pair of hyperplanes for each sufficiently large k, giving a contradiction. Thus $\mathbf{P}_0$ must be a transverse pair of hyperplanes. We may assume without loss of generality that the axis of $\mathbf{P}_0$ is $\{0\} \times \mathbf{R}^{n-1}$, and let us label $\mathbf{P}_0^i$, $i = 1, \ldots, 4$, the four half-spaces whose union is $\mathbf{P}_0 \setminus (\{0\} \times \mathbf{R}^{n-1})$. We claim that in this case, for sufficiently large k, $\mathbf{C}_k$ must be a transverse pair of hyperplanes, giving the necessary contradiction again.

To prove this claim, we only need to see that, the blow up argument, with appropriate changes, of Section 6 (up to the estimates (6.53)–(6.56)), can be carried out in the present setting with a suitably chosen sequence of smooth, stable immersions M_k approximating $\mathbf{C}_k$ (in the sense described in item (5) of HYPOTHESES ($\star$), Section 1), with the sequence $\mathbf{H}_k$ replaced by a sequence $\mathbf{F}_k$. Here, for each k, $\mathbf{F}_k$ is the union of any four distinct, n-dimensional half-spaces $\mathbf{F}_k^i$, $i = 1, \ldots, 4$, (which respectively replace $\mathbf{H}_k^i$, $i = 1, \ldots, 4$, of Section 6) meeting along a common $(n-1)$-dimensional axis, with

$$d_{\mathcal{H}}\left(\mathbf{F}_k \cap B_2^{n+1}(0), \mathbf{P}_0 \cap B_2^{n+1}(0)\right) \le c\, d_{\mathcal{H}}\left(\operatorname{spt}\|\mathbf{C}_k\| \cap B_2^{n+1}(0), \mathbf{P}_0 \cap B_2^{n+1}(0)\right)$$

where c is an arbitrary, fixed, positive constant independent of k. By repeating the argument of Section 6 after making the necessary modifications (listed below), we then see that by Lemma 4.2 (case $l = 1$) of [**11**], the blow-up of the M_k's, where the blow up constants β_k are now given by

$$\beta_k = \sqrt{\int_{M_k \cap B_1^{n+1}(0)} \operatorname{dist}^2(x, \mathbf{F}_k) d\mathcal{H}^n(x)},$$

is the union of four distinct n-dimensional half spaces meeting along a common axis. This is a contradiction, implying that for sufficiently large k, each M_k is itself a union of four half spaces meeting along a common axis, and hence, by stationarity, a pair of hyperplanes. This contradiction follows by exactly the argument in the paragraph entitled "**Completion of the Proof of Theorem 1.1 in Case** (a)" at the end of Section 6.

Thus, to complete the proof of case (b) of the theorem, we only need to observe the necessary modifications to various lemmas of Section 6 in order to be able to carry out the blow up argument in the current setting. We list them below:

(1) First notice that since $M_k \to \mathbf{P}_0$, (e.g. in the sense of Hausdorff distance), it follows that for sufficiently large k, $M_k \cap (B_1^{n+1}(0) \setminus T_k)$ is equal to four disjoint, embedded sheets smoothly converging, respectively, to the four half-spaces $\mathbf{P}_0^i$, $i = 1, \ldots, 4$, that make up $\mathbf{P}_0$. Here $T_k = B^2_{\gamma(\beta_k)}(0) \times \mathbf{R}^{n-1}$ where γ is a function satisfying $\gamma(t) \downarrow 0$ as $t \downarrow 0$.

(2) Observe the necessary notational differences. The $\mathbf{F}_k$ are a sequence converging to $\mathbf{P}_0$, whereas previously, the $\mathbf{H}_k$ were converging to a hyperplane which we took to be $\mathbf{R}^n$. Thus, anything that was written as a graph over $\mathbf{R}^{n\,+}$ or $\mathbf{R}^{n\,-}$ before (e.g. graph of each of the functions $\tilde{g}_k^i$, w_k^i and w^i, $i = 1, \ldots, 4$) must now be written as a graph over the appropriate half space of $\mathbf{P}_0$.

(3) In the statement of Lemma 6.19, τ_k must be replaced by $\gamma(\beta_k)$, where γ is as in item (1) above. In the proof of this lemma, in place of equation (6.19), we now have

$$|\nu_k(\gamma_y^k(L)) - \nu_k(\gamma_y^k(0))| \geq c(\mathbf{P}_0)$$

where $c(\mathbf{P}_0)$ is a fixed positive constant independent of k and depending only on the angle between the two hyperplanes of $\mathbf{P}_0$. The rest of the proof needs to be modified accordingly. The right hand side of equation (6.26) becomes an absolute constant independent of k. This follows from the stability inequality (2.10).

(4) The inequality in part (i) of Lemma 6.21 should be replaced by

$$|\xi|^2 \leq c\,\beta_k^2.$$

To see this, observe that the reason there is a factor of ϵ_k^2 multiplying the term $|\xi|^2$ in that inequality is that the angle between $\mathbf{H}_k^i$ and $\mathbf{R}^n$, for each i, is equal to $c\,\epsilon_k$, which is converging to 0. However, in the present setting, the angles between $\mathbf{F}_k^i$ and $\mathbf{R}^n$ remain bounded away from zero by a constant depending only on $\mathbf{P}_0$, allowing the modification indicated. (This modification in fact amounts to "undoing" the modification already made to the corresponding result of [**11**] in obtaining Lemma 6.21. Thus, the version of the lemma needed for the present section is exactly Theorem 3.1 of [**11**].)

(5) In Lemma 6.22, the inequality

$$\sup\,\left((\kappa_k^1)^2 + \epsilon_k^2((\kappa_k^1)^2 + (\kappa_k^2)^2)\right) \leq c\,\beta_k^2$$

must be replaced by

$$\sup\,\left((\kappa_k^1)^2 + (\kappa_k^2)^2\right) \leq c\,\beta_k^2.$$

This is a direct consequence of the modification in (3) above. (Again, this means that we are simply reverting to the exact version of corollary 3.2 of [**11**].)

(6) In place of the L^2 conditions (6.53)–(6.56), we have in the present setting the conditions

$$\int_0^{1/16}\int_{B^{n-1}_{1/16}(0)} \frac{|w^{(i)}(r\omega_i, y) - (\kappa^1(r,y)e_1^\perp + \kappa^2(r,y)e_2^\perp)|^2}{r^{3/2}} < \infty$$

for $i = 1, \ldots, 4$, where ω_i is such that $\mathbf{P}_0^i = \{(r\omega_i, y) : r > 0,\, y \in \mathbf{R}^{n-1}\}$, κ^1, κ^2 are bounded functions and $e_j^\perp$, $j = 1, 2$, means the orthogonal projection of e_j onto the direction normal to $\mathbf{P}_0^i$. We also have that

$$\lim_{r\to 0^+} \frac{\partial^2}{\partial r\,\partial y^j} \sum_{i=1}^{4} w^{(i)}(r\omega_i, y) = 0$$

for each $j = 1, \ldots, (n-1)$, uniformly for $|y| \leq 1$. The proof of this is exactly as in [**11**], pp. 635–639.

(7) The proof that graph w^i are four n-dimensional half-spaces meeting along a common axis is now completed as in Lemma 4.2, case $l = 1$ of [**11**].

8. Acknowledgment

The present work was carried out as part of my thesis at Stanford University. I would like to thank my advisor Leon Simon for his constant encouragement, invaluable help and guidance during the preparation of my thesis. I would also like to thank Richard Schoen and Brian White for their help and encouragement.

References

[1] W.K. Allard, *On the first variation of a varifold*, Ann. Math. **95** (1972) 417–491, MR 0307015, Zbl 0252.49028.

[2] F.J. Almgren, Jr., *Almgren's big regularity paper: Q-valued functions minimizing Dirichlet's integral and the regularity of area minimizing rectifiable currents up to codimension two*, (V. Scheffer, J. Taylor, eds), World Scientific Monograph Ser. Math., Vol. 1, 2000, MR 1777737, Zbl 0985.49001.

[3] J. Duggan, *Regularity theorems for varifolds with mean curvature*, Thesis, Australian National University, 1984.

[4] D. Gilbarg & N. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer-Verlag, Berlin–Heidelberg–New York, 1983, MR 0737190, Zbl 0562.35001.

[5] R. Hardt & L. Simon, *Boundary regularity and embedded solutions for the oriented Plateau problem*, Ann. Math. **110** (1979) 439–486, MR 0554379, Zbl 0457.49029.

[6] J.H. Michael & L. Simon, *Sobolev and mean-value inequalities on generalized submanifolds of* R^n, Commun. Pure Appl. Math. **26** (1973) 361–379, MR 0344978, Zbl 0256.53006.

[7] R. Schoen, *Existence and regularity theorems for some geometric variational problems*, Thesis, Stanford University, 1977.

[8] R. Schoen & L. Simon, *Regularity of stable minimal hypersurfaces*, Commun. Pure Appl. Math. **34** (1981) 741–797, MR 0634285, Zbl 0497.49034.

[9] R. Schoen, L. Simon & S.-T. Yau, *Curvature estimates for minimal hypersurfaces*, Acta Math. **134** (1975) 275–288, MR 0423263, Zbl 0323.53039.

[10] L. Simon, *Lectures on geometric measure theory*, Proc. Centre Math. Anal., Austral. Nat. Univ., Canberra **3** (1983), MR 0756417, Zbl 0546.49019.

[11] L. Simon, *Cylindrical tangent cones and the singular set of minimal submanifolds*, J. Differential Geometry **38** (1993) 585–652, MR 1243788, Zbl 0819.53029.

[12] L. Simon, *Theorems on regularity and singularity of energy minimizing maps*, Birkhauser, 1996, MR 1399562, Zbl 0864.58015.

[13] N. Wickramasekera, *On the singularities and a Bernstein property of immersed stable minimal hypersurfaces*, to appear.

Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, MA 02139

E-mail address: neshan@math.mit.edu