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RIGIDITY FOR FAMILIES OF POLARIZED CALABI–YAU VARIETIES

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Abstract

In this paper, we study the analogue of the Shafarevich conjecture for polarized Calabi–Yau varieties. We use variations of Hodge structures and Higgs bundles to establish a criterion for the *rigidity* of families. We then apply the criterion to obtain that some important and typical families of Calabi–Yau varieties are rigid, for examples., Lefschetz pencils of Calabi–Yau varieties, *strongly degenerated* families (not only for families of Calabi–Yau varieties), families of Calabi–Yau varieties admitting a degeneration with *maximal unipotent monodromy*.

0. Introduction

Throughout this paper, the base field k is the complex number field $\mathbb{C}$.

Shafarevich conjecture over function field. At the 1962 ICM in Stockholm, Shafarevich conjectured: *"There exists only a finite number of fields of algebraic functions K/k of a given genus $g \geq 1$, the critical prime divisors of which belong to a given finite set S"* (cf. [**32**]). Shafarevich proved his own conjecture in the setting of hyperelliptic curves in one unpublished work. The conjecture was confirmed by Paršin for $S = \emptyset$, by Arakelov in general.

Let C be a smooth projective curve of genus $g(C)$ and $S \subset C$ be a finite subset. An algebraic family over C is called *isotrivial* if over an open dense subscheme of C any two smooth fibers are isomorphic. The geometric description of the Shafarevich conjecture is:

Fix (C, S) and an integer $q \geq 2$, there are only finitely many non-isotrivial smooth families of curves of genus q over $C \setminus S$.

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Let $\mathcal{M}_q$ be the coarse moduli space of smooth projective curves of genus $q \geq 2$. Adding stable curves at the boundary of $\mathcal{M}_q$, one has the Deligne–Mumford compactification $\overline{\mathcal{M}_q}$. Fix (C, S) and $q \geq 2$, a smooth family $(f : X \to C \setminus S)$ of curves of genus q induces naturally a unique moduli morphism $\eta_f : C \setminus S \to \mathcal{M}_q$. η_f can be extended to $\overline{\eta}_f : C \to \overline{\mathcal{M}}_q$ because of the smoothness of C. Hence, parameterizing families is same as parameterizing these morphisms which can be characterized by their graphs. The graph $\Gamma_{\overline{\eta}_f}$ is a projective curve contained in $C \times \overline{\mathcal{M}}_q$ with the first projection mapping itself isomorphically onto C, hence the problem is translated into looking for a parametrization $\mathbb{T}$ in the Hilbert scheme of $C \times \overline{\mathcal{M}}_q$. In general, the Hilbert scheme is an infinite union of schemes of finite type, each component has a different Hilbert polynomial and represents a deformation type of family. Fortunately, this parameterizing scheme $\mathbb{T}$ is of finite type over $\mathbb{C}$, i.e., only finite Hilbert polynomials can actually occur. The original conjecture was reformulated by Arakelov and Paršin into four problems (cf. [**1**], [**29**]):

Conjecture 0.1 (Shafarevich Conjecture)**.** Fix (C, S) and $q \geq 2$.

- (**B**) The elements of the set of non-isotrivial families of curves of genus q over C with singular locus S are parameterized by points of a scheme $\mathbb{T}$ of finite type over $\mathbb{C}$ (*Boundedness*).
- (**R**) Any deformation of a non-isotrivial family of curves of genus q over C with singular locus S is trivial, i.e., $\dim \mathbb{T} = 0$ (*Rigidity*).
- (**H**) No non-isotrivial family of curves of genus q exists if $2g(C) - 2 + \#S \leq 0$, i.e., $\mathbb{T} \neq \emptyset \Rightarrow 2g(C) - 2 + \#S > 0$ (*Hyperbolicity*).
- (**WB**) For a non-isotrivial family $f : X \to C$, $\deg f_* \omega_{X/C}^m$ is bounded above in term only of $g(C), \#S, q, m$ (*Weak Boundedness*).

The Hilbert polynomial of $\Gamma_{\overline{\eta}_f}$ is determined by its first term $\deg \overline{\eta}_f^* \mathcal{L}$ where $\mathcal{L}$ is a fixed ample line on $\overline{\mathcal{M}_q}$. Thus, (**B**) is equivalent to the boundedness of $\deg \overline{\eta}_f^* \mathcal{L}$ due to Mumford's works on moduli spaces of curves, i.e., that it is sufficient to prove (**WB**). On the other hand, (**WB**) is obtained by the well-known Arakelov inequality. It was shown recently that (**WB**) $\Rightarrow$ (**H**) if the general fiber is a smooth curve of $q \geq 2$ (cf. [**22**]). (**R**) follows directly from the positivity of relative dual sheafs of non-isotrivial families.

Shafarevich problems for higher dimensional varieties. Define $Sh(C, S, K)$ to be the set of all equivalent classes of non-isotrivial family $\{f : \mathcal{X} \to C\}$ such that $\mathcal{X}_b$ is a smooth projective variety with type 'K' for any $b \in C \setminus S$. Two such families are equivalent if they are isomorphic over $C \setminus S$. The general Shafarevich problem is to find 'K' and the data (C, S) such that the set $Sh(C, S, K)$ is finite.

Example 0.2. Faltings dealt with the case of Abelian varieties, and he formulated a Hodge theoretic condition (the Deligne–Faltings $(*)$-condition) for a fiber space to be rigid (cf. [**12**]):

A smooth family $f : \mathcal{X}_0 \to C \setminus S$ of Abelian varieties is said to satisfy $(*)$-condition if any anti-symmetric endomorphism σ of $\mathbb{V}_{\mathbb{Z}} = R^1 f_* \mathbb{Z}$ defines an endomorphism of $\mathcal{X}_0$ (so σ is of type $(0,0)$).

By the global Torelli theorem, a polarization of the Abelian scheme induces a sympletic bilinear form Q on $R^1 f_* \mathbb{Z}$. Faltings showed that the $(*)$-condition is equivalent to

$$\mathrm{End}^Q(\mathbb{V}_{\mathbb{Q}}) \otimes \mathbb{C} = (\mathrm{End}^Q(\mathbb{V}_{\mathbb{Q}}) \otimes \mathbb{C})^{0,0}.$$

On the other hand, the Zariski tangent space of the moduli space of Abelian schemes over $C - S$ with a fixed polarization is isomorphic to

$$(\mathrm{End}^Q(\mathbb{V}_{\mathbb{Q}}) \otimes \mathbb{C})^{-1,1}.$$

Example 0.3. If a family is not rigid, one should have non-rigid VHSs. By utilizing the differential geometry of period maps and Hodge metrics on period domains, Peters generalized the result of Faltings to polarized variations of Hodge structure of arbitrary weight (cf. [**30**]): *A polarized variation of Hodge structure $(\mathbb{V}, Q)$ underlying $\mathbb{V}_{\mathbb{Z}}$ is rigid if and only if*

$$(\mathrm{End}^Q(\mathbb{V}_{\mathbb{Q}}) \otimes \mathbb{C})^{-1,1} = 0.$$

Example 0.4. As studying deformations of a family can be reduced to studying deformations of the corresponding period map, Jost and Yau analyzed $Sh(C, S, K)$ for a large class of varieties by harmonic maps (cf. [**17**]). They provided analytic methods to solve the *rigidity* and gave differential geometric proofs of Shafarevich conjectures. They started to study Higgs bundles with singular Hermitian metrics and their applications to Shafarevich problems.

Eyssidieux and Mok also have many results on deformations of period maps in case that period domains are Hermitian symmetric (cf. [**26**],[**10**, **11**],[**9**]). They showed the *gap rigidity* (which is stronger than the *rigidity* in our case) for locally bounded symmetric domains of certain type (including all tube domains).

Example 0.5 (Non-rigid Family)**.** Faltings constructed an example to show that the set $Sh(C, S, K)$ is infinite for Abelian varieties of dimension ≥ 8 with some type 'K' (cf. [**12**]). Saito and Zucker generalized the construction of Faltings to the setting in case that the condition 'K' is an algebraic polarized $K3$ surface, and they were able to classify all cases if the set $Sh(C, S, K)$ is infinite (cf. [**37**]).

All these examples did not require that families are polarized. For curves, the condition $q \geq 2$ is equivalent to that the canonical line bundle $\omega_{\mathcal{X}_{gen}}$ of a general fiber is ample, hence that fixing $g(\mathcal{X}_{gen})$ can be replaced by that fixing the Hilbert polynomial $h_{\omega_{\mathcal{X}_{gen}}}$ of $\omega_{\mathcal{X}_{gen}}$, and these families automatically become families of canonical polarized curves. For higher dimensional varieties, instead of fixing $g(\mathcal{X}_{gen})$, we should consider any polarized projective variety (X, L) such that L is an ample line bundle on X with $\chi(X, L^{\nu}) \equiv$ a fixed Hilbert polynomial $h(\nu)$ for $\nu >> 0$.

Fix a pair (C, S) such that C is a non-singular projective curve and S is a set of finite points of C. Consider the polarization, $Sh(C, S, K)$ is now defined to be the set of all equivalent classes of non-isotrivial smooth polarized family $\{(f : \mathcal{X} \to C \setminus S, \mathcal{L})\}$ satisfying: $\mathcal{X}_b = f^{-1}(b)$ is a smooth projective variety with type 'K'; $\mathcal{L}$ is an invertible sheaf on $\mathcal{X}$, relatively ample over $C \setminus S$ with a fixed Hilbert polynomial $\chi(\mathcal{L}^{\nu}|_{\mathcal{X}_b}) \equiv h(\nu)$ for any $b \in C \setminus S$ and $\nu >> 0$; two families are equivalent if they are isomorphic over $C \setminus S$ as polarized families (see the following theorem 0.8). Therefore, the analogue of the Shafarevich conjecture for higher dimensional varieties is formulated as follows:

Conjecture 0.6 (The Shafarevich problem for higher dimensional polarized varieties)**.** Fix a pair (C, S) and a Hilbert polynomial $h(\nu)$.

(**B**) The elements of $Sh(C, S, K)$ are parameterized by points of a scheme $\mathbb{T}$ of finite type over $\mathbb{C}$.

(**R**) When does one have $\dim \mathbb{T} = 0$?

(**H**) $\mathbb{T} \neq \emptyset \Rightarrow 2g(C) - 2 + \#S > 0$.

(**WB**) For a family $(f : X \to C) \in Sh(C, S, K)$, $\deg f_*\omega^m_{X/C}$ is bounded above in term of $g(C), \#S, h, m$. In particular, the bound is independent of f.

Remark. (**WB**) $\Rightarrow$ (**H**) holds for any canonically polarized family.

Example 0.7. (Recent Results).

1. Migliorni, Kovác and Zhang proved that families of minimal algebraic surfaces of general type over a curve of genus g and $\#S$ singular fibers such that $2g(C)-2+\#S \leq 0$ are isotrivial (cf. [**25**][**21**][**49**]). Oguiso and Viehweg proved it for families of elliptic surfaces (cf. [**28**]).
2. Jost–Zuo and Viehweg–Zuo recently made contributions to the *weak boundedness* (cf. [**20**][**46**]), and obtained Arakelov–Yau type inequalities for families of higher dimension varieties over a curve.
3. Consider an algebraic family over a fixed smooth curve, let F be a general fiber. Bedulev–Viehweg proved **(B)** if F is an algebraic surface of general type. They also proved **(WB)** if F is a canonically

polarized variety (cf. [**2**]). As explained in [**43**], one actually obtains **(B)**. Precisely, Viehweg–Zuo obtained that (**WB**) $\Rightarrow$ (**B**) if F is a minimal model of general type or if the family is not degenerate and F is a minimal model of Kodaira dimension zero (cf. [**43**, **45**]), and they also showed that **(WB)** holds if F is a minimal model of general type or F has semi-ample ω_F.

4. Liu–Todorov–Yau–Zuo gave another proof of the *boundedness* of the analogue of Shafarevich conjecture for Calabi–Yau manifolds by originally using the *Schwarz–Yau lemma* and the *Bishop compactness*. They also constructed a non-rigid family of Calabi–Yau manifolds and related *Yukawa couplings* with the *rigidity* ([**24**]).

Moduli spaces of Calabi–Yau manifolds play a pivotal role in the classification theory of Calabi–Yau varieties and in Mirror Symmetry. Unfortunately, we know little about their structures. As studying moduli stacks can be reduced to studying families of manifolds, we begin to study the analogue of the Shafarevich conjecture for families of high dimensional polarized varieties and understand part of the structure of moduli stacks. The existence of coarse moduli spaces of polarized manifolds was proven by Mumford, Gieseker and Viehweg, it is the fundamental theorem for us to study moduli problems for manifolds.

Theorem 0.8 (cf. [**41**])**.** *Let h be a fixed polynomial of degree n with $h(\mathbb{Z}) \subset \mathbb{Z}$. Define moduli functor*

$\mathcal{M}_h(Y) := \{(f : \mathcal{X} \to Y, \mathcal{L})$; *$f$ flat,projective and $\mathcal{L}$ invertible,relatively ample over Y, such that: for all $p \in Y(\mathbb{C})$ $\mathcal{X}_p = f^{-1}(p)$ is a projective manifold with semi-ample canonical bundle and* $\chi(\mathcal{L}|_{\mathcal{X}_p}) = h\,\}/\sim$.

Then, the moduli functor $\mathcal{M}_h$ is bounded by the Matsusaka Big theorem, and there exists a quasi-projective coarse moduli scheme M_h for $\mathcal{M}_h$, of finite type over $\mathbb{C}$. Moreover, if $\omega_\Gamma^\delta = \mathcal{O}_\Gamma \forall\, \Gamma \in \mathcal{M}_h(\mathbb{C})$ for one integer $\delta > 0$, then for some $p > 0$, there exists an ample line bundle $\lambda^{(p)}$ on M_h such that $\phi_g^* \lambda^{(p)} = g_* \omega_{\mathcal{X}/Y}^{\delta \cdot p}$ for any family $(g : \mathcal{X} \to Y, \mathcal{L}) \in \mathcal{M}_h(Y)$ with moduli morphism $\phi_g : Y \to M_h$.

Remarks. A line bundle $\mathcal{B}$ is semi-ample if for some $\mu > 0$ the sheaf $\mathcal{B}^\mu$ is generated by global sections. $(f : \mathcal{X} \to Y, \mathcal{L}) \sim (f' : \mathcal{X}' \to Y, \mathcal{L}')$ if there exist a Y-isomorphism $\tau : \mathcal{X} \to \mathcal{X}'$ and an invertible sheaf $\mathcal{F}$ on Y such that $\tau^* \mathcal{L}' \cong \mathcal{L} \otimes f^* \mathcal{F}$. The statement is also true if "$\cong$" is replaced by the numerical equivalence "$\equiv$".

Recently, Viehweg–Zuo obtained remarkable results for the Shafarevich problem.

Theorem 0.9 (cf. [**42**, **43**, **44**])**.** *Brody hyperbolicity holds for moduli spaces of canonically polarized complex manifolds, thus the boundedness* *of $Sh(C, S, Z)$ for arbitrary Z with ω_Z semi-ample holds. Moreover, the automorphism group of moduli stacks of polarized manifolds is finite and the rigidity holds for a general family.*

Remark. A complex analytic space $\mathcal{N}$ is called *Brody hyperbolic* if every holomorphic map $\mathbb{C} \to \mathcal{N}$ is constant. Its algebraic version is called *algebraic hyperbolic*. One has that (**H**) is true if the moduli space is *algebraic hyperbolic*.

Rigidity for families of Calabi–Yau varieties. In this paper, a *Calabi–Yau manifold* is a smooth projective variety X (of dimension n) such that the canonical line bundle K_X is trivial and $h^i(X, \mathcal{O}_X) = 0 \, \forall i$ with $0 < i < n$. We study the *rigidity* problem of the analogue Shafarevich conjecture for Calabi–Yau manifolds. Before we cite our main results, we shall point out that the *rigidity* for the analogue of the Shafarevich conjecture fails for general condition by the following key observation.

Example 0.10. (cf. [47]) Very recently, Viehweg–Zuo constructed a non-trivial family of Calabi–Yau manifolds such that the closed fibers are Calabi–Yau manifolds endowed with *complex multiplication* over a dense set. Precisely, they obtained a family $g : \mathcal{Z} \to S$ of quintic hypersurfaces in $\mathbb{CP}^4$ such that S is finite dominant over a ball quotient (which is a Shimura variety) and S has a dense set of *CM* points. Furthermore, they got an important counter example for the *rigidity* part of the Shafarevich problem by showing that there exists a product of moduli spaces of hypersurfaces of degree d in $\mathbb{P}^n$ and that this product can be embedded into the moduli space of hypersurfaces of degree d in $\mathbb{P}^N$ for one $N > n$.

Altogether, the remaining step for the Shafarevich problem is to find nice conditions for rigidity. We obtain three main results as follows:

(I) We prove that any non-isotrivial Lefschetz pencil $f : \mathfrak{X} \to \mathbb{P}^1$ of Calabi–Yau varieties of odd dimension n is rigid (Lefschetz pencils of even dimensional Calabi–Yau varieties are automatically trivial). The proof depends on the construction of Lefschetz pencils in Deligne's Weil conjecture I. Let f^0 be the maximal smooth subfamily of f and $\mathbb{V}$ be the $\mathbb{Z}$-local system of vanishing cycles space. Actually, we obtain that the pieces of $(n, 0)$-type and $(0, n)$-type of the VHS $R^n f^0_*(\mathbb{C})$ are both in $\mathbb{V}_{\mathbb{C}} = \mathbb{V} \otimes \mathbb{C}$. On the other hand, if the family f is non-rigid we would have a non-zero $(-1, 1)$-type endomorphism σ of $\mathbb{V}_{\mathbb{C}}$ which is flat under the Gauss–Manin connection. The action of σ induces a non-trivial splitting of the local system $\mathbb{V}_{\mathbb{C}}$, but it contradicts to that $\mathbb{V}$ is absolutely irreducible.

(II) We obtain a general result that any non-isotrivial *strongly degenerated* family (not only for families of Calabi–Yau varieties) must be rigid. A family over projective smooth curve is called *strongly degenerate* if it has a singular fiber with only pure Hodge type cohomology. If the family is non-rigid, we then have a non-zero $(-1,1)$-type endomorphism σ which is flat under the Gauss–Manin connection as same as in (I). The Künneth formula says that we can identify this σ with a monodromy-invariant section of a VHS from the self-product family. We compare the Hodge type of σ with the Hodge types of cohomology groups of the singular fiber of the self-product family. Then, we have a contradiction that σ must be zero. As a corollary, we obtain the *Weakly Arakelov theorem* for high dimensional varieties.

(III) We introduce a general criterion of Viehweg–Zuo for rigidity. From the criterion, we deduce a result of Liu–Todorov–Yau–Zuo and Viehweg–Zuo: a family of Calabi–Yau varieties over an algebraic curve is rigid if its *Yukawa coupling* is non-zero. Together with the results of Schmid and Simpson on residues of holomorphic vector bundles over singularizes, we prove that any family of Calabi–Yau varieties over an algebraic curve admitting a degeneration with *maximal unipotent monodromy* must be rigid.

1. Higgs Bundles over Quasi-Projective Manifolds

The generalized Donaldson–Simpson–Uhlenbeck–Yau correspondence. Let the base M be a quasi-projective manifold such that there is a smooth projective completion $\overline{M}$ with a reduced normal crossing divisor $D_\infty = \overline{M} - M$.

Let (V, ∇) be a flat $\mathrm{GL}(n,\mathbb{C})$ vector bundle on M, i.e., a fundamental representation $\rho : \pi_1(M) \to \mathrm{GL}(n,\mathbb{C})$. A Hermitian metric H on V leads to a decomposition $\nabla = D_H + \vartheta$, which corresponds to the Cartan decomposition of Lie algebra $gl(n,\mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{p}$. D_H is a unitary connection preserving the metric H, so ρ is unitary if and only if $\vartheta = 0$. The Hermitian metric H can be regarded as a ρ-equivariant map

$$h : \widetilde{M} \to \mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n) \text{ with } dh = \vartheta$$

where $\widetilde{M}$ is the universal covering of M. With respect to the complex structure of M, one has the decomposition

$$D_H = D_H^{1,0} + D_H^{0,1},\ \vartheta = \vartheta^{1,0} + \vartheta^{0,1}.$$

The following three conditions are equivalent:

- $\nabla_H^*(\vartheta) = 0$ (∇_H^* is defined by $(e, \nabla_H^*(f))_H := (\nabla(e), f)_H$);

- $(D_H^{0,1})^2 = 0,\ D_H^{0,1}(\vartheta^{1,0}) = 0,\ \vartheta^{1,0} \wedge \vartheta^{1,0} = 0$;
- h is a harmonic map, i.e., H is a *harmonic metric* (or V is *harmonic*).

Altogether, suppose that H is *harmonic*, $(E, \overline{\partial}_E, \theta)$ is a Higgs bundle with respect to the holomorphic structure $\overline{\partial}_E := D_H^{0,1}$ where E takes the underlying bundle as V and $\theta := \vartheta^{1,0}$; D_H is the unique metric connection with respect to $\overline{\partial}_E$; and H is the *Hermitian–Yang–Mills metric* on the Higgs bundle $(E, \overline{\partial}_E, \theta)$, i.e.,

$$D_H^2 = -(\theta_H^* \wedge \theta + \theta \wedge \theta_H^*).$$

The existence of the ρ-equivariant harmonic map was proven by Simpson in case that M is a curve (cf. [**34**]), by Jost–Zuo in case that M is a higher dimensional manifold (cf. [**18**, **19**]). In fact, Jost–Zuo obtained some useful estimates of curvatures at the infinity as Simpson did for punctured curves (cf. Section 2, Main Estimate, Theorem 1 in [**34**]). The ρ-equivariant harmonic map is unique and depends only on ρ if M is compact, but the uniqueness does not hold if M is not compact. On the other hand, a theorem of Cornalba and Griffiths for extensions of analytic sheaves (cf. [**3**]) shows that the induced Higgs bundle E can extend to a coherent sheaf $\overline{E}$ on $\overline{M}$ and θ can extend to $\overline{\theta} \in \Gamma(\overline{M}, \mathcal{E}nd(\overline{E}) \otimes \Omega^1_{\overline{M}}(\log D_\infty))$. The extension of (E, θ) is not unique, but one can treat this non-uniqueness by taking filtered extensions $(E, \theta)_\alpha$, and obtains a filtered Higgs bundle $\{(E, \theta)_\alpha\}$ (cf. [**50**]).

Conversely, let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle equipped with a Hermitian metric H. One has a unique metric connection D_H on (E, θ) with respect to the holomorphic structure $\overline{\partial}_E$, and a $(0,1)$-form θ_H^* is determined by $(\theta e, f)_H = (e, \theta_H^* f)_H$ where e, f are arbitrary sections of E. Denote $\partial_E := D_H - \overline{\partial}_E$, $\nabla' := \partial_E + \theta_H^*$, and $\nabla'' := \overline{\partial}_E + \theta$. One has $\nabla'' \circ \nabla'' = 0$, and $(\partial_E)^2 = 0$ as $D_H^2 = \pi^{1,1}(D_H^2)$. Denote $\nabla = \nabla' + \nabla''$. Then, $\nabla = D_H + \theta + \theta_H^*$. Let (V, ∇'') be another holomorphic vector bundle with respect to ∇'' where V takes the underlying bundle as E. The metric H on (E, θ) is called *Hermitian-Yang-Mills* if ∇ is a flat connection on V, i.e., H is a *harmonic metric* on V (cf. [**17**],[**50**]). If M is a compact Kähler manifold or a quasi-projective curve, one has the *Donaldson–Simpson–Uhlenbeck–Yau correspondence* (DSUY correspondence), i.e., that the *Hermitian-Yang-Mills metric* exists (cf. [**4**],[**33**],[**34**],[**40**]). Suppose that (V, ∇) is flat, as $(\nabla^{1,0})^2 = \pi^{2,0}(\nabla^2) = 0$ (with respect to $\overline{\partial}_E$), then $\partial_E(\theta) = \overline{\partial}_E(\theta_H^*) = 0$; hence $\nabla' \circ \nabla' = 0$ and ∇' is just the Gauss–Manin connection of the holomorphic bundle (V, ∇'').

Corollary 1.1. *Let* $(E, \overline{\partial}_E, \theta, H)$ *be a Higgs bundle induced from a harmonic bundle. Then, one has*

(1.1.1)

$\overline{\partial}_E(\theta) = 0$	$\partial_E(\theta) = 0$	$\theta \wedge \theta = 0$
$\overline{\partial}_E(\theta^*_H) = 0$	$\partial_E(\theta^*_H) = 0$	$\theta^*_H \wedge \theta^*_H = 0$
$(\overline{\partial}_E)^2 = 0$	$(\partial_E)^2 = 0$	$F^H_{D_H} = -(\theta^*_H \wedge \theta + \theta \wedge \theta^*_H)$

An algebraic vector bundle E over M is said to have a *parabolic structure* if there is a collection of algebraic bundles E_α extending E over $\overline{M}$ such that the extensions form a decreasing left continuous filtration and $E_{\alpha+1} = E_\alpha \otimes \mathcal{O}_M(-D_\infty)$. It is sufficient to consider the index $0 \leq \alpha < 1$. As the set of values where the filtration jumps is discrete, it is a finite set. Over a punctured curve $C_0 = C - S$, the *parabolic degree* of E is then defined by

$$\text{par.deg}(E) := \deg \overline{E} + \sum_{s \in S} \sum_{0 \leq \alpha < 1} \alpha \dim(\text{Gr}_\alpha \overline{E}(s))$$

where $\overline{E} := E_0 = \cup E_\alpha$; the *parabolic degree* of a filtered bundle $\{E_\alpha\}$ on a higher dimensional M is defined just by taking the *parabolic degree* of the restriction $\{(E_\alpha)|_C\}$ over a general curve C_0 in M (see the choice of C_0 in the following theorem 1.2). As any subsheaf of *parabolic* vector bundle E has a *parabolic* structure induced from E, one then has the definition of the *stability* for *parabolic* bundles (cf. [**34**]).

A *harmonic bundle* (V, H, ∇) is called *tame* if the metric H has at most polynomial growth near the infinity. The *tameness* of (V, H, ∇) is equivalent to that all eigenvalues of the Higgs field of the induced Higgs bundle (E, θ) have poles of order at most one at the infinity D_∞. In other words, a *harmonic bundle* (V, H, ∇) is *tame* if and only if (V, ∇) has only regular singularity at D_∞. Hence, any *tame harmonic bundle* and its induced Higgs bundle over M are algebraic. Simpson and Jost–Zuo proved that any $\mathbb{C}$-local system on M has a *tame harmonic metric*. Moreover, if (E, θ) is a Higgs bundle induced from a *tame harmonic bundle*, E has a *parabolic structure* compatible with the extensive Higgs field, i.e., that one has a filtered regular Higgs bundle $\{(E, \theta)_\alpha\}$ on $\overline{M}$. Consider (E, θ) over Δ^* as an example. Let E_α be an extensive vector bundle generated by all sections $e \in E|_{\Delta^*}$ with $|e(t)|_H \leq C|t|^{\alpha+\varepsilon}$ for $\forall \varepsilon > 0$. Then, $\theta_\alpha : E_\alpha \to E_\alpha \otimes \Omega^1_\Delta(\log\ 0)$. Denote $\overline{E} = \cup E_\alpha$ and $\overline{\theta} = \cup \theta_\alpha$. If all monodromies are quasi-unipotent, the extension $\overline{E} = E_0$ can be chosen to be the Deligne quasi-unipotent extension.

If M is a punctured curve, the main theorem of Simpson in [**34**] shows that a filtered Higgs bundle $\{(E, \theta)_\alpha\}$ is ploy-stable of *parabolic degree* zero if and only if it corresponds to a ploy-stable local system of *degree* zero. It can be generalized to higher dimensional bases (cf. [**18**],[**50**]):

Theorem 1.2 (The generalized Donaldson–Simpson–Uhlenbeck–Yau correspondence)**.** *Let M be a quasi-projective manifold such that it has a smooth projective completion $\overline{M}$ and $D_\infty = \overline{M} \setminus M$ is a normal crossing divisor. Let (V, H, ∇) be a tame harmonic bundle on M and $\{(E, \theta)_\alpha\}$ be the induced filtered Higgs bundle. Then, one has:*

1. *(V, H, ∇) is a direct sum of irreducible ones and $\{(E, \theta)_\alpha\}$ is a poly-stable filtered Higgs bundle of parabolic degree zero.*
2. *If (V, H, ∇) is irreducible, $\{(E, \theta)_\alpha\}$ is a stable filtered Higgs bundle of parabolic degree zero.*

Remark. Given a stable filtered Higgs bundle $\{(E, \theta)_\alpha\}$ of *parabolic degree* zero, it is still an open question whether there exists an irreducible *tame harmonic bundle* such that $\{(E, \theta)_\alpha\}$ is induced from it. The difficulty is the existence of *Hermitian–Yang–Mills metrics* on vector bundles over quasi-projective manifolds.

Sketch of the proof. In the higher dimensional algebraic manifold M, one can choose a smooth punctured curve C_0 such that its smooth completion $C \subset \overline{M}$ is a complete intersection of very ample divisors and it intersects D_∞ transversally. The homomorphism of fundamental groups $\pi_1(C_0) \twoheadrightarrow \pi_1(M) \to 0$ is then surjective by the quasi-projective version of the *Lefschetz hyperplane theorem* (cf. [**16**]). The restriction $(V, H, \nabla)|_{C_0}$ is a *tame harmonic bundle* as Jost–Zuo showed that the metric H could be chosen to be *pluriharmonic*, i.e., the metric on the restricted bundle over any subvariety of M is always *harmonic* [**18**, **19**]). The statement follows directly from Simpson's results on non-compact curve and the subjectivity of $\pi_1(C_0) \twoheadrightarrow \pi_1(M) \to 0$. q.e.d.

Stability of Higgs bundle. As an application, we have

Theorem 1.3. *Let M be a quasi-projective manifold such that it has a smooth projective completion $\overline{M}$ and $D_\infty = \overline{M} \setminus M$ is a simply normal crossing divisor in $\overline{M}$. Let (V, H, ∇) be a tame harmonic bundle on M and $(E, \overline{\partial}_E, \theta)$ be the induced Higgs bundle. Then, e is a non-trivial holomorphic section of $(E, \overline{\partial}_E)$ with $\theta(e) = 0$ if and only if e is a non-zero flat section of (V, ∇).*

Proof. The "⇒" part is obvious. We only show the "⇐" part.

Let e be a non-trivial holomorphic section of $(E, \overline{\partial}_E)$ with $\theta(e) = 0$, it corresponds to the non-zero sheaf morphism $\mathcal{O}_M \to \mathcal{O}(E)$. Let F be the saturation sheaf generated by e, F is a holomorphic subbundle of E. Actually, $0 \to (F, 0) \to (E, \theta)$ is a Higgs subsheaf.

Step 1. If M is a curve, then $\overline{F} = \mathcal{O}_M(D)$ with $D \geq 0$ where $\overline{F}$ is the extension of F to $\overline{M}$. Thus, $\mathrm{par.deg}(F) \geq 0$. On the other hand, Simpson showed in [**34**, Lemma 6.2] that

$$\mathrm{par.deg}(F) = \int_M \mathrm{Tr}(\Theta(F, H_F))$$

where H is the *Hermitian–Yang–Mills metric* on E and H_F is the restricted metric on F. H induces a C^∞ splitting $E = F \oplus F^\perp$, then

$$\Theta(F, H_F) = \Theta(E, H)|_F + \overline{A} \wedge A = -(\theta \wedge \theta_H^*)|_F - (\theta_H^* \wedge \theta)|_F + \overline{A} \wedge A$$

where $A \in A^{1,0}(\mathrm{Hom}(F, F^\perp))$ is the second fundamental form of subbundle $F \subset E$. As $\theta(F) = 0$, $\Theta(F, H_F) = -(\theta \wedge \theta_H^*)|_F + \overline{A} \wedge A$. Hence,

$$\mathrm{par.deg}(F) = \int_M \mathrm{Tr}(\Theta(F, H_F)) \leq 0,$$

It is obvious that

$$\mathrm{par.deg}(F) = 0 \Longleftrightarrow A = 0 \text{ and } \theta_H^*|_F = 0.$$

Therefore, the splitting $E = F \oplus F^\perp$ is holomorphic. In fact, there is a splitting of Higgs bundle

$$(E, \theta) = (E_1, \theta) \oplus (F, 0),$$

and two filtered sub Higgs bundles are both polystable. The *tame harmonic bundle* $\mathbb{U}$ corresponds to the Higgs bundle $(F, 0)$, so it is unitary. Thus, the metric connection on $\mathbb{U}$ is flat and e is a non-zero flat section of $\mathbb{U}$ (cf. [**27**] if M is a compact curve).

Step 2. If M is higher dimensional, we can take a generic projective curve $C \subset \overline{M}$ such that C is a complete intersection of very ample divisors intersecting D_∞ transversally and $\pi_1(C \cap M) \twoheadrightarrow \pi_1(M)$ is surjective. Let $C_0 = C \cap M$, the restriction $H|_{C_0}$ is also a *tame harmonic* metric. As in Step 1, we have a Higgs splitting over C_0

$$(E|_{C_0}, \theta_{C_0}) = (E_1, \theta_{C_0}) \oplus (F|_{C_0}, 0)$$

where (E_1, θ_{C_0}) and $(F|_{C_0}, 0)$ are Higgs subbundles (one should be careful that the restricted Higgs field $\theta|_{C_0}$ is not θ_{C_0}). The Higgs splitting corresponds to a local system splitting $\mathbb{V} = \mathbb{W} \oplus \mathbb{U}$ over C_0 where $\mathbb{U}$ is unitary part corresponding to $(F|_{C_0}, 0)$. On the other hand, by the subjectivity of $\pi_1(C_0) \twoheadrightarrow \pi_1(M)$, we have a splitting of a harmonic bundle over M

$$\widetilde{\mathbb{V}} = \widetilde{\mathbb{W}} \oplus \widetilde{\mathbb{U}}$$

where $\widetilde{\mathbb{V}}|_{C_0} = \mathbb{V}$, $\widetilde{\mathbb{W}}|_{C_0} = \mathbb{W}$ and $\widetilde{\mathbb{U}}|_{C_0} = \mathbb{U}$. $\widetilde{\mathbb{U}}$ corresponds to the Higgs subbundle $(F, 0)$ by the *generalized DSUY correspondence*, so that it is unitary. Then, e is a flat section of $\widetilde{\mathbb{U}}$. q.e.d.

Remark. In fact, we generalize a result of Simpson on compact manifolds (cf. [**35**, Lemma 1.2]). The result can also follow from the Bochner method (cf. [**31**]) by using the non-positivity of curvature forms and the estimates of curvatures at the infinity (cf. [**18**, **19**]).

2. The geometry of Lefschetz pencils

Lefschetz pencils.

Definition 2.1 (cf. [**7**])**.** Let $X \subset \mathbb{P}^N$ be a projective manifold and L be a fixed hyperplane in $\mathbb{P}^N$ of dimension $N-2$. The set of hyperplanes $H_s \subset \mathbb{P}^N$ passing through L is parameterized by a projective line $\mathbb{P}^1$ in the dual space $(\mathbb{P}^N)^* \simeq \mathbb{P}^N$. L can be chosen to satisfy that:

(a) L intersects X transversely so that $Y = X \cap L$ is a smooth subvariety of X.

(b) There exists a finite subset $S = \{s_1, \cdots s_k\} \subset \mathbb{P}^1$ such that
  - for any $s_j \in S$, the variety X_{s_j} has one and only one ordinary double singularity $x_j \in Y \cap X_{s_j}$;
  - if $s \notin S$, H_s intersects X transversely (then, $X_s = X \cap H_s$ is a non-singular variety).

Such a family $\{X_s\}_{s \in \mathbb{P}^1}$ is called *Lefschetz pencil*.

Remark. For a general projective line $C \subset (\mathbb{P}^N)^*$, $L = \bigcap_{s \in C}(X \cap H_s)$ actually satisfies all conditions in the definition.

Lemma 2.2 (cf. [**7**])**.** *Any Lefschetz pencil is a proper flat family with a section.*

Example 2.3 (A typical Lefschetz pencil of Calabi–Yau varieties)**.** Let $(Y, \mathcal{L})$ be a projective manifold with a very ample line bundle. The Veronese embedding $\iota : Y \hookrightarrow \mathbb{P}|\mathcal{L}| = \mathbb{P}$ is given by the complete linear system $|\mathcal{L}| = H^0(Y, \mathcal{L})$, then $\mathcal{L} = \iota^*\mathcal{O}_{\mathbb{P}}(1)$ and $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) = H^0(Y, \mathcal{L})$. Suppose that $Y = \mathbb{P}^{n+1}$ and $\mathcal{L} = \mathcal{O}_Y(n+2)$. The Veronese embedding becomes $\iota : \mathbb{P}^{n+1} \hookrightarrow \mathbb{P}^N$ where $N = \begin{pmatrix} 2n+3 \\ n+1 \end{pmatrix} - 1$. Blowing up $X := \iota(\mathbb{P}^{n+1})$ centered at a certain codimension 2 subvariety, one obtains a non-singular projective variety $\widetilde{X}$ and a Lefschetz pencil $f : \widetilde{X} \to \mathbb{P}^1$ of n-dimensional Calabi–Yau varieties.

Let F, H be smooth degree 5 homogenous polynomials with H is not in the Jacobian ideal of F. The family $t_0F + t_1H$ by the parameter $[t_0, t_1] \in \mathbb{P}^1$ may not be Lefschetz pencil, but is it rigid?

Proposition 2.4. *Let $f : \mathfrak{X} \to \mathbb{P}^1$ be a family. Assume that the infinitesimal Torelli theorem holds for a general fiber and f is not a local constant analytic family. Then, f has at least 3 singular fibers.*

Sketch of the proof. It is sufficient to show that the restricted family $f : \mathfrak{X} \to \mathbb{C} \setminus \{0\}$ is not smooth. Replacing $\mathbb{C} \setminus \{0\}$ by an unramified covering:

$$(\mathbb{C} \setminus \{0\}, z) \xrightarrow{t=z^k} (\mathbb{C} \setminus \{0\}, t),$$

we have that the pull back family f' has only unipotent monodromy VHS. Then, the monodromy of the VHS must be trivial because the global monodromy is semi-simple by Deligne's complete reducible theorem (cf. [6]). Thus, f is a local constant analytic family. q.e.d.

Corollary 2.5. *Let $f : \mathfrak{X} \to \mathbb{P}^1$ be a Lefschetz pencil of Calabi–Yau varieties such that the period map is injective at one point of $\mathbb{P}^1$, i.e., non-isotrivial, then f has at least 3 singular fibers.*

Remark. Actually, one has a general result: Consider a non-isotrivial family $(f : \mathcal{X} \to \mathbb{P}^1)$. If X is a projective manifold of non-negative Kodaira dimension, then f has at least 3 singular fibres (cf. [42]).

Vanishing cycles spaces of Lefschetz pencils. Let $f : \mathfrak{X} \to \mathbb{P}^1$ be a Lefschetz pencil of n-dimensional varieties degenerate at $S = \{s_1, \cdots, s_k\}$. Let Δ_i $(1 \le i \le k)$ be a set of disjoint disks centered at the points s_i $(1 \le i \le k)$ each other and let $\Delta_i^* = \Delta_i \setminus \{s_i\}$. Let $f_1 : \mathfrak{X}_1 = f^{-1}(\Delta_1) \to \Delta_1$ be the restricted family, there is a unique singularity $x \in f_1^{-1}(0)$ which is simple. If one chooses a holomorphic local coordinates $z = (z_0, \cdots, z_n)$ in a suitable neighborhood of $x = (0, \cdots, 0)$, f_1 can be written as

$$f_1(z) = z_0^2 + \cdots + z_n^2.$$

Let $B_\epsilon(x)$ be a small ball in $\mathfrak{X}_1$ centered at x. If $s \in \Delta$ is sufficiently closed to 0, the homotopy type of $V_s := B_s \cap \mathfrak{X}_s$ is same as a real $2n$-dimensional sphere, so $H_c^n(V_s, \mathbb{Z}) \simeq \mathbb{Z}$ and there is a natural inclusion

$$\iota : H_c^n(V_s, \mathbb{Z}) \to H^n(\mathfrak{X}_s, \mathbb{Z}).$$

The image $\delta = \iota(1)$ (up to multiplying -1) is called *vanishing cycle*. Similarly, one has $f_i : \mathfrak{X}_i \to \Delta_i$ and vanishing cycles δ_i in $H^n(\mathfrak{X}_{s_i'})$

where s'_i is near s_i for $2 \leq i \leq k$. Fixing a point $s_0 \in \mathbb{P}^1 \setminus S$, there is a monodromy transformation identifying $H^n(\mathfrak{X}_{s'_i}, \mathbb{Z})$ with $H^n(\mathfrak{X}_{s_0}, \mathbb{Z})$ for each $1 \leq i \leq k$. Hence, in this way, $\delta_i \in H^n(\mathfrak{X}_{s_0}, \mathbb{Z})$ for all i.

The $\mathbb{Q}$-liner space V generated by all $\delta_i \in H^n(\mathfrak{X}_{s_0}, \mathbb{Z})$ is called *vanishing cycles space*. One has the Picard–Lefschetz formula (cf. [**15**]):

$$T_i(\upsilon) = \upsilon + (-1)^{(n+1)(n+2)/2}(\upsilon, \delta_i)\delta_i, \ \upsilon \in H^n(\mathfrak{X}_{s_0}, \mathbb{Q}) \tag{2.5.1}$$

with

$$(\delta_i, \delta_i) = \begin{cases} 0 & n \text{ odd} \\ 2(-1)^{n/2} & n \text{ even.} \end{cases}$$

The transform $T_i : H^n(\mathfrak{X}_{s_0}, \mathbb{C}) \to H^n(\mathfrak{X}_{s_0}, \mathbb{C})$ is generated by a loop around s_i and is called a *Picard–Lefschetz transformation*. The non-degenerated intersection form $(,)$ is preserved by all *Picard–Lefschetz transformations*. If one considers only a local Lefschetz pencil $f_1 : \mathfrak{X}_1 \to \Delta_1$, the local monodromy transform $T : H^n(\mathfrak{X}_t, \mathbb{C}) \to H^n(\mathfrak{X}_t, \mathbb{C})$ satisfies that:

$$\begin{aligned} (T - \mathrm{id})^2 = 0 & \quad n \text{ odd} \\ T^2 = id & \quad n \text{ even.} \end{aligned}$$

Thus, the Landman theorem holds naturally for Lefschetz pencils.

Lemma 2.6 (Lefschetz in the classical case cf. [**7**])**.** *Vanishing cycles are conjugate under the action of $\pi_1(\mathbb{P}^1 \setminus S, s_0)$ up to sign. In particular, the vanishing cycles space V is stable under the action of $\pi_1(\mathbb{P}^1 \setminus S, s_0)$.*

Proposition 2.7. *Let $f : \mathfrak{X} \to \mathbb{P}^1$ be a Lefschetz pencil. Assume that the infinitesimal Torelli theorem holds for generic fiber. If the Torelli map of f is injective at one point of $\mathbb{P}^1$, i.e., f is non-isotrivial, the vanishing cycles space V would be non-trivial.*

Proof. Let $f^0 : \mathfrak{X}_0 = f^{-1}(\mathbb{P}^1 \setminus S) \to \mathbb{P}^1 \setminus S$ be the maximal smooth subfamily of f. The set S is non-empty and $\#S \geq 3$ by 2.4. Suppose $V = 0$, then $T_j \equiv id \ \forall j$ by the Picard–Lefschetz formula and the VHS $R^n_{prim} f^0_* \mathbb{Q}$ on $\mathbb{P}^1 \setminus S$ can be extended to a VHS on $\mathbb{P}^1$. But the Torelli map becomes a constant map because $\mathbb{P}^1$ is simply connected. q.e.d.

Invariant subspace of cohomology group. Let $f : \mathfrak{X} \to Y$ be a smooth proper family over an algebraic manifold Y and $t_0 \in Y$ be a fixed point. Denote $X = f^{-1}(t_0) = \mathfrak{X}_{t_0}$. For each $t \in Y$, $\pi_1(Y, t_0)$ acts on $H^n(X_t, \mathbb{Q})$ naturally and one has a $\pi_1(Y, t_0)$-invariant subspace

$$H^n(\mathfrak{X}_t, \mathbb{Q})^{\pi_1(M, t_0)} \hookrightarrow H^n(\mathfrak{X}_t, \mathbb{Q})$$

such that the inclusion is a Hodge morphism of type $(0,0)$. One then has a constant sheaf $(R^n f_*(\mathbb{Q}))^{\pi_1(Y,t_0)}$ by gluing $\{H^n(\mathfrak{X}_t,\mathbb{Q})^{\pi_1(Y,t_0)}\}_{t\in Y}$ together. Moreover, for $\forall t \in Y$ there is a natural Hodge isomorphism

$$\kappa_t : H^0(Y, R^n f_*(\mathbb{Q})) \xrightarrow{\simeq} H^n(\mathfrak{X}_t,\mathbb{Q})^{\pi_1(Y,t_0)}.$$

Theorem 2.8 (Deligne [6]). *Let $f : \mathfrak{X} \to Y$ be a smooth proper family over non-singular algebraic variety Y. One has:*

1) *Leray's spectral sequence for f*

$$E_2^{p,q} = H^p(Y, R^q f_*(\mathbb{Q})) \Rightarrow H^{p+q}(\mathfrak{X},\mathbb{Q})$$

degenerates at E_2.

2) *Let $\overline{\mathfrak{X}}$ be any smooth compactification of $\mathfrak{X}$ and $i : \mathfrak{X} \hookrightarrow \overline{\mathfrak{X}}$ be the inclusion. Then, the composite Hodge morphism*

$$H^n(\overline{\mathfrak{X}},\mathbb{Q}) \to H^n(\mathfrak{X},\mathbb{Q}) \to H^0(Y, R^n f_*(\mathbb{Q}))$$

is surjective.

3) $(R^n f_*(\mathbb{Q}))^{\pi_1(Y,t_0)}$ *is the maximal constant sub $\mathbb{Q}$-VHS of $R^n f_*(\mathbb{Q})$.*

The theorem 2.8 says that the (p,q)-component $\omega^{p,q}$ of a global section ω of a VHS is invariant under $\pi_1(Y,t_0)$, so $\omega^{p,q}$ is also a global section. Therefore, if a global section ω of $(R^n f_*(\mathbb{C}))^{\pi_1(Y)}$ is of (p,q)-type at one point of a connected manifold Y, it is of (p,q)-type everywhere.

Corollary 2.9. *Let $f : \mathfrak{X} \to \mathbb{P}^1$ be a Lefschetz pencil of n-folds smooth over $C_0 = \mathbb{P}^1 \setminus S$. Let $s_0 \in C_0$ be a fixed point and V be the vanishing circle space of $H^n(\mathfrak{X}_{s_0},\mathbb{Q})$ and $V^\perp$ be the orthogonal complement of V in $H^n(\mathfrak{X}_{s_0},\mathbb{Q})$ under the non-degenerated intersection $(,)$. Assume that V is not trivial, then $V^\perp = H^n(\mathfrak{X}_{s_0},\mathbb{Q})^{\pi_1(C_0,s_0)}$.*

The proof depends on the Picard–Lefschetz formula. Altogether,

Theorem 2.10 (Deligne [7]). *Let $f : \mathfrak{X} \to \mathbb{P}^1$ be a Lefschetz pencil of n-dimensional varieties and $f^0 : \mathfrak{X}_0 \to \mathbb{P}^1 \setminus S$ be the maximal smooth subfamily of f. Let $\mathbb{V}$ be a Z-local system generated by the stable action of $\pi_1(\mathbb{P}^1 \setminus S, s_0)$ on the vanishing cycles space V where $s_0 \in \mathbb{P}^1 \setminus S$ is a fixed base point. Then, one has a $\mathbb{Q}$-VHS splitting*

$$R^n f^0_*(\mathbb{Q}) = \mathbb{V}_{\mathbb{Q}} \oplus (R^n f^0_*(\mathbb{Q}))^{\pi_1(\mathbb{P}^1\setminus S, s_0)}.$$

Moreover, if $\mathbb{V}$ is non-trivial then it is absolutely irreducible, i.e., for any algebraic field extension $\mathbb{K}/\mathbb{Q}$, $\mathbb{V}_{\mathbb{K}}$ is an irreducible $\pi_1(\mathbb{P}^1\setminus S, s_0)$-module.

Remark. The complete irreducible theorem of Deligne induces a $\mathbb{Q}$-local system decomposition $R^n f^0_*\mathbb{Q} = \oplus_i \mathbb{V}_{i\mathbb{Q}}$. Each $\mathbb{V}_{i\mathbb{Q}}$ has an induced Hodge filtration from the Hodge filtration of $R^n f^0_*\mathbb{Q}$. Hence, the decomposition is actually of VHS, but in general, it is not compatible with polarizations.

Lefschetz pencils of Calabi–Yau varieties.

Lemma 2.11. *Let $f : \mathfrak{X} \to Y$ be a smooth family of Calabi–Yau n-folds and $\mathcal{I}$ be the Higgs bundle induced from the $\mathbb{Q}$-VHS $(R^n f_*(\mathbb{Q}))^{\pi_1(Y)}$. If the differential of the Torelli map of f is injective at some points in Y, then*

$$\dim_{\mathbb{C}} \mathcal{I}^{n,0} = \dim_{\mathbb{C}} \mathcal{I}^{0,n} = 0.$$

Proof. Suppose that the differential of the Torelli map of f is injective at $y_0 \in Y$. Let $\phi : (\mathfrak{X}, X) \to (S, 0)$ be the Kuranishi family of $X = f^{-1}(y_0)$. The theorem of Bogomolov–Todorov–Tian says that the deformation of X is unobstructed, so S is smooth (cf. [**38**],[**39**]). We then have a commutative diagram over a neighborhood U (in the topology of complex analytic spaces) of y_0 in Y

$$\begin{array}{ccc} (U, t_0) & \xrightarrow{\quad\pi\quad} & (S,0) \\ & {\scriptstyle\lambda_U}\searrow \quad \swarrow{\scriptstyle\lambda} & \\ & \prod \mathrm{Gr}(h^p, H_{\mathbb{C}}) & \end{array}$$

where λ, λ_U are Torelli maps and π is the Kuranishi map. Moreover, we can assume that λ_U is an embedding by contracting U sufficiently.

We claim that

$$h^{n,0}(H^n(\mathfrak{X}_{t_0}, \mathbb{C})^{\pi_1(Y,t_0)}) < h^{n,0}(\mathfrak{X}_{t_0}) = 1.$$

Otherwise, $h^{n,0}(H^n(\mathfrak{X}_{t_0}, \mathbb{C})^{\pi_1(Y,t_0)}) = h^{n,0}(\mathfrak{X}_{t_0}) = 1$. Let $\overline{\mathfrak{X}}$ be the smooth Haronaka compactification of $\mathfrak{X}$, there are composite morphisms:

$$\bar{i}_t^* : H^n(\overline{\mathfrak{X}}, \mathbb{C}) \xrightarrow{j \circ i^*} H^0(Y, R^n f_* \mathbb{C}) \xrightarrow{\kappa_t} H^n(\mathfrak{X}_t, \mathbb{C})^{\pi_1(M,t)} \hookrightarrow H^n(\mathfrak{X}_t, \mathbb{C}) \, \forall t \in Y,$$

and each $\bar{i}_t^*$ is a morphism of Hodge structure of type $(0,0)$ (cf. [**6**]). By 2.8,

$$h^{n,0}(H^n(\mathfrak{X}_t, \mathbb{C})^{\pi_1(Y,t)}) = h^{n,0}(\mathfrak{X}_t) = 1 \, \forall t \in Y.$$

Therefore, each $\bar{i}_t^*$ has to be surjective and all $(n,0)$ holomorphic forms of $H^n(\mathfrak{X}_t, \mathbb{C})$ lift to $H^n(\overline{\mathfrak{X}}, \mathbb{C})$, i.e., that we have a Hodge isomorphism

$$H^n(\mathfrak{X}_t, \mathbb{Q}) \xrightarrow{\simeq} H^n(\mathfrak{X}_t, \mathbb{Q})^{\pi_1(Y,t_0)} \xrightarrow{\simeq} H^0(Y, R^n f_*(\mathbb{Q})).$$

It is a contradiction to that λ_U is an embedding over U. q.e.d.

Corollary 2.12. *Let $f : \mathfrak{X} \to Y$ be a proper smooth family of Calabi–Yau n-folds. If the differential of the Torelli map is injective at some points, then $f_*(\omega_{\mathfrak{X}/Y})$ and its multi-tensors are geometric non-trivial.*

Proof. Suppose that the differential of the Torelli map of f is injective at $y_0 \in Y$. Let $\pi : \mathcal{X} \to \mathfrak{M}$ be the maximal subfamily of the Kuranishi family for $X = f^{-1}(y_0)$ with respect to a fixing polarization. The polarized Kuranishi base $\mathfrak{M}$ is smooth and the Kodaira–Spencer map $\rho(t) : T_{\mathfrak{M},t} \to H^1(\mathcal{X}_t, T_{\mathcal{X}_t})$ is injective for $\forall t \in \mathfrak{M}$. In the category of complex analytic spaces, the polarized Kuranishi family is universal. Actually, under the assumptions we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{X}_U & \xrightarrow{\hookrightarrow} & \mathcal{X} \\ \downarrow{\scriptstyle f_U} & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{\hookrightarrow} & \mathfrak{M} \end{array}$$

where $f_U : \mathfrak{X}_U = \mathfrak{X} \times_Y U \to U$ is the restricted family over a small open neighborhood $U \subset Y$ of y_0. Hence, we have

$$\pi_*(\omega_{\mathcal{X}/\mathfrak{M}})|_U = \pi_*(\omega_{\mathcal{X}/\mathfrak{M}}|_{\mathfrak{X}_U}) = f_*\omega_{\mathfrak{X}/Y}|_U.$$

On the other hand, on $\mathfrak{M}$

$$\omega_{WP} = -\frac{\sqrt{-1}}{2}\partial\overline{\partial}\log h = c_1(\pi_*\omega_{\mathcal{X}/\mathfrak{M}}, h)$$

where h is the Hodge metric and ω_{WP} is the Kähler form of the Weil-Petersson metric (cf. [**38**],[**39**]). Therefore,

$$\int_U (c_1(f_*\omega_{\mathfrak{X}/Y}, h))^{\dim Y} = \int_U (c_1(\pi_*(\omega_{\mathcal{X}/\mathfrak{M}}), h))^{\dim Y} > 0.$$

q.e.d.

A polarized VHS over Y is *isotrivial* if it becomes a constant VHS after a finite étale base change. It is obvious that a polarized VHS is *isotrivial* if and only if the Hodge filtration is local constant (cf. [**6**]).

Proposition 2.13. *Let $f : \mathfrak{X} \to \mathbb{P}^1$ be a Lefschetz pencil of n-folds smooth over $P^1 \backslash S$ and $f^0 : \mathfrak{X}_0 \to \mathbb{P}^1 \backslash S$ be the maximal smooth subfamily of f. Assume that n is even, then the VHS $R^n f^0_*(\mathbb{Q})$ must be isotrivial and*

$$(R^n f^0_*(\mathbb{C}))^{n,0}, (R^n f^0_*(\mathbb{C}))^{0,n} \subset (R^n f^0_*(\mathbb{C}))^{\pi_1(\mathbb{P}^1\backslash S)}.$$

Proof. All Picard–Lefschetz transforms are of order 2 as n is even. Since $\mathbb{P}^1$ is topologically simply-connected, the global monodromy group Γ is a finite commutative group. Actually, $\Gamma \cong (\prod_1^{\#S} \mathbb{Z}_2, +)$. The VHS $R^n f^0_*(\mathbb{Q})$ must be *isotrivial* by [**14**, Theorem 9.8]. Then,

$$T_i(H^{n,0}(\mathfrak{X}_{s_0},\mathbb{C})) \subset H^{n,0}(\mathfrak{X}_{s_0},\mathbb{C})\ \forall i$$

where s_0 is a fixed base point in $\mathbb{P}^1 \setminus S$. By the Picard–Lefschetz formula, we have $(\eta,\delta_i)\delta_i \in H^{n,0}(\mathfrak{X}_{s_0},\mathbb{C})\ \forall i$ for any $\eta \in H^{n,0}(\mathfrak{X}_{s_0},\mathbb{C})$.

We claim that $(\eta,\delta_j) = 0\ \forall j$ for any $\eta \in H^{n,0}(\mathfrak{X}_{s_0},\mathbb{C})$. Otherwise, $(\eta,\delta_j) \neq 0$ for one j and one $\eta \neq 0$, then $\delta_j \in H^{n,0}(\mathfrak{X}_{s_0},\mathbb{C})$. But it is a contradiction to that $(\delta_j,\delta_j) \neq 0$. From the formula 2.5.1, we then have:

$$H^{n,0}(\mathfrak{X}_{s_0},\mathbb{C}) \subset H^n(\mathfrak{X}_{s_0},\mathbb{C})^{\pi_1(\mathbb{P}^1\setminus S,s_0)}.$$

q.e.d.

Corollary 2.14. *Let $f:\mathfrak{X}\to\mathbb{P}^1$ be a Lefschetz pencil of Calabi–Yau varieties. Assume that the Torelli map is injective at some points, the global algebraic monodromy group is infinite.*

Altogether, from 2.10, 2.11 and 2.13 we obtain a key result for the *rigidity* of Lefschetz pencils of Calabi–Yau varieties.

Theorem 2.15. *Let $f:\mathfrak{X}\to\mathbb{P}^1$ be a Lefschetz pencil of n-dimensional Calabi–Yau varieties and $f^0:\mathfrak{X}_0 = f^{-1}(\mathbb{P}^1\setminus S)\to\mathbb{P}^1\setminus S$ be its maximal smooth subfamily. Let $\mathbb{V}$ be the $\mathbb{Z}$-local system generated by the vanishing cycles space. If the differential of the Torelli map of f^0 is injective at some points (for example, non-isotrivial families). Then, we have:*

1. *The integer n must be odd.*
2. *$\mathbb{V}$ is a non-trivial absolutely irreducible $\pi_1(\mathbb{P}^1\setminus S)$-module and*
$$(R^n_{prim}f^0_*(\mathbb{C}))^{n,0},(R^n_{prim}f^0_*(\mathbb{C}))^{0,n}\subset\mathbb{V}_{\mathbb{C}}.$$
3. *$\mathbb{V}_{\mathbb{Q}}$ is the $\mathbb{Q}$-sub local system of $R^n_{prim}f^0_*(\mathbb{Q})$. Moreover,*
$$R^n_{prim}f^0_*(\mathbb{Q}) = \mathbb{V}_{\mathbb{Q}}\oplus(R^n_{prim}f^0_*(\mathbb{Q}))^{\pi_1(\mathbb{P}^1\setminus S)}$$
by the relative Lefschetz decomposition.

3. A criterion for rigidity and its applications

Endomorphisms of Higgs bundles over a product variety. Let S_0, T_0 be quasi-projective manifolds such that they have smooth projective completions S, T and $D_S = S - S_0, D_T = T - T_0$ are normal crossing divisors. Let $(\mathbb{V},\nabla)$ be an arbitrary polarized $\mathbb{R}$-VHS over $S^0\times T^0$ such that local monodromies around the divisor D at infinity are quasi unipotent. Let (E,θ) be the Higgs bundle induced from $\mathbb{V}$. Extending the Higgs bundle to the infinity, we have the quasi canonical extension $(\overline{E},\overline{\theta})$ with

$$\theta:\overline{E}\to\overline{E}\otimes\Omega^1_{S\times T}(\log D).$$

We then obtain a flat endomorphism of $\mathbb{V}|_{S_t}$ with Hodge type $(-1,1)$ (cf. [17] and [51]) in the following way.

- First, one has a decomposition

$$\Omega^1_{S\times T}(\log D) = p_S^*\Omega^1_S(\log D_S) \oplus p_T^*\Omega^1_T(\log D_T)$$

where $p_S : S\times T \to S$ and $p_T : S\times T \to T$ are projections. The Higgs map

$$\theta : p_S^*\Theta_S(-\log D_S) \oplus p_T^*\Theta_T(-\log D_T) \to End(\overline{E}) \tag{3.0.1}$$

is just the sheaf map of the differential of the period map. Fix a $t \in T^0$, the restricted Higgs map over S_t is

$$\theta|_{S_t} : (p_S^*\Theta_S(-\log D_S) \oplus p_T^*\Theta_T(-\log D_T))|_{S_t} \to End(\overline{E})|_{S_t}. \tag{3.0.2}$$

Note that

$$p_T^*\Theta_T(-\log D_T)|_{S_t} \simeq \bigoplus^{l} \mathcal{O}_{S_t}$$

where l is the dimension of T. Let 1_T be a constant section of $p_T^*\Theta_T(-\log D_T)|_{S_t}$. Then, we obtain an endomorphism

$$\sigma := \theta|_{S_t}(1_T) : E|_{S_t^0} \to E|_{S_t^0}. \tag{3.0.3}$$

and σ must be of $(-1,1)$-type. Moreover, σ is a morphism of Higgs sheaf, i.e., there is a commutative diagram

$$\begin{array}{ccc} E|_{S_t^0} & \xrightarrow{\theta_{S_t^0}} & E|_{S_t^0} \otimes \Omega^1_{S_t^0} \\ \big\downarrow{\scriptstyle\sigma} & & \big\downarrow{\scriptstyle\sigma\otimes id} \\ E|_{S_t^0} & \xrightarrow{\theta_{S_t^0}} & E|_{S_t^0} \otimes \Omega^1_{S_t^0} \end{array}$$

by $\theta_{S_t^0} \wedge \theta_{S_t^0} = 0$, where

$$\theta_{S_t} : \Theta_{S_t}(-\log D_{S_t}) \to (p_S^*\Theta_S(-\log D_S)\oplus p_T^*\Theta_T(-\log D_T))|_{S_t} \to \mathrm{End}(\overline{E})|_{S_t}$$

is the Higgs field of $E|_{S_t}$.

- Let M be any quasi-projective manifold with a smooth compactification $\overline{M}$ and a normal crossing divisor $D_\infty = \overline{M} - M$. Let (E,θ) be a Higgs bundle on M. If E carries $\mathbb{R}$-structure, same as [35, Lemma 2.11] for compact M, the dual Higgs bundle is

$$(E^\vee = \bigoplus_{p+q=n} E^{\vee -p,-q}, \theta_\vee)$$

where $E^{\vee -p,-q} = (E^{p,q})^\vee = E^{q,p}$, $\theta_\vee^{-p,-q} = -\theta^{q,p}$ and

$$\theta_\vee^{-p,-q} : E^{\vee -p,-q} \to E^{\vee -p-1,-q+1} \otimes \Omega^1_M.$$

(However, for a polarized $\mathbb{R}$-VHS $\mathbb{V}_{\mathbb{R}}$ of weight n one has a natural non-degenerated pairing $\mathbb{V}_{\mathbb{R}} \times \mathbb{V}_{\mathbb{R}} \to \mathbb{R}(-n)$, hence $\mathbb{V}_{\mathbb{R}}^{\vee} = \mathbb{V}_{\mathbb{R}}(n)$). Therefore, $\mathrm{End}(E)$ is a Higgs bundle, i.e.,

$$(\mathrm{End}(E), \theta^{end}) = (\bigoplus_{(p-p')+(q-q')=0} E^{p,q} \otimes E^{\vee -p',-q'}, \theta^{end})$$

with $\theta^{end}(u \otimes v^{\vee}) = \theta(v) \otimes v^{\vee} + u \otimes \theta_{\vee}(v^{\vee})$. All local monodromies for $\mathrm{End}(E)$ are quasi unipotent as so are for E, and one has Deligne's quasi-canonical extension for $\mathrm{End}(E)$. Moreover, one has:

Lemma 3.1 (Proposition 2.1 in [**51**]). *Let* $(\mathrm{End}(E), \theta^{end})$ *be the Hodge bundle corresponding to a polarized VHS on* $\mathrm{End}(\mathbb{V}_{\mathbb{R}})$ *which is induced by the polarized* $\mathbb{R}$*-VHS* $\mathbb{V}_{\mathbb{R}}$. *Then,*

$$\theta^{end}(d\phi(T_{\overline{M}})(-\log D_{\infty})) = 0.$$

- The image of the map 3.0.2 is a trivial Higgs subsheaf as it is contained in the kernel of a Higgs field, and σ is a flat section by the poly-stability of the Higgs sheaf. Precisely,

Proposition 3.2. *The endomorphism σ obtained above is a flat $(-1,1)$-type section of* $\mathrm{End}(E)|_{S_t^0}$. *Therefore, σ is an endomorphism of a $\mathbb{C}$-local system, i.e.,*

$$\sigma : \mathbb{V}_{\mathbb{C}}|_{S_t^0} \to \mathbb{V}_{\mathbb{C}}|_{S_t^0}. \tag{3.2.1}$$

Proof. First, $\theta|_{S^0 \times T^0} = d\phi$ where $\phi : S^0 \times T^0 \to D/\Gamma$ is the period map associated to $\mathbb{V}$. Denote $\mathrm{Image}(\theta)$ is the image sheaf. We have

$$\sigma_{s,t} \in (d\phi(\Theta_{S^0 \times T^0}))_{s,t} = (\mathrm{Image}(\theta))_{s,t}$$

and $\theta^{end}_{s,t}(\sigma_{s,t}) = 0$ for all $(s,t) \in S_t^0$ by 3.1.

On the other hand, we can else get σ if we fix t and vary $s \in S^0$. Thus, σ is of type $(-1,1)$ and $\theta^{end}_{S_t^0}(\sigma) = 0$. By 1.3, σ is a flat section of $(\mathrm{End}(\mathbb{V}_{\mathbb{C}})|_{S_t^0}, \nabla)$. q.e.d.

A criterion for infinitesimal rigidity. In the following sections, we study the manifolds for which the *infinitesimal Torelli theorem* holds, and we assume that there is a natural condition for a smooth family $f : \mathfrak{X} \to Y$ of projective n-folds:

(**) *The differential of the period map for the VHS $R^n_{prim} f_*(\mathbb{Q})$ is injective at some points of Y.*

By the *infinitesimal Torelli theorem*, the condition is equivalent to that the induced moduli map $\eta_f : Y \to \mathfrak{M}_h$ is a generic finite morphism, i.e., f contains no isotrivial subfamily such that the base is a subvariety passing through a general point of Y. Suppose that Y is a curve, the $(**)$-condition is then equivalent to that f is a non-isotrivial family.

A smooth family f is *rigid* if there exists no non-trivial *deformation* of f over a non-singular quasi-projective curve. A *deformation* of a smooth family f over a quasi-projective variety T^0 with base point $0 \in T^0$ is a smooth projective morphism $g : \mathcal{X} \to Y \times T^0$ with a commutative diagram

$$\begin{array}{ccccc} \mathfrak{X} & \xrightarrow{\simeq} & g^{-1}(Y \times \{0\}) & \xrightarrow{\subset} & \mathcal{X} \\ f\downarrow & & \downarrow & & \downarrow g \\ Y & \xrightarrow{\simeq} & Y \times \{0\} & \xrightarrow{\subset} & Y \times T^0 \end{array} .$$

A family of varieties is *rigid* if its maximal smooth subfamily is *rigid*.

In the category of complex analytic spaces, we replace T^0 by a small disk. Then, we should say *infinitesimal rigidity* instead of *rigidity*. It is obvious that an algebraic family is automatically *rigid* if it is *infinitesimal rigid* in the category of complex analytic spaces. If there is no ambiguity, we often do not distinguish between the two notations.

Let $f : \mathfrak{X} \to Y$ be a smooth polarized family of projective n-folds. If a deformation g of f over a quasi-projective smooth curve T^0 is not trivial, then the period map for g is not degenerate along T^0-direction at some points of $Y \times \{0\}$ by the *infinitesimal Torelli theorem*. Therefore, with our methods in studying Higgs bundles over a product variety, we reprove a theorem of Faltings, Peters and Jost–Yau (cf. [**12**],[**30**],[**17**]).

Theorem 3.3 (A criterion for rigidity). *Let $f : \mathfrak{X} \to Y$ be a smooth family of polarized projective n-folds satisfying the $(**)$ condition. If f is non-rigid, there exists a flat non-zero section σ of*

$$\mathrm{End}(R^n_{prim} f_*(\mathbb{C}))^{-1,1}.$$

Moreover, the Zariski tangent space at $[f]$ of the deformation space of f is into $\mathrm{End}(R^n_{prim} f_*(\mathbb{C}))^{-1,1}$.

Saito–Zucker and Zuo also obtained similar criterions (cf. [**37**],[**51**]). It is not difficult for us to generalize the criterion to non-rigid polarized VHSs with this method.

Notably the non-positivity of curvatures in the horizontal directions of period domains will underly the validity of this criterion of the rigidity. Here, we explain the role played by the differential geometry of period domains: Consider the simplest case. Let $f_t : \mathfrak{X}_t \to Y$ be one parameter non-trivial holomorphic deformations of families of polarized Calabi–Yau manifolds satisfying the $(**)$-condition. We then have one parameter period maps g_t from Y into a fixed period domain. The infinitesimal deformation of g_t gives rise to a holomorphic section of a Hermitian holomorphic vector bundle with non-positive curvature in the sense of Griffiths. By the Bochner method and the estimates of curvatures at the infinity, this holomorphic section must be parallel.

It is the original ideal of Jost–Yau to deal with the *rigidity* of Shafarevich problems (cf. [**17**]). We shall point out that the $(-1,1)$-type of this flat section is a key point in proving the *rigidity* throughout this paper, and it seems one cannot obtain these results only by the pure techniques of the differential geometry. It is successful for us to use the theory of algebraic Higgs bundles.

Applications of the criterion. Let $f:(\mathfrak{X},\mathcal{L})\to Y$ be a smooth family of polarized Calabi–Yau n-folds with $(X,L)=f^{-1}(0)$ and let $\pi:\mathfrak{Y}\to\mathfrak{M}_{c_1(L)}$ be the maximal subfamily of Kuranishi family of $\pi^{-1}(0)=X$ with a fixed polarization L. For each $t\in\mathfrak{M}_{c_1(L)}$, the Kodaira–Spencer map

$$\rho(t):T_{\mathfrak{M},t}\xrightarrow{\simeq}H^1(\mathfrak{X}_t,T_{\mathfrak{X}_t})_{c_1(L)}$$

is isomorphic. Let Φ be the period map of π and U be a sufficient small neighborhood (in topology of complex analytic spaces) of 0 in M. If the differential of the period map $\Psi:Y\longrightarrow D/\Gamma$ is injective at $0\in Y$, one has a locally commutative diagram

$$\begin{array}{ccc}(U,0) & \xrightarrow{\pi} & (\mathfrak{M}_{c_1(L)},0)\\ & {\scriptstyle\Psi}\searrow \quad \swarrow{\scriptstyle\Phi} & \\ & (D,0) & \end{array}$$

where π is an embedding over U. Let (E,θ) be the Higgs bundle induced from $R^n_{prim}f_*(\mathbb{Q})$, the horizonal tangent space $T^h_{D,0}$ is contained in

$$\bigoplus_{p=0}^{n}\mathrm{Hom}(E^{n-p,p}|_0,E^{n-p-1,p+1}|_0)$$

and $d\Psi|_0=(\theta^{n,0}|_0,\cdots,\theta^{0,n}|_0)$ where

$$\theta^{p,q}:E^{p,q}\to E^{p-1,q-1}\otimes\Omega^1_Y \text{ with } p+q=n$$

is the (p,q)-component of the Higgs field θ.

Proposition 3.4 (Infinitesimal Torelli theorem)**.** *The differential of the period map of a family of Calabi–Yau n-folds is injective at one point if and only if $\theta^{n,0}:E^{n,0}\to E^{n-1,1}$ is injective at this point.*

If f is non-rigid, one would has a non-trivial deformation g over $Y \times T$ such that g is not degenerate along the orientation of T at some points $\{s_0, \cdots, s_k\}$ of $(Y, 0) \subset Y \times T^0$. We then have a non-zero $(-1, 1)$-type endomorphism σ of E which is flat under the Gauss–Manin connection. As σ is obtained from the period map over $Y \times T^0$ along the T^0-direction, σ is not degenerate at points $\{s_0, \cdots, s_k\}$, i.e., the morphism $\sigma^{n,0} : E^{n,0} \to E^{n,0}$ is injective at $\{s_0, \cdots, s_k\}$. Because T is of dimension one and σ is flat, we actually have

Corollary 3.5. *σ is not degenerate at each point in Y, i.e., the morphism $\sigma^{n,0} : E^{n,0} \to E^{n-1,1}$ is injective everywhere. Hence, the duality map $\sigma^{1,n-1} : E^{1,n-1} \to E^{0,n}$ is surjective at each point of Y. Furthermore, $\sigma^{n+1} \equiv 0$ because $\sigma(E^{n,0}) \equiv 0$.*

4. The rigidity of Lefschetz pencils of Calabi–Yau varieties

Theorem 4.1. *Let $f : \mathfrak{X} \to \mathbb{P}^1$ be a non-isotrivial Lefschetz pencil of n-dimensional Calabi–Yau varieties. Then, the family f must be rigid.*

Proof. Let $f^0 : \mathfrak{X}_0 = f^{-1}(C_0) \to C_0 = \mathbb{P}^1 \setminus S$ be the maximal smooth subfamily of f. We have shown that n must be odd and there is a decomposition of the $\mathbb{Q}$-VHS

$$R^n_{prim} f^0_*(\mathbb{Q}) = \mathbb{V} \oplus (R^n_{prim} f^0_*(\mathbb{Q}))^{\pi_1(\mathbb{P}^1 \setminus S)}.$$

with $(R^n_{prim} f^0_*(\mathbb{Q}))^{n,0}$ and $(R^n_{prim} f^0_*(\mathbb{Q}))^{0,n}$ are in $\mathbb{V}_{\mathbb{C}}$. Assume the statement is not true, we obtain a non-trivial deformation

$$\begin{array}{ccc} \mathfrak{X}_0 & \xrightarrow{\subset} & \mathcal{X} \\ {\scriptstyle f^0}\downarrow & & \downarrow{\scriptstyle g} \\ C_0 \times \{0\} & \xrightarrow{\subset} & C_0 \times T^0 \end{array} .$$

where T^0 is a smooth quasi-projective curve. We then have a non-zero flat $(-1, 1)$-type endomorphism σ of $R^n_{prim} f^0_*(\mathbb{C})$. As $R^n_{prim} f^0_*(\mathbb{C})$ determines a Higgs bundle (E, θ) on C_0, we have:

Lemma 4.2. *Any non-zero flat $(-1, 1)$-type endomorphism $\sigma \in \mathrm{End}(E)$ induces a splitting of the Higgs bundle*

$$(E, \theta) = \mathrm{Ker}(\sigma) \bigoplus (\mathrm{Ker}(\sigma))^{\perp}.$$

The statement is also true if C_0 is replaced by a higher dimensional quasi-projective variety Y.

Proof. The statement is a special case of the *generalized DSUY correspondence* 1.2. As the polarization Q and the endomorphism σ both are flat under the Gauss–Manin connection, we obtain a $\mathbb{C}$-splitting of the local system

$$R^n_{prim} f^0_*(\mathbb{C}) = \mathrm{Ker}(\sigma) \oplus (\mathrm{Ker}(\sigma))^\perp$$

which is compatible with the polarization Q. $(\mathrm{Ker}(\sigma))^\perp$ is the orthogonal component of $\mathrm{Ker}(\sigma)$ in $R^n_{prim} f^0_*(\mathbb{C})$, i.e., the complete reducibility. Therefore, if we regard σ as a $\mathcal{O}_C$-linear map, we have

$$R^n_{prim} f^0_*(\mathbb{C}) \otimes \mathcal{O}_C = \mathrm{Ker}(\sigma) \oplus (\mathrm{Ker}(\sigma))^\perp.$$

Restricting the Hodge filtration of the VHS $R^n_{prim} f^0_*(\mathbb{C})$ to these sub local systems and taking the grading of the Hodge filtration, we actually have a decomposition of the Higgs bundles. Moreover, it is really a splitting of a *complex variation of Hodge structure* (cf. [**34**]). q.e.d.

Continue to prove the theorem. $E^{0,n} \subset \mathrm{Ker}(\sigma)$ along M. Because of non-triviality of the deformation of the family, there exists a point $s_0 \in \mathbb{P}^1 \setminus S$ such that $\sigma : E^{n,0}|_{C_0} \to E^{n-1,1}|_{C_0}$ is injective at s_0 by 3.5. Thus,

$$E^{n,0} \not\subseteq \mathrm{Ker}(\sigma) \ \text{ and } \ E^{0,n} \subset \mathrm{Ker}(\sigma)$$

at s_0. On the other hand,

$$\rho : \pi_1(\mathbb{P}^1 \setminus S) \to \mathrm{Aut}(\mathbb{V}_{\mathbb{C},s_0})$$

is irreducible and both $E^{n,0}$ and $E^{0,n}$ are in $\mathbb{V}_\mathbb{C}$ by 2.15. We have a contradiction. q.e.d.

Actually, we use two ideas in proving the theorem. The first idea is that *a $(-1,1)$-type endomorphism of a complex polarized VHS induces a splitting of the underlying $\mathbb{C}$-local system*, it is implied in Zuo's theorem on the negativity of kernels of Kodaira–Spencer maps of Hodge bundles (cf. [**51**]). The second idea is the well-known *Kazhdan–Margulis theorem* (cf. [**7**]): *For any Lefschetz pencil of odd dimensional varieties, the image $\pi_1(\mathbb{P}^1 \setminus S, s_0)$ is a Zariski open set in $\mathrm{Sp}(\mathbb{V}_{\mathbb{C},s_0}, (,))$ where $\mathbb{V}$ is the $\mathbb{Z}$-local system of the vanishing cycles space.*

Altogether, the theorem 4.1 explores that there are deep relations between two important objects in the algebraic geometry: *rigidity* and *algebraic monodromy*. We start to study these relations in [**48**].

Definition 4.3. Let $\mathbb{V}$ be a $\mathbb{Q}$-local system on a quasi-projective manifold Y with monodromy representation

$$\rho : \pi_1(Y, y_0) \to \mathrm{GL}(V),\ V := \mathbb{V}_{y_0}.$$

where y_0 is a fixed base point in Y.

1. The monodromy group $\pi_1(Y, y_0)^{mon}$ is defined to be the Zariski closure of the smallest algebraic subgroup of $\mathrm{GL}(V)$ containing the monodromy representation $\rho(\pi_1(Y, y_0))$.
2. Assume that V carries a non-degenerated bilinear form Q which is symmetric (or anti-symmetric) and preserved by the monodromy group. We call the monodromy is *big* if the connected component (including the identity) of $\pi_1(Y, y_0)^{mon}$ acts irreducibly on $\mathbb{V}_{\mathbb{C}}$.

Theorem 4.4. *Let $f : X \to Y$ be a non-isotrivial smooth family of Calabi–Yau n-folds. Assume that $R^n_{prim} f_*(\mathbb{Q})$ has a sub $\mathbb{Q}$-VHS $\mathbb{V}$ with big monodromy and the bottom Hodge filtration of the VHS $R^n_{prim} f_*(\mathbb{Q})$ is in $\mathbb{V}$. Then, the family f must be rigid.*

5. The rigidity of strongly degenerated families

Definition 5.1 (Strongly degenerated family)**.** Let $f : \mathfrak{X} \to C$ be a family over a smooth projective curve C with singular values $\{c_0, \cdots, c_k\}$. f is called *strongly degenerate* at c_0 if f satisfies following conditions:

1. $X_i = f^{-1}(c_i) = X_{i1} + \cdots + X_{ir_i}$ is reduced and it is a union of transversely crossing smooth divisors for all i, i.e., f is semistable.
2. The cohomology of each component of the singular fiber $X_0 = f^{-1}(c_0)$ has pure type (p, p).

Let $f : \mathfrak{X} \to C$ be a *strongly degenerated* family of n-folds. Set $C_0 = C \setminus \{c_0, \cdots, c_k\}$, we have the maximal smooth subfamily $f^0 : \mathfrak{X}_0 \to C_0$ of f. Define the fiber product family by $\pi : \mathcal{Y} \triangleq \mathfrak{X} \times_C \mathfrak{X} \to C$. π is only degenerate at $\{c_0, \cdots, c_k\}$, and $\pi^0 : \mathcal{Y}_0 = \mathfrak{X} \times_{C_0} \mathfrak{X} \to C_0$ is the maximal smooth subfamily of π. By the Künneth formula, we obtain that

Corollary 5.2. *Let $f : \mathfrak{X} \to C$ be a family strongly degenerate at $c_0 \in C$. Then, the fiber product family π is also strongly degenerate at c_0.*

Now, we study endomorphisms of Higgs bundles over a quasi-projective variety M. Let (E, θ) be a Higgs bundle with positive Hermitian metric H. Under the condition that (E, θ) carries $\mathbb{R}$-structure, we describe $\mathrm{End}(E)^{-1,1}$ precisely in the following way. As in section 3, $\mathrm{End}(E)$ is also a Higgs bundle:

$$(\mathrm{End}(E) = \bigoplus_{r+s=0} \mathrm{End}(E)^{r,s}, \theta^{end}) = (\bigoplus_{(p-p')+(q-q')=0} E^{p,q} \otimes E^{\vee -p', -q'}, \theta^{end})$$

with $\theta^{end}(u \otimes v^\vee) = \theta(v) \otimes v^\vee + u \otimes \theta_\vee(v^\vee)$. Thus,

$$\mathrm{End}(E)^{-1,1} = \bigoplus_{p+q=n} E^{p,q} \otimes E^{q-1,p+1}, \tag{5.2.1}$$

and $\mathrm{End}(E)^{-1,1} \subset (E \otimes E)^{n-1,n+1}$ as vector spaces.

Lemma 5.3. *Let (E, θ) be the Higgs bundle associated to $R^n_{prim} f^0_*(\mathbb{C})$. Then, by the Künneth formula $H^*(X, \mathbb{C}) \otimes H^*(Z, \mathbb{C}) = H^*(X \times Z, \mathbb{C})$, we have the inclusion*

$$R^n_{prim} f^0_*(\mathbb{C}) \otimes R^n_{prim} f^0_*(\mathbb{C}) \subset R^{2n} \pi^0_*(\mathbb{C}), \tag{5.3.1}$$

which is compatible with Hodge structures. Therefore, as vector spaces

$$\mathrm{End}(E)^{-1,1} \subset R^{n+1} \pi^0_*(\Omega^{n-1}_{\mathcal{Y}_0/C_0}). \tag{5.3.2}$$

Theorem 5.4. *Any non-isotrivial strongly degenerated family (not only for families of Calabi–Yau varieties) must be rigid.*

Proof. Let $f : \mathfrak{X} \to C$ be a *strongly degenerated* family of n-folds smooth over $C_0 = C - \{c_0, \cdots, c_k\}$. We have the maximal smooth subfamily f^0 as before and π, π^0 respectively. Suppose that $f^0 : \mathfrak{X}_0 \to C_0$ is non-rigid, we then have a non-zero $(-1,1)$-type global section σ of $\mathrm{End}(R^n_{prim} f^0_*(\mathbb{C}))$. By 5.3,

$$0 \neq \sigma \in (R^{n+1} \pi^0_*(\Omega^{n-1}_{\mathcal{Y}_0/C_0}))^{\pi_1(C_0)}.$$

On the other hand, there is a commutative diagram due to 2.8

$$\begin{array}{ccc} H^n(\mathcal{Y}, \mathbb{C}) & \xrightarrow{\quad i^* \quad} & H^n(\mathcal{Y}_0, \mathbb{C}) \\ & {\scriptstyle \overline{i}_t^*} \searrow \qquad \swarrow {\scriptstyle i_t^*} & \\ & H^n(\mathcal{Y}_t, \mathbb{C})^{\pi_1(C_0, t)} & \end{array}$$

where $i_t : \mathcal{Y}_t \hookrightarrow \mathcal{Y}_0$, $\overline{i}_t : \mathcal{Y}_t \hookrightarrow \mathcal{Y}$ are natural embedding; and

$$H^n(\mathcal{Y}_t, \mathbb{C})^{\pi_1(C_0, t)} \cong H^0(C_0, R^n \pi^0_*(\mathbb{C}))$$

is an isomorphism of Hodge structure. For each $t \in C_0$, as $\overline{i}_t^*$ is a surjective Hodge morphism, we have the restriction maps

$$r_t^{p,q} : H^q(\mathcal{Y}, \Omega^p_{\mathcal{Y}}) \hookrightarrow H^n(\mathcal{Y}, \mathbb{C}) \xrightarrow{\overline{i}_t^*} H^n(\mathcal{Y}_t, \mathbb{C}) \to H^q(\mathcal{Y}_t, \Omega^p_{\mathcal{Y}_t}) \tag{5.4.1}$$

where $p + q = 2n$. Thus, we obtain:

The (p,q)-component group $H^n(\mathcal{Y}_t, \mathbb{C})^{\pi_1(C_0,t)}$ is the image of $H^q(\mathcal{Y}, \Omega^p_{\mathcal{Y}})$ under $r_t^{p,q}$.

Let $B \subset \mathcal{Y}$ be a $2n$-dimensional reduced non-singular algebraic cycle. Then,

$$\int_B \alpha \wedge \overline{\alpha} = \int_B \alpha|_B \wedge \overline{\alpha}|_B \, \forall \alpha \in H^q(\mathcal{Y}, \Omega^p_{\mathcal{Y}})$$

and that $\int_B \alpha \wedge \overline{\alpha} = 0$ is equivalent to $\alpha|_B = 0$. For all $\alpha \in H^q(\mathcal{Y}, \Omega^p_{\mathcal{Y}})$, we have

$$\alpha|_{D_j} = 0 \, \forall j \Longleftrightarrow \int_{Y_0} \alpha \wedge \overline{\alpha} = 0$$

where $Y_0 = \pi^{-1}(c_0) = \sum D_j$ is a semistable singular fiber. On the other hand, because all smooth fibers are homological equivalent to Y_0 we have

$$\int_{Y_0} (\alpha \wedge \overline{\alpha})|_{Y_0} = \int_{\mathcal{Y}_t} (\alpha \wedge \overline{\alpha})|_{\mathcal{Y}_t} \ \forall t \in C_0.$$

Hence, if $H^{p,q}(D_j) = 0 \ \forall j$, then $\alpha|_{\mathcal{Y}_t} = 0 \ \forall \alpha \in H^q(\mathcal{Y}, \Omega^p_{\mathcal{Y}})$, i.e., the restriction map

$$r_t^{p,q} : H^q(\mathcal{Y}, \Omega^p_{\mathcal{Y}}) \to H^q(\mathcal{Y}_t, \Omega^p_{\mathcal{Y}_t}) \ \forall t \in C_0$$

is a zero map.

Now, π is *strongly degenerate* at c_0, the cohomology of each component of Y_0 has only pure Hodge type. In particular,

$$H^{n-1,n+1}(D_j) = 0 \, \forall j.$$

Therefore, the restriction map

$$r_t^{n-1,n+1} : H^{n+1}(\mathcal{Y}, \Omega^{n-1}_{\mathcal{Y}}) \to H^{n+1}(\mathcal{Y}_t, \Omega^{n-1}_{\mathcal{Y}_t}) \, \forall t \in C_0$$

has to be zero, i.e.,

$$(R^{n+1}\pi^0_*(\Omega^{n-1}_{\mathcal{Y}_0/C_0}))^{\pi_1(C_0)} = 0.$$

q.e.d.

From the proof the theorem, we actually obtain a stronger property: *A non-trivial deformation of a family gives arise to a non-trivial Betti-cohomology class in the cohomology group of the total space of the self-product of the original family.* This class is very special. It is not of Hodge type and does not vanish along fibers. Thus, we have shown that there are relations between the geometry of total spaces of families and the *rigidity*.

Corollary 5.5 (Weakly Arakelov theorem). *Let $f : \mathcal{X} \to C$ be a semistable family over a smooth projective curve C with at least one singular fiber, which is a normal crossing divisor and each of its components is dominated by a projective space. Then, f is rigid.*

Remark. Our result is weaker than the original Arakelov theorem, but it is for arbitrary higher dimensional varieties.

Example 5.6. Let F be a smooth homogenous polynomial with degree d. Any one parameter family in $\mathbb{P}^n$ of type

$$F(X_0, \cdots, X_n) + t \prod_{i=1}^{d} X_{\tau(i)} = 0$$

is rigid, where $\tau : \{1, \cdots, d\} \to \{1, \cdots, n\}$ is an injective map.

6. Yukawa couplings and rigidity

Yukawa couplings. It is better to understand more about the endomorphism σ in section 3 which has deep background in string theory. Let σ^l be the l-iterated of σ-operator on E, then $\sigma^l \equiv 0$ for $l >> 0$.

Proposition 6.1. *Let $f^0 : \mathcal{X}_0 \to Y_0$ be a smooth family of polarized Calabi–Yau n-folds satisfying the $(**)$-condition. Then, if f is non-rigid*

$$\sigma^n \equiv 0.$$

Proof. Otherwise, we have a non-zero flat endomorphism σ^n. Denote $L := f^0_* \Omega^n_{\mathcal{X}_0/Y_0}$. The endomorphism σ^n in fact is a global holomorphic section of the line bundle $(L^*)^{\otimes 2}$ (under the holomorphic structure $\overline{\partial}_E$ of the induced Higgs bundle), then $(L^*)^{\otimes 2}$ is trivial. But it is impossible by the results of 2.11 and 2.12. q.e.d.

Definition 6.2. Let $g : \mathcal{X} \to Y$ be a smooth family of Calabi–Yau n-folds over a quasi-projective manifold Y and (E, θ) be the Higgs bundle induced from the VHS $R^n_{prim} g_*(\mathbb{C})$. The *Yukawa coupling* is just the n-iterated Higgs field $\theta^n : E \to E \otimes \mathrm{Sym}^n \Omega^1_Y$.

The definition is a little different from classic literatures, but they are compatible with each other. As the Higgs bundle (E, θ) can be splitting into

$$(E, \theta) = (\bigoplus_{p+q=n} E^{p,q}, \bigoplus \theta^{p,q})$$

with $\theta^{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_Y$, the n-iterated Higgs field

$$\theta^{n,0} \circ \cdots \circ \theta^{0,n} : E^{n,0} \longrightarrow E^{0,n} \otimes (\otimes^n \Omega^1_Y)$$

can factor through $\theta^n : E^{n,0} \longrightarrow E^{0,n} \otimes \mathrm{Sym}^n \Omega^1_Y$ because $\theta \wedge \theta = 0$. So, we can formulate the *Yukawa coupling* as

$$\theta^n : \mathrm{Sym}^n \Theta_Y \to \mathcal{H}om(E^{n,0}, E^{0,n}) = ((R^0 g_* \Omega^n_{\mathcal{X}/Y})^*)^{\otimes 2}.$$

Assume that Y has a smooth compactification $\overline{Y}$ such that $D_\infty = \overline{Y} \setminus Y$ is a normal crossing divisor and the Higgs bundle (E, θ) over Y has regular singularities at D_∞. Then, (E, θ) is algebraic and the *Yukawa coupling* is a global algebraic section of

$$((R^0 g_*(\Omega^1_{\mathcal{X}/Y}))^*)^{\otimes 2} \otimes \mathrm{Sym}^n \Omega^1_Y.$$

Let $f^0 : \mathfrak{X}_0 \to C_0$ be a non-isotrivial smooth family over a smooth quasi-projective curve $C_0 = C \setminus \{s_1, \cdots, s_k\}$. Suppose that f^0 is non-rigid. Then, we have a deformation $F : \mathcal{X} \to C_0 \times T$ of f^0 where $\mathcal{X}|_{\{C_0\}\times 0} = \mathfrak{X}_0$, and a non-zero σ on the induced Higgs bundle over C^0. By 3.4, all restricted families

$$F_s : \mathcal{X}|_{\{s\}\times T} \to \{s\} \times T \ \forall s \in C_0$$

satisfy the $(**)$-condition, and by 6.1 there are endomorphisms

$$\sigma'_s : E|_{\{s\}\times T} \to E|_{\{s\}\times T} \text{ (maybe zero)}$$

with $(\sigma'_s)^n \equiv 0 \ \forall s \in C_0$. On the other hand, we have Higgs maps

$$\theta_t : E|_{C_0 \times \{t\}} \to E|_{C_0 \times \{t\}} \otimes \Omega^1_{C_0}$$

along C_0. Because $\sigma'_s|_t$ comes from $\theta_t|_s$ for each (s,t), we also can obtain σ'_s by fixing $s \in C_0$ and varying $t \in T$. Altogether, we have another proof of the criterion of Viehweg–Zuo and Liu–Todorov–Yau–Zuo from the construction of σ.

Proposition 6.3 ([**43**, **45**],[**24**])**.** *Let $f^0 : \mathfrak{X}_0 \to C_0$ be a smooth family of Calabi–Yau n-folds. If the Yukawa coupling is not zero at some points, then the family f^0 must be rigid.*

Remarks. That the *Yukawa coupling* is non-zero at some points implies that the family f satisfies the $(**)$-condition, so f is automatically non-isotrivial. Denote $\mathcal{Z} = \{s \in C_0 \mid \theta^n_s = 0\}$. $\mathcal{Z}$ is either a set of finite points or the total C_0. The criterion actually says that if $\mathcal{Z}$ is a finite set, then f is rigid. We shall point out that the converse statement is not always true: Using the covering trick, Viehweg–Zuo recently constructed some rigid families of Calabi–Yau varieties with the *Yukawa coupling* identifying with zero (cf. [**47**]).

We would like to introduce a stronger result of Viehweg–Zuo, they dealt with more general cases: not only for Calabi–Yau manifolds but also for projective manifolds with semi-ample canonical line bundle or minimal models of general type.

Criterion 6.4. [**43**, Corollary 6.5] or [**45**, Corollary 8.4]. Let h be a fixed Hilbert polynomial of degree n and $\mathfrak{M}_h$ be a coarse moduli space of polarized n-folds with semi-ample canonical line bundle (including Calabi–Yau manifolds) or of general type. Assume $\mathfrak{M}_h$ has a nice compactification and carries a universal family $\pi : \mathfrak{X} \to \mathfrak{M}_h$ (In the real situation, one needs to work on stacks). Let $k_{\mathfrak{M}}$ be the largest integer such that the $k_{\mathfrak{M}_h}$-times iterated Kodaira–Spencer map of this universal family is not zero (obviously, $1 \leq k_{\mathfrak{M}_h} \leq n$). If the $k_{\mathfrak{M}_h}$-times iterated Kodaira–Spencer map for a family $f : X \to Y$ is not zero, then the family f must be rigid.

For higher dimensional base, one always considers the following condition (good partial compactification cf. [**43**]): Let U be a manifold and Y be a smooth projective compactification of U with a reduced normal crossing divisor boundary $D_\infty = Y \setminus U$. Starting with a smooth family $f : V \to U$, we first choose a smooth projective compactification X of V, such that $f : V \to U$ extends to $f : X \to Y$. Then, one leaves out codimension 2 subschemes of Y such that the restricted morphism is flat (cf. [**43**, **45**]).

A theorem of Landman–Katz–Borel says that local monodromies for the VHS $R^n f_* \mathbb{Q}_V$ around D_∞ are all quasi-unipotent (cf. [**31**]), so we have Deligne's quasi-canonical extension of the VHS, i.e., that the real part of the eigenvalues of the residues around the components of S lies in $[0, 1)$). Taking grading of the filtration, we get the quasi-canonical extension of the Higgs bundle

$$\Big(\bigoplus_{p+q=n} \overline{E}^{p,q}, \ \bigoplus \overline{\theta}^{p,q} \Big)$$

where $\overline{\theta}^{p,q} : \overline{E}^{p,q} \to \overline{E}^{p-1,q+1} \otimes \Omega^1_Y(\log S)$. We still call

$$\overline{\theta}^n : \overline{E}^{n,0} \longrightarrow \overline{E}^{0,n} \otimes \mathrm{Sym}^n \Omega^1_Y(\log S)$$

the *Yukawa coupling*. If there is no confusion, we also formulate the *Yukawa coupling* as

$$\overline{\theta}^n : \mathrm{Sym}^n(T_Y(-\log S)) \to \overline{E}^{0,n} \otimes (\overline{E}^{n,0})^\vee.$$

One has $\overline{E}^{p,q} = R^q f_* \Omega^p_{\mathcal{X}/Y}(\log \Delta)$ and

$$\Omega^n_{\mathcal{X}/Y}(\log \Delta) = \omega_{\mathcal{X}/Y}(\Delta_{\mathrm{red}} - \Delta)$$

where $\Delta = f^*S$ (cf. [**13**]). Denote $\mathcal{L} := \Omega^n_{\mathcal{X}/Y}(\log\Delta)$. The *Yukawa coupling* is

$$\overline{\theta}^n : R^0 f_*\Omega^n_{\mathcal{X}/Y}(\log\Delta) \longrightarrow R^n f_*\mathcal{O}_{\mathcal{X}} \otimes \mathrm{Sym}^n\Omega^1_Y(\log S),$$

or it is

$$\overline{\theta}^n : \mathrm{Sym}^n T^1_Y(-\log S) \longrightarrow R^n f_*\mathcal{O}_{\mathcal{X}} \otimes (f_*\mathcal{L})^{-1}.$$

If Δ is reduced, then $\mathcal{L} = \omega_{X/Y}$ and the *Yukawa coupling* then is

$$\overline{\theta}^n : \mathrm{Sym}^n T^1_Y(-\log S) \longrightarrow R^n f_*\mathcal{O}_{\mathcal{X}} \otimes (f_*\omega_{X/Y})^{-1}.$$

We have a generalization of 6.3 from the Viehweg–Zuo criterion.

Theorem 6.5. *Under assumptions made above, let $f : \mathcal{X} \to Y$ be a family of n-dimensional projective varieties with a general fiber having semi-ample canonical line bundle or being of general type. If the Yukawa coupling $\overline{\theta}^n(f)$ is not zero, then the family f must be rigid.*

Proof. First, we have the tautological sequence

$$0 \longrightarrow f^*\Omega^1_Y(\log S) \longrightarrow \Omega^1_{\mathcal{X}}(\log\Delta) \longrightarrow \Omega^1_{\mathcal{X}/Y}(\log\Delta) \longrightarrow 0,$$

and the wedge product sequences

$$\begin{aligned}(6.5.1)\quad 0 \longrightarrow f^*\Omega^1_Y(\log S) \otimes \Omega^{p-1}_{\mathcal{X}/Y}(\log\Delta) \longrightarrow \\ \mathfrak{gr}(\Omega^p_{\mathcal{X}}(\log\Delta)) \longrightarrow \Omega^p_{\mathcal{X}/Y}(\log\Delta) \longrightarrow 0\end{aligned}$$

where

$$\mathfrak{gr}(\Omega^p_{\mathcal{X}}(\log\Delta)) = \Omega^p_{\mathcal{X}}(\log\Delta)/f^*\Omega^2_Y(\log S) \otimes \Omega^{p-2}_{\mathcal{X}/Y}(\log\Delta).$$

We study various sheaves

$$F^{p,q} = R^q f_*(\Omega^p_{\mathcal{X}/Y}(\log\Delta) \otimes \mathcal{L}^{-1})$$

for $\mathcal{L} = \Omega^n_{\mathcal{X}/Y}(\log\Delta)$, together with edge morphisms

$$\tau_{p,q} : F^{p,q} \to F^{p-1,q+1} \otimes \Omega^1_Y(\log S)\, \forall (p,q) \text{ with } p+q=n$$

induced by the exact sequences (6.5.1), tensored with $\mathcal{L}^{-1}$. Explained in [**43**, 4.4 iii],

$$R^q f_*(\wedge^{n-p} T_{\mathcal{X}/Y}(-\log\Delta)) = R^q f_*(\Omega^p_{\mathcal{X}/Y}(\log\Delta) \otimes \mathcal{L}^{-1})$$

for $\forall (p,q)$ with $p+q=n$, and all $F^{p,q}$ are indeed from the deformations of the family $f : \mathcal{X} \to Y$.

Over $U = Y \setminus S$, edge morphisms $\tau_{p,q}$ also can be obtained in the following way. Consider the exact sequence

$$0 \to T_{V/U} \to T_V \to f^*T_U \to 0.$$

It induces sequences

$$0 \longrightarrow \bigwedge^{n-p+1} T_{V/U} \longrightarrow \tilde{T}_V^{n-p+1} \longrightarrow \bigwedge^{n-p} T_{V/U} \otimes f^*T_U \longrightarrow 0,$$

where $\tilde{T}_V^{n-p+1}$ is a subsheaf of $\bigwedge^{n-p+1} T_V$ for each p. The edge morphisms are

$$\tau_{p,q}^{\vee} : (R^q f_*(\wedge^{n-p} T_{V/U})) \otimes T_U \to R^{q+1} f_*(\wedge^{n-p+1} T_{V/U}),$$

they are the wedge product with the Kodaira–Spencer class. Tensoring with Ω_U^1, we get back $\tau_{p,q}|_U$. Thus,

$$\tau_{p,q} : F^{p,q} \to F^{p-1,q+1} \otimes \Omega_Y^1(\log S) \,\forall (p,q)$$

are the *log Kodaira–Spencer maps*, and

$$\begin{aligned} \tau^m : F^{n,0} = \mathcal{O}_Y &\xrightarrow{\tau_{n,0}} F^{n-1,1} \otimes \Omega_Y^1(\log S) \\ &\xrightarrow{\tau_{n-1,1}} F^{n-2,2} \otimes \mathrm{Sym}^2(\Omega_Y^1(\log S)) \longrightarrow \\ &\cdots \xrightarrow{\tau_{n-m+1,m-1}} F^{n-m,m} \otimes \mathrm{Sym}^m(\Omega_Y^1(\log S)) \end{aligned}$$

is the *m-times iterated log Kodaira–Spencer class.* Finally, we have a factor map by

$$\begin{array}{ccc} & \mathrm{Sym}^n(T_Y(-\log S)) & \\ {}^{\tau^n(f)}\swarrow & & \searrow{}^{\overline{\theta}^n(f)} \\ R^n f_*(\mathcal{L}^{-1}) & \xrightarrow{\qquad\qquad} & R^n f_*\mathcal{O}_{\mathcal{X}} \otimes (f_*\mathcal{L})^{-1} \end{array}$$

and

$$\overline{\theta}^n(f) \neq 0 \Longrightarrow \tau^n(f) \neq 0 \Longrightarrow k_{\mathfrak{M}} = n \Longrightarrow \tau_{\mathfrak{M}}^{k_{\mathfrak{M}}} \neq 0.$$

Hence, the family f must be rigid. q.e.d.

Remark. Except if $\mathrm{rank}\overline{E}^{n,0} = 1$, it is difficult to prove a similar result only by regarding variations of Hodge structures. So, one should study geometric deformations more precisely.

Example 6.6. Let F, H be two homogenous polynomials of degree $n+2$ such that F defines a non-singular hypersurface in $\mathbb{P}^{n+1}$ and H is not in the Jacobian ideal of F. Consider a special family $F(t) = F + tH, t \in \mathbb{P}^1$ which satisfies that H^n is not in the Jacobian ideal of

$F + \mu H$ for some $\mu \in \mathbb{P}^1$. Then, the *Yukawa coupling* at $\mu \in \mathbb{P}^1$ is not zero (cf. [**24**]), and the family is rigid.

Residues of Higgs fields. Let $f : \mathfrak{X} \to C$ be a family smooth over C_0. Let $(\mathcal{V}, \nabla)$ be a locally free sheaf with the Gauss–Manin connection given by the weight n polarized VHS $R^n_{prim} f_*(\mathbb{Q}_{X_0})$ where $\mathfrak{X}_0 = f^{-1}(C_0)$. As all monodromies for the VHS are quasi-unipotent, one has Deligne's quasi-canonical extension $(\overline{\mathcal{V}}, \overline{\nabla})$ over C with $\overline{\nabla} : \overline{\mathcal{V}} \to \overline{\mathcal{V}} \otimes \Omega^1_C(\log S)$. $(\mathcal{V}, \nabla)$ corresponds to a regular filtered Higgs bundle $\{(E, \theta)_\alpha\}$. For convenience, we restrict f to a unit disk Δ and study the local degenerated family $f_{loc} : \mathfrak{X}_{loc} \to (\Delta, t)$ which is smooth over $\Delta^* = \Delta - \{0\}$. $\overline{\mathcal{V}}|_0$ can be represented by $\mathcal{V}|_{\Delta^*}$ and the local monodromy T extends naturally to be an endomorphism on $\overline{E}$. $T_0 = T|_{\mathcal{V}|_0} : \mathcal{V}|_0 \to \mathcal{V}|_0$ is the restriction endomorphism (cf. [**34**]).

Definition 6.7 (Residues of $(\overline{\mathcal{V}}, \overline{\nabla})$ [**36**])**.** For an integrable logarithmic connection $\overline{\nabla} : \overline{\mathcal{V}} \to \overline{\mathcal{V}} \otimes \Omega^1_\Delta(\log 0)$, one has a composite map

$$(id_{\overline{\mathcal{V}}} \otimes R_0) \circ \overline{\nabla} : \overline{\mathcal{V}} \xrightarrow{\overline{\nabla}} \overline{\mathcal{V}} \otimes \Omega^1_\Delta(\log 0) \xrightarrow{id_{\overline{\mathcal{V}}} \otimes R_0} \overline{\mathcal{V}}|_0 \ ,$$

where $R_0 : \Omega^1_\Delta(\log 0) \to \mathbb{C}$ is defined by $R_0(hdt/t) = h(0)$. The composite map is zero at $t\overline{\mathcal{V}}$, then it defines the *residue morphism*

$$\mathrm{Res}_0(\overline{\nabla}) : \overline{\mathcal{V}}|_0 \to \overline{\mathcal{V}}|_0.$$

The local monodromy T around 0 is called *unipotent* if $(T-1)^{k+1} = 0$ and $(T-1)^k \neq 0$ for a fixed integer $k \in [0, n]$. If $k = n$, the monodromy T is called *maximal unipotent* (cf. [**8**]).

Suppose that T is *unipotent.* Let

$$N := \log T = \sum_{j=1}^{k} (1/j)(-1)^j (T - \mathrm{Id})^j.$$

Then, one has $N^{k+1} = 0$ and $N^k \neq 0$. The canonical extension $(\overline{\mathcal{V}}, \overline{\nabla})$ is generated by all sections

$$\widetilde{v}(t) = \exp\left(\frac{-\log t}{2\pi\sqrt{-1}} N\right) \cdot v$$

where v is a flat section (multiplicative values) of $\mathcal{V}$ over Δ^*. Moreover,

$$\widetilde{v}(te^{2\pi\sqrt{-1}}) = \widetilde{v}(t).$$

Lemma 6.8. [**5**, Theorem II 3.11] *Let $(\mathcal{V}, \nabla)$ be a locally free sheaf with the Gauss–Manin connection over Δ^* given by a polarized VHS with unipotent monodromy T. Then,*

$$\mathrm{Res}_0(\overline{\nabla}) = \frac{-1}{2\pi\sqrt{-1}} \log T = \frac{-1}{2\pi\sqrt{-1}} N.$$

Definition 6.9 (Residues of Higgs fields [**34**])**.** Let (V, H, D) be a *tame harmonic bundle* over Δ^* and $\{(E, \theta)_\alpha\}$ be the induced regular filtered Higgs sheaf on Δ. Denote $\overline{E} = \cup E_\alpha$ and $\overline{\theta} = \cup \theta_\alpha$. The composite map

$$(id_{\overline{E}} \otimes R_0) \circ \overline{\theta} : \overline{E} \xrightarrow{\overline{\theta}} \overline{E} \otimes \Omega^1_\Delta(\log 0) \xrightarrow{id_{\overline{E}} \otimes R_0} \overline{E}|_0$$

induces the residue $\mathrm{Res}_0(\overline{\theta}) : \overline{E}|_0 \to \overline{E}|_0$. Precisely, the residue is from

$$0 \longrightarrow \Omega^1_\Delta \otimes \mathcal{E}nd(\overline{E}) \longrightarrow \Omega^1_\Delta(\log 0) \otimes \mathcal{E}nd(\overline{E}) \xrightarrow{R_0 \otimes id} \mathcal{E}nd(\overline{E})|_0 \longrightarrow 0.$$

Remark. If a Higgs bundle (E, θ) is induced from a polarized VHS, $\mathrm{Res}_0(\overline{\theta})$ is automatically nilpotent.

Proposition 6.10 (Schmid–Simpson [**31**][**34**])**.** *Under the isomorphism* $\overline{E}|_0 \cong \overline{\mathcal{V}}|_0$, *the nilpotent part of* $\mathrm{Res}_0(\overline{\nabla})$ *is isomorphic to the nilpotent part of* $\mathrm{Res}_0(\overline{\theta})$.

The proof is simple if the local monodromy T around 0 of the VHS $(\mathcal{V}, \nabla)$ over Δ^* is *unipotent*: The filtration of extensive Higgs bundles is simple and jumps only at $\alpha = 0$, so one has

$$\mathrm{Res}_0(\overline{\theta}) \cong \mathrm{Res}_0(\overline{\nabla}) = \frac{-1}{2\pi\sqrt{-1}} \log T.$$

Theorem 6.11. *Let $f : \mathfrak{X} \to C$ be a family of n-dimensional Calabi–Yau varieties. If f admits a degeneration with maximal unipotent monodromy, then the family f must be rigid.*

Proof. By 6.8 and 6.10, the *Yukawa coupling* is non-zero if the local family $f_{loc} : \mathfrak{X}_{loc} \to \Delta$ is degenerate at 0 with a *maximal unipotent monodromy* (cf. [**48**]). q.e.d.

Example 6.12 (cf. [**23**])**.** Lian-Todorov-Yau recently studied a family of Calabi–Yau varieties in $\mathbb{P}^{n+k}$ with $n \geq 4, k \geq 1$ determined by

$$G_{1,t} = tF_1 - \prod_{i=0}^{n_1} x_i = 0, \ldots, G_{k,t} = tF_k - \prod_{j=n_1+..+n_{k-1}}^{n_k} x_j = 0$$

for $t \in \mathbb{P}^1$ and $[x_0, \cdots, x_{n+k}] \in \mathbb{P}^{n+k}$, where the system

$$F_1 = \ldots = F_k = 0 \text{ with } n_i = \deg F_i \geq 2 \text{ and } \sum n_i = n + k + 1$$

defines a smooth Calabi–Yau variety. This family must admit a degeneration with *maximal unipotent monodromy*, hence it is a rigid family.

Example 6.13 (cf. [8])**.** As a special case of Lian-Todorov-Yau, Morrison showed that the *Yukawa coupling* is non-zero at 0 for a family defined by

$$X_0^5+X_1^5+X_2^5+X_3^5+X_4^5-5\lambda X_0X_1X_2X_3X_4=0, [X_0,X_1,X_2,X_3,X_4]\in\mathbb{P}^4$$

where $\lambda\in\mathbb{P}^1$ is a parameter. By 6.12, this family admits a degeneration with *maximal unipotent monodromy* at 0. On the other hand, this family is rigid also by 6.6, because $(X_0X_1X_2X_3X_4)^3$ is not in the Jacobian ideal of $X_0^5+X_1^5+X_2^5+X_3^5+X_4^5$.

Remark. The two families in examples 6.12 and 6.13 are all *strongly degenerated* (after a semistable reduction), so that their rigidity also follows from 5.4.

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