J. DIFFERENTIAL GEOMETRY
68 (2004) 73-98

A GEOMETRIC ANALOGUE OF THE BIRCH AND SWINNERTON–DYER CONJECTURE OVER THE COMPLEX NUMBER FIELD

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Abstract

We will define a Ruelle–Selberg type zeta function for a certain lomathcal system over a Riemann surface whose genus is greater than or equal to three. Also, we will investigate its property, especially their special values. As an application, we will show that a geometric analogue of BSD conjecture is true for a family of abelian varieties which has only semi-stable reductions defined over the complex number field.

1. Introduction

Suppose we are given an abelian variety A defined over a number field K. Then, it is associated to an L-function $L_{A/K}(s)$ which absolutely converges on $\mathrm{Re}\, s > (3/2)$. It is conjectured that $L_{A/K}(s)$ can be analytically continued to an entire function throughout the s-plane. Moreover, the Birch and Swinnerton–Dyer conjecture predicts that its order of zero at $s = 1$ is equal to the rank of the Mordell–Weil group of A over K [**2**], [**3**]. In the following, we will abbreviate the Birch and Swinnerton–Dyer conjecture to the BSD conjecture.

Artin and Tate considered a geometric analogue of the BSD conjecture over a finite field [**15**], [**16**]. Let X be a smooth projective surface over a finite field which has an elliptic fibration $X \xrightarrow{f} S$ on a complete smooth curve S. Suppose that the moduli of $X \xrightarrow{f} S$ is not a constant. Using the Frobenius action on $H^1(\bar{X}_s, \mathbf{Q}_l)$, they associate to it an L-function $L_{X/S}(s)$, which is an analogue of $L_{A/K}(s)$. Here, X_s is a special fibre at $s \in S$ and $\bar{X}_s$ is the base extension of X_s to the separable closure $k(s)^{\mathrm{sep}}$ of the residue field $k(s)$. Artin and Tate conjectured that its order of zero at $s = 1$ is equal to the rank of the Mordell–Weil group of X/S. Moreover, Tate has shown their conjecture is equivalent to the

Received 10/10/2003.

statement that the l-primary part $Br(X)(l)$ of the Brauer group of X is finite [16].

We want to consider a geometric analogue of the BSD conjecture over $\mathbf{C}$.

Let

$$X \xrightarrow{f} S$$

be a commutative group scheme defined over a smooth projective curve whose generic fiber is an abelian variety of dimension d. Moreover, we assume the fibration satisfies all of the following conditions:

Condition 1.1.

(1) Let Σ be a subset of S where the fibration degenerates. The fibration is the Neron's model of the generic fibre which has a semi-stable reduction at each point of Σ.

(2) We set
$$S_0 = S \setminus \Sigma.$$
Then the Euler–Poincaré characteristic of S_0 is negative.

(3) There is a discrete subgroup Γ in $SL_2(\mathbf{R})$ such that $-1_2 \notin \Gamma$ and $S_0 = \Gamma \backslash \mathcal{H}$. Let us fix a base point x_0 of S_0 and we will identify $\pi_1(S_0, x_0)$ and Γ.

(4) We have a monodromy representation
$$\Gamma \simeq \pi_1(S_0, x_0) \xrightarrow{\rho_X} \mathrm{Aut}(V), \quad V = H^1(f^{-1}(x_0), \mathbf{R}).$$
Then, there is a positive constant α and C such that
$$|\mathrm{Tr}\rho_X(\gamma)| \le Ce^{\alpha l(\gamma)}$$
is satisfied for any hyperbolic $\gamma \in \Gamma$.

(1) The moduli of the fibration is not a constant. More precisely, it satisfies
$$H^0(S, R^1 f_* \mathcal{O}_X) = 0.$$

By the monodromy theorem [11], (1) implies Γ has no elliptic element. The Conditions (3) and (4) are not so restrictive. For example, if necessary taking a subgroup of finite index, the Condition (3) will be always satisfied. Also, it is easy to see that the Condition (4) is satisfied if the monodromy representation is a restriction of an algebraic group homomorphism from $SL_2(\mathbf{R})$ to $GL_{2d}(\mathbf{R})$ to Γ.

In order to define the Selberg and the Ruelle zeta functions of the fibration, we fix our notations.

Let Γ^*_{conj} be the set of non-trivial conjugacy classes of Γ and let $\Gamma^*_{h,\mathrm{conj}}$ be its subset consisting of hyperbolic conjugacy classes. There is a natural bijection between $\Gamma^*_{h,\mathrm{conj}}$ and the set of non-trivial closed geodesics

$\pi(M)^*$, and we will identify them. Then, $\gamma \in \Gamma^*_{h,\mathrm{conj}}$ is uniquely written as

$$\gamma = \gamma_0^{\mu(\gamma)},$$

where γ_0 is a primitive closed geodesic and $\mu(\gamma)$ is a positive integer, which will be called the multiplicity of γ. The subset of $\Gamma^*_{h,\mathrm{conj}}$ consisting of primitive closed geodesics will be denoted by $\Gamma^*_{\mathrm{pr,conj}}$. The length $l(\gamma)$ of $\gamma \in \Gamma^*_{h,\mathrm{conj}}$ is defined to be the length of the corresponding closed geodesic. Finally, we set

$$D(\gamma) = e^{\frac{1}{2}l(\gamma)} - e^{-\frac{1}{2}l(\gamma)}.$$

Now the Selberg zeta function $\zeta_{S,f}(s)$ of the fibration is defined to be

$$\zeta_{S,f}(s) = \exp\left(-\sum_{\gamma\in\Gamma^*_{h,\mathrm{conj}}} \frac{2\mathrm{Tr}\rho_X(\gamma)}{D(\gamma)\mu(\gamma)} e^{-sl(\gamma)}\right)$$

and we define the Ruelle zeta function $\zeta_{R,f}(s)$ to be

$$\zeta_{R,f}(s) = \frac{\zeta_{S,f}(s-\frac{1}{2})}{\zeta_{S,f}(s+\frac{1}{2})}.$$

It is easy to see that $\zeta_{S,f}(s)$ absolutely converges on $\{s \in \mathbf{C} \mid \mathrm{Re}\, s > (1/2)+\alpha\}$ and we will show it can be meromorphically continued throughout the whole plane. Also, it will be shown that $\zeta_{S,f}(s)$ (resp. $\zeta_{R,f}(s)$) is regular at $s = 0$ (resp. $s = 1/2$). Our interest is $\mathrm{ord}_{s=0}\zeta_{S,f}(s)$ and $\mathrm{ord}_{s=(1/2)}\zeta_{R,f}(s)$.

Theorem 1.2. *Let $X(S)$ be the Mordell–Weil group of the fibration. Then we have*

$$2\dim_{\mathbf{Q}} X(S)\otimes\mathbf{Q} \le \mathrm{ord}_{s=0}\zeta_{S,f}(s) = \mathrm{ord}_{s=\frac{1}{2}}\zeta_{R,f}(s).$$

Moreover, suppose $H^2(X, \mathcal{O}_X) = 0$. Then, we have equality in the above formula.

We will show that the Ruelle zeta function has an Euler product:

$$\zeta_{R,f}(s) = c_0 \prod_{\gamma_0\in\Gamma^*_{\mathrm{pr,conj}}} (\det[1_{2d} - \rho_X(\gamma_0)e^{-sl(\gamma_0)}])^2,$$

where c_0 is a certain constant and 1_{2d} be the $2d \times 2d$ identity matrix. Let f be a meromorphic function on a domain and let m be a positive integer. We will say that f has a virtual zero (resp. a virtual pole) at $a \in \mathbf{C}$ of order m if its logarithmic derivative is meromorphically continued throughout the plane and has a simple pole of residue m (resp. $-m$) at a. Now, Theorem 1.2 implies the following:

Theorem 1.3. *(A geometric analogue of the BSD conjecture over* **C***) The Euler product*

$$L_{X\backslash S}(s) = \prod_{\gamma_0 \in \Gamma^*_{\mathrm{pr,conj}}} \det[1_{2d} - \rho_X(\gamma_0) e^{-sl(\gamma_0)}]$$

virtually has a zero at $s = 1/2$ *whose order is greater than or equal to the rank of the Mordell–Weil group. Moreover, if* $H^2(X, \mathcal{O}_X) = 0$*, then they are equal.*

We will show that the condition $H^2(X, \mathcal{O}_X) = 0$ corresponds to the finiteness of l-part of the Brauer group of Artin's theorem.

The author appreciates the referee who pointed out mistakes in the first draft.

2. A formula for the Laplacian

To begin with, we will fix our notations. Let G be $SL_2(\mathbf{R})$ and let $\mathbf{g}$ be its Lie algebra. We set

$$R = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \qquad L = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \qquad H = \frac{1}{i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$X_0 = \frac{1}{2}H, \qquad X_1 = \frac{1}{2i}(R - L), \qquad X_2 = \frac{1}{2}(R + L).$$

They satisfy relations

$$[H, R] = 2R, \qquad [H, L] = -2L, \qquad [R, L] = H \tag{1}$$

and

$$[X_1, X_2] = \frac{1}{i}X_0, \qquad X_1^2 + X_2^2 = \frac{1}{2}(RL + LR).$$

Note that $\{iH, X_1, X_2\}$ forms an orthogonal basis of $\mathbf{g}$ with respect to the Killing form. For reals x, θ and for a positive y, we set

$$n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad a(y) = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix}.$$

According to the Iwasawa decomposition $G = NAK$, any element g of G is uniquely written as

$$g = n(x(g))a(y(g))k(\theta(g)),$$

where $x(g) \in \mathbf{R}$, $y(g) > 0$, and $\theta(g) \in [0, 2\pi)$. We consider G endowed with a coordinate by this parametrization and we normalize our Haar measure dg of G to be

$$dg = \frac{dx\,dy}{y^2}d\theta,$$

which is used in [4] and [8]. For an element X of $\mathbf{g}_{\mathbf{C}} = \mathbf{g} \otimes_{\mathbf{R}} \mathbf{C}$, we define its (right) Lie derivation $r(X)$ on $C^\infty(G)$ by

$$(r(X)f)(g) = \frac{d}{dt} f(g\exp(tX)) \mid_{t=0}, \quad f \in C^\infty(G).$$

For example, we have

$$r(X_0) = \frac{1}{2i}\frac{d}{d\theta}.$$

Let Ω be the Casimir element of $\mathbf{g}$:

$$\Omega = \frac{1}{4}(H^2 + 2RL + 2LR) = X_0^2 + X_1^2 + X_2^2.$$

It is known that the center of the universal enveloping algebra $u(\mathbf{g}_{\mathbf{C}})$ of $\mathbf{g}_{\mathbf{C}}$ is generated by Ω. Let $\mathbf{k}$ be the Lie subalgebra of $\mathbf{g}$ corresponding to the maximal compact subgroup $K(= U(1))$ of G and let

$$\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$$

be the Cartan decomposition. Then, $\mathbf{p}$ is identified with the tangent plane of the Poincaré upper half plane $\mathcal{H}$ at i and $\{X_1, X_2\}$ forms an orthogonal basis of $\mathbf{p}$. Here, we always consider $\mathcal{H}$ is given the metric of the constant curvature $\equiv -1$. Let $\{\omega_1, \omega_2\}$ be its dual basis.

For an integer m, let $\mathbf{C}(m)$ be a unitary representation of the maximal compact subgroup $K = U(1)$ of G whose action is given by

$$\rho_m(k(\theta))\mathbf{1}_m = e^{im\theta}\mathbf{1}_m,$$

where $\mathbf{1}_m$ is a base of $\mathbf{C}(m)$. Note that K acts on $\mathbf{p}_{\mathbf{C}} = \mathbf{p} \otimes \mathbf{C}$ and its dual space $\mathbf{p}_{\mathbf{C}}^*$ by the adjoint action and they have an irreducible decomposition as a K-module,

$$\mathbf{p}_{\mathbf{C}} \simeq \mathbf{p}_{\mathbf{C}}^* \simeq \mathbf{C}(-2) \oplus \mathbf{C}(2). \tag{2}$$

Note that ρ_m induces a homogeneous complex line bundle $\mathcal{L}_m$ over $\mathcal{H}$,

$$\mathcal{L}_m = G \times_{(K,\rho_m)} \mathbf{C}(m).$$

Then, the space of smooth sections of $\mathcal{L}_m$ (resp. $\mathcal{L}_m \otimes_{\mathbf{C}} (T^*\mathcal{H})_{\mathbf{C}}$, $\mathcal{L}_m \otimes_{\mathbf{C}} \wedge^2(T^*\mathcal{H})_{\mathbf{C}}$, where $\cdot_{\mathbf{C}}$ denotes the complexification) is naturally identified with the K-invariant part $(C^\infty(G) \otimes \mathbf{C}(m))^K$ of $C^\infty(G) \otimes \mathbf{C}(m)$ (resp. $(C^\infty(G) \otimes \mathbf{p}_{\mathbf{C}}^* \otimes \mathbf{C}(m))^K$, $(C^\infty(G) \otimes \wedge^2\mathbf{p}_{\mathbf{C}}^* \otimes \mathbf{C}(m))^K$). By this identification, the negative Hodge Laplacian

$$\Delta = -(d\delta + \delta d)$$

acts on the latter spaces, where δ is the formal adjoint of d. In order to write down this action, we need some notations.

By definition, we have

$$R = X_2 + iX_1, \qquad L = X_2 - iX_1.$$

Hence, $\{R_0 = \frac{1}{\sqrt{2}}R,\ L_0 = \frac{1}{\sqrt{2}}L\}$ forms a unitary basis of $\mathbf{g}_{\mathbf{C}}$. Note that they satisfy

$$X_1^2 + X_2^2 = R_0 L_0 + L_0 R_0.$$

Let $\{\xi_1, \xi_2\}$ be its dual basis. According to (1), we have

$$\mathrm{Ad}(k(\theta))\xi_1 = e^{-2i\theta}, \qquad \mathrm{Ad}(k(\theta))\xi_2 = e^{2i\theta}. \tag{3}$$

Hence, (2) is nothing but the obvious identity

$$\mathbf{p}_{\mathbf{C}}^* = \mathbf{C}\xi_1 \oplus \mathbf{C}\xi_2.$$

Now, taking account of identities

$$X_1 \cdot \mathbf{1}_m = X_2 \cdot \mathbf{1}_m = 0,$$

the following proposition will be obtained by a simple computation.

Proposition 2.1. *Let F_1, F_2 and f be smooth functions on G. Suppose each $\alpha = f \otimes \mathbf{1}_m$, $\beta = F_1 \otimes \mathbf{1}_m \otimes \xi_1 + F_2 \otimes \mathbf{1}_m \otimes \xi_2$ and $\gamma = f \otimes \mathbf{1}_m \otimes (\xi_1 \wedge \xi_2)$ is K-invariant. Then, we have*

(1)

$$\Delta(\alpha) = \left(r(\Omega) - \frac{m^2}{4} \right) f \otimes \mathbf{1}_m.$$

(2)

$$\begin{aligned}\Delta(\beta) = &\left(r(\Omega) - \left(1 - \frac{m}{2}\right)^2 + \left(1 - \frac{m}{2}\right) \right) F_1 \otimes \mathbf{1}_m \otimes \xi_1 \\ &+ \left(r(\Omega) - \left(1 + \frac{m}{2}\right)^2 + \left(1 + \frac{m}{2}\right) \right) F_2 \otimes \mathbf{1}_m \otimes \xi_2.\end{aligned}$$

(3)

$$\Delta(\gamma) = \left(r(\Omega) - \frac{m^2}{4} \right) f \otimes \mathbf{1}_m \otimes (\xi_1 \wedge \xi_2).$$

Example 2.2. Let α, β, and γ be the same as above. Suppose that $m = \pm 1$. Then, we have

(1)

$$\Delta(\alpha) = \left(r(\Omega) - \frac{1}{4} \right) f \otimes \mathbf{1}_m, \quad m = \pm 1.$$

(2)

(a)

$$\Delta(\beta) = \left(r(\Omega) + \frac{1}{4} \right) F_1 \otimes \mathbf{1}_m \otimes \xi_1 + \left(r(\Omega) - \frac{3}{4} \right) F_2 \otimes \mathbf{1}_m \otimes \xi_2, \quad m = 1.$$

(b)

$$\Delta(\beta)=\left(r(\Omega)-\frac{3}{4}\right)F_1\otimes\mathbf{1}_m\otimes\xi_1+\left(r(\Omega)+\frac{1}{4}\right)F_2\otimes\mathbf{1}_m\otimes\xi_2,\quad m=-1.$$

(3)

$$\Delta(\gamma)=\left(r(\Omega)-\frac{1}{4}\right)f\otimes\mathbf{1}_m\otimes(\xi_1\wedge\xi_2),\quad m=\pm 1.$$

3. Hodge decomposition of locally homogeneous vector bundles

Let Γ be a discrete subgroup of G such that the volume of $M=\Gamma\backslash\mathcal{H}$ is finite and that the Euler–Poincaré characteristic $\chi(M)$ is negative. We always assume that Γ has no elliptic element and $-1_2\notin\Gamma$. It is known that every hyperbolic element $\gamma\in G$ is conjugate to an element of a form

$$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}^{m(\gamma)}\begin{pmatrix}\sqrt{y_\gamma}&0\\0&\sqrt{y_\gamma}^{-1}\end{pmatrix},$$

in G, where $m(\gamma)\in\{0,1\}$ and y_γ is a positive. We will determine $m(\gamma)$ for $\gamma\in\Gamma^*_{h,conj}$ after Fried [6]. Note that

$$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}^{m(\gamma)}$$

is the holonomy when one parallel transports a normal vector around the closed geodesic corresponding to γ. But since M is oriented, $m(\gamma)$ should be equal to 0.

Let V be a $2d$ dimensional vector space over $\mathbf{R}$ such that $V_{\mathbf{C}}=V\otimes_{\mathbf{R}}\mathbf{C}$ has a decomposition as a K-module

$$V_{\mathbf{C}}=V_{\mathbf{C}}^{1,0}\oplus V_{\mathbf{C}}^{0,1},\qquad V_{\mathbf{C}}^{1,0}=\mathbf{C}(1)^{\oplus d},\qquad V_{\mathbf{C}}^{0,1}=\mathbf{C}(-1)^{\oplus d}.$$

Suppose V has a Γ-action ρ. Then, the complexification $\mathcal{V}_{\mathbf{C}}$ of the vector bundle

$$\mathcal{V}=\mathcal{H}\times_{(\Gamma,\rho)}V$$

on M and its tensor product with the cotangent bundle $(T^*M\otimes\mathcal{V})_{\mathbf{C}}$ or the vector bundle of two forms $(\wedge^2T^*M\otimes\mathcal{V})_{\mathbf{C}}$ have a Hodge decomposition in Zucker's sense, which we will recall [20].

Since the cotangent bundle $T^*\mathcal{H}$ of $\mathcal{H}$ is equal to $G\times_{(K,\mathrm{Ad})}\mathbf{p}^*$, its complexification has a decomposition

$$(T^*\mathcal{H})_{\mathbf{C}}=G\times_K\mathbf{p}_{\mathbf{C}}^{1,0}\oplus G\times_K\mathbf{p}_{\mathbf{C}}^{0,1},$$

by (2), where $\mathbf{p}_{\mathbf{C}}^{1,0}=\mathbf{C}(2)$ and $\mathbf{p}_{\mathbf{C}}^{0,1}=\mathbf{C}(-2)$. $G\times_K\mathbf{p}_{\mathbf{C}}^{1,0}$ (resp. $G\times_K\mathbf{p}_{\mathbf{C}}^{0,1}$) will be denoted by $(T^*\mathcal{H})_{\mathbf{C}}^{1,0}$ (resp. $(T^*\mathcal{H})_{\mathbf{C}}^{0,1}$) and they are pushed down to subbundles $(T^*M)^{1.0}_{\mathbf{C}}$ and $(T^*M)^{0.1}_{\mathbf{C}}$ of $(T^*M)_{\mathbf{C}}$ respectively. On the other hand, a homogeneous vector bundle on $\mathcal{H}$

$$\tilde{\mathcal{V}}_{\mathbf{C}} = G \times_K V_{\mathbf{C}}$$

descends to $\mathcal{V}_{\mathbf{C}}$ and the image of

$$\tilde{\mathcal{V}}^{1.0}_{\mathbf{C}} = G \times_K V^{1.0}_{\mathbf{C}}, \qquad \tilde{\mathcal{V}}^{0.1}_{\mathbf{C}} = G \times_K V^{0.1}_{\mathbf{C}}$$

will be denoted by $\mathcal{V}^{1.0}_{\mathbf{C}}$ and $\mathcal{V}^{0.1}_{\mathbf{C}}$, respectively. Now, the (p,q)-*part* $(T^*M \otimes \mathcal{V})^{p.q}_{\mathbf{C}}$ of $(T^*M \otimes \mathcal{V})_{\mathbf{C}}$ is defined to be

$$(T^*M \otimes \mathcal{V})^{p.q}_{\mathbf{C}} = \oplus_{a+c=p,b+d=q}(T^*M)^{a.b}_{\mathbf{C}} \otimes \mathcal{V}^{c.d}_{\mathbf{C}}.$$

Note that this is nothing but the descent of

$$\tilde{\mathcal{W}}^{p.q} = G \times_K W^{p.q}, \quad W^{p.q} = \oplus_{a+c=p,b+d=q}(\mathbf{p}^{a.b}_{\mathbf{C}} \otimes V^{c.d}_{\mathbf{C}}).$$

Since d-copies of $\{\mathbf{1}_1 \otimes \xi_1, \mathbf{1}_{-1} \otimes \xi_2\}$ (resp. $\{\mathbf{1}_1 \otimes \xi_2, \mathbf{1}_{-1} \otimes \xi_1\}$) forms a basis of $W^{1.1}$ (resp. $W^{2.0} \oplus W^{0.2}$), Example 2.2 implies the following formulae.

Proposition 3.1. *Let $f \otimes w \in (C^\infty \otimes W^{p.q})^K$.*

(1) *If $(p.q) = (1.1)$,*

$$\Delta(f \otimes w) = \left(\left(r(\Omega) + \frac{1}{4}\right) f\right) \otimes w.$$

(2) *If $(p.q) = (2.0)$, or (0.2),*

$$\Delta(f \otimes w) = \left(\left(r(\Omega) - \frac{3}{4}\right) f\right) \otimes w.$$

Proposition 3.2. (1) *If $f \otimes v \in (C^\infty \otimes V_{\mathbf{C}})^K$,*

$$\Delta(f \otimes v) = \left(\left(r(\Omega) - \frac{1}{4}\right) f\right) \otimes v.$$

(2) *If $f \otimes v \in (C^\infty \otimes V_{\mathbf{C}} \otimes \wedge^2 T^*(M))^K$,*

$$\Delta(f \otimes v \otimes (\xi_1 \wedge \xi_1)) = \left(\left(r(\Omega) - \frac{1}{4}\right) f\right) \otimes v \otimes (\xi_1 \wedge \xi_1).$$

4. The heat kernel of the positive Hodge Laplacian

In the sequel to the paper, we shall only treat the vector bundle $(T^*M \otimes \mathcal{V})^{1.1}_{\mathbf{C}}$. Let μ be the action of K on $W^{1.1} \simeq (\mathbf{C}(1) \oplus \mathbf{C}(-1))^{\oplus d}$. As we have seen in the previous section, this is the descent of the homogeneous vector bundle

$$\tilde{\mathcal{W}}^{1.1} = G \times_{(K,\mu)} W^{1.1}$$

on $\mathcal{H}$ to M. After Barbasch and Moscovici [1], we will regard $C^\infty(G)\otimes \mathrm{End}(W^{1.1})$ as $K\times K$-module via the action

$$(\rho(k_1,k_2)f\otimes A)(g) = f(k_1gk_2)\otimes \mu(k_1)^{-1}A\mu(k_2)^{-1},$$

where k_i(resp. g) is an element of K(resp. G) and $f\otimes A\in C^\infty(G)\otimes \mathrm{End}(W^{1.1})$. So, $h\in C^\infty(G)\otimes \mathrm{End}(W^{1.1})$ is $K\times K$-invariant if and only if it satisfies the covariance property

$$h(k_1gk_2) = \mu(k_1)h(g)\mu(k_2). \tag{4}$$

Let $\Delta_H = -\Delta$ be the positive Hodge Laplacian and let Δ_G be the Laplacian of G

$$\Delta_G = -r(\Omega) + 2r(X_0)^2.$$

It is known that the heat kernel p_t of G is contained in $C^\infty(G)\cap L^1\cap L^2$ and that it satisfies

$$(e^{-t\Delta_G}u)(x) = \int_G p_t(x^{-1}y)u(y)dy, \quad u\in L^2(G)$$

for any $t>0$ [1]. We want to construct the heat kernel of Δ_H. Let R be the right regular representation of G and we set

$$Q_R = \int_K R(k)\otimes \mu(k)dk.$$

Then, Q_R is a projection from $C^\infty(G)\otimes W^{1.1}$ to its K-invariant part $(C^\infty(G)\otimes W^{1.1})^K$. By Proposition 3.1(1), Δ_H satisfies

$$\Delta_H = Q_R\circ(\Delta_G\otimes \mathrm{Id}_{W^{1.1}})\circ Q_R - \frac{3}{4},$$

and the kernel function $h_t^H (t>0)$ of $e^{-t\Delta_H}$ is given by

$$h_t^H(x) = e^{\frac{3}{4}t}\int_K dk_1\mu(k_1)^{-1}\int_K p_t(k_1xk_2)\mu(k_2)^{-1}dk_2.$$

Note that h_t^H is contained in $[(C^\infty(G)\cap L^1\cap L^2)\otimes \mathrm{End}(\mathrm{W}^{1.1})]^{K\times K}$ and since K is commutative, it satisfies

$$\mu(k)h_t^H(x) = h_t^H(x)\mu(k), \quad x\in G, k\in K. \tag{5}$$

The following lemma is an immediate consequence of Proposition 3.1.

Lemma 4.1. *Let π be an irreducible unitary representation of G and let $\mathcal{H}(\pi)$ be its representation space. Then*

$$\pi(h_t^H) = e^{t(\pi(\Omega)+\frac{1}{4})}\mathrm{Id}$$

on $(\mathcal{H}(\pi)\otimes W^{1.1})^K$.

Let π be an irreducible unitary representation of G. By the Frobenius reciprocity law [10], [18], we have $\pi(h_t^H) = [\pi|_K{:}\mu]e^{t(\pi(\Omega)+(1/4))}\mathrm{Id}_{W^{1.1}}$.

For example, let us take π to be the principal series $\pi_{\nu,1}$ of a parameter $(s,\epsilon) = ((1/2)+i\nu, 1)$ [4]. Using the explicit description of $\pi_{\nu,1}$, one obtains $[\pi_{\nu,1}|_K{:}\mu] = 1$ and $\pi_{\nu,1}(\Omega) = -((1/4)+\nu^2)$. Therefore, we have

$$\pi_{\nu,1}(h_t^H) = e^{-t\nu^2}\mathrm{Id}_{W^{1.1}}. \tag{6}$$

For $p > 0$, let $\mathcal{C}^p(G)$ be the Harish–Chandra's $\mathcal{C}^p$-space of G. Since it is known $h_t^H(t>0)$ is contained in $\mathcal{C}^p(G)\otimes\mathrm{End}(W^{1.1})$ [1], [7], we may apply it the Selberg trace formula. This will be the main object of the following sections.

5. The Selberg trace formula

For brevity, we set $\mathcal{W}^{p.q} = (T^*M\otimes\mathcal{V})^{p.q}_{\mathbf{C}}$ and let $L^2(M,\mathcal{W}^{p.q})$ be the space of square integrable sections with respect to the Poincaré metric. We will only treat the case of $(p.q) = (1.1)$.

There is an orthogonal decomposition

$$L^2(M,\mathcal{W}^{1.1}) = L^2(M,\mathcal{W}^{1.1})_{\mathrm{disc}} \oplus L^2(\mathcal{W}^{1.1})_{\mathrm{cont}},$$

according to the type of spectra of Δ_H. The trace

$$\mathrm{Tr}[e^{-t\Delta_H}|L^2(M,\mathcal{W}^{1.1})_{\mathrm{disc}}]$$

can be computed by the Selberg trace formula and consists of three main terms [1], [7], [8], [13],

$$\mathrm{Tr}[e^{-t\Delta_H}|L^2(M,\mathcal{W}^{1.1})_{\mathrm{disc}}] = I(t)+G(t)+P(t).$$

$I(t)$ (resp. $G(t)$) is the orbital integral over 1_2 (resp. hyperbolic orbits). $P(t)$ is the orbital integral on parabolic orbits minus a contribution of continuous spectra. Also, $G(t)$ will be concerned with the Ruelle and Selberg zeta functions. Each term will be explicitly computed in the later sections.

If 0 is a spectrum of Δ_H, it is known that it is a discrete one and $\ker[\Delta_H|L^2(M,\mathcal{W}^{1.1})]$ is a subspace of $L^2(M,\mathcal{W}^{1.1})_{\mathrm{disc}}$[12][20]. We set

$$b = \dim_{\mathbf{C}}\ker[\Delta_H|L^2(M,\mathcal{W}^{1.1})]$$

and let

$$0 = \lambda_1 = \cdots = \lambda_b < \lambda_{b+1} \le \lambda_{b+2} \le \cdots,$$

be the discrete spectra of Δ_H. The following proposition is a consequence of an easy computation.

Proposition 5.1. *The integral*

$$2s\int_0^\infty e^{-s^2t}\mathrm{Tr}[e^{-t\Delta_H}|L^2(M,\mathcal{W}^{1.1})_{\mathrm{disc}}]dt$$

exists for $-\frac{\pi}{4} < \arg s < \frac{\pi}{4}$ *and it can be meromorphically continued throughout the* s*-plane as*

$$\frac{2b}{s} + \sum_{n=b+1}^{\infty} \left(\frac{1}{s + \sqrt{\lambda_n} i} + \frac{1}{s - \sqrt{\lambda_n} i} \right).$$

6. Ruelle and Selberg zeta functions

Taking account of $m(\gamma) = 0$ for $\gamma \in \Gamma^*_{h,\mathrm{conj}}$ (cf. Section 3) and using the identity (6), the orbital integral $G(t)$ of h_t^H on hyperbolic orbits (cf. Section 5) can be computed in the same way as [6], [10], [19]:

$$G(t) = \sum_{\gamma \in \Gamma^*_{h,\mathrm{conj}}} \frac{l(\gamma_0) \mathrm{Tr} \rho(\gamma)}{\pi D(\gamma)} \int_{-\infty}^{\infty} e^{-t\nu^2 - il(\gamma)\nu} d\nu.$$

(Also, we have used the fact that the centralizer of

$$A = \left\{ \left(\begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right) \,\middle|\, a > 0 \right\}$$

in K is $\{\pm 1\}$.) Here, $D(\gamma)$ is defined to be

$$D(\gamma) = e^{\frac{1}{2} l(\gamma)} - e^{-\frac{1}{2} l(\gamma)}.$$

Inserting the formula

$$\int_{-\infty}^{\infty} e^{-t\nu^2 - il(\gamma)\nu} d\nu = \sqrt{\frac{\pi}{t}} e^{-\frac{l(\gamma)^2}{4t}},$$

to the above identity, we get the following proposition.

Proposition 6.1.

$$G(t) = \sum_{\gamma \in \Gamma^*_{h,\mathrm{conj}}} \frac{l(\gamma_0) \mathrm{Tr} \rho(\gamma)}{\sqrt{\pi t} D(\gamma)} e^{-\frac{l(\gamma)^2}{4t}}, \quad t > 0.$$

Now, we define our Selberg zeta function.

Definition 6.2. The Selberg zeta function $\zeta_S(s)$ is defined to be

$$\zeta_S(s) = \exp\left(- \sum_{\gamma \in \Gamma^*_{h,\mathrm{conj}}} \frac{2 \mathrm{Tr} \rho(\gamma)}{D(\gamma) \mu(\gamma)} e^{-sl(\gamma)} \right).$$

Let $Z_S(s)$ be its logarithmic derivative,

$$Z_S(s) = \frac{\zeta_S'(s)}{\zeta_S(s)}.$$

For the sake of convergence of the Selberg zeta function, we always assume ρ satisfies the following technical condition.

Condition 6.3. There exists positive constants C and α such that

$$|\mathrm{Tr}\rho(\gamma)| \leq Ce^{\alpha l(\gamma)}$$

is satisfied for any hyperbolic $\gamma \in \Gamma$.

Under the condition, it is easy to see $\zeta_S(s)$ absolutely converges for $\mathrm{Re}\, s > \alpha + (1/2)$ [9]. Note that $Z_S(s)$ has a form

$$Z_S(s) = \sum_{\gamma \in \Gamma^*_{h,\mathrm{conj}}} \frac{2l(\gamma_0)\mathrm{Tr}\rho(\gamma)}{D(\gamma)} e^{-sl(\gamma)}.$$

There is an intimate relation between $G(t)$ and $Z_S(s)$.

Proposition 6.4. *For* $s \in \mathbf{C}$ *with* $-\frac{\pi}{4} < \arg s < \frac{\pi}{4}$ *and* $\mathrm{Re}\, s$ *is sufficiently large, we have the identity*

$$2s \int_0^\infty e^{-s^2 t} G(t) dt = Z_S(s).$$

Proof. It is easy to see the integral converges for such s. Since both sides are analytic functions with respect to s, it is sufficient to check the identity for $s > 0$. Applying a change of variables

$$x = s\sqrt{t}$$

to the formula

$$\int_0^\infty e^{-(x^2 + \frac{a^2}{x^2})} dx = \frac{\sqrt{\pi}}{2} e^{2a}, \quad \mathrm{Re}\, a^2 > 0,$$

we get

$$\frac{1}{2} e^{-sl(\gamma)} = s \int_0^\infty \frac{1}{\sqrt{4\pi t}} e^{-s^2 t - \frac{l(\gamma)^2}{4t}} dt.$$

Combining this with Proposition 6.1, the assertion will be proved. q.e.d.

Definition 6.5. The Ruelle zeta function $\zeta_R(s)$ is defined to be

$$\zeta_R(s) = \frac{\zeta_S(s - \frac{1}{2})}{\zeta_S(s + \frac{1}{2})}.$$

In the later sections, we will show that both Selberg and Ruelle zeta functions can be meromorphically continued throughout the s-plane. Also, $\zeta_R(s)$ has *an Euler product*.

Proposition 6.6. *We have an identity*

$$\zeta_R(s) = c_0 \prod_{\gamma_0 \in \Gamma^*_{\mathrm{pr,conj}}} (\det[1_{2d} - \rho(\gamma_0) e^{-sl(\gamma_0)}])^2.$$

Proof. We will compare both sides taking their logarithmic derivatives. The logarithmic derivative of $\zeta_R(s)$ is

$$\begin{aligned}\frac{\zeta_R'(s)}{\zeta_R(s)} &= \sum_{\gamma\in\Gamma^*_{h,\mathrm{conj}}} \frac{2l(\gamma_0)\mathrm{Tr}\rho(\gamma)}{D(\gamma)}\left(e^{-(s-\frac{1}{2})l(\gamma)} - e^{-(s+\frac{1}{2})l(\gamma)}\right)\\ &= 2\sum_{\gamma\in\Gamma^*_{h,\mathrm{conj}}} l(\gamma_0)\mathrm{Tr}\rho(\gamma)e^{-sl(\gamma)}\\ &= 2\sum_{\gamma\in\Gamma^*_{\mathrm{pr},\mathrm{conj}}} l(\gamma_0)\sum_{k=1}^{\infty}\mathrm{Tr}\rho(\gamma_0^k)e^{-skl(\gamma_0)}.\end{aligned}$$

Note that the last equation is just the logarithmic derivative of the RHS of the required identity. q.e.d.

7. Orbital integrals

First of all, we will fix our notations.

According to the Iwasawa decomposition

$$G = NAK,$$

every element g of G may be uniquely written to be

$$g = n(g)a(g)k(g), \quad n(g)\in N, \quad a(g)\in A, \quad k(g)\in K.$$

We give a coordinate on A by

$$\mathbf{R}_{>0} = \{x\in\mathbf{R} \mid x>0\} \simeq A, \quad y \leftrightarrow \begin{pmatrix}\sqrt{y} & 0\\ 0 & \sqrt{y}^{-1},\end{pmatrix}$$

and we set

$$z(g) = n(g)a(g).$$

By the map

$$G \xrightarrow{\rho} \mathcal{H}, \quad \rho(g) = g\cdot i,$$

$z(g)$ may be considered as a point of $\mathcal{H}$. For example, $z(n(x)a(y))$ is identified with $z = x+iy$. For $s\in\mathbf{C}$ and an irreducible representation μ of K, a smooth function y^s_μ on G is defined to be

$$y^s_\mu(nak) = \mu(k)y^s(a).$$

Note that y^s_μ satisfies

$$y^s_\mu(gk) = \mu(k)y^s_\mu(g), \quad k\in K, \quad g\in G.$$

Every irreducible representation of K is parametrized by an integer (i.e., its weight), we will sometimes identify them. Let Σ be the parameter space of irreducible unitary representations of G:

- $\Sigma_{\mathrm{pr}} = \{\frac{1}{2}+i\nu \mid \nu\in\mathbf{R}\}$,

- $\Sigma_{\text{comp}} = \{s \in \mathbf{R} \mid 0 < s < 1\}$,
- For an irreducible representation μ of G, we set

$$\Sigma^{\mu}_{\text{disc}} = \left\{ \frac{k}{2} \mid k \in \mathbf{Z}, k \geq 1, k \equiv \mu \,(\text{mod}\, 2) \right\},$$

- $\Sigma_{\mu} = \Sigma_{\text{pr}} \cup \Sigma_{\text{comp}} \cup \Sigma^{\mu}_{disc}$.

Suppose we are given a $(\mathbf{g}, K)$-module whose K-type is μ. Then, we may choose its basis from $\{y^s_\mu\}_{s\in\Sigma_\mu}$. Let p^h_t be the heat kernel of the positive hyperbolic Laplacian

$$\Delta = -y^2 \left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} \right).$$

By the formula

$$\Delta y^s = s(1-s)y^s, \tag{7}$$

we have

$$(e^{-ts(1-s)}y^s)(z) = (e^{-t\Delta}y^s)(z) = \int_{\mathcal{H}} p^h_t(z, z')y^s(z')dz'.$$

Here, the integral is taken with respect to the Poincaré metric. Under the isomorphism as K-modules,

$$W^{1.1} \simeq \mathbf{C}(1)^{\oplus d} \oplus \mathbf{C}(-1)^{\oplus d},$$

a smooth function

$$Y^s_\alpha(g) = (\alpha_{-1}y^s_{-1}(g), \alpha_1 y^s_1(g)), \quad \alpha = (\alpha_{-1}, \alpha_1), \quad \alpha_{\pm 1} \in \mathbf{C}^{\oplus d}$$

may be considered as an element of $(C^\infty(G) \otimes W^{1.1})^K$. Using Example 2.2, Lemma 7.1 is easily seen by a simple computation.

Lemma 7.1.

(1) *Y^s_α is an eigenfunction of Δ_H whose eigenvalue is*

$$\lambda = s(1-s) - \frac{1}{4}.$$

(2) *For $t > 0$, we define a smooth function h_t on $G \times G$ as*

$$h_t(g, g') = e^{\frac{t}{4}} p^h_t(z(g),\, z(g')) \cdot \begin{pmatrix} \mu_{-1}(k(g))\mu_{-1}(k(g'))^{-1}1_d & 0 \\ 0 & \mu_1(k(g))\mu_1(k(g'))^{-1}1_d \end{pmatrix}.$$

Then, we have

$$\int_G h_t(g,\, g')Y^s_\alpha(g')dg' = e^{-t\lambda}Y^s_\alpha(g).$$

Corollary 7.2. *h_t is the kernel function of $e^{-t\Delta_H}$.*

Now, we will compute

$$2s\int_0^\infty e^{-s^2 t} I(t)dt,$$

where $I(t)$ is the orbital integral of the identity element:

$$I(t) = \mathrm{vol}\,(M)\mathrm{Tr}\, h_t(1_2,\, 1_2).$$

The Corollary 7.2 and Mckean's computation [9] show $I(t)$ is given by

$$I(t) = \frac{2d\mathrm{vol}\,(M)}{(4\pi t)^{\frac{3}{2}}}\int_0^\infty \frac{\nu e^{-\frac{\nu^2}{4t}}}{\sinh\frac{\nu}{2}} d\nu.$$

Also, he has shown

$$\int_0^\infty dt \frac{e^{-(\sigma(\sigma-1)+\frac{1}{4})t}}{(4\pi)^{\frac{1}{2}} t^{\frac{3}{2}}}\int_0^\infty \frac{\nu e^{-\frac{\nu^2}{4t}}}{\sinh\frac{\nu}{2}} d\nu = \sum_{n=0}^{\infty}\frac{1}{\sigma+n}, \quad \sigma\in\mathbf{C}.$$

Now, by the change of variables

$$s + \frac{1}{2} = \sigma,$$

and by the generalized Gauss–Bonnet formula [17]:

$$\mathrm{vol}\,(M) = -2\pi\chi(M),$$

we have

Theorem 7.3.

$$2s\int_0^\infty e^{-s^2 t} I(t)dt = -2sd\chi(M)\sum_{n=0}^{\infty}\frac{1}{s+\frac{1}{2}+n}.$$

Note that the poles of the integral are located on

$$\left\{-\frac{1}{2}-n\right\}_{n\in\mathbf{Z},n\geq 0},$$

and the residue at $s = -(1/2) - n$ is equal to $d(1+2n)\chi(M)$. In particular, they are all integers.

Next, we will compute the orbital integral over parabolic orbits. Let γ be a parabolic element which conjugate to $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$. The orbital integral associated to the conjugacy class of γ is defined to be [14]

$$P_t(\gamma) = \frac{1}{2\pi}\int_{AK}\mathrm{Tr}\, h_t(g,\, \gamma g) da\, dk.$$

Since γ is parabolic, $k(\gamma g) = k(g)$. Therefore, we have

$$h_t(g,\, \gamma g) = e^{\frac{t}{4}} p_t^h(z(g),\, z(\gamma g)) 1_{2d},$$

and in particular,

$$\operatorname{Tr} h_t(g,\, \gamma g) = 2d e^{\frac{t}{4}} p_t^h(z(g),\, z(\gamma g)).$$

It is known that p_t^h can be expressed by a smooth function k_t on $\mathbf{R}_{>0}$ [8]:

$$p_t^h(z,\, z') = k_t\left(\frac{|z - z'|^2}{yy'}\right), \quad z = x + iy, \quad z' = x' + iy'.$$

So, we have

$$P_t(\gamma) = 2d e^{\frac{t}{4}} \int_0^\infty k_t\left(\frac{m^2}{y^2}\right) \frac{dy}{y^2}.$$

For $Y > 0$, we set

$$P_t(\gamma)_Y = 2d e^{\frac{t}{4}} \int_0^Y k_t\left(\frac{m^2}{y^2}\right) \frac{dy}{y^2}.$$

The formula

$$\int_{-\infty}^\infty e^{-t\nu^2} d\nu = \sqrt{\frac{\pi}{t}}$$

and Kubota's computation [8] imply the following proposition.

Proposition 7.4. *Let h be the number of cusps for Γ. Then, we obtain*

$$\sum_\gamma P_t(\gamma)_Y$$
$$= dh\left(\frac{\log Y}{\sqrt{\pi t}} - \frac{1}{\pi}\int_{-\infty}^\infty e^{-t\nu^2} \frac{\Gamma'(1+i\nu)}{\Gamma(1+i\nu)} d\nu - \frac{\log 2}{\sqrt{\pi t}} + \frac{e^{\frac{t}{4}}}{2}\right) + o_Y(1),$$

where γ runs through conjugacy classes of parabolic elements and $\lim_{Y\to\infty} o_Y(1) = 0$.

Next, we will compute a contribution from the continuous spectra. Each of $L^2(M,\, \mathbf{C})$ and $L^2(M, \mathcal{W}^{1.1})$ is a G-module by the right regular representation. For an irreducible representation π of G, the isotypical component of $L^2(M,\, \mathbf{C})$ (resp. $L^2(M, \mathcal{W}^{1.1})$) will be denoted by $L^2(M,\, \mathbf{C})(\pi)$ (resp. $L^2(M, \mathcal{W}^{1.1})(\pi)$). It is known $L^2(M, \mathcal{W}^{1.1})_{\text{cont}}$ is the direct integral over principal series $\pi_\nu = \pi_{\nu,1}$ [12]:

$$L^2(M, \mathcal{W}^{1.1})_{\text{cont}} = \int_{\mathbf{R}}^{\oplus} L^2(M, \mathcal{W}^{1.1})(\pi_\nu) d\nu.$$

Let k_t^ν be the kernel function of $e^{-t\Delta}\mid_{L^2(M,\mathbf{C})(\pi_\nu)}$. Since we know

$$e^{-t\Delta}\mid_{L^2(M,\mathbf{C})(\pi_\nu)}= e^{-t(\frac{1}{4}+\nu^2)}\mathrm{Id},$$

by (8), the equation (6) implies the kernel function of $e^{-t\Delta_H}\mid_{L^2(M,\mathcal{W}^{1.1})(\pi_\nu)}$ is equal to $e^{\frac{t}{4}}k_t^\nu(z,\, z')1_{2d}$. Hence, we get

$$\mathrm{Tr}[e^{-t\Delta_H}\mid_{L^2(M,\mathcal{W}^{1.1})_{\mathrm{cont}}}] = 2de^{\frac{t}{4}}\int_{-\infty}^{\infty}d\nu\int_{D_\Gamma}k_t^\nu(z,\, z)dz,$$

where D_Γ is the fundamental domain for Γ. Let $\{\kappa_1,\ldots,\kappa_h\}$ be the cusps for Γ and let $E_j(z,\, s)$ be the Eisenstein series corresponding to κ_j. Since it is known that

$$k_t^\nu(z,\, z') = \frac{e^{-(\frac{1}{4}+\nu^2)t}}{4\pi}\sum_{j=1}^{h}E_j\left(z,\,\frac{1}{2}+i\nu\right)\overline{E_j\left(z',\,\frac{1}{2}+i\nu\right)},$$

we have

$$\begin{aligned}&\mathrm{Tr}[e^{-t\Delta_H}\mid_{L^2(M,\mathcal{W}^{1.1})_{\mathrm{cont}}}]\\ &\quad= \frac{d}{2\pi}\sum_{j=1}^{h}\int_{D_\Gamma}dz\int_{-\infty}^{\infty}e^{-t\nu^2}E_j\left(z,\,\frac{1}{2}+i\nu\right)\overline{E_j\left(z,\,\frac{1}{2}+i\nu\right)}d\nu.\end{aligned}$$

For $Y>0$, we set

$$D_\Gamma(Y) = \{z\in D_\Gamma \mid \mathrm{Im}\, z\le Y\}.$$

The following identity is proved in [8] Appendix:

$$\begin{aligned}&\frac{1}{4\pi}\sum_{j=1}^{h}\int_{D_\Gamma(Y)}dz\int_{-\infty}^{\infty}e^{-t(\frac{1}{4}+\nu^2)}E_j\left(z,\,\frac{1}{2}+i\nu\right)\overline{E_j\left(z,\,\frac{1}{2}+i\nu\right)}d\nu\\ &\quad= \frac{he^{\frac{-t}{4}}}{2\sqrt{\pi t}}\log Y - \frac{1}{4\pi}\int_{-\infty}^{\infty}e^{-t(\frac{1}{4}+\nu^2)}\frac{\Psi'(\frac{1}{2}+i\nu)}{\Psi(\frac{1}{2}+i\nu)}d\nu + \frac{1}{4}\mathrm{Tr}\Phi\left(\frac{1}{2}\right)+o_Y(1).\end{aligned}$$

We will explain the terminologies.

The constant term $a_{ij,0} = a_{ij,0}(y,s)$ of the Fourier expansion of $E_i(z,\, s)$ at κ_j can be written as

$$a_{ij,0} = \delta_{ij}y^s + \varphi_{ij,0}(s)y^{1-s},$$

where $\varphi_{ij,0}(s)$ is a meromorphic function on $\mathbf{C}$. We set

$$\varphi_{ij}(s) = \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)}\varphi_{ij,0}(s).$$

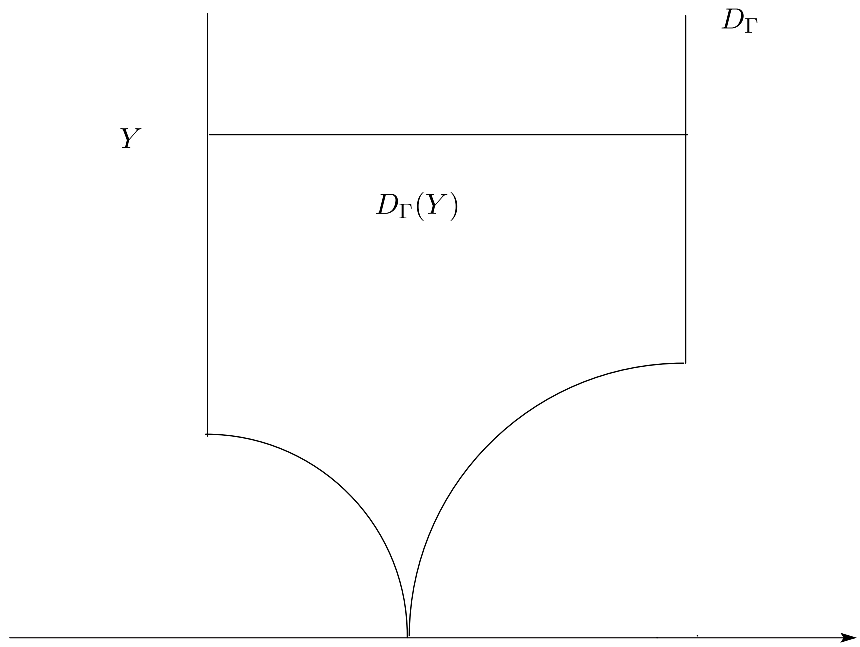

Figure 1.

and the matrix valued function $\Phi(s)$ is defined as

$$\Phi(s) = (\varphi_{ij}(s))_{1 \le i,j \le h}.$$

It is known that $\Phi(s)$ satisfies

$$\Phi(s)\Phi(1-s) = 1_h$$

and that $\Phi(s)$ is a unitary matrix for $s \in (1/2) + i\mathbf{R}$ [8]. In particular, $\Phi(1/2)$ is conjugate to

$$\begin{pmatrix} \epsilon_1 & 0 & \dots & 0 \\ 0 & \epsilon_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \dots & 0 & \epsilon_h \end{pmatrix}, \quad \epsilon_i \in \{\pm 1\}.$$

Let ν_1 (resp. ν_{-1}) be the cardinal of $\{\epsilon_i \mid \epsilon_i = 1\}$ (resp. $\{\epsilon_i \mid \epsilon_i = -1\}$). Then, we have

$$\frac{1}{2}\left(h - \operatorname{Tr}\Phi\left(\frac{1}{2}\right)\right) = \nu_{-1}.$$

$\Psi(s)$ is defined to be the determinant of $\Phi(s)$. It is known that $\Psi(1/2)+\nu)$ $(\nu \in \mathbf{R})$ satisfies the following properties [7].

Fact 7.5.

(1) $\Psi(\frac{1}{2}+\nu)\Psi(\frac{1}{2}-\nu)=1$.

(2) $\Psi(\frac{1}{2}+\nu)$ is a ratio of entire functions p and q of finite order,
$$\Psi\left(\frac{1}{2}+\nu\right)=\frac{p(\nu)}{q(\nu)}.$$

(3) We have
$$\frac{|\Psi'(\frac{1}{2}+\nu)|}{|\Psi(\frac{1}{2}+\nu)|}\leq \text{Const}\cdot(\log|\nu|)^2.$$
for $|\nu|\geq 2$.

(4) There exists an entire function $r(\nu)$ such that
$$p(\nu)=q(-\nu)e^{r(\nu)}.$$

(5) $\frac{\Psi'(\frac{1}{2}+\nu)}{\Psi(\frac{1}{2}+\nu)}$ is regular on the imaginary axis and its poles are located on $\{\pm q_k\}_k(\operatorname{Re} q_k>0)$. Moreover, their residues satisfy
$$b_k=\text{Res}_{\nu=q_k}\frac{\Psi'(\frac{1}{2}+\nu)}{\Psi(\frac{1}{2}+\nu)}=-\text{Res}_{\nu=-q_k}\frac{\Psi'(\frac{1}{2}+\nu)}{\Psi(\frac{1}{2}+\nu)}.$$

Now, $P(t)$ is defined to be
$$P(t)=\lim_{Y\to\infty}\left\{\sum_{\gamma}P_t(\gamma)_Y-\frac{d}{2\pi}\sum_{j=1}^{h}\int_{D_\Gamma(Y)}dz\right.$$
$$\left.\cdot\int_{-\infty}^{\infty}e^{-t\nu^2}E_j\left(z,\,\frac{1}{2}+i\nu\right)\overline{E_j\left(z,\,\frac{1}{2}+i\nu\right)}d\nu\right\},$$
which may be considered as
$$\sum_{\gamma}P_t(\gamma)-\text{Tr}[e^{-t\Delta_H}\,|_{L^2(M,\mathcal{W}^{1.1})_{\text{cont}}}].$$
Putting altogether, we have

Theorem 7.6.
$$P(t)=d\Bigg(\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-t\nu^2}\frac{\Psi'(\frac{1}{2}+i\nu)}{\Psi(\frac{1}{2}+i\nu)}d\nu$$
$$-\frac{h}{\pi}\int_{-\infty}^{\infty}e^{-t\nu^2}\frac{\Gamma'(1+i\nu)}{\Gamma(1+i\nu)}d\nu-\frac{h}{\sqrt{\pi t}}\log 2+\nu_{-1}e^{\frac{t}{4}}\Bigg).$$

Next, we will compute
$$2s\int_0^{\infty}e^{-s^2t}P(t)dt$$

for suitable $s \in \mathbf{C}$. First of all, the integral

$$2s \int_0^\infty e^{-(s^2-\frac{1}{4})t} dt$$

absolutely converges for $\operatorname{Re} s^2 > \frac{1}{4}$ and can be meromorphically continued throughout the whole plane as

$$\frac{1}{s+\frac{1}{2}} + \frac{1}{s-\frac{1}{2}}.$$

On the other hand, since we have

$$s \int_0^\infty \frac{e^{-s^2 t}}{\sqrt{t}} dt = \sqrt{\pi}$$

for $s > 0$, LHS may be continued over the whole plane as the constant function $\sqrt{\pi}$.

Proposition 7.7. *For $s \in \mathbf{C}$ with $-\frac{\pi}{4} < \arg s < \frac{\pi}{4}$, we have*

(1)

$$\frac{s}{\pi} \int_0^\infty dt\, e^{-s^2 t} \int_{-\infty}^\infty e^{-t\nu^2} \frac{\Gamma'(1+i\nu)}{\Gamma(1+i\nu)} d\nu = \frac{\Gamma'(1+s)}{\Gamma(1+s)},$$

(2)

$$\frac{s}{\pi} \int_0^\infty dt\, e^{-s^2 t} \int_{-\infty}^\infty e^{-t\nu^2} \frac{\Psi'(\frac{1}{2}+i\nu)}{\Psi(\frac{1}{2}+i\nu)} d\nu = \frac{\Psi'(\frac{1}{2}-s)}{\Psi(\frac{1}{2}-s)} - \sum_k b_k \left(\frac{1}{s+q_k} + \frac{1}{s-q_k} \right).$$

Proof. $\frac{\Gamma'(z)}{\Gamma(z)}$ has poles on non-positive integers and we obtain an estimate

$$\frac{|\Gamma'(1+i\nu)|}{|\Gamma(1+i\nu)|} \leq \text{Const} \cdot \log |\nu|, \quad |\nu| \geq 2.$$

For $s \in \mathbf{C}$ with $-\frac{\pi}{4} < \arg s < \frac{\pi}{4}$, a simple computation shows

(8)

$$s \int_0^\infty dt\, e^{-s^2 t} \int_{-\infty}^\infty e^{-t\nu^2} \frac{\Gamma'(1+i\nu)}{\Gamma(1+i\nu)} d\nu = s \int_{-\infty}^\infty \frac{1}{\nu^2+s^2} \frac{\Gamma'(1+i\nu)}{\Gamma(1+i\nu)} d\nu.$$

Note that the poles of $\frac{\Gamma'(1+i\nu)}{\Gamma(1+i\nu)}$ are $\nu = i, 2i, 3i, \ldots$ Using the estimate above, one may apply the residue theorem for the contour in Figure 2. Then, one will find

$$\int_{-\infty}^\infty \frac{s}{\nu^2+s^2} \frac{\Gamma'(1+i\nu)}{\Gamma(1+i\nu)} d\nu = \pi \frac{\Gamma'(1+s)}{\Gamma(1+s)}.$$

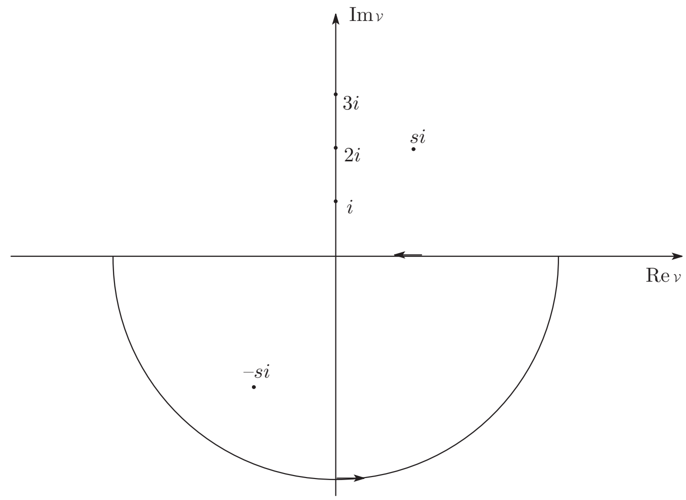

Figure 2.

Thus, (1) is proved. Taking account of Fact 7.5 (3) and (5), (2) will be proved by the same way. q.e.d.

We set

$$F(s) = \frac{\Psi'(\frac{1}{2} - s)}{\Psi(\frac{1}{2} - s)} - \sum_k b_k \left(\frac{1}{s + q_k} + \frac{1}{s - q_k} \right).$$

Then, Fact 7.5(5) implies the poles of $F(s)$ are located on $\{\pm q_k\}_k$ and each of them is simple. Moreover, we have

$$\mathrm{Res}_{s=q_k} F(s) = 0, \qquad \mathrm{Res}_{s=-q_k} F(s) = -2b_k.$$

Hence, $F(s)$ is regular on $\{s \in \mathbf{C} \mid \mathrm{Re}\, s \geq 0\}$. Combining altogether, we obtain the following theorem.

Theorem 7.8. *The integral $2s\int_0^\infty e^{-s^2t}P(t)dt$ can be meromorphically continued to the whole plane as*

$$\begin{aligned}2s\int_0^\infty e^{-s^2t}P(t)dt = d\Bigg(&\frac{\Psi'(\frac{1}{2}-s)}{\Psi(\frac{1}{2}-s)} - \sum_k b_k\left(\frac{1}{s+q_k}+\frac{1}{s-q_k}\right)\\ &+\nu_{-1}\left(\frac{1}{s+\frac{1}{2}}+\frac{1}{s-\frac{1}{2}}\right)\\ &-2h\frac{\Gamma'(1+s)}{\Gamma(1+s)}-2h\log 2\Bigg).\end{aligned}$$

It has only simple poles whose residues are integers. Moreover, it is regular at $s=0$.

Now, the analytic continuation of $Z_S(s)$ is proved.

Theorem 7.9. *$Z_S(s)$ can be meromorphically continued throughout the s-plane as*

$$\begin{aligned}Z_S(s) = \frac{2b}{s} &+ \sum_{n=b+1}^{\infty}\left(\frac{1}{s+\sqrt{\lambda_n}i}+\frac{1}{s-\sqrt{\lambda_n}i}\right)\\ &+2sd\chi(M)\sum_{n=0}^{\infty}\frac{1}{s+\frac{1}{2}+n} - d\Bigg(\frac{\Psi'(\frac{1}{2}-s)}{\Psi(\frac{1}{2}-s)}\\ &\quad-\sum_k b_k\left(\frac{1}{s+q_k}+\frac{1}{s-q_k}\right)+\nu_{-1}\left(\frac{1}{s+\frac{1}{2}}+\frac{1}{s-\frac{1}{2}}\right)\Bigg)\\ &+2dh\left(\frac{\Gamma'(1+s)}{\Gamma(1+s)}+\log 2\right).\end{aligned}$$

RHS has only simple poles and every residue is an integer. Its poles are located in $\{s\in\mathbf{C}\mid \operatorname{Re}s\le 0\}$ except for $s=1/2$. Moreover, we have

$$\mathrm{Res}_{s=0}Z_S(s) = 2b.$$

Since $Z_S(s)$ is the logarithmic derivative of $\zeta_S(s)$, Theorem 7.9 implies

Theorem 7.10. *$\zeta_S(s)$ can be meromorphically continued to the whole plane. Also, it satisfies*

$$\mathrm{ord}_{s=0}\zeta_S(s) = 2b.$$

By definition, the logarithmic derivative of $\zeta_R(s)$ is

$$\frac{\zeta_R'(s)}{\zeta_R(s)} = Z_S\left(s-\frac{1}{2}\right) - Z_S\left(s+\frac{1}{2}\right).$$

and Theorem 7.9 tells us that

$$\mathrm{Res}_{s=\frac{1}{2}} \frac{\zeta_R'(s)}{\zeta_R(s)} = 2b.$$

Theorem 7.11. *The Ruelle zeta function $\zeta_R(s)$ can be meromorphically continued throughout the s-plane and it has a zero at $s = 1/2$ of order $2b$.*

Taking account of Proposition 6.6, this implies the *logarithmic derivative* of the "Euler product"

$$\prod_{\gamma_0 \in \Gamma^*_{\mathrm{pr,conj}}} \det[1_{2d} - \rho(\gamma_0) e^{-sl(\gamma_0)}]$$

can be meromophically continued throughout the whole plane and its residue at $s = 1/2$ is equal to b.

8. A geometric application

Let X be a smooth quasi-projective variety with a fibration

$$X \xrightarrow{f} S$$

as in the introduction. We will use the notation of the Condition 1.1.

Let $\mathcal{V}_{\mathbf{C}}$ be the flat vector bundle over S_0 which associates to the locally constant sheaf $R^1 f_* \mathbf{C}$. $V_{\mathbf{C}} = V \otimes_{\mathbf{R}} \mathbf{C}$ has a direct decomposition

$$V_{\mathbf{C}} = (\mathbf{C}(1) \oplus \mathbf{C}(-1))^{\oplus d}$$

as a K-module and we may form a homogeneous vector bundle on $\mathcal{H}$

$$\tilde{\mathcal{V}}_{\mathbf{C}} = G \times_K V_{\mathbf{C}}.$$

Then $\mathcal{V}_{\mathbf{C}}$ is the descent of $\tilde{\mathcal{V}}_{\mathbf{C}}$ using the monodromy representation. Let $\mathcal{W}^{p.q}$ be the same as in Section 5. The Condition 1.1 (5) implies the rational Mordell–Weil group $X(S) \otimes \mathbf{Q}$ is of finite dimension and by the cycle map it may be considered as a subspace of $H^1(S_0, R^1 f_* \mathbf{Q})$. Note that the Hodge–Lefshetz theorem implies

$$X(S) \otimes \mathbf{Q} = H^1(S_0, R^1 f_* \mathbf{Q}) \cap \mathrm{Ker}[\Delta_H \mid L^2(S_0, \mathcal{W}^{1.1})].$$

Moreover, we have

$$H^1(S_0, R^1 f_* \mathbf{Q}) \cap \mathrm{Ker}[\Delta_H \mid L^2(S_0, \mathcal{W}^{2.0} \oplus \mathcal{W}^{0.2})] \subset H^{2.0}(X) \oplus H^{0.2}(X)$$

by the compatibility of the Hodge decomposition and the Leray spectral sequence [**5**], [**20**]. Thus, we obtain

Theorem 8.1. *The rank of the Mordell–Weil group is less than or equal to the dimension of* $\mathrm{Ker}[\Delta_H \mid L^2(S_0, \mathcal{W}^{1.1})]$. *Moreover, if $H^2(X, \mathcal{O}_X) = 0$, they are equal.*

We define the Selberg zeta function $\zeta_{S,f}(s)$ (resp. Ruelle zeta function $\zeta_{R,f}(s)$) of the fibration f to be one associated to ρ_X. Theorems 7.10, 7.11 and 8.1 imply a solution of geometric analogue of the BSD conjecture.

Theorem 8.2. *We have*

$$2\dim_{\mathbf{Q}} X(S)\otimes\mathbf{Q} \leq \mathrm{ord}_{s=0}\zeta_{S,f}(s) = \mathrm{ord}_{s=\frac{1}{2}}\zeta_{R,f}(s).$$

Moreover, if $H^2(X,\mathcal{O}_X)=0$, we have the equality in the above formula.

The Proposition 6.6 and the above theorem imply the following:

Theorem 8.3 (A geometric analogue of the BSD conjecture over $\mathbf{C}$)**.** *The Euler product*

$$L_{X\backslash S}(s) = \prod_{\gamma_0\in\Gamma^*_{\mathrm{pr,conj}}} \det[1_{2d} - \rho_X(\gamma_0)e^{-sl(\gamma_0)}]$$

has a virtual zero at $s=1/2$ whose order is greater than or equal to the rank of the Mordell–Weil group. Moreover, if $H^2(X,\mathcal{O}_X)=0$, they are equal.

Let us define the topological Brauer group $Br_{\mathrm{top}}(X)$ of X as

$$Br_{\mathrm{top}}(X) = H^2(X,\mathcal{O}_X^*),$$

where the cohomology is taken with respect to the classical topology.

Proposition 8.4. *The topological Brauer group of X is finitely generated if and only if $H^2(X,\mathcal{O}_X)$ vanishes.*

Proof. The exponential sequence

$$0\to\mathbf{Z}\to\mathcal{O}_X\to\mathcal{O}_X^*\to 0,$$

implies an exact sequence

$$H^2(X,\mathbf{Z})\to H^2(X,\mathcal{O}_X)\to Br_{\mathrm{top}}(X)\to H^3(X,\mathbf{Z}).$$

Since X is quasi-projective, both $H^2(X,\mathbf{Z})$ and $H^3(X,\mathbf{Z})$ are finitely generated and our assertion is clear. q.e.d.

Thus, we know that the condition $H^2(X,\mathcal{O}_X)=0$ corresponds to finiteness of l-part of the Brauer group of Artin's theorem.

References

[1] D. Barbasch & H. Moscovici, *L^2-index theorem and the Selberg trace formula*, J. Functional Anal. **53** (1983) 151–201, MR 0722507, Zbl 0537.58039.

[2] B.J. Birch & H.P.F. Swinnerton-Dyer, *Notes on elliptic curves* I, J. reine u. angewandte Math. **212** (1963) 7–25, MR 0179168, Zbl 0118.27601.

[3] B.J. Birch and H.P.F. Swinnerton-Dyer, *Notes on elliptic curves* II, J. reine u. angewandte Math. **218** (1965) 79–108, MR 0146143, Zbl 0147.02506.

[4] D. Bump. *Automorphic Forms and Representation*, Number 55 in Cambridge Stud. Adv. Math. Cambridge University Press, 1997, MR 1431508, Zbl 0868.11022.

[5] D.A. Cox & S. Zucker, *Intersection numbers of sections of elliptic surfaces*, Invent. math. **53** (1979) 1–44, MR 0538682, Zbl 0444.14004.

[6] D. Fried, *Analytic torsion and closed geodesics on hyperbolic manifolds*, Invent. Math. **84** (1986) 523–540, MR 0837526, Zbl 0621.53035.

[7] R. Gangolli & G. Warner, *Zeta functions of Selberg's type for some noncompact quotient of symmetric space of rank one*, Nagoya Math. J. **78** (1980) 1–44, MR 0571435.

[8] T. Kubota, *Elementary Theory of Eisenstein Series*, Kodansha, Tokyo, 1973, MR 0429749, Zbl 0268.10012.

[9] H.P. McKean. *Selberg's trace formula as applied to a compact Riemann surface*, Comm. Pure Appl. Math. **25** (1972) 225–246, MR 0473167, Zbl 0317.30018.

[10] J.J. Millson, *Closed geodesics and the η-invariant*, Ann. Math. **108** (1978) 1–39, MR 0501204, Zbl 0399.58020.

[11] D.R. Morrison, *The Clemens-Schmid exact sequence and applications*, in 'Topics in Transcendental Algebraic Geometry' (P. Griffith, ed.), Ann. Math. Stud., **106**, 101–119, Princeton University Press, 1984, MR 0756848, Zbl 0576.32034.

[12] W. Müller, *Manifolds with Cusps of Rank One*, **1244**, Springer-Verlag, 1987, MR 0891654, Zbl 0632.58001.

[13] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric spaces with application to Dirichlet series*, J. Indian Math. Soc. **20** (1956) 47–87, MR 0088511, Zbl 0072.08201.

[14] S. Shokranian, *The Selberg-Authur Trace Formula*, Lecture Notes in Math. **1503**, Springer-Verlag, 1992, MR 1176101, Zbl 0769.11026.

[15] J. Tate. *Algebraic cycles and poles of zeta functions*, in 'Proc. of the Purdue Conference on Arithmetic Algebraic Geometry', Harper and Row, 1964, MR 0225778, Zbl 0213.22804.

[16] J. Tate. *On the conjecture of Birch and Swinnerton-Dyer and a geometric analog*, in 'Seminaire Bourbaki', **306**, 1–26, 1966, MR 1610977, Zbl 0199.55604.

[17] G. van der Geer, *Hilbert Modular Surfaces*, A series of Modern Surveys in Mathematics, **16**, Springer-Verlag, 1988, MR 0930101, Zbl 0634.14022.

[18] N. Wallach. *An asymptotic formula of Gelfand and Gangolli for spectrum of $\Gamma\backslash G$*, J. Differential Geom. **11** (1976) 91–101, MR 0417340, Zbl 0341.43009.

[19] N. Wallach. *On the Selberg trace formula in the case of compact quotient*, Bull. AMS **82** (1976) 171–195, MR 0404533, Zbl 0350.22008.

[20] S. Zucker. *Hodge theory with degenerating coefficients*, Ann. Math. **109** (1979) 415–476, MR 0534758, Zbl 0446.14002.

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