

**THE TOPOLOGY, GEOMETRY
AND CONFORMAL STRUCTURE OF
PROPERLY EMBEDDED MINIMAL SURFACES**

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Abstract

This paper develops new tools for understanding surfaces with more than one end and infinite topology which are properly minimally embedded in Euclidean three-space. On such a surface, the set of ends forms a totally disconnected compact Hausdorff space, naturally ordered by the relative heights of the ends in space. One of our main results is that the middle ends of the surface have quadratic area growth, and are thus not limit ends. This implies that the surface can have at most two limit ends, which occur at the top and bottom of the ordering, and thus only a countable number of ends, which is a strong topological restriction. There are also restrictions on the asymptotic geometry and conformal structure of such a surface: for example, we prove that if the surface has exactly two limit ends, then it is recurrent (that is, almost all Brownian paths are dense in the surface), and in particular, any positive harmonic function on the surface is constant. These results have played an important role in several recent advances in the theory, including the uniqueness of the helicoid, the invariance of flux for a coordinate function on a properly immersed minimal surface, and the topological classification of properly embedded minimal surfaces of finite genus.

1. Introduction

Let \mathcal{M} denote the set of connected properly embedded minimal surfaces in \mathbb{R}^3 with at least two ends. At the beginning of the past decade,

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there were two outstanding conjectures on the asymptotic geometry of the ends of an $M \in \mathcal{M}$ that were known to lead to topological restrictions on M . The first of these conjectures, the generalized Nitsche conjecture, stated that an annular end of such a $M \in \mathcal{M}$ is asymptotic to a plane or to the end of a catenoid. Based on earlier work in [18], Collin [3] proved the generalized Nitsche conjecture. In the case $M \in \mathcal{M}$ has finite topology, the solution of this conjecture implies that M has finite total Gaussian curvature, which by previous work in [4, 8, 9, 21] led to topological obstructions for such a minimal surface M .

Our paper deals with the case where $M \in \mathcal{M}$ has infinite topology. Before stating the second conjecture, we recall some definitions. For any connected manifold M , an end of M is an equivalence class of proper arcs on M , where two such arcs are equivalent if for any compact domain D in M , the ends of these arcs are contained in the same non-compact component of $M - D$. The set \mathcal{E}_M of all the ends of M has a natural topology that makes \mathcal{E}_M into a compact Hausdorff space. The limit points in \mathcal{E}_M are by definition the *limit ends* of M ; an end $e \in \mathcal{E}_M$, which is not a limit end will be called a *simple end*. To every $M \in \mathcal{M}$ is associated a unique plane passing through the origin in \mathbb{R}^3 called the *limit tangent plane at infinity* of M (see [2]). The existence of such a limit plane at infinity depends strongly on the property that M has at least two ends. For convenience, we will always assume that the limit tangent plane at infinity is horizontal or, equivalently, is the x_1x_2 -plane P .

A result of Frohman and Meeks [6] states that the ends of M are linearly ordered by their relative heights over P . Furthermore, they prove that this linear ordering, up to reversing it, depends only on the proper ambient isotopy class of M in \mathbb{R}^3 . Since the space of ends \mathcal{E}_M is compact and the ordering is linear, for any $M \in \mathcal{M}$, there exists a unique *top end* which is the highest end in the ordering on \mathcal{E}_M . Similarly, the *bottom end* of M is defined to be the end of M , which is lowest in the associated ordering. The ends of M that are neither top nor bottom ends are called *middle ends* of M .

The second conjecture, motivated by analogy with the finite topology setting, asserted that the middle ends of an $M \in \mathcal{M}$ are simple ends which are C^0 -asymptotic to a plane or to an end of a catenoid. This conjecture was verified [7] for middle ends of finite genus, but remained open in the case of an infinite genus middle end, where it was further conjectured that the limit to a plane or a catenoid end must have finite integer multiplicity greater than one. One consequence of this second

conjecture is that the middle ends of $M \in \mathcal{M}$ can be represented by proper subdomains with compact boundary whose area in the ball B_R of radius R centered at the origin is approximately equal to $n\pi R^2$ for some integer n when R is large. (Recall that a proper subdomain $E \subset M$ with compact boundary is said to represent an end $e \in \mathcal{E}_M$ if E contains a proper arc representing e .)

In this paper, we will develop new fundamental theoretical tools for understanding the topology, asymptotic geometry and conformal structures of examples in \mathcal{M} . These tools are powerful enough to prove that the middle ends of an $M \in \mathcal{M}$ are simple ends and have quadratic area growth $n\pi R^2$. An important consequence of these methods is that the topology of examples in \mathcal{M} with an infinite number of ends is very restrictive.

Theorem 1.1. *If $M \in \mathcal{M}$, then a limit end of M is a top or bottom end. Thus, M has at most two limit ends, and in particular, M can have only a countable number of ends.*

Note that the above theorem gives strong topological restrictions that a properly embedded minimal surface with an infinite number of ends must satisfy. (For example, the plane with a Cantor set removed has an uncountable number of limit ends and so cannot properly minimally embed in \mathbb{R}^3 .) It is a consequence of the following geometric result on the middle ends.

Theorem 1.2. *Suppose $M \in \mathcal{M}$. For a middle end e of M , there is an associated positive integer multiplicity $n(e)$. The multiplicity $n(e)$ is defined by choosing a proper subdomain $E(e)$ with compact boundary that represents e such that the area $A(R) = \text{Area}(B_R \cap E(e))$ divided by πR^2 converges to $n(e)$ as $R \rightarrow \infty$. Furthermore, $E(e)$ can be chosen so that for any other representative $\widetilde{E}(e) \subset E(e)$ of $e \in \mathcal{E}_M$ the associated area function $\widetilde{A}(R)$ divided by πR^2 also converges to $n(e)$ as $R \rightarrow \infty$. In particular, a limit end of $M \in \mathcal{M}$ must be a top or a bottom end.*

Classical examples by Riemann [20] and more recent examples by Callahan, Hoffman and Meeks [1] demonstrate that there exist many 1-periodic examples in \mathcal{M} with two limit ends.

Theorem 1.2 is a crucial initial ingredient in the complete topological classification theorem [5] for properly embedded minimal surfaces in \mathbb{R}^3 . Specifically, $M_1, M_2 \in \mathcal{M}$ differ by a diffeomorphism of \mathbb{R}^3 if

and only if they have the same genus and, up to reversing the order of the ends, the corresponding ends have the same genus (either 0 or ∞) and the corresponding integer multiplicities of the middle ends given in the statement of Theorem 1.2 are the same modulo 2. Theorem 1.2 has also played an essential role in the recent classification of properly embedded non-simply connected periodic minimal surfaces of genus zero [12].

We will also show that some of the examples in \mathcal{M} have strong restrictions on their conformal structure as well. Recall that a Riemannian surface M is *recurrent* if almost all Brownian paths are dense in M . An important conformal property for recurrent Riemannian surfaces is that positive harmonic functions are constant.

Theorem 1.3. *If $M \in \mathcal{M}$ and M has two limit ends, then M is recurrent.*

Since triply-periodic minimal surfaces have one end and are never recurrent, some restriction on the number of ends is necessary for the conclusion of Theorem 1.3 to hold. Embeddedness is also a necessary hypothesis in Theorem 1.3, since at the end of Section 3, we will construct properly immersed minimal surfaces with two ends in \mathbb{R}^3 that have non-constant bounded harmonic functions. Indeed, it has been conjectured that if $M \in \mathcal{M}$, then M is recurrent [11].

The proof of Theorem 1.3 depends on a basic result (Theorem 3.1) on the conformal structure of a properly immersed minimal surface with boundary contained in a closed halfspace of \mathbb{R}^3 . Theorem 3.1 has played an important role in the proof of uniqueness of the helicoid [17] and also in the proof of the invariance of flux for a coordinate function of a properly immersed minimal surface (see [11]). Theorems 1.1–1.3 and 3.1 are also used in the topological and geometric classification of properly embedded minimal surfaces of finite genus in \mathbb{R}^3 (see [13]–[15]).

Our paper is organized as follows. In Section 2, we apply the classical Weierstrass representation of minimal surfaces to derive some special proper superharmonic functions defined on certain regions of a properly immersed minimal surface. Next, we use these special functions, together with the divergence theorem, to prove that middle ends of an $M \in \mathcal{M}$ have quadratic area growth. In Section 3, we again use these special functions to derive some of our basic theorems on conformal structure.

2. Quadratic area growth of middle ends

In the proof of the ordering theorem [6], one shows that every middle end of a surface $M \in \mathcal{M}$ is contained between two catenoids in the sense of the following definition.

Definition 2.1. Suppose M is a properly immersed minimal surface with compact boundary in \mathbb{R}^3 . We will say that M is contained between two catenoids if for some $c_1 > 0$, $M \subset \{(x_1, x_2, x_3) \mid |x_3| \leq c_1 \ln r, r^2 = x_1^2 + x_2^2, r \geq 2\}$.

Since the middle ends of a properly embedded minimal surface are contained between two catenoids, the following lemma implies that the middle ends of a properly embedded minimal surface are never limit ends.

Lemma 2.2. *If M is a properly immersed minimal surface with compact boundary and M is contained between two catenoids, then M has quadratic area growth. This means that the area of M in the ball $B_R = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 \leq R^2\}$ is at most $C\pi(R+1)^2$ for some positive C . Furthermore, such an M has at most C ends.*

The proof of Lemma 2.2 depends on a fundamental inequality given in Lemma 2.3.

Lemma 2.3. *Let M be a minimal surface and assume $r = \sqrt{x_1^2 + x_2^2} \neq 0$ on M . Then, $|\Delta_M \ln r| \leq |\nabla_M x_3|^2 / r^2$.*

Proof. Assume M is not a plane and note that the points where the gradient of x_3 is zero are isolated on M ; so it suffices to prove the inequality stated in the lemma on the complement of these horizontal points.

Let g denote the stereographic projection of the Gauss map of M to the extended complex plane $\mathbb{C} \cup \infty$. Then, by the classical Weierstrass representation, the coordinates $(x_1, x_2, x_3)(z)$ are given by

$$\operatorname{Re} \int \left(\frac{1}{2} \left(\frac{1}{g} - g \right), \frac{i}{2} \left(\frac{1}{g} + g \right), 1 \right) dz,$$

where $z = x_3 + ix_3^*$. We will let Δ_z denote the planar Laplacian in z -coordinates.

Then

$$(x_1 + ix_2)(z) = \overline{\int \frac{1}{2g} dz} - \int \frac{g}{2} dz = \bar{\xi} - \mu,$$

where $\xi' = \frac{1}{2g}$, $\mu' = \frac{g}{2}$, $\xi'\mu' = \frac{1}{4}$. Letting $\ln w = \ln r + i\theta$ for $x_1 + ix_2 = w = re^{i\theta}$, we have

$$\begin{aligned} \Delta_z \ln(x_1 + ix_2) &= 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} \ln(\bar{\xi} - \mu) = 4 \frac{\partial}{\partial \bar{z}} \left(\frac{-\mu'}{\bar{\xi} - \mu} \right) \\ &= \frac{4\mu'\bar{\xi}'}{(\bar{\xi} - \mu)^2} \\ &= \Delta_z \ln r + i \Delta_z \theta. \end{aligned}$$

Thus, $\Delta_z \ln r = \operatorname{Re} \left(4 \frac{(\mu'\bar{\xi}')}{(\bar{\xi} - \mu)^2} \right) = \operatorname{Re} \left(\frac{g}{g} \cdot \frac{1}{(r^2 e^{2i\theta})} \right)$. Since $\left| \frac{g}{g} \cdot \frac{1}{e^{2i\theta}} \right| = 1$, then $|\Delta_z \ln r| \leq \frac{1}{r^2}$. Since $\Delta_M = |\nabla_M x_3|^2 \Delta_z$, this completes the proof of Lemma 2.3. q.e.d.

(Later, Meeks [10] modified our calculations in the proof of Lemma 2.3 to obtain related results for ends of periodic minimal surfaces.)

Proof of the Lemma 2.2. Let $C_t = \{p \in \mathbb{R}^3 \mid r(p) = t\}$ be the vertical cylinder of radius t and let M_t be the part of M inside C_t . Since the part of M inside the ball of radius R centered at the origin is contained in M_R , it suffices to prove that M_R has quadratic area growth as a function of R .

In the complement of the x_3 -axis, one has the ordered orthonormal basis $(\nabla r, \nabla x_3, r\nabla\theta) = (A_1, A_2, A_3)$. Let $B_i = A_i - (\vec{n} \cdot A_i)\vec{n}$ be the tangent part of A_i (here, \vec{n} is the unit normal to M), so

$$(\vec{n} \cdot A_1)^2 + (\vec{n} \cdot A_2)^2 + (\vec{n} \cdot A_3)^2 = 1$$

and

$$|B_i|^2 = |A_i|^2 - (\vec{n} \cdot A_i)^2 = 1 - (\vec{n} \cdot A_i)^2.$$

Hence,

$$|B_1|^2 + |B_2|^2 = 1 + (\vec{n} \cdot A_3)^2 \geq 1.$$

Since $B_1 = \nabla_M r$ and $B_2 = \nabla_M x_3$,

$$|\nabla_M r|^2 + |\nabla_M x_3|^2 \geq 1.$$

Thus,

$$\int_{M_R} dA \leq \int_{M_R} (|\nabla_M r|^2 + |\nabla_M x_3|^2) dA.$$

Therefore, it remains to prove that both $\int_{M_R} |\nabla_M r|^2 dA$ and $\int_{M_R} |\nabla_M x_3|^2 dA$ grow at most quadratically in R .

Without loss of generality, after removing a compact subset and homothetically scaling M , we may assume that the third coordinate function on M satisfies the inequality $|x_3| \leq \frac{1}{2} \ln r$. Consider the function $f: M \rightarrow \mathbb{R}$ defined by $-x_3 \arctan(x_3) + \frac{1}{2} \ln(x_3^2 + 1)$. A calculation yields $\Delta_M(f) = \frac{-|\nabla_M x_3|^2}{x_3^2 + 1}$.

By Lemma 2.3, $\Delta_M \ln r \leq |\nabla_M x_3|^2 / r^2$ and so the function $h = \ln r + f(x_3)$ is superharmonic on M . Since $\ln r$ is proper in a closed region of \mathbb{R}^3 containing M and M is proper, it follows that $\ln r$ is a proper function on M . And $h \geq \frac{1}{10} \ln r$, so h is a proper non-negative superharmonic function on M .

Now, for any positive proper C^2 -function H on M and $T \geq \sup(H(\partial M))$,

$$\int_{H^{-1}([0, T])} \Delta_M H = - \int_{\partial M} \nabla_M H \cdot \eta + \int_{H^{-1}(T)} |\nabla_M H|,$$

where η is the outward pointing conormal to the boundary. Hence, if $\Delta_M H \leq 0$, then $\int_{H^{-1}(T)} |\nabla_M H|$ is positive monotonically decreasing as $T \rightarrow \infty$. So, $\Delta_M H \in L^1(M)$ and, choosing $H = h$, we see that $\Delta_M h \in L^1(M)$. Since for r large, $|\Delta_M h| \geq \frac{1}{2} |\Delta_M f|$, we note that $\Delta_M f \in L^1(M)$ as well. Hence,

$$\int_{M_R} |\Delta_M f| dA = \int_{M_R} \frac{|\nabla_M x_3|^2}{x_3^2 + 1} dA \leq c_2$$

for some constant c_2 . But $|x_3| \leq \ln(R)$ on M_R , so we have

$$\int_{M_R} |\nabla_M x_3|^2 dA \leq \int_{M_R} \left(\frac{(\ln R)^2 + 1}{x_3^2 + 1} \right) |\nabla_M x_3|^2 dA \leq [(\ln R)^2 + 1] c_2.$$

This completes the proof that $\int_{M_R} |\nabla_M x_3|^2 dA$ grows at most quadratically in R .

Since $\Delta_M f \in L^1(M)$ and $|\Delta_M f| \geq |\Delta_M \ln r|$, we also have $\Delta_M \ln r \in L^1(M)$. Because

$$\int_{M_R} \Delta_M \ln r = - \int_{\partial M} \frac{\nabla_M r \cdot \eta}{r} + \int_{C_R \cap M} \frac{|\nabla_M r|}{R} = c_3 + \frac{1}{R} \int_{C_R \cap M} |\nabla_M r|$$

and $\int_{M_R} |\Delta_M \ln r|$ converges, $\frac{1}{R} \int_{C_R \cap M} |\nabla_M r|$ has a finite limit as $R \rightarrow \infty$. Hence,

$$\int_{C_R \cap M} |\nabla_M r| \leq c_4 R$$

for some constant c_4 . In fact, $\int_{C_\rho \cap M} |\nabla_M r| \leq c_4 \rho$ for any $\rho \in [1, R]$, so the co-area formula implies

$$\int_{M_R} |\nabla_M r|^2 \leq \int_1^R c_4 \rho d\rho \leq \frac{c_4 R^2}{2},$$

which means that $\int_{M_R} |\nabla_M r|^2$ grows quadratically in R . It follows that the area of M grows quadratically in R .

We now check that M has a finite number of ends. If E is a proper non-compact subdomain of M with ∂E compact, then we know that the area of E grows at most quadratically. But, by the monotonicity formula for area [22], the area of E must grow asymptotically at least as quickly as the area of a plane, which means that for large R the area of E inside the ball B_R is at least $\pi(R - 1)^2$, and if M has at least n ends, then, for every $\varepsilon > 0$, the area of M_R must be greater than $(n - \varepsilon)\pi(R + 1)^2$ for large R . In particular, the last sentence in the statement of Lemma 2.2 holds, which completes our proof. q.e.d.

Remark 2.4. In the statement of Lemma 2.2, the hypothesis that M lies between two catenoids can be weakened to the property that M lies above a catenoid end and intersects some positive vertical cone in a compact set. Under this weaker hypothesis, the conclusion of Lemma 2.2 that M has quadratic area growth still holds. To prove this more general result, one uses the function $h_c = h + cx_3$ for some positive c in place of the function h defined in the proof of the lemma. Indeed, under this weaker hypothesis, h_c is again proper and positive for sufficiently large c ; since M is minimal, $\Delta_M x_3 = 0$ and so $|\Delta_M h_c| = |\Delta_M h| \geq \frac{1}{2} |\Delta_M f|$, implying $\Delta_M f \in L^1(M)$ as before. We also have $|x_3| \leq aR$ on M_R for some positive a , and so, with constant c_2 chosen as above

$$\int_{M_R} |\nabla_M x_3|^2 dA \leq \int_{M_R} \left(\frac{(aR)^2 + 1}{x_3^2 + 1} \right) |\nabla_M x_3|^2 dA \leq [(aR)^2 + 1]c_2.$$

The rest of the argument is unchanged.

We now explain how Theorem 1.2 stated in the Introduction follows from Lemma 2.2. Let M be as in the statement of Lemma 2.2 and let P

denote the x_1x_2 -plane. The monotonicity formula for area [22] implies that $\lim_{r \rightarrow \infty} A(r)/\pi r^2$ exists and is a finite number $n(M)$. Since M can be viewed as a locally finite integral varifold with compact boundary and M has quadratic area growth, standard compactness theorems (see [22]) imply that the sequence of integral varifolds $\frac{1}{k}M = \{(\frac{x_1}{k}, \frac{x_2}{k}, \frac{x_3}{k}) \mid (x_1, x_2, x_3) \in M\}$ converges to the locally finite integral varifold $n(M)P$ as $k \rightarrow \infty$. Hence, the area-multiplicity $n(M)$ of M is a positive integer. The integer $n(M)$ can be easily identified from the proof of Lemma 2.2 as $n(M) = \lim_{R \rightarrow \infty} \frac{1}{2\pi R} \int_{M \cap C_R} |\nabla_M r|$. (This identification will be used later in the proof of Theorem 3.5 and Lemma 3.6.) Then, Theorem 1.2 stated in the Introduction follows immediately from these comments and Lemma 2.2.

3. Parabolicity

We now apply the results of Section 2 to derive some global results on the conformal structure of properly immersed or properly embedded minimal surfaces. We will say that a Riemannian surface M with boundary is *parabolic* if bounded harmonic functions on the surface are determined by their boundary values. We recall that given a point p on a Riemannian surface M with boundary, then there is an associated measure μ_p on ∂M , called the “hitting” or *harmonic* measure, such that $\mu_p(I)$, for an interval $I \subset \partial M$, is the probability that a Brownian path beginning at p “hits” the boundary a first time at a point in I . Note that harmonic measure enjoys a domain monotonicity property: if $M' \subset M$ is a subdomain containing p , and if also $I \subset \partial M'$, then the corresponding harmonic measures satisfy $\mu'_p(I) \leq \mu_p(I)$; this is because the family of Brownian paths from p to I within M' is contained in the corresponding family of paths within M .

It is well known that M is parabolic if and only if the harmonic measure μ_p for any $p \in \text{Int}(M)$ is full, that is, $\int_{\partial M} d\mu_p = 1$. In fact, if μ_p is full and $f: M \rightarrow \mathbb{R}$ is a bounded harmonic function, then for any $p \in \text{Int}(M)$, $f(p) = \int_{\partial M} f(x)d\mu_p$. It is easy to check that if μ_p is full for some point $p \in \text{Int}(M)$, then μ_q is full for any other point $q \in \text{Int}(M)$.

In order to verify whether a Riemannian surface M with boundary is parabolic, it is sufficient to find a proper non-negative superharmonic function $h: M \rightarrow [0, \infty)$. To see this, suppose $f_1, f_2: M \rightarrow \mathbb{R}$ are two bounded harmonic functions on M with the same boundary values and $f_1(p) > f_2(p)$ for some $p \in \text{Int}(M)$. Then, consider the proper function

$H_t: M \rightarrow \mathbb{R}$ defined by $H_t(x) = h(x) - t(f_1(x) - f_2(x))$. For t sufficiently large, $H_t(p) < 0$ and hence H_t has a minimum at some interior point of M , contradicting the minimum principle for superharmonic functions.

With this preliminary discussion in mind, we now state the first theorem of this section.

Theorem 3.1. *If M is a connected properly immersed minimal surface in \mathbb{R}^3 , possibly with boundary, then $M(+) = \{(x_1, x_2, x_3) \in M \mid x_3 \geq 0\}$ is parabolic.*

Proof. Let $M(n) = \{(x_1, x_2, x_3) \in M \mid 0 \leq x_3 \leq n\}$. We first prove that

$$M(n, *) = \{(x_1, x_2, x_3) \in M(n) \mid 1 \leq x_1^2 + x_2^2\}$$

is parabolic. Let $r(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2}$ and define $h: M(n, *) \rightarrow [0, \infty)$ by $h(p) = \ln(r(p)) - x_3^2(p)$. By Lemma 2.3, h is a superharmonic function on $M(n, *)$ and is proper since $\ln r$ is proper and x_3^2 is bounded. Because h is eventually positive, $M(n, *)$ is parabolic. As $M(n)$ is the union of $M(n, *)$ and the compact surface $M(n) \cap \{r \leq 1\}$, $M(n)$ is also parabolic.

We now check that $M(+)$ is parabolic by proving that each component C of $M(+)$ is parabolic. Let $p \in C$ be a point with positive third coordinate; by rescaling, we will assume $x_3(p) = 1$. Since $M(n)$ is parabolic, so is $C(n) = C \cap M(n) = C \cap x_3^{-1}([0, n])$, and thus the relation $1 = x_3(p)$ can be evaluated as an integral

$$\begin{aligned} 1 &= \int_{\partial(C(n))} x_3(x) d\mu_p(n) \\ &= 0 \cdot \int_{\partial C \cap x_3^{-1}(0)} d\mu_p(n) + \int_{\partial C(n) \cap x_3^{-1}((0, n))} x_3(x) d\mu_p(n) \\ &\quad + n \cdot \int_{\partial C(n) \cap x_3^{-1}(n)} d\mu_p(n), \end{aligned}$$

where $\mu_p(n)$ is the harmonic measure on the boundary of $C(n)$. The middle term is non-negative, so

$$\int_{\partial C(n) \cap x_3^{-1}(n)} d\mu_p(n) \leq \frac{1}{n}.$$

Since $\mu_p(n)$ is full on $C(n)$,

$$\int_{\partial C(n) - x_3^{-1}(n)} d\mu_p(n) \geq 1 - \frac{1}{n}.$$

Taking limits (using domain monotonicity of $\mu_p(n)$) as $n \rightarrow \infty$, one obtains $\int_{\partial C} d\mu_p = 1$, which proves the theorem. q.e.d.

Remark 3.2. Recently Theorem 3.1 has been applied [11] to prove that the flux of a coordinate function of a properly immersed minimal surface is well defined. Also, Meeks and Rosenberg [16] have used this Theorem 3.1 to prove that if M is a finite topology properly immersed minimal surface in \mathbb{R}^3 such that a plane intersects M transversely in a finite number of component curves, then M is a conformally finitely punctured Riemann surface. This result and Theorem 3.1 should be compared with the theorem of Morales [19], which proves the existence of a proper conformal minimal immersion of the open unit disk into \mathbb{R}^3 .

Corollary 3.3. *Suppose D is a proper domain in \mathbb{R}^2 and M is a minimal graph over D which is bounded from below. Then M is parabolic.*

Recall that a complete Riemannian surface M is called *recurrent* for Brownian motion if, with probability one, a Brownian path starting at a point $p \in M$ will enter every neighborhood of any other point $q \in M$ for a divergent sequence of times. The notion of being recurrent is closely related to the notion of parabolicity for surfaces with boundary. If M is the union of two subdomains that intersect in a compact subset of M , then M is recurrent if these subdomains are parabolic. Since a properly immersed minimal surface M in \mathbb{R}^3 can be expressed as $M = M(+) \cup M(-)$, where $M(-) = \{(x_1, x_2, x_3) \in M \mid x_3 \leq 0\}$, and $M(+)$ and $M(-)$ are parabolic, then if $M(+) \cap M(-) = M \cap x_3^{-1}(0)$ is compact, M is recurrent. We restate this result as a corollary.

Corollary 3.4. *If M is a properly immersed minimal surface in \mathbb{R}^3 and some plane intersects M in a compact set, then M is recurrent for Brownian motion.*

In certain cases, it can be shown that a properly embedded minimal surface has a compact intersection with some plane. The final theorem of this section gives an important instance of this compact intersection property.

Theorem 3.5. *If M is a properly embedded minimal surface with two limit ends, then between any two middle ends of M , there is a plane that intersects M transversely in a compact set. In other words, given two distinct middle ends of M , there is a plane P that intersects*

M transversely in a compact set and the representatives of these ends in $M - P$ lie on opposite sides of P . In particular, M is recurrent for Brownian motion.

Theorem 3.5 will follow immediately from Lemma 3.6.

Lemma 3.6. *Suppose that M is a non-compact properly immersed minimal surface with compact boundary contained between vertical catenoid ends C_1 and C_λ , where C_1 has logarithmic growth 1 and C_λ has logarithmic growth $\lambda \geq 1$ and C_λ lies above C_1 . If the asymptotic area growth of M is $n\pi r^2$, then the vertical flux $F = \int_{\partial M} |\nabla_M x_3|$ satisfies $2\pi n \leq F \leq 2\pi n\lambda$.*

Proof. Recall the function $f = -x_3 \arctan(x_3) + \frac{1}{2} \ln(x_3^2 + 1)$ defined in the proof of Lemma 2.2. In that proof, it was shown that $\Delta_M f, \Delta_M \ln r \in L^1(M)$. It also follows from that proof for every real number c , $h_c = \ln r + cx_3 + f$ is superharmonic outside of a compact subdomain of M . Under the hypotheses given in the statement of Lemma 3.6, for a divergent sequence of points $\{p_i\}$ in M , $\lim_{i \rightarrow \infty} h_c(p_i) = +\infty$ for $c > \frac{\pi}{2} - \frac{1}{\lambda}$ and $\lim_{i \rightarrow \infty} h_c(p_i) = -\infty$ for $c < \frac{\pi}{2} - 1$.

Assume now that $c \frac{\pi}{2} - \frac{1}{\lambda}$. The type of calculations carried out in the proof of Lemma 2.2 implies that for T sufficiently large, $\int_{h_c^{-1}(T)} \nabla_M h_c \cdot \eta$ is a positive monotonically decreasing function of T , where η is the outward pointing unit conormal to $h_c^{-1}((-\infty, T])$. (In the case $c < \frac{\pi}{2} - 1$, $\int_{h_c^{-1}(T)} \nabla_M h_c \cdot \eta$ is negative and monotonically increasing in norm as a function of $-T$.) Hence, for $c \frac{\pi}{2} - \frac{1}{\lambda}$ and T large,

$$0 \leq \int_{h_c^{-1}(T)} \nabla_M h_c \cdot \eta = \int_{h_c^{-1}(T)} \frac{\nabla_M r}{r} \cdot \eta + \int_{h_c^{-1}(T)} \nabla_M (cx_3 + f) \cdot \eta.$$

Since $\nabla_M f = -\nabla_M x_3 \cdot \arctan(x_3)$ and $x_3 \rightarrow \infty$ as $T \rightarrow \infty$,

$$\begin{aligned} \left(c - \frac{\pi}{2}\right) \int_{\partial M} \nabla_M x_3 \cdot \eta &= \left(\frac{\pi}{2} - c\right) \lim_{T \rightarrow \infty} \int_{h_c^{-1}(T)} \nabla_M x_3 \cdot \eta \\ &\leq \lim_{T \rightarrow \infty} \int_{h_c^{-1}(T)} \frac{\nabla_M r}{r} \cdot \eta. \end{aligned}$$

Let $c = \frac{\pi}{2} - \frac{1}{\lambda}$ and we obtain

$$F = \int_{\partial M} |\nabla_M x_3| = - \lim_{T \rightarrow \infty} \int_{h_c^{-1}(T)} \nabla_M x_3 \cdot \eta \leq \lambda \cdot \lim_{T \rightarrow \infty} \int_{h_c^{-1}(T)} \frac{\nabla_M r}{r} \cdot \eta.$$

But since $\Delta_M \ln r \in L^1(M)$, the divergence theorem implies

$$\lim_{T \rightarrow \infty} \int_{h_c^{-1}(T)} \frac{\nabla_M r}{r} \cdot \eta = \lim_{R \rightarrow \infty} \int_{M \cap C_R} \frac{\nabla_M r}{R} \cdot \eta = \lim_{R \rightarrow \infty} \frac{1}{R} \int_{M \cap C_R} |\nabla_M r|,$$

where C_t is the cylinder of radius t centered along the x_3 -axis. From the discussion immediately following the proof of Lemma 2.2:

$$\lim_{R \rightarrow \infty} \frac{1}{R} \int_{M \cap C_R} |\nabla_M r| = 2\pi n.$$

Hence, for $c = \frac{\pi}{2} - \frac{1}{\lambda}$, we obtain $F \leq 2\pi n \lambda$. Making similar calculations in the case $c < \frac{\pi}{2} - 1$, we obtain the inequality $F \geq 2\pi n$, which completes the proof of the lemma. q.e.d.

Proof of Theorem 3.5. Suppose M has two limit ends with horizontal limit tangent plane at infinity. From the proof of the ordering theorem [6], there exists an end E of a vertical catenoid or of a horizontal plane between any two middle ends of M . If E were a vertical catenoid with positive logarithmic growth between middle ends e_1 and e_2 , where e_2 is the next middle end of M above e_1 , then between e_2 and the end e_3 just above e_2 , there would be an end \tilde{E} of a catenoid between e_2 and e_3 such that the logarithmic growth of \tilde{E} is at least equal to the logarithmic growth of E .

After a homothety of \mathbb{R}^3 , we may assume that the logarithmic growth of E is 1. By Lemma 3.6, the flux of $\nabla_M x_3$ across the boundary of any proper domain $M(e_2)$ representing e_2 must be at least 2π . Similarly, for the end e_3 , the flux of $\nabla_M x_3$ across the boundary of any proper domain $M(e_3)$ representing e_3 must also be at least 2π . In fact, if we let $\{e_2, e_3, \dots, e_n, \dots\} \subset \mathcal{E}_M$ be the infinite set of middle ends above e_1 , and ordered so that $e_i < e_j$ if $i < j$, then for any such end e_j there exist proper disjoint subdomains $M(e_i)$ representing the e_i such that the flux of $\nabla_M x_3$ across $\partial M(e_i)$ is at least 2π .

Assume now that P is a horizontal plane that intersects $E_1 = M(e_1)$ in a circle $S^1 \subset P$, which we may assume is the boundary of E_1 . Let

D be the disk in P with $\partial D = S^1$. Without loss of generality, we may assume that D intersects M transversely in a finite number of simple closed curves which separate M into a finite number of components. Let M' be one of the components above $D \cup E_1$ that contains an infinite number of middle ends of M . Then, $x_3: M' \rightarrow [x_3(P), \infty)$ is proper and so the flux of $\nabla_M x_3$ across $\partial M'$ is at least as big as the sum of the fluxes of $\nabla_M x_3$ coming from the middle ends of M' . Since each middle end of M' contributes at least 2π of flux, the total flux of $\nabla_M x_3$ across $\partial M'$ must be infinite. But $\partial M'$ is compact, and the flux is no more than the total length of the boundary of M' , which is finite. This contradiction proves E must be a horizontal plane, from which Theorem 3.5 follows. q.e.d.

3.7. An example with non-constant bounded harmonic functions. It is conjectured that every $M \in \mathcal{M}$ is recurrent, and that, for a properly embedded minimal surface M with one end, every positive harmonic function on M is constant. About 20 years ago, Dennis Sullivan asked whether a positive harmonic function on a properly embedded minimal surface in \mathbb{R}^3 must be constant. Let N be a properly embedded triply-periodic minimal surface in \mathbb{R}^3 . It is known that a positive harmonic function $f: N \rightarrow \mathbb{R}$ is constant, but such an N is never recurrent. We will now construct a two sheeted covering space with $p: M \rightarrow N$ which has non-constant harmonic functions, and so $j \circ p: M \rightarrow \mathbb{R}^3$ is a properly immersed minimal surface (j is the inclusion of N into \mathbb{R}^3) with two ends and with non-constant bounded harmonic functions.

First, recall that N is topologically an infinite genus surface with one end. Let $\Gamma = \{a_1, a_2, \dots, a_n, \dots\}$ be a countable proper collection of closed curves on N , which generate the first homology group of N . Let $\sigma: \pi_1(N) \rightarrow \mathbb{Z}_2$ be the homomorphism which factors through $\tilde{\sigma}: H_1(N) \rightarrow \mathbb{Z}_2$, where $\tilde{\sigma}([a_1]) = 1$ and $\tilde{\sigma}([a_i]) = 0$ for $i \neq 1$. Let $p: M \rightarrow N$ be the \mathbb{Z}_2 -cover of N corresponding to $\text{Ker}(\sigma) \subset \pi_1(N)$. Let $D_1 \subset D_2 \subset \dots$ be a proper compact exhaustion of N by subdomains with one boundary curve and such that $a_1 \subset D_1$. Let $E = N - \text{Int}(D_1)$ and $E(+)$ and $E(-)$ be the two components of $p^{-1}(E)$. Note that $E(+)$ and $E(-)$ are proper domains which represent the two ends of M . Let $\tilde{D}_1 \subset \tilde{D}_2 \subset \dots$ be the associated pullback compact exhaustion of M and note that each domain \tilde{D}_i has two boundary curves, $\partial(i, -) \subset E(-)$ and $\partial(i, +) \subset E(+)$, respectively.

Let $h_n: \tilde{D}(n) \rightarrow [-1, 1]$ be the harmonic function with boundary value -1 on $\partial(n, -)$ and $+1$ on $\partial(n, +)$. Since $\{h_n\}$ is a uniformly bounded, increasing sequence, it converges to a harmonic function $h: M \rightarrow [-1, 1]$. We will now prove that h is non-constant by showing h has -1 as an asymptotic limiting value on the end of $E(-)$ and has $+1$ as an asymptotic limiting value on the end of $E(+)$.

Consider a divergent sequence $p(i) \in N$ such that $p(i) \in D(i) - D(i - 1)$. Let $W(n) = D(n) - \text{Int}(D_1)$. For $n > i$, consider $P = \int_{\partial D_1} \mu(p(i), n)$, where $\mu(p(i), n)$ is the hitting measure for $p(i)$ considered to lie in $W(n)$. Since N is not recurrent for Brownian motion, for every $\varepsilon > 0$, there exists an $N(\varepsilon)$ such that if $n > i > N(\varepsilon)$, then the probability P of a Brownian path starting at $p(i)$ in $W(n)$ of exiting a first time at ∂D_1 is less than ε . This implies that if $p(i, +) \in E(+)$ and $p(i, -) \in E(-)$ are the two lifts of $p(i)$ to M , then $h_n(p(i, +)) \geq 1 - 2\varepsilon$ and $h_n(p(i, -)) \leq -1 + 2\varepsilon$, which implies $h(p(i), +) \geq 1 - 2\varepsilon$ and $h(p(i, -)) \leq -1 + 2\varepsilon$. By letting $\varepsilon \rightarrow 0$, we obtain our earlier claim that h is asymptotic to -1 on the end of $E(-)$ and asymptotic to $+1$ on the end of $E(+)$.

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