J. DIFFERENTIAL GEOMETRY
67 (2004) 289-333

MOMENTUM MAPS AND MORITA EQUIVALENCE

Ping Xu

Dedicated to Alan Weinstein on the occasion of his 60th birthday

Abstract

We introduce quasi-symplectic groupoids and explain their relation with momentum map theories. This approach enables us to unify into a single framework various momentum map theories, including ordinary Hamiltonian G-spaces, Lu's momentum maps of Poisson group actions, and the group-valued momentum maps of Alekseev–Malkin–Meinrenken. More precisely, we carry out the following program:

(1) We define and study properties of quasi-symplectic groupoids.

(2) We study the momentum map theory defined by a quasi-symplectic groupoid $\Gamma \rightrightarrows P$. In particular, we study the reduction theory and prove that $J^{-1}(\mathcal{O})/\Gamma$ is a symplectic manifold for any Hamiltonian Γ-space $(X \xrightarrow{J} P, \omega_X)$ (even though $\omega_X \in \Omega^2(X)$ may be degenerate), where $\mathcal{O} \subset P$ is a groupoid orbit. More generally, we prove that the intertwiner space $(X_1 \times_P \overline{X_2})/\Gamma$ between two Hamiltonian Γ-spaces X_1 and X_2 is a symplectic manifold (whenever it is a smooth manifold).

(3) We study Morita equivalence of quasi-symplectic groupoids. In particular, we prove that Morita equivalent quasi-symplectic groupoids give rise to equivalent momentum map theories. Moreover the intertwiner space $(X_1 \times_P \overline{X_2})/\Gamma$ depends only on the Morita equivalence class. As a result, we recover various well-known results concerning equivalence of momentum maps including the Alekseev–Ginzburg–Weinstein linearization theorem and the Alekseev–Malkin–Meinrenken equivalence theorem between quasi-Hamiltonian spaces and Hamiltonian loop group spaces.

The author's research was partially supported by NSF grants DMS00-72171 and DMS03-06665.

Received 10/29/2003.

1. Introduction

"Momentum" usually refers to quantities whose conservation under the time evolution of a physical system is related to some symmetry of the system. Noether [28], in the course of developing ideas of Einstein and Klein in general relativity theory, found a very general equivalence between symmetries and conservation laws in field theory; this is now known as Noether's theorem. Focusing on the relation between symmetries and conserved quantities, the study of momentum maps has received much attention in the last three decades, continuing to the present day with the formulation of new notions of symmetry. In geometric terms, a phase space with a symmetry group consists of a symplectic (or Poisson) manifold X and an Hamiltonian action of a Lie group G. By the latter, we mean a symplectic (or Poisson) action of G on X together with an equivariant map $J : X \to \mathfrak{g}^*$ such that for each $\xi \in \mathfrak{g}$, the one-parameter group of transformations of X generated by ξ is the flow of the Hamiltonian vector field with Hamiltonian $\langle J(x), \xi\rangle \in C^\infty(X)$. The map J is called the momentum (or moment) map of the Hamiltonian action. One very important aspect of the momentum map theory is the study of Marsden–Weinstein (or symplectic) reduction, which is the simultaneous use of symmetries and conserved quantities to reduce the dimension of a Hamiltonian system.

With the advance of physics and mathematics, new notions of symmetry and momentum have appeared. For instance, a Poisson group symmetry is the classical limit of a "quantum group symmetry" in quantum group theory [12]. Lu's momentum map theory [19] for Poisson Lie group actions is a theory adapted from the usual Hamiltonian theory which incorporates the Poisson structure on the symmetry group G. Computations of the symplectic structures on moduli spaces of flat connections on surfaces have led to another notion of Hamiltonian symmetry known as quasi-Hamiltonian symmetry. In this new theory, the 2-form ω on the phase space is neither closed nor non-degenerate, but these "defects" are compensated for by the presence of an auxiliary structure on the group. This is the starting point of the theory of quasi-Hamiltonian G-spaces with group-valued momentum maps of Alekseev–Malkin–Meinrenken (AMM) [2]. All these momentum map theories share many similarities, but involve different techniques and proofs. It is also known that some of these momentum theories are equivalent to one another. For instance, for compact groups, the AMM

group-valued momentum map theory is equivalent to the Hamiltonian momentum map theory of loop groups of Meinrenken–Woodward [**23**, **24**, **25**], and for compact Bruhat–Poisson groups, Lu's momentum map theory is equivalent to the usual Hamiltonian momentum map theory [**1**]. However, these results are fragmentary and their geometric significance remains unclear. It is therefore natural to investigate the relations between these theories, and to seek a uniform framework, which is an open question raised by Weinstein [**34**]. A unified approach would seek to develop a single momentum map theory which reduces to the theories already established under special circumstances. While necessarily generalizing the problem, this would allow a direct comparison of the features of the various momentum maps in a more intrinsic manner. The importance of such a single momentum map theory is not merely to give another interpretation of these existing momentum map theories, but rather to explore the intrinsic ingredients of these theories so that techniques in one theory can be applied to another. This is particularly important in the study of group-valued momentum map theory where there are still many open problems, including the quantization problem which we believe will be the main application of our approach [**17**].

The approach taken in this paper involves extending the notion of symmetry from actions of groups to actions of groupoids. This was motivated by the work of Mikami–Weinstein [**26**] who showed that the usual Hamiltonian momentum map is in fact equivalent to the symplectic action of the symplectic groupoid $T^*G \rightrightarrows \mathfrak{g}^*$, which integrates the Lie–Poisson structure on $\mathfrak{g}^*$. Similarly, in [**35**], Weinstein and the author proved that the momentum map theory of Lu for an Hamiltonian Poisson group G-space is equivalent to the symplectic action of the symplectic groupoid $G \times G^* \rightrightarrows G^*$ integrating the dual Poisson group G^* [**20**]. By a symplectic action of a symplectic groupoid $\Gamma \rightrightarrows P$ on a symplectic manifold X, we mean a map $J : X \to P$ equipped with a Γ-action $\Gamma \times_P X \to X$ which is compatible with the symplectic structures [**26**]. In this case, X is called an Hamiltonian-Γ space.

There is strong evidence that the AMM group-valued momentum map is closely related to the transformation groupoid $G \times G \rightrightarrows G$. Here, G acts on itself by conjugation. However, $G \times G \rightrightarrows G$ is no longer a symplectic groupoid since the closed 3-form, i.e., the Cartan form Ω on G, must now play a role. In fact, one can show that the standard AMM 2-form $\omega \in \Omega^2(G \times G)$ together with $\Omega \in \Omega^3(G)$ gives a 3-cocycle

of the total de Rham complex of the groupoid and defines a non-trivial class in the equivariant cohomology $H^3_G(G)$ [6].

This example suggests that one must enrich the notion of a symplectic groupoid in order to include such "twisted" symplectic structures on the groupoids. Thus, we arrive at quasi-symplectic groupoids, the main subject of the present paper. A quasi-symplectic groupoid is a Lie groupoid $\Gamma \rightrightarrows P$ equipped with a 2-form $\omega \in \Omega^2(\Gamma)$ and a 3-form $\Omega \in \Omega^3(P)$ such that $\omega + \Omega$ is a 3-cocycle of the de Rham complex of the groupoid, where ω must satisfy a weak non-degeneracy condition. When ω is honestly non-degenerate, this is the so-called twisted symplectic groupoid studied by Cattaneo and the author [10] as the global object integrating a twisted Poisson structure of Severa–Weinstein [30]. In particular, when Ω vanishes, it reduces to an ordinary symplectic groupoid.

It turns out that much of the theory of Hamiltonian Γ-spaces of a symplectic groupoid Γ can be generalized to the present context of quasi-symplectic groupoids. In particular, one can perform reduction and prove that $J^{-1}(\mathcal{O})/\Gamma$ is a symplectic manifold (even though $\omega_X \in \Omega^2(X)$ may be degenerate), where $\mathcal{O} \subset P$ is an orbit of the groupoid. More generally, one can introduce the classical intertwiner space $(X_1 \times_P \overline{X_2})/\Gamma$ between two Hamiltonian Γ-spaces X_1 and X_2, generalizing the same notion studied by Guillemin–Sternberg [14] for the ordinary Hamiltonian G-spaces. One shows that this is a symplectic manifold (whenever it is a smooth manifold).

As for symplectic groupoids, one can also introduce Morita equivalence for quasi-symplectic groupoids. In particular, we prove the following main result. (i) Morita equivalent quasi-symplectic groupoids give rise to equivalent momentum map theories in the sense that there is an equivalence of categories between their Hamiltonian Γ-spaces; (ii) the symplectic manifold $(X_1 \times_P \overline{X_2})/\Gamma$ depends only on the Morita equivalence class of Γ. As a result, we recover various well-known results concerning equivalence of momentum maps including the Alekseev–Ginzburg–Weinstein linearization theorem and Alekseev–Malkin–Meinrenken equivalence theorem for group-valued momentum maps. They are essentially due to the Morita equivalence between the Lu–Weinstein symplectic groupoid $G \times G^* \rightrightarrows G^*$ and the standard cotangent symplectic groupoid $T^*G \rightrightarrows \mathfrak{g}^*$, where G is a compact simple Lie group equipped with the Bruhat–Poisson group structure, and the Morita equivalence

between the symplectic groupoid $(LG \times L\mathfrak{g} \rightrightarrows L\mathfrak{g}, \omega_{LG\times L\mathfrak{g}})$ and the AMM quasi-symplectic groupoid $(G \times G \rightrightarrows G, \omega + \Omega)$.

Another main motivation of the present work is the quantization problem. It is natural to study the geometric quantization of the symplectic reduced space $J^{-1}(\mathcal{O})/\Gamma$ or more generally the symplectic intertwiner space $(X_1 \times_P \overline{X_2})/\Gamma$, and prove the Guillemin–Sternberg conjecture that "$[Q, R] = 0$": quantization commutes with reduction, for Hamiltonian Γ-spaces. As an application, our uniform framework naturally leads to the following construction of prequantizations. A prequantization of the quasi-symplectic groupoid $(\Gamma \rightrightarrows P, \omega + \Omega)$ is an S^1-gerbe over the stack corresponding to the groupoid $\Gamma \rightrightarrows P$, while a prequantization of an Hamiltonian Γ-space is a line bundle L on which the gerbe acts. A prequantization of the symplectic intertwiner space $(X_1 \times_P \overline{X_2})/\Gamma$ can be constructed using these data. For symplectic groupoids, such a prequantization was studied in [**37**]. Details of this construction for quasi-symplectic groupoids appear elsewhere [**17**]. Note that in the usual Hamiltonian case, since the symplectic 2-form defines a zero class in the third cohomology group of the groupoid $T^*G \rightrightarrows \mathfrak{g}^*$, which in this case is the equivariant cohomology $H^3_G(\mathfrak{g}^*)$, gerbes do not enter explicitly. However, for a general quasi-symplectic groupoid (for instance the AMM quasi-symplectic groupoid), since the 3-cocycle $\omega+\Omega$ may define a non-trivial class, gerbes are inevitable in the construction.

Recently, Zung proved the convexity theorem for Hamiltonian Γ-spaces of proper quasi-symplectic groupoids, which encompasses many classical convexity theorems in the literature [**38**]. Finally, we note that recently Bursztyn–Crainic–Weinstein–Zhu showed that infinitesimally quasi-symplectic groupoids (which are called twisted presymplectic groupoids in [**8**]) correspond to twisted Dirac structures. They also studied the infinitesimal version of our Hamiltonian Γ-spaces. We refer to [**8**] for details.

2. Quasi-symplectic groupoids

In this section, we introduce quasi-symplectic groupoids and discuss their basic properties.

2.1. Pre-quasi-symplectic groupoids. A simple and compact way to define a pre-quasi-symplectic groupoid is to use the de-Rham double complex of a Lie groupoid. First, let us recall its definition below.

Let $\Gamma \rightrightarrows \Gamma_0$ be a Lie groupoid with source and target maps $s, t : \Gamma \to \Gamma_0$. Define for all $p \geq 0$

$$\Gamma_p = \underbrace{\Gamma \times_{\Gamma_0} \cdots \times_{\Gamma_0} \Gamma}_{p \text{ times}},$$

i.e., Γ_p is the manifold of composable sequences of p arrows in the groupoid $\Gamma \rightrightarrows \Gamma_0$. We have $p+1$ canonical maps $\Gamma_p \to \Gamma_{p-1}$ giving rise to a diagram

$$\cdots \Gamma_2 \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} \Gamma_1 \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \Gamma_0 \,. \tag{1}$$

In fact, $\Gamma_\bullet$ is a simplicial manifold. Consider the double complex $\Omega^\bullet(\Gamma_\bullet)$:

$$\begin{tikzcd}
\cdots & \cdots & \cdots & \\
\Omega^1(\Gamma_0) \arrow[u, "d"] \arrow[r, "\partial"] & \Omega^1(\Gamma_1) \arrow[u, "d"] \arrow[r, "\partial"] & \Omega^1(\Gamma_2) \arrow[u, "d"] \arrow[r, "\partial"] & \cdots \\
\Omega^0(\Gamma_0) \arrow[u, "d"] \arrow[r, "\partial"] & \Omega^0(\Gamma_1) \arrow[u, "d"] \arrow[r, "\partial"] & \Omega^0(\Gamma_2) \arrow[u, "d"] \arrow[r, "\partial"] & \cdots
\end{tikzcd} \tag{2}$$

Its boundary maps are $d : \Omega^k(\Gamma_p) \to \Omega^{k+1}(\Gamma_p)$, the usual exterior derivative of differentiable forms and $\partial : \Omega^k(\Gamma_p) \to \Omega^k(\Gamma_{p+1})$, the alternating sum of the pull-back maps of (1). We denote the total differential by $\delta = (-1)^p d + \partial$. The cohomology groups of the total complex $\Omega^\bullet(\Gamma_\bullet)$

$$H^k_{DR}(\Gamma_\bullet) = H^k\big(\Omega^\bullet(\Gamma_\bullet)\big)$$

are called the de Rham cohomology groups of $\Gamma \rightrightarrows \Gamma_0$. We now introduce the following;

Definition 2.1. A pre-quasi-symplectic groupoid is a Lie groupoid $\Gamma \rightrightarrows P$ equipped with a two-form $\omega \in \Omega^2(\Gamma)$ and a three-form $\Omega \in \Omega^3(P)$ such that

$$d\Omega = 0, \quad d\omega = \partial\Omega, \quad \partial\omega = 0. \tag{3}$$

In other words, $\omega + \Omega$ is a 3-cocycle of the total de-Rham complex of the groupoid $\Gamma \rightrightarrows P$.

Remark 2.2. It is simple to see that the last condition $\partial\omega = 0$ is equivalent to that of the graph of the multiplication $\Lambda \subset \Gamma \times \Gamma \times \overline{\Gamma}$ is isotropic. In this case, ω is said to be multiplicative.

By $A \to P$, we denote the Lie algebroid of $\Gamma \rightrightarrows P$, where the anchor map is denoted by $a : A \to TP$. For any $\xi \in \Gamma(A)$, by $\overrightarrow{\xi}$ and $\overleftarrow{\xi}$ we denote its corresponding right- and left-invariant vector fields on Γ respectively. The following properties can be easily verified (see also [10]).

Proposition 2.3. *Let* $(\Gamma \rightrightarrows P, \omega + \Omega)$ *be a pre-quasi-symplectic groupoid.*

(1) $\epsilon^*\omega = 0$, *where* $\epsilon : P \to \Gamma$ *is the unit map;*
(2) $i^*\omega = -\omega$, *where* $i : \Gamma \to \Gamma$ *is the groupoid inversion;*
(3) *for any* $\xi, \eta \in \Gamma(A)$,

$$\omega(\overrightarrow{\xi}, \overrightarrow{\eta}) = -\omega(\overleftarrow{\xi}, \overleftarrow{\eta}), \quad \omega(\overrightarrow{\xi}, \overleftarrow{\eta}) = 0;$$

(4) *for any* $\xi, \eta \in \Gamma(A)$, $\omega(\overrightarrow{\xi}, \overrightarrow{\eta})$ *is a right invariant function on* Γ, *and* $\omega(\overleftarrow{\xi}, \overleftarrow{\eta})$ *is a left invariant function on* Γ.

Proof. Let $\Lambda = \{(x, y, z) | z = xy,\ (x, y) \in \Gamma_2\} \subset \Gamma \times \Gamma \times \overline{\Gamma}$ be the graph of groupoid multiplication. Thus, Λ is isotropic with respect to $(\omega, \omega, -\omega)$.

(1) For any $\delta'_m,\ \delta''_m \in T_mP$, since $(\delta'_m, \delta'_m, \delta'_m),\ (\delta''_m, \delta''_m, \delta''_m) \in T\Lambda$, it follows that $\omega(\delta'_m, \delta''_m) = 0$.

(2) $\forall x \in \Gamma$ and $\forall \delta'_x,\ \delta''_x \in T_x\Gamma$, it is clear that $(\delta'_x, i_*\delta'_x, s_*\delta'_x)$, $(\delta''_x, i_*\delta''_x,\ s_*\delta''_x) \in T\Lambda$. Thus, using (1), we have

$$\omega(\delta'_x, \delta''_x) + \omega(i_*\delta'_x, i_*\delta''_x) = 0,$$

and therefore (2) follows.

(3) Since $i_*\overrightarrow{\xi} = -\overleftarrow{\xi}$ and $i_*\overrightarrow{\eta} = -\overleftarrow{\eta}$, from (2) it follows that $\omega(\overrightarrow{\xi}, \overrightarrow{\eta}) = -\omega(\overleftarrow{\xi}, \overleftarrow{\eta})$. Now, for any $x \in \Gamma$, since both vectors $(\overrightarrow{\xi}(x),\ 0_{t(x)},\ \overrightarrow{\xi}(x))$ and $(0_x, \overleftarrow{\eta}(t(x)), \overleftarrow{\eta}(x))$ are tangent to Λ, we thus have $\omega(\overrightarrow{\xi}(x), \overleftarrow{\eta}(x)) = 0$.

(4) It is simple to see that, for any $\xi, \eta \in \Gamma(A)$ and any composable pair $(x, y) \in \Gamma_2$, $(\overrightarrow{\xi}(x), 0_y,\ \overrightarrow{\xi}(xy))$, $(\overrightarrow{\eta}(x), 0_y,\ \overrightarrow{\eta}(xy)) \in T\Lambda$. Thus

$$\omega(\overrightarrow{\xi}(x), \overrightarrow{\eta}(x)) - \omega(\overrightarrow{\xi}(xy), \overrightarrow{\eta}(xy)) = 0.$$

Hence, $\omega(\overrightarrow{\xi}, \overrightarrow{\eta})$ is a right invariant function on Γ. Similarly, one proves that $\omega(\overleftarrow{\xi}, \overleftarrow{\eta})$ is a left invariant function on Γ. q.e.d.

We next investigate the kernel of ω along the unit space P. For any $m \in P$, there are two ways to identify elements of A_m as tangent vectors of Γ, namely vectors tangent to the t-fiber $\xi \to \overrightarrow{\xi}(m)$, or to the s-fiber $\xi \to \overleftarrow{\xi}(m)$. Write

$$\overrightarrow{A}|_m = \{\overrightarrow{\xi}(m) | \forall \xi \in A_m\}, \qquad \overleftarrow{A}|_m = \{\overleftarrow{\xi}(m) | \forall \xi \in A_m\}. \tag{4}$$

Thus, we have the following decomposition of the tangent space:

$$T_m\Gamma = \overrightarrow{A}|_m \oplus T_mP = \overleftarrow{A}|_m \oplus T_mP, \quad \forall m \in P. \tag{5}$$

Corollary 2.4. *Under the same hypothesis as in Proposition 2.3, we have, for any $m \in P$,*

(1) $\ker \omega_m = (\ker \omega_m \cap \overrightarrow{A}|_m) \oplus (\ker \omega_m \cap T_mP)$,
(2) *if* $\overrightarrow{\xi}(m) \in \ker \omega_m$, *then* $a(\xi) \in \ker \omega_m$,
(3) *for any* $\xi \in A_m$, $\overrightarrow{\xi}(m) \in \ker \omega_m$ *if and only if* $\overleftarrow{\xi}(m) \in \ker \omega_m$.

Proof. To prove (1), it suffices to show that if $\overrightarrow{\xi}(m) + v \in \ker \omega_m$, where $\xi \in A_m$ and $v \in T_mP$, then both $\overrightarrow{\xi}(m)$ and v belong to $\ker \omega_m$. According to Proposition 2.3 (1), for any $u \in T_mP$, we have

$$\omega(\overrightarrow{\xi}(m), u) = \omega(\overrightarrow{\xi}(m) + v, u) = 0.$$

On the other hand, for any $\eta \in A_m$, we have $\omega(\overrightarrow{\xi}(m), \overleftarrow{\eta}(m)) = 0$ according to Proposition 2.3 (3). Thus, it follows that $\overrightarrow{\xi}(m) \in \ker \omega_m$, which also implies that $v \in \ker \omega_m$.

(2) Note that $a(\xi) = \overrightarrow{\xi}(m) - \overleftarrow{\xi}(m)$. Hence, for any $\eta \in A_m$, we have

$$\begin{aligned}\omega(a(\xi), \overrightarrow{\eta}(m)) &= \omega(\overrightarrow{\xi}(m) - \overleftarrow{\xi}(m), \overrightarrow{\eta}(m)) \\ &= \omega(\overrightarrow{\xi}(m), \overrightarrow{\eta}(m)) - \omega(\overleftarrow{\xi}(m), \overrightarrow{\eta}(m)) = 0.\end{aligned}$$

It thus follows that $a(\xi) \in \ker \omega_m$ since $\epsilon^*\omega = 0$ according to Proposition 2.3 (1).

(3) follows from (2) since $a(\xi) = \overrightarrow{\xi}(m) - \overleftarrow{\xi}(m)$. q.e.d.

2.2. Quasi-symplectic groupoids. Let us set

$$\ker \omega_m \cap A_m = \{\xi \in A_m | \overrightarrow{\xi}(m) \in \ker \omega_m\}. \tag{6}$$

Corollary 2.4 implies that the anchor induces a well-defined map from $\ker \omega_m \cap A_m$ to $\ker \omega_m \cap T_mP$. Now, we are ready to introduce the non-degeneracy condition.

Definition 2.5. A pre-quasi-symplectic groupoid $(\Gamma \rightrightarrows P, \omega + \Omega)$ is said to be quasi-symplectic if the following non-degeneracy condition is satisfied: the anchor

$$a : \ker \omega_m \cap A_m \to \ker \omega_m \cap T_m P$$

is an isomorphism.

Given a pre-quasi-symplectic groupoid $(\Gamma \rightrightarrows P, \omega + \Omega)$, the two-form ω induces a well-defined linear map:

$$\omega^b : T_m P \longrightarrow A_m^*, \quad \langle \omega^b(v), \xi \rangle = \omega(v, \overrightarrow{\xi}(m)), \quad \forall v \in T_m P, \ \xi \in A_m.$$

Indeed, one easily sees that ω^b induces a well-defined map:

$$\phi : \frac{T_m P}{\ker \omega_m \cap T_m P} \longrightarrow \left(\frac{A_m}{\ker \omega_m \cap A_m} \right)^*, \tag{7}$$

$$\langle \phi[v], [\xi] \rangle = \langle \omega^b(v), \xi \rangle = \omega(v, \overrightarrow{\xi}(m)) \quad \forall v \in T_m P, \quad \xi \in A_m.$$

The following result plays an essential role in understanding the non-degeneracy condition.

Proposition 2.6. *Assume that* $(\Gamma \rightrightarrows P, \omega + \Omega)$ *is a pre-quasi-symplectic groupoid. Then,* ϕ *is a linear isomorphism.*

Proof. Assume that $\phi[v] = 0$ for $v \in T_m P$. Then, $\omega(v, \overrightarrow{\xi}(m)) = 0, \forall \xi \in A_m$, which implies that $v \in \ker \omega_m$ since $\epsilon^* \omega = 0$. Hence, $[v] = 0$. So ϕ is injective.

Conversely, assume that $\xi \in A_m$ satisfies the property that $\langle \phi[v], [\xi] \rangle = 0, \forall v \in T_m P$. Hence, $\omega(\overrightarrow{\xi}(m), v) = 0, \forall v \in T_m P$. This implies that $\overrightarrow{\xi}(m) \in \ker \omega_m$. Therefore, $\xi \in \ker \omega_m \cap A_m$, or $[\xi] = 0$. This implies that ϕ is surjective. q.e.d.

An immediate consequence is the following result, which gives a useful way of characterizing a quasi-symplectic groupoid.

Proposition 2.7. *A pre-quasi-symplectic groupoid* $(\Gamma \rightrightarrows P, \omega + \Omega)$ *is a quasi-symplectic groupoid if and only if*

(1) *the anchor* $a : \ker \omega_m \cap A_m \to \ker \omega_m \cap T_m P$ *is injective, and*
(2) $\dim \Gamma = 2 \dim P$.

Proof. By Proposition 2.6 and using dimension counting, we have

$$\dim(\ker \omega_m \cap A_m) - \dim(\ker \omega_m \cap T_m P) = \dim \Gamma - 2 \dim P. \tag{8}$$

Assume that $(\Gamma \rightrightarrows P, \omega+\Omega)$ is a quasi-symplectic groupoid. Equation (8) implies that $\dim \Gamma = 2 \dim P$. The converse is proved by working backwards. q.e.d.

A special class of quasi-symplectic groupoids are the so called twisted symplectic groupoids [**10**], which are pre-quasi-symplectic groupoids $(\Gamma \rightrightarrows P, \omega + \Omega)$ such that ω is honestly non-degenerate. In particular, symplectic groupoids [**32**] are always quasi-symplectic. In the next subsection, we will discuss another class of quasi-symplectic groupoids motivated by the Lie group valued momentum map theory of Alekseev–Malkin–Meinrenken [**2**].

2.3. AMM quasi-symplectic groupoids. First of all, let us fix some notations. Assume that a Lie group G acts smoothly on a manifold M from the left. By a transformation groupoid, we mean the groupoid $G \times M \rightrightarrows M$, where the source and target maps are given, respectively, by $s(g,x) = gx,\ t(g,x) = x$, $\forall (g,x) \in G \times M$, and the multiplication is $(g_1, x) \cdot (g_2, y) = (g_1 g_2, y)$, where $x = g_2 y$.

Let G be a Lie group equipped with an ad-invariant non-degenerate symmetric bilinear form $(\cdot,\cdot)$. Consider the transformation groupoid $G \times G \rightrightarrows G$, where G acts on itself by conjugation. Following [**2**], we denote by θ and $\bar{\theta}$ the left and right Maurer–Cartan forms on G respectively, i.e., $\theta = g^{-1}dg$ and $\bar{\theta} = dgg^{-1}$. Let $\Omega \in \Omega^3(G)$ denote the bi-invariant 3-form on G corresponding to the Lie algebra 3-cocycle $(1/12)(\cdot, [\cdot,\cdot]) \in \wedge^3 \mathfrak{g}^*$:

$$\Omega = \frac{1}{12}(\theta, [\theta, \theta]) = \frac{1}{12}(\bar{\theta}, [\bar{\theta}, \bar{\theta}]) \tag{9}$$

and $\omega \in \Omega^2(G \times G)$ the two-form:

$$\omega|_{(g,x)} = -\frac{1}{2}[(Ad_x \operatorname{pr}_1^* \theta, \operatorname{pr}_1^* \theta) + (\operatorname{pr}_1^* \theta, \operatorname{pr}_2^*(\theta + \bar{\theta}))], \tag{10}$$

where (g,x) denotes the coordinate in $G \times G$, and pr_1 and $\operatorname{pr}_2 : G \times G \to G$ are the natural projections.

Proposition 2.8. *Let G be a Lie group equipped with an ad-invariant non-degenerate symmetric bilinear form $(\cdot,\cdot)$. Then, the transformation groupoid $(G \times G \rightrightarrows G, \omega + \Omega)$ is a quasi-symplectic groupoid, called the AMM quasi-symplectic groupoid.*

Proof. First, one needs to check that $\omega + \Omega$ is a 3-cocycle. This can be done by a tedious computation, and is left for the reader.

It remains to check the non-degeneracy condition, which is in fact embedded in the proof of Proposition 3.2 [2]. For completeness, let us sketch a proof below.

The Lie algebroid A of $G \times G \rightrightarrows G$ is a transformation Lie algebroid: $\mathfrak{g} \times G \to G$, where the anchor map $a : \mathfrak{g} \times G \to TG$ is given by $a(\xi, x) = r_x(\xi) - l_x(\xi)$, $\forall \xi \in \mathfrak{g}$. Therefore, $a(\xi, x) = 0$ if and only if $Ad_x\xi = \xi$. On the other hand, for any $\xi \in \mathfrak{g}$ being identified with an element in A_x, we have $\overrightarrow{\xi}|_{(1,x)} = (\xi, 0) \in T^t_{(1,x)}(G \times G)$. For any $\delta_x \in T_xG$, let $\delta_{(1,x)} = (0, \delta_x) \in T_{(1,x)}(G \times G)$. Clearly, $\delta_{(1,x)}$ is a tangent vector to the unit space.

It follows from Equation (10) that

$$\omega(\overrightarrow{\xi}|_{(1,x)}, \delta_{(1,x)}) = \omega((\xi, 0), (0, \delta_x)) = -\frac{1}{2}\delta_x \lrcorner (\xi, \theta + \bar{\theta}).$$

Therefore, we have $\epsilon^*(\overrightarrow{\xi}|_{(1,x)} \lrcorner \omega) = \frac{1}{2}(\xi, \theta + \bar{\theta})$. Hence, $\overrightarrow{\xi}|_{(1,x)} \lrcorner \omega = 0$ if and only if $(Ad_x + 1)\xi = 0$. This implies that $a : \ker \omega_x \cap A_x \to \ker \omega_x \cap T_xG$ is injective. Therefore, it follows from Proposition 2.7 that $(G \times G \rightrightarrows G, \omega + \Omega)$ is indeed a quasi-symplectic groupoid. q.e.d.

Remark 2.9. From the above proposition, we see that $[\omega + \Omega]$ defines a class in the equivariant cohomology $H^3_G(G)$. When G is a compact simple Lie group with the basic form $(\cdot, \cdot)$, $[\omega + \Omega]$ is a generator of $H^3_G(G)$. In Cartan model, it corresponds to the class defined by the d_G-closed equivariant 3-form $\chi_G(\xi) = \Omega - \frac{1}{2}(\theta + \bar{\theta}, \xi) : \mathfrak{g} \longrightarrow \Omega^*(G)$, $\forall \xi \in \mathfrak{g}$ (see [8, 22]).

3. Hamiltonian Γ-spaces

3.1. Definitions and properties. In this section, we introduce the notion of Hamiltonian Γ-spaces for a quasi-symplectic groupoid $\Gamma \rightrightarrows P$, which generalizes the usual notion of Hamiltonian spaces of symplectic groupoids in the sense of Mikami–Weinstein [26].

First, we need the following:

Definition 3.1. Given a quasi-symplectic groupoid $(\Gamma \rightrightarrows P, \omega + \Omega)$, let $J : X \to P$ be a left Γ-space, i.e., Γ acts on X from the left. By a compatible two-form on X, we mean a two-form $\omega_X \in \Omega^2(X)$ satisfying

(1) $d\omega_X = J^*\Omega$ and

(2) the graph of the action $\Lambda = \{(r, x, rx) | t(r) = J(x)\} \subset \Gamma \times X \times X$ is isotropic with respect to the two-form $(\omega, \omega_X, -\omega_X)$.

Then, $(X \xrightarrow{J} P, \omega_X)$ is called a pre-Hamiltonian Γ-space.

In the sequel, we simply refer to the second condition as to "the graph of the action $\Lambda \subset \Gamma \times X \times \overline{X}$ is isotropic", where the bar on the last factor X indicates that the opposite two-form is used.

To illustrate the intrinsic meaning of the above compatibility condition, let us elaborate it in terms of groupoids. Let $Q := \Gamma \times_P X \rightrightarrows X$ denote the transformation groupoid corresponding to the Γ-action, and, by abuse of notation, $J : Q \to \Gamma$ the natural projection. It is simple to see that

$$\begin{tikzcd} Q \arrow[r, "J"] \arrow[d, shift left] \arrow[d, shift right] & \Gamma \arrow[d, shift left] \arrow[d, shift right] \\ X \arrow[r, "J"] & P \end{tikzcd} \tag{11}$$

is a Lie groupoid homomorphism. Therefore, it induces a map, i.e., the pull-back map, on the level of de-Rham complex

$$J^* : \quad \Omega^\bullet(\Gamma_\bullet) \to \Omega^\bullet(Q_\bullet).$$

Proposition 3.2. *Let $(\Gamma \rightrightarrows P, \omega + \Omega)$ be a quasi-symplectic groupoid and $J : X \to P$ a left Γ-space. Then, $\omega_X \in \Omega^2(X)$ is a compatible two-form if and only if*

$$J^*(\omega + \Omega) = \delta\omega_X. \tag{12}$$

Proof. Note that

$$\delta\omega_X = (s^*\omega_X - t^*\omega_X) + d\omega_X,$$

where $s, t : \Gamma \times_P X \to X$ are the source and target maps of the groupoid $\Gamma \times_P X \rightrightarrows X$. So, Equation (12) is equivalent to

$$s^*\omega_X - t^*\omega_X = J^*\omega \qquad \text{and} \qquad d\omega_X = J^*\Omega.$$

It is simple to see that the first equation above is equivalent to that the graph of the action $\Lambda \subset \Gamma \times X \times \overline{X}$ is isotropic by using the source and target maps $s(r, x) = r \cdot x$ and $t(r, x) = x$, $\forall (r, x) \in \Gamma \times_P X$. q.e.d.

Remark 3.3. As a consequence, $J^* : H^3_{DR}(\Gamma_\bullet) \to H^3_{DR}(Q_\bullet)$ maps $[\omega + \Omega]$ into zero. When $[\omega + \Omega]$ is of integral class, it defines an S^1-gerbe over the stack $\mathfrak{X}_\Gamma$ corresponding to the groupoid $\Gamma \rightrightarrows P$, the above proposition implies that the pull-back S^1-gerbe over $\mathfrak{X}_Q$ is always trivial.

If Γ is the symplectic groupoid $T^*G \rightrightarrows \mathfrak{g}^*$, Q can be identified with the transformation groupoid $G \times X \rightrightarrows X$ and the groupoid homomorphism $J : Q\ (\cong G\times X) \to \Gamma\ (\cong G\times\mathfrak{g}^*)$ is simply $id\times J$. In this case, $H^3_{DR}(\Gamma_\bullet) \cong H^3_G(\mathfrak{g}^*)$ and $H^3_{DR}(Q_\bullet) \cong H^3_G(X)$. In Cartan model, Equation (12) is equivalent to

$$d_G\omega_X = J^*\chi_G(\xi).$$

Here, $\chi_G \in \Omega^3_G(\mathfrak{g}^*)$ is the equivariant closed 3-form defined as $\chi_G(\xi) = -d\langle a, \xi\rangle$, where $a : \mathfrak{g}^* \to \mathfrak{g}^*$ is the identity map. Similarly, if Γ is the AMM quasi-symplectic groupoid $G \times G \rightrightarrows G$, Q is isomorphic to the transformation groupoid $G \times X \rightrightarrows X$. Then, the relevant d_G-closed equivariant 3-form $\chi_G \in \Omega^3_G(G)$ is

$$\chi_G(\xi) = \Omega - \frac{1}{2}(\theta + \bar{\theta}, \xi).$$

See Remark 2.1 of [**2**].

Note that in the first case, $\chi_G \in \Omega^3_G(\mathfrak{g}^*)$ defines a zero class in $H^3_G(\mathfrak{g}^*)$, while in the case of the AMM quasi-symplectic groupoid, $\chi_G \in \Omega^3_G(G)$ defines a non-zero class in $H^3_G(G)$. This fact is the key ingredient for explaining the difference of their quantization theories, while in the latter case, S^1-gerbes are inevitable in the construction [**17**].

As is well known, a Lie groupoid action induces a Lie algebroid action, called the infinitesimal action, which can be described as follows. For any $x \in X$ and any $\xi \in A_m$, where $J(x) = m$, let $\gamma(t)$ be a path in the t-fiber $t^{-1}(m)$ of Γ through the point m such that $\dot{\gamma}(0) = \overrightarrow{\xi}(m)$, and define $\hat{\xi}(x) \in T_xX$ to be the tangent vector corresponding to the curve $\gamma(t) \cdot x$ through the point x. In this way, one obtains a linear map

$$A_m \longrightarrow T_xX, \qquad \xi \to \hat{\xi}(x)$$

called the infinitesimal action. In particular, this action induces a Lie algebra homomorphism $\Gamma(A) \to \mathfrak{X}(X)$. One also easily checks that

$$a(\xi) = J_*\hat{\xi}(x), \quad \forall \xi \in A_m.$$

The following lemma follows easily from the compatibility condition in Definition 3.1 (2).

Lemma 3.4. *Let* $(\Gamma \rightrightarrows P, \omega + \Omega)$ *be a quasi-symplectic groupoid. If a* Γ*-space* $J : X \to P$ *equipped with a two-form* ω_X *satisfies the compatibility condition in Definition 3.1* (2)*, then for any* $x \in X$ *such that* $J(x) = m$ *and any* $\xi \in A_m$*, we have*

$$J^*\epsilon^*(\overrightarrow{\xi}(m) \lrcorner \omega) = \hat{\xi}(x) \lrcorner \omega_X. \tag{13}$$

Proof. It is simple to see that for any $\xi \in A_m$, $(\overrightarrow{\xi}(m), 0, \hat{\xi}(x))$ is tangent to Λ. On the other hand, $\forall \delta_x \in T_xX$, $(J_*\delta_x, \delta_x, \delta_x)$ is also tangent to Λ. Thus, it follows that

$$\omega(\overrightarrow{\xi}(m), J_*\delta_x) - \omega_X(\hat{\xi}(x), \delta_x) = 0.$$

Equation (13) thus follows immediately. q.e.d.

From this lemma, one easily sees that if $\overrightarrow{\xi}(m) \in \ker \omega$, then $\hat{\xi}(x)$ automatically belongs to the kernel of ω_X. As in [**2**], we impose the following minimal non-degeneracy condition.

Definition 3.5. Let $(\Gamma \rightrightarrows P, \omega + \Omega)$ be a quasi-symplectic groupoid. A Hamiltonian Γ-space is a left Γ-space $X \to P$ equipped with a compatible two-form ω_X such that $\forall x \in X$,

$$\ker \omega_X|_x = \{\hat{\xi}(x) | \xi \in A_{J(x)} \text{ such that } \overrightarrow{\xi}(J(x)) \in \ker \omega\}. \tag{14}$$

For any $x \in X$, by A^x_x, we denote the linear subspace of $A_{J(x)}$ consisting of those vectors $\xi \in A_{J(x)}$ such that $\hat{\xi}(x) = 0$.

Lemma 3.6. *Assume that* $(\Gamma \rightrightarrows P, \omega + \Omega)$ *is a quasi-symplectic groupoid and* $J : X \to P$ *is a* Γ*-space equipped with a compatible two-form* ω_X*. Then*

(1) $\dim J_*(T_xX) \leq \operatorname{rank} A - \dim A^x_x$,

(2) *if, moreover,* $(X \xrightarrow{J} P, \omega_X)$ *is an Hamiltonian* Γ*-space, then*
   (a) $\ker \omega_{J(x)} \cap A_{J(x)} \to \ker \omega_X|_x$, $\xi \to \hat{\xi}(x)$ *is an isomorphism and*
   (b) $\ker J_* \cap \ker \omega^b_X = 0$.

Proof.
(1) $\forall \delta_x \in T_xX$ and $\xi \in A^x_x$, we have

$$\langle \phi[J_*\delta_x], [\xi] \rangle = \omega(J_*\delta_x, \overrightarrow{\xi}(J(x))) = -\omega_X(\hat{\xi}(x), \delta_x) = 0,$$

where ϕ is the linear isomorphism defined by Equation (7). This implies that

$$\phi[\mathrm{pr}_1(J_*(T_xX))] \subseteq (\mathrm{pr}_2 A^x_x)^\perp,$$

where

$$\mathrm{pr}_1 : T_{J(x)}P \to \frac{T_{J(x)}P}{\ker \omega_{J(x)} \cap T_{J(x)}P}$$

and

$$\mathrm{pr}_2 : A_{J(x)} \to \frac{A_{J(x)}}{\ker \omega_{J(x)} \cap A_{J(x)}}$$

are projections.

Second, we note that pr_2 is injective when being restricted to A_x^x. To see this, we only need to show that $A_x^x \cap (\ker \omega_{J(x)} \cap A_{J(x)}) = 0$. Assume that $\xi \in A_x^x \cap (\ker \omega_{J(x)} \cap A_{J(x)})$. Then, we have $\hat{\xi}(x) = 0$ and $\overrightarrow{\xi}(J(x)) \lrcorner \omega = 0$. Hence, $a(\xi) = J_*\hat{\xi}(x) = 0$, which implies that $\xi = 0$ by Definition 2.5. As a consequence, we have $\dim(\mathrm{pr}_2 A_x^x) = \dim A_x^x$. Hence,

$$\begin{aligned}
&\dim J_*(T_xX) - \dim(\ker \omega_{J(x)} \cap T_{J(x)}P) \\
&\quad \le \dim \mathrm{pr}_1(J_*(T_xX)) \quad (\text{since } \phi \text{ is a linear isomorphism}) \\
&\quad = \dim \phi[\mathrm{pr}_1(J_*(T_xX))] \\
&\quad \le \dim(\mathrm{pr}_2 A_x^x)^\perp \\
&\quad = [\operatorname{rank} A - \dim(\ker \omega_J(x) \cap A_J(x))] - \dim(\mathrm{pr}_2 A_x^x) \\
&\quad = [\operatorname{rank} A - \dim(\ker \omega_J(x) \cap A_J(x))] - \dim A_x^x.
\end{aligned}$$

Thus (1) follows immediately since Γ is a quasi-symplectic groupoid.

2a) By the minimal non-degeneracy assumption, we know that the map

$$\ker \omega_{J(x)} \cap A_{J(x)} \to \ker \omega_X|_x, \qquad \xi \to \hat{\xi}(x)$$

is surjective. To show that it is injective, assume that $\xi \in \ker \omega_{J(x)} \cap A_{J(x)}$ such that $\hat{\xi}(x) = 0$. Then, $a(\xi) = J_*\hat{\xi}(x) = 0$. Since ω is non-degenerate in the sense of Definition 2.5, we have $\xi = 0$.

2b) Assume that $\delta_x \in \ker J_* \cap \ker \omega_X^b$. Since $J : X \longrightarrow P$ is an Hamiltonian Γ-space, by assumption, we have $\delta_x = \hat{\xi}(x)$, where $\xi \in A_{J(x)}$ such that $\overrightarrow{\xi}(J(x)) \lrcorner \omega = 0$. Hence, $a(\xi) = J_*\hat{\xi}(x) = J_*\delta_x = 0$, and therefore, $\xi = 0$ since Γ is a quasi-symplectic groupoid.

This completes the proof. q.e.d.

For a subspace $V \subseteq T_xX$, by V^{ω_X}, we denote its ω_X-orthogonal subspace of V. As a consequence, we have the following proposition which plays a key role in our reduction theory.

Proposition 3.7. *Assume that* $(\Gamma \rightrightarrows P, \omega + \Omega)$ *is a quasi-symplectic groupoid, and* $(X \xrightarrow{J} P, \omega_X)$ *an Hamiltonian* Γ*-space. Then*

$$(\ker J_*)^{\omega_X} = \{\hat{\xi}(x) | \forall\ \xi \in A_{J(x)}\}. \tag{15}$$

Proof. It is simple to see that $(\ker J_*)^{\omega_X} = [\omega_X^b(\ker J_*)]^{\perp}$. Therefore, it follows that

$$\begin{aligned}
&\dim(\ker J_*)^{\omega_X} \\
&\quad = \dim X - \dim[\omega_X^b(\ker J_*)] \\
&\qquad\qquad \text{(since } \omega_X^b \text{ is injective when being restricted to } \ker J_*) \\
&\quad = \dim X - \dim \ker J_* \\
&\quad = \dim J_*(T_x X) \qquad \text{(by Lemma 3.6)} \\
&\quad \leq \operatorname{rank} A - \dim A_x^x \\
&\quad = \dim\{\hat{\xi}(x) | \forall\ \xi \in A_{J(x)}\}.
\end{aligned}$$

On the other hand, clearly, we have

$$\{\hat{\xi}(x) | \forall\ \xi \in A_{J(x)}\} \subseteq (\ker J_*)^{\omega_X}$$

according to Equation (13). Thus, Equation (15) follows immediately. q.e.d.

3.2. Two fundamental examples. Below, we study two fundamental examples of Hamiltonian Γ-spaces, which are naturally associated to a quasi-symplectic groupoid.

Proposition 3.8. *Assume that* $(\Gamma \rightrightarrows P, \omega + \Omega)$ *is a quasi-symplectic groupoid. Then*

(1) $J : \Gamma \to P \times P$ *is an Hamiltonian* $\Gamma \times \overline{\Gamma}$*-space, where* $J(r) = (s(r), t(r)),\ \forall r \in \Gamma$*, and the action is defined by*

$$(r_1, r_2) \cdot x = r_1 x r_2^{-1}, \quad t(r_1) = s(x),\ t(x) = t(r_2).$$

(2) *Given any orbit* $\mathcal{O} \subset P$*, there is a natural two-form* $\omega_{\mathcal{O}} \in \Omega^2(\mathcal{O})$ *so that the natural inclusion* $i : \mathcal{O} \to P$ *defines an Hamiltonian* Γ*-space under the natural* Γ*-action.*

Proof.

(1) It is clear, from definition, that $d\omega = J^*\Omega$. To check the second compatibility condition of Definition 3.1, it suffices to show that

$$\{(r_1, r_2, x, r_1 x r_2^{-1}) |\ t(r_1) = s(x),\ t(x) = t(r_2)\} \subset \Gamma \times \overline{\Gamma} \times \Gamma \times \overline{\Gamma}$$

is isotropic. This can be proved using the multiplicativity assumption on ω, i.e., $\partial\omega = 0$, as in [**33**]. To check the minimal non-degeneracy condition, note that for any $\xi, \eta \in \Gamma(A)$, the vector field on Γ generated by the infinitesimal action of (ξ, η) is given by $\overrightarrow{\xi}(x) - \overleftarrow{\eta}(x)$. Next, note that for any $\delta_x \in T_x\Gamma,\ \xi \in \Gamma(A)$, we have

(16)
$$\omega(\overleftarrow{\xi}(x), \delta_x) = \omega(\overleftarrow{\xi}(t(x)), t_*\delta_x), \qquad \omega(\overrightarrow{\xi}(x), \delta_x) = \omega(\overrightarrow{\xi}(s(x)), s_*\delta_x).$$

These equations follow essentially from Equation (13) since $s : \Gamma \to P$ equipped with the natural left Γ-action (or $t : \Gamma \to P$ with the left Γ-action: $r \cdot x = xr^{-1}$, respectively) satisfies the hypothesis of Lemma 3.4.

Now, assume that $\delta_x \in \ker\omega$. Then, $t_*\delta_x \in \ker\omega$ by Equation (16), since P is isotropic. By the non-degeneracy assumption, we have $t_*\delta_x = a(\eta)$ for some $\eta \in A|_{t(x)}$ such that $\overrightarrow{\eta}(t(x)) \in \ker\omega$. Hence, $\overleftarrow{\eta}(t(x)) \in \ker\omega$ by Corollary 2.4 (3), which in turn implies that $\overleftarrow{\eta}(x) \in \ker\omega$ according to Equation (16). Let $\delta'_x = \delta_x + \overleftarrow{\eta}(x)$. Then,

$$t_*\delta'_x = t_*\delta_x + t_*\overleftarrow{\eta}(x) = t_*\delta_x - a(\eta) = 0.$$

Also, we know that $\delta'_x \in \ker\omega$. Therefore, one can write $\delta'_x = \overrightarrow{\xi}(x)$ where $\xi \in A_{s(x)}$ such that $\overrightarrow{\xi}(s(x)) \in \ker\omega$. We thus have proved that $\delta_x = \overrightarrow{\xi}(x) - \overleftarrow{\eta}(x)$, where $\overrightarrow{\eta}(t(x)) \in \ker\omega$ and $\overrightarrow{\xi}(s(x)) \in \ker\omega$.

(2) Let $\mathcal{O} \subset P$ be the groupoid orbit through the point $m_0 \in P$. It is standard that $t^{-1}(m_0) \xrightarrow{s} \mathcal{O}$ is a $\Gamma^{m_0}_{m_0}$-principal bundle, where $\Gamma^{m_0}_{m_0}$ denotes the isotropy group at m_0. From the multiplicativity assumption on ω, it is simple to see that $\omega|_{t^{-1}(m_0)}$, the pull-back of ω to the t-fiber $t^{-1}(m_0)$, is indeed basic with respect to the $\Gamma^{m_0}_{m_0}$-action. Hence, it descends to a two-form $\omega_{\mathcal{O}}$ on $\mathcal{O}$. That is, $\omega|_{t^{-1}(m_0)} = s^*\omega_{\mathcal{O}}$. It thus follows that

$$s^*d\omega_{\mathcal{O}} = (s^*\Omega - t^*\Omega)|_{t^{-1}(m_0)},$$

which implies that $d\omega_{\mathcal{O}} = i^*\Omega$. It is also clear that the two-form $\omega_{\mathcal{O}}$ is compatible with the groupoid Γ-action since ω is multiplicative. To show the minimal non-degeneracy condition, assume that $x \in t^{-1}(m_0)$ is an arbitrary point, and $\delta_x \in T_xt^{-1}(m_0)$ such that $[\delta_x] = s_*\delta_x \in \ker\omega_{\mathcal{O}}|_m$, where $m = s(x)$. By definition, $\omega(\delta_x, \delta'_x) = 0,\ \forall\delta'_x \in T_xt^{-1}(m_0)$. It thus follows that $\omega(r_{x^{-1}}\delta_x, r_{x^{-1}}\delta'_x) = 0$. Let $\xi, \eta \in A_m$ such that $r_{x^{-1}}\delta_x = \overrightarrow{\xi}(m)$ and $r_{x^{-1}}\delta'_x = \overrightarrow{\eta}(m)$. Thus, we have $\omega(\overrightarrow{\xi}(m), \overrightarrow{\eta}(m)) = 0,\ \forall\eta \in$

A_m. Therefore

$$\begin{aligned}\omega(a(\xi), \overrightarrow{\eta}(m)) &= \omega(\overrightarrow{\xi}(m) - \overleftarrow{\xi}(m),\ \overrightarrow{\eta}(m)) \\ &= \omega(\overrightarrow{\xi}(m), \overrightarrow{\eta}(m)) = 0,\ \forall \eta \in A_m.\end{aligned}$$

It thus follows that $a(\xi) \in \ker \omega$ since $\omega(a(\xi), T_m P) = 0$. That is, $a(\xi) \in \ker \omega_m \cap T_m P$. By the non-degeneracy assumption on ω (see Definition 2.5), we deduce that there exists $\xi_1 \in A_m$ such that $\overrightarrow{\xi_1}(m) \in \ker \omega$ and $a(\xi_1) = a(\xi)$. So $\xi - \xi_1$ belongs to the isotropy Lie algebra at m. As a result, it follows that the minimal non-degeneracy condition is indeed satisfied since $[\delta_x] = s_* \delta_x = \hat{\xi}(m) = \hat{\xi_1}(m)$. q.e.d.

3.3. Examples of Hamiltonian Γ-spaces. In this subsection, we list various examples of momentum maps appeared in the literature, which can be considered as special cases of our Hamiltonian Γ-spaces. In fact, our definition is a natural generalization of Hamiltonian Γ-spaces of a symplectic groupoid of Mikami–Weinstein [**26**], which include the usual Hamiltonian momentum maps and Lu's momentum maps of Poisson group actions as special cases.

Example 3.9. Consider the symplectic groupoid $(T^*G \rightrightarrows \mathfrak{g}^*, \omega)$, where ω is the standard cotangent symplectic structure. Then, its Hamiltonian spaces are exactly the Hamiltonian G-spaces $J : X \to \mathfrak{g}^*$ in the ordinary sense.

Example 3.10. When $P = G^*$, the dual of a simply connected complete Poisson Lie group G, its symplectic groupoid Γ is a transformation groupoid: $G \times G^* \rightrightarrows G^*$, where G acts on G^* by left dressing action [**20**]. In this case, Hamiltonian Γ-spaces can be described in terms of the so-called Poisson G-spaces. Recall that a symplectic (or more generally a Poisson) manifold X with a left G-action is called a Poisson G-space if the action map $\overline{G} \times X \to X$ is a Poisson map. A Poisson morphism $J : X \to G^*$ is said to be a momentum map for the Poisson G-space [**19**], if

$$X \in \mathfrak{g} \mapsto -\pi_X^{\#}(J^*(X^r)) \in \mathfrak{X}(X) \tag{17}$$

is the infinitesimal generator of the G-action, where X^r denotes the right-invariant one-form on G^* with value $X \in \mathfrak{g}^*$ at the identity, and π_X is the Poisson tensor on X. An explicit relation between Hamiltonian Γ-spaces and Poisson G-spaces can be established as follows [**35**]. If

$J : X \to G^*$ is an Hamiltonian Γ-space, then X is a Poisson G-space with the action:

$$gx = (g, J(x)) \cdot x \tag{18}$$

for any $g \in G$ and $x \in X$, where $(g, J(x))$ is considered as an element in $\Gamma = G \times G^*$ and the dot on the right-hand side refers to the groupoid Γ-action on X. Then, J is the momentum map of the induced Poisson G-action, in the sense of Lu [19]. Conversely, if a symplectic manifold X is a Poisson G-space with a momentum mapping $J : X \to G^*$, Equation (18) defines an Hamiltonian Γ-space.

Example 3.11. Let $(\cdot, \cdot)$ be an ad-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$. It is well-known that $(\cdot, \cdot)$ induces a Lie algebra 2-cocycle $\lambda \in \wedge^2(L\mathfrak{g}^*)$ on the loop Lie algebra defined by [29]:

$$\lambda(X, Y) = \frac{1}{2\pi} \int_0^{2\pi} (X(s), Y'(s)) ds \ \forall X(s), Y(s) \in L\mathfrak{g} \tag{19}$$

and therefore defines an affine Poisson structure on $L\mathfrak{g}$. Its symplectic groupoid Γ can be identified with the transformation groupoid $LG \times L\mathfrak{g} \rightrightarrows L\mathfrak{g}$, where LG acts on $L\mathfrak{g}$ by the gauge transformation [6]:

$$g \cdot \xi = Ad_g \xi + g' g^{-1} \quad \forall g \in LG, \ \xi \in L\mathfrak{g}. \tag{20}$$

This is the standard gauge transformation when $L\mathfrak{g}$ is identified with the space of connections on the trivial G-bundle over the unit circle S^1. The symplectic structure on $LG \times L\mathfrak{g}$ can be obtained as follows. By $\widetilde{L\mathfrak{g}}$, we denote the corresponding Lie algebra central extension. Assume that λ satisfies the integrality condition (i.e., the corresponding closed two-form $\omega_{LG} \in \Omega^2(LG)^{LG}$ is of integer class). It defines a loop group central extension $S^1 \longrightarrow \widetilde{LG} \xrightarrow{\pi} LG$. Consider $\widetilde{\pi} : \widetilde{LG} \times L\mathfrak{g} \to LG \times L\mathfrak{g}$, where $\widetilde{\pi} = \pi \times id$. Let i denote the embedding $\widetilde{LG} \times L\mathfrak{g} \cong \widetilde{LG} \times (L\mathfrak{g} \times \{1\}) \subset \widetilde{LG} \times \widetilde{L\mathfrak{g}} \cong T^*\widetilde{LG}$. Then

$$\widetilde{\pi}^* \omega_{LG \times L\mathfrak{g}} = i^* \omega_{T^*\widetilde{LG}}.$$

In this case, the corresponding Hamiltonian Γ-spaces are exactly Hamiltonian loop group spaces studied extensively by Meinrenken–Woodward [23, 24, 25].

Example 3.12. Let Γ be the AMM quasi-symplectic groupoid $(G \times G \rightrightarrows G, \omega + \Omega)$. It is simple to see that Hamiltonian Γ-spaces correspond exactly to quasi-Hamiltonian G-spaces with a group valued momentum map $J : X \to G$ in the sense of [2], namely those G-spaces X

equipped with a G-invariant two-form $\omega_X \in \Omega(X)^G$ and an equivariant map $J \in C^\infty(X, G)^G$ such that

(B1) The differential of ω_X is given by

$$d\omega_X = J^*\Omega.$$

(B2) The map J satisfies

$$\hat{\xi} \lrcorner \omega_X = \frac{1}{2} J^*(\xi, \theta + \bar{\theta}), \forall \xi \in \mathfrak{g}.$$

(B3) At each $x \in X$, the kernel of ω_X is given by

$$\ker \omega_X|_x = \{\hat{\xi}(x) |\ \xi \in \ker(\mathrm{Ad}_{J(x)} + 1)\}.$$

3.4. Hamiltonian bimodules. A useful way to study Hamiltonian Γ-spaces is via the Hamiltonian bimodules.

Definition 3.13. Given quasi-symplectic groupoids $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ and $(H \rightrightarrows H_0, \omega_H + \Omega_H)$, an Hamiltonian G-H-bimodule is a manifold X equipped with a two-form $\omega_X \in \Omega^2(X)$ such that

(1) $G_0 \xleftarrow{\rho} X \xrightarrow{\sigma} H_0$ is a left G-space and a right H-space, and the two actions commute;
(2) $X \xrightarrow{\rho \times \sigma} G_0 \times H_0$ is an Hamiltonian $G \times \overline{H}$-space, where the action is given by $(g, h) \cdot x = gxh^{-1}, \ \ \forall g \in G, \ h \in H, \ x \in X$ such that $t(g) = \rho(x)$ and $t(h) = \sigma(x)$.

In particular, an Hamiltonian Γ-space can be considered as an Hamiltonian Γ-$\cdot$-bimodule, where $\cdot$ denotes the trivial quasi-symplectic groupoid $\cdot \rightrightarrows \cdot$.

Given an Hamiltonian G–H-bimodule $G_0 \xleftarrow{\rho} X \xrightarrow{\sigma} H_0$, let $Q \rightrightarrows X$ be the transformation groupoid

$$Q := (G \times H) \times_{(G_0 \times H_0)} X \rightrightarrows X.$$

Then, the natural projections $\mathrm{pr}_1 : Q \to G$ and $\mathrm{pr}_2 : Q \to H$ are groupoid homomorphisms. As an immediate consequence of Proposition 3.2, we have the following

Proposition 3.14. *If $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ and $(H \rightrightarrows H_0, \omega_H + \Omega_H)$ are quasi-symplectic groupoids, and $G_0 \xleftarrow{\rho} X \xrightarrow{\sigma} H_0$ is an Hamiltonian G-H-bimodule, then*

$$\mathrm{pr}_1^*(\omega_G + \Omega_G) - \mathrm{pr}_2^*(\omega_H + \Omega_H) = \delta\omega_X.$$

Therefore, on the level of cohomology, we have

$$\mathrm{pr}_1^*[\omega_G + \Omega_G] = \mathrm{pr}_2^*[\omega_H + \Omega_H],$$

where $\mathrm{pr}_1^* : H^3_{DR}(G_\bullet) \to H^3_{DR}(Q_\bullet)$ and $\mathrm{pr}_2^* : H^3_{DR}(H_\bullet) \to H^3_{DR}(Q_\bullet)$ are the homomorphisms of cohomology groups induced by the groupoid homomorphisms $\mathrm{pr}_1 : Q \to G$ and $\mathrm{pr}_2 : Q \to H$, respectively.

Let $(G \rightrightarrows G_0, \omega_G + \Omega_G)$, $(H \rightrightarrows H_0, \omega_H + \Omega_H)$, and $(K \rightrightarrows K_0, \omega_K + \Omega_K)$ be quasi-symplectic groupoids. Assume that $G_0 \overset{\rho_1}{\leftarrow} X \overset{\sigma_1}{\to} H_0$ is an Hamiltonian G-H-bimodule, and $H_0 \overset{\rho_2}{\leftarrow} Y \overset{\sigma_2}{\to} K_0$ an Hamiltonian H-K-bimodule. Moreover, we assume that the fiber product $X \times_{H_0} Y$ is a manifold (for instance, this is true if $\sigma_1 \times \rho_2 : X \times Y \to H_0 \times H_0$ is transversal to the diagonal) and the diagonal H-action on $X \times_{H_0} Y$, $h \cdot (x, y) = (x \cdot h^{-1}, h \cdot y)$, is free and proper so that the quotient space is a smooth manifold, which is denoted by $X \times_H Y$. That is

$$X \times_H Y := \frac{X \times_{H_0} Y}{H}.$$

Let $\rho_3 : X \times_H Y \to G_0$ and $\sigma_3 : X \times_H Y \to K_0$ be the maps given by $\rho_3([x, y]) = \rho_1(x)$ and $\sigma_3([x, y]) = \sigma_2(y)$, respectively. Define a left G-action and a right K-action on $X \times_H Y$ by

$$g \cdot [x, y] = [g \cdot x, y] \quad \text{and} \quad [x, y] \cdot k = [x, y \cdot k], \tag{21}$$

whenever they are defined. It is clear that $G_0 \overset{\rho_3}{\leftarrow} X \times_H Y \overset{\sigma_3}{\to} K_0$ becomes a left G- and right K-space, and that these two actions commute with each other.

To continue our discussion, we need to make a technical assumption.

Definition 3.15. We say that two smooth maps $\tau_i : X_i \to M$, $i = 1, 2$, are clean, if

(1) the fiber product $X_1 \times_M X_2$ is a smooth manifold;
(2) for any $(x_1, x_2) \in X_1 \times_M X_2$, $f_* T_{(x_1,x_2)}(X_1 \times_M X_2)$ is equal to either $\tau_{1*} T_{x_1} X_1$ or $\tau_{2*} T_{x_2} X_2$, where $f : X_1 \times_M X_2 \to M$ is defined as $f(x_1, x_2) = \tau_1(x_1) = \tau_2(x_2)$.

For instance, two maps are clean if one of them is a submersion. The main result of this subsection is the following

Theorem 3.16. *Let $(G \rightrightarrows G_0, \omega_G + \Omega_G)$, $(H \rightrightarrows H_0, \omega_H + \Omega_H)$, and $(K \rightrightarrows K_0, \omega_K + \Omega_K)$ be quasi-symplectic groupoids. Assume that $G_0 \overset{\rho_1}{\leftarrow} X \overset{\sigma_1}{\to} H_0$ is an Hamiltonian G–H-bimodule, and $H_0 \overset{\rho_2}{\leftarrow} Y \overset{\sigma_2}{\to} K_0$ is an Hamiltonian H–K-bimodule. If $Z := X \times_H Y$ is a manifold, then*

(1) *the two-form* $i^*(\omega_X \oplus \omega_Y) \in \Omega^2(X \times_{H_0} Y)$, *where* $i : X \times_{H_0} Y \to X \times Y$ *is the natural inclusion, descends to a two-form* ω_Z *on* Z *and*
(2) *if, moreover, it is assumed that* σ_1 *and* ρ_2 *are clean, then* (Z, ω_Z), *equipped with the left* G*- and right* K*-actions as in Equation* (21), *is an Hamiltonian* G*-*K*-bimodule.*

Proof. First, note that for any $(x, y) \in X \times_{H_0} Y$, the tangent space to the H-orbit is spanned by vectors of the form $(\hat{\xi}(x), \hat{\xi}(y))$, $\forall \xi \in A_H|_m$, where A_H is the Lie algebroid of H, and $m = \sigma_1(x) = \rho_2(y)$. Here, we let H act on X from the left: $h \cdot x = xh^{-1}$, and $\hat{\xi}(x)$ denotes the infinitesimal vector field generated by this action. Now

$$\begin{aligned}
&(\hat{\xi}(x), \hat{\xi}(y)) \lrcorner\, i^*(\omega_X \oplus \omega_Y) \\
&\quad = \hat{\xi}(x) \lrcorner\, \omega_X + \hat{\xi}(y) \lrcorner\, \omega_Y \\
&\quad = -\sigma_1^* \epsilon^*(\overrightarrow{\xi}(m) \lrcorner\, \omega_H) + \rho_2^* \epsilon^*(\overrightarrow{\xi}(m) \lrcorner\, \omega_H) \\
&\quad = 0.
\end{aligned}$$

Secondly, let $\mathcal{L}$ be any local bisection of $H \rightrightarrows H_0$. Then, $\mathcal{L}$ induces a local diffeomorphism on both X and Y, denoted by $\Phi_{\mathcal{L}}$. By the left multiplication, $\mathcal{L}$ also induces a local diffeomorphism on H itself, which again, by abuse of notation, is denoted by $\Phi_{\mathcal{L}}$. We need to prove that

$$\Phi_{\mathcal{L}}^*[i^*(\omega_X \oplus \omega_Y)] = i^*(\omega_X \oplus \omega_Y). \tag{22}$$

Given any tangent vectors $(\delta_x^i, \delta_y^i) \in T_{(x,y)}(X \times_{H_0} Y)$, $i = 1, 2$, let $u^i = \sigma_{1*}\delta_x^i = \rho_{2*}\delta_y^i \in T_m H_0$, where $m = \sigma_1(x) = \rho_2(y)$, and $\delta_h^i = \Phi_{\mathcal{L}*} u^i \in T_h H$. It is simple to see that $(0, \delta_h^i, \delta_x^i, \Phi_{\mathcal{L}*}\delta_h^i) \in T\Lambda_1$ and $(\delta_h^i, 0, \delta_y^i, \Phi_{\mathcal{L}*}\delta_y^i) \in T\Lambda_2$, where $\Lambda_1 \subset G \times \overline{H} \times X \times \overline{X}$ and $\Lambda_2 \subset H \times \overline{K} \times Y \times \overline{Y}$ are the corresponding graphs of the groupoid actions. From the compatibility condition, it follows that

$$-\omega_H(\delta_h^1, \delta_h^2) + \omega_X(\delta_x^1, \delta_x^2) - \omega_X(\Phi_{\mathcal{L}*}\delta_x^1, \Phi_{\mathcal{L}*}\delta_x^2) = 0$$

and

$$\omega_H(\delta_h^1, \delta_h^2) + \omega_Y(\delta_y^1, \delta_y^2) - \omega_Y(\Phi_{\mathcal{L}*}\delta_y^1, \Phi_{\mathcal{L}*}\delta_y^2) = 0.$$

Thus, we have

$$\begin{aligned}
&(\omega_X \oplus \omega_Y)((\delta_x^1, \delta_y^1),\ (\delta_x^2, \delta_y^2)) \\
&\quad = (\omega_X \oplus \omega_Y)((\Phi_{\mathcal{L}*}\delta_x^1, \Phi_{\mathcal{L}*}\delta_y^1),\ (\Phi_{\mathcal{L}*}\delta_x^2, \Phi_{\mathcal{L}*}\delta_y^2)).
\end{aligned}$$

Equation (22) thus follows. Therefore, we conclude that there is a two-form ω_Z on $Z := X \times_H Y$ such that

$$\pi^*\omega_Z = i^*(\omega_X \oplus \omega_Y),$$

where $\pi : X \times_{H_0} Y \to Z$ is the projection.

It is straightforward to check that

$$d\omega_Z = (\rho_3 \times \sigma_3)^*[\Omega_G \oplus \overline{\Omega_K}]$$

and the two-form ω_Z is compatible with the action of the quasi-symplectic groupoid $G \times \overline{K} \rightrightarrows G_0 \times \overline{K_0}$.

It remains to prove the minimal non-degeneracy condition. First, we need the following

Lemma 3.17. *Let $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ and $(H \rightrightarrows H_0, \omega_H + \Omega_H)$ be quasi-symplectic groupoids, $G_0 \xleftarrow{\rho} X \xrightarrow{\sigma} H_0$ an Hamiltonian G–H-bimodule with $\omega_X \in \Omega^2(X)$. Then*

$$(\ker \rho_*)^{\omega_X} = \{\hat{\xi}(x) | \xi \in A_G|_{\rho(x)}\} + \ker \omega_X \quad \textit{and} \tag{23}$$

$$(\ker \sigma_*)^{\omega_X} = \{\hat{\xi}(x) | \xi \in A_H|_{\sigma(x)}\} + \ker \omega_X, \tag{24}$$

where A_G and A_H denote the Lie algebroids of G and H, respectively.

Proof. It is obvious that $\{\hat{\xi}(x) | \xi \in A_G|_{\rho(x)}\} + \ker \omega_X \subseteq (\ker \rho_*)^{\omega_X}$. Now

$$\begin{aligned}
&\dim(\ker \rho_*)^{\omega_X} \\
&\quad = \dim X - \dim \omega_X^b(\ker \rho_*) \\
&\quad = \dim \rho_*(T_x X) + \dim \ker \rho_* - \dim \omega_X^b(\ker \rho_*) \\
&\quad = \dim \rho_*(T_x X) + \dim(\ker \omega_X \cap \ker \rho_*) \quad \text{(by Lemma 3.6 (1))} \\
&\quad \le \operatorname{rank} A_G - \dim(A_G|_x^x) + \dim(\ker \omega_X \cap \ker \rho_*) \\
&\quad = \dim\{\hat{\xi}(x) | \xi \in A_G|_{\rho(x)}\} + \dim(\ker \omega_X \cap \ker \rho_*).
\end{aligned}$$

On the other hand, using Lemma 3.6 (2), it is easy to check that

$$(\ker \omega_X \cap \ker \rho_*) \oplus (\ker \omega_X \cap \{\hat{\xi}(x) | \xi \in A_G|_{\rho(x)}\}) = \ker \omega_X. \tag{25}$$

To prove this equation, first, one easily sees that $\ker \omega_X$ can be written as the sum of the two subspaces on the left-hand side. To show that this is a direct sum, it suffices to show that the intersection of these two

subspaces is zero. This is because

$$\begin{aligned}
&(\ker \omega_X \cap \ker \rho_*) \cap \{\hat{\xi}(x) | \xi \in A_G|_{\rho(x)}\} \\
&\quad \subseteq \ker \omega_X \cap \ker \rho_* \cap \ker \sigma_* \\
&\quad = \ker \omega_X \cap \ker(\rho \times \sigma)_* \quad \text{(by Lemma 3.6 (2)b)} \\
&\quad = 0.
\end{aligned}$$

From Equation (25), it follows that

$$\begin{aligned}
&\dim(\{\hat{\xi}(x) | \xi \in A_G|_{\rho(x)}\} + \ker \omega_X) \\
&\quad = \dim\{\hat{\xi}(x) | \xi \in A_G|_{\rho(x)}\} + \dim \ker \omega_X \\
&\qquad - \dim(\ker \omega_X \cap \{\hat{\xi}(x) | \xi \in A_G|_{\rho(x)}\}) \\
&\quad = \dim\{\hat{\xi}(x) | \xi \in A_G|_{\rho(x)}\} + \dim(\ker \omega_X \cap \ker \rho_*).
\end{aligned}$$

Thus Equation (23) follows immediately. Similarly, Equation (24) can be proved. This concludes the proof of the lemma. q.e.d.

Assume that $[(\delta_x, \delta_y)] \in T_{[(x,y)]}Z$, where $(\delta_x, \delta_y) \in T_{(x,y)}(X \times_{H_0} Y)$, is in the kernel of ω_Z. Then

$$\omega_X(\delta_x, \delta'_x) + \omega_Y(\delta_y, \delta'_y) = 0 \quad \forall (\delta'_x, \delta'_y) \in T_{(x,y)}(X \times_{H_0} Y). \tag{26}$$

By letting $\delta'_y = 0$, it follows that $\omega_X(\delta_x, \delta'_x) = 0$ for any $\delta'_x \in \ker \sigma_{1*}$. Therefore, according to Lemma 3.17, we have

$$\delta_x \in (\ker \sigma_{1*})^{\omega_X} = \{\hat{\eta}(x) | \eta \in A_H|_{\sigma_1(x)}\} + \ker \omega_X.$$

It thus follows that we can always write $\delta_x = \hat{\xi}(x) + \hat{\eta_1}(x)$ for some $\xi \in A_G|_{\rho_1(x)}$ and $\eta_1 \in A_H|_{\sigma_1(x)}$ such that $\overrightarrow{\xi}(\rho_1(x)) \in \ker \omega_G$.

Similarly, one shows that $\delta_y = \hat{\eta_2}(y) + \hat{\zeta}(y)$, for some $\eta_2 \in A_H|_{\rho_2(y)}$ and $\zeta \in A_K|_{\sigma_2(y)}$ such that $\overrightarrow{\zeta}(\sigma_2(y)) \in \ker \omega_K$.

Now, $\sigma_{1*}\delta_x = -a_{A_H}(\eta_1)$ and $\rho_{2*}\delta_y = a_{A_H}(\eta_2)$. Thus, we have $\eta_1 - \eta_2 \in \ker a_{A_H}$. From Equations (26) and (13), it follows that

$$\begin{aligned}
&\omega_G(\overrightarrow{\xi}(m), \rho_{1*}\delta'_x) - \omega_H(\overrightarrow{\eta_1}(n), \sigma_{1*}\delta'_x) \\
&\qquad + \omega_H(\overrightarrow{\eta_2}(n), \rho_{2*}\delta'_y) - \omega_K(\overrightarrow{\zeta}(p), \sigma_{2*}\delta'_y) = 0,
\end{aligned}$$

where $m = \rho_1(x)$, $n = \sigma_1(x) = \rho_2(y)$ and $p = \sigma_2(y)$. Hence, $\omega_H(\overrightarrow{\eta_1 - \eta_2}(n), \delta_n) = 0$ for any $\delta_n \in f_*T_{(x,y)}(X \times_{H_0} Y)$, where $f : X \times_{H_0} Y \to H_0$ is the map $f(x, y) = \sigma_1(x)$. By the clean assumption, we may assume that $f_*T_{(x,y)}(X \times_{H_0} Y) = \sigma_{1*}(T_xX)$ (or $\rho_{2*}(T_yY)$,

in which case, a similar proof can be carried out). Thus, we have $\omega_H(\overrightarrow{\eta_1-\eta_2}(n), \sigma_{1*}(T_xX))=0$, which implies that $\hat{\eta_1}(x)-\hat{\eta_2}(x) \in \ker \omega_X^b$. On the other hand, since $(\rho_1 \times \sigma_1)_*(\hat{\eta_1}(x)-\hat{\eta_2}(x)) = (0, a_{A_H}(\eta_1-\eta_2)) = 0$, we have $\hat{\eta_1}(x) - \hat{\eta_2}(x) = 0$ according to Lemma 3.6 (2)b. It thus follows that

$$\begin{aligned}[(\delta_x, \delta_y)] &= [(\hat{\xi}(x) + \hat{\eta_1}(x), \hat{\eta_2}(y) + \hat{\zeta}(y)] \\ &= [(\hat{\xi}(x) + \hat{\eta_2}(x), \hat{\eta_2}(y) + \hat{\zeta}(y)] = [(\hat{\xi}(x), \hat{\zeta}(y))],\end{aligned}$$

which implies the minimal non-degeneracy condition. This completes the proof. q.e.d.

3.5. Reduction. Theorem 3.16 has many important consequences. As an immediate consequence, we have the following reduction theorem.

Theorem 3.18. *Let $(\Gamma \rightrightarrows P, \omega + \Omega)$ be a quasi-symplectic groupoid, and $(X \xrightarrow{J} P, \omega_X)$ an Hamiltonian Γ-space. Assume that $m \in P$ is a regular value of J and Γ_m^m acts on $J^{-1}(m)$ freely and properly, where Γ_m^m denotes the isotropy group at m. Then, $J^{-1}(m)/\Gamma_m^m$ is a symplectic manifold. More generally, if $(\Gamma_i \rightrightarrows P_i, \omega_i + \Omega_i)$, $i = 1, 2$, are quasi-symplectic groupoids, $(X \xrightarrow{J_1 \times J_2} P_1 \times P_2, \omega_X)$ is an Hamiltonian $\Gamma_1 \times \Gamma_2$-space, $m \in P_2$ is a regular value for $J_2 : X \longrightarrow P_2$, and $(\Gamma_2)_m^m$ acts on $J_2^{-1}(m)$ freely and properly, then $J_2^{-1}(m)/(\Gamma_2)_m^m$ is naturally an Hamiltonian Γ_1-space.*

Proof. Note that $(X \xrightarrow{J_1 \times J_2} P_1 \times P_2, \omega_X)$ being an Hamiltonian $\Gamma_1 \times \Gamma_2$-space is equivalent to X being a Γ_1-$\overline{\Gamma_2}$-bimodule by considering X as a right Γ_2-space. Let $\mathcal{O} \subset P_2$ be the groupoid orbit of Γ_2 through m. Then, $P_2 \xleftarrow{i} \bar{\mathcal{O}} \to \cdot$ is an Hamiltonian $\overline{\Gamma_2}$-$\cdot$-bimodule according to Proposition 3.8. The clean assumption is satisfied since $J_2 : J_2^{-1}(\mathcal{O}) \to \mathcal{O}$ is a submersion. By Theorem 3.16, $X \times_{\overline{\Gamma_2}} \bar{\mathcal{O}}$ is an Hamiltonian Γ_1-$\cdot$-Hamiltonian bimodule, i.e., a Hamiltonian Γ_1-space. It is easy to see that $X \times_{\overline{\Gamma_2}} \bar{\mathcal{O}}$ is naturally diffeomorphic to $J_2^{-1}(m)/(\Gamma_2)_m^m$. q.e.d.

Remark 3.19. As a consequence, X/Γ (assuming being a smooth manifold) is naturally a Poisson manifold. One should also be able to see this using the reduction of Dirac structures, as an Hamiltonian Γ-space infinitesimally corresponds to some particular Dirac structure [**8**].

Various reduction theorems in the literature are indeed consequences of Theorem 3.18. In particular, applying Theorem 3.18 to the AMM quasi-symplectic groupoids, we recover the Hamiltonian reduction theorem of quasi-Hamiltonian G-spaces of Alekseev–Malkin–Meinrenken [**2**].

Corollary 3.20. *Let X be a quasi-Hamiltonian $G_1 \times G_2$-space and let $f \in G_2$ be a regular value of the momentum map $J_2 : X \to G_2$. Then, the pull-back of the 2-form ω to $J_2^{-1}(f)$ descends to the* reduced space

$$X_f = J_2^{-1}(f)/(G_2)_f$$

(*assuming it is a smooth manifold*) *and makes it into a quasi-Hamiltonian G_1-space. Here, $(G_2)_f$ is the isotropy group of G_2 at f. In particular, if $G_1 = \{e\}$ is trivial, then X_f is a symplectic manifold.*

Another immediate consequence of Theorem 3.16 is the following

Theorem 3.21. *Let $(\Gamma \rightrightarrows P, \omega + \Omega)$ be a quasi-symplectic groupoid, and $(X \xrightarrow{J_1} P, \omega_X)$, and $(Y \xrightarrow{J_2} P, \omega_Y)$ be Hamiltonian Γ-spaces. Assume that $J_1 : X \to P$ and $J_2 : Y \to P$ are clean. Then, $X \times_\Gamma \overline{Y}$ is a symplectic manifold.*

We will call $X \times_\Gamma \overline{Y}$ the classical intertwiner space between X and Y. When $\Gamma \rightrightarrows P$ is the symplectic groupoid $T^*G \rightrightarrows \mathfrak{g}^*$, this reduces to the classical intertwiner space $(X \times \overline{Y})_0$ of Hamiltonian G-spaces [**14**]. We refer the reader to [**37**] for the detailed study of classical intertwiner spaces of symplectic groupoids.

4. Morita equivalence

This section is devoted to the study of Morita equivalence of quasi-symplectic groupoids. The main result is that Morita equivalent quasi-symplectic groupoids define equivalent momentum map theories. See Theorem 4.19 and Corollary 4.20.

4.1. Morita equivalence of quasi-symplectic groupoids. Morita equivalence is an important equivalence relation for groupoids. Indeed, groupoids moduli Morita equivalence can be identified with the so called stacks, which are useful in the study of singular spaces such as moduli spaces. Morita equivalence of symplectic groupoids were studied in [**36**]. Here, we will generalize this notion to quasi-symplectic groupoids. Let us first recall the definition of Morita equivalence of Lie groupoids [**18**, **36**].

Definition 4.1. Lie groupoids $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ are said to be Morita equivalent if there exists a manifold X together with two surjective submersions

$$G_0 \xleftarrow{\rho} X \xrightarrow{\sigma} H_0$$

a left action of G with respect to ρ, a right action of H with respect to σ such that

(1) the two actions commute with each other;
(2) X is a locally trivial G-principal bundle over $X \xrightarrow{\sigma} H_0$,
(3) X is a locally trivial H-principal bundle over $G_0 \xleftarrow{\rho} X$.

In this case, $G_0 \xleftarrow{\rho} X \xrightarrow{\sigma} H_0$ is called an equivalence bimodule between the Lie groupoids G and H.

It is known that de-Rham cohomology groups are invariant under Morita equivalence.

Proposition 4.2 ([5, 7, 15])**.** *If $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ are Morita equivalent Lie groupoids, then*

$$H^k_{DR}(G_\bullet) \xrightarrow{\sim} H^k_{DR}(H_\bullet).$$

Definition 4.3. Quasi-symplectic groupoids $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ and $(H \rightrightarrows H_0, \omega_H + \Omega_H)$ are said to be Morita equivalent if there exists a Morita equivalence bimodule $G_0 \xleftarrow{\rho} X \xrightarrow{\sigma} H_0$ between the Lie groupoids G and H, together with a two-form $\omega_X \in \Omega^2(X)$ such that X is also an Hamiltonian G-H-bimodule.

Suppose that $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ are Morita equivalent Lie groupoids with equivalence bimodule $G_0 \xleftarrow{\rho} X \xrightarrow{\sigma} H_0$. We say that $m \in G_0$ and $n \in H_0$ are related if $\rho^{-1}(m) \cap \sigma^{-1}(n) \neq \emptyset$. The following are basic properties [36].

Proposition 4.4. *If $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ are Morita equivalent Lie groupoids with equivalence bimodule $G_0 \xleftarrow{\rho} X \xrightarrow{\sigma} H_0$. Assume that $m_0 \in G_0$ and $n_0 \in H_0$ are related. Then*

(1) $\dim G + \dim H = 2 \dim X$,
(2) *an element $n \in H_0$ is related to m_0 if and only if n lies in the same groupoid orbit as n_0; and conversely, $m \in G_0$ is related to n_0 if and only if m lies in the same groupoid orbit as m_0, and*
(3) *the isotropy groups at m_0 and n_0 are isomorphic.*

Theorem 4.5. *Morita equivalence is indeed an equivalence relation for quasi-symplectic groupoids.*

Proof. From Proposition 3.8 (1), we know that Morita equivalence is reflective. If $G_0 \overset{\rho}{\leftarrow} X \overset{\sigma}{\rightarrow} H_0$ is an Hamiltonian bimodule defining the Morita equivalence between $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ and $(H \rightrightarrows H_0, \omega_H + \Omega_H)$, then $H_0 \overset{\sigma}{\leftarrow} \overline{X} \overset{\rho}{\rightarrow} G_0$, with the reversed actions, is an Hamiltonian bimodule defining the Morita equivalence between $(H \rightrightarrows H_0, \omega_H + \Omega_H)$ and $(G \rightrightarrows G_0, \omega_G + \Omega_G)$. So, the symmetry follows. As for the transitivity, let $(G \rightrightarrows G_0, \omega_G + \Omega_G)$, $(H \rightrightarrows H_0, \omega_H + \Omega_H)$, and $(K \rightrightarrows K_0, \omega_K + \Omega_K)$ be quasi-symplectic groupoids. Assume that $G_0 \overset{\rho_1}{\leftarrow} X \overset{\sigma_1}{\rightarrow} H_0$ is a G-H equivalence bimodule, and $H_0 \overset{\rho_2}{\leftarrow} Y \overset{\sigma_2}{\rightarrow} K_0$ an H-K-equivalence bimodule, respectively. It is known that $Z = X \times_H Y$ is a bimodule defining the Morita equivalence between the groupoids $G \rightrightarrows G_0$ and $K \rightrightarrows K_0$. According to Theorem 3.16, Z is also an Hamiltonian G-K-bimodule. Thus, $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ and $(K \rightrightarrows K_0, \omega_K + \Omega_K)$ are Morita equivalent quasi-symplectic groupoids. q.e.d.

In what follows, we describe some useful constructions of producing Morita equivalent quasi-symplectic groupoids.

Let $\Gamma \rightrightarrows P$ be a Lie groupoid, and $\omega_i + \Omega_i \in \Omega^2(\Gamma) \oplus \Omega^3(P)$, $i = 1, 2$, be two cohomologous 3-cocycles. This means that there are $B \in \Omega^2(P)$ and $\theta \in \Omega^1(\Gamma)$ such that

$$(\omega_1 + \Omega_1) - (\omega_2 + \Omega_2) = \delta(B + \theta).$$

Following [**9**], we say that $\omega_1 + \Omega_1$ and $\omega_2 + \Omega_2$ differ by a gauge transformation of the first type if $(\omega_1 + \Omega_1) - (\omega_2 + \Omega_2) = \delta B$, i.e.,

$$\omega_1 - \omega_2 = s^*B - t^*B, \quad \Omega_1 - \Omega_2 = dB.$$

And we say that $\omega_1 + \Omega_1$ and $\omega_2 + \Omega_2$ differ by a gauge transformation of the second type if $(\omega_1 + \Omega_1) - (\omega_2 + \Omega_2) = \delta\theta$, i.e.,

$$\Omega_1 = \Omega_2, \quad \omega_1 = \omega_2 - d\theta \quad \text{for some } \theta \in \Omega^1(\Gamma) \text{ such that } \partial\theta = 0.$$

It is simple to see that gauge transformations of the first type transform quasi-symplectic groupoids into quasi-symplectic groupoids (see also [**8**]). Below, we see that the resulting quasi-symplectic groupoids are indeed Morita equivalent (see [**9**] for the case of symplectic groupoids).

Proposition 4.6. *Assume that* $(\Gamma \rightrightarrows P, \omega + \Omega)$ *is a quasi-symplectic groupoid. Then,* $(\Gamma \rightrightarrows P, \omega' + \Omega')$*, where* $\omega' = \omega + s^*B - t^*B$ *and* $\Omega' = \Omega + dB$*, for any* $B \in \Omega^2(P)$*, is a Morita equivalent quasi-symplectic groupoid.*

Proof. First, we need to show that ω' is non-degenerate in the sense of Definition 2.5. By Proposition 2.7, it suffices to show that $a : \ker \omega'_m \cap A_m \to \ker \omega'_m \cap T_mP$ is injective. Assume that $\xi \in \ker \omega'_m \cap A_m$ such that $a(\xi) = 0$. Then, we have for any $v \in T_mP$,

$$0 = \omega'(\overrightarrow{\xi}, v) = (\omega + s^*B - t^*B)(\overrightarrow{\xi}, v) = \omega(\overrightarrow{\xi}, v) + B(a(\xi), v) = \omega(\overrightarrow{\xi}, v).$$

Thus, we have $\xi \in \ker \omega_m \cap A_m$, which implies that $\xi = 0$.

To prove the Morita equivalence, let $X = \Gamma$ and $\omega_X = \omega + s^*B$. We let $(\Gamma \rightrightarrows P, \omega' + \Omega')$ act on X from the left by left multiplications and let $(\Gamma \rightrightarrows P, \omega + \Omega)$ act on X from the right by right multiplications. It is simple to check that these actions are compatible with the quasi-symplectic structures. It remains to check the minimal non-degeneracy condition. Assume that $\delta_x \in \ker \omega_X$. Then, for any $\zeta \in A_{t(x)}$, we have,

$$0 = \omega_X(\delta_x, \overleftarrow{\zeta}(x)) = \omega(\delta_x, \overleftarrow{\zeta}(x)) + B(s_*\delta_x, s_*\overleftarrow{\zeta}) = \omega(\delta_x, \overleftarrow{\zeta}(x))$$

since $s_*\overleftarrow{\zeta}(x) = 0$. Hence, $\omega(t_*\delta_x, \overleftarrow{\zeta}(t(x))) = 0$ according to Equation (16), which implies that $t_*\delta_x \in \ker \omega$. Therefore, $t_*\delta_x = a(\eta)$ for some $\eta \in A_{t(x)}$ such that $\overrightarrow{\eta}(t(x)) \in \ker \omega$. Hence, $\overleftarrow{\eta}(t(x)) \in \ker \omega$ by Corollary 2.4 (3), which implies that $\overleftarrow{\eta}(x) \in \ker \omega$. Set $\delta'_x = \delta_x + \overleftarrow{\eta}(x)$. Thus, $t_*\delta'_x = t_*\delta_x + t_*\overleftarrow{\eta}(x) = t_*\delta_x - a(\eta) = 0$. Hence, we may write $\delta'_x = \overrightarrow{\xi}(x)$ for some $\xi \in A_{s(x)}$. Moreover, a simple computation yields that

$$\begin{aligned} \overrightarrow{\xi}(x) \lrcorner \omega' &= \delta'_x \lrcorner \omega' = \delta'_x \lrcorner \omega_X - \delta'_x \lrcorner t^*B \\ &= \delta'_x \lrcorner \omega_X = \delta_x \lrcorner \omega_X + \overleftarrow{\eta}(x) \lrcorner \omega + \overleftarrow{\eta}(x) \lrcorner s^*B = 0. \end{aligned}$$

Thus, $\overrightarrow{\xi}(s(x)) \in \ker \omega'$ according to Equation (16). This concludes the proof. q.e.d.

Remark 4.7. Note that quasi-symplectic groupoids are in general not preserved under gauge transformations of the second type. For instance, the symplectic structure ω on the symplectic groupoid $T^*G \rightrightarrows \mathfrak{g}^*$ is $d\theta$, where $\theta \in \Omega^1(T^*G)$ is the Liouville one-form. It is simple to see that θ satisfies the condition $\partial\theta = 0$. However, $T^*G \rightrightarrows \mathfrak{g}^*$ with the zero two-form is clearly not quasi-symplectic.

For a Lie groupoid $\Gamma \rightrightarrows P$ and a surjective submersion $\phi : Y \to P$, we denote by $\Gamma[Y]$ the subgroupoid of $(Y \times Y) \times \Gamma$ consisting of $\{(y_1, y_2, r)|\ s(r) = \phi(y_1),\ t(r) = \phi(y_2)\}$, called the pull-back groupoid. Clearly, the projection $\mathrm{pr} : \Gamma[Y] \to \Gamma$ defines a groupoid homomorphism. By abuse of notations, we also use pr to denote the corresponding map on the unit spaces $\phi : Y \to P$.

Proposition 4.8. *Assume that* $(\Gamma \rightrightarrows P, \omega + \Omega)$ *is a quasi-symplectic groupoid, and* $\phi : Y \to P$ *a surjective submersion. Then,* $(\Gamma[Y] \rightrightarrows Y, \mathrm{pr}^*\omega + \mathrm{pr}^*\Omega)$ *is a quasi-symplectic groupoid Morita equivalent to* $(\Gamma \rightrightarrows P, \omega + \Omega)$.

Proof. It is obvious that $\mathrm{pr}^*\omega + \mathrm{pr}^*\Omega$ is a 3-cocycle since pr is a Lie groupoid homomorphism. By A_Y, we denote the Lie algebroid of $\Gamma[Y] \rightrightarrows Y$. It is simple to see that $\forall y \in Y$,

$$A_Y|_y = \{(\delta_y, \xi)|\delta_y \in T_yY,\ \xi \in A_{\phi(y)} \ \text{ such that } \ \phi_*\delta_y = a(\xi)\}$$

with the anchor a_Y: $A_Y \to TY$ being given by the projection $(\delta_y, \xi) \to \delta_y$, where A is the Lie algebroid of Γ. Therefore, an element $(\delta_y, \xi) \in A_Y|_y$, where $\phi_*\delta_y = a(\xi)$, belongs to $\ker(\mathrm{pr}^*\omega)|_y \cap A_Y|_y$ if and only if $\xi \in \ker\omega_{\phi(y)} \cap A_{\phi(y)}$. This implies that

$$a_Y : \ker(\mathrm{pr}^*\omega) \cap A_Y \to \ker(\mathrm{pr}^*\omega) \cap TY$$

is indeed injective. It thus follows that $(\Gamma[Y] \rightrightarrows Y, \mathrm{pr}^*\omega + \mathrm{pr}^*\Omega)$ is a quasi-symplectic groupoid by dimension counting.

To show the Morita equivalence, let $X := \Gamma \times_{t,P,\phi} Y$ and $\omega_X = p^*\omega$, where $p : X \to \Gamma$ is the natural projection. It is standard that $P \xleftarrow{\rho} X \xrightarrow{\sigma} Y$ is a Γ-$\Gamma[Y]$-bimodule defining a Morita equivalence between these two Lie groupoids, where

$$\rho(r, y) = s(r) \quad \text{and} \quad \sigma(r, y) = y$$

and the left Γ-action is

$$\widetilde{r} \cdot (r, y) = (\widetilde{r}r, y), \qquad t(\widetilde{r}) = s(r), \tag{27}$$

while the right $\Gamma[Y]$-action is

$$(r, y) \cdot (y_1, y_2, \widetilde{r}) = (r\widetilde{r}, y_2), \quad y = y_1, \ \ t(r) = \phi(y) = \phi(y_1) = s(\widetilde{r}). \tag{28}$$

It is also simple to check that ω_X is compatible with the $\Gamma \times \overline{\Gamma[Y]}$-action. For the minimal non-degeneracy condition, assume that $(\delta_r, \delta_y) \in T_{(r,y)}X$ such that $(\delta_r, \delta_y) \lrcorner\, \omega_X = 0$, which is equivalent to that

$\delta_r \lrcorner\, \omega = 0$. By Proposition 3.8, we have $\delta_r = \overrightarrow{\xi}(r) - \overleftarrow{\eta}(r)$, where $\xi \in A_{s(r)}$ and $\eta \in A_{t(r)}$ such that $\overrightarrow{\xi}(s(r))$ and $\overrightarrow{\eta}(t(r)) \in \ker \omega$. Thus, $(\delta_r, \delta_y) = \hat{\xi}(r,y) - \hat{\eta'}(r,y)$, where $\eta' = (\delta_y, \eta) \in A_Y|_y$ clearly satisfies the condition that $\overrightarrow{\eta'}(y) \in \ker(\mathrm{pr}^* \omega)$. This concludes the proof. q.e.d.

Combining Propositions 4.6 and 4.8, we are lead to the following:

Theorem 4.9. *Let $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ and $(H \rightrightarrows H_0, \omega_H + \Omega_H)$ be pre-quasi-symplectic groupoids, which are Morita equivalent as Lie groupoids with an equivalence bimodule $G_0 \overset{\rho}{\leftarrow} X \overset{\sigma}{\rightarrow} H_0$. If $\rho^*(\omega_G + \Omega_G)$ and $\sigma^*(\omega_H + \Omega_H)$, as de-Rham 3-cocycles of the groupoid $G[X] \cong H[X] \rightrightarrows X$, differ by a gauge transformation of the first type, then if one is quasi-symplectic, so is the other. Moreover, they are Morita equivalent as quasi-symplectic groupoids.*

4.2. Generalized homomorphisms of quasi-symplectic groupoids. Recall that a *generalized homomorphism* from a Lie groupoid $G \rightrightarrows G_0$ to $H \rightrightarrows H_0$ is given by a manifold X, two smooth maps $G_0 \overset{\rho}{\leftarrow} X \overset{\sigma}{\rightarrow} H_0$, a left action of G with respect to ρ, a right action of H with respect to σ, such that the two actions commute, and X is a locally trivial H-principal bundle over $G_0 \overset{\rho}{\leftarrow} X$ [**18**]. In particular, $\rho : X \to G_0$ must be a surjective submersion, and the (right) H-action on X is free and proper.

Generalized homomorphisms can be composed just like the usual groupoid homomorphisms; thus, there is a category $\mathcal{G}$ whose objects are Lie groupoids and morphisms are generalized homomorphisms [**15**, **16**, **31**], where isomorphisms in the category $\mathcal{G}$ are just Morita equivalences [**27**, **36**].

Similarly, we can introduce the notion of generalized homomorphisms between quasi-symplectic groupoids.

Definition 4.10. A generalized homomorphism from a quasi-symplectic groupoid $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ to a quasi-symplectic groupoid $(H \rightrightarrows H_0, \omega_H + \Omega_H)$ is an Hamiltonian G–H-bimodule $G_0 \overset{\rho}{\leftarrow} X \overset{\sigma}{\rightarrow} H_0$, which is, in the same time, also a generalized homomorphism from G to H.

Theorem 3.16 implies the following theorem.

Theorem 4.11. *There is a category, whose objects are quasi-symplectic groupoids, and morphisms are generalized homomorphisms of*

quasi-symplectic groupoids. The isomorphisms in this category correspond exactly to Morita equivalences of quasi-symplectic groupoids.

It is known that a strict homomorphism of Lie groupoids must be a generalized homomorphism. For quasi-symplectic groupoids, one can also introduce the notion of strict homomorphisms.

Definition 4.12. A strict homomorphism of quasi-symplectic groupoids from $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ to $(H \rightrightarrows H_0, \omega_H + \Omega_H)$ is a Lie groupoid homomorphism $\phi : G \to H$ satisfying

(1) $\phi^*(\omega_H + \Omega_H) = \omega_G + \Omega_G$ and
(2) if $\xi \in A_H|_m$ satisfies the properties that $a_H(\xi) = 0$ and $\phi^*(\overrightarrow{\xi}(m) \lrcorner \omega_H) = 0$, then $\xi = 0$, where A_H is the Lie algebroid of $H \rightrightarrows H_0$ and $a_H : A_H \to TH_0$ denotes its anchor map.

Proposition 4.13. *For quasi-symplectic groupoids, strict homomorphisms imply generalized homomorphisms.*

Proof. Assume that $\phi : G \to H$ is a strict homomorphism of quasi-symplectic groupoids from $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ to $(H \rightrightarrows H_0, \omega_H + \Omega_H)$. Let $X = G_0 \times_{\phi, H_0, s} H$, and set $\rho(g_0, h) = g_0$, $\sigma(g_0, h) = t(h)$. Define a left G- and a right H-action on X, respectively, by

$$g \cdot (g_0, h) = (s(g), \phi(g)h) \quad \text{and} \quad (g_0, h) \cdot h' = (g_0, hh').$$

One checks that this defines a generalized homomorphism from $G \rightrightarrows G_0$ to $H \rightrightarrows H_0$ [18]. Let $\omega_X = i^*(0, \omega_H)$, where $i : G_0 \times_{\phi, H_0, s} H \subset G_0 \times H$ is the inclusion. It is simple to see that ω_X is compatible with the G-H-bi-actions. It remains to prove the minimal non-degeneracy condition. Note that for any $\xi \in \Gamma(A_G)$ and $\eta \in \Gamma(A_H)$, the vector field on X generated by the infinitesimal action of (ξ, η) is given by

$$\widehat{(\xi \oplus \eta)}(g_0, h) = (a_G(\xi)(g_0), \overrightarrow{\phi_* \xi}(h) - \overleftarrow{\eta}(h)) \quad \forall (g_0, h) \in X.$$

Assume that $\delta_x = (\delta_{g_0}, \delta_h) \in \ker \omega_X$, where $x = (g_0, h) \in X$. This implies that

$$\phi_* \delta_{g_0} = s_* \delta_h \quad (\text{thus } \phi(g_0) = s(h)) \quad \text{and} \tag{29}$$

$$\omega_H(\delta_h, \delta'_h) = 0, \quad \forall \delta'_h \in T_h H \ \text{ such that } \ s_* \delta'_h \in \mathrm{Im}(\phi_*). \tag{30}$$

In particular, for any $\zeta \in A_H|_{t(h)}$, since $s_* \overleftarrow{\zeta}(h) = 0$, which is always in the image of ϕ_*, we have $\omega_H(\delta_h, \overleftarrow{\zeta}(h)) = 0$. From Equation (16), it thus follows that $\omega_H(t_* \delta_h, \overleftarrow{\zeta}(t(h))) = 0$, which implies that $t_* \delta_h \in$

$\ker\omega_H$. By the non-degeneracy assumption of Definition 2.5, we have $t_*\delta_h = a_H(\eta)$, where $\eta \in A_H|_{t(h)}$ such that $\overrightarrow{\eta}(t(h)) \in \ker\omega_H$. Hence $\overleftarrow{\eta}(t(h)) \in \ker\omega_H$ according to Corollary 2.4 (3), which implies that $\overleftarrow{\eta}(h)$ belongs to $\ker\omega_H$ by Equation (16). Let $\widetilde{\delta}_h = \delta_h + \overleftarrow{\eta}(h)$. Then, we have $t_*\widetilde{\delta}_h = t_*\delta_h + t_*\overleftarrow{\eta}(h) = t_*\delta_h - a_H(\eta) = 0$. Thus, $\widetilde{\delta}_h = \overrightarrow{\xi_1}(h)$ for some $\xi_1 \in A_H|_{s(h)}$, and hence we have $\delta_h = \overrightarrow{\xi_1}(h) - \overleftarrow{\eta}(h)$. On the other hand, for any $\chi \in A_G|_{g_0}$, since $s_*\overrightarrow{\phi_*\chi}(h) = a_H(\phi_*\chi) = \phi_* a_G(\chi) \in \operatorname{Im}\phi_*$, we have $\omega_H(\delta_h, \overrightarrow{\phi_*\chi}(h)) = 0$ by Equation (30). Now

$$\begin{aligned}\omega_H(\delta_h, \overrightarrow{\phi_*\chi}(h)) &= \omega_H(\overrightarrow{\xi_1}(h) - \overleftarrow{\eta}(h), \overrightarrow{\phi_*\chi}(h))\\ &= \omega_H(\overrightarrow{\xi_1}(h), \overrightarrow{\phi_*\chi}(h))\\ &= \omega_H(s_*\overrightarrow{\xi_1}(h), \overrightarrow{\phi_*\chi}(s(h))),\end{aligned}$$

where we used Equation (16) in the last equality.

Now, $s_*\overrightarrow{\xi_1}(h) = s_*(\delta_h + \overleftarrow{\eta}(h)) = s_*\delta_h = \phi_*\delta_{g_0}$. Therefore, we have $\omega_H(\phi_*\delta_{g_0}, \overrightarrow{\phi_*\chi}(s(h))) = 0$, $\forall\chi \in A_G|_{g_0}$. It thus follows that $\delta_{g_0} \in \ker(\phi^*\omega_H)$. Since $(G \rightrightarrows G_0, \phi^*\omega_H + \phi^*\Omega_H)$ is quasi-symplectic, by the non-degeneracy assumption, we conclude that $\delta_{g_0} = a_G(\xi)$ for some $\xi \in A_G|_{g_0}$ such that $\overrightarrow{\xi}(g_0) \in \ker(\phi^*\omega_H)$. Therefore, for any $\delta'_{g_0} \in T_{g_0}G_0$, $\omega_H(\overrightarrow{\phi_*\xi}(s(h)), \phi_*\delta'_{g_0}) = (\phi^*\omega_H)(\overrightarrow{\xi}(g_0), \delta'_{g_0}) = 0$. Hence, $\omega_H(\overrightarrow{\xi_1}(s(h)) - \overrightarrow{\phi_*\xi}(s(h)), \phi_*\delta'_{g_0}) = \omega_H(\overrightarrow{\xi_1}(s(h)), \phi_*\delta'_{g_0})$. Since $s : H \to H_0$ is a submersion, we may assume that $\phi_*\delta'_{g_0} = s_*\delta''_h$ for some $\delta''_h \in T_hH$, and therefore

$$\begin{aligned}\omega_H(\overrightarrow{\xi_1}(s(h)), \phi_*\delta'_{g_0}) &= \omega_H(\overrightarrow{\xi_1}(s(h)), s_*\delta''_h) = \omega_H(\overrightarrow{\xi_1}(h), \delta''_h)\\ &= \omega_H(\delta_h + \overleftarrow{\eta}(h), \delta''_h) = \omega_H(\delta_h, \delta''_h) = 0\end{aligned}$$

by Equation (30) since $s_*\delta''_h = \phi_*\delta'_{g_0} \in \operatorname{Im}\phi_*$. On the other hand, since $a_H(\xi_1 - \phi_*\xi) = s_*\overrightarrow{\xi_1}(h) - a_H(\phi_*\xi) = s_*\delta_h - \phi_*(a_G(\xi)) = s_*\delta_h - \phi_*\delta_{g_0} = 0$, it follows that $\xi_1 - \phi_*\xi = 0$. Therefore, we conclude that $\delta_x = (\delta_{g_0}, \delta_h) = \widehat{(\xi \oplus \eta)}(g_0, h)$, where $\overrightarrow{\xi}(g_0) \in \ker(\phi^*\omega_H)$ and $\overrightarrow{\eta}(t(h)) \in \ker\omega_H$. q.e.d.

Remark 4.14. Note that the second condition in Definition 4.12 is necessary for Proposition 4.13 to hold. For instance, given a quasi-symplectic groupoid $H \rightrightarrows H_0$ and a fixed point in H_0, one may always think of this point as a groupoid homomorphism from $\cdot \rightrightarrows \cdot$ to $H \rightrightarrows$

H_0. The first condition is satisfied automatically. However, $\cdot \xleftarrow{\rho} H \xrightarrow{\sigma} H_0$ is, in general, not a generalized homomorphism of quasi-symplectic groupoids since H is not, in general, an Hamiltonian H-space under the right H-action.

The following proposition describes the precise relation between generalized homomorphisms and strict homomorphisms for quasi-symplectic groupoids.

Proposition 4.15. *Any generalized homomorphism of quasi-symplectic groupoids is equivalent to the composition of a Morita equivalence with a strict homomorphism.*

Proof. The inverse direction follows from Proposition 4.13 and Theorem 4.11, so it remains to prove the other direction.

Assume that $G_0 \xleftarrow{\rho} X \xrightarrow{\sigma} H_0$ is a generalized homomorphism of quasi-symplectic groupoids from $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ to $(H \rightrightarrows H_0, \omega_H + \Omega_H)$.

Consider the transformation groupoid $Q := (G \times H) \times_{(G_0 \times H_0)} X \rightrightarrows X$ as in Proposition 3.14. One easily checks that $Q \rightrightarrows X$ is isomorphic to $G[X] \rightrightarrows X$, where the isomorphism is given by $(g, h, x) \to (x, g^{-1}xh, g)$, $\forall (g, h, x) \in (G \times H) \times_{(G_0 \times H_0)} X$. Therefore, we have two groupoid homomorphisms $\mathrm{pr}_1 : G[X] \to G$ and $\mathrm{pr}_2 : G[X] \to H$. Equip $G[X] \rightrightarrows X$ with the 3-cocycle

$$\omega_{G[X]} + \Omega_{G[X]} := \mathrm{pr}_1^*(\omega_G + \Omega_G) - \delta\omega_X.$$

By Theorem 4.9, we know that $(G[X] \rightrightarrows X, \omega_{G[X]} + \Omega_{G[X]})$ is Morita equivalent to $(G \rightrightarrows G_0, \omega_G + \Omega_G)$. On the other hand, according to Proposition 3.14, we have $\omega_{G[X]} + \Omega_{G[X]} = \mathrm{pr}_2^*(\omega_H + \Omega_H)$. It thus follows from Theorem 3.16 that $X \xleftarrow{\rho} X \times_{\sigma, H_0, s} H \xrightarrow{\sigma} H_0$ is an Hamiltonian $G[X]$-H bimodule defining a generalized homomorphism of quasi-symplectic groupoids from $(G[X] \rightrightarrows X, \omega_{G[X]} + \Omega_{G[X]})$ to $(H \rightrightarrows H_0, \omega_H + \Omega_H)$. Here, the two-form ω_Z on $Z := X \times_{\sigma, H_0, s} H$ is given by $\omega_Z = i^*(0, \omega_H)$, where $i : Z \to X \times H$ is the inclusion. By Lemma 3.6 2(b), one easily sees that Condition (2) in Definition 4.12 is satisfied so that $\mathrm{pr}_2 : G[X] \to H$ is indeed a strict homomorphism of quasi-symplectic groupoids. This completes the proof. q.e.d.

The proof of the above proposition also yields the following:

Corollary 4.16. *If $f : G_0 \xleftarrow{\rho} X \xrightarrow{\sigma} H_0$ is a generalized homomorphism of quasi-symplectic groupoids from $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ to*

$(H \rightrightarrows H_0, \omega_H + \Omega_H)$, then $f^*[\omega_H + \Omega_H] = [\omega_G + \Omega_G]$, where $f^* : H^3_{DR}(H_\bullet) \to H^3_{DR}(G_\bullet)$ is the induced homomorphism of the de Rham cohomology groups.

In particular, if $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ and $(H \rightrightarrows H_0, \omega_H + \Omega_H)$ are Morita equivalent quasi-symplectic groupoids, then $[\omega_G + \Omega_G]$ and $[\omega_H + \Omega_H]$ define the same class under the isomorphism $H^3_{DR}(G_\bullet) \simeq H^3_{DR}(H_\bullet)$.

4.3. Hamiltonian spaces for Morita equivalent quasi-symplectic groupoids.

Definition 4.17. Assume that $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ and $(H \rightrightarrows H_0, \omega_H + \Omega_H)$ are Morita equivalent quasi-symplectic groupoids with an equivalence bimodule $G_0 \xleftarrow{\rho} X \xrightarrow{\sigma} H_0$. Let $\phi : F \to G_0$ be an Hamiltonian G-space, and $\psi : E \to H_0$ an Hamiltonian H-space. We say that F and E are a *pair of related Hamiltonian spaces* if there is an isotropy submanifold $\Omega \subset X \times \overline{F} \times E$, such that

(1) Ω is a graph over both $X \times_{G_0} \overline{F}$ and $X \times_{H_0} E$ and
(2) $(yx^{-1}) \cdot f = y(x^{-1}(f))$ and $(x^{-1}z) \cdot e = x^{-1}(z(e))$, whenever either side is defined for any $x, y, z \in X, e \in E$ and $f \in F$, where by $x^{-1}(f)$ (or $z(e)$ resp.), we denote the unique element in E (or F resp.) such that $(x, f, x^{-1}(f)) \in \Omega$ (or $(z, z(e), e) \in \Omega$ resp.) and yx^{-1} (or $x^{-1}z$ resp.) denotes the corresponding element $[y, x]$ (or $[x, z]$ resp.) in the groupoid G (or H resp.) under the identification: $G \cong (X \times_{H_0} X)/H$ (or $H \cong G \setminus (X \times_{G_0} X)$ resp.)

The following property follows immediately from definition 4.17

Proposition 4.18.

(1) *$x^{-1}(x(e)) = e$ and $x(x^{-1}(f)) = f$ for all composable $x \in X, e \in E$ and $f \in F$;*
(2) *for all composable $g \in G, x, y \in X, h \in H, f \in F$ and $e \in E$,*

$$(g \cdot x)^{-1}(f) = x^{-1}(g^{-1} \cdot f),\ (g \cdot y)(e) = g \cdot y(e),$$
$$(x \cdot h)^{-1}(f) = h^{-1} \cdot (x^{-1}(f)),\ (y \cdot h)(e) = y(h \cdot e).$$

We are now ready to prove the main result of this section.

Theorem 4.19. *Suppose that $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ and $(H \rightrightarrows H_0, \omega_H + \Omega_H)$ are Morita equivalent quasi-symplectic groupoids with an equivalence bimodule $G_0 \xleftarrow{\rho} X \xrightarrow{\sigma} H_0$. Then,*

(1) *corresponding to any Hamiltonian G-space $\phi : F \to G_0$, there is a unique (up to isomorphism) Hamiltonian H-space $\psi : E \to H_0$ such that F and E are a pair of related Hamiltonian spaces and vice versa.*

(2) *let $\phi_i : F_i \to G_0$, $i = 1, 2$, be Hamiltonian G-spaces and $\psi_i : E_i \to H_0$, $i = 1, 2$, their related Hamiltonian H-spaces. If ϕ_1 and ϕ_2 are clean, then ψ_1 and ψ_2 are clean, and the classical intertwiner spaces $F_1 \times_G \overline{F_2}$ and $E_1 \times_H \overline{E_2}$ are symplectically diffeomorphic.*

Proof. The proof is a simple modification of Theorem 4.2 in [**36**].

(1) Suppose that $\phi : F \to G_0$ is an Hamiltonian G-space. Then, $G_0 \xleftarrow{\phi} F \to \cdot$ is an Hamiltonian G-$\cdot$-bimodule. Since $H_0 \xleftarrow{\sigma} \overline{X} \xrightarrow{\rho} G_0$ is an Hamiltonian H–G-bimodule, from Theorem 3.16, it follows that $E := \overline{X} \times_G F$ is an Hamiltonian H-$\cdot$-bimodule, i.e., an Hamiltonian H-space. Here, $\psi : E \to H_0$ and the H-action on E are defined by

$$\psi([x, f]) = \sigma(x)$$

and

$$h \cdot [x, f] = [x \cdot h^{-1}, f].$$

Let $\Omega = \{(x, f, [x, f]) | \forall (x, f) \in X \times_{G_0} F\} \subset X \times \overline{F} \times E$. It is straightforward to check that Ω is an isotropy submanifold, and is indeed a graph over both $X \times_{G_0} F$ and $X \times_{H_0} E$. Hence, $\phi : F \to G_0$ and $\psi : E \to H_0$ are a pair of related Hamiltonian spaces. Conversely, one easily sees that $F \cong X \times_H E$ by working backwards.

(2) Let $\Omega_i \subset X \times \overline{F_i} \times E_i$, $i = 1, 2$, be as in (1). Then, $\Omega_1 \times \overline{\Omega_2} \subset X \times \overline{F_1} \times E_1 \times \overline{X} \times F_2 \times \overline{E_2}$ is an isotropy submanifold, which is a graph over $X \times_{G_0} \overline{F_1} \times \overline{X} \times_{G_0} F_2$. Given any $[(f_1, f_2)] \in F_1 \times_G F_2$, take any element $x \in X$ such that $\rho(x) = \phi_1(f_1) = \phi_2(f_2)$. Let $e_1 \in E_1$, and $e_2 \in E_2$ such that $(x, f_1, e_1, x, f_2, e_2) \in \Omega_1 \times \Omega_2$. Then, it is simple to see that $(e_1, e_2) \in E_1 \times_{H_0} E_2$ and $[e_1, e_2] \in E_1 \times_H E_2$ is independent of the choice of x and (f_1, f_2). Thus, we obtain a well-defined map:

$$\Phi : F_1 \times_G \overline{F_2} \to E_1 \times_H \overline{E_2}, \qquad [f_1, f_2] \to [e_1, e_2].$$

It is simple to check that Φ is a bijection, which is indeed a symplectic diffeomorphism by using the fact that $\Omega_1 \times \overline{\Omega_2}$ is isotropic. q.e.d.

Corollary 4.20. *Assume that* $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ *and* $(H \rightrightarrows H_0, \omega_H + \Omega_H)$ *are Morita equivalent quasi-symplectic groupoids, and* $\phi : F \to G_0$ *and* $\psi : E \to H_0$ *are a pair of related Hamiltonian* G*- and* H*-spaces respectively. Let* $n \in H_0$ *and* $m \in G_0$ *be a pair of related points. Then, the reduced spaces* $\phi^{-1}(m)/G_m^m$ *and* $\psi^{-1}(n)/H_n^n$ *are symplectically diffeomorphic.*

Remark 4.21. Corollary 4.20 indicates that the reduction of Hamiltonian spaces of quasi-symplectic groupoids is of stack natural.

In fact, the same argument in the proof of Theorem 4.19 leads to the following more general result.

Theorem 4.22. *Assume that* $f : G_0 \xleftarrow{\rho} X \xrightarrow{\sigma} H_0$ *is a generalized homomorphism of quasi-symplectic groupoids from* $(G \rightrightarrows G_0, \omega_G + \Omega_G)$ *to* $(H \rightrightarrows H_0, \omega_H + \Omega_H)$*. Then*

(1) *if* $\phi : E \to H_0$ *is an Hamiltonian* H*-space, and the maps* ϕ *and* σ *are clean, then* $\psi : F \to G_0$*, where* $F = X \times_H E$*, is an Hamiltonian* G*-space, called the pull-back Hamiltonian space and denoted by* f^*E*,*

(2) *let* $\phi_i : E_i \to H_0$*,* $i = 1, 2$*, be Hamiltonian* H*-spaces and* $\psi_i : F_i \to G_0$*,* $i = 1, 2$*, their pull-back Hamiltonian* G*-spaces. If* ϕ_1 *and* ϕ_2 *are clean, then* ψ_1 *and* ψ_2 *are clean, and moreover, there exists a natural symplectic immersion between their classical intertwiner spaces* $F_1 \times_G \overline{F_2} \to E_1 \times_H \overline{E_2}$*.*

4.4. Examples. In this section, we will discuss various examples of Morita equivalent quasi-symplectic groupoids and derive some familiar corollaries as a consequence. We start with a general set-up.

Let $(\Gamma \rightrightarrows P, \omega + \Omega)$ be a quasi-symplectic groupoid and $\phi : Y \to P$ a surjective submersion. Consider $(\Gamma[Y] \rightrightarrows Y, \omega' + \Omega')$, where $\omega' = (\overline{B}, B, \omega) \in \Omega^2(\Gamma[Y])$ and $\Omega' = \phi^*\Omega - dB$ (in applications, normally $\phi^*\Omega = dB$ for some $B \in \Omega^2(Y)$, so $\Omega' = 0$). According to Propositions 4.6 and 4.8, this is a quasi-symplectic groupoid Morita equivalent to $(\Gamma \rightrightarrows P, \omega + \Omega)$. Applying Theorem 4.19, we obtain the following

Proposition 4.23.

(1) *There is a bijection between Hamiltonian* Γ*-spaces and Hamiltonian* $\Gamma[Y]$*-spaces.*

More precisely, if $(M \xrightarrow{J} P, \omega_M)$ *is an Hamiltonian* Γ*-space, then* $(N \xrightarrow{\widetilde{J}} Y, \omega_N)$ *is an Hamiltonian* $\Gamma[Y]$*-space, where* N *is the*

fiber product $Y \times_P M$, $\widetilde{J} : N \to Y$ *is the projection to the first component, and* $\omega_N = -\widetilde{J}^* B + p^*\omega_M$. *Here,* $p : N \to M$ *is the projection to the second component.*

Conversely, if $(N \xrightarrow{\widetilde{J}} Y, \omega_N)$ *is an Hamiltonian* $\Gamma[Y]$*-space, its corresponding* Γ*-space* $(M \xrightarrow{J} P, \omega_M)$ *is given as follows.* M *is the quotient space* $N/\Gamma[Y]'$, *where* $\Gamma[Y]'$ *is the subgroupoid of* $\Gamma[Y]$ *consisting of all elements* (y_1, y_2, u) *with* $y_1, y_2 \in Y$ *and* $u \in P$ *satisfying* $\phi(y_1) = \phi(y_2) = u$, $J : M \to P$ *is given by* $J([n]) = (\phi \circ \widetilde{J})(n)$, *and the two-form* ω_M *on* M *is defined by the equation:*

$$\pi^*\omega_M = \omega_N + \widetilde{J}^* B.$$

Here, $\pi : N \to M$ *denotes the natural projection map.*

(2) *If* $(M \xrightarrow{J} P, \omega_M)$ *and* $(N \xrightarrow{\widetilde{J}} Y, \omega_N)$ *are a pair of Hamiltonian* Γ*- and* $\Gamma[Y]$*-spaces as above, and* $\mathcal{O} \subset P$ *and* $\mathcal{O}_Y \subset Y$ *are a pair of related groupoid orbits, then the reduced spaces* $J^{-1}(\mathcal{O})/\Gamma$ *and* $\widetilde{J}^{-1}(\mathcal{O}_Y)/\Gamma[Y]$ *are symplectic diffeomorphic.*

Proof. As in the proof of Propositions 4.6 and 4.8, the Morita equivalence Hamiltonian bimodule is given by $P \xleftarrow{\rho} X \xrightarrow{\sigma} Y$, where $X = \Gamma \times_{t,P,\phi} Y$ and $\omega_X = (\omega, B)$. The left Γ- and right- $\Gamma[Y]$-actions are given by Equations (27) and (28), respectively.

Now, we are ready to apply Theorem 4.19. If $J : M \to P$ is an Hamiltonian Γ-space, then its corresponding Hamiltonian $\Gamma[Y]$-space is $N = \overline{X} \times_\Gamma M$, which is the quotient by Γ of the space $\{(r, y, m) | t(r) = \phi(y),\ J(m) = s(r)\}$. It is simple to see that the latter is diffeomorphic to the fiber product $Y \times_P M$, and, under this diffeomorphism, the two-form on $\overline{X} \times_\Gamma M$ goes to $-\widetilde{J}^* B + p^*\omega_M$.

Conversely, assume that $\widetilde{J} : N \to Y$ is an Hamiltonian $\Gamma[Y]$-space. Then, $M = X \times_{\Gamma[Y]} N \cong (X \times_Y N)/\Gamma[Y]$. Now, $X \times_Y N = \{(r, y, n) | t(r) = \phi(y),\ \widetilde{J}(n) = y\}$. It is simple to see that, under the $\Gamma[Y]$-action, any element in $X \times_Y N$ is equivalent to (u, y, n) where $y = \widetilde{J}(n)$ and $u = \phi(\widetilde{J}(n))$. Any two such elements (u, y, n) and (u', y', n') are equivalent if and only if $n' = \gamma' \cdot n$ where $\gamma' \in \Gamma[Y]'$. As a result, M can be identified with $N/\Gamma[Y]'$, and the two-form (ω, B, ω_N) on $X \times_Y N$ goes to $\omega_N + \widetilde{J}^* B$ under the identification

$$\{(u, y, n) | \forall n \in N,\ y = \widetilde{J}(n),\ u = \phi(\widetilde{J}(n))\} \xrightarrow{\sim} N.$$

Therefore, we have $\pi^*\omega_M = \omega_N + \widetilde{J}^*B$.

The rest of the claims follows easily from Theorem 4.19. q.e.d.

We now consider various special cases of the above proposition.

Let G be a compact connected Lie group equipped with the Bruhat–Poisson group structure [**21**], and $\mathfrak{g}$ be its Lie algebra. By G^*, we denote its simply-connected dual Poisson group. It is known that there exists a diffeomorphism [**1**, **3**]:

$$E : \mathfrak{g}^* \to G^*,$$

which is G-equivariant with respect to the coadjoint action on $\mathfrak{g}^*$ and the left dressing action on G^*. Let us recall the construction briefly. Here, we follow the presentation of [**3**].

Let $\kappa : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ be the Cartan involution given by the complex conjugation, and let $\dagger : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ be the anti-involution $\xi^\dagger = -\kappa(\xi)$. We also denote by $\dagger$ the induced anti-involution of $G^{\mathbb{C}}$, considered as a real group. Let $B^\sharp : \mathfrak{g}^* \to \mathfrak{g}$ be the isomorphism induced by the Killing form B. For any $\mu \in \mathfrak{g}^*$, the element $g = \exp(iB^\sharp(\mu)) \in G^{\mathbb{C}}$ admits a unique decomposition $g = ll^\dagger$, for some $l \in G^*$. Then, E is defined by $E(\mu) = l$.

Let $\beta \in \Omega^1(\mathfrak{g}^*)$ be the one-form [**3**]

$$\beta = \frac{1}{2i}\mathcal{H}\big(E^*B^{\mathbb{C}}(\theta, \theta^\dagger)\big), \tag{31}$$

where $\theta \in \Omega^1(G^*) \otimes \mathfrak{g}^*$ is the left-invariant Maurer–Cartan form, and $\theta^\dagger$ its image under the map $\dagger : \mathfrak{g}^* \subset \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$, $\mathcal{H} : \Omega^\star(\mathfrak{g}^*) \to \Omega^{\star-1}(\mathfrak{g}^*)$ is the standard homotopy operator for the de Rham differential. Let $B = d\beta \in \Omega^2(\mathfrak{g}^*)$.

The following proposition also follows from Ginzburg–Weinstein theorem [**13**].

Proposition 4.24. *The Lu–Weinstein symplectic groupoid $G \times G^* \rightrightarrows G^*$ is Morita equivalent to the standard cotangent symplectic groupoid $T^*G \rightrightarrows \mathfrak{g}^*$.*

Proof. Since $E : \mathfrak{g}^* \to G^*$ is G-equivariant, the pull-back groupoid $(G \times G^*)[\mathfrak{g}^*] \rightrightarrows \mathfrak{g}^*$ is clearly isomorphic to the transformation groupoid $G \times \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$, which is naturally isomorphic to $T^*G \rightrightarrows \mathfrak{g}^*$. Moreover, from Lemma 2 (2) in [**1**] (or Proposition 3.1 in [**3**]), it follows that

$$E^*\omega' - \omega = \partial B.$$

Therefore, these two symplectic groupoids are Morita equivalent since $dB = 0$. q.e.d.

As an application, we are lead to the following Alekseev–Ginzburg–Weinstein linearization theorem [1].

Corollary 4.25. *Let G be a connected compact Lie group equipped with the Bruhat–Poisson group structure. Then*

(1) *(M, ω_M) is an Hamiltonian Poisson group G-space with the momentum map $J : M \to G^*$ if and only if (M, ω'_M) is a usual Hamiltonian G-space with the momentum map $\widetilde{J} : M \to \mathfrak{g}^*$, where*

$$J = E \circ \widetilde{J}, \quad \omega'_M = \omega_M - \widetilde{J}^* B.$$

(2) *If $\widetilde{\mathcal{O}}$ is a coadjoint orbit in $\mathfrak{g}^*$ and $\mathcal{O} = E(\widetilde{\mathcal{O}})$ is its corresponding dressing orbit in G^*, then the reduced spaces $\widetilde{J}^{-1}(\widetilde{\mathcal{O}})/G$ and $J^{-1}(\mathcal{O})/G$ are symplectically diffeomorphic.*

Next, we consider the AMM quasi-symplectic groupoid $(G \times G \rightrightarrows G, \omega + \Omega)$. Let $\mathrm{Hol} : L\mathfrak{g} \longrightarrow G$ be the holonomy map, i.e., the time-1 map of the differential equation:

$$\mathrm{Hol}_s(r)^{-1} \frac{\partial}{\partial s} \mathrm{Hol}_s(r) = r, \ \mathrm{Hol}_0(r) = e.$$

Then, we have $\mathrm{Hol}^* \Omega = d\mu$, where μ is the two-from on $L\mathfrak{g}$ [2].

$$\mu = \frac{1}{2} \int_0^1 (\mathrm{Hol}_s^* \bar{\theta}, \frac{\partial}{\partial s} \mathrm{Hol}_s^* \bar{\theta}) ds,$$

where $\bar{\theta} \in \Omega^1(G) \otimes \mathfrak{g}$ is the right Maurer–Cartan form.

The pull-back groupoid of the AMM-groupoid under the holonomy map is isomorphic to the transformation groupoid $LG \times L\mathfrak{g} \rightrightarrows L\mathfrak{g}$, where LG acts on $L\mathfrak{g}$ by the gauge transformation (20). To see this, note that

$$(G \times G)[L\mathfrak{g}] \cong \Big\{ (r_1(s), r_2(s), g) | r_1(s), r_2(s) \in L\mathfrak{g}, g \in G \\ \text{such that } g^{-1} \mathrm{Hol}(r_1) g = \mathrm{Hol}(r_2) \Big\}.$$

Define

$$\tau : \ (G \times G)[L\mathfrak{g}] \to LG \times L\mathfrak{g}, \quad (r_1(s), r_2(s), g) \longrightarrow (g(s), r_1(s)), \tag{32}$$

where $g(s)$ is defined by

$$Ad_{g(s)^{-1}}r_1(s) - g(s)^{-1}\frac{dg(s)}{ds} = r_2(s), \quad g(0) = g. \tag{33}$$

It is simple to see that τ is indeed a diffeomorphism, under which the groupoid structure on $(G\times G)[L\mathfrak{g}]$ becomes the transformation groupoid $LG \times L\mathfrak{g} \rightrightarrows L\mathfrak{g}$.

Proposition 4.26. [6] *The symplectic groupoid* $(LG \times L\,\mathfrak{g} \rightrightarrows L\mathfrak{g}, \omega_{LG\times L\mathfrak{g}})$ *is Morita equivalent to the AMM quasi-symplectic groupoid* $(G \times G \rightrightarrows G, \omega + \Omega)$.

Proof. From the above discussion, we know that $LG\times L\mathfrak{g} \rightrightarrows L\mathfrak{g}$ is the pull-back groupoid of $G\times G \rightrightarrows G$ under the holonomy map Hol. Denote by f the groupoid homomorphism from $LG \times L\mathfrak{g} \rightrightarrows L\mathfrak{g}$ to $G \times G \rightrightarrows G$, where on the space of morphisms and the space of objects, f is given, respectively, by $f(g(s), r(s)) = (g(0), \mathrm{Hol}(r))$ and $f(r(s)) = \mathrm{Hol}(r)$, $\forall g(s) \in LG,\ r(s) \in L\mathfrak{g}$. Then, a simple computation yields that

$$\omega_{LG\times L\mathfrak{g}} - f^*(\omega + \Omega) = \delta\mu.$$

Thus, the conclusion follows from Propositions 4.6 and 4.8 immediately. q.e.d.

Remark 4.27. The above result was used in [6] to construct an equivariant S^1-gerbe over the stack G/G.

An immediate consequence is the following equivalence theorem of Alekseev–Malkin–Meinrenken [2].

Corollary 4.28.

(1) *There is a bijection between Hamiltonian LG-spaces and quasi-Hamiltonian G-spaces.*

More precisely, if $(M \xrightarrow{J} G, \omega_M)$ *is a quasi-Hamiltonian G-space, then* $(N \xrightarrow{\widetilde{J}} L\mathfrak{g}, \omega_N)$ *is an Hamiltonian LG-space, where* N *is the fiber product* $L\mathfrak{g} \times_G M$, $\widetilde{J} : N \to L\mathfrak{g}$ *is the projection map to the first component, and* $\omega_N = -\widetilde{J}^*\mu + p^*\omega_M$. *Here,* $p : N \to M$ *is the projection to the second component.*

Conversely, if $(N \xrightarrow{\widetilde{J}} L\mathfrak{g}, \omega_N)$ *is an Hamiltonian LG-space, its corresponding quasi-Hamiltonian G-space* $(M \xrightarrow{J} G, \omega_M)$ *is given as follows.* M *is the quotient space* $N/\Omega G$, *where* ΩG *is the based*

loop group $\Omega G \subset LG$, $J : M \to G$ *is given by* $J([n]) = (Hol \circ \widetilde{J})(n)$, *and the two-form on* M *is defined by*

$$\pi^* \omega_M = \omega_N + \widetilde{J}^* \mu,$$

where $\pi : N \to M$ *denotes the projection.*

(2) *Let* $(M \xrightarrow{J} G, \omega_M)$ *and* $(N \xrightarrow{\widetilde{J}} L\mathfrak{g}, \omega_N)$ *be as above. Then, the reduced spaces* $J^{-1}(e)/G$ *and* $\widetilde{J}^{-1}(0)/LG$ *are symplectically diffeomorphic.*

Proof. This essentially follows from Proposition 4.23. Note that under the isomorphism (32), the subgroupoid $(G \times G)[L\mathfrak{g}]'$ of $(G \times G)[L\mathfrak{g}]$ corresponds to the transformation groupoid $L\mathfrak{g} \times \Omega G \rightrightarrows L\mathfrak{g}$. q.e.d.

Remark 4.29. For a quasi–Manin triple $(d, \mathfrak{g}, \mathfrak{h})$, Alekseev and Kosmann–Schwarzbach introduced a momentum map theory with target space D/G [4]. It would be interesting to investigate what the corresponding quasi-symplectic groupoid is. In particular, different choices of complements $\mathfrak{h}$ should give rise to Morita equivalent quasi-symplectic groupoids.

5. Acknowledgments

The author would like to thank several institutions for their hospitality while work on this project was being done: RIMS/Kyoto University, Ecole Polytechnique, Erwin Schrödinger Institute, and University of Geneva. He also wishes to thank many people for useful discussions and comments, including Anton Alekseev, Philip Boalch, Henrique Bursztyn, Eckhard Meinrenken, Jim Stasheff, and Alan Weinstein. Some results of the paper were presented in Poisson 2002, Lisbon. He would like to thank the organizers for inviting him and giving him the chance to present the work. Special thanks go to Laurent–Gengoux family, who generously provided a fantastic working environment during the Séminaire Itinérant Géométrie et Physique I, Normandie 2003, where this work was completed.

References

[1] A. Alekseev, *On Poisson actions of compact Lie groups on symplectic manifolds*, J. Differential Geom. **45** (1997) 241–256, MR 1449971, Zbl 0912.53018.

[2] A. Alekseev, A. Malkin & E. Meinrenken, *Lie group valued moment maps*, J. Differential Geom. **48** (1998) 445–495, MR 1638045, Zbl 0948.53045.

[3] A. Alekseev, E. Meinrenken & C. Woodward, *Linearization of Poisson actions and singular values of matrix products*, Ann. Inst. Fourier (Grenoble) **51** (2001) 1691–1717, MR 1871286, Zbl 1012.53064.

[4] A. Alekseev & Y. Kosmann–Schwarzbach, *Manin pairs and moment maps*, J. Differential Geom. **56** (2000) 133–165, MR 1863024.

[5] K. Behrend, B. Edidin, B. Fantechi, W. Fulton, L. Gottsche & A. Kresch, *Introduction to stacks*, book in writing.

[6] K. Behrend, P. Xu & B. Zhang, *Equivariant gerbes over compact simple Lie groups*, C.R. Acad. Sci. Paris Serie I **336** (2003) 251–256, MR 1968268.

[7] K. Behrend & P. Xu, *S^1-bundles and gerbes over differentiable stacks*, C.R. Acad. Sci. Paris Serie I **336** (2003) 163–168, MR 1969572.

[8] H. Bursztyn, M. Crainic, A. Weinstein & C. Zhu, *Integration of twisted Dirac brackets*, Duke Math. J., **123** (2004) 549–607, MR 2068969.

[9] H. Bursztyn & O. Radko, *Gauge equivalence of Dirac structures and symplectic groupoids*, Ann. Inst. Fourier (Grenoble) **53** (2003) 309–337, MR 1973074, Zbl 1026.58019.

[10] A. Cattaneo & P. Xu, *Integration of twisted Poisson structures*, J. Geom. Phys. **49** (2004) 187–196.

[11] A. Coste, P. Dazord & A. Weinstein, *Groupoïdes symplectiques*, Publications du Département de Mathématiques de l'Université de Lyon, I, **2/A** (1987) 1–65, MR 0996653.

[12] V.G. Drinfel'd, *Quantum groups*, Proc. ICM, Berkeley, 1986, Vol. 1, 789–820, MR 0934283, Zbl 0667.16003.

[13] V. Ginzburg & A. Weinstein, *Lie-Poisson structure on some Poisson Lie groups*, J. Amer. Math. Soc. **5** (1992) 445–453, MR 1126117, Zbl 0766.58018.

[14] V. Guillemin & S. Sternberg, *Geometric quantization and multiplicities of group representations*, Invent. Math. **67** (1982) 515–538, MR 0664118, Zbl 0503.58018.

[15] A. Haefliger, *Groupoïdes d'holonomie et classifiants*, Structure transverse des feuilletages, Toulouse 1982, Astérisque **116** (1984) 70–97, MR 0755163, Zbl 0562.57012.

[16] M. Hilsum & G. Skandalis, *Morphismes K-orientés d'espaces de feuilles et fonctorialité en théorie de Kasparov* (d'après une conjecture d'A. Connes), Ann. Sci. École Norm. Sup. **20** (1987) 325–390, MR 0925720, Zbl 0656.57015.

[17] C. Laurent-Gengoux & P. Xu, *Quantization of pre-quasi-symplectic groupoids and their Hamiltonian spaces, Weinstein Festschrift*, Progress in Math., to appear, math. SG/0311154.

[18] C. Laurent-Gengoux, J.-L. Tu & P. Xu, *Chern-Weil map and characteristic classes of principal G-bundles over groupoids*, math.DG/0401420.

[19] J.-H. Lu, *Momentum mappings and reduction of Poisson actions*, Séminaire Sud-Rhodanien à Berkeley, Symplectic geometry, groupoids, and integrable systems, Springer-MSRI publications, 1991, 209–226, MR 1104930, Zbl 0735.58004.

[20] J.-H. Lu & A. Weinstein, *Groupoïdes symplectiques doubles des groupes de Lie-Poisson*, C.R. Acad. Sc. Paris **309** (1989) 951–954, MR 1054741, Zbl 0701.58025.

[21] J.-H. Lu, & A. Weinstein, *Poisson Lie groups, dressing transformations, and the Bruhat decomposition*, J. Differential Geom. **31** (1990) 501–526, MR 1037412, Zbl 0673.58018.

[22] E. Meinrenken, *The basic gerbe over a compact simple Lie group*, Enseign. Math. (2) **49** (2003) 307–333, MR 2026898.

[23] E. Meinrenken & C. Woodward, *Hamiltonian loop group actions and Verlinde factorization*, J. Differential Geom. **50**(**3**) (1998) 417–469, MR 1690736, Zbl 0949.37031.

[24] E. Meinrenken & C. Woodward, *Cobordism for Hamiltonian loop group actions and flat connections on the punctured two-sphere*, Math. Z. **231** (1999) 133–168, MR 1696761, Zbl 0951.53051.

[25] E. Meinrenken & C. Woodward, *Canonical bundles for Hamiltonian loop group manifolds*, Pacific J. Math. **198** (2001) 477–487, MR 1835519.

[26] K. Mikami & A. Weinstein, *Moments and reduction for symplectic groupoid actions*, Publ. RIMS Kyoto Univ. **24** (1988) 121–140, MR 0944869, Zbl 0659.58016.

[27] P. Muhly, J. Renault & D. Williams, *Equivalence and isomorphism for groupoid C^*-algebras*, J. Operator Theory **17** (1987) 3–22, MR 0873460, Zbl 0645.46040.

[28] E. Noether, *Invariante Variationsprobleme*, Nachr. v.d. Ges. d. Wiss. zu Gottingen, 1918, 235–257.

[29] A. Pressley & G. Segal, *Loop groups*, Oxford Mathematical Monographs, Oxford University Press, New York, 1986, MR 0900587, Zbl 0618.22011.

[30] P. Severa & A. Weinstein, *Poisson geometry with a* 3*-form background*, Prog. Theor. Phys. Suppl. **144** (2001) 145–154, MR 2023853, Zbl 1029.53090.

[31] J.-L. Tu, P. Xu & C. Laurent-Gengoux, *Twisted K-theory of differentiable stacks*, Annales Scientifiques de l'ENS, to appear, math.KT/0306138.

[32] A. Weinstein, *Symplectic groupoids and Poisson manifolds*, Bull. AMS **16** (1987) 101–104, MR 0866024, Zbl 0618.58020.

[33] A. Weinstein, *Affine Poisson structures*, Internat. J. Math. **1** (1990) 343–360, MR 1078517, Zbl 0725.58014.

[34] A. Weinstein, *The geometry of momentum*, math.SG/0208108.

[35] A. Weinstein & P. Xu, *Classical solutions of the quantum Yang-Baxter equations*, Commun. Math. Phys. **148** (1992) 309–343, MR 1178147, Zbl 0849.17015.

[36] P. Xu, *Morita equivalent symplectic groupoids*, Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989), Math. Sci. Res. Inst. Publ. **20** (1991) 291–311, MR 1104935, Zbl 0733.58013.

[37] P. Xu, *Classical intertwiner spaces and quantization*, Commun. Math. Phys. **164** (1994) 473–488, MR 1291241, Zbl 0840.58021.

[38] N.T. Zung, *Proper groupoids and momentum maps: linearization, affinity, and convexity*, math.SG/0407208.

Department of Mathematics
Pennsylvania State University
University Park, PA 16802
E-mail address: ping@math.psu.edu