J. DIFFERENTIAL GEOMETRY
67 (2004) 257-288

MODULAR INVARIANCE CHARACTERISTIC NUMBERS AND η INVARIANTS

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Abstract

We extend the "miraculous cancellation" formulas of Alvarez–Gaumé, Witten and Kefeng Liu to a twisted version where an extra complex line bundle is involved. We apply our result to discuss intrinsic relations between the higher dimensional Rokhlin type congruences due to Ochanine, Finashin and Zhang. In particular, an analytic proof of the Finashin congruence is given.

1. Introduction

A well-known theorem of Rokhlin [**23**] states that the Signature of an oriented closed smooth spin 4-manifold is divisible by 16. It has two kinds of higher dimensional generalizations. One is due to Atiyah and Hirzebruch [**3**] stating that the $\widehat{A}$-genus of an $8k+4$ dimensional oriented closed smooth spin manifold is an even integer. The other one, due to Ochanine (cf. [**21**]), states that the Signature of such an $8k+4$ dimensional manifold is divisible by 16.

Recall that Atiyah and Hirzebruch's result in [**3**] goes beyond the $\widehat{A}$-genus. In fact, they proved that if E is a real vector bundle over an $8k+4$ dimensional oriented closed smooth spin manifold M, then $\langle \widehat{A}(TM)\mathrm{ch}(E\otimes\mathbb{C}),[M]\rangle \in 2\mathbb{Z}$.

It turns out that these two generalizations of the original Rokhlin divisibility are closely related: in [**17**], Landweber shows how one can use the ideas of elliptic genus to deduce the Ochanine divisibility directly from the divisibility results of Atiyah and Hirzebruch.

This work was supported in part by MOEC and the 973 project of MOSTC..
Received 06/13/2004.

Now let M be a closed oriented smooth 4-manifold, not necessarily spin. Let B be an orientable characteristic submanifold of M, that is, B is a compact two dimensional submanifold of M such that $[B] \in H_2(M, \mathbb{Z}_2)$ is dual to the second Stiefel–Whitney class of TM.

In this case, Rokhlin established in [**24**] a congruence formula of the type

$$\frac{\mathrm{Sign}\,(M) - \mathrm{Sign}\,(B \cdot B)}{8} \equiv \phi(B) \mod 2\mathbb{Z}, \tag{1.1}$$

where $B \cdot B$ is the self-intersection of B in M and $\phi(B)$ is a spin cobordism invariant associated to (M, B).

Clearly, when M is spin and $B = \emptyset$, (1.1) reduces to the original Rokhlin divisibility.

In the case where B might be non-orientable, an extension of (1.1) was proved by Guillou and Marin in [**11**] (see [**15**] for a comprehensive account).

In [**21**], Ochanine generalizes (1.1) to $8k+4$ dimensional closed Spin^c manifolds. His formula is also of type (1.1) and thus extends his divisibility result to Spin^c manifolds.

In [**10**], Finashin generalizes the Ochanine congruence to the case where the characteristic submanifold B is non-orientable. His formula also extends the Guillou–Marin congruence to $8k+4$ dimensional closed oriented manifolds.

On the other hand, Zhang [**25**, **26**] proves another type of congruence formulae for $8k + 4$ dimensional manifolds. His results generalize the Atiyah–Hirzebruch divisibilities and, when restricted to 4-manifolds, also recover (1.1) as well as the Guillou–Marin extension of it.

In view of Landweber's elliptic genus proof of the Ochanine divisibility, it is natural to ask whether there exist similar intimate relations between these higher dimensional generalizations of (1.1) as well as its extension by Guillou and Marin.

In [**26**, Appendix], this question was briefly dealt with for the case of 12 dimensional Spin^c manifolds, with the help of the so called "miraculous cancellation" formula first proved by the physicists Alvarez–Gaumé and Witten [**1**] (cf. (2.43) in the text). In particular, an intrinsic analytic interpretation of Ochanine's spin cobordism invariant $\phi(B)$ is given. Moreover, it was pointed out that a higher dimensional generalization of the "miraculous cancellation" formula of Alvarez–Gaumé and

Witten would lead to a better understanding of the general congruence formulae due to Ochanine [21] and Finashin [10].

This required higher dimensional "miraculous cancellation" formula was established by Kefeng Liu [18] by developing modular invariance properties of characteristic forms. In some sense, Liu's formula refines the argument of Landweber [17] to the level of differential forms.

In [19], by combining Liu's formula with the analytic arguments in [26], Liu and Zhang were able to give an intrinsic analytic interpretation of the Ochanine invariant $\phi(B)$ for any $8k+2$ dimensional closed spin manifold B (compare with [22]), as well as the Finashin invariant [10] for any $8k+2$ dimensional closed pin$^-$ manifold appearing in the Finashin congruence formula. In particular, this leads to analytic versions, stated in [19, Theorems 4.1 and 4.2], of the Finashin and Ochanine congruences.

Now, the remaining question is whether these analytic versions of the Finashin and Ochanine congruences could be deduced directly from the Rokhlin type congruences in [25, 26].

The purpose of this paper is to give a positive answer to this question. To be more precise, what we get is a generalization of the "miraculous cancellation" formulas of Alvarez–Gaumé, Witten and Liu to a twisted version where an extra complex line bundle is involved. Modular invariance properties developed in [18] still play an important role in the proof of such an extended cancellation formula. When applying our formula to Spin^c manifolds, we are led directly to an unexpected refined version of [19, Theorem 4.2] (cf. (3.2) in the text). Moreover, by combining this twisted cancellation formula with the analytic Rokhlin type congruences proved in [26], we are able to give a direct analytic proof of [19, Theorem 4.1], which by [19, Theorem 3.1] is equivalent to the original Finashin congruence formula.

The rest of the article is organized as follows. In Section 2, we establish our twisted extension of the "miraculous cancellation" formulas of Alvarez–Gaumé, Witten and Liu. We include also an Appendix to this section where we state an analogous twisted cancellation formula for $8k$ dimensional manifolds. In Section 3, we apply the twisted cancellation formula proved in Section2 to $8k+4$ dimensional Spin^c manifolds and show how it leads directly to a refinement of the Ochanine congruence formula. Finally, in Section4, we combine the results in Section 2 with the Rokhlin type congruences of Zhang [26, Theorem 3.2] to give a direct proof of the analytic version of the Finashin congruence stated in [19, Theorem 4.1].

Some of the results of this article have been announced in [13].

2. Modular invariance and a twisted "miraculous cancellation" formula

In this section, we generalize the "miraculous cancellation" formulas of Alvarez–Gaumé, Witten and Kefeng Liu ([1], [18]) to a twisted version where an extra complex line bundle (or, equivalently, a rank two real oriented vector bundle) is involved.

This section is organized as follows. In Section 2.1, we present the basic geometric data and recall the definitions of the characteristic forms to be discussed. In Section 2.2, we state the main result of this section, which is a twisted extension of the "miraculous cancellation" formulas of Alvarez–Gaumé, Witten and Liu. In Section 2.3, we recall some basic facts about modular forms which will be used in Section 2.4 to give a proof of the main result stated in Section 2.2. Finally, in Section 2.5, we specialize the main result stated in Section 2.2 to the tangent bundle case and give an explicit expression of it in the 12 dimensional case. There is also an appendix to this section where we include an analogous twisted cancellation formula for $8k$ dimensional manifolds.

2.1. Some characteristic forms. Let M be an $8k+4$ dimensional Riemannian manifold. Let ∇^{TM} be the associated Levi–Civita connection and $R^{TM}=\nabla^{TM,2}$ the curvature of ∇^{TM}.

Let $\widehat{A}(TM,\nabla^{TM})$, $\widehat{L}(TM,\nabla^{TM})$ be the Hirzebruch characteristic forms defined by

$$\widehat{A}(TM,\nabla^{TM})=\det{}^{1/2}\left(\frac{\frac{\sqrt{-1}}{4\pi}R^{TM}}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^{TM}\right)}\right),\tag{2.1}$$
$$\widehat{L}(TM,\nabla^{TM})=\det{}^{1/2}\left(\frac{\frac{\sqrt{-1}}{2\pi}R^{TM}}{\tanh\left(\frac{\sqrt{-1}}{4\pi}R^{TM}\right)}\right).$$

Let E, F be two Hermitian vector bundles over M carrying Hermitian connections ∇^{E}, ∇^{F} respectively. Let $R^{E}=\nabla^{E,2}$ (resp. $R^{F}=\nabla^{F,2}$) be the curvature of ∇^{E} (resp. ∇^{F}). If we set the formal difference

$G = E - F$, then G carries with an induced Hermitian connection ∇^G in an obvious sense. We define the associated Chern character form as

$$\mathrm{ch}(G, \nabla^G) = \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right] - \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^F\right)\right]. \tag{2.2}$$

In the rest of this paper, when there will be no confusion about the Hermitian connection ∇^E on a Hermitian vector bundle E, we will write simply $\mathrm{ch}(E)$ for the associated Chern character form.

For any complex number t, let

$$\Lambda_t(E) = \mathbb{C}|_M + tE + t^2\Lambda^2(E) + \cdots,$$
$$S_t(E) = \mathbb{C}|_M + tE + t^2S^2(E) + \cdots$$

denote respectively the total exterior and symmetric powers of E.

We recall the following relations between these two operations (cf. [**2**, Chapter 3]),

$$S_t(E) = \frac{1}{\Lambda_{-t}(E)}, \quad \Lambda_t(E - F) = \frac{\Lambda_t(E)}{\Lambda_t(F)}. \tag{2.3}$$

Therefore, we have the following formulas for Chern character forms,

$$\mathrm{ch}(S_t(E)) = \frac{1}{\mathrm{ch}(\Lambda_{-t}(E))}, \quad \mathrm{ch}(\Lambda_t(E - F)) = \frac{\mathrm{ch}(\Lambda_t(E))}{\mathrm{ch}(\Lambda_t(F))}. \tag{2.4}$$

If W is a real Euclidean vector bundle over M carrying an Euclidean connection ∇^W, then its complexification $W_{\mathbb{C}} = W \otimes \mathbb{C}$ is a complex vector bundle over M carrying a canonically induced Hermitian metric from that of W, as well as a Hermitian connection from ∇^W.

We refer to [**27**, Section 1.6] for the definitions and notations of other Pontrjagin (resp. Chern) forms associated to real (resp. complex) vector bundles with connections.

2.2. A twisted "miraculous cancellation" formula. We make the same assumptions and use the same notations as in Section 2.1.

Let V be a rank $2l$ real Euclidean vector bundle over M carrying a Euclidean connection ∇^V.

Let ξ be a rank two real oriented Euclidean vector bundle over M carrying a Euclidean connection ∇^ξ.

If E is a complex vector bundle over M, set $\widetilde{E} = E - \mathbb{C}^{\mathrm{rk}(E)}$.

Let $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in \mathbb{H}$, the upper half complex plane.

Set

$$\begin{aligned}\Theta_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) = &\bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\widetilde{V_{\mathbb{C}}} - 2\widetilde{\xi_{\mathbb{C}}}) \\ &\otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_{\mathbb{C}}}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{-q^{s-\frac{1}{2}}}(\widetilde{\xi_{\mathbb{C}}}), \\ \Theta_2(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) = &\bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{-q^{m-\frac{1}{2}}}(\widetilde{V_{\mathbb{C}}} - 2\widetilde{\xi_{\mathbb{C}}}) \\ &\otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_{\mathbb{C}}}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{q^s}(\widetilde{\xi_{\mathbb{C}}}).\end{aligned} \tag{2.5}$$

Clearly, $\Theta_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})$ and $\Theta_2(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})$ admit formal Fourier expansions in $q^{1/2}$ as

(2.6)

$$\Theta_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) = A_0(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) + A_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})q^{1/2} + \cdots,$$

$$\Theta_2(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) = B_0(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) + B_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})q^{1/2} + \cdots,$$

where the A_j's and B_j's are elements in the semi-group formally generated by Hermitian vector bundles over M. Moreover, they carry canonically induced Hermitian connections.

Let $c = e(\xi, \nabla^{\xi})$ be the Euler form of ξ canonically associated to ∇^{ξ} (cf. [**27**, Section 3.4]). Let $R^V = \nabla^{V,2}$ denote the curvature of ∇^V.

If ω is a differential form over M, we denote by $\omega^{(8k+4)}$ its top degree component.

We can now state our main result of this section as follows.

Theorem 2.1. *If the equality $p_1(TM, \nabla^{TM}) = p_1(V, \nabla^V)$ for the first Pontrjagin forms holds, then one has*

(2.7)

$$\left(\frac{1}{2}\right)^{l+2k+1} \left\{ \frac{\widehat{A}(TM, \nabla^{TM}) \det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^V\right)\right)}{\cosh^2(\frac{c}{2})} \right\}^{(8k+4)}$$

$$= \sum_{r=0}^{k} 2^{-6r} \left\{ \widehat{A}(TM, \nabla^{TM}) \mathrm{ch}(b_r(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})) \cosh\left(\frac{c}{2}\right) \right\}^{(8k+4)},$$

where each $b_r(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})$, $0 \le r \le k$, is a canonical integral linear combination of $B_j(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})$, $0 \le j \le r$.

Certainly, when $\xi = \mathbf{R}^2$ and $c = 0$, Theorem 2.1 is exactly Liu's result in [**18**, Theorem 1], which, in the $(TM, \nabla^{TM}) = (V, \nabla^V)$ and $k = 1$ case, recovers the original "miraculous cancellation" formula of Alvarez–Gaumé and Witten [**1**].

Theorem 2.1 will be proved in Section 2.4 by using Liu's argument in [**18**], taking into account the appearance of ξ and c. In the next subsection, for the sake of completeness, we will recall some of the materials concerning modular forms which will be used in Section 2.4.

2.3. Some properties about the Jacobi theta functions and modular forms. Recall that the four Jacobi theta functions are defined by (cf. [**8**]):

(2.8)
$$\theta(v,\tau) = 2q^{\frac{1}{8}}\sin(\pi v)\prod_{j=1}^{\infty}\left[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^j)(1-e^{-2\pi\sqrt{-1}v}q^j)\right],$$

(2.9)
$$\theta_1(v,\tau) = 2q^{\frac{1}{8}}\cos(\pi v)\prod_{j=1}^{\infty}\left[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^j)(1+e^{-2\pi\sqrt{-1}v}q^j)\right],$$

(2.10)
$$\theta_2(v,\tau) = \prod_{j=1}^{\infty}\left[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^{j-\frac{1}{2}})(1-e^{-2\pi\sqrt{-1}v}q^{j-\frac{1}{2}})\right],$$

(2.11)
$$\theta_3(v,\tau) = \prod_{j=1}^{\infty}\left[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^{j-\frac{1}{2}})(1+e^{-2\pi\sqrt{-1}v}q^{j-\frac{1}{2}})\right],$$

where $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in \mathbb{H}$. Set

$$\theta'(0,\tau) = \left.\frac{\partial\theta(v,\tau)}{\partial v}\right|_{v=0}. \tag{2.12}$$

We refer to [**8**, Chapter 3] for a proof of the following Jacobi identity.

Proposition 2.2. *The following identity holds,*

$$\theta'(0,\tau) = \pi\,\theta_1(0,\tau)\theta_2(0,\tau)\theta_3(0,\tau). \tag{2.13}$$

Let as usual

$$SL_2(\mathbb{Z}) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \,:\, a,b,c,d\in\mathbb{Z},\ ad-bc=1\right\}.$$

Let S, T be the two generators of $SL_2(\mathbb{Z})$ such that they act on $\mathbb{H}$ by $S\tau = -1/\tau$, $T\tau = \tau + 1$. One has the following transformation laws of theta functions under S and T (cf. [8]),

$$\theta(v, \tau + 1) = e^{\frac{\pi\sqrt{-1}}{4}}\theta(v, \tau),$$

$$\theta\left(v, -\frac{1}{\tau}\right) = \frac{1}{\sqrt{-1}}\left(\frac{\tau}{\sqrt{-1}}\right)^{1/2} e^{-\tau v^2}\theta\left(\tau v, \tau\right), \tag{2.14}$$

$$\theta_1(v, \tau + 1) = e^{\frac{\pi\sqrt{-1}}{4}}\theta_1(v, \tau),$$

$$\theta_1\left(v, -\frac{1}{\tau}\right) = \left(\frac{\tau}{\sqrt{-1}}\right)^{1/2} e^{-\tau v^2}\theta_2(\tau v, \tau), \tag{2.15}$$

$$\theta_2(v, \tau + 1) = \theta_3(v, \tau),$$

$$\theta_2\left(v, -\frac{1}{\tau}\right) = \left(\frac{\tau}{\sqrt{-1}}\right)^{1/2} e^{-\tau v^2}\theta_1(\tau v, \tau), \tag{2.16}$$

$$\theta_3(v, \tau + 1) = \theta_2(v, \tau),$$

$$\theta_3\left(v, -\frac{1}{\tau}\right) = \left(\frac{\tau}{\sqrt{-1}}\right)^{1/2} e^{-\tau v^2}\theta_3(\tau v, \tau). \tag{2.17}$$

Let Γ be a subgroup of $SL_2(\mathbb{Z})$.

Definition 2.3. A modular form over Γ is a holomorphic function $f(\tau)$ on $\mathbb{H} \cup \{\infty\}$ such that

(2.18)
$$f(g\tau) := f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(g)(c\tau + d)^k f(\tau) \text{ for any } g \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

where $\chi : \Gamma \to \mathbb{C}^*$ is a character of Γ and k is called the weight of f.

Let $\mathcal{M}_{\mathbb{R}}(\Gamma)$ denote the ring of modular forms over Γ with real Fourier coefficients. Following [18], denote by $\theta_j = \theta_j(0, \tau)$, $1 \le j \le 3$, and define

$$\delta_1(\tau) = \frac{1}{8}(\theta_2^4 + \theta_3^4), \quad \varepsilon_1(\tau) = \frac{1}{16}\theta_2^4\theta_3^4, \tag{2.19}$$

$$\delta_2(\tau) = -\frac{1}{8}(\theta_1^4 + \theta_3^4), \quad \varepsilon_2(\tau) = \frac{1}{16}\theta_1^4\theta_3^4. \tag{2.20}$$

They admit Fourier expansions

$$\delta_1(\tau) = \frac{1}{4} + 6q + \cdots, \qquad \varepsilon_1(\tau) = \frac{1}{16} - q + \cdots, \tag{2.21}$$

$$\delta_2(\tau) = -\frac{1}{8} - 3q^{1/2} + \cdots, \qquad \varepsilon_2(\tau) = q^{1/2} + \cdots, \tag{2.22}$$

where the "$\cdots$" terms are higher degree terms all having integral coefficients. They also satisfy the transformation laws (cf. [**17**] and [**18**]),

$$\delta_2\left(-1/\tau\right) = \tau^2\delta_1(\tau), \qquad \varepsilon_2\left(-1/\tau\right) = \tau^4\varepsilon_1(\tau). \tag{2.23}$$

Let $\Gamma_0(2)$, $\Gamma^0(2)$ be two subgroups of $SL_2(\mathbb{Z})$ defined by

$$\Gamma_0(2) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \mod 2\mathbb{Z} \right\},$$
$$\Gamma^0(2) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \,\middle|\, b \equiv 0 \mod 2\mathbb{Z} \right\}.$$

Then T, ST^2ST are the two generators of $\Gamma_0(2)$, while STS, T^2STS are the two generators of $\Gamma^0(2)$.

Lemma 2.4 ([**18**, Lemma 2])**.** *One has that δ_2 (resp. ε_2) is a modular form of weight 2 (resp. 4) over $\Gamma^0(2)$. Moreover, one has $\mathcal{M}_{\mathbb{R}}(\Gamma^0(2)) = \mathbb{R}[\delta_2(\tau), \varepsilon_2(\tau)]$.*

2.4. A proof of Theorem 2.1. Without loss of generality, we will adopt the Chern roots formalism as in [**18**] in the computation of characteristic forms.

Recall that if $\{w_i\}$ are the formal Chern roots of a Hermitian vector bundle E carrying a Hermitian connection ∇^E, then one has the following formula for the Chern character form of the exterior power of E (Compare with [**14**]),

$$\mathrm{ch}\left(\Lambda_t(E)\right) = \prod_i (1 + e^{w_i} t). \tag{2.24}$$

For $\tau \in \mathbb{H}$ and $q = e^{2\pi\sqrt{-1}\tau}$, set

$$P_1(\tau) = \left\{ \frac{\widehat{A}(TM, \nabla^{TM})\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^V\right)\right)}{\cosh^2(\frac{c}{2})} \cdot \mathrm{ch}\left(\Theta_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}), \nabla^{\Theta_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})}\right) \right\}^{(8k+4)}, \tag{2.25}$$

$$P_2(\tau) = \left\{ \widehat{A}(TM, \nabla^{TM})\mathrm{ch}\left(\Theta_2(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}), \nabla^{\Theta_2(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})}\right) \cdot \cosh\left(\frac{c}{2}\right) \right\}^{(8k+4)}, \tag{2.26}$$

where $\nabla^{\Theta_i(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})}$, $i = 1,\ 2$, are the Hermitian connections with $q^{j/2}$-coefficients on $\Theta_i(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})$ induced from those on the A_j's and B_j's (Compare with (2.6)).

Since (2.7) is a local formula over M, without loss of generality, we may assume that both TM and V are oriented. Let $\{\pm 2\pi\sqrt{-1}y_v\}$ (resp. $\{\pm 2\pi\sqrt{-1}x_j\}$) be the formal Chern roots for $(V_{\mathbb{C}}, \nabla^{V_{\mathbb{C}}})$ (resp. $(T_{\mathbb{C}}M, \nabla^{T_{\mathbb{C}}M})$). Let $c = 2\pi\sqrt{-1}u$.

From (2.1) and (2.25), one finds,

$$P_1(\tau) = 2^l \left\{ \left(\prod_{j=1}^{4k+2} \frac{\pi x_j}{\sin(\pi x_j)}\right) \left(\prod_{v=1}^{l} \cos(\pi y_v)\right) \cdot \frac{\mathrm{ch}\left(\Theta_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})\right)}{\cos^2(\pi u)} \right\}^{(8k+4)}. \tag{2.27}$$

By (2.4), (2.5), one can write $\mathrm{ch}(\Theta_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}))$ as the product,

$$\begin{aligned} &\mathrm{ch}(\Theta_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})) \\ &\quad = \prod_{n=1}^{\infty} \frac{\mathrm{ch}(\Lambda_{-q^n}(\mathbb{C}^{8k+4}))}{\mathrm{ch}(\Lambda_{-q^n}(T_{\mathbb{C}}M))} \prod_{m=1}^{\infty} \frac{\mathrm{ch}(\Lambda_{q^m}(V_{\mathbb{C}}))}{\mathrm{ch}(\Lambda_{q^m}(\mathbb{C}^{2l}))} \\ &\qquad \cdot \prod_{t=1}^{\infty} \left(\frac{\mathrm{ch}(\Lambda_{q^t}(\mathbb{C}^2))}{\mathrm{ch}(\Lambda_{q^t}(\xi_{\mathbb{C}}))}\right)^2 \prod_{r=1}^{\infty} \frac{\mathrm{ch}(\Lambda_{q^{r-\frac{1}{2}}}(\xi_{\mathbb{C}}))}{\mathrm{ch}(\Lambda_{q^{r-\frac{1}{2}}}(\mathbb{C}^2))} \prod_{s=1}^{\infty} \frac{\mathrm{ch}(\Lambda_{-q^{s-\frac{1}{2}}}(\xi_{\mathbb{C}}))}{\mathrm{ch}(\Lambda_{-q^{s-\frac{1}{2}}}(\mathbb{C}^2))}. \end{aligned} \tag{2.28}$$

From (2.8), (2.13) and (2.24), one deduces directly that

$$\begin{aligned}(2.29)\qquad &\prod_{j=1}^{4k+2}\frac{\pi x_j}{\sin(\pi x_j)}\prod_{n=1}^{\infty}\frac{\mathrm{ch}(\Lambda_{-q^n}(\mathbb{C}^{8k+4}))}{\mathrm{ch}(\Lambda_{-q^n}(T_{\mathbb{C}}M))}\\ &=\prod_{j=1}^{4k+2}x_j\frac{\pi\theta_1(0,\tau)\theta_2(0,\tau)\theta_3(0,\tau)}{\theta(x_j,\tau)}\\ &=\prod_{j=1}^{4k+2}x_j\frac{\theta'(0,\tau)}{\theta(x_j,\tau)}.\end{aligned}$$

Similarly, from (2.9) to (2.11) and (2.24), one deduces that

$$\begin{aligned}(2.30)\qquad &\prod_{v=1}^{l}\cos(\pi y_v)\prod_{m=1}^{\infty}\frac{\mathrm{ch}(\Lambda_{q^m}(V_{\mathbb{C}}))}{\mathrm{ch}(\Lambda_{q^m}(\mathbb{C}^{2l}))}=\prod_{v=1}^{l}\frac{\theta_1(y_v,\tau)}{\theta_1(0,\tau)},\\ &\prod_{r=1}^{\infty}\frac{\mathrm{ch}(\Lambda_{q^{r-\frac{1}{2}}}(\xi_{\mathbb{C}}))}{\mathrm{ch}(\Lambda_{q^{r-\frac{1}{2}}}(\mathbb{C}^2))}=\frac{\theta_3(u,\tau)}{\theta_3(0,\tau)}\end{aligned}$$

and

$$\begin{aligned}(2.31)\qquad &\frac{1}{\cos^2(\pi u)}\prod_{t=1}^{\infty}\left(\frac{\mathrm{ch}(\Lambda_{q^t}(\mathbb{C}^2))}{\mathrm{ch}(\Lambda_{q^t}(\xi_{\mathbb{C}}))}\right)^2=\frac{\theta_1^2(0,\tau)}{\theta_1^2(u,\tau)},\\ &\prod_{s=1}^{\infty}\frac{\mathrm{ch}(\Lambda_{-q^{s-\frac{1}{2}}}(\xi_{\mathbb{C}}))}{\mathrm{ch}(\Lambda_{-q^{s-\frac{1}{2}}}(\mathbb{C}^2))}=\frac{\theta_2(u,\tau)}{\theta_2(0,\tau)}.\end{aligned}$$

Putting (2.27)–(2.31) together, one finds that the first part of the following result holds.

Proposition 2.5. *The following two identities hold,*

$$\begin{aligned}(2.32)\qquad P_1(\tau)=2^l\Bigg\{&\prod_{j=1}^{4k+2}\left(x_j\frac{\theta'(0,\tau)}{\theta(x_j,\tau)}\right)\left(\prod_{v=1}^{l}\frac{\theta_1(y_v,\tau)}{\theta_1(0,\tau)}\right)\\ &\cdot\frac{\theta_1^2(0,\tau)}{\theta_1^2(u,\tau)}\frac{\theta_3(u,\tau)}{\theta_3(0,\tau)}\frac{\theta_2(u,\tau)}{\theta_2(0,\tau)}\Bigg\}^{(8k+4)},\end{aligned}$$

$$P_2(\tau) = \left\{ \prod_{j=1}^{4k+2} \left(x_j \frac{\theta'(0,\tau)}{\theta(x_j,\tau)} \right) \left(\prod_{v=1}^{l} \frac{\theta_2(y_v,\tau)}{\theta_2(0,\tau)} \right) \cdot \frac{\theta_2^2(0,\tau)}{\theta_2^2(u,\tau)} \frac{\theta_3(u,\tau)}{\theta_3(0,\tau)} \frac{\theta_1(u,\tau)}{\theta_1(0,\tau)} \right\}^{(8k+4)}. \tag{2.33}$$

Proof. Formula (2.32) has been proved above. By a similar computation, one also gets (2.33). q.e.d.

Next, by direct verifications in applying the transformation laws from (2.14)–(2.17) to (2.32), (2.33), respectively, one gets

Proposition 2.6. *If the equation $p_1(TM, \nabla^{TM}) = p_1(V, \nabla^V)$ for the first Pontrjagin forms holds, then $P_1(\tau)$ is a modular form of weight $4k+2$ over $\Gamma_0(2)$; while $P_2(\tau)$ is a modular form of weight $4k+2$ over $\Gamma^0(2)$. Moreover, the following identity holds,*

$$P_1\left(-\frac{1}{\tau}\right) = 2^l \tau^{4k+2} P_2(\tau). \tag{2.34}$$

We can now proceed to prove Theorem 2.1 as follows.

Observe that at any point $x \in M$, up to the volume form determined by the metric on T_xM, both $P_i(\tau)$, $i = 1,\ 2$, can be viewed as power series of $q^{1/2}$ with real Fourier coefficients. Thus, one can combine Lemma 2.4 and Proposition 2.6 to get, at x, that

$$P_2(\tau) = h_0(8\delta_2)^{2k+1} + h_1(8\delta_2)^{2k-1}\varepsilon_2 + \cdots + h_k(8\delta_2)\varepsilon_2^k, \tag{2.35}$$

where each h_j, $0 \le j \le k$, is a real multiple of the volume form at x.

By (2.23), (2.34) and (2.35), one deduces that

$$\begin{aligned} P_1(\tau) &= \frac{2^l}{\tau^{4k+2}} P_2\left(-\frac{1}{\tau}\right) \\ &= \frac{2^l}{\tau^{4k+2}} \left[h_0 \left(8\delta_2\left(-\frac{1}{\tau}\right)\right)^{2k+1} + h_1 \left(8\delta_2\left(-\frac{1}{\tau}\right)\right)^{2k-1} \varepsilon_2\left(-\frac{1}{\tau}\right) \right. \\ &\qquad \left. + \cdots + h_k \left(8\delta_2\left(-\frac{1}{\tau}\right)\right) \left(\varepsilon_2\left(-\frac{1}{\tau}\right)\right)^k \right] \\ &= 2^l \left[h_0(8\delta_1)^{2k+1} + h_1(8\delta_1)^{2k-1}\varepsilon_1 + \cdots + h_k(8\delta_1)\varepsilon_1^k \right]. \end{aligned} \tag{2.36}$$

By (2.5), (2.21), (2.25) and by setting $q = 0$ in (2.36), one deduces that

$$\left\{ \frac{\widehat{A}(TM, \nabla^{TM}) \det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^V\right)\right)}{\cosh^2\left(\frac{c}{2}\right)} \right\}^{(8k+4)} = 2^{l+2k+1} \sum_{r=0}^{k} 2^{-6r} h_r. \tag{2.37}$$

Now, in order to prove (2.7), one needs to show that each h_r, $0 \leq r \leq k$, can be expressed as a canonical integral linear combination of $\{\widehat{A}(TM, \nabla^{TM}) \mathrm{ch}(B_j, \nabla^{B_j}) \cosh(\frac{c}{2})\}^{(8k+4)}$, $0 \leq j \leq r$, with coefficients not depending on $x \in M$.

As in [**17**], one can use the induction method to prove this fact easily by comparing the coefficients of $q^{j/2}$, $j \geq 0$, between the two sides of (2.35). We leave the details to the interested reader.

Here, for convenience, we write out the explicit expressions for h_0 and h_1 as follows.

$$h_0 = -\left\{ \widehat{A}(TM, \nabla^{TM}) \cosh\left(\frac{c}{2}\right) \right\}^{(8k+4)}, \tag{2.38}$$

$$h_1 = \left\{ \widehat{A}(TM, \nabla^{TM}) \left[24(2k+1) - \mathrm{ch}(B_1, \nabla^{B_1})\right] \cosh\left(\frac{c}{2}\right) \right\}^{(8k+4)}. \tag{2.39}$$

Remark 2.7. By (2.6), (2.7), (2.22), (2.26) and (2.35), one sees that for any integer $r \geq 0$, $\{\widehat{A}(TM, \nabla^{TM}) \mathrm{ch}(B_r, \nabla^{B_r}) \cosh(\frac{c}{2})\}^{(8k+4)}$ can be expressed as a canonical integral linear combination of the terms $\{\widehat{A}(TM, \nabla^{TM}) \mathrm{ch}(B_j, \nabla^{B_j}) \cosh(\frac{c}{2})\}^{(8k+4)}$, $0 \leq j \leq k$. This fact is by no means trivial for $r \geq k+1$. It depends heavily on the modular invariance of $P_2(\tau)$.

2.5. The case of $V = TM$. In this subsection, we apply Theorem 2.1 to the case where $V = TM$ and $\nabla^V = \nabla^{TM}$. In this case, in view of

(2.1), (2.7) becomes

$$\left\{\frac{\widehat{L}(TM,\nabla^{TM})}{\cosh^2(\frac{c}{2})}\right\}^{(8k+4)} = 8\sum_{r=0}^{k} 2^{6k-6r}\left\{\widehat{A}(TM,\nabla^{TM})\mathrm{ch}\left(b_r(T_{\mathbb{C}}M,\xi_{\mathbb{C}})\right)\cosh\left(\frac{c}{2}\right)\right\}^{(8k+4)}, \tag{2.40}$$

where $b_r(T_{\mathbb{C}}M,\xi_{\mathbb{C}})$ is the simplified notation for $b_r(T_{\mathbb{C}}M,T_{\mathbb{C}}M,\xi_{\mathbb{C}})$.

Now, we assume $k=1$, that is, $\dim M = 12$. Then, by concentrating on the coefficients of $q^{1/2}$, we can identify the B_1 term as follows. We have, by (2.5), that

$$\begin{aligned}\Theta_2(T_{\mathbb{C}}M,T_{\mathbb{C}}M,\xi_{\mathbb{C}}) &= \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M})\otimes\bigotimes_{m=1}^{\infty}\Lambda_{-q^{m-\frac{1}{2}}}(\widetilde{T_{\mathbb{C}}M}-2\widetilde{\xi_{\mathbb{C}}})\\ &\quad\otimes\bigotimes_{r=1}^{\infty}\Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_{\mathbb{C}}})\otimes\bigotimes_{s=1}^{\infty}\Lambda_{q^s}(\widetilde{\xi_{\mathbb{C}}})\\ &= \left(1-(T_{\mathbb{C}}M-12-2\xi_{\mathbb{C}}+4)q^{\frac{1}{2}}\right)\otimes\left(1+(\xi_{\mathbb{C}}-2)q^{\frac{1}{2}}\right)+\cdots\\ &= 1+(-T_{\mathbb{C}}M+3\xi_{\mathbb{C}}+6)q^{\frac{1}{2}}+\cdots,\end{aligned} \tag{2.41}$$

where the "$\cdots$" terms are the terms involving $q^{j/2}$'s with $j\geq 2$.

From (2.6) and (2.37)–(2.41), one finds

$$\begin{aligned}\left\{\frac{\widehat{L}(TM,\nabla^{TM})}{\cosh^2(\frac{c}{2})}\right\}^{(12)} &= \left\{\left[8\widehat{A}(TM,\nabla^{TM})\mathrm{ch}(T_{\mathbb{C}}M,\nabla^{T_{\mathbb{C}}M})-32\widehat{A}(TM,\nabla^{TM})\right.\right.\\ &\quad\left.\left.-24\widehat{A}(TM,\nabla^{TM})\left(e^c+e^{-c}-2\right)\right]\cosh\left(\frac{c}{2}\right)\right\}^{(12)}.\end{aligned} \tag{2.42}$$

If we set $\xi=\mathbb{R}^2$ and $c=0$ in (2.43), then we get

$$
\begin{aligned}
(2.43)\quad & \left\{\widehat{L}(TM,\nabla^{TM})\right\}^{(12)} \\
& = \left\{8\widehat{A}(TM,\nabla^{TM})\mathrm{ch}(T_{\mathbb{C}}M,\nabla^{T_{\mathbb{C}}M}) - 32\widehat{A}(TM,\nabla^{TM})\right\}^{(12)},
\end{aligned}
$$

which is exactly the "miraculous cancellation" formula first proved by Alvarez–Gaumé and Witten [**1**].

3. Spin^c manifolds and Rokhlin congruences for characteristic numbers

In this section, we apply our twisted cancellation formula (2.7) to Spin^c manifolds to give a direct proof of the analytic version of the Ochanine congruence [**21**] stated in [**19**, Theorem 4.2]. In fact, the result we obtain is stronger than [**19**, Theorem 4.2] (see (3.2) for a precise statement).

This section is organized as follows. In Section 3.1, we apply Theorem 2.1 to Spin^c manifolds to get a congruence formula for characteristic numbers. In Section 3.2, we recall the analytic version of the Ochanine congruence stated in [**19**, Theorem 4.2] and show that it can be proved directly as a consequence of the congruence formula stated in Section 3.1.

In this section, we will use the same notations as in Section 2.

3.1. A congruence formula for Spin^c manifolds. Let M be an $8k+4$ dimensional Riemannian manifold as in Section 2. In this section, we also assume that M is closed and oriented. Moreover, we make the assumption that there is an $8k+2$ dimensional closed oriented submanifold B such that if $\widetilde{c}\in H^2(M,\mathbb{Z})$ is the Poincaré dual of $[B]\in H_{8k+2}(M,\mathbb{Z})$, then

$$
\widetilde{c}\equiv w_2(TM) \qquad \mathrm{mod}\ 2\mathbb{Z}, \tag{3.1}
$$

where $w_2(TM)\in H^2(M,\mathbb{Z}_2)$ is the second Stiefel–Whitney class of TM.

Thus, M now is a Spin^c manifold. One can also show that there exists an oriented real rank two Euclidean vector bundle ξ over M, carrying a Euclidean connection ∇^ξ, such that if $c=e(\xi,\nabla^\xi)$ is the Euler form associated to (ξ,∇^ξ), then $\widetilde{c}=[c]$ in $H^2(M,\mathbb{Z})$.

Remark 3.1. If we view ξ as a complex line bundle, then $\widetilde{c}$ is the first Chern class of ξ.

Now, we set $V = TM$ and $\nabla^V = \nabla^{TM}$ as in Section 2.5). And for simplification, we write $\Theta_2(T_{\mathbb{C}}M, \xi_{\mathbb{C}})$ for $\Theta_2(T_{\mathbb{C}}M, T_{\mathbb{C}}M, \xi_{\mathbb{C}})$, etc. Then, $\Theta_2(T_{\mathbb{C}}M, \mathbb{C}^2)$ is exactly the (complexification of) $\Theta_2(TM)$ in [**19**].

Let $B \cdot B$ be the self-intersection of B in M. It can be thought of as an $8k$ dimensional closed oriented manifold.

We can now state the main result of this section as follows.

Theorem 3.2. *The following congruence formula holds,*

$$\begin{aligned} &\frac{\mathrm{Sign}(M) - \mathrm{Sign}(B \cdot B)}{8} \\ &\qquad \equiv \int_M \widehat{A}(TM, \nabla^{TM}) \mathrm{ch}\left(b_k(T_{\mathbb{C}}M, \xi_{\mathbb{C}})\right) \cosh\left(\frac{c}{2}\right) \mod 64\mathbb{Z}, \end{aligned} \tag{3.2}$$

where $b_k(T_{\mathbb{C}}M, \xi_{\mathbb{C}})$ is the same term appearing in the right-hand side of (2.40), and can be canonically expressed as an integral linear combination of $B_j(T_{\mathbb{C}}M, \xi_{\mathbb{C}})$, $0 \le j \le k$.

Proof. By the fact that $[c] \in H^2(M, \mathbb{Z})$ is the Poincaré dual of $[B] \in H_{8k+2}(M, \mathbb{Z})$, a direct computation shows that [**21**]

$$\int_M \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2(\frac{c}{2})} = \mathrm{Sign}(M) - \mathrm{Sign}(B \cdot B). \tag{3.3}$$

Formula (3.2) follows directly from (2.40), (3.3) and the integrality result of Atiyah and Hirzebruch [**3**] stating that in the current Spin^c situation, one has

$$\int_M \widehat{A}(TM, \nabla^{TM}) \mathrm{ch}\left(b_j(T_{\mathbb{C}}M, \xi_{\mathbb{C}})\right) \cosh\left(\frac{c}{2}\right) \in \mathbb{Z} \tag{3.4}$$

for any $0 \le j \le k$. q.e.d.

3.2. A proof of the Ochanine congruence formula.

We need only prove the analytic version of the Ochanine congruence [**19**], an equivalent version of which is stated as [**19**, Theorem 4.2]. So, we first recall the statement of [**19**, Theorem 4.2] in our notation.

Theorem 3.3 (Liu and Zhang [19, Theorem 4.2])**.** *The following congruence formula holds,*

$$\begin{aligned}\frac{\mathrm{Sign}(M)-\mathrm{Sign}(B\cdot B)}{8} \\ \equiv \int_M \widehat{A}(TM,\nabla^{TM})\mathrm{ch}\left(b_k(T_{\mathbb{C}}M+\mathbb{C}^2-\xi_{\mathbb{C}},\mathbb{C}^2)\right)\cosh\left(\frac{c}{2}\right) \bmod 2\mathbb{Z}.\end{aligned} \tag{3.5}$$

Proof. By (3.2), one need only prove that

$$\begin{aligned}\int_M \widehat{A}(TM,\nabla^{TM})\mathrm{ch}\left(b_k(T_{\mathbb{C}}M,\xi_{\mathbb{C}})\right)\cosh\left(\frac{c}{2}\right) \\ -\int_M \widehat{A}(TM,\nabla^{TM})\mathrm{ch}\left(b_k(T_{\mathbb{C}}M+\mathbb{C}^2-\xi_{\mathbb{C}},\mathbb{C}^2)\right)\cosh\left(\frac{c}{2}\right)\in 2\mathbb{Z}.\end{aligned} \tag{3.6}$$

To prove (3.6), we first compare $\Theta_2(T_{\mathbb{C}}M,\xi_{\mathbb{C}})$ and $\Theta_2(T_{\mathbb{C}}M+\mathbb{C}^2-\xi_{\mathbb{C}},\mathbb{C}^2)$.

By (2.3) and (2.5), one deduces that

$$\begin{aligned}\Theta_2(T_{\mathbb{C}}M+\mathbb{C}^2-\xi_{\mathbb{C}},\mathbb{C}^2) \\ =\bigotimes_{n=1}^{\infty}S_{q^n}(\widetilde{T_{\mathbb{C}}M}-\widetilde{\xi}_{\mathbb{C}})\otimes\bigotimes_{m=1}^{\infty}\Lambda_{-q^{m-\frac{1}{2}}}(\widetilde{T_{\mathbb{C}}M}-\widetilde{\xi}_{\mathbb{C}}) \\ =\Theta_2(T_{\mathbb{C}}M,\mathbb{C}^2)\otimes\frac{\bigotimes_{n=1}^{\infty}\Lambda_{-q^n}(\widetilde{\xi}_{\mathbb{C}})}{\bigotimes_{m=1}^{\infty}\Lambda_{-q^{m-\frac{1}{2}}}(\widetilde{\xi}_{\mathbb{C}})},\end{aligned} \tag{3.7}$$

$$\Theta_2(T_{\mathbb{C}}M,\xi_{\mathbb{C}}) =\Theta_2(T_{\mathbb{C}}M,\mathbb{C}^2)\otimes\frac{\bigotimes_{r=1}^{\infty}\Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi}_{\mathbb{C}})\otimes\bigotimes_{s=1}^{\infty}\Lambda_{q^s}(\widetilde{\xi}_{\mathbb{C}})}{\left(\bigotimes_{m=1}^{\infty}\Lambda_{-q^{m-\frac{1}{2}}}(\widetilde{\xi}_{\mathbb{C}})\right)^2}. \tag{3.8}$$

From (3.7) and (3.8), one finds,

$$\begin{aligned}\Theta_2(T_{\mathbb{C}}M,\xi_{\mathbb{C}})=\Theta_2(T_{\mathbb{C}}M+\mathbb{C}^2-\xi_{\mathbb{C}},\mathbb{C}^2) \\ \otimes\frac{\bigotimes_{r=1}^{\infty}\Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi}_{\mathbb{C}})\otimes\bigotimes_{s=1}^{\infty}\Lambda_{q^s}(\widetilde{\xi}_{\mathbb{C}})}{\bigotimes_{m=1}^{\infty}\Lambda_{-q^{m-\frac{1}{2}}}(\widetilde{\xi}_{\mathbb{C}})\otimes\bigotimes_{n=1}^{\infty}\Lambda_{-q^n}(\widetilde{\xi}_{\mathbb{C}})}.\end{aligned} \tag{3.9}$$

From (2.3) and the fact that $\mathrm{rk}(\xi) = 2$, one verifies directly that for any integer $r \geq 1$,

$$\begin{aligned}\Lambda_{q^r}(\widetilde{\xi_{\mathbb{C}}}) &\equiv \Lambda_{-q^r}(\widetilde{\xi_{\mathbb{C}}}) \mod 2q^r\widetilde{\xi_{\mathbb{C}}}\mathbb{Z}[[q^r]], \\ \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_{\mathbb{C}}}) &\equiv \Lambda_{-q^{r-\frac{1}{2}}}(\widetilde{\xi_{\mathbb{C}}}) \mod 2q^{r-\frac{1}{2}}\widetilde{\xi_{\mathbb{C}}}\mathbb{Z}[[q^{r-\frac{1}{2}}]].\end{aligned} \tag{3.10}$$

Let $\mathbb{Z}[T_{\mathbb{C}}M, \xi_{\mathbb{C}}]$ denote the ring with integral coefficients generated by the exterior as well as symmetric powers of $T_{\mathbb{C}}M$ and $\xi_{\mathbb{C}}$.

From (3.9) and (3.10), one gets,

$$\Theta_2(T_{\mathbb{C}}M, \xi_{\mathbb{C}}) \equiv \Theta_2(T_{\mathbb{C}}M + \mathbb{C}^2 - \xi_{\mathbb{C}}, \mathbb{C}^2) \mod 2q^{\frac{1}{2}}\widetilde{\xi_{\mathbb{C}}}\mathbb{Z}[T_{\mathbb{C}}M, \xi_{\mathbb{C}}][[q^{\frac{1}{2}}]]. \tag{3.11}$$

Thus, if one expands $\Theta_2(T_{\mathbb{C}}M + \mathbb{C}^2 - \xi_{\mathbb{C}}, \mathbb{C}^2)$ as

$$\begin{aligned}&\Theta_2(T_{\mathbb{C}}M + \mathbb{C}^2 - \xi_{\mathbb{C}}, \mathbb{C}^2) \\ &\quad = B_0(T_{\mathbb{C}}M + \mathbb{C}^2 - \xi_{\mathbb{C}}, \mathbb{C}^2) + B_1(T_{\mathbb{C}}M + \mathbb{C}^2 - \xi_{\mathbb{C}}, \mathbb{C}^2)q^{\frac{1}{2}} + \cdots ,\end{aligned} \tag{3.12}$$

one gets that for any $j \geq 1$,

$$B_j(T_{\mathbb{C}}M, \xi_{\mathbb{C}}) \equiv B_j(T_{\mathbb{C}}M + \mathbb{C}^2 - \xi_{\mathbb{C}}, \mathbb{C}^2) \mod 2\widetilde{\xi_{\mathbb{C}}}\mathbb{Z}[T_{\mathbb{C}}M, \xi_{\mathbb{C}}]. \tag{3.13}$$

By (3.13) and an induction argument as in the proof of Theorem 2.1 (Compare with [**17**]), one then sees easily that for any integer r such that $0 \leq r \leq k$, one has

$$b_r(T_{\mathbb{C}}M, \xi_{\mathbb{C}}) = b_r(T_{\mathbb{C}}M + \mathbb{C}^2 - \xi_{\mathbb{C}}, \mathbb{C}^2) + 2\widetilde{\xi_{\mathbb{C}}}C_r, \tag{3.14}$$

for some $C_r \in \mathbb{Z}[T_{\mathbb{C}}M, \xi_{\mathbb{C}}]$.

Formula (3.6) follows from (3.2), (3.14) and the Atiyah–Hirzebruch integrality [**3**] stating that $\int_M \widehat{A}(TM, \nabla^{TM})\mathrm{ch}(\widetilde{\xi_{\mathbb{C}}}C_r)\cosh(\frac{c}{2}) \in \mathbb{Z}$ for any $0 \leq r \leq k$.

The proof of Theorem 3.3 is complete. q.e.d.

Remark 3.4. The factor $\widetilde{\xi_{\mathbb{C}}}$ appearing in (3.10) and (3.14) will play an important role in the discussion of the Finashin congruence formula in the next section.

4. η invariants and a proof of the Finashin congruence formula

In this section, we combine the results in Sections 2 and 3 with the results in [19] and [26] to give a direct analytic proof of the Finashin congruence formula [10].

This section is organized as follows. In Section 4.1, we recall the original statement of the Finashin congruence [10] as well as an equivalent analytic version given in [19, Theorem 4.1]. In Section 4.2, we recall a Rokhlin type congruence formula from [26, Theorem 3.2] in the form which is useful for the current situation. In Section 4.3, we establish a cancellation formula for certain characteristic numbers on $8k+2$ dimensional manifolds. This cancellation formula will be used in Section 4.4, where we complete the proof of the analytic version of the Finashin congruence recalled in Section 4.1.

4.1. The Finashin congruence and an analytic version of it. Let M be an $8k+4$ dimensional closed smooth oriented manifold. Let B be an $8k+2$ dimensional closed smooth submanifold in M such that $[B] \in H_{8k+2}(M, \mathbb{Z}_2)$ is Poincaré dual to $w_2(TM) \in H^2(M, \mathbb{Z}_2)$. In this section, we make the assumption that B is non-orientable, as the case where B is orientable has been discussed in Section 3.

Under the above assumptions, $M \setminus B$ is oriented and spin. We fix a spin structure on $M \setminus B$. Then, it induces canonically a pin$^-$ structure on B (cf. [15, Lemma 6.2]).

Let $o(TB)$ be the orientation bundle of TB. Let L be the rank two real vector bundle over B defined by $L = o(TB) + \mathbb{R}$ (here "+" stands for direct sum). Let g^L be a Euclidean metric on L and denoted by $L_1 = \{l \in L : \|l\| \leq 1\}$ the associated unit disc bundle. Then ∂L_1 carries a canonically induced spin structure from the pin$^-$ structure on B (cf. [10] and [15]).

As an $8k+3$ dimensional oriented spin manifold, $-\partial L_1$ bounds an $8k+4$ dimensional oriented spin manifold Z. Following [10], one defines

$$\Phi(B) \equiv \frac{\mathrm{Sign}(Z)}{8} \mod 2\mathbb{Z}. \tag{4.1}$$

It is clear from the Ochanine divisibility (cf. [21, 22]) that $\Phi(B)$ is well-defined. In [10], Finashin shows that it is a pin$^-$ cobordism invariant of B.

Let $B \cdot B$ be the self-intersection of B in M. Then, $B \cdot B$ can be thought of as an $8k$ dimensional closed oriented manifold (cf. [**10**], see also Section 4.4).

We can now state the original Finashin congruence as follows.

Theorem 4.1 (Finashin [**10**]). *The following congruence formula holds,*

$$\frac{\mathrm{Sign}(M) - \mathrm{Sign}(B \cdot B)}{8} \equiv \Phi(B) \mod 2\mathbb{Z}. \tag{4.2}$$

In [**19**], by combining the higher dimensional "miraculous cancellation" formula of Liu [**18**] with the analytic arguments in [**26**], Liu and Zhang give an intrinsic analytic interpretation of the Finashin invariant $\Phi(B)$. We recall their result as follows.

Let g^{TB} be a metric on TB. Let ∇^{TB} be the associated Levi–Civita connection. Let ∇^L be a Euclidean connection on $L = o(TB) + \mathbb{R}$.

Let $\pi : B' \to B$ be the orientable double cover of B. We fix an orientation on B'. Then B' is spin and carries an induced spin structure from the pin$^-$ structure on B.

When we pull back the bundles and the associated metrics and connections from B to B', we will use an extra notation "$'$" as indication.

Let $P : B' \to B'$ be the canonical involution on B' with respect to the double covering $\pi : B' \to B$.

Let $b_r(T_{\mathbb{C}}B + o(TB) \otimes \mathbb{C} + \mathbb{C}, \mathbb{C}^2)$, $0 \le r \le k$, be the virtual Hermitian vector bundles defined in the same way as in (2.40). They lift to virtual vector bundles $b_r(T_{\mathbb{C}}B' + \mathbb{C}^2, \mathbb{C}^2)$, $0 \le r \le k$, over B', carrying the canonically induced P-invariant Hermitian connections.

Let $S(TB')$ be the bundle of spinors associated to $(TB', g^{TB'})$. For any integer r such that $0 \le r \le k$, let

$$\begin{aligned} D_{B'}^{b_r(T_{\mathbb{C}}B' + \mathbb{C}^2, \mathbb{C}^2)} : \Gamma(S(TB') \otimes b_r(T_{\mathbb{C}}B' + \mathbb{C}^2, \mathbb{C}^2)) \\ \longrightarrow \Gamma(S(TB') \otimes b_r(T_{\mathbb{C}}B' + \mathbb{C}^2, \mathbb{C}^2)) \end{aligned} \tag{4.3}$$

be the corresponding Dirac operator. It is a P-equivariant first order elliptic differential operator and is formally self-adjoint. As explained in [**19**] and [**26**], $\frac{1}{2}(1+P) D_{B'}^{b_r(T_{\mathbb{C}}B' + \mathbb{C}^2, \mathbb{C}^2)}$ determines a formally self-adjoint elliptic differential operator $\widetilde{D}_B^{b_r(T_{\mathbb{C}}B + o(TB) \otimes \mathbb{C} + \mathbb{C}, \mathbb{C}^2)}$ (called the twisted Dirac operator) on B. When there is no confusion, we will use the brief notation $\widetilde{D}_B^{b_r}$ to denote this twisted Dirac operator.

Let $\overline{\eta}(\widetilde{D}_B^{b_r})$, $0 \leq r \leq k$, be the reduced η invariant of $\widetilde{D}_B^{b_r}$ in the sense of Atiyah, Patodi and Singer [**4**]. By using the Atiyah–Patodi–Singer index theorem for manifolds with boundary [**4**], one knows that each $\overline{\eta}(\widetilde{D}_B^{b_r})$ is a pin$^-$ cobordism invariant of B.

The following analytic interpretation of $\Phi(B)$ is proved in [**19**, Theorem 3.2],

$$\Phi(B) \equiv \sum_{r=0}^{k} 2^{6k-6r}\overline{\eta}(\widetilde{D}_B^{b_r}) \mod 2\mathbb{Z}. \tag{4.4}$$

From (4.2) and (4.4), one gets the following analytic version of the Finashin congruence formula.

Theorem 4.2 (Liu and Zhang [**19**, Theorem 4.1])**.** *The following congruence formula holds,*

$$\frac{\mathrm{Sign}(M) - \mathrm{Sign}(B \cdot B)}{8} \equiv \sum_{r=0}^{k} 2^{6k-6r}\overline{\eta}(\widetilde{D}_B^{b_r}) \mod 2\mathbb{Z}. \tag{4.5}$$

In the rest of this section, we will give a direct proof of (4.5) by combining the twisted cancellation formula (2.7) with the analytic Rokhlin congruence formula for KO characteristic numbers proved in [**26**], which we recall in the next subsection.

4.2. Rokhlin congruences for KO characteristic numbers associated to b_r, $0 \leq r \leq k$. We make the same assumptions and use the same notations as before.

Let $N \to B$ be the normal bundle to B in M. Let g^{TM} be a metric on TM, with the associated Levi–Civita connection denoted by ∇^{TM}. Without loss of generality, we assume that the following orthogonal splitting on B holds,

$$TM|_B = TB \oplus N, \qquad g^{TM} = g^{TB} \oplus g^N, \tag{4.6}$$

where g^N is the induced metric on N. Let ∇^N be the Euclidean connection on N induced from $\nabla^{TM}|_B$.

Since M is oriented, one has the equality for the first Stiefel–Whitney classes,

$$w_1(TB) = w_1(N). \tag{4.7}$$

In particular, if $o(N)$ is the orientation bundle of N, then $o(N) = o(TB)$. Let $N' = \pi^* N$ be the pull back of $N \to B$ to B'. Then, N' is a rank two real orientable vector bundle over B', carrying a pull back Euclidean structure, as well as a pull back Euclidean connection $\nabla^{N'}$. Since we have fixed an orientation on TB' in the previous subsection, we see that N' carries an induced orientation from the pull back of the splitting (4.6) and from the orientation on $\pi^*(TM|_B)$.

Let $e' = e\,(N', \nabla^{N'}) \in \Omega^2\,(B')$ be the Euler form associated to $(N', \nabla^{N'})$. Then, $\frac{1}{2}(1+P)e'$ determines an element $e \in \Omega^2(B) \otimes o(TB)$, which is the Euler form associated to (N, ∇^N).

The following Rokhlin type congruence formula for KO characteristic numbers associated to $b_r(T_{\mathbb{C}}M, \mathbb{C}^2)$'s is a direct consequence of the general congruence formula proved in [**26**, Section 3].

Theorem 4.3 (Zhang [**26**, Theorem 3.2])**.** *For any integer r such that $0 \le r \le k$, the following congruence formula holds,*

(4.8)
$$\begin{aligned}
&\int_M \widehat{A}(TM, \nabla^{TM})\mathrm{ch}(b_r(T_{\mathbb{C}}M, \mathbb{C}^2)) \\
&\quad \equiv \overline{\eta}(\widetilde{D}_B^{b_r}) + \int_B \widehat{A}(TB, \nabla^{TB})\Big(\mathrm{ch}\left(b_r\left(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2\right)\right) \\
&\qquad - \cosh\left(\frac{e}{2}\right)\mathrm{ch}\left(b_r(T_{\mathbb{C}}B + o(TB)\otimes\mathbb{C} + \mathbb{C}, \mathbb{C}^2)\right)\Big)\frac{1}{2\sinh(\frac{e}{2})} \mod 2\mathbb{Z}.
\end{aligned}$$

Remark 4.4. The more precise expression of the integration in the right-hand side of (4.8) is

$$\begin{aligned}
\frac{1}{2}\int_{B'} \widehat{A}(TB', \nabla^{TB'})&\Big(\mathrm{ch}\left(b_r\left(T_{\mathbb{C}}B' + N'_{\mathbb{C}}, \mathbb{C}^2\right)\right) \\
&- \cosh\left(\frac{e'}{2}\right)\mathrm{ch}\left(b_r\left(T_{\mathbb{C}}B' + \mathbb{C}^2, \mathbb{C}^2\right)\right)\Big)\frac{1}{2\sinh(\frac{e'}{2})}.
\end{aligned} \tag{4.9}$$

Remark 4.5. The proof of Theorem 3.2 in [**26**] uses in an essential way the techniques developed by Bismut–Cheeger [**6**] and Dai [**9**] on the computation of the adiabatic limits of η invariants of Dirac operators.

4.3. A cancellation formula for characteristic numbers on $8k+2$ dimensional manifolds. We assume for a moment that B is oriented so that we are in the situation discussed in Section 3.

We first set $\xi = \mathbb{R}^2$ and $c = 0$ in (2.40) to get

$$\left\{\widehat{L}(TM, \nabla^{TM})\right\}^{(8k+4)} = 8\sum_{r=0}^{k} 2^{6k-6r}\left\{\widehat{A}(TM, \nabla^{TM})\mathrm{ch}(b_r(T_{\mathbb{C}}M, \mathbb{C}^2))\right\}^{(8k+4)}, \tag{4.10}$$

which was first proved in [**18**].

From (2.40) and (4.10), one gets

$$\frac{1}{8}\int_M \left(1 - \frac{1}{\cosh^2\left(\frac{c}{2}\right)}\right)\widehat{L}(TM, \nabla^{TM}) = \sum_{r=0}^{k} 2^{6k-6r}\int_M \widehat{A}(TM, \nabla^{TM})\Big(\mathrm{ch}(b_r(T_{\mathbb{C}}M, \mathbb{C}^2)) - \cosh\left(\frac{c}{2}\right)\mathrm{ch}(b_r(T_{\mathbb{C}}M, \xi_{\mathbb{C}}))\Big). \tag{4.11}$$

Now since $[c]$ is Poincaré dual to $[B] \in H_{8k+2}(M, \mathbb{Z})$, one sees that if $i : B \hookrightarrow M$ denotes the canonical embedding, then $i^*\xi = N$ and $i^*[c] = [e]$, the Euler class of N.

From (2.1), [**14**, (9.3)] and the Chern–Weil theorem (cf. [**27**, Chapter 1]), one deduces that

$$\begin{aligned}\frac{1}{8}\int_M \left(1 - \frac{1}{\cosh^2\left(\frac{c}{2}\right)}\right)\widehat{L}(TM, \nabla^{TM}) &= \frac{1}{8}\int_B \widehat{L}(TB, \nabla^{TB})\frac{e}{\tanh\left(\frac{e}{2}\right)}\frac{\sinh^2\left(\frac{e}{2}\right)}{\cosh^2\left(\frac{e}{2}\right)}\frac{1}{e} \\ &= \frac{1}{8}\int_B \widehat{L}(TB, \nabla^{TB})\frac{\sinh\left(\frac{e}{2}\right)}{\cosh\left(\frac{e}{2}\right)},\end{aligned} \tag{4.12}$$

and that for any integer r such that $0 \le r \le k$,

(4.13)
$$\begin{aligned}\int_M \widehat{A}(TM, \nabla^{TM}) &\left(\mathrm{ch}(b_r(T_{\mathbb{C}}M, \mathbb{C}^2)) - \cosh(\frac{c}{2})\mathrm{ch}(b_r(T_{\mathbb{C}}M, \xi_{\mathbb{C}}))\right) \\ &= \int_B \widehat{A}(TB, \nabla^{TB})\Bigg(\mathrm{ch}(b_r(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2)) \\ &\quad - \cosh\left(\frac{e}{2}\right)\mathrm{ch}(b_r(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}}))\Bigg)\frac{1}{2\sinh\left(\frac{e}{2}\right)}.\end{aligned}$$

From (4.11)–(4.13), one gets,

$$\begin{aligned}\frac{1}{8}\int_B \widehat{L}(TB, \nabla^{TB})\frac{\sinh\left(\frac{e}{2}\right)}{\cosh\left(\frac{e}{2}\right)} &= \sum_{r=0}^{k} 2^{6k-6r}\int_B \widehat{A}(TB, \nabla^{TB})\Bigg(\mathrm{ch}(b_r(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2)) \\ &\quad - \cosh\left(\frac{e}{2}\right)\mathrm{ch}(b_r(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}}))\Bigg)\frac{1}{2\sinh\left(\frac{e}{2}\right)}.\end{aligned} \tag{4.14}$$

In fact, (4.14) holds for any $8k+2$ dimensional closed oriented spin manifold B and a rank two real vector bundle N over it, for one can always take the unit disc bundle N_1 of N. Then, ∂N_1 is an $8k+3$ dimensional oriented spin manifold and $-\partial N_1$ bounds an $8k+4$ dimensional oriented spin manifold Z. One may take $M = N_1 \cup_{\partial N_1} Z$ to get (4.14).

In the next subsection, we will apply (4.14) to the pair (B', N') discussed in Section 4.2.

Remark 4.6. Formula (4.14) actually can be refined to the level of differential forms, and one can prove this directly without passing to the cobordism argument. See [**12**] for more details.

4.4. A proof of Theorem 4.2. We now come back to the situation of Sections 4.1 and 4.2.

Recall that $B \cdot B$ denotes the self-intersection of B in M. It can be constructed as follows: take a transversal section X of N, then

$$B \cdot B = \{b \in B \,:\, X(b) = 0\}. \tag{4.15}$$

Let $X' = \pi^* X$ be the pull back of X over B'. Then, X' is a transversal section of N' and

$$B' \cdot B' = \{b' \in B' \,:\, X'(b') = 0\} \tag{4.16}$$

is a double cover of $B \cdot B$. Let $N'_{B'\cdot B'}$ be the normal bundle to $B' \cdot B'$ in B', then $N'_{B'\cdot B'} = N'|_{B'\cdot B'}$. Thus, $B' \cdot B'$ carries a canonically induced orientation from those of $TB'|_{B'\cdot B'}$ and $N'|_{B'\cdot B'}$. Moreover, the canonical involution P preserves the orientation on $B' \cdot B'$ and thus induces an orientation on $B \cdot B$.

By a simple application of the Hirzebruch Signature theorem (cf. [**14**]) to $B \cdot B$ and $B' \cdot B'$, one gets

$$\operatorname{Sign}(B' \cdot B') = 2\operatorname{Sign}(B \cdot B). \tag{4.17}$$

Let $e(N'_{B'\cdot B'})$ denote the Euler class of $N'_{B'\cdot B'}$.

From (2.1), [**14**, (9.3)], the Chern–Weil theorem (cf. [**27**]) and the Hirzebruch Signature theorem (cf. [**14**]), one deduces that

$$\begin{aligned}
&\int_{B'} \widehat{L}(TB', \nabla^{TB'}) \frac{\sinh\left(\frac{e'}{2}\right)}{\cosh\left(\frac{e'}{2}\right)} \\
&\quad = \left\langle \frac{\widehat{L}(T(B'\cdot B'))}{e(N'_{B'\cdot B'})} \frac{e(N'_{B'\cdot B'})}{\tanh\left(\frac{e(N'_{B'\cdot B'})}{2}\right)} \frac{\sinh\left(\frac{e(N'_{B'\cdot B'})}{2}\right)}{\cosh\left(\frac{e(N'_{B'\cdot B'})}{2}\right)}, [B'\cdot B'] \right\rangle \\
&\quad = \left\langle \widehat{L}(T(B'\cdot B')), [B'\cdot B'] \right\rangle = \operatorname{Sign}(B'\cdot B').
\end{aligned} \tag{4.18}$$

From (4.8)–(4.10), (4.17), (4.18) and (4.14) (when applied to the pair (B', N')), one deduces that when mod $2\mathbb{Z}$, one has

$$\begin{aligned}
&\frac{\operatorname{Sign}(M) - \operatorname{Sign}(B\cdot B)}{8} - \sum_{r=0}^{k} 2^{6k-6r} \overline{\eta}(\widetilde{D}_B^{b_r}) \\
&\quad = -\frac{1}{16} \int_{B'} \widehat{L}(TB', \nabla^{TB'}) \frac{\sinh\left(\frac{e'}{2}\right)}{\cosh\left(\frac{e'}{2}\right)} \\
&\qquad + \frac{1}{2} \sum_{r=0}^{k} 2^{6k-6r} \int_{B'} \widehat{A}(TB', \nabla^{TB'}) \Big(\operatorname{ch}\left(b_r(T_{\mathbb{C}}B' + N'_{\mathbb{C}}, \mathbb{C}^2)\right) \\
&\qquad\quad - \cosh\left(\frac{e'}{2}\right) \operatorname{ch}\left(b_r(T_{\mathbb{C}}B' + \mathbb{C}^2, \mathbb{C}^2)\right) \Big) \frac{1}{2\sinh\left(\frac{e'}{2}\right)}
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
&= \frac{1}{2}\sum_{r=0}^{k} 2^{6k-6r}\int_{B'}\widehat{A}(TB',\nabla^{TB'})\Big(\mathrm{ch}\left(b_r(T_{\mathbb{C}}B'+N'_{\mathbb{C}},\mathbb{C}^2)\right)\\
&\qquad -\cosh\left(\frac{e'}{2}\right)\mathrm{ch}\left(b_r(T_{\mathbb{C}}B'+\mathbb{C}^2,\mathbb{C}^2)\right)\Big)\frac{1}{2\sinh\left(\frac{e'}{2}\right)}\\
&\quad -\frac{1}{2}\sum_{r=0}^{k} 2^{6k-6r}\int_{B'}\widehat{A}(TB',\nabla^{TB'})\Big(\mathrm{ch}\left(b_r(T_{\mathbb{C}}B'+N'_{\mathbb{C}},\mathbb{C}^2)\right)\\
&\qquad -\cosh\left(\frac{e'}{2}\right)\mathrm{ch}\left(b_r(T_{\mathbb{C}}B'+N'_{\mathbb{C}},N'_{\mathbb{C}})\right)\Big)\frac{1}{2\sinh\left(\frac{e'}{2}\right)}\\
&= \frac{1}{2}\sum_{r=0}^{k} 2^{6k-6r}\int_{B'}\widehat{A}(TB',\nabla^{TB'})\frac{\cosh\left(\frac{e'}{2}\right)}{2\sinh\left(\frac{e'}{2}\right)}\Big(\mathrm{ch}\left(b_r\left(T_{\mathbb{C}}B'+N'_{\mathbb{C}},N'_{\mathbb{C}}\right)\right)\\
&\qquad -\mathrm{ch}\left(b_r(T_{\mathbb{C}}B'+\mathbb{C}^2,\mathbb{C}^2)\right)\Big).
\end{aligned}$$

From (4.19), one sees that in order to prove Theorem 4.2, one need only prove the following result.

Lemma 4.7. *For any integer r such that $0\le r\le k$, one has*

$$\begin{aligned}
\frac{1}{2}\int_{B'}\widehat{A}(TB',\nabla^{TB'})\frac{\cosh\left(\frac{e'}{2}\right)}{2\sinh\left(\frac{e'}{2}\right)}&\Big(\mathrm{ch}(b_r(T_{\mathbb{C}}B'+N'_{\mathbb{C}},N'_{\mathbb{C}}))\\
&-\mathrm{ch}\left(b_r(T_{\mathbb{C}}B'+\mathbb{C}^2,\mathbb{C}^2)\right)\Big)\in 2\mathbb{Z}.
\end{aligned}\tag{4.20}$$

Proof. Let $\mathbb{Z}[T_{\mathbb{C}}B',N'_{\mathbb{C}}]$ be the ring with integral coefficients generated by the exterior and symmetric powers of $T_{\mathbb{C}}B'$ and $N'_{\mathbb{C}}$.

By an obvious analogue of (3.14), one has that for any integer r such that $0\le r\le k$,

$$b_r(T_{\mathbb{C}}B'+N'_{\mathbb{C}},N'_{\mathbb{C}})-b_r(T_{\mathbb{C}}B'+\mathbb{C}^2,\mathbb{C}^2)=2\widetilde{N}'_{\mathbb{C}}T'_r\tag{4.21}$$

for some $T'_r\in\mathbb{Z}[T_{\mathbb{C}}B',N'_{\mathbb{C}}]$.

From (4.21), one deduces that

$$\begin{aligned}(4.22)\quad &\frac{\cosh\left(\frac{e'}{2}\right)}{2\sinh\left(\frac{e'}{2}\right)}\left(\mathrm{ch}\left(b_r(T_{\mathbb{C}}B'+N'_{\mathbb{C}},N'_{\mathbb{C}})\right)-\mathrm{ch}\left(b_r(T_{\mathbb{C}}B'+\mathbb{C}^2,\mathbb{C}^2)\right)\right)\\
&=\frac{\cosh\left(\frac{e'}{2}\right)}{\sinh\left(\frac{e'}{2}\right)}\left(\exp(e')+\exp(-e')-2\right)\mathrm{ch}(T'_r)\\
&=\frac{\cosh\left(\frac{e'}{2}\right)}{\sinh\left(\frac{e'}{2}\right)}4\sinh^2\left(\frac{e'}{2}\right)\mathrm{ch}(T'_r)\\
&=2\sinh(e')\mathrm{ch}(T'_r).\end{aligned}$$

From (4.22), [**14**, (9.3)] and the Chern–Weil theorem (cf. [**27**]), one finds,

$$\begin{aligned}(4.23)\quad &\frac{1}{2}\int_{B'}\widehat{A}(TB',\nabla^{TB'})\frac{\cosh\left(\frac{e'}{2}\right)}{2\sinh\left(\frac{e'}{2}\right)}\Big(\mathrm{ch}(b_r(T_{\mathbb{C}}B'+N'_{\mathbb{C}},N'_{\mathbb{C}}))\\
&\qquad -\mathrm{ch}\left(b_r(T_{\mathbb{C}}B'+\mathbb{C}^2,\mathbb{C}^2)\right)\Big)\\
&=\int_{B'}\widehat{A}(TB',\nabla^{TB'})\mathrm{ch}(T'_r)\sinh(e')\\
&=\Big\langle\widehat{A}(T(B'\cdot B'))\mathrm{ch}(i^*_{B'\cdot B'}T'_r)\frac{\sinh(e(N'_{B'\cdot B'}))}{e(N'_{B'\cdot B'})}\\
&\qquad\cdot\frac{e(N'_{B'\cdot B'})}{2\sinh\left(\frac{e(N'_{B'\cdot B'})}{2}\right)},[B'\cdot B']\Big\rangle\\
&=\left\langle\widehat{A}(T(B'\cdot B'))\mathrm{ch}(i^*_{B'\cdot B'}T'_r)\cosh\left(\frac{e(N'_{B'\cdot B'})}{2}\right),[B'\cdot B']\right\rangle,\end{aligned}$$

where $i^*_{B'\cdot B'}: B'\cdot B'\hookrightarrow B'$ denotes the canonical embedding.

It is clear that $i^*_{B'\cdot B'}T'_r$ is invariant under the canonical involution P and thus induces a virtual complex vector bundle over $B\cdot B$. We denote it by $T_r(B\cdot B)$.

Let $N_{B\cdot B}$ be the normal bundle to $B\cdot B$ in B. Then, $e(N'_{B'\cdot B'})$ is the pull back of the Euler class $e(N_{B\cdot B})$ of $N_{B\cdot B}$ through the covering map $B'\cdot B'\to B\cdot B$. Moreover, it is clear that the total Pontrjagin class of the oriented real vector bundle $N_{B\cdot B}\oplus o(N_{B\cdot B})$ is given by

$$p(N_{B\cdot B}\oplus o(N_{B\cdot B}))=1+(e(N_{B\cdot B}))^2\,. \tag{4.24}$$

With these notations, one gets

$$\begin{aligned}\left\langle \widehat{A}(T(B'\cdot B'))\mathrm{ch}(i^*_{B'\cdot B'}T'_r)\cosh\left(\frac{e(N'_{B'\cdot B'})}{2}\right),[B'\cdot B']\right\rangle \\ =2\left\langle \widehat{A}(T(B\cdot B))\mathrm{ch}(T_r(B\cdot B))\cosh\left(\frac{e(N_{B\cdot B})}{2}\right),[B\cdot B]\right\rangle .\end{aligned} \tag{4.25}$$

Now as B is pin$^-$, one has the following equality for the Stiefel–Whitney classes (cf. [**15**] and [**10**]),

$$w_2(TB)+(w_1(TB))^2=0. \tag{4.26}$$

By (4.7) and (4.26), one gets

$$w_2(TB)+(w_1(N))^2=0. \tag{4.27}$$

Pulling back (4.27) to $B\cdot B$, one gets

$$w_2(T(B\cdot B))+w_2(N_{B\cdot B})+(w_1(N_{B\cdot B}))^2=0. \tag{4.28}$$

On the other hand, one has

$$w_2(N_{B\cdot B}\oplus o(N_{B\cdot B}))=w_2(N_{B\cdot B})+(w_1(N_{B\cdot B}))^2. \tag{4.29}$$

From (4.28) and (4.29), one finds

$$w_2(T(B\cdot B))=w_2(N_{B\cdot B}\oplus o(N_{B\cdot B})). \tag{4.30}$$

From (4.24), (4.30) and the obvious fact that $T_r(B\cdot B)$ is indeed the complexification of some (virtual) real vector bundle over $B\cdot B$, one then applies a result of Mayer [**20**, Satz 3.2(vi)] to conclude that

$$2\left\langle \widehat{A}(T(B\cdot B))\mathrm{ch}(T_r(B\cdot B))\cosh\left(\frac{e(N_{B\cdot B})}{2}\right),[B\cdot B]\right\rangle\in 2\mathbb{Z}. \tag{4.31}$$

Formula (4.20) then follows from (4.23), (4.25) and (4.31).

The proof of Lemma 4.7 is complete. q.e.d.

The proof of Theorem 4.2 is thus also complete. q.e.d.

Remark 4.8. It is remarkable that Mayer's result is needed here in the B non-orientable case. The basic reason is that $T(B\cdot B)+N_{B\cdot B}+o(N_{B\cdot B})$ is a rank $8k+3$ real oriented spin vector bundle. Thus, the associated spinor bundle carries a quarternionic structure. Mayer's result is then a direct consequence of the Atiyah–Singer index theorem [**5**].

Remark 4.9. Recall that (4.4) gives an analytic interpretation of the Finashin invariant of an $8k+2$ dimensional closed pin$^-$ manifold. In view of [**16**], where it is shown that the Ochanine invariant of an $8k+2$ dimensional closed spin manifold is a Brown–Kervaire invariant [**7**], we believe that the Finashin invariant of an $8k+2$ dimensional closed pin$^-$ manifold M should also be a Brown–Kervaire invariant. This would give an analytic interpretation of the later, generalizing the two dimensional result in [**26**, (4.12)].

Appendix A. A twisted cancellation formula in $8k$ dimension

In this appendix, we present a twisted cancellation formula for $8k$ dimensional manifolds, which can be seen as a direct analogue of the $8k+4$ dimensional formula (2.7). Since the statement is parallel and the proof is almost the same, we only indicate the necessary modifications. In particular, we will use the same notation as in Section 2.

Let M be an $8k$ dimensional Riemannian manifold with Levi–Civita connection ∇^{TM}. Let V be a rank $2l$ real Euclidean vector bundle over M carrying a Euclidean connection ∇^V. Let ξ be a rank two real oriented Euclidean vector bundle over M carrying a Euclidean connection ∇^ξ. Let $R^V = \nabla^{V,2}$ be the curvature of ∇^V and $c = e(\xi, \nabla^\xi)$ be the Euler form associated to (ξ, ∇^ξ).

Let $\Theta_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})$, $\Theta_2(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})$ be two elements defined in the same way as in (2.5) and assume they admit Fourier expansions in the same way as in (2.6).

We can state the main result of this appendix as follows.

Theorem A.1. *If the equality $p_1(TM, \nabla^{TM}) = p_1(V, \nabla^V)$ for the first Pontrjagin forms holds, then one has*

(A.1)
$$\left\{ \frac{\widehat{A}(TM, \nabla^{TM})\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^V\right)\right)}{\cosh^2(\frac{c}{2})} \right\}^{(8k)}$$
$$= 2^{l+2k}\sum_{r=0}^{k} 2^{-6r}\left\{\widehat{A}(TM, \nabla^{TM})\mathrm{ch}(b_r(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}))\cosh\left(\frac{c}{2}\right)\right\}^{(8k)},$$

where each $b_r(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})$, $0 \le r \le k$, is a canonical integral linear combination of $B_j(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})$, $0 \le j \le r$.

If $\xi = \mathbb{R}^2$ and $c = 0$, (A.2) reduces to the formula in [**18**, p. 32].
If we take $V = TM$ and $\nabla^V = \nabla^{TM}$, we have by (A.2) that

(A.2)
$$\left\{\frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2\left(\frac{c}{2}\right)}\right\}^{(8k)} = \sum_{r=0}^{k} 2^{6k-6r}\left\{\widehat{A}(TM, \nabla^{TM})\mathrm{ch}(b_r(T_{\mathbb{C}}M, T_{\mathbb{C}}M, \xi_{\mathbb{C}}))\cosh\left(\frac{c}{2}\right)\right\}^{(8k)}.$$

Now, we assume $k = 1$, that is, $\dim M = 8$. In this case, by proceeding similarly as in Section 2(e), one finds,

(A.3)
$$\left\{\frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2(\frac{c}{2})}\right\}^{(8)} = \left\{\left[-\widehat{A}(TM, \nabla^{TM})\mathrm{ch}(T_{\mathbb{C}}M, \nabla^{T_{\mathbb{C}}M}) + 24\widehat{A}(TM, \nabla^{TM}) + 3\widehat{A}(TM, \nabla^{TM})\left(e^c + e^{-c} - 2\right)\right]\cosh\left(\frac{c}{2}\right)\right\}^{(8)}.$$

5. Acknowledgements

The authors would like to thank one of the referees for helpful comments.

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