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MORSE THEORY ON HAMILTONIAN G-SPACES AND EQUIVARIANT K-THEORY

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Abstract

Let G be a torus and M a compact Hamiltonian G-manifold with finite fixed point set M^G. If T is a circle subgroup of G with $M^G = M^T$, the T-moment map is a Morse function. We will show that the associated Morse stratification of M by unstable manifolds gives one a canonical basis of $K_G(M)$. A key ingredient in our proof is the notion of local index $I_p(a)$ for $a \in K_G(M)$ and $p \in M^G$. We will show that corresponding to this stratification there is a basis τ_p, $p \in M^G$, for $K_G(M)$ as a module over $K_G(\mathrm{pt})$ characterized by the property: $I_q\tau_p = \delta_p^q$. For M a GKM manifold we give an explicit construction of these τ_p's in terms of the associated GKM graph.

1. Introduction

Let M^{2d} be a compact symplectic manifold, G an n-dimensional torus and $\sigma : G \times M \to M$ a Hamiltonian action. Assume the fixed point set M^G is finite. Let T be a circle subgroup of G with the property that $M^T = M^G$, and let $\phi : M \to \mathbb{R}$ be the T moment map. This function is a Morse function and all its critical points are of even index; so, by standard Morse theory, the unstable manifolds of ϕ with respect to a G-invariant Riemannian metric define a basis of $H_*(M, \mathbb{R})$ and by Poincare duality a basis for $H^*(M, \mathbb{R})$ consisting of the Thom classes of the closures of unstable manifolds. Moreover, these unstable manifolds are G-invariant so they also define a basis for $H^*_G(M)$ as a module over $H^*_G(\mathrm{pt})$.

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In K-theory the situation is a little more complicated. The critical points of ϕ carry a natural partial order, which is defined by setting $p \leq q$ if q is inside the closure of the unstable manifold of ϕ at p and then completing this order by transitivity. So, for any unstable manifold U of ϕ at p, one can consider the union

$$W_U = \bigcup U_q$$

of unstable manifolds U_q for $q \geq p$. It is known that there exist classes in K-theory which are supported on this set. However, except in certain special cases (e.g., algebraic torus actions), it is not known whether there is a genuine (Thom) class in K-theory associated with U. (For algebraic torus actions such classes can be defined using the structure sheaf of the closure of U, see [**4**] for details.)

We will show in this paper, however, that there is another way of attaching to the Morse decomposition of M a basis of $K_G(M)$ which works even in the case of non-algebraic torus actions. (As will be explained later, in the algebraic case our classes will be different from those constructed using structure sheaves.) The key idea in our approach is a notion of *local index* for a K-class $a \in K_G(M)$ at a critical point p of ϕ. This is defined as follows: Let S be the stable manifold of ϕ at p, and for small $\varepsilon > 0$ let S_ε be the compact symplectic orbifold obtained from S by the symplectic cutting operation of Lerman [**13**]. We recall that S_ε is obtained from the manifold with boundary

$$\tilde{S}_\varepsilon = \{x \in S, \phi(x) \geq \phi(p) - \varepsilon\} \tag{1.1}$$

by collapsing to points the T-orbits on the boundary. In particular, there is a projection $\rho : \tilde{S}_\varepsilon \to S_\varepsilon$ and an inclusion $i : \tilde{S}_\varepsilon \to M$; so a K-class $a \in K_G(M)$ defines a class $\kappa_\varepsilon(a) = \rho_! i^* a \in K_G(S_\varepsilon)$, where $\rho_!$ is the pushforward map (we will define the map κ_ε in more detail in Section 5).

Now let *the local index* of a at p

$$I_p(a) \in K_G(\mathrm{pt})$$

be the Atiyah–Segal index of $\kappa_\varepsilon(a)$, that is, the pushforward of $\kappa_\varepsilon(a)$ with respect to the map $S_\varepsilon \to \mathrm{pt}$. Recall that $K_G(\mathrm{pt})$ is just the representation ring $R(G)$ of the torus G, so that each local index is just a virtual representation of G. One of the main results of this paper is the following theorem.

Theorem 1.1. *Let p be a critical point of ϕ and U the unstable manifold of ϕ at p. Then there exists a unique K-theory class $\tau_p \in K_G(M)$ with the properties:*

(i) $I_p(\tau_p) = 1$,
(ii) $I_q(\tau_p) = 0$ *for all critical points q of ϕ except p,*
(iii) *The restriction of τ_p to a critical point q is zero unless $q \in W_U$.*

Moreover, the τ_p's generate $K_G(M)$ freely as a module over $K_G(\mathrm{pt})$.

Let $\mathcal{I} : K_G(M) \to K_G(M^G)$ be the map which takes the value I_p, at p. This we will call the *total index map*. As explained in Remark 5.1 the total index is not an $R(G)$-module homomorphism but it is a homomorphism with respect to the subring, $R(G/T)$, of $R(G)$. Theorem 1.1 implies the following:

Corollary 1.2. *The total index map, $\mathcal{I}$, is an $R(G/T)$ module isomorphism.*

Remark 1.3. Notice that we can define local indices even if the fixed point set of the action is not finite. Namely, let F be a connected component of M^G, not necessarily consisting of one point. Then if S is the stable manifold of ϕ at F we can still define S_ε, the projection ρ, the inclusion i, and the map κ_ε. Moreover, there is a fibration $\mathbf{P} : S_\varepsilon \to F$ whose fibers are weighted projective spaces. So, we can define the local index at F

$$I_F : K_G(M) \to K_G(F)$$

to be $\mathbf{P}_! \kappa_\varepsilon$, the composition of the pushforward $\mathbf{P}_!$ with κ_ε. Then the total local index map

$$\mathcal{I} : K_G(M) \to K_G(M^G)$$

is well defined and is an $R(G/T)$-module homomorphism; so, it is natural to pose the following question whose answer, we believe, depends on whether or not ϕ is K-theoretically perfect.

Question. *When is $\mathcal{I}$ an isomorphism*?

Remark 1.4. Notice that local indices can also be defined in the setting of equivariant cohomology. Namely, for $a \in H^*_G(M)$, we let $I_p(a)$ be the pushforward (or integral) of $\kappa'_\varepsilon(a)$, where $\kappa'_\varepsilon : H^*_G(M) \to H^*_G(S_\varepsilon)$ is defined in exactly the same way as κ_ε. It will be clear from the proof of Theorem 1.1 that its analogue for equivariant cohomology is also true, and that the cohomological analogues of the τ_p's are just "the equivariant Poincare duals" of the closures of the unstable manifolds.

The other main result of this paper is a constructive version of Theorem 1.1 for GKM spaces, that is, an explicit computation of the classes τ_p. We start by recalling some facts about GKM spaces. The *one-skeleton* of M

$$\{x \in M,\ \dim G \cdot x = 1\} \tag{1.2}$$

is a union of symplectic submanifolds of M. The action σ is defined to be *a GKM action* and M *a GKM space* if each connected component of the one-skeleton is exactly of dimension 2. It is easy to see that if σ is GKM the fixed point set M^G has to be finite. Let

$$V = \{p_1, \dots, p_\ell\}$$

be the points of M^G. To each connected component e_i° of (1.2) let e_i be its closure, and let

$$E = \{e_1, \dots, e_N\}$$

be the set of e_i's. We claim:

(i) e_i is an imbedded copy of $\mathbb{CP}^1$,
(ii) $e_i - e_i^\circ$ is a two element subset of V,
(iii) for $i \neq j$ the intersection $e_i \cap e_j$ is empty or is a one-element subset of V,
(iv) for $p \in V$ the set $\{e_i \in E, p \in e_i\}$ is d-element subset of E.

For the proof of these assertions see, for instance, [7]. These assertions can be interpreted as saying that V and E are the vertices and edges of a d-valent graph Γ.

One can describe the action of G on the one-skeleton (1.2) by means of a labeling function which labels each oriented edge of this graph by an element of the weight lattice $\mathbb{Z}_G^*$ of G. Explicitly, let e be an edge of Γ joining a vertex p and a vertex q. To e we can associate two oriented edges e_p and e_q pointing from p to q and from q to p respectively. As a geometric object e is a G invariant imbedded $\mathbb{CP}^1$ with fixed points at p and q; if we denote by α_{e_p} the weight of the isotropy representation of G on the tangent space to e at p (and by $\alpha_{e_q} = -\alpha_{e_p}$ the weight of the isotropy representation at q) we get a labeling function α which describes how each connected component of the one-skeleton is rotated about its axis of symmetry by G.

GKM theory is concerned with reconstructing, in so far as possible, the geometry of M from the combinatorics of the pair (Γ, α). It is known for instance that the ring structure of $H_G^*(M)$ and $K_G(M)$ are determined by (Γ, α). (See [**6**, **1**, **5**, **11**, **18**] for versions of this result. We will explain below how $K_G(M)$ is determined by (Γ, α).) It was also shown in [**9**] that if T is generic circle subgroup of G, and $\tau \in H^*_G(M)$ the cohomology class dual to an unstable manifold of T, the restriction of τ to M^G is completely determined by (Γ, α). In this paper, we will prove analogous results for $K_G(M)$.

Let us recall how the ring structure of $K_G(M)$ is determined by (Γ, α). One knows that the restriction map

$$K_G(M) \to K_G(M^G) \tag{1.3}$$

is an injection, so $K_G(M)$ is a subring of the much simpler ring

$$K_G(M^G) = \bigoplus_{i=1}^{\ell} K_G(p_i) \tag{1.4}$$

Since $K_G(\mathrm{pt}) = R(G)$, an element of the ring (1.4) is just a map

$$\chi : V \to R(G) \tag{1.5}$$

and one has:

Theorem 1.5. [**1**, **11**] *For each $e \in E$ connecting $p, q \in V$ the homomorphisms*

$$e^{2\pi\sqrt{-1}\alpha_{e_p}} : G \to \ S^1 \ \textit{and} \ \ e^{2\pi\sqrt{-1}\alpha_{e_q}} : G \to S^1$$

have the same kernel. Denote this kernel by G_e. Then the element (1.5) of $K_G(M^G)$ is in the image of (1.3) if and only if for every $e \in E$

$$r_e(\chi_p) = r_e(\chi_q) \tag{1.6}$$

p and q being the vertices of e and r_e the restriction map $R(G) \to R(G_e)$.

For GKM manifolds, one can also translate some aspects of Morse theory into the language of graphs. Recall that ϕ is the moment map on M with respect to the circle T action. Think of each edge e of the graph connecting vertices p and q as two oriented edges e_p and e_q. Then if $\phi(p) > \phi(q)$, we say that the edge e_p going from p to q is *descending* and e_q from q to p *ascending*. If U is the unstable manifold of ϕ at p, then every fixed point, q, inside W_U^G is the terminal point of a path on Γ starting at p and consisting of ascending edges; and this gives one a way of describing W_U in terms of Γ. In particular, we will present below an explicit formula for the image of τ_p under the imbedding (1.3), which expresses the restriction of τ_p to $q \in M^G$ as a sum of combinatorial expressions associated with the ascending paths in Γ going from p to q. (An analogous formula for the cohomological counterpart of τ_p can be found in [**9**].) Our results will follow from the following theorem, which allows one to compute local indices in terms of restrictions of K-theory classes to fixed points and vice versa.

Theorem 1.6. *For $p \in V = M^G$, let $e_1, \dots, e_m$ be the descending edges with initial vertex at p. Let the edge e_i connect p to q_i and be labeled by the weight α_i. Then for any $a \in K_G(M)$ we have*

$$I_p(a) = \sum_{i=1}^{m} \tilde{\pi}_i \tilde{r}_i \left(\frac{a_{q_i}}{(1-\zeta)\prod_{j\neq i}\left(1 - e^{2\pi\sqrt{-1}\alpha_j}\right)} \right) + \frac{a_p}{\prod_{i=1}^{m}(1 - e^{2\pi\sqrt{-1}\alpha_i})}, \tag{1.7}$$

where a_q is the restriction of a to q, ζ is the generator of the character ring $R(T)$, $\tilde{r}_i$ is the restriction $R(G \times T) \to R(G_{e_i} \times T)$ and $\tilde{\pi}_i : R(G_{e_i} \times T) \to R(G)$ is the Gysin map defined in (4.3).

We conclude this introduction with a section-by-section summary of the contents of the paper. In Section 2, we prove Theorem 1.1 by adopting certain arguments from classical Morse theory to the setting of equivariant K-theory. In Section 3, we prove an algebraic counterpart of Theorem 1.6, namely, an analogue for the ring $R(G)$ of the "Lagrange interpolation formula" of [**9**]. More specifically, let T be a circle subtorus of G and $H \subset G$ a complementary subtorus, so that $G = T \times H$. Let $\hat{R}(G)$ be the ring of finite sums $\sum_k c_k z^k$ where c_k is in the quotient ring $Q(H)$ of $R(H)$. Let $w : T \to S^1$ be an isomorphism. Via the splitting $G = T \times H$, we can extend w to a homomorphism of G onto S^1 by setting w equal to 1 on H. Let ξ be the infinitesimal generator of T (chosen so that it corresponds under w to the standard generator $\frac{\partial}{\partial\theta}$ of S^1). The interpolation in question is with respect to weights $\alpha_i \in \mathbb{Z}_G^*$, $i = 1, \dots, m$. Letting G_i be the kernel of the homomorphism

$$e^{2\pi\sqrt{-1}\alpha_i} : G \to S^1$$

it describes to what extent an element f of $R(G)$ is determined by its restrictions to the $R(G_i)$'s. More explicitly, it asserts:

Theorem 1.7. *Assume the weights α_i are pairwise linearly independent and $\alpha_i(\xi) \neq 0$. For an element f of $R(G)$, define $f_i \in \hat{R}(G)$,*

$i = 1, \dots, m$, by

$$f_i(z) = \pi_i r_i \left(\frac{f(w,h)}{(1-\frac{z}{w})\prod_{j\neq i}(1-e^{2\pi\sqrt{-1}\alpha_j})} \right), \tag{1.8}$$

where r_i is the restriction map $R(G) \to R(G_i)$, and π_i is the Gysin map $R(G_i) \to R(H)$ associated with the projection $G_i \to G/T \cong H$ (see (3.3) for definitions). Then

$$f_0 = \frac{f}{\prod_{i=1}^m (1-e^{2\pi\sqrt{-1}\alpha_i})} - \sum_{i=1}^m \operatorname{sgn}(\alpha_i(\xi)) f_i \tag{1.9}$$

is an element of $R(G)$.

In Section 4, we will apply the Atiyah–Segal localization theorem to equivariant K-classes on twisted projective spaces to obtain a formula for the equivariant index of such a class, and in Section 5, we will apply this formula to the twisted projective space, S_ε, and show that the formulas (1.7) and (1.9) are essentially the same formula viewed from different perspectives, i.e., are the topological and algebraic versions of this formula. More specifically, we will show that if p is a vertex of the GKM graph Γ, f the restriction to p of an element a of $K_G(M)$ and the α_i's the weights associated with the descending edges of Γ with initial vertex at p, then the f_0 in (1.9) is just the local index $I_p(a)$. We will then use this result to prove Theorem 1.6.

In Section 6, we will obtain explicit formulas for the τ_p's in terms of their restrictions to the fixed points. These will be proved by a repeated iteration of (1.7), or equivalently of (1.9). (We recall that $I_q(\tau_p) = 0$ if $p \neq q$ and $I_p(\tau_p) = 1$; so (1.7) gives one an effective way of computing τ_p at q in terms of the values of τ_p at the points in M^G lying below q in W_U.) In Section 7, we give a more general definition of the notion of "local index", for which many of the results above still hold with minor modifications. This new definition involves choosing, for each $p \in M^G$, a circle subgroup, T_p, depending on p, and replacing the space, S_ε, by the space obtained by cutting the stable manifold, S of ϕ at p by T_p. If these spaces are manifolds, i.e., don't have orbifold singularities, the formulas (1.7) and (1.9) become considerably simpler (for instance the Gysin maps in these formulas are all identity maps). In particular these formulas are now very similar to the analogous formulas in equivariant cohomology. (See [**9**].) As an application of these results, we discuss this generalized index map for the Grassmannian and explain some tie-ins of our results with recent work of Lenart [12] on Schur and Grothendieck polynomials.

2. Morse theory and equivariant K-theory

We will deduce Theorem 1.1 from the series of lemmas below. These lemmas are K-theoretic analogs of classical results in equivariant Morse theory [2].

As before, M is a compact symplectic manifold with a Hamiltonian G action, T is a generic circle subgroup of G with $M^T = M^G$ and ϕ is the moment map of the T action. For a critical point p of ϕ, that is, $p \in M^G$, pick a G-invariant complex structure on the tangent space T_p at p compatible with the symplectic structure. Then T_p splits into the negative and positive components T_p^- and T_p^+ on which the circle T acts with negative and positive weights respectively. Let Λ_p^- be the virtual vector space $\sum(-1)^k\Lambda^k(T_p^-)$ with its given G action. By definition, this is an element of $R(G) \cong K_G(p)$. Moreover, if $\alpha_1, \dots, \alpha_m$ are the (possibly repeating) weights of the G action on T_p^-, then, as a virtual character of G

$$\Lambda_p^- = \prod_i (1 - e^{2\pi\sqrt{-1}\alpha_i}). \tag{2.1}$$

Recall that $K_G^n(M)$ is the compactly supported K-group, $K_{G,c}(M \times \mathbb{R}^n)$ and that $K_G(M) = K_G^0(M)$. Moreover, by Bott periodicity

$$K_G^n(M) \cong K_G^{n+2}(M).$$

For a critical point p, let $\phi(p) = c$. Assume there is only one critical point p in $\phi^{-1}(c)$. (We may do so without loss of generality since there is always a small G-invariant perturbation of ϕ which is a Morse function with the same stable and unstable manifolds and which satisfies the above property.) For a small $\varepsilon > 0$, let

$$M_p^+ = \{x \in M | \phi(x) \leq c + \varepsilon\} \quad \text{and} \quad M_p^- = \{x \in M | \phi(x) \leq c - \varepsilon\}.$$

Lemma 2.1. *The* K*-theory long exact sequence for the pair* (M_p^+, M_p^-) *splits into short exact sequences*

$$0 \to K_G^*(M_p^+, M_p^-) \to K_G^*(M_p^+) \to K_G^*(M_p^-) \to 0. \tag{2.2}$$

Proof. Let S be the stable manifold at p and $S^- = S \cap M_p^-$. Then there are isomorphisms

$$K_G^*(p) \xrightarrow{\mathcal{T}} K_G^*(S, S^-) \xrightarrow{\mathcal{H}} K_G^*(M_p^+, M_p^-),$$

where $\mathcal{T}$ is the Thom isomorphism and $\mathcal{H}$ comes from homotopy equivalence.

To show that the long exact sequence splits, it is enough to show that the maps

$$\mathcal{J} : K_G^*(M_p^+, M_p^-) \to K_G^*(M_p^+)$$

are all injective.

Let ι_p be the inclusion of p into M^+. It is well known that for $* = 0$, the map

$$\iota_p^* \circ \mathcal{J} \circ \mathcal{H} \circ \mathcal{T} : K_G^0(p) \to K_G^0(p)$$

is a multiplication by Λ_p^- and hence injective, since by (2.1) Λ_p^- is not a zero divisor in $R(G)$. Therefore, the map $\mathcal{J}$ must also be injective for $* = 0$.

For $* = 1$, notice that by the Thom isomorphism

$$K_G^1(S, S^-) = K_{G,c}^0(\mathbb{R}),$$

where the action of G on $\mathbb{R}$ is trivial. Since the non-equivariant K-space, $K^1(\mathrm{pt}) = K_c^0(\mathbb{R})$ is known to be trivial, it is easy to conclude that $K_{G,c}^0(\mathbb{R})$ is trivial as well. Indeed, by splitting each vector bundle on $\mathbb{R}$ into bundles for which the action of G on the fibers is given by a single weight, we reduce the calculation of $K_{G,c}^0(\mathbb{R})$ to the calculation of $K_c^0(\mathbb{R})$. This implies that $K_G^1(S, S^-)$ is trivial, and that $\mathcal{J}$ is injective, and hence finishes the proof. q.e.d.

Corollary 2.2. *The restriction*

$$K_G(M) \to K_G(M^G)$$

is injective.

Proof. For every critical point p, it is enough to show that if the restriction

$$K_G(M_p^-) \to K_G(M_p^- \cap M^G) \tag{2.3}$$

is injective then

$$K_G(M_p^+) \to K_G(M_p^+ \cap M^G) \tag{2.4}$$

is also injective.

The long exact sequence of the pair $(M_p^+ \cap M^G, M_p^- \cap M^G)$ obviously splits into short exact sequences. By Lemma 2.1, the short exact sequence (2.2) maps into the corresponding short exact sequence for $(M_p^+ \cap M^G, M_p^- \cap M^G)$. The restriction

$$K_G(M_G^+, M_G^-) \to K_G(p)$$

is an injection, since Λ_p^- is not a zero divisor. So, the injectivity of (2.3) together with the five-lemma implies the injectivity of (2.4). q.e.d.

Recall that the elements of M^G are partially ordered with $p \leq q$ if q lies in the closure of the unstable manifold at p. Completing this relation by transitivity, one gets a partial order on M^G. The proof of the following lemma is postponed until Section 5, where we will deduce it from the Atiyah–Segal localization theorem for weighted projective spaces.

Lemma 2.3. *Assume $\tau \in K_G(M)$ restricts to zero at every $q \in M^G$ with $q < p$. Then*

$$\tau(p) = I_p(\tau)\Lambda_p^-$$

where $\tau(p)$ is the restriction of τ to p.

Lemma 2.4. *For every critical point p of ϕ, there exists an element τ of $K_G(M)$ which restricts to zero at every $q \in M^G$ with $\phi(q) < \phi(p)$ (in other words, τ is supported above p), such that $I_p(\tau) = 1$.*

Proof. By Lemma 2.3, it is enough to construct a K-class, τ, supported above p whose restriction to the point p is Λ_p^-. By Lemma 2.1, such an element exists in $K_G(M_p^+)$. Moreover, by induction such an element exists in $K_G(M_q^+)$ for every q with $\phi(q) > \phi(p)$. Indeed, because of the short exact sequence (2.2) for the pair (M_q^+, M_q^-) if such an element exists in $K_G(M_q^-)$ it can be lifted to an element in $K_G(M_q^+)$. q.e.d.

Proof of Theorem 1.1. Let $p_1, \dots, p_r$ be the points of M^G ordered such that $\phi(p_1) \geq \phi(p_2) \geq \cdots \geq \phi(p_r)$. Let us prove by induction on k that classes τ_{p_k} satisfying (1.1), (1.1) and (1.1) exist. For $k = 1$, this follows from Lemmas 2.3 and 2.4. Assume we can construct classes $\tau_{p_1}, \dots, \tau_{p_{k-1}}$ satisfying these properties. By Lemma 2.4, we can choose a class τ supported above p with $I_{p_k}(\tau) = 1$. By Lemma 2.3, we conclude that $I_{p_\ell}(\tau) = 0$ for every $\ell > k$. Choose the largest $\ell < k$ with $I_{p_\ell}(\tau) \neq 0$. Then change τ to $\tau - I_{p_\ell}(\tau)\tau_{p_\ell}$. It is clear that the new τ satisfies

$I_{p_\ell}(\tau) = \delta_k^\ell$ for $m \geq \ell$. Hence by induction, we can find a class τ_{p_k} such that $I_{p_\ell}(\tau_{p_k}) = \delta_k^m$ for all m, implying (1.1) and (1.1).

To show that such τ_{p_k} satisfies (1.1), assume that for every $m < \ell$ either $p_m \geq p_k$, or $\tau_{p_k}(p_m) = 0$. We will now show that if $p_\ell \not\geq p_k$, then $\tau_{p_k}(p_\ell) = 0$. Notice that every $p < p_\ell$ is among the points $p_{\ell+1}, \ldots, p_r$ satisfying $p \not\geq p_k$ so that $\tau_{p_k}(p) = 0$ by the above assumption. Hence $\tau_{p_k}(p_\ell) = 0$ by Lemma 2.3.

To prove uniqueness of the classes τ_p, it remains to be shown that if both τ_p and τ'_p satisfy (1.1)–(1.1), then they must be equal. Set $\delta = \tau_p - \tau'_p$. Then δ is a K-theory class whose local indices are all zero. By Corollary 2.2, if δ were not zero, there would exist a critical point q such that the restriction of δ to q is non-zero. Pick such q with minimal $\phi(q)$. Then by Lemma 2.3, the local index $I_q(\delta)$ is not zero.

Finally, we need to show that the τ_p's generate $K_G(M)$ freely as a $K_G(\mathrm{pt})$ module. We first prove that every $a \in K_G(M)$ can be expressed as a linear combination of τ_p's with coefficients in $R(G)$. To do so, order the points of M^G as above. We will prove by induction on k that if a restricts to zero at $p_{k+1}, \ldots, p_r$, then a is a linear combination of $\tau_{p_1}, \ldots, \tau_{p_k}$. Clearly this holds for $k = 1$. To prove the induction step for $k > 1$, notice that if a is supported above p_k the class $a - I_{p_k}(a)\tau_{p_k}$ restricts to zero at $p_k, \ldots, p_r$ and by the induction assumption is a linear combination of $\tau_{p_1}, \ldots, \tau_{p_{k-1}}$. This proves the induction step. It remains to be shown that no non-trivial linear combination, $\gamma = \sum c_k \tau_{p_k}$, with $c_k \in R(G)$ is zero. Because of Corollary 2.2, it is enough to show that γ restricts non-trivially to M^G whenever one of the c_k's in not zero. Pick the largest k with $c_k \neq 0$. Then by Lemma 2.3, the restriction of δ to p_k is not zero. This finishes the proof of Theorem 1.1. q.e.d.

Proof of Corollary 1.2. The injectivity of $\mathcal{I}$ follows from the fact that $a \in K_G(M)$ must be equal to $a' \in K_G(M)$ whenever $I_p(a) = I_p(a')$ for all $p \in M^G$, which is true, since by the argument used in the proof of Theorem 1.1 the class $a - a'$ must vanish.

To prove surjectivity we must show that for any choice of $i_p \in R(G)$ there exists a class $a \in K_G(M)$ with $I_p(a) = i_p$. Order the points of M^G as in the proof of Theorem 1.1. Take $a = \sum_{k=1}^r c_k \tau_{p_k}$ with

$$c_k = i_{p_k} - I_{p_k}\left(\sum_{\ell=k+1}^{r} c_\ell \tau_{p_\ell}\right).$$

Then it is easy to see that $I_p(a) = i_p$ for every p. q.e.d.

3. Lagrange interpolation

We recall the statement of the classical Lagrange interpolation formula for the ring of polynomials in one variable and then describe how to generalize this formula to the representation ring $R(G)$.

Let $f(z) = \sum_{k=0}^{N} c_k z^k$ be a polynomial in z with complex coefficients. Given d distinct complex numbers, $a_i \in \mathbb{C}$, $i = 1, \ldots, d$ one has

Lemma 3.1.

$$f_0(z) = \frac{f(z)}{\prod_i (z - a_i)} - \sum_i \frac{c_i}{z - a_i} \tag{3.1}$$

is a polynomial of degree $N - d$, where

$$c_i = \frac{f(a_i)}{\prod_{j \neq i}(a_i - a_j)}. \tag{3.2}$$

Proof. The function

$$g(z) = \frac{\sum_i f(a_i) \prod_{j \neq i}(z - a_j)}{\prod_{j \neq i}(a_i - a_j)}$$

is a polynomial of degree at most $d - 1$ which takes the same values as $f(z)$ at the points $z = a_i$, so $f(z) - g(z)$ is divisible by $\prod_i (z - a_i)$. q.e.d.

A slightly more complicated variant of this identity is the following. Let $k_1, \ldots, k_d$ be positive integers. Given d non-zero complex numbers a_i, satisfying $a_i^{k_j} \neq a_j^{k_i}$ for $i \neq j$, define the "Gysin map" π_i, which acts on rational functions in two variable z and w by

$$\pi_i(h(z, w)) = \frac{1}{k_i} \sum_{\ell=1}^{k_i} h(z, w_{i,\ell}) \tag{3.3}$$

summed over the roots $w_{i,\ell}$ of $z^{k_i} - a_i$. Then, we have the following:

Lemma 3.2.

$$f_0(z) = \frac{f(z)}{\prod_i (z^{k_i} - a_i)} - \sum_i \frac{f_i(z)}{z^{k_i} - a_i} \tag{3.4}$$

is a polynomial of degree $N - \sum k_i$*, where* f_i *is the following polynomial of degree at most* $k_i - 1$

$$\frac{z^{k_i} - a_i}{a_i}\pi_i\left(\frac{f(w)w}{(z-w)\prod_{j\neq i}(w^{k_j} - a_j)}\right). \tag{3.5}$$

Moreover, if $f = \sum_{i=0}^{N} c_i z^i$ *then* $f_0 = \sum_{j=0}^{N-k} d_j z^j$*, where* $k = \sum k_i$ *and*

$$d_j = \sum_{i=k+j}^{N} c_i \left(\sum_{\ell_1 k_1 + \cdots + \ell_d k_d = i-k-j} a_1^{\ell_1} \ldots a_d^{\ell_d}\right). \tag{3.6}$$

Proof. Factoring

$$z^{k_i} - a_i = \prod_{\ell=1}^{k_i}(z - w_{i,\ell})$$

and applying (3.1) we get

$$f_0 = \frac{f(z)}{\prod_i (z^{k_i} - a_i)} - \sum_{i=1}^{d} h_i,$$

where

$$h_i(z) = \sum_{\ell=1}^{k_i} \frac{f(w_{i,\ell})}{z - w_{i,\ell}} \cdot \frac{1}{\left(\prod_{m\neq\ell}(w_{i,\ell} - w_{i,m})\right)\left(\prod_{j\neq i, 1\leq m\leq k_j}(w_{i,\ell} - w_{j,m})\right)}.$$

But

$$\prod_{m\neq\ell}(w_{i,\ell} - w_{i,m}) = \lim_{w\to w_{i,\ell}} \frac{w^{k_i} - a_i}{w - w_{i,\ell}} = k_i w^{k_i - 1}|_{w_{i,\ell}} = \frac{k_i a_i}{w_{i,\ell}}$$

and

$$\prod_{j\neq i, 1\leq m\leq k_j}(w_{i,\ell} - w_{j,m}) = \prod_{j\neq i}(w_{i,\ell}^{k_j} - a_j).$$

So

$$h_i(z) = \frac{1}{k_i}\sum_{\ell=1}^{k_i} g_i(z, w_{i,\ell}) = \pi_i(g_i(z,w)),$$

where

$$g_i(z,w) = \frac{1}{a_i}\frac{f(w)w}{(z-w)\prod_{i\neq j}(w^{k_j} - a_j)}.$$

This proves (3.4) and (3.5). The fact that f_i is of degree at most $k_i - 1$ is an easy consequence of (3.5) and (3.3).

To prove (3.6), expand both sides of (3.4) in powers of z. In particular, note that

$$\frac{f(z)}{\prod_i (z^{k_i}-a_i)} = \sum_{i=0}^{N} c_i z^{i-k} \prod_{i=1}^{d}\Big(1-\frac{a_i}{z^{k_i}}\Big)^{-1} = \sum_{i=0}^{N} c_i \left(\sum_{\ell_1,\dots,\ell_d=0}^{\infty} a_1^{\ell_1}\dots a_d^{\ell_d}\right) z^{i-k-\sum \ell_i k_i}. \tag{3.7}$$

It is clear from (3.4) that f_0 is the "polynomial part" of this expression. Indeed the expansion of $(f_i(z))/(z^{k_i}-a_i)$ only involves negative powers of z and f_0 is a polynomial in z. Therefore, f_0 is the sum of the terms in the expression (3.7) for which the exponent $i-k-\sum \ell_i k_i$ is greater than or equal to zero. Hence its coefficients are given by (3.6). q.e.d.

Remark 3.3. Notice that if we let f to be a Laurent polynomial $f=\sum_{i=-N}^{M} c_k z^k$ with $c_k\in\mathbb{C}$, then Lemmas 3.1 and 3.2 hold after substituting the statements that f_0 and f_i's are polynomials of certain degree by claims that f_0 and f_i's are Laurent polynomials.

A third variant of this formula involves the character ring of the group $\mathbb{C}^*=\mathbb{C}-\{0\}$, i.e., finite sums of the form $f(z)=\sum_{i=-N}^{M} c_k z^k$ with $c_k\in\mathbb{C}$. It asserts that if $k_1,\dots,k_d$ are non-zero integers and $b_1,\dots,b_d$ are non-zero complex numbers satisfying $b_i^{k_j}\neq b_j^{k_i}$ for $i\neq j$, then

Lemma 3.4.

$$f_0(z)=\frac{f(z)}{\prod_i(1-b_iz^{k_i})}+\sum_i \operatorname{sgn}(k_i)\frac{f_i(z)}{1-b_iz^{k_i}} \tag{3.8}$$

as well as $f_1,\dots,f_d$ are in the character ring of $\mathbb{C}^*$, where

$$f_i(z)=(1-b_iz^{k_i})\pi_i\left(\frac{f(w)w}{(z-w)\prod_{j\neq i}(1-b_jw^{k_j})}\right). \tag{3.9}$$

Here, if $k_i>0$ then π_i is defined as in (3.3) with $a_i=1/b_i$, and if $k_i<0$ then π_i for a rational function $h(z,w)$ is given by

$$\pi_i(h(z,w)) = -\frac{1}{k_i}\sum_{\ell=1}^{-k_i} h(z,w_{i,\ell}) \tag{3.10}$$

summed over the roots $w_{i,\ell}$ of $(z^{-k_i}-b_i)$.

Proof. This is easily deduced from (3.4) and (3.5) by setting

$$\begin{aligned} 1 - b_i z^{k_i} &= z^{k_i}(z^{-k_i} - b_i) \quad \text{for } k_i \text{ negative and} \\ &= -b_i(z^{k_i} - \frac{1}{b_i}) \quad \text{for } k_i \text{ positive} \end{aligned}$$

and applying (3.4) to the function

$$g = f \prod_{k_i > 0} \left(\frac{-1}{b_i}\right) \prod_{k_i < 0} z^{-k_i}$$

instead of f, as allowed by Remark 3.3. q.e.d.

Our last version of Lagrange interpolation is a multidimensional generalization of the character formula (3.8) which leads to a proof of Theorem 1.7. As in Theorem 1.7, let G be an n-dimensional torus, $R(G)$ the character ring of G and α_i, $i = 1, \dots, d$ elements of the weight lattice of G. Corresponding to each α_i one has a homomorphism

$$e^{2\pi\sqrt{-1}\alpha_i} : G \to S^1.$$

Let G_i be the kernel of this homomorphism and let T be a circle subgroup of G. Assume ξ is the infinitesimal generator of T. Fixing a complimentary subtorus H to T in G we have

$$G = T \times H = S^1 \times H \subseteq \mathbb{C}^* \times H.$$

Hence one can regard elements of $R(G)$ as a finite sum $f = \sum_{k=-N}^{M} c_k z^k$ with $c_k \in R(H)$.

Applying Lemma 3.4 to the sum $\sum_{k=-N}^{M} c_k z^k$, we get an identity of the form (3.8), f_0 and $f_1, \dots, f_d$ being polynomials with coefficients in the quotient field of $R(H)$. Moreover, it is clear from (3.6) that the coefficients of f_0 are actually in the ring $R(H)$ itself; i.e. modulo the splitting, $G = T \times H$, f_0 is in $R(G)$.

Proof of Theorem 1.7. Let

$$1 - b_i z^{k_i} = 1 - z^{k_i} e^{2\pi\sqrt{-1}\beta_i} = 1 - e^{2\pi\sqrt{-1}\alpha_i}$$

where $b_i = e^{2\pi\sqrt{-1}\beta_i}$, β_i is the restriction of α_i to H, and $k_i = \alpha_i(\xi)$. Notice that pairwise linear independence of the weights α_i guarantees that $b_i^{k_j} \neq b_j^{k_i}$ for $i \neq j$. Theorem 1.7 then follows from the identity of the form (3.8), with f_0 and $f_1, \dots, f_d$ being polynomials with coefficients in the quotient field of $R(H)$, and the identity

$$\pi_i(b_j w^{k_j}) = \pi_i r_i(e^{2\pi\sqrt{-1}\alpha_j}), \tag{3.11}$$

whose left-hand side is defined by either (3.3) or (3.10) depending on $\operatorname{sgn}(k_i)$. q.e.d.

4. Atiyah–Segal localization for twisted projective spaces

This section describes in detail the twisted projective spaces which arise as symplectic cuts of stable manifolds. It also discusses the Atiyah–Segal localization theorem on these twisted projective spaces.

Let $\alpha_1, \dots, \alpha_m$ be weights of the torus G, such that if ξ is the infinitesimal generator of the circle subgroup T, then $k_i = \alpha_i(\xi) \neq 0$ for $i = 1, \dots, m$. Assume all k_i's are negative. (At the end of this section, we discuss the case when some k_i's are positive.) Let T act on $\mathbb{C}^{m+1}$ with weights $k_1, \dots, k_m, k_{m+1} = -1$. Let $T_{\mathbb{C}}$ be the complexification of T which acts on $\mathbb{C}^{m+1}$ with the same weights. The twisted projective space we are interested in is the orbifold

$$\widetilde{\mathbb{CP}}^m = \mathbb{C}^{m+1}//T_{\mathbb{C}} = (\mathbb{C}^{m+1} - \{0\})/T_{\mathbb{C}}.$$

We will not review orbifold theory, but refer the reader to [**17**] for an exposition of orbifold theory and further references.

To define local orbifold charts on $\widetilde{\mathbb{CP}}^m$, let $\tilde{U}_i$ be the m-dimensional affine space with coordinates $(z_1, \dots, \hat{z}_i, \dots, z_{m+1})$. Denote by U_i the open subset of $\widetilde{\mathbb{CP}}^m$ where the projective coordinate z_i is not zero. Define the map $\phi_i : \tilde{U}_i \to U_i$ by

$$\phi_i(z_1, \dots, \hat{z}_i, \dots, z_{m+1}) = [z_1, \dots, 1, \dots, z_{m+1}].$$

Let Γ_i be the finite abelian group of k_ith roots of unity. Then $w \in \Gamma_i$ acts on $\tilde{U}_i$ by

$$w \cdot (z_1, \dots, \hat{z}_i, \dots, z_{m+1}) = (w^{k_1} z_1, \dots, \hat{z}_i, \dots, w^{k_{m+1}} z_{m+1})$$

so that $\tilde{U}_i/\Gamma_i = U_i$. The triples $(\tilde{U}_i, \Gamma_i, \phi_i)$ are the orbifold charts of $\widetilde{\mathbb{CP}}^m$.

Assume that G acts on $\mathbb{C}^{m+1}$ with weights $\alpha_1, \dots, \alpha_m, \alpha_{m+1} = 0$, so that this action descends to an action on $\widetilde{\mathbb{CP}}^m$. To make sure that this action has a finite fixed point set, we assume that the weights $\alpha_1, \dots, \alpha_m$ are pairwise linearly independent. Then the fixed points of the G-action on $\widetilde{\mathbb{CP}}^m$ are points $p_i = [0, \dots, 1, \dots, 0]$ with 1 in the ith place. An element $g = \exp(\eta) \in G$ acts on U_i by

$$\begin{aligned} g \cdot [z_1, \dots, 1, \dots, z_{m+1}] &= \left[e^{2\pi\sqrt{-1}\alpha_1(\eta)} z_1, \dots, e^{2\pi\sqrt{-1}\alpha_i(\eta)}, \dots, e^{2\pi\sqrt{-1}\alpha_m(\eta)} z_m, z_{m+1} \right] \\ &= \Big[e^{2\pi\sqrt{-1}(\alpha_1(\eta) - \frac{k_1\alpha_i(\eta)}{k_i})} z_1, \dots, 1, \dots, \\ &\qquad e^{2\pi\sqrt{-1}(\alpha_m(\eta) - \frac{k_m\alpha_i(\eta)}{k_i})} z_m, e^{\frac{2\pi\sqrt{-1}\alpha_i(\eta)}{k_i}} z_{m+1} \Big]. \end{aligned}$$

So, the isotropy action of G at p_i is given by the rational weights $\alpha_j - (k_j\alpha_i/k_i)$ for $j \neq i$, $1 \leq j \leq m$ and the rational weight α_i/k_i. Another way to think about those weights is the following. Notice that if G_i is the kernel of the map $e^{2\pi\sqrt{-1}\alpha_i} : G \to S^1$ and r_i is the restriction $R(G) \to R(G_i)$, then

$$e^{2\pi\sqrt{-1}(\alpha_j - \frac{k_j\alpha_i}{k_i})} = r_i(e^{2\pi\sqrt{-1}\alpha_j}).$$

In particular, the rational weights $\alpha_j - (k_j\alpha_i/k_i)$ are genuine integer weights of G_i. A similar computation shows that the weights of the action of G at the fixed point $p_{m+1} = [0, \dots, 0, 1]$ are $\alpha_1, \dots, \alpha_m$.

We will now explicitly compute the pushforward map in equivariant K-theory,

$$\mathrm{ind}_G : K_G(\widetilde{\mathbb{CP}}^m) \to K_G(\mathrm{pt})$$

using the Atiyah–Segal localization theorem [**3**]; however, before we do this in general, we will first consider the situation when all k_i's are equal to minus one, and $\widetilde{\mathbb{CP}}^m$ is just the standard projective space $\mathbb{CP}^m$.

Recall that on a G-manifold X, the index map (or K-theoretic pushforward)

$$\mathrm{ind}_G : K_G(X) \to K_G(\mathrm{pt}) \simeq R(G)$$

is defined by imbedding X into a linear complex representation space V of G, applying the Thom isomorphism to map $K_G(X)$ to $K_G(V)$ and then using Bott periodicity to identify $K_G(V)$ with $K_G(\mathrm{pt})$. We also recall that by the Atiyah–Segal localization theorem, the restriction map

$$K_G(X) \to K_G(X^G)$$

becomes an isomorphism after localizing with respect to a certain prime ideal of $R(G)$ (see [**3**]), and that it is possible to write an explicit formula for $\mathrm{ind}_G(\delta)$ in terms of this restriction. In the case $X = \mathbb{CP}^m$, this formula says that

$$\text{(4.1)} \qquad \mathrm{ind}_G(\delta) = \sum_{i=1}^{m} \frac{\delta_i}{(1-e^{2\pi\sqrt{-1}\alpha_i})\prod_{j\neq i}(1-e^{2\pi\sqrt{-1}(\alpha_j-\alpha_i)})} + \frac{\delta_{m+1}}{\prod_{j=1}^{m}(1-e^{2\pi\sqrt{-1}\alpha_j})},$$

where δ_i is the restriction of δ to p_i, and the denominators in the formula are just the virtual characters of the exterior algebra complexes $\sum(-1)^k\Lambda^k T_{p_i}$ of the tangent spaces T_{p_i} at the fixed points.

We now drop the assumption that all k_j's are equal to minus one, so that $\widetilde{\mathbb{CP}}^m$ may have orbifold singularities. Let us recall the definition of the equivariant index map for orbifolds (for more details see [**10**] or [**16**]). Assume an orbifold X is presented as a quotient Y/K of a manifold Y by a locally free action of a compact group K action. (We will describe such a presentation of $\widetilde{\mathbb{CP}}^m$ shortly.) Moreover, assume G acts on Y and commutes with the K action, then the action of G descends to an action on X. It is well known that there exists an isomorphism (which for purposes of this paper will be treated as a definition of $K_G(X)$)

$$\Psi : K_G(X) \xrightarrow{\simeq} K_{G\times K}(Y).$$

Then for $\delta \in K_G(X)$, we define its index by

$$\mathrm{ind}_G^X \delta = \Big(\mathrm{ind}_{G\times K}^Y\big(\Psi(\delta)\big)\Big)^K \in R(G),$$

the K invariant part of the $G\times K$ index of $\Psi(\delta)$. The relative version of the localization theorem on Y produces a localization formula for orbifolds, which expresses $\mathrm{ind}_G^X(\delta)$ in terms of the restriction of δ to X^G. We will not give the general version of this formula, since we will only apply it to the case of the weighted projective space $\widetilde{\mathbb{CP}}^m$. So from this point on, we specialize to the case $X = \widetilde{\mathbb{CP}}^m$.

The twisted projective space $\widetilde{\mathbb{CP}}^m$ can be realized as a the symplectic reduction of $\mathbb{C}^{m+1}$ by the action of T, which is just the quotient S^{2m+1}/T, where S^{2m+1} is the sphere

$$S^{2m+1} = \{(z_1,\dots,z_{m+1}) \in \mathbb{C}^{m+1} | \sum k_i|z_i|^2 = -1\}$$

on which the circle T acts locally freely. So, as mentioned above, there is an isomorphism,

$$\Psi : K_G(\widetilde{\mathbb{CP}}^m) \xrightarrow{\simeq} K_{G\times T}(S^{2m+1}).$$

For $\delta \in K_G(\widetilde{\mathbb{CP}}^m)$, we are interested in computing the K-theoretic index

$$\mathrm{ind}_G^{\widetilde{\mathbb{CP}}^m} \delta = \left(\mathrm{ind}_{G\times T}^{S^{2m+1}} \Psi(\delta)\right)^T \in R(G)$$

by means of Atiyah–Segal localization. Recall that the fixed points of the G action on $\widetilde{\mathbb{CP}}^m$ are the points $p_1, \dots, p_{m+1}$. Denote by ι_i the inclusion $p_i \to \widetilde{\mathbb{CP}}^m$. Let s_i be the circle inside S^{2m+1} which after dividing by T becomes p_i and let $\tilde{\iota}_i : s_i \to S^{2m+1}$ be the natural inclusion. The stabilizer of the $G \times T$ action on s_i is the group $G_i \times T'$, where T' is the subgroup $\{(t, t^{-1}) | t \in T\}$ of $G \times T$ and G_i is the kernel of $e^{2\pi\sqrt{-1}\alpha_i} : G \to S^1$. Then

$$K_{G\times T}(s_i) = K_{G_i\times T'}(p_i) \cong R(G_i \times T').$$

We now define maps $\tilde{r}_i$ and $\tilde{\pi}_i$, which, as we will explain below, are extensions of the maps, r_i and π_i of Section 3. The map $\tilde{r}_i$ will be the restriction homomorphism

$$R(G \times T') \to R(G_i \times T'), \tag{4.2}$$

which clearly extends r_i, the restriction map from $R(G)$ to $R(G_i)$.

Consider the covering map $\rho : G_i \times T' \to G$ which sends $(g, (t, t^{-1})) \in G \times T'$ to $gt \in G$. The kernel of this map is the group $\Gamma_i \cong G_i \cap T$ which can be identified with the k_ith roots of unity, so that

$$G_i \times T'/\Gamma_i = G.$$

The Gysin map

$$\begin{aligned} \tilde{\pi}_i : K_{G\times T}(s_i) = K_{G_i\times T'}(\mathrm{pt}) = R(G_i \times T') \\ \to R(G) = K_G(\mathrm{pt}) = K_G(p_i) \end{aligned} \tag{4.3}$$

is defined as follows. Let V be a virtual representation for $G_i \times T'$, and let $\tilde{\pi}_i(V)$ be the subspace V^{Γ_i} of V fixed by Γ_i. Since $G = G_i \times T'/\Gamma_i$, the space V^{Γ_i} is a virtual representation space for G, so $\tilde{\pi}_i$ is well defined. Again, this map is clearly an extension of the map $\pi_i : K_{G_i}(\mathrm{pt}) \to K_H(\mathrm{pt})$ defined in (3.11), where H is a subtorus for which $G = H \times T$.

The key ingredient in the Atiyah–Segal localization formula is the fact that the composition of the pushforward map $\tilde{\iota}_{i!}$ and the restriction $\tilde{\iota}^*$

$$K_{G_i\times T'}(\mathrm{pt}) = K_{G\times T}(s_i) \xrightarrow{\tilde{\iota}_{i!}} K_{G\times T}(S^{2m+1}) \xrightarrow{\tilde{\iota}_i^*} K_{G\times T}(s_i) = K_{G_i\times T'}(\mathrm{pt})$$

is multiplication by the character of the exterior algebra complex

$$\sum (-1)^k \Lambda^k T_{p_i}$$

of the tangent space at p_i, which is just

$$\tilde{r}_i\left((1-\zeta)\prod_{j\neq i}\left(1-e^{2\pi\sqrt{-1}\alpha_j}\right)\right), \tag{4.4}$$

where ζ is the generator of the character ring $R(T')$. (Recall that G acts on T_{p_i} with the rational weights, but the group $G_i \times T'$, a cover of G, acts on T_{p_i} with integer weights. So, in (4.4) ζ is the character of the representation of T' associated with the rational weight α_i/k_i of G.)

The Atiyah–Segal localization theorem states that the map $\tilde{\iota}_! = \sum \tilde{\iota}_{i!}$ becomes an isomorphism after localization. Thus, together with the fact that $\tilde{\iota}^*\tilde{\iota}_{i!}$ is multiplication by (4.4), it implies that

$$\begin{aligned}\mathrm{ind}_{G\times T}^{S^{2m+1}}(\Psi(\delta)) = \sum_{i=1}^m &\frac{\tilde{\iota}_i^*\Psi(\delta)}{\tilde{r}_i\Big((1-\zeta)\prod_{j\neq i}\left(1-e^{2\pi\sqrt{-1}\alpha_j}\right)\Big)}\\ &+ \frac{\tilde{\iota}_{m+1}^*\Psi(\delta)}{\prod_{i=1}^m(1-e^{2\pi\sqrt{-1}\alpha_i})}.\end{aligned}$$

It remains to take T invariants of both part of this formula. Notice that taking the T-invariant part of a K-class in $K_{G\times T}(s_i) = K_{G_i\times T'}(\mathrm{pt})$ is equivalent to applying the map $\tilde{\pi}_i$ to this class. Hence we get

Lemma 4.1. *For $\delta \in K_G(\widetilde{\mathbb{CP}}^m)$*

$$\begin{aligned}\mathrm{ind}_G^{\widetilde{\mathbb{CP}}^m}(\delta) = \sum_{i=1}^m \tilde{\pi}_i &\left(\frac{\tilde{\iota}_i^*\Psi(\delta)}{\tilde{r}_i\left((1-\zeta)\prod_{j\neq i}\left(1-e^{2\pi\sqrt{-1}\alpha_j}\right)\right)}\right)\\ &+ \frac{\tilde{\iota}_{m+1}^*\Psi(\delta)}{\prod_{i=1}^m(1-e^{2\pi\sqrt{-1}\alpha_i})}.\end{aligned} \tag{4.5}$$

In case not all the numbers k_i are negative, assume that for the first r weights, $\alpha_1, \dots, \alpha_r$, these numbers are positive and for the others are negative. Then apply Lemma 4.1 to the twisted projective space defined for the weights $-\alpha_1, \dots, -\alpha_r, \alpha_{r+1}, \dots, \alpha_m$ and the equivariant K-theory class $\delta \prod_{i=1}^r e^{-2\pi\sqrt{-1}\alpha_i}$. This, after an easy computation, yields

Lemma 4.2. *For* $\delta \in K_G(\widetilde{\mathbb{CP}}^m)$

$$\begin{aligned}(-1)^r \mathrm{ind}_G^{\widetilde{\mathbb{CP}}^m}&(\delta \prod_{i=1}^r e^{-2\pi\sqrt{-1}\alpha_i}) \\ &= \sum_{i=1}^m \mathrm{sgn}(-k_i)\tilde{\pi}_i \left(\frac{\tilde{\iota}_i^* \Psi(\delta)}{\tilde{r}_i \left((1-\zeta) \prod_{j \neq i} \left(1 - e^{2\pi\sqrt{-1}\alpha_j}\right)\right)} \right) \\ &\quad + \frac{\tilde{\iota}_{m+1}^* \Psi(\delta)}{\prod_{i=1}^m (1 - e^{2\pi\sqrt{-1}\alpha_i})}.\end{aligned}$$

5. Calculation of local indices for GKM spaces

In this section, we apply Lemma 4.1 to the twisted projective spaces, S_ε, to prove Lemma 2.3 and Theorem 1.6. We also describe the relationship between formulas (1.9) and (4.5).

Let us recall the definition of symplectic cuts of stable manifolds. Assume (M, ω) is a Hamiltonian G-space. Choose a generic circle subgroup T of G such that $M^T = M^G$. Let p be an isolated fixed point and S the stable manifold of p. Let ω_S be the restriction of ω to S, and consider the space $S \times \mathbb{C}$ with the symplectic form $(w_S, -\sqrt{-1}dy \wedge d\bar{y})$, where y is the complex coordinate on $\mathbb{C}$. If T acts on $\mathbb{C}$ with the weight -1, the action of T on $S \times \mathbb{C}$ is Hamiltonian. Restrict the moment map ϕ of the T action to S. If $\phi(p) = c$, the moment map for the action of T on $S \times \mathbb{C}$ is

$$\tilde{\phi}(x, y) = \phi(x) - c - |y|^2 \,,$$

and the symplectic cut S_ε is the symplectic reduction of $S \times \mathbb{C}$ at $-\varepsilon$. For more details and an explanation of why this definition of S_ε coincides with definition (1.1) see [**13**].

The Kirwan map

$$\tilde{\kappa}_\varepsilon : K_{G\times T}(S \times \mathbb{C}) \to K_G(S_\varepsilon)$$

is the composition of the restriction of a class in $K_{G\times T}(S\times\mathbb{C})$ to $\tilde{\phi}^{-1}(\varepsilon)$ and the identification of $K_{G\times T}(\tilde{\phi}^{-1}(\varepsilon))$ with $K_G(S_\varepsilon)$. As before, let T' be the circle subgroup of $G\times T$ given by $\{(t,t^{-1})|t\in T\}$, so that there is a canonical identification $G\times T\cong G\times T'$. Notice that T' acts trivially on S, hence

$$K_{G\times T}(S\times\mathbb{C}) = K_{G\times T'}(S\times\mathbb{C}) = K_G(S)\otimes K_{T'}(\mathbb{C}).$$

Using this identification, define, for a class, $a\in K_G(M)$

$$\kappa_\varepsilon(a) = \tilde{\kappa}_\varepsilon(a_S\otimes 1),$$

where a_S is the restriction of a to S. Then $I_p(a)$ is just the G-equivariant index of $\tilde{\kappa}_\varepsilon(a_S\otimes 1)$.

Remark 5.1. We will explain why I_p is an $R(G/T)$ (rather than $R(G)$) module homomorphism. There are two $R(G)$ module structures on $K_{G\times T}(S\times\mathbb{C})$: one coming from the projection of $G\times T$ onto the first factor and the other coming from the multiplication map $G\times T\to G$, which maps $(g,t)\mapsto g\cdot t^{-1}$, and the map $\tilde{\kappa}_\varepsilon$ is an $R(G)$-module homomorphism with respect to the second (not the first) $R(G)$-module structure. However, for I_p to be an $R(G)$-module homomorphism, $\tilde{\kappa}_\varepsilon$ has to be an $R(G)$-module homomorphism with respect to the first module structure. Because of this, I_p is just an $R(G/T)$-module homomorphism. (This is also clear, by the way, from the algebraic description of the map, I_p, in Theorem 1.7. Namely, formulas (3.4) and (3.6) make clear that the map, $f\to f_0$ in Theorem 1.7 is *not* an $R(G)$ morphism.)

Let T_p be the tangent space at p. This space decomposes into a direct sum

$$T_p = T_p^-\oplus T_p^+,$$

where T acts on $T_p^\pm$ with positive and negative weights respectively. The exponential map identifies a neighborhood of p in S with a neighborhood of the origin inside T_p^-. Let G act with weights $\alpha_1,\dots,\alpha_m$ (so far we allow linear dependencies) on T_p^-. Then, it is obvious that S_ε can be identified with the projective space $\widetilde{\mathbb{CP}}^m$ defined in the previous section.

Before we specialize to GKM spaces we will prove Lemma 2.3.

Proof of Lemma 2.3. As above identify the symplectic cut of the stable manifold S at p with $\widetilde{\mathbb{CP}}^m$. Assume for the moment that we are in the GKM setting so that every pair of α_i's are linearly independent. Apply Lemma 4.1 to $\kappa_\varepsilon(\tau)$, where τ is a class with $\tau(q)=0$ whenever $q<p$. In

particular, if a descending edge goes from p to q, then $\tau(q) = 0$. Hence $\tilde{\iota}_i^* \Psi(\kappa_\varepsilon(\tau)) = 0$, unless $i = m+1$ and

$$\tilde{\iota}_{m+1}^* \Psi(\kappa_\varepsilon(\tau)) = \tau_p.$$

Thus Lemma 4.1 yields

$$I_p(\tau) = \mathrm{ind}_G^{\widetilde{\mathbb{CP}}^m}(\kappa_\varepsilon(\tau)) = \frac{\tilde{\iota}_{m+1}^* \Psi(\kappa_\varepsilon(\tau))}{\prod_{i=1}^m (1 - e^{2\pi\sqrt{-1}\alpha_i})} = \frac{\tau_p}{\Lambda_p^-},$$

which finishes the proof in the case α_i's are pairwise linearly independent.

If we allow linear dependencies among pairs of α_i's, then a very similar proof will go through. Atiyah–Segal localization gives a formula analogous to Lemma 4.1, the difference being that the fixed point set of the G action on $\widetilde{\mathbb{CP}}^m$ contains an isolated point p_{m+1} and other (possibly non-isolated) fixed points. Here, the first term on the right-hand side of (4.5) (which is the contribution to the index of these other points) is more complicated. However, if the class τ is supported above p, that is $\tau(q) = 0$ whenever $\phi(q) < \phi(p)$, then it is clear that the first term on the right-hand side of (4.5) vanishes, since restrictions to the corresponding fixed point sets are zeros.

To finish the proof, we will explain why we can assume without loss of generality that $\phi(q) < \phi(p)$ if and only if $q < p$, so that the condition that $\tau(q) = 0$ for $q < p$ implies that τ is supported above p. This follows from the equivariant version of the following standard result in classical Morse theory (see [**15**, Theorem 4.1]). Given two real numbers $a < b$, assume $\phi^{-1}([a, b])$ contains two critical points p and q whose stable and unstable manifolds do not intersect. Then for any $a < c$, $c' < b$, there exists another Morse function ϕ' which coincides with ϕ outside of $\phi^{-1}([a, b])$, and has the same critical points as ϕ and the same stable and unstable manifolds as ϕ, and $\phi'(p) = c$ and $\phi'(q) = c'$. It is clear from the proof of [**15**, Theorem 4.1] that the proof of this fact does not rely on any transversality arguments and can be restated and proved in the equivariant setting. q.e.d.

Let us now turn to the proof of Theorem 1.6. We assume that M is a GKM manifold with respect to the G action, so that the one-skeleton Γ with its G-action carries all the information we need to compute the equivariant cohomology and the equivariant K-theory of M.

Let $p \in M$ be a fixed point under the G action, or, in other words, a vertex of Γ, and let T act on T_p^- with the weights $\alpha_1, \dots, \alpha_m$. The GKM assumption implies that every pair of weights is linearly independent, and in terms of the graph Γ it means that the descending edges $e_1, \dots, e_m$ coming out of p are labeled by the weights $\alpha_1, \dots, \alpha_m$.

For any class $a \in K_G(M)$, let us apply Lemma 4.1 to the class $\kappa_\epsilon(a)$ to compute $I_p(a)$. We can make Lemma 4.1 more specific by computing the restrictions of $\kappa_\varepsilon(a)$ to the fixed points of $\widetilde{\mathbb{CP}}^m$. It is easy to see that

$$\tilde{\iota}_i^* \Psi(\kappa_\varepsilon(a)) = r_i(a_p) = r_i(a_{q_i}),$$

where q_i is the second vertex of the edge e_i. By inserting these identities in (4.5), one immediately gets the identity (1.7) of Theorem 1.6.

We also observe that if we apply the Lagrange interpolation formula (1.9) to the virtual character $a_p \in R(G)$ with $\alpha_1, \dots, \alpha_m$ being the labels of the edges of Γ pointing down from p, we get a slightly weaker version of Theorem 1.6. Namely, we get a formula whose right-hand side is identical to the right-hand side of (1.7), but has the term f_0 (instead of $I_p(a)$) on the left-hand side. So, the proof of Theorem 1.6 presented above gives a geometric interpretation of the term f_0 in the Lagrange interpolation formula. Similarly, Lemma 4.2 can be interpreted as the geometric analogue of the formula (1.9).

Notice also that the above discussion can be adapted to the setting of equivariant cohomology, with the pushforward in K-theory, or index, replaced by the pushforward in cohomology, or integration.

6. The restriction of the classes τ_p to fixed points

Theorem 1.6 enables one to express local indices in terms of restrictions of equivariant K-classes to fixed points. In this section, we will do the opposite. Namely, we will compute the restrictions of equivariant K-theory classes to fixed points in terms of local indices. Since the τ_p's generate $K_G(M)$, it is enough to do this for the classes τ_p's. Hence, we will derive an explicit graph-theoretic formula for the restriction of τ_p to a fixed point $q \in M^G$.

To state this result, we recall some notation: Assume that an edge e of Γ is equipped with an orientation. Denote by $i(e)$ and $t(e)$ the initial and terminal vertices of e, and we denote by $\bar{e}$ the edge obtained from e by reversing its orientation. Thus $i(\bar{e}) = t(e)$ and $t(\bar{e}) = i(e)$. We call e an ascending edge if $\alpha_e(\xi) > 0$ and a descending edge if $\alpha_e(\xi) < 0$.

Let H be a complimentary torus to T in G. As in Section 3, we will make use of the splitting $G = T \times H$ to identify the ring $R(G)$ with the ring of finite sums

$$\sum_{k=-N}^{M} c_k z^k, \quad c_k \in R(H), \tag{6.1}$$

and we will denote by $Q(H)$ the quotient field of the ring $R(H)$ and by $\hat{R}(G)$ the ring of finite sums

$$\sum_{k=-N}^{M} c_k z^k, \quad c_k \in Q(H). \tag{6.2}$$

Thus by (6.1) $\hat{R}(G)$ contains $R(G)$ as a subring.

Let e be an ascending edge of Γ. The key ingredient in our combinatorial formula for $\tau_p(q)$ is a $Q(H)$ module endomorphism

$$Q_e : \hat{R}(G) \to \hat{R}(G).$$

To define this endomorphism, let $p = t(e)$ and let $e_1, \ldots, e_{r+1}$ be the descending edges of Γ with $i(e_j) = p$. We will order these edges so that $e_{r+1} = \bar{e}$. Let $\alpha_j = \alpha_{e_j}$. Then $e^{2\pi\sqrt{-1}\alpha_j} = z^{k_j} e^{2\pi\sqrt{-1}\beta_j}$, where k_j is a negative integer and β_j is an element of the weight lattice of H. Denote by G_e the kernel of $e^{2\pi\sqrt{-1}\alpha_e} : G \to S^1$. Given an element f of $\hat{R}(G)$ and $h \in H$, we define $Q_e f$ to be the expression

$$\begin{aligned} Q_e f(z,h) = &\left(\prod_{j=1}^{r+1} (1 - z^{k_j} e^{2\pi\sqrt{-1}\beta_j}) \right) \\ &\times \pi_i r_i \, \frac{f(w,h)}{(1-\frac{z}{w}) \prod_{j=1}^{r} (1 - e^{2\pi\sqrt{-1}\alpha_j})}, \end{aligned} \tag{6.3}$$

or alternatively the sum

$$\begin{aligned} Q_e f(z,h) = &\left(\prod_{j=1}^{r+1} (1 - z^{k_j} e^{2\pi\sqrt{-1}\beta_j}) \right) \\ &\times \frac{1}{k} \sum_{i=1}^{r} \frac{f(w_i,h)}{(1-\frac{z}{w_i}) \prod_{j=1}^{r} (1 - w_i^{k_j} e^{2\pi\sqrt{-1}\beta_j})}, \end{aligned} \tag{6.4}$$

where $w_1, \dots, w_k$ are the preimages of h in G_e with respect to the projection $G_e \to H$, and r_i and π_i are the restriction and Gysin maps. By Theorem 1.7, this expression is a finite sum of the form (6.2) and hence an element of $\hat{R}(G)$.

Notice that with $e = e_i$, $Q_e f(z,h)$ is the term f_i in formula (1.8) of Theorem 1.7, multiplied by $\prod_{j=1}^{r+1}(1 - z^{k_j} e^{2\pi\sqrt{-1}\beta_j})$. From the geometric interpretation of Theorem 1.7, as Atiyah–Segal localization on twisted projective spaces, it is possible to interpret Q_e as a purely topological operation. This is an easy exercise, and we omit the details.

Now let γ be a path in Γ joining p to q; i.e., a sequence of oriented edges $e_1, \dots, e_s$ with $i(e_1) = p_1$, $t(e_s) = q$ and $t(e_j) = i(e_{j+1})$. We will call γ an ascending path if all the e_j's are ascending, and we will denote by

$$Q_\gamma : \hat{R}(G) \to \hat{R}(G)$$

the composition $Q_{e_s} Q_{e_{s-1}} \dots Q_{e_1}$.

Theorem 6.1. *The restriction $\tau_p(q)$ of τ_p to $q \in M^G$ is equal to the sum*

$$\sum Q_\gamma(\tau_p(p)) \tag{6.5}$$

over all ascending paths γ in Γ joining p to q.

Proof. For $p = q$, this formula is obvious, so we assume $q \neq p$. Let $e_1, \dots, e_r$ be the ascending edges of Γ with terminal vertex q and let $q_j = i(e_j)$. Since $I_q(\tau_p) = 0$, we get from Theorems 1.6 and 1.7

$$\tau_p(q) = \sum Q_{e_i} \tau_p(q_i). \tag{6.6}$$

If one of the q_i's, say q_1 is equal to p, one can incorporate it as a term in the sum (6.5), since e_1 is a path of length one joining p to q. For the q_i's which are not equal to p, let e_{ij} be the ascending edges of Γ with $t(e_{ij}) = q_i$ and let $q_{ij} = i(e_{ij})$. By iterating (6.6), we get for the contribution of these q_i's to the sum (6.6)

$$\sum_{i,j} Q_{e_i} Q_{e_{ij}} \tau_p(q_{ij}). \tag{6.7}$$

Again, if one of the q_{ij}'s is equal to p, the summand in (6.7) corresponding to it gets incorporated in the sum (6.5), since the path with edges e_{ij} and e_i is an ascending path on Γ joining p to q. For the remaining $q_{i,j}$'s, we iterate this argument again. It is clear that after sufficiently many iterations, we obtain the formula (6.5). q.e.d.

An analogue of Theorem 6.1 for the equivariant cohomology ring $H^*_G(M)$ can be found in [**9**]. In equivariant cohomology, the τ_p's are defined as the equivariant Thom classes of the unstable manifolds, and it was shown in [**9**] that the restriction of these classes to the fixed points are given by a formula analogous to (6.5) with operators, Q_e, defined by formulas similar to (6.3). Moreover, Catalin Zara was able to show that, for this combinatorial version of (6.5), a lot of summands cancel each other out making this formula an effective tool for computational purposes. We conjecture (or at least hope) that the same will be true in K-theory.

7. Example: The case of Grassmannian

In this section, we present a generalization of the definition of local index. This allows us to single out a class of GKM manifolds for which the computation of the restriction of the class, τ_p, to M^G is not much more complicated in K-cohomology than in ordinary cohomology. (We will show that one example of such a manifold is the Grassmannian.)

For a compact symplectic manifold (M, ω) with a Hamiltonian G action, the total index map $\mathcal{I} : K_G(M) \to K_G(M^G)$ depends on a choice of a circle subgroup $T \in G$ with $M^G = M^T$. Recall that the local index of $a \in K_G(M)$ at a fixed point p depends on the symplectic cut of the stable manifold S_p of ϕ at p with respect to the T action. We now modify this definition of local index by considering symplectic cuts of these stable manifolds with respect to other circle subgroups of G.

Namely, besides choosing a circle subgroup T of G with $M^G = M^T$, choose for each fixed point p a circle subgroup T_p of G, such that the T_p-moment map $\mu_p : M \to \mathbb{R}$ restricted to S_p attains its maximum at p. This allows us to define this local symplectic cut using the T_p action instead of the T action. So, now let S_ε be the symplectic cut of the stable manifold of ϕ at p with respect to the circle T_p action. As before, we have a map $\kappa_\varepsilon : K_G(M) \to K_G(S_\varepsilon)$ and we define the local index $\tilde{I}_p$ by

$$\tilde{I}_p(a) = \mathrm{ind}_G(\kappa_\varepsilon(a)).$$

Notice that all the results in Section 2 hold for the new local indices $\tilde{I}_p$. In particular, Theorem 1.1 is still true. (Notice, however, that the classes τ_p may be different from those defined using the old definition of local indices.) Also Lemmas 4.1, 4.2 as well as Theorem 1.6 hold with the following minor changes. If ξ_p is the infinitesimal generator of T_p, then the numbers k_i have to be defined as $\alpha_i(\xi_p)$, the maps $\tilde{\pi}_i$ and $\tilde{r}_i$ are defined using the circle T_p instead of T, and ζ is the generator of the character ring $R(T_p)$.

We now define the new total index $\tilde{\mathcal{I}} : K_G(M) \to K_G(M^G)$ to be the sum of all $\tilde{I}_p$. (Notice that the new total index is no longer an $R(G/T)$-module homomorphism.) Let us call $\tilde{\mathcal{I}}$ *torsion-free* if we can pick the circles T and T_p's in such a way that all the numbers k_i's for all the fixed points are equal to minus one. Then, Theorem 1.6 simplifies to:

Theorem 7.1. *Let M be a GKM space. For $p \in V = M^G$, let $e_1, \dots, e_m$ be the descending edges with initial vertex at p. Let e_i connect p to q_i and be labeled by weight α_i. If $\tilde{\mathcal{I}}$ is torsion-free, then for any $a \in K_G(M)$, we have*

$$\tag{7.1} \begin{aligned} \tilde{I}_p(a) = &\sum_{i=1}^{m} \frac{r_i(a_{q_i})}{(1 - e^{2\pi\sqrt{-1}\alpha_i}) \prod_{j \neq i} \left(1 - e^{2\pi\sqrt{-1}(\alpha_j - \alpha_i)}\right)} \\ &+ \frac{a_p}{\prod_{i=1}^{m}(1 - e^{2\pi\sqrt{-1}\alpha_i})}, \end{aligned}$$

Proof. This immediately follows from (1.7) with all k_i's equal to minus one, or from a straight forward application of (4.1). q.e.d.

As mentioned above, local indices can be defined in the setting of equivariant cohomology by using integration instead of the index map for the pushforward. Then Theorem 7.1 has a counterpart in the equivariant cohomology which states that for $a \in H^*_G(M)$,

$$\tag{7.2} \tilde{I}_p(a) = \sum_{i=1}^{m} \frac{r_i(a_{q_i})}{\alpha_i \prod_{j \neq i} (\alpha_j - \alpha_i)} + \frac{a_p}{\prod_{i=1}^{m} \alpha_i},$$

where $\tilde{I}_p$ is the local index in equivariant cohomology. Notice that (7.2) is obtained from (7.1) by the formal substitution of every expression of the form $(1 - e^{2\pi\sqrt{-1}\alpha})$ by α.

Without giving details, we remark that it is possible to prove an analogue of Theorem 6.1 for this new local index. However, the operators Q_γ will be defined differently and will no longer map $\hat{R}(G)$ into itself, where $\hat{R}(G)$ is the ring of the sums of the form (6.2). In the torsion free case, the operators Q_γ will simplify and after formal substitution of $(1 - e^{2\pi\sqrt{-1}\alpha})$ by α will resemble the operators of [**9**] appearing in their "path integral formula".

We will conclude this section by showing that the Grassmannian, $Gr(k,n)$, of k-planes in $\mathbb{C}^n$ is an example of a space for which one can define a torsion-free total index map. To see this, identify the n-dimensional torus $\tilde{G}$ with the product of n circles $S_1 \times \cdots \times S_n$. Let S_i act on the ith component of $\mathbb{C}^n$ with weight 1 and with weight 0 on the other components. This action induces an action of $\tilde{G}$ on $Gr(k,n)$. If S is the diagonal of the torus $\tilde{G}$, then its action on $Gr(k,n)$ is trivial, and the action of $G = \tilde{G}/S$ on $Gr(k,n)$ is effective. The G action on $Gr(k,n)$ is known to be GKM and its one-skeleton Γ is the Johnson graph.

Let $\xi_1, \dots, \xi_n$ be the infinitesimal generators of the circles $S_1, \dots, S_n$. They form a basis of the Lie algebra $\tilde{\mathfrak{g}}$ of $\tilde{G}$. Let $\alpha_1, \dots, \alpha_n$ be the dual basis of $\tilde{\mathfrak{g}}^*$. Then $\sum c_i\alpha_i$ with $c_i \in \mathbb{Z}$ is a weight of G as long as $\sum c_i = 0$.

Let v_i be a non-zero vector in the ith component of $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$. The fixed points of the G action on $Gr(k,n)$ are indexed by the k-element subset of $\{1, \dots, n\}$. Namely, if I is such a set, the fixed point p_I is the span of vectors v_{i_k} with $i_k \in I$. The weights of the isotropy action at p_I are $\alpha_i - \alpha_j$ with $i \notin I$ and $j \in I$. Let $\tilde{T}_I$ be the diagonal subcircle of the torus $\prod_{i\in I} S_i$ and T_I its image inside $(\prod_{i\in I} S_i)/S$. Then the isotropy action of T_I at p_I is given by the weight -1, and the moment map associated to the T_I action attains its maximum at p_I.

So, if we pick any generic circle subgroup T of G and then define local indices $\tilde{I}_{p_I}$ using symplectic cuts with respect to the actions of T_I, the total index will be torsion-free. In particular, Theorem 7.1 applies to $Gr(k,n)$.

Now let us specialize to the case of the ordinary (non-equivariant) cohomology and K-cohomology rings of $Gr(k,n)$. They are known to be isomorphic, where the isomorphism $\Phi : K(Gr(k,n)) \to H^*(Gr(k,n))$ is given by sending the Chern classes in K-theory of the dual of the tautological vector bundle on $Gr(k,n)$ to the corresponding Chern classes in cohomology. For more details concerning this isomorphism and the discussion below, see [**12**].

Pick a generic circle, T, in G, such that its moment map attains its minimum at p_I with $I = \{1, \dots, k\}$. Call the closure of the stable manifold at p_I the Schubert variety X_I. Then we can define two different bases of $H^*(Gr(k,n))$ one basis, s_I, being given by the Poincare duals, or Thom classes, of the X_I's, and the other basis, g_I, being given by the images under Φ of the structure sheaves of the X_I's. (The topological K-theory of $Gr(k, n)$ can be identified with its algebraic K-theory, so that we can use coherent sheaves, in particular the structure sheaves of Schubert varieties, to define K-classes, see [**4**] for details.) These bases are not the same and the transition matrix between these two bases was worked out in [**12**] using the combinatorics of Schur and Grothendieck polynomials.

Notice that the classes τ_{p_I} constructed in Theorem 1.1 descend to a basis $\hat{\tau}_{p_I}$ of $K(Gr(k, n))$. From the discussion above it is clear that $\Phi(\hat{\tau}_{p_I}) = s_I$. So this allows one to interpret the coefficients computed in [**12**] geometrically. Namely, these are the coefficients appearing when we express K-theory classes of the structure sheaves of the X_I's as linear combinations of the $\hat{\tau}_p$'s. Hence, the problem of computing these coefficients can be reduced to the computation of the (non-equivariant) local indices of the K-theory classes of the structure sheaves of the X_I's. It would be very interesting to reprove the results of [**12**] using this geometric approach.

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