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# SCALAR CURVATURE AND STABILITY OF TORIC VARIETIES

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## Abstract

We define a stability condition for a polarised algebraic variety and state a conjecture relating this to the existence of a Kahler metric of constant scalar curvature. The main result of the paper goes some way towards verifying this conjecture in the case of toric surfaces. We prove that, under the stability hypothesis, the Mabuchi functional is bounded below on invariant metrics, and that minimising sequences have a certain convergence property. In the reverse direction, we give new examples of polarised surfaces which do not admit metrics of constant scalar curvature. The proofs use a general framework, developed by Guillemin and Abreu, in which invariant Kahler metrics correspond to convex functions on the moment polytope of a toric variety.


## 1. Introduction

This paper is a step towards the solution of the general problem of finding conditions under which a complex projective variety admits a Kahler metric of constant scalar curvature. The pattern of the answer one expects is that this differential geometric condition should be equivalent to some notion of "stability" in the sense of Geometric Invariant Theory. This expectation is probably now an item of folklore: going back to suggestions put forward by Yau in the case of Kahler-Einstein metrics, and the many results of Tian and others in this case; reinforced by a detailed formal picture which makes clear the analogy with the well-established relation between the stability of vector bundles and Yang-Mills connections [5]. Here, we begin the investigation of

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the problem in the special case of toric varieties, working within a general differential-geometric framework developed by Guillemin [11], [12] and Abreu [1]. While we are not able to achieve a decisive result, the geometry in this comparatively simple situation is surprisingly rich and displays very clearly the interplay between the analysis and the stability condition.

While the general idea that constant scalar curvature should be related to stability has been mentioned by a number of authors a precise formulation of a conjecture, with the appropriate stability condition, has not yet appeared in the literature. Thus we begin by addressing this question, in Section 2 below, and define a condition on a polarised variety $(V, L)$ which, following Tian, we call "K-stability". (Tian gives a very similar definition in the particular case of Fano varieties.) Thus, we believe, the general expectation alluded to above can be given precise form in the following conjecture.

**Conjecture.** *A smooth polarised projective variety* $(V, L)$ *admits a Kahler metric of constant scalar curvature in the class* $c_1(L)$ *if and only if it is K-stable.*

Turning to our main topic, toric varieties: our principle result bears on the two-dimensional case. To state our result we should recall that for any compact Kahler manifold $(V, \omega_0)$ the *Mabuchi functional* is a functional $\mathcal{M}$, defined on the metrics in the class $[\omega_0]$, whose critical points are precisely the metrics of constant scalar curvature [14]. Expressing a metric via a Kahler potential, i.e., as $\omega_0 + 2i\overline{\partial}\partial\psi$, we can regard $\mathcal{M}$ as a functional on the set $\mathcal{H}$ of potentials. Now suppose that $(V, L)$ is a polarised toric surface, with an effective action of the torus $T^c = (\mathbf{C}^*)^2$ on $L$, covering an action on $V$. Working in the Kahler class $c_1(L)$, we may regard the potentials, more invariantly as Hermitian metrics on the line bundle $L$. Thus $T^c$ acts naturally on $\mathcal{H}$. Let $\mathcal{H}^T$ be the set of potentials which are fixed by the action of the compact subgroup $T = (S^1)^2$ of $T^c$. A critical point of $\mathcal{M}$ on $\mathcal{H}^T$ is a metric of constant scalar curvature, by the principle of symmetric criticality. There is a natural action of the quotient $T^c/T$ on $\mathcal{H}^T$. To state our main result we introduce a piece of special terminology: we say that a sequence of potentials $\psi_\alpha$ is *K-convergent* if there is a sequence of scalars $c_\alpha$ and elements $g_\alpha$ of $T^c/T$ such that the potentials $\psi'_\alpha = g_\alpha(\psi_\alpha) + c_\alpha$ satisfy:

(1) The $\psi'_\alpha$ converge pointwise on $V \setminus V^T$, where $V^T$ is the finite subset of fixed points of the $T$-action;

(2) On a Zariski-open neighbourhood of any fixed point we can introduce local co-ordinates $(w_1, w_2)$ in which the action is linear and diagonal and for any $\eta_1, \eta_2 > 0$ there is a constant $c_{\underline{\eta}}$ such that

$$\psi'_\alpha(w_1, w_2) \geq \eta_1 \log |w_1| + \eta_2 \log |w_2| + c_{\underline{\eta}},$$

for all $\alpha$.

(We do not envisage that this notion of convergence has any very general significance: it merely summarises what we are conveniently able to prove.)

Our main result is:

**Theorem 1.1.** *If a polarised toric surface is K-stable then the Mabuchi functional $\mathcal{M}$ is bounded below on $\mathcal{H}^T$ and any minimising sequence has a K-convergent subsequence.*

(We will also obtain partial converses to this Theorem (Propositions 7.1.2, 7.1.3), which give, by the way, new examples of complex surfaces which do not admit metrics of constant scalar curvature.)

Of course one would like to go beyond this result to prove that the limit of the minimising sequence is a smooth potential, minimising the Mabuchi functional and hence defining a metric of constant scalar curvature. A slightly different, but related, issue is to show uniqueness: that one gets the same limit for any minimising sequence. We leave these questions, which clearly involve substantial further analytical and PDE problems, for now. We should point out that the arguments in the present paper are all of a fairly elementary nature and we hope they can be seen as preparing the way for an attack on these PDE questions.

We give a brief outline of the proof of Theorem 1.1 and the organisation of this paper. In Section 3 we review the differential geometry of toric varieties, following Guillemin and Abreu. This involves the well-known correspondence between an $n$-dimensional toric variety and a polytope $\overline{P}$ in $\mathbf{R}^n$. A $T$-invariant Kahler metric corresponds to a convex "symplectic potential" function $u$ on $\overline{P}$. The scalar curvature is given by a beautiful formula of Abreu

$$S(u) = -\sum_{ij} \frac{\partial^2 u^{,ij}}{\partial x_i \partial x_j},$$

where $\left(u^{,ij}\right)$ is the inverse of the Hessian matrix $u_{,ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$. Thus the constant scalar curvature equation which motivates our study, $S(u) =$

constant, is a fully nonlinear, fourth order, PDE which stands in a similar relation to the elliptic Monge-Ampere equation, $\det(u_{ij}) = 1$ as, in linear PDE, the biharmonic equation stands to the Laplace equation. Our main concern in this paper is not directly with this equation but with the corresponding functional. For any function $A$ on the polytope $\overline{P}$ we define

$$\mathcal{F}_A(u) = -\int_P \log\det(u_{,ij})\ d\mu + \mathcal{L}_A(u),$$

where $\mathcal{L}_A$ is the linear functional

$$\mathcal{L}_A(u) = \int_{\partial P} u\ d\sigma - \int_P Au\ d\mu.$$

Here $d\mu$ is standard Lebesgue measure on $\mathbf{R}^n$ and $d\sigma$ is a certain measure on the boundary $\partial P$. The Euler-Lagrange equation $\delta\mathcal{F}_A = 0$ is the PDE $S(u) = A$. We are primarily interested in the case when $A$ is a constant, and then the functional $\mathcal{F}_A$ corresponds to the Mabuchi functional (restricted to $T$-invariant metrics), as considered in Theorem 1.1.

In Section 4 we take up the algebro-geometric point of view and we show that any *rational, piecewise-linear* convex function on $P$ defines a "test configuration" in the sense of our definition of K-stability in Section 2. We show that, if the toric variety is $K$-stable, then for any nontrivial function $u$ of this kind $\mathcal{L}_A(u) \geq 0$ (here $A$ is a constant): in fact $\mathcal{L}_A(u)$ is essentially the "Futaki invariant" which enters into the definition of K-stability.

Section 5 is the heart of the paper. Here we relate the nonlinear functional $\mathcal{F}_A$ to its linear component $\mathcal{L}_A$. We show that if $\mathcal{L}_A$ is positive on a suitable class of convex functions then $\mathcal{F}_A$ is bounded below. All of the discussion up to this point applies equally well in any dimension $n$. The restriction to the case $n = 2$ in our main Theorem enters here in the analysis of the semi-positive case, where we are able to show that the extremal function is piecewise-linear. We expect that a similar result will be true in higher dimensions, but the arguments become substantially harder. Finally, in Section 6, we show that the extremal function is actually a *rational* piecewise linear function: again we expect this to be true in higher dimensions, but an extension of the direct attack on the algebra that we use in the two dimensional case seems to become very complicated. In Section 7 we put together the various components to complete the proof of Theorem 1.1, and conclude with a discussion of a number of ideas related to the work in the paper.

There is a substantial existing literature, going back to [15], on the existence of Kahler-Einstein metrics on toric Fano varieties. Of course in the Fano case—with the anticanonical polarisation—the constant scalar curvature and Kahler-Einstein conditions are equivalent. It will be interesting to relate the techniques we develop here to the results already known in the Fano case.

## 2. K-stability of polarised varieties

### 2.1 The definition

In this section we define a notion of stability for a pair $(V, L)$ where $L$ is an ample line bundle over a compact complex manifold $V$. The main point of the definition is that it involves *another* space $V_0$. In simple cases this will given by another complex structure on the differentiable manifold underlying $V$, but we also need to admit singularities, so we allow $V_0$ to be a general scheme. Suppose $\Lambda$ is an ample line bundle over a projective scheme $W$, and suppose we have a fixed $\mathbf{C}^*$-action on the pair $(W, \Lambda)$. For each positive integer $k$ we have a vector space

$$H_k = H^0(W; \Lambda^k)$$

with a $\mathbf{C}^*$-action. From this we obtain integers $d_k = \dim H_k$ and $w_k$, the weight of the induced action on the highest exterior power of $H_k$. By general theory the integers $d_k, w_k$ are, for large $k$, given by polynomial functions of $k$, with rational co-efficients: $d_k = Q(k), w_k = P(k)$ say. We write $F(k) = w_k/kd_k$. The ring $\bigoplus H_k$ is finitely generated, and it follows easily that $w_k/kd_k$ is bounded: hence the degree of $P$ is at most the degree of $Q$ plus 1. So for large enough $k$ we have an expansion

$$F(k) = F_0 + F_1 k^{-1} + F_2 k^{-2} + \cdots,$$

with rational co-efficients $F_i$. We define the *Futaki invariant* of the $\mathbf{C}^*$-action on $(W, \Lambda)$ to be the co-efficient $F_1$.

**Definition 2.1.1.** A *test configuration* for $(V, L)$, of exponent $r$ consists of:

(1) a scheme $\mathcal{V}$ with a $\mathbf{C}^*$-action;

(2) a $\mathbf{C}^*$-equivariant line bundle $\mathcal{L} \to \mathcal{V}$;

(3) a flat $\mathbf{C}^*$-equivariant map $\pi : \mathcal{V} \to \mathbf{C}$, where $\mathbf{C}^*$ acts on $\mathbf{C}$ by multiplication in the standard way;

such that any fibre $V_t = \pi^{-1}(t)$ for $t \neq 0$ is isomorphic to $V$ and the pair $(V, L^r)$ is isomorphic to $(V_t, \mathcal{L}|_{V_t})$.

There are a few obvious points to note about this definition. First, all the fibres $V_t$ for nonzero $t$ are necessarily isomorphic, due to the action. Second, the $\mathbf{C}^*$-action on $\mathcal{V}$ induces an action on the central fibre: the scheme $V_0 = \pi^{-1}(0)$. Third, a special class of test configurations arises when $(V, L)$ has a $\mathbf{C}^*$-action and we take the *product configuration* $\mathcal{V} = V \times \mathbf{C}$.

**Definition 2.1.2.** The pair $(V, L)$ is *K-stable* if for each test configuration for $(V, L)$ the Futaki invariant of the induced action on $(V_0, \mathcal{L}|_{V_0})$ is less than or equal to zero, with equality if and only if the configuration is a product configuration.

This definition is close to that given by Tian in [16], but there are a number of differences. Tian considers Fano manifolds and works with vector fields rather than group actions. More importantly, Tian restricts attention to configurations with smooth, or at least normal, central fibres. Despite these differences of detail we use the same terminology "K-stable" introduced by Tian, since it does not seem worth introducing new terminology into the literature. A semantic point is that what we have called K-stable would be called "properly K-semistable" by Tian.

## 2.2 Differential-geometric formula

Here we explain the relation between the "Futaki invariant" of the previous section and the usual differential geometric formulation, in the case when the central fibre $V_0$ is smooth. Thus we suppose that $(V_0, L)$ is a smooth polarised variety with a fixed $\mathbf{C}^*$-action. Let $\omega_0$ be any Kahler metric on $V_0$ in the class $2\pi c_1(L)$, induced by a choice of Hermitian metric on $L$. Let $\hat{v}$ be the vector field on the total space of $L$ which generates the $\mathbf{C}^*$ action, covering a vector field $v$ on $V_0$. We can write

$$\hat{v} = \overline{v} + if\underline{t},$$

where $\overline{v}$ is the horizontal lift of $v$, $\underline{t}$ is the canonical vector field on the total space of $L$, defined by the action of scalar multiplication, and $f$ is a smooth function on $V_0$. The condition that $\hat{v}$ is holomorphic gives

$$\overline{\partial} f = -\left(i_v(\omega_0)\right)^{0,1}.$$

(See [7], p. 490). Let $S$ be the scalar curvature of $\omega_0$ and $a$ be the average value of $S$ over $V_0$. Set

$$\nu = \int_V (S-a)f \, \frac{\omega_0^n}{n!}.$$

**Proposition 2.2.1.** *The number $\nu$ is independent of the choice of metric $\omega_0$, and hence is an invariant of the $\mathbf{C}^*$-action on $(V_0, L)$.*

This is essentially the original result of Futaki [8]: see also the exposition of Calabi [3]. In fact if we let $g = G(S-a)$, where $G$ is the Green's operator, we can rewrite the integral as

$$\int_V (\partial g, v)\omega_0^n$$

which is the form given in [3]. This formulation also has the advantage that it uses only the holomorphic vector field $v$, and in fact the condition that the action lifts to $L$ is not needed.

**Proposition 2.2.2.** *The invariant $F_1$ is given by*

$$F_1 = -\frac{1}{4\mathrm{Vol}(V_0)}\nu.$$

To prove this we use Riemann-Roch formulae to compute the dimensions $d_k$ and weights $w_k$ of the action on $H^0(V_0, L^k)$; this is very similar to the discussion in Chapter 5 of [9]. We can suppose that $k$ is large enough for all the higher cohomology groups to vanish so $d_k$ is given by the Riemann-Roch formula

$$d_k = \int_{V_0} \mathrm{ch}(L^k)\mathrm{Td}(V_0) = \int_{V_0} e^{k\omega_0}\mathrm{Td}(V_0).$$

The Todd class is represented by the differential form $\mathrm{Td}(V_0) = 1 + \frac{1}{2}\rho + \dots$ where $\rho$ is the Ricci form, so we have

$$d_k = Ck^n + Dk^{n-1} + O(k^{n-2}),$$

where $C = \int_{V_0} \frac{\omega_0^n}{n!}$ is the volume of $V_0$ and

$$D = \frac{1}{2(n-1)!}\int_{V_0} \rho\omega_0^{n-1} = \int_{V_0} \frac{S}{4}\frac{\omega_0^n}{n!}.$$

To compute the weights we use the equivariant Riemann-Roch formula. Here we suppose that $\omega_0$ is preserved by the circle subgroup of $\mathbf{C}^*$

and let $v$ be the vector field generating this action. Recall that the equivariant cohomology $H^*_{S^1}(V_0)$ is the cohomology of the homotopy quotient $V_0 \times_{S^1} ES^1$ and there is an integration-over-the-fibre map

$$\int_{V_0} : H^*_{S^1}(V_0) \to H^*(BS^1) = \mathbf{Q}[t],$$

where $t$ is a generator in $H^2$. The equivariant Riemann-Roch formula asserts that $w_k$ is given by the co-efficient of $t$ in

$$\int_{V_0} \mathrm{ch}(L^k)\mathrm{Td}(V_0),$$

where now the Chern and Todd classes are regarded as equivariant cohomology classes. To make the calculation we use the de Rham model of equivariant cohomology, with the complex $\left(\Omega^*_{V_0}\right)^{S^1}[t]$ and differential $d+ti_v$, see [2]. The equivariant Chern class of $L$ is represented by $\omega_0+tf$ in this model. Let $\rho + tR$ be the representative for $c_1(V_0)$, where $R$ is a function on $V_0$. Then we obtain

$$w_k = Ak^{n+1} + Bk^n + O(k^{n-1}),$$

where

$$A = \int_{V_0} f\frac{\omega^n}{n!}, \; B = \int_{V_0} \left(\frac{fS}{4} + R\right)\frac{\omega^n}{n!}.$$

**Lemma 2.2.3.** *The function $R$ is the divergence of the vector field $Iv$ on $V$.*

To prove this we use the characterisation

$$L_v(\theta) = \nabla_v(\theta) + iR\theta,$$

for any local section $\theta$ of the canonical bundle $K_{V_0}$. In particular we may consider a local holomorphic section $\theta$, in which case we obtain

$$L_{Iv}(\theta) = \nabla_{Iv}(\theta) - R\theta. \tag{2.2.4}$$

Write $\theta \wedge \overline{\theta} = \sigma\frac{\omega^n}{n!}$, then

$$\begin{aligned} L_{Iv}(\theta\wedge\overline{\theta}) &= (\nabla_{Iv}\sigma + \sigma\mathrm{div}(Iv))\frac{\omega^n}{n!} \\ &= \nabla_{Iv}(\theta\wedge\overline{\theta}) + \sigma\mathrm{div}(Iv) \end{aligned}$$

On the other hand (2.2.4) gives

$$L_{Iv}(\theta \wedge \overline{\theta}) = (\nabla_{Iv}\theta \wedge \overline{\theta}) + \theta \wedge \nabla_{Iv}\overline{\theta}) - 2R\theta \wedge \overline{\theta},$$
$$= \nabla_{Iv}(\theta \wedge \overline{\theta}) - 2R\theta \wedge \overline{\theta},$$

and the result follows from these.

It follows from Lemma (2.2.3) that the integral of $R$ over $V_0$ vanishes so

$$B = \int_{V_0} \frac{fS}{4}\frac{\omega^n}{n!}.$$

Then

$$\frac{w_k}{kd_k} = \frac{A + Bk^{-1} + O(k^{-2})}{C + Dk^{-1} + O(k^{-2})} = \frac{A}{C} + \frac{AD - BC}{C^2}k^{-1} + O(k^{-2}),$$

and $AD - BC = -\frac{C}{4}\nu$ as required.

## 2.3 Relation with Hilbert-Mumford stability

We will discuss briefly the relation between the notion of K-stability defined above and "Hilbert-Mumford" stability, as studied in the algebraic geometry literature. This discussion is very similar to the corresponding one in [16]. Suppose $(V, L)$ is a polarised variety as usual and that the linear system $|L^r|$ gives a projective embedding of $V$ as a projective variety with Hilbert polynomial $P(n) = \chi(L^{rn})$. The Hilbert scheme $H_P$ parametrising projective schemes in $\mathbf{CP}^N$ with the given Hilbert polynomial $P$ is constructed as a subscheme of a Grassmannian of $m$-planes in the symmetric power $s^n(\mathbf{C}^{N+1})$ by assigning to a scheme $X$ the subspace of polynomials of degree $n$ which vanish on $X$. In turn this Grassmannian is embedded in a projective space by the Plucker embedding, so we have a point in

$$\mathbf{P}\left(\Lambda^m(s^n(\mathbf{C}^{N+1}))\right)$$

where $m$ is dim $s^n(\mathbf{C}^{N+1}) - P(n)$. The group $\mathrm{SL}(N+1, \mathbf{C})$ acts on $\Lambda^m s^n \mathbf{C}^{N+1}$ and the construction naturally assigns to $(V, L^r)$ an orbit under the induced action on the Hilbert scheme in the projective space. The pair $(V, L)$ is said to be Hilbert-Mumford stable, with the given values of $r$ and $n$, if this is a stable orbit in the sense of Geometric Invariant theory i.e if the corresponding orbit $\mathcal{O}$ in the vector space is closed. The Hilbert criterion asserts that this is true if and only if for any

1-parameter subgroup $\mathbf{C}^* \subset \mathrm{SL}(N+1,\mathbf{C})$ and any $z \in \mathcal{O}$ the $\mathbf{C}^*$-orbit of $z$ is closed. This readily translates into the following numerical criterion. For any 1-parameter subgroup $\rho : \mathbf{C}^* \to \mathrm{SL}(N+1,\mathbf{C})$ the points $[\rho(t)(z)]$ in the projective space converge to some limit $[z_0]$ as $t \to 0$, where $[z_0]$ is fixed by $\rho(\mathbf{C}^*)$. Thus there is an integer weight $W(\rho) \in \mathbf{Z}$ with $\rho(t)(z_0) = t^W z_0$. The stability condition is that $W(\rho) \leq 0$ for all 1-parameter subgroups $\rho$, with equality if and only $\rho$ fixes $z$.

The connection with our previous discussion arises from the fact that there is essentially a one-to-one correspondence between test configurations in the sense of 2, with exponent $r$, and 1-parameter subgroups in $\mathrm{SL}(H^0(L^r))$. Starting with a test configuration we take a point $[z_0]$ in the Hilbert scheme representing the central fibre $V_0$. The $\mathbf{C}^*$-action on $V_0$ induces an action on $\mathbf{C}^{N+1} = H^0(V_0, \mathcal{L})$. Making a base change if necessary we can suppose that this is the product of a scalar action and a 1-parameter subgroup in $\mathrm{SL}(N+1,\mathbf{C})$ and we are in just the position considered above. In the other direction, starting with a 1-parameter subgroup, the limit point $[z_0]$ lies in the Hilbert scheme so the orbit extends to a map from $\mathbf{C}$ to the Hilbert scheme and we get a test configuration by pulling back the universal family. The essential task then is to see what the numerical criterion boils down to in this case, in terms of the $\mathbf{C}^*$-action on $(V_0, \mathcal{L})$. The line spanned by $z_0$ is, by definition, the highest exterior power of the kernel of the surjection

$$s^n(H^0(V_0, \mathcal{L})) \to H^0(V_0, \mathcal{L}^n).$$

Thus it can be identified with

$$\Lambda^{\max} s^n(H^0(V_0, \mathcal{L})) \otimes \Lambda^{\max} H^0(V_0, \mathcal{L}^n)^{-1}.$$

For any vector space $E$ there is a canonical isomorphism

$$\Lambda^{\max} s^n(E) \cong (\Lambda^{\max} E)^{n \dim s^n(E)/\dim E},$$

so the line in question can be identified (writing $s^n$ for $s^n(H^0(\mathcal{L}))$) with

$$\Lambda^{\max} H^0(V_0, \mathcal{L})^{n \dim s^n/\dim H^0(\mathcal{L})} \otimes \Lambda^{\max} H^0(\mathcal{L}^n)^{-1}.$$

However, we must take care of the fact that the 1-parameter subgroup we need is obtained by writing the natural action as a product of a scalar part and a part in $\mathrm{SL}(N+1)$. The scalar part is given by the the weight of the natural action on $\Lambda^{\max} H^0(\mathcal{L})$ to the power $1/\dim H^0(\mathcal{L})$. Thus the action of this scalar part on the line spanned by $z_0$ has weight equal to $u^{mn/\dim H^0(\mathcal{L})}$ where $u$ is the weight of the natural $\mathbf{C}^*$-action on $\Lambda^{\max} H^0(\mathcal{L})$. (This is, in general, rational rather than integral, corresponding to the fact that we may need to lift to a covering of the $\mathbf{C}^*$-action. However we can ignore this covering if we are only interested in finding the sign of $W$.) Putting this together, the weight $W$ we need is the weight of the natural action on the line

$$(\Lambda^{\max} H^0(\mathcal{L}))^{(n \dim s^n - mn)/\dim H^0(\mathcal{L})} \otimes \Lambda^{\max} H^0(\mathcal{L}^n)^{-1}.$$

Since $\dim s^n - m = P(n) = d_n$ we can write this weight as

$$W = \frac{n d_n w_1}{d_1} - w_n = n d_n (F(1) - F(n)).$$

So we conclude that, with the given values of $r, n$, the pair $(V, L)$ is Hilbert-Mumford stable if and only if for each test configuration with exponent $r$ the central fibre satisfies the numerical condition $F(n) - F(1) \geq 0$, with equality if and only if the configuration is a product. To relate this to the K-stability condition, note that any configuration of exponent $r_0$ defines configurations of exponent $pr_0$ for all positive integers $p$— replacing $\mathcal{L}$ by $\mathcal{L}^p$. Suppose we know that, for some $r_0$ and all large $p$, all configurations of exponent $r = pr_0$ arise in this way. Then the condition becomes that

$$F(pn) - F(n) \geq 0$$

where $F$ is the function defined by a configuration of exponent $r_0$. Clearly, if the variety is $K$-stable this condition will hold, for each such configuration, once $n$ and $p$ are large enough. Thus the essential difference between the notions of Hilbert-Mumford stability, and $K$-stability is that the first depends on the values of the function $F$ at specific values of the parameter, while the second involves the asymptotics as the parameter becomes large. This difference is analogous to the difference between the notions of "Gieseker stability" and "Mumford stability" in the theory of vector bundles.

## 3. Toric geometry

### 3.1 The symplectic potential

In this subsection we review standard material on toric differential geometry, following Guillemin [11], [12] and Abreu [1]. Another paper which adopts a similar point of view is [10]. The main theme is the interplay between symplectic and complex structures. We begin on the symplectic side, so we consider a compact symplectic manifold $V, \omega$ of real dimension $2n$ and a line bundle $L \to V$ with a connection whose curvature is $-2\pi i\omega$. We suppose that the torus $T^n = (S^1)^n$ acts effectively on this data with a moment map $m : V \to \mathbf{R}^n$. The image of the moment map is an integral polytope $\overline{P}$ in $\mathbf{R}^n$ i.e., it is the convex hull of a finite set of points in the lattice $\mathbf{Z}^n$. The polytope is defined by a finite number of linear inequalities

$$h_k(x) \geq c_k, \tag{3.1.1}$$

where $h_k$ are linear maps from $\mathbf{R}^n$ to $\mathbf{R}$ which induce primitive maps from the integer lattice $\mathbf{Z}^n$ to $\mathbf{Z}$. We denote the interior of the polytope by $P$. The fibres of the moment map $m$ are orbits under the $T^n$-action. The action is free on the dense open subset $V_0 = m^{-1}(P)$ and the universal cover of $V_0$ can be identified with $P \times \mathbf{R}^n$. We introduce standard coordinates $(x_i, \eta_i),\ i = 1, \dots n$ on this universal cover (which, in the familiar way we also think of as coordinates on $V_0$) in which the symplectic form is given by $\omega = \sum dx_i d\eta_i$; the moment map is given by projection to the $x$-coordinates and the group acts by translation in the $\eta$-coordinates.

We now turn to the complex side, so we suppose instead that $V$ is a compact complex $n$-manifold with an action of the complex torus $T^n_C = (\mathbf{C}^*)^n$ which has a dense, free, open orbit $V_0$, and the action lifts to a positive holomorphic line bundle $L \to V$. Thus we can identify $V_0$ with $(\mathbf{C}^*)^n$ and we have standard complex co-ordinates $w_1, \dots, w_n \in \mathbf{C} \setminus \{0\}$. Again, we work in the covering space where we can take co-ordinates $z_i = \log w_i = \xi_i + \sqrt{-1}\eta_i$ in $\mathbf{C}$, so the action is represented by translation in the $z_i$ variables. We consider Kahler metrics on $V$ which are invariant under the action of $T^n \subset T^n_C$. Such a metric can be represented by a Kahler potential over $V_0$; $\omega = 2i\overline{\partial}\partial\phi$ so $\phi$ can be viewed as a function of the complex variables $z_1, \dots z_n$. The $T$-invariance means that we may restrict attention to functions which only depend on the real parts $\xi_1, \dots, \xi_n$. In sum, our Kahler metric is given by

$$\omega = \frac{\sqrt{-1}}{2} \sum_{ij} \phi_{,ij} dz_i d\overline{z}_j, \tag{3.1.2}$$

where

$$\phi_{,ij} = \frac{\partial^2 \phi}{\partial \xi_i \partial \xi_j},$$

and $\phi = \phi(\xi_1, \dots, \xi_n)$ is a strictly *convex* function on $\mathbf{R}^n$.

To match up these points of view one observes that the moment map for the $T^n$ action on the Kahler metric (3.1.2) is given by

$$m(z_1, \dots z_n) = \left( \frac{\partial \phi}{\partial \xi_i} \right).$$

We define the *symplectic potential* $u$ to be the *Legendre dual* of $\phi$. Recall that this is defined in the following way. For each point $\underline{x}$ in the image, $P$, of the moment map there is a unique point $\underline{\xi} = \underline{\xi}(\underline{x})$ in $\mathbf{R}^n$ where $\frac{\partial \phi}{\partial \xi_i} = x_i$. We let

$$u(\underline{x}) = \sum x_i \xi_i - \phi(\underline{\xi}). \tag{3.1.3}$$

Then $u$ is a convex function on $P$ and $\phi$ can be recovered from $u$ by the symmetrical relation between Legendre dual functions. Thus the passage between the complex and symplectic viewpoints amounts to the change from the $\xi_i$ to $x_i$ coordinates and the metric data is encoded in either the Kahler potential $\phi(\underline{\xi})$ on $\mathbf{R}^n$ or the symplectic potential $u(\underline{x})$ on $P$. In the symplectic coordinates the metric is given by

$$g = \sum u_{,ij} dx_i dx_j + \sum u^{,ij} d\xi_i d\xi_j,$$

where the matrix $\left(u^{,ij}\right)$ is the inverse of the Hessian matrix $u_{,ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$. According to Abreu [1], the scalar curvature of this metric is

$$S(u) = -\sum_{ij} \frac{\partial^2 u^{,ij}}{\partial x_i \partial x_j}. \tag{3.1.4}$$

We digress to explain how this expression for the scalar curvature fits into the picture developed in [5], considering the action of the symplectomorphism group of a symplectic manifold on the space of compatible almost-complex structures. In the presence of the $T$-action on the symplectic manifold $V$ we restrict attention to the space $\mathcal{J}^T$ of $T$-invariant almost-complex structures, with the action of the group $\mathcal{G}^T$, the identity component of the group of symplectomorphisms of $V$ which commute with $T$. The Lie algebra of $\mathcal{G}^T$ can be identified with certain smooth functions on $\overline{P}$ modulo constants i.e., the symplectomorphisms are generated by Hamiltonian functions of the $x_i$ variables, Elements of $\mathcal{J}^T$ are represented by maps from $P$ to the Siegel generalised upper half-space

$H_n = \mathrm{Sp}(2n, \mathbf{R})/\mathrm{U}(n)$ of linear complex structures on $\mathbf{R}^{2n}$, compatible with the standard symplectic form. The crucial point is that $H_n$ has an invariant Kahler structure. Explicitly, we can represent any almost-complex structure at a point by a pair of matrices $A, B$, describing the action of $J$ on the tangent space to the $T$-orbits:

$$J\left(\frac{\partial}{\partial \eta_i}\right) = \sum A_{ij}\frac{\partial}{\partial \eta_j} + \sum B_{ij}\frac{\partial}{\partial x_j}.$$

This defines a unique complex structure provided that $B$ is symmetric and positive definite and $B^{-1}A$ is symmetric. We write

$$X = U + iV = B^{-1}A + iB^{-1}.$$

This gives the standard representation of $H_n$ as the set of complex symmetric matrices $X$ with positive definite imaginary part, making evident the complex structure on $H_n$. The invariant Kahler form $\Omega$ is defined by the formula:

$$\begin{aligned}(3.1.5) \qquad & \Omega(\delta_1 U + i\delta_1 V, \delta_2 U + i\delta_2 V)\\ &= \mathrm{Tr}\,(\delta_1 U V^{-1}\delta_2 V V^{-1} - \delta_2 U V^{-1}\delta_1 V V^{-1})\\ &= -\mathrm{Tr}(\delta_1 U \delta_2 B - \delta_2 U \delta_1 B).\end{aligned}$$

We see then that elements of $\mathcal{J}^T$ can be represented by matrix-valued functions $U, V$ as above on $P$. The action of a symplectomorphism generated by a Hamiltonian function $f$ on $P$ is just

$$(3.1.6) \qquad X \mapsto X + f_{,ij}$$

where $f_{,ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. We define a map $\mu$ from $\mathcal{J}^T$ to functions on $P$ by

$$\mu(U, V) = -\sum \frac{\partial^2 V^{ij}}{\partial x_i \partial x_j},$$

where $(V^{ij}) = V^{-1}$. It follows from (3.1.5) and (3.1.6) that $\mu$ is a *moment map* for the action of the compactly supported elements of $\mathcal{G}^T$ on $\mathcal{J}^T$, where the latter space is given the symplectic form induced in a natural way by $\Omega$, and integration over $P$. This just amounts to the integration-by-parts formula

$$\int_P f \sum \frac{\partial^2 \beta^{ij}}{\partial x_i \partial x_j} = \int_P \sum f_{ij}\beta^{ij},$$

for compactly supported $f$ and arbitary $\beta$.

In [5] we considered a formal complexification of the action of the symplectomorphisms on the space of almost-complex structures. In the present situation, this extension can be made in a completely straightforward way—we just use the same formula (3.1.6) with a complex-valued function $f$ on $P$. The integrability condition for an almost-complex structure represented by a complex matrix-valued function $X_{ij}$ is just

$$\frac{\partial X_{ij}}{\partial x_k} = \frac{\partial X_{ik}}{\partial x_j},$$

and all the structures for which this holds lie in the same orbit of the complexified action: i.e., we can write $X_{ij} = f_{ij}$ for some complex-valued function $f$ on $P$. In particular

$$V_{ij} = u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j},$$

where $u$ is the imaginary part of $f$. Taking inverses, we see that the moment map $\mu$ on this complex orbit is given by Abreu's formula $\sum \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j}$. This ties up Abreu's calculation with the discussion in [5] where it is shown, in general, that the scalar curvature furnishes a moment map for the action of the symplectomorphisms on the space of almost-complex structures; and concludes our digression.

Throughout this subsection we have focussed attention on the interior of $P$, corresponding to the free orbits of the $T$-action. We turn now, following Guillemin and Abreu, to the question of the boundary behaviour of the symplectic potential. Recall that the polytope $P$ is defined by linear inequalities $h_k(x) > c_k$. We write $\delta_k(x) = h_k(x) - c_k$, so the $\delta_k$ are positive functions on $P$. Let

$$u_0(x) = \sum_k \delta_k(x) \log \delta_k(x),$$

so $u_0$ is continuous function on $\overline{P}$, smooth on the interior. This gives the model for the boundary behaviour required to generate a smooth metric on the compact manifold $V$. We define $\mathcal{S}$ to be the set of continuous, convex, functions $u$ on $\overline{P}$ such that $u - u_0$ is smooth on $\overline{P}$. Then the discussion of this section can be summed up the following Proposition.

**Proposition 3.1.7.** *Let $V$ be a symplectic toric manifold defined by a polytope $\overline{P}$ in $\mathbf{R}^n$. There is a one-to-one correspondence between the $T$-invariant Kahler potentials $\psi$ on $V$; and the symplectic potentials $u$ in $\mathcal{S}$. Changing $\psi$ to $\psi + c$ for $c \in \mathbf{R}$ changes $u$ to $u - c$. The action of $T^c/T$ on the Kahler potentials coresponds to the action of the linear functions, by addition, on the symplectic potentials.*

Of course the Kahler potentials $\phi$ that we are considering here, corresponding to symplectic potential $u$, are not quite the same as those discussed in Section 1 where we represent the metrics in a cohomology class as $\omega_0 + 2i\overline{\partial}\partial\psi$. However it is easy to pass between the two points of view. The potentials $\phi$, regarded as distributions on the compact manifold $V$, satisfy an equation $2i\overline{\partial}\partial\phi = \omega + H$, where $H$ is a fixed current, independent of $\phi$, supported on the divisor at infinity. Thus if we have a reference metric $\omega_0$ we can represent another metric $\omega$ as $\omega_0 + i\overline{\partial}\partial\psi$ where $\psi$ is the smooth function $\phi - \phi_0$.

One point to notice about this correspondence is that it behaves well under restriction to the faces of the polytope $P$. These faces correspond to toric subvarities of $V$ and the restriction of any function in $\mathcal{S}$ to the interior of a face is smooth and yields a symplectic potential for the corresponding subvariety.

## 3.2 The Mabuchi functional

We begin by considering a general compact Kahler manifold $(V, \omega_0)$ of complex dimension $n$. The Mabuchi functional $\mathcal{M}$ is a real-valued function on the set of Kahler metrics in the same Kahler class $[\omega_0]$, defined up to the addition of an overall constant [14]. The functional is defined through the formula for its variation at a metric $\omega = \omega_0 + 2i\overline{\partial}\partial\psi$ with respect to an infinitesimal change $\delta\psi$ in the Kahler potential:

$$\delta\mathcal{M} = -\int_V (S - a)\delta\psi \frac{\omega^n}{n!}, \tag{3.2.1}$$

where $S$ is the scalar curvature of $\omega$ and $a$ is the average value of the scalar curvature, a topological invariant of the data. Thus $\delta\mathcal{M}$ is not changed if one adds a constant to $\delta\psi$, and so depends only on the variation of the metric. From a more abstract point of view, the formula (3.2.1) defines a 1-form on the space of Kahler metrics in the cohomology class and one checks that this 1-form is closed, hence is the derivative of a function $\mathcal{M}$. The Mabuchi functional is related to the Futaki invariant in the following way. Let $v$ be a holomorphic vector field on $V$ and the $(0,1)$ component of $i_v(\omega)$ be $-\overline{\partial} f$ as in (2.2). Then the infinitesimal change in the Kahler potential generated by the action of $v$ is just the imaginary part of $f$. It follows that the infinitesimal change in the Mabuchi functional, under the action of $v$ is given by the imaginary part of the Futaki invariant. That is, if $\sigma_t : V \to V$ is the 1-parameter group of holomorphic automorphisms of $V$ generated by $v$,

$$\frac{d}{dt}\mathcal{M}(\sigma_t^*(\omega)) = \mathrm{Im}(\nu(v)). \tag{3.2.2}$$

We also want to recall the definition of another functional $I_V$ on the set of Kahler potentials. This is defined in the same manner as above by the formula

$$\delta I_V = 2\int_V \delta\psi \frac{\omega^n}{n!}, \tag{3.2.3}$$

see [6]. In this case the variation does not vanish when $\delta\psi$ is constant, so $I$ is defined on the potentials rather than the metrics.

Now suppose that $D$ is a positive divisor in $V$ representing $c_1(V)$. Thus there is a meromorphic $n$-form $\chi$ on $V$ with no zeros and with a simple pole along $D$. We assume that $D$ is a sum of smooth subvarieties $D_r$, meeting with normal crossings in $V$. Thus there are functionals $I_{D_r}$ defined on the restriction of potentials to $D_r$ and we set $I_D = \sum I_{D_r}$. Given a Kahler metric $\omega$ we define a function $\nu$ on $V$ by $\nu = |\chi|^{-2}$ so

$$C_n\omega^n = \nu\ \chi \wedge \overline{\chi},$$

for a universal numerical factor $C_n$ which will not be important.

**Proposition 3.2.4.** *For any metric $\omega = \omega_0 + 2i\overline{\partial}\partial\psi$ on $V$*

$$\mathcal{M}(\omega) = -I_D(\psi) + aI_V(\psi) + \int_V \log\nu \frac{\omega^n}{n!}.$$

Note here that the combination $I_D(\psi) - aI_V(\psi)$ is unchanged if $\psi$ is changed by a constant, so it is well-defined on the Kahler metrics, rather than potentials. In the formula of the Proposition both sides are well-defined up to overall constants.

To prove the Proposition, we write

$$S\frac{\omega^n}{n!} = \frac{2}{(n-1)!}\rho \wedge \omega^{n-1},$$

where $\rho$ is the Ricci form. We have an equation of currents

$$\rho = D - i\overline{\partial}\partial \log \nu,$$

so

$$\delta\mathcal{M} = -2\int_D \delta\psi \ \frac{\omega^{n-1}}{(n-1)!} + 2\int_V i\overline{\partial}\partial\log\nu \ \delta\psi \ \frac{\omega^{n-1}}{(n-1)!} + 2a\int_V \delta\psi \ \frac{\omega^n}{n!}.$$

Hence

$$\delta\mathcal{M} = \delta(-I_D(\psi) + aI_V(\psi)) + 2\int_V i\overline{\partial}\partial\log\nu \ \delta\psi \ \frac{\omega^{n-1}}{(n-1)!}.$$

Integrating by parts, this gives

$$\delta\mathcal{M} = \delta(-I_D(\psi) + aI_V(\psi)) + n\int_V \log\nu(\delta\omega)\frac{\omega^{n-1}}{n!}.$$

On the other hand, differentiating the equation $C_n\omega^n = \nu\chi\wedge\overline{\chi}$, we have

$$n(\delta\omega)\omega^{n-1} = C_n^{-1}\delta\nu \ \chi\wedge\overline{\chi},$$

and

$$\delta\left(\int_V \nu\log\nu\chi\wedge\overline{\chi}\right) = \int_V (\log\nu + 1)\delta\nu \ \chi\wedge\overline{\chi}.$$

The contribution

$$\int_V \delta\nu \ \chi\wedge\overline{\chi}$$

is

$$C_n^{-1}\delta\int_V \omega^n,$$

which vanishes; so we have

$$\delta\int_V \nu\log\nu \ \chi\wedge\overline{\chi} = C_n\int_V \log\nu(\delta\omega)\omega^{n-1},$$

and hence

$$\delta\mathcal{M} = \delta\left(-I_D(\psi) + aI_V(\psi) + \int_V \log\nu\frac{\omega^n}{n!}\right),$$

which completes the proof.

Proposition (3.2.4) is related to a formula of Chen [4]. Indeed one gets a similar formula given any current representing $c_1(V)$: in Chen's case this is the Ricci form of some fixed metric while in our case it is the divisor $D$.

We now specialise to the case when $V$ is a toric variety, as before. Up to scale, there is a unique $T$-invariant meromorphic $n$-form $\chi$ on $V$ given in our co-ordinates $z_i$ by $\chi = dz_1 dz_2 \dots dz_n$. This has a simple pole along the divisor $D = m^{-1}(\partial P)$ and the constant $a$ is

$$a = \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(V)}. \tag{3.2.5}$$

We define a measure $d\sigma$ on $\partial P$ as follows. On the face defined by the equation $h_r(x) = c_r$ we let $d\sigma$ be the constant $(n-1)$-form such that $dh_r \wedge d\sigma$ is, up to sign, the standard Euclidean volume form $d\mu$. For example, if $P$ is a polyhedron in $\mathbf{R}^2$ and an edge of $P$ has slope $p/q$, where $p, q$ are coprime integers, then the measure on this edge is given by

$$d\sigma = \frac{|dx_2|}{p} = \frac{|dx_1|}{q}.$$

Now let $\omega_0$ be some invariant metric on $V$ and let $\phi_0$ be a Kahler potential defining $\omega_0$ in our standard co-ordinates on the open $T^c$-orbit, as in 3.1. Any other invariant metric $\omega_0 + i\overline{\partial}\partial\psi$ is given by a Kahler potential $\phi = \phi_0 + \psi$ on the open orbit, which corresponds to a symplectic potential $u$ on $P$.

**Lemma 3.2.6.**

$$I_V(\psi) = -(2\pi)^n \int_P u d\mu; \tag{1}$$

$$I_D(\psi) = -(2\pi)^n \int_{\partial P} u \; d\sigma. \tag{2}$$

Notice here that $I_V$ and $I_D$ are *a priori* defined only up to an overall constant, and this is the way in which the formulae should be read. However the expressions on the right-hand side of the formulae have no such ambiguity; thus in this toric situation there is a natural way to normalise the functionals.

To prove the Lemma we recall first that the push forward of the volume form under the moment map $m$ is $(2\pi)^n$ times the standard volume form $d\mu$ on $\mathbf{R}^n$. Now recall that the symplectic potential $u$ is defined by

$$u(\underline{x}) = \sum x_i \xi_i - \phi(\underline{\xi}), \tag{3.2.7}$$

where $\underline{\xi} = \underline{\xi}(\underline{x})$ is implicity defined by the equation $\frac{\partial \phi}{\partial \xi_i} = x_i$. Consider an infinitesimal variation $\widetilde{\phi}$ in the Kahler potential, leading to variations $\underline{\widetilde{\xi}}$ and $\widetilde{u}$ in $\underline{\xi}$ and $u$. Differentiating (3.2.7), we have

$$\widetilde{u}(\underline{x}) = \sum x_i \widetilde{\xi_i} - \widetilde{\phi}(\xi_i) - \sum \frac{\partial \phi}{\partial \xi_i} \widetilde{\xi}_k,$$

and the defining equation for $\underline{\xi}$ gives simply $\widetilde{u}(\underline{x}) = -\widetilde{\phi}(\underline{\xi})$. Thus the variation in the functional $I_V$, which is by definition,

$$\delta I_V = \int_V \widetilde{\phi} \frac{\omega^n}{n!},$$

can be written as

$$\delta I_V = -(2\pi)^n \int_P \widetilde{u} d\mu,$$

by the property of the push-forward measure. This is the same as the variation in

$$-(2\pi)^n \int_P u d\mu,$$

thus proving item (1) of the Lemma. Item (2) follows in just the same way, when one replaces $P$ by its codimension 1-faces.

**Proposition 3.2.8.** *The Mabuchi functional is given by* $\mathcal{M}(\omega) = (2\pi)^n \mathcal{F}_a(u)$ *where*

$$\mathcal{F}_a(u) = -\int_P \log\det(u_{ij}) + \int_{\partial P} u d\sigma - a \int_P u d\mu.$$

Given the preceding Lemma, this is just a matter of transforming the term

$$\int_V \log \nu \frac{\omega^n}{n!}$$

in (3.2.4) into the symplectic setting. Differentiating the equations defining $\underline{\xi}(\underline{x})$ and $u$ we find that the Hessian $u_{ij}$ of $u$ is the inverse of the Hessian $\frac{\partial^2 \phi}{\partial \xi_i \partial \xi_j}$ of $\phi$ (at the points $\underline{x}$ and $\underline{\xi}(\underline{x})$ respectively). Thus the function $\log \nu$ is $\log \nu = \log\det(\phi_{ij}) = -\log\det(u_{ij})$, and

$$\int_V \log \nu \frac{\omega^n}{n!} = -(2\pi)^n \int_P \log\det(u_{ij}) d\mu$$

by the property of the push-forward measure.

Notice that the constant $a$, given by (3.2.5), can also be written as

$$a = \frac{\mathrm{Vol}(\partial P, d\sigma)}{\mathrm{Vol}(P, d\mu)}.$$

Thus $a$ is fixed so that $\mathcal{F}_a(u)$ is unchanged if one adds a constant to $u$. In a similar vein, recall from Section 2 that the Futaki invariant gives, in our situation, a linear map from the Lie algebra of the holomorphic automorphism group $T^c$ to $\mathbf{C}$. This Lie algebra can be represented by the $\mathbf{R}$-linear complex valued functions on $P \subset \mathbf{R}^n$.

**Lemma 3.2.9.** *The Futaki invariant maps a linear function $f$ to*

$$\int_{\partial P} f d\sigma - a \int_P f d\mu.$$

This follows immediately from (3.2.8) and the relation (3.2.2) between the Mabuchi functional and the Futaki invariant. Thus the toric variety has Futaki invariant zero precisely when the centre of mass of $P$ in $\mathbf{R}^n$, with the measure $a d\mu$ coincides with that of $\partial P$, with the measure $d\sigma$, and in this case the functional $\mathcal{F}_a(u)$ is unchanged if one adds any affine-linear function to $u$.

## 3.3 General properties of the functional

In this section we will consider a more general functional $\mathcal{F}_A$ defined on suitable convex functions $u$ on $P$, with

$$\mathcal{F}_A(u) = -\int_P \log\det(u_{ij}) + \mathcal{L}_A(u), \tag{3.3.1}$$

where $A$ is a bounded function on $P$ and

$$\mathcal{L}_A(u) = \int_{\partial P} u d\sigma - \int_P A u d\mu. \tag{3.3.2}$$

From the point of view of toric varieties it is natural to consider the functions $u$ in the space $\mathcal{S}$, which correspond to smooth Kahler metrics. But it also useful and interesting to extend the domain to include more general convex functions. Let $\mathcal{C}_\infty$ denote the set of continuous convex functions on $\overline{P}$ which are smooth in the interior. Clearly $\mathcal{L}_A$ is well-defined on $\mathcal{C}_\infty$, but the situation for the nonlinear term is not so clear. For any function $u$ in $\mathcal{C}_\infty$ let $\log^+\det(u_{ij})$ be $\max(0, \log\det(u_{ij}))$, so in particular $\log^+\det(u_{ij})$ is defined to be zero if $\det(u_{ij})$ vanishes.

**Lemma 3.3.3.** *For any $u$ in $\mathcal{C}_\infty$, $\log^+ \det u_{ij}$ is integrable on $P$.*

Given this, we can define $\mathcal{F}_A(u)$ for any $u \in \mathcal{C}_\infty$, taking values in $(-\infty, \infty]$.

**Proposition 3.3.4.** *Suppose $A \in L^\infty(P)$ and $v \in \mathcal{S}$ satisfies the equation*

$$\sum \frac{\partial^2 v^{ij}}{\partial x_i \partial x_j} = -A$$

*in $P$. Then $v$ is an absolute minimiser for $\mathcal{F}_A$ on $\mathcal{C}_\infty$ i.e., for any $u \in \mathcal{C}_\infty$, $\mathcal{F}_A(u) \geq \mathcal{F}_A(v)$ .*

The proofs of (3.3.3) and (3.3.4) hinge on the following Lemma. Note first that for any $u \in \mathcal{S}$ the function $(u^{,ij})_{,ij}$ is bounded on $P$: this is clear from its identification with the scalar curvature on the toric variety, or by direct calculation.

**Lemma 3.3.5.** *Suppose that $u \in \mathcal{S}$ and $f \in \mathcal{C}_\infty$. Then $u^{,ij} f_{,ij}$ is integrable on $P$ and*

$$\int_P u^{,ij} f_{,ij} d\mu = \int_P (u^{,ij})_{,ij} f d\mu - \int_{\partial P} f d\sigma.$$

For $\delta > 0$ let $P_\delta$ be the interior polytope with faces parallel to those of $P$ and separated by a distance $\delta$. The function $f$ is smooth over the closure of $P_\delta$ so we can integrate by parts

$$\int_{P_\delta} \left( u^{,ij} f_{,ij} - (u^{,ij})_{,ij} f \right) d\mu = \int_{\partial P_\delta} u^{,ij}_{,i} f + u^{,ij} f_{,i}. \tag{3.3.6}$$

Here the integrand on the right-hand side is written as a vector field, which defines an $(n-1)$-form by contraction with the volume form $d\mu$. We want to show that

$$\int_{\partial P_\delta} u^{,ij}_{,i} f \to \int_{\partial P} f d\sigma, \tag{3.3.7}$$

and

$$\int_{\partial P_\delta} u^{ij} f_i \to 0, \tag{3.3.8}$$

as $\delta \to 0$. Consider the second term. Let $x$ be a point of $\partial P_\delta$, in the interior of an $(n-1)$-dimensional face, and $v = v(x)$ be the vector $v^i = u^{,ij} \nu_j$ where $\nu$ is the unit normal to the face containing $x$. Thus the integrand in (3.3.8) can be written as $\nabla_v f$; the derivative of $f$ at $x$ in the direction $v$. We let $y = y(x)$ be the closest point to $x$ on the intersection of the ray $\{x + tv : t > 0\}$ and the boundary $\partial P$. We claim that there are constants $c_1, c_2$ such that for all small $\delta$ and all $x$,

$$|v| \leq c_1 |y - x| \ ; \ |y - x| \leq c_2 \delta. \tag{3.3.9}$$

We assume this for the moment. The convexity of $f$ implies that

$$|\nabla_v f| \leq c_1 |f(y) - f(x)|,$$

using the first inequality of (3.3.9). So

$$\begin{aligned} \left| \int_{\partial P_\delta} u^{ij} f_{,i} \right| &\leq c_1 \int_{\partial P_\delta} |f(y(x)) - f(x)| \\ &\leq c_1 \mathrm{Vol}(\partial P_\delta) \ \max_{x \in \partial P_\delta} |f(y(x)) - f(x)|. \end{aligned}$$

Then the second inequality of (3.3.9), and the fact that $f$ is uniformly continous in $\overline{P}$ implies the desired result (3.3.8). To prove (3.3.9) we consider first the model case when $u = \sum x_i \log x_i$ on the region $x_1, \dots x_n > 0$ and $x$ is a point $(\delta, x_2, \dots, x_n)$. Then $u^{ij}$ is the diagonal matrix $\mathrm{diag}(\delta, x_2, \dots, x_n)$ and the normal is $\nu = (1, 0, \dots, 0)$. Thus the vector $v$ is $(\delta, 0, \dots, 0)$ so $y = (0, x_2, \dots, x_n)$ and

$$|v| = |y - x| = \delta.$$

The proof in the general case is straightforward, checking that changing $u$ by adding a smooth function makes a small change to this model case.

We now turn to (3.3.7). Here we want to show that the $(n-1)$-form $u_i^{ij}$ converges to the measure $d\sigma$ (defined in the obvious way) on $\partial P_\delta$. Exact equality holds for the model case and it is straightforward to deduce the assertion from this.

**Corollary 3.3.10.** *The same assertion as in Lemma* (3.3.5) *holds when $f$ is the difference of two functions in $\mathcal{C}_\infty$.*

This is obvious by linearity.

The other ingredient in the proofs of (3.3.3) and (3.3.4) is the convexity of the function $-\log \det J$ on the space of positive definite matrices. Thus if $j(t) = -\log \det((1-t)J_0 + tJ_1)$ we have

$$j'(0) = -\mathrm{Tr}(J_0^{-1}(J_1 - J_0)), \ j''(0) = \mathrm{Tr}(J_0^{-1}(J_1 - J_0))^2.$$

We now give the proof of (3.3.3). First, by considering the function $u+|x|^2$ we may reduce to the case where $\det(u_{ij}) \geq 1$ so $\log^+ \det(u_{ij}) = \log\det(u_{ij})$. We choose any function $v \in \mathcal{S}$ and write $f = u - v$. Then we apply (3.3.10) to $f$, so that $v^{ij} f_{ij}$ is integrable on $P$. However the convexity of $-\log\det$ implies that

$$\log\det(u_{ij}) = \log\det(v_{ij} + f_{ij}) \leq \log\det(v_{ij}) + v^{ij} f_{ij}.$$

So $\log\det(u_{ij})$ is integrable.

To prove (3.3.4), the convexity of $-\log\det$ immediately implies that the functional $\mathcal{F}_A$ is convex on $\mathcal{C}\infty$. Let $u = v + f$ as above and let $g(t) = \mathcal{F}_A(v + tf)$. Thus $g$ is a convex function on $[0, 1]$. We claim that $g$ is differentiable at $t = 0$. For

$$\frac{\partial}{\partial t} \log\det(v_{ij} + t f_{ij}) = v^{ij} f_{ij}$$

which is integrable over $P$, thus the result follows from the standard test for differentiation under the integral and

$$g'(0) = -\int_P ((v^{ij})_{ij} + A) f d\mu,$$

which vanishes by the hypothesis of (3.3.4). Now the convexity of $g$ implies that $g(1) \geq g(0)$, as required.

**Proposition 3.3.11.** *For any $A \in L^\infty$ the infimum of $\mathcal{F}_A$ on $\mathcal{S}$ is equal to the infimum of $\mathcal{F}_A$ on $\mathcal{C}_\infty$.*

Since $\mathcal{S} \subset \mathcal{C}_\infty$ we need to show that for any $v \in \mathcal{C}_\infty$ and $\epsilon > 0$ there is a $u \in \mathcal{S}$ with $\mathcal{F}_A(u) \leq \mathcal{F}_A(v) + \epsilon$. First we define $v_\eta$, for $\eta < 1$, by $v_\eta(x) = v(\eta x)$. Thus $v_\eta$ is smooth on $\overline{P}$ and it is easy to see that $\mathcal{F}_A(v_\eta) \to \mathcal{F}_A(v)$ as $\eta \to 1$. Thus it suffices to consider the case when $v$ itself is smooth on $\overline{P}$. We can choose a family of smooth functions $l_\delta$ on $[0, \infty)$, for $\delta > 0$, with $l_\delta(x) = x \log x$ if $x \leq \delta$, $l''_\delta \geq 0$ and $0 \geq |l_\delta(x)| \geq 2\delta \log \delta$ for all $x$. Then we can define a function $U_\delta$ on $P$ by

$$U_\delta = \sum_{k=1}^{r} l_\delta(h_k(x) - c_k).$$

Thus $U_\delta \in \mathcal{S}$ but $\|U_\delta\|_{L^\infty} \leq 2r\delta \log(\delta^{-1})$. So

$$\mathcal{F}_A(v + U_\delta) - \mathcal{F}_A(v) = O(r\delta \log(\delta^{-1})) \to 0$$

as $\delta \to 0$.

# 4. K-stability for toric varieties

## 4.1 Basic theory

In this section we take up the algebro-geometric point of view. Our goal is to give a criterion for a toric variety to be K-stable. We begin by recalling the basic facts about the algebraic geometry of toric varieties.

Let $P$ be any integral polytope in $\mathbf{R}^n$. For positive integers $k$ let

$$B_{k,P} = k\overline{P} \cap \mathbf{Z}^n$$

and let $U_{k,P}$ be a complex vector space with a basis labelled by the elements of $B_{k,P}$. This space carries an obvious representation of the $n$-torus $T^c$, since the integer points label the one-dimensional representations. The basic fact we need is:

**Proposition 4.1.1.** *There is a projective toric variety $V_P$ associated to $P$ with a positive, $T^c$-equivariant line bundle $L \to V_P$ such that*

$$H^0(V_P; L^k) = U_{k,P}$$

*as representations of $T^c$.*

To define $V_P$ one can just take the closure of a generic $T^c$-orbit in the projectivization of $U^*_{1,P}$. Note that $V_P$ will only be a smooth variety if $P$ satisfies the "Delzant" condition, but this will not be important for us.

By Proposition (4.1.1), the dimension of the space $H^0(V_P, L^k)$ is given by counting the number of lattice points, $N(kP)$ say, in $k\overline{P}$. We need a simple result describing the asymptotics of $N(kP)$ as $k$ tends to infinity. The most basic fact is that

$$N(kP) = k^n \mathrm{Vol}(P) + O(k^{n-1}), \tag{4.1.2}$$

the refinement we need is:

**Proposition 4.1.3.** *The number $N(kP)$ of lattice points in $k\overline{P}$ satisfies*

$$N(kP) = k^n \mathrm{Vol}(P) + \frac{k^{n-1}}{2}\mathrm{Vol}(\partial P) + O(k^{n-2})$$

*as $k \to \infty$. Here the volume of $\partial P$ is computed using the measure $d\sigma$.*

Proposition (4.1.3) is almost a standard fact—compare the much more general results of [13], for example. We give a proof, for completeness, in the Appendix.

## 4.2 Constructing degenerations

Continuing with notation as above, suppose that $f$ is a convex, rational, piecewise-linear, function on $\mathbf{R}^n$. That is

$$f = \max(\lambda_1, \ldots \lambda_p),$$

where $\lambda_r$ are affine-linear functions with rational co-efficients. Fix an integer $R$ such that $f \leq R$ on $\overline{P}$. Given this data, we define a polytope $Q \subset \mathbf{R}^{n+1} = \mathbf{R}^n \times \mathbf{R}$ to be

$$Q = \{(x,t) : x \in P, 0 < t < R - f(x)\}.$$

The polytope $kQ$ is defined by integral equations provided $k$ is a multiple of a suitable denominator $d$. Thus, taking $k = d$, we get an $(n+1)$-dimensional toric variety $W$ with a line bundle $\mathcal{L} \to W$. The face $\overline{Q} \cap (\mathbf{R}^n \times \{0\}) = P \times \{0\}$ is a copy of $P$ so we get a natural embedding of $i : V \to W$ such that the restriction of $\mathcal{L}$ to $V$ is isomorphic to $L^d$. We write the torus $T^{n+1}$ acting on $W$ as $T^n \times \mathbf{C}^*$, where the $T^n$ action restricts in the obvious way to $i(V)$. The result we need can be summed up as follows.

**Proposition 4.2.1.** *There is a $\mathbf{C}^*$-equivariant map $p : W \to \mathbf{CP}^1$ with $p^{-1}(\infty) = i(V)$ such that the restriction of $p$ to $W \setminus i(V)$ is a test configuration for $(V, L)$ with Futaki invariant*

$$F_1 = -\frac{1}{2\mathrm{Vol}(P)} \left( \int_{\partial P} f d\sigma - a \int_P f d\mu \right),$$

*where $a = \frac{\mathrm{Vol}(\partial P)}{\mathrm{Vol}(P)}$.*

First, there is no loss in supposing that the functions $\lambda_r$ defining $f$ have integral co-efficients, so $d = 1$. We consider the sections $H^0(\mathcal{L})$ over $W$. This space has a basis $s_{I,i}$ where $I$ is a lattice point in $P \cap \mathbf{Z}^n$ and $0 \leq i \leq R - f(i)$, and the $\mathbf{C}^*$ action acts with weight $i$ on $s_{I,i}$. It follows that the ratios $s_{I,i}/s_{I,i+1}$ give $\mathbf{C}^*$-equivariant maps from the region where they are defined (outside the common zeros of $s_{I,i}, s_{I,i+1}$) to $\mathbf{CP}^1$. By rescaling the basis elements we can suppose that these maps all agree on the intersections of their domains of definition. For each point in $W$ there is some section $s_{I,i}$ which does not vanish, so we get a well-defined equivariant map $p : W \to \mathbf{CP}^1$. Now the set $i(V)$ is fixed by the $\mathbf{C}^*$ action, and the action is trivial on the restriction of $\mathcal{L}$ to $i(V)$. Thus all the sections $s_{I,i}$ for $i > 0$ must vanish on $i(V)$ and so

$p$ maps $i(V)$ to $\infty \in \mathbf{CP}^1$. We claim that there are no other points in $p^{-1}(\infty)$. First, if $w$ is a point with a nontrivial stabiliser then by the same argument as above there is a section $s_{I,i}$ which does not vanish at $w$ with $(I,i)$ in the boundary of $Q$ but $i \neq 0$. This implies that $p(w) \neq \infty$. Second if $w$ is a point in the free $T^{n+1}$-orbit in $W$ then the closure of the $\mathbf{C}^*$-orbit of $p(w)$ must be the whole of the image of $p$, and so $p(w)$ cannot be the fixed point $\infty$. Now $\mathbf{C}^*$ acts with same weight $-1$ on the normal bundle to $i(V)$ and the tangent bundle to $\mathbf{CP}^1$ at $\infty$. It follows that the derivative of $p$ cannot vanish on $i(V)$, so the nearby fibres $p^{-1}(t)$ for large $t$ are diffeomorphic to $V$. However these fibres are toric varieties and so by the rigidity of toric varieties we see that $p^{-1}(t)$ is in fact isomorphic to $V$ for large $t$, and hence for all nonzero $t$ because of the group action.

The only remaining task is to compute the Futaki invariant of this test configuration. The divisors $V_0 = p^{-1}(0)$ and $i(V)$ are each defined by the vanishing of sections $\sigma_0, \sigma_1$ of the line bundle $p^*(\mathcal{O}(1))$ over $W$. Hence, at least when $k$ is large enough, we have exact sequences:

$$0 \to H^0(W, \mathcal{L}^k(-1)) \to H^0(W, \mathcal{L}^k) \to H^0(V_0, \mathcal{L}^k) \to 0$$
$$0 \to H^0(W, \mathcal{L}^k(-1)) \to H^0(W, \mathcal{L}^k) \to H^0(i(V), \mathcal{L}^k) \to 0$$

where the inclusion maps are multiplication by $\sigma_0, \sigma_1$. From this we see first that the dimension $d_k$ of $H^0(V_0, \mathcal{L}^k)$ is the same as that of $H^0(V, \mathcal{L}^k)$. The $\mathbf{C}^*$-action acts with weight 0 on $\sigma_0$ and weight 1 on $\sigma_1$ so it follows that the weight $w_k$ of the action on $\Lambda^{d_k} H^0(V_0, \mathcal{L}^k)$ is given by the weight of the action on $\Lambda^{d_k} H^0(i(V), \mathcal{L}^k)$ plus the dimension of $H^0(W, \mathcal{L}^k(-1))$. But the action on $H^0(i(V), \mathcal{L}^k)$ is trivial, so we have

(4.2.2)
$$d_k = \dim\, H^0(W, \mathcal{L}^k(-1)) = \dim\, H^0(W, \mathcal{L}^k) - \dim\, H^0(i(V), \mathcal{L}^k).$$

Thus
$$d_k = N(kP), w_k = N(kQ) - N(kP).$$

For the first term we immediately have from (4.1.3):

$$d_k = k^n \mathrm{Vol}(P) + \frac{k^{n-1}}{2} \mathrm{Vol}(\partial P) + O(k^{n-2}).$$

For the second term, we divide the boundary of $Q$ into three parts:

(1) points $(x, 0)$ for $x \in \overline{P}$;

(2) points $(x, R - f(x))$ for $x \in \overline{P}$;

(3) points $(x, t)$ with $x \in \partial P$ and $0 < t < R - f(x)$.

Clearly the number of lattice points in each of the first two parts is just $N(kP)$. The number $\nu_k$ of lattice points in the third class is estimated, applying (4.1.2) to each face, by the volume so

$$\nu_k = k^n \int_{\partial P} R - f d\sigma + O(k^{n-1}).$$

Then, applying (4.1.3) to $Q$, we have

$$w_k = k^{n+1} \int_P (R - f) d\mu + \frac{k^n}{2} \int_{\partial P} (R - f) d\sigma. \tag{4.2.3}$$

The formulae (4.2.2) and (4.2.3) imply that

$$\frac{w_k}{k d_k} = \frac{A}{C} + \frac{1}{2kC^2}(BC - AD) + O(k^{-2}),$$

where

$$\begin{aligned} A &= \int_P R - f d\mu; \\ B &= \int_{\partial P} R - f d\sigma; \\ C &= \mathrm{Vol}(P); \\ D &= \mathrm{Vol}(\partial P). \end{aligned}$$

By definition, the Futaki invariant is the co-efficient $(BC - AD)/2C^2$ of the $k^{-1}$ term, which is

$$-\frac{1}{2\mathrm{Vol}(P)} \left( \int_{\partial P} f d\sigma - \frac{\mathrm{Vol}(\partial P)}{\mathrm{Vol}(P)} \int_P f d\mu \right),$$

as asserted in the Proposition.

## 5. The main argument

### 5.1 Reduction to the linear functional

We will now relate the nonlinear functional $\mathcal{F}_A$ to its linear part $\mathcal{L}_A$. Here we can take any bounded function $A$ on $P$ which satisfies the moment condition, for each affine-linear function $f$,

$$\int_P fA d\mu = \int_{\partial P} f d\sigma. \tag{5.1.1}$$

When this condition holds the functionals $\mathcal{L}_A(u), \mathcal{F}_A(u)$ are not changed if one adds an affine-linear function to $u$, and it is convenient to work with functions which are normalised in the following way. We fix a point $p \in P$ and say $u$ is normalised if $u \geq 0$ and $u(p) = 0$. The main result is:

**Proposition 5.1.2.** *Suppose there is a constant $\lambda > 0$ such that for all normalised functions $u \in \mathcal{C}_\infty$ we have*

$$\mathcal{L}_A(u) \geq \lambda \int_{\partial P} u \; d\sigma.$$

*Then $\mathcal{F}_A$ is bounded below on $\mathcal{C}_\infty$.*

The proof of this is surprisingly easy. First, observe the following:

**Lemma 5.1.3.** *There is a constant $C$ such that*

$$\int_P u \; d\mu \leq C \int_{\partial P} u \; d\sigma$$

*for all normalised functions $u$.*

This is clear when one works in polar co-ordinates centred at the point $p$. Now fix some arbitrary smooth invariant metric on $V$, thus some function $v$ in $\mathcal{S}$. We define a function $B$ on $P$ by

$$B = -\sum \frac{\partial^2 v^{ij}}{\partial x_i \partial x_j}.$$

We know that $B$ represents the scalar curvature on $V$, hence is certainly bounded (a fact we could readily verify directly). According to Proposition (3.3.4), the function $v$ gives an absolute minimum for $\mathcal{F}_B$ on $\mathcal{C}_\infty$, so for all functions $u \in \mathcal{C}_\infty$

$$-\int_P \log\det(u_{ij}) d\mu + \mathcal{L}_B(u) \geq C,$$

where $C = \mathcal{F}_B(v)$. Now replace $u$ in this inequality by $ru$, for a constant $r > 0$. The inequality becomes

$$-\int_P \log\det(u_{ij})d\mu + r\mathcal{L}_B(u) \geq C_r, \tag{5.1.4}$$

where $C_r = C + \log r\mathrm{Vol}(P)$. The hypothesis of the Proposition gives

$$\mathcal{L}_A(u) \geq \lambda \int_{\partial P} u\ d\sigma, \tag{5.1.5}$$

for all normalised $u$, and Lemma (5.1.3) implies that

$$|\mathcal{L}_A(u) - \mathcal{L}_B(u)| \leq C\|A - B\|_{L^\infty} \int_{\partial P} u d\sigma. \tag{5.1.6}$$

Combining (5.1.5) and (5.1.6) we have

$$|\mathcal{L}_A(u) - \mathcal{L}_B(u)| \leq C'\mathcal{L}_A(u), \tag{5.1.7}$$

where $C' = \lambda^{-1}C\|A - B\|_{L^\infty}$. Thus

$$\mathcal{L}_B(u) \leq (C' + 1)\mathcal{L}_A(u)$$

and (5.1.4) gives

$$-\int_P \log\det(u_{ij})d\mu + r(C' + 1)\mathcal{L}_A(u) \geq C_r.$$

Finally, choose $r = r_0 = (C' + 1)^{-1}$, to get $\mathcal{F}_A(u) \geq C_{r_0}$.

The point here is that $\mathcal{C}_\infty$ is preserved by multiplication by positive scalars. The same is not true for $\mathcal{S}$ and this is the only reason for introducing the larger domain for $\mathcal{F}_A$ in Section (3.3). We use the same idea to prove

**Proposition 5.1.8.** *Under the same hypothesis as* (5.1.2), *there is constant $K$ such that if $u^{(\alpha)}$ is any sequence of normalised functions in $\mathcal{C}_\infty$ which is a minimising sequence for $\mathcal{F}_A$ on $\mathcal{C}_\infty$ then*

$$\int_{\partial P} u^{(\alpha)} \leq K,$$

*for large enough $\alpha$.*

To prove this we consider the scaling of any function $v$ in $\mathcal{C}_\infty$;

$$\mathcal{F}_A(rv) = -\int_P \log\det(v_{ij})d\mu - \log r \mathrm{Vol}(P) + r\mathcal{L}_A(v).$$

Fixing $v$, this is a function of $r$ which is minimised when $r = r_1 = \mathcal{L}_A(u)/\mathrm{Vol}(P)$, and $\mathcal{L}_A(r_1 u) = \mathrm{Vol}(P)$. Given any minimising sequence of normalised functions $u^{(\alpha)}$ we can write $u^{(\alpha)} = r_\alpha v^{(\alpha)}$ where

$$\mathcal{L}_A(v^{(\alpha)}) = \mathrm{Vol}(P).$$

Then

$$\mathcal{F}_A(u^{(\alpha)}) = \mathcal{F}_A(v^{(\alpha)}) + \mathrm{Vol}(P)(r_\alpha - \log r_\alpha - 1).$$

It follows that $r_\alpha - \log r_\alpha - 1 \to 0$ as $\alpha \to \infty$, and thence that $r_\alpha \to 1$. This means that $\mathcal{L}_A(u^{(\alpha)}) \to \mathrm{Vol}(P)$ and then our result follows from (5.1.3).

## 5.2 Compactness

Here we study the linear functional $\mathcal{L}_A$, for an arbitrary bounded function $A$ on $P$. Our goal is to obtain a criterion under which the bound in (5.1.2) of the previous subsection will hold. Let $P^*$ be the union of $P$ and its codimension-one faces.

**Definition 5.2.1.** The set $\mathcal{C}_1$ is the set of positive convex functions $f$ on $P^*$ such that

$$\int_{\partial P} f \; d\sigma < \infty.$$

Note that the integral of any function in $\mathcal{C}_1$ over the boundary makes sense, even though $f$ is not defined on the whole boundary, because the measure $d\sigma$ is supported on the codimension-one faces. The same proof as for Lemma (5.1.3) shows that any $f \in \mathcal{C}_1$ is integrable on $P$, so $\mathcal{L}_A(f)$ is defined. Our main result is:

**Proposition 5.2.2.** *Either there is a $\lambda > 0$ such that*

$$\mathcal{L}_A(f) \geq \lambda \int_{\partial P} f d\sigma$$

*for all normalised functions $f$ in $\mathcal{C}_\infty$ or there is a function $f$ in $\mathcal{C}_1$ which is not affine linear and such that $\mathcal{L}_A(f) \leq 0$.*

The general scheme of proof is a compactness argument using gradient bounds derived from the following Lemma. For any convex function $f$ on the polytope $P$ and any point $x$ in $P$ we let $D_x(f)$ be the supremum of the slope of supporting hyperplanes to $f$ at $x$; i.e.,

$$D_x(f) = \sup\{|\lambda| : f(y) \geq \lambda(y-x) + f(x) \text{ for all } y \in P\}.$$

**Lemma 5.2.3.** *There is a universal constant $\kappa$ such that for any positive convex function $f$ on $P$ with*

$$\int_P f d\mu < \infty$$

*and for any point $x \in P$ we have*

$$D_x f \leq \kappa d_x^{-(n+1)} \int_P f d\mu,$$

*where $d_x$ is the distance from $x$ to the boundary of $P$.*

To prove this, we may suppose that $x$ is the origin and that $f(y) \geq Dy_1 + f(0)$, where $y_1$ denotes the first component of $y \in P \subset \mathbf{R}^n$. Let $B^+(d_0)$ denote the intersection of the ball of radius $d_0$ with the half-space $\{y_1 \geq 0\}$ so $B^+(d_0) \subset P$ and $f \geq Dy_1$ on $B^+(d_0)$. Thus

$$\int_P f d\mu \geq D \int_{B^+(d_0)} y_1 d\mu = D\kappa^{-1} d_0^{n+1},$$

for a constant $\kappa = (n+1)/2\mathrm{Vol}(B^{n-1})$.

**Lemma 5.2.4.** *For any convex function $f$ on $P$ and any two points $x, y$ in $P$:*

$$|f(x) - f(y)| \leq \max(D_x(f), D_y(f))\ |x-y|.$$

This is clear in the one-dimensional case and the general case reduces to this by considering restriction to the line segment between $x$ and $y$.

Now for small $d > 0$ let $P_d$ denote the set of points in $P$ such that $d_x \geq d$. Clearly

$$\min_{P_d} f \leq \frac{1}{\mathrm{Vol}(P_d)} \int_P f d\mu,$$

and Lemmas (5.2.3) and (5.2.4) imply that

$$\max_{P_d} f \leq \min_{P_d} f + R\kappa d^{-(n+1)} \int_P f d\mu,$$

where $R$ is the diameter of $P$. Thus a bound on the integral of $f$ over $P$ gives a Lipschitz bound on $f$ in any interior region, and Ascoli-Arzela implies:

**Corollary 5.2.5.** *Any sequence of positive convex functions $f_n$ on $P$ with*

$$\int_P f_n d\mu \leq C$$

*has a subsequence which converges uniformly over compact subsets of $P$.*

**Proposition 5.2.6.** *Suppose that $f_n$ is a sequence of normalised functions in $\mathcal{C}_\infty$ with*

$$\int_{\partial P} f_n \leq C.$$

*Then there is a subsequence which converges, uniformly over compact subsets of $P$, to a convex function which has a continuous extension to a function $f^*$ on $P^*$, defining an element of $\mathcal{C}_1$ with*

$$\int_{\partial P} f^* d\sigma \leq \liminf \int_{\partial P} f_n d\sigma.$$

In the proof we will assume our base point $p \in P$ is the origin.

First, Lemma (5.1.3) yields a bound on the integrals of the $f_n$ over $P$, so we are in the situation of Corollary (5.2.5) and we can suppose that the $f_n$ converge uniformly on compact subsets of $P$ to a limit $f$. Next, we can apply Corollary (5.2.5), replacing $P$ by one of its codimension 1 faces, so we may suppose that the $f_n$ converge pointwise over all of $P^*$. We must beware however because we may not be able to take $f^* = \lim f_n$. Instead, we proceed as follows. It is a simple fact that any convex function on an open interval has a well-defined limit, possibly infinite, at each end point. We apply this to the restriction of $f$ to the rays through the origin, so for any point $z$ in a codimension-1 face of the boundary of $P$ we define $f^*(z) \in [0, \infty]$ to be the limit of $f(tz)$ as $t$ tends to 1 from below. For each $\eta < 1$ consider the set $\eta\partial P \subset P$, with the obvious measure induced by $d\sigma$. It is clear that the integral of $f_n$ over $\eta\partial P$ is at most $C$ so Lemmas (5.2.3), (5.2.4) give a Lipschitz bound on the restriction of $f$ to $\eta K$ for any compact subset $K$ of a codimension face of $P$. It follows then that $f^*(z)$ is the limit of $f(x_n)$ for *any* sequence $x_n$ in $P$ converging to $z$.

We claim now that, for $z$ in a codimension-one face,

$$f^*(z) \leq \lim f_n(z). \tag{5.2.7}$$

(This statement includes the assertion that $f^*(z)$ is finite.) Indeed, since the $f_n$ are convex and normalised,

$$f_n(tz) \leq t f_n(z),$$

so taking limits as $n \to \infty$

$$f(tz) \leq t \lim f_n(z),$$

and now (5.2.7) follows by taking limits as $t \to 1$. Thus $f^*$ is a continuous function on $P^*$, convex on $P$, and hence convex on $P^*$ by continuity. We also have

$$\int_{\partial P} f^* d\sigma \leq \int_{\partial P} \lim f_n d\sigma \leq \liminf \int_{\partial P} f_n d\sigma,$$

using (5.2.7) for the first inequality and Fatou's Lemma for the second. This completes the proof.

We can now prove our main result (5.2.2). Suppose the first alternative does not hold, so there is a sequence of normalised functions $f_n \in \mathcal{C}_\infty$ with

$$\int_{\partial P} f_n d\sigma = 1$$

and $\mathcal{L}_A(f_n) \to 0$ as $n \to \infty$; that is

$$\int_P A f_n d\mu \to 1.$$

We apply Proposition (5.2.6), so we can suppose that there is a function $f^*$ in $\mathcal{C}_1$ such that $f_n$ converges to $f^*$, uniformly on compact subsets of $P$. For $\eta < 1$, we consider the "collar" region $P \setminus \eta P$. It is clear that there is a constant $c$ such that for any normalised convex function $g$

$$\int_{P \setminus \eta P} g d\mu \leq c(1-\eta) \int_{\partial P} g d\sigma.$$

Thus

$$\int_{P \setminus \eta P} A f_n d\mu \leq c \|A\|_{L^\infty} (1-\eta).$$

It follows from this, and the uniform convergence over compact subsets, that

$$\int_P A f^* d\mu = \lim \int_P A f_n d\mu = 1.$$

On the other hand we know that

$$\int_{\partial P} f d\sigma \leq \lim \int_{\partial P} f_n d\sigma = 1,$$

so we have $\mathcal{L}_A(f^*) \leq 0$, as desired.

Recall that a piecewise linear convex function on $\mathbf{R}^n$ is a function of the form

$$\max\,(g_1, \ldots, g_N)$$

where the $g_j$ are affine-linear functions.

**Proposition 5.2.8.** *For any $f \in \mathcal{C}_1$ there is a sequence of piecewise linear convex functions $f_n$ such that*

$$\int_{\partial P} |f_n - f| d\sigma \to 0; \quad \int_P |f_n - f| d\mu \to 0,$$

*as $n \to \infty$.*

We will construct $f_n$ such that $0 \leq f_n \leq f$ everywhere, so the conditions we need are

$$\int_{\partial P} f - f_n \to 0, \quad \int_P f - f_n \to 0.$$

First note that it suffices to satisfy these conditions separately, since if the sequence $f_n^{\partial P}$ satisfies the first condition, with $f_n^{\partial P} \leq f$, and the sequence $f_n^P$ satisfies the second condition, with $f_n^P \leq f$; the sequence of functions $f_n = \max(f_n^P, f_n^{\partial P})$, are still PL convex functions and satisfies both conditions. Now consider a decomposition

$P = \cup_i \Sigma_i$ where the $\Sigma_i$ are the intersections of $P$ with a standard mesh of closed cubes in $\mathbf{R}^n$ with side length $\delta$. Let $\sigma_i \in \Sigma_i$ be a point where $f$ is minimal. Let $\underline{f}_\delta$ be the function on $P$ with $\underline{f}_\delta(x) = f(\sigma_i)$ for $x$ in the interior $\Sigma_i$, defined arbitrarily on the boundaries. The integral of this function corresponds to the usual "lower sum" in the theory of the Riemann integral, and it is clear using (5.2.3) and the continuity of $f$ in $P$ that

$$\int_P f - \underline{f}_\delta d\mu \to 0$$

as $\delta \to 0$.

The convex sets in $\mathbf{R}^{n+1} = \mathbf{R}^n \times \mathbf{R}$

$$\Sigma_i \times (-\infty, f(\sigma)), \{(x,t) \in P \times \mathbf{R} : t \geq f(x)\},$$

have disjoint interiors. It follows from a standard separation theorem that there is a hyperplane in $\mathbf{R}^{n+1}$ such that the two sets lie in different closed half-spaces. In other words, this means that there is an affine-linear function $g_i$ such that

$$f(x) \geq g_i(x)$$

for all $x \in P$, and

$$g_i(x) \geq f(\sigma_i)$$

for all $x \in \Sigma_i$. Thus if we define $g$ to be $\max g_i$ we have $f \geq g \geq \underline{f}_\delta$ and so

$$\int f - g \leq \int f - \underline{f}_\delta$$

and $\int f - g \to 0$ with $\delta$.

To handle the boundary integral we first consider the restriction of $f$ to $\eta\partial P$ for $\eta < 1$. We may consider this as a function $f_\eta$ on $\partial P$ in an obvious way and the dominated convergence theorem implies that $f_\eta$ tends to $f|_{\partial P}$ in $L^1(\partial P)$ as $\eta \to 1$. Let $g$ be any PL function with $0 \leq g \leq f$ on $P$. The fact that $f(0) = 0$ implies that $g(x) \geq g(\eta x)$ for all $x \in \partial P$. Thus

$$\int_{\partial P} f - g d\sigma \leq \int_{\partial P} f - f_\eta d\sigma + \int_{\eta \partial P} f - g d\sigma.$$

Since the first term can be made arbitrarily small by taking $\eta$ close to 1, it suffices to show that we can find a PL function $g$ of this kind with

$$\int_{\eta\partial P} f - g \, d\sigma$$

as small as we please. To see this we follow the same argument as before. We subdivide $\eta\partial P$ into small convex pieces and use the separation theorem to find a PL function $g \leq f$ everywhere on $P$ but with $g$ bounded below by the Riemann lower sum function on $\eta\partial P$. Note there is no loss in supposing $g \geq 0$, as above, because we can always replace $g$ by $\max(g, 0)$.

## 5.3 Extremal functions

In this subsection we analyse the extremals of the linear functional in the semi-positive case. We suppose the dimension $n$ is 2 throughout the subsection (although some of the arguments go over to higher dimensions). We define a *simple piecewise linear function* on $\mathbf{R}^2$ to be a function of the form

$$f(x) = \max(\lambda(x) + c, 0),$$

where $\lambda : \mathbf{R}^2 \to \mathbf{R}$ is a linear function with $|\lambda| = 1$. We call the line $\lambda(x) = -c$ on which $f$ is not smooth the *crease* of $f$. Let $A$ be a continuous, strictly positive function on $\overline{P} \subset \mathbf{R}^2$ satisfying the moment conditions (5.1.1). The goal of this subsection is to prove:

**Proposition 5.3.1.** *Suppose $\mathcal{L}_A(u) \geq 0$ for all $u \in \mathcal{C}_1$ but there is a function $u \in \mathcal{C}_1$, which is not affine linear, with $\mathcal{L}_A(u) = 0$. Then there is a simple piecewise linear function $f$ whose crease meets $P$ and with $\mathcal{L}_A(f) = 0$.*

The proof will require a number of elementary lemmas. First note that, while $u$ is only defined *a priori* on $P^*$—the complement of the vertices in $P$—it is easy to see that it extends to a continous map from $\overline{P}$ to $(-\infty, \infty]$, i.e., $u(x)$ has a well-defined limit, finite or infinite, as $x$ approaches a vertex. Now suppose that $u \geq 0$ on $\overline{P}$ and that $u = 0$ somewhere in $P$. Let $K = u^{-1}(0) \subset \overline{P}$. The convexity of $u$ implies that $K$ is a closed convex set.

**Lemma 5.3.2.** *$K$ is the convex hull of $K \cap \partial P$.*

To prove the Lemma we need to show that any point $x_0$ of $K$ lies inside the convex hull of $K \cap \partial P$. Suppose not, then there is a line separating $x_0$ from $K \cap \partial P$. It follows that there is an affine-linear function $g$ with $g(x_0) > 0$ and $g(y) \leq -c < 0$ for all $y \in K \cap \partial P$. For small $\epsilon > 0$ consider the convex function

$$u_\epsilon(x) = \max(u(x), \epsilon g(x)).$$

A straightforward compactness argument shows that when $\epsilon$ is sufficiently small, $u(y) \geq \epsilon g(y)$ for all $y \in \partial P$. Thus $u_\epsilon = u$ on $\partial P$ while $u_\epsilon(x) > u(x)$. It follows that $\mathcal{L}_A(u_\epsilon) < \mathcal{L}_A(u)$ (because the boundary contributions is not changed but the interior contribution is strictly smaller, since $A > 0$). This gives a contradiction.

Of course, adding an affine-linear function to $u$, we deduce an analogue of (5.3.2) for any supporting hyperplane of $u$. One approach to proving (5.3.1) would be to show that this implies that $u$ can be written as an integral

$$u = \int_T f_t d\tau,$$

where $T$ is some parameter space, $d\tau$ is a positive measure on $T$, and for each $t$ the function $f_t$ is a simple PL function. The hypothesis of (5.3.1) would then imply that $\mathcal{L}_A(f_t) = 0$ for each $t$ which would imply the desired result. However this line of argument seems to involve some relatively sophisticated analysis, so instead we will follow a more elementary line in the proof below.

**Lemma 5.3.3.** *Let $f$ be a continous function on an interval $[a, b]$ in $\mathbf{R}$ and suppose $a \leq c \leq c' \leq b$.*

(1) *Suppose that $f \geq 0$ on $[a, b]$, that $f$ is convex on the sub-intervals $[a, c]$ and $[c', b]$ and that $f = 0$ on the interval $[c, c']$. Then $f$ is convex on $[a, b]$.*

(2) *Suppose that $f = \mu_1$ on the segment $[a, c]$ and $f = \mu_2$ on the segment $[c', b]$, where $\mu_1, \mu_2$ are affine-linear functions. Suppose that $f$ is convex on $[c, c']$ and that $f \geq \mu_1, \mu_2$ throughout $[a, b]$. Then $f$ is convex on $[a, b]$.*

The proof is left to the reader.

**Lemma 5.3.4.** *Either the conclusion of Proposition* (5.3.1) *is true or for each point $x_0$ in $P$ the supporting hyperplane to $u$ at $x_0$ is unique; that is, there is exactly one affine-linear function $g$ with $g(x_0) = u(x_0)$ and $g \leq u$ on $P$.*

Suppose that there are two distinct supporting hyperplanes at $x_0$. By adding an affine-linear function to $u$, we may suppose that one of them is the zero hyperplane, i.e., that $u(x_0) = 0$ and $u \geq 0$ on $\overline{P}$. Thus there is a simple piecewise linear convex function $f$ with crease a line $L$ through $x_0$ such that $u \geq \epsilon f$ for some fixed $\epsilon > 0$. The crease $L$ meets $\partial P$ in two points $p, q$ say. The set $K = u^{-1}(0)$ is contained in the half-space where $f \geq 0$ bounded by $L$, and so Lemma (5.3.2) implies that $p$ and $q$ are both in $K$ (because $x_0$ is in $K$). Hence $K$ contains the line segment $L \cap \overline{P}$. Define a function $v$ on $P$ by $v = u - \epsilon f$. We claim that $v$ is convex. For $v$ is certainly convex on the intersection of $\overline{P}$ with each of the two half-spaces defined by the crease (since it differs from $u$ by an affine linear function on each piece); $v \geq 0$ and $v$ vanishes on the intersection $L \cap P$ of the two pieces. Then the convexity of $v$ follows from (1) of Lemma (5.3.3) (with $c = c'$), applied to any segment in $\overline{P}$.

Now we write $u = \epsilon f + v$. The hypothesis of Proposition (5.3.1) implies that $\mathcal{L}_A(v), \mathcal{L}_A(f) \geq 0$ so we must have $\mathcal{L}_A(v) = \mathcal{L}_A(f) = 0$ and the Lemma is proved.

Let us now suppose–arguing for a contradiction–that the conclusion of Proposition (5.3.1) does *not* hold. The conclusion of Lemmas (5.3.2) and (5.3.3) can be expressed as follows. We define an equivalence relation on points of $P$ by saying $x$ is related to $x'$ if $u$ has the same supporting hyperplane at $x$ and $x'$. Then each equivalence class is of the form $K \cap P$ where $K \subset \overline{P}$ is the convex hull of $K \cap \partial P$. It follows then that each equivalence class in $P$ is either a line segment or intersection of a closed polygon with $P$. Let $K_1 \cap P, K_2 \cap P$ be two distinct equivalence classes. Thus there are distinct affine-linear functions $g_1, g_2$ such that $u = g_i$ on $K_i$. There are open half-spaces $H_1, H_2$ in $\mathbf{R}^2$ with the following properties:

(1) $H_1 \cap \overline{P}, H_2 \cap \overline{P}$ are disjoint convex subsets of $\overline{P}$.

(2) $K_i$ is contained in the closure of $H_i$.

(3) The topological boundary of $H_i \cap \overline{P}$ in $\overline{P}$ is contained in $K_i$.

We define a function $\underline{u}$ on $\overline{P}$ by

$$\begin{aligned} \underline{u}(x) &= g_1(x) \quad \text{if } x \in H_1 \cap \overline{P}; \\ &= g_2(x) \quad \text{if } x \in H_2 \cap \overline{P}; \\ &= u(x) \quad \text{otherwise.} \end{aligned}$$

The conditions on the half-planes, stated above, show that $\underline{u}$ is continuous on $P$. We let $v = u - \underline{u}$.

**Lemma 5.3.5.** *The functions $\underline{u}$ and $v$ are convex.*

As for Lemma (5.3.4), this follows from Lemma (5.3.3), by restricting to segments in $P$. (Item (1) of (5.3.3) applies to $v$ and item (2) to $\underline{u}$.) Clearly the functions $\underline{u}$ and $v$ both lie in $\mathcal{C}_1$ so, as before, the extremal condition implies:

**Corollary 5.3.6.** $\mathcal{L}_A(\underline{u}) = \mathcal{L}_A(v) = 0$.

Return now to to consider the decomposition of $\overline{P}$ into closed, convex equivalence classes defined by the extremal function $u$. Choose any equivalence class $K \cap P$ and a point $p$ in $K \cap P$ which does not lie in the interior of $K$. Thus $p$ lies in a line segment in $K$ with end points $a, b$ in $\partial P$. We consider a point $q$ in $P$, close to $p$ but not in $K$. Let $K_q$ be the equivalence class containing $q$. Clearly, if $q$ is close enough to $p$ it must lie on a line segment in $K_q$ with end points $a', b'$ where $a'$ and $a$ lie on some common edge $E$ of $\partial P$ and $b'$ and $b$ lie on another common edge $F$ of $\partial P$. It may happen that $a'$ is equal to $a$ or $b'$ is equal to $b$ (both cannot occur since $p$ and $q$ are in different equivalence classes). Let $I$ be the line segment in $P$ with end points $p, q$.

**Lemma 5.3.7.** *If the conclusion of Proposition* (5.3.1) *does not hold, there is a constant* $c_1 > 0$ *such that* $\mathcal{L}_A(f) \geq c_1$ *for all simple convex piecewise linear functions whose crease meets* $I$.

The proof is easy: $\mathcal{L}_A$ is a continuous function on the compact set of simple piecewise-linear functions whose crease meets $I$.

Suppose that $K_1, K_2$ are distinct equivalence classes as above, and each meets the line segment$I$. Let $d$ be the minimal distance between points of the sets $K_1 \cap I, K_2 \cap I$. Continuing with the notation as above, if we define $h = \max(g_1, g_2)$ then $u \geq h$ and we can write $h = g_1 + \epsilon f$ where $\epsilon > 0$ and $f$ is a simple piecewise-linear function. Thus $\mathcal{L}_A(h) = \epsilon \mathcal{L}_A(f)$, since $\mathcal{L}_A$ vanishes on the affine-linear functions, and

$$\mathcal{L}_A(h) \geq c_1 \epsilon \tag{5.3.8}$$

by Lemma (5.3.7).

**Lemma 5.3.9.** *There is a constant* $c_2 > 0$ *such that for any* $K_1, K_2$ *as above,* $(\underline{u} - h)(x) \leq c_2 \epsilon d$.

Assuming this Lemma we can complete the proof of (5.3.1). The Lemma implies that, for any $K_1, K_2$

$$|\mathcal{L}_A(\underline{u} - h)| \leq c_3 \epsilon d,$$

where

$$c_3 = c_2 \left( \int_P |A| d\mu + \mathrm{Vol}(\partial P, d\sigma) \right).$$

Thus, assuming the conclusion of Proposition (5.3.1) is false,

$$\mathcal{L}_A(\underline{u}) \geq c_1 \epsilon - c_3 d \epsilon$$

by (5.3.8). On the other hand, by (5.3.6), $\mathcal{L}_A(\underline{u}) = 0$, so we must have $d \geq d_I = c_1/c_3$. Now to finish the proof: we subdivide the segment $I$ into $N$ equal pieces by a chain of points $p_0 = p, p_1, \dots p_N = q$ and choose $N$ so large that the distance between consecutive points $p_i, p_{i+1}$ is less than $d_I$. It follows then that consecutive points must lie in the same equivalence class, hence the supporting hyperplanes at $p$ and $q$ are in fact equal, a contradiction.

It only remains to give the proof of Lemma (5.3.9). We suppose that neither $K_1$ nor $K_2$ is equal to $K$ or $K_q$—the argument works equally well in the case of equality, except for a change of notation. Likewise we asssume that $a, a', b, b'$ are all distinct—again the argument works just as well if $a = a'$ with a change of notation. The complement $\overline{P} \setminus (K \cup K_q)$ has a connected component which is a quadrilateral with two opposite edges given by segments $\overline{aa'}$ and $\overline{bb'}$ in the edges $E, F$ of $P$, but otherwise disjoint from the the boundary of $P$. It follows easily that $K_1 \cap \partial P$ has exactly two connected components—one contained in $E$ and the other in $F$—and likewise for $K_2$. The function $\underline{u} - h$ is supported in $Q = \overline{P} \setminus (H_1 \cup H_2)$ and it follows in turn that $Q$ is another quadrilateral with two opposite edges given by segments $\overline{\alpha\alpha'} \subset \overline{aa'}$, $\overline{\beta\beta'} \subset \overline{bb'}$ say, and that (after possibly interchanging $K_1, K_2$) the other edges $\overline{\alpha\beta}$ and $\overline{\alpha'\beta'}$ lie in $K_1, K_2$ respectively. This means that any point of $Q$ lies on a line segment $J$ in $Q$ with one endpoint in $K_1$ and one endpoint in $K_2$ and with the length of $J$ at most equal to $D$, where

$$D = \max(d(\alpha, \alpha'), d(\beta, \beta')).$$

Elementary geometry shows that there is a constant $C$, depending only on the line segments $I, E, F$ such that

$$D \leq Cd.$$

To finish the proof we again use a simple lemma about convex functions of one real variable.

**Lemma 5.3.10.** *Let $f$ be a positive convex function on an interval $[0, L]$ with $f(0) = 0$. Let*

$$\widetilde{h}(x) = \max(0, \lambda(x - L) + f(a)),$$

*where $\lambda > 0$ is such that $f(x) \geq \widetilde{h}(x)$ for all $x \in [0, L]$. Then for all $x$ in $[0, L]$ we have*

$$f(x) - \widetilde{h}(x) \leq \frac{\lambda L}{4}.$$

We leave the proof of this to the reader. Lemma (5.3.9), with $c_2 = C/4$, follows from (5.3.10) and the preceding discussion, by considering restriction to the line segment $J$.

## 6. Rationality of extremal functions

In this section we consider an integer polytope $P \subset \mathbf{R}^2$ with the canonical boundary measure, which satisfies the moment conditions

(5.1.1). We say a simple PL function $f$ is *rational* if it has the form

$$f(x) = \rho \max(0, \lambda(x) + c)$$

where $\rho$ is real, $c$ is rational and $\lambda$ is a linear function on $\mathbf{R}^2$ with rational co-efficients. This is the same as saying that the crease of $f$ is a rational line. Here we prove:

**Proposition 6.1.** *If there is a simple* PL *function $g_0$, whose crease meets $P$, such that $\mathcal{L}_a(g_0) \leq 0$ then there is a rational simple* PL *function $f$, whose crease meets $P$, with $\mathcal{L}_a(f) \leq 0$.*

To prove this we consider three cases—clearly the crease of $g_0$ meets $\partial P$ in precisely two points $p, q$ and the cases are:

(1) one or both of $p, q$ is a vertex of $P$;

(2) neither $p$ nor $q$ is a vertex of $P$ and the faces of $P$ containing $p$ and $q$ are parallel;

(3) neither $p$ nor $q$ is a vertex of $P$ and the faces of $P$ containing $p, q$ are not parallel.

We begin with the third case. We consider the function $\mathcal{L}_a$ on the manifold of simple PL functions. Under the hypothesis (3) this is a smooth function in a neighbourhood of $g_0$. The rational simple PL functions form a dense subset, so the desired result will obviously hold if either $\mathcal{L}_a(g_0) < 0$ or $\mathcal{L}_a(g_0) = 0$ but the derivative of $\mathcal{L}_a$ does not vanish at $g_0$. So we need to show that if $\mathcal{L}_a$ and its derivative both vanish at $g_0$ then $g_0$ is rational. Let $P^+$ be the support of $g_0$ in $P$ and $P^-$ be the complement in $P$: likewise let $(\partial P)^+$ be the support of $g_0$ in $\partial P$ and $(\partial P)^-$ be its complement in $\partial P$.

**Lemma 6.2.** *The function $\mathcal{L}_a$ and its derivative both vanish at $g_0$ if and only if $P^+$ and $(\partial P)^+$ have the same mass and centre of mass: that is*

$$\int_{(\partial P)^+} \alpha d\sigma = a \int_{P^+} \alpha d\mu$$

*for all affine-linear functions $\alpha$ on $\mathbf{R}^2$.*

We leave the proof to the reader. Notice that the hypothesis (5.1.1) implies that we get the same condition if we replace $P^+, (\partial P)^+$ by $P^-, (\partial P)^-$.

We will now write down this condition more explicitly. By making a rational affine transformation of $\mathbf{R}^2$ we may suppose that the faces of $P$ containing $p$ and $q$ are segments of the co-ordinate axes in $\mathbf{R}^2$ and that $P$ lies in the positive quadrant $\{(x_1, x_2) : x_1, x_2 > 0\}$. Let $p = (t_1, 0)$ and the face containing $p$ be

$$I_1 = \{(\tau_1, 0) : a_1 < \tau_1 < b_1\},$$

and let $q = (0, t_2)$ and the face containing $q$ be

$$I_2 = \{(0, \tau_2) : a_2 < \tau_2 < b_2\}.$$

The crease of $g_0$ is the line with equation $\frac{x_1}{t_1} + \frac{x_2}{t_2} = 1$. Let $\Delta$ be the triangular region defined by the inequalities

$$x_1 > 0, \quad x_2 > 0, \quad \frac{x_1}{t_1} + \frac{x_2}{t_2} < 1.$$

We may suppose, interchanging $P^+$ and $P^-$ if necessary, that $P^+$ is contained in $\Delta$. Thus $P^+ = Q \cup R$ say, where $R$ is the region defined by the inequalities

$$x_1 > 0, \quad x_2 > 0, \quad \frac{x_1}{t_1} + \frac{x_2}{t_2} < 1, \quad \frac{x_1}{a_1} + \frac{x_2}{a_2} > 1,$$

and $Q = P^+ \setminus R$. The set $Q$ is a union of rational triangles: likewise $\Delta = S \cup R$ where $S = \Delta \setminus R$ is a rational triangle. The mass $M(\Delta)$ of $\Delta$, with the measure $a d\mu$, is $\frac{1}{2} a t_1 t_2$. The moments

$$M_i(\Delta) = \int_\Delta x_i a d\mu$$

are

$$M_i(\Delta) = \frac{1}{6} a t_1 t_2 t_i.$$

The mass $M(P^+)$ differs from $M(\Delta)$ by $M(Q) - M(S)$ and each of $M(Q), M(S)$ is rational, since $a$ is rational and $Q$ and $S$ are made up of rational triangles. Thus

$$M(P^+) \cong M(\Delta), \quad M_i(P^+) \cong M_i(\Delta) \bmod \mathbf{Q}.$$

We can argue in the same way for the boundary. Let the measure $d\sigma$ be $\lambda_i d\tau_i$ on the face $I_i$, where $\lambda_1, \lambda_2$ are positive rational numbers. Let $J_1, J_2$ be the segments

$$J_1 = \{(\tau_1, 0) : 0 < \tau_1 < t_1\}, \quad J_2 = \{(0, \tau_2) : 0 < \tau_2 < t_2\}.$$

The mass of $J_1 \cup J_2$, with the measure $\lambda_i d\tau_i$ on $J_i$, is

$$M(J_1 \cup J_2) = \lambda_1 t_1 + \lambda_2 t_2$$

and the moments are

$$M_i(J_1 \cup J_2) = \frac{1}{2}\lambda_i t_i^2.$$

The same argument as before shows that the mass and moments of $(\partial P)^+$ differ from those of $J_1 \cup J_2$ by rational numbers. Thus the condition that $P^+$ and $(\partial P)^+$ have the same mass and moments gives

$$\begin{aligned} \frac{1}{2}at_1t_2 - (\lambda_1 t_1 + \lambda_2 t_2) &= r_0 \\ \frac{1}{6}at_1^2t_2 - \frac{1}{2}\lambda_1 t_1^2 &= r_1 \\ \frac{1}{6}at_1t_2^2 - \frac{1}{2}\lambda_2 t_2^2 &= r_2, \end{aligned} \tag{6.3}$$

where $r_0, r_1, r_2$ are rational.

**Lemma 6.4.** *Let $a, \lambda_1, \lambda_2, r_0, r_1, r_2$ be rational numbers with $a, \lambda_1, \lambda_2 \neq 0$. If $(t_1, t_2)$ is a solution of the equations* (6.3) *then either $t_1, t_2$ are both rational or $\lambda_1 r_1 = \lambda_2 r_2$ but $\lambda_1 t_1 \neq \lambda_2 t_2$.*

To prove the Lemma, we may first suppose that $a = 2$ (multiplying the equations by an overall factor). Now put $s_i = \lambda_i t_i$ and $L = \lambda_1\lambda_2$. The equations become:

$$\begin{aligned} s_1s_2 &= Ls_1 + Ls_2 + r_0' \\ \frac{1}{3}s_1^2s_2 &= \frac{1}{2}Ls_1^2 + r_1' \\ \frac{1}{3}s_2^2s_1 &= \frac{1}{2}Ls_2^2 + r_2' \end{aligned} \tag{6.5}$$

for rational $r_i'$. Now multiply the first equation by $s_1/3$ and use the second equation to get:

$$\frac{1}{6}Ls_1^2 = \frac{1}{3}Ls_1s_2 + \frac{r_0'}{3}s_1 - r_1'.$$

Substitute for $s_1s_2$ using the first equation and divide by $L/3$ to obtain

$$\frac{1}{2}s_1^2 = \left(L + \frac{r_0'}{L}\right)s_1 + Ls_2 + k_1,$$

where $k_1 = r_0' - \frac{3r_1'}{L}$ is rational. Symmetrically

$$\frac{1}{2}s_2^2 = Ls_1 + \left(L + \frac{r_0'}{L}\right)s_2 + k_2,$$

where $k_2 = r_0' - \frac{3r_1'}{L}$. Thus if we let $\alpha = L + r_0'/L$ we have

$$\frac{1}{2}s_1^2 = \alpha s_1 + Ls_2 + k_1$$
$$\frac{1}{2}s_2^2 = Ls_1 + \alpha s_2 + k_2$$
$$s_1 s_2 = Ls_1 + Ls_2 + r_0'.$$

Now put $u = s_1 + s_2$ and $v = s_1 - s_2$. The equations above become

$$\frac{1}{2}u^2 = (2L + \alpha)u + (K + L(\alpha - L))$$
$$\frac{1}{2}v^2 = \alpha u + (K - L(\alpha - L))$$
$$\frac{1}{2}uv = (\alpha - L)v + J$$

where $K = k_1 + k_2, J = k_1 - k_2$. The first equation is quadratic in $u$ and we have

$$u = (2L + \alpha) \pm \sqrt{D},$$

where

$$D = (2L + \alpha)^2 + 2(K + L(\alpha - L)).$$

We want to prove that $\sqrt{D}$ is rational. Suppose the contrary; the second and third equations above yield:

$$v^2 = 2\alpha u + (D - 8\alpha L - \alpha^2) \tag{6.6}$$
$$uv = 2(\alpha - L)v + 2J.$$

So

$$v = \frac{2J}{u - 2(\alpha - L)}. \tag{6.7}$$

Note that the denominator here cannot vanish, since $u$ is not rational, by hypothesis. Now

$$u - 2(\alpha - L) = (4L - \alpha) + \sqrt{D}$$

so

$$(u - 2(\alpha - L))^{-1} = \mu \left( (4L - \alpha) - \sqrt{D} \right),$$

where $\mu = (4L - a)^2 - D$ is rational. Thus (6.7) implies

$$v = \nu \left( (4L - \alpha) - \sqrt{D} \right),$$

where $\nu = 2J\mu \in \mathbf{Q}$. This gives

$$v^2 = \nu^2 \left( (4L - \alpha)^2 + D - 2(4L - \alpha)\sqrt{D} \right),$$

whereas (6.6) gives

$$v^2 = 2\alpha \left( (2L + \alpha) + \sqrt{D} \right) + (D - 8\alpha L - \alpha^2).$$

If $\sqrt{D}$ is irrational we may equate the co-efficients of $\sqrt{D}$ in these two expressions for $v^2$ so we get two equations:

$$\alpha = -\nu^2(4L - \alpha)$$
$$2\alpha(2L + \alpha) + D - 8\alpha L - \alpha^2 = \nu^2 \left( (4L - \alpha)^2 + D \right).$$

The second of these simplifies to

$$\alpha^2 - 4\alpha L + D = \nu^2((4L - \alpha)^2 + D).$$

Thus if we put $\beta = \alpha - 4L$ our equations become

$$\alpha = \nu^2 \beta$$
$$\alpha\beta + D = \nu^2(\beta^2 + D)$$

from which we deduce $D = \nu^2 D$. Thus $\nu^2 = 1$ and $\alpha = \beta$, which implies $L = 0$, a contradiction to our hypothesis (since $L$ was defined to be $\lambda_1\lambda_2$).

We have now shown that $\sqrt{D}$ and hence $u$ is rational. Equation (6.7) shows that $v$ is rational so long as $u \neq 2(\alpha - L)$. If $u$ and $v$ are both rational then so are $t_1, t_2$. On the other hand if $u = 2(\alpha - L)$ and $v$ is irrational then $J = 0$—which gives $k_1 = k_2$ and so $\lambda_1 r_1 = \lambda_2 r_2$—while $\lambda_1 t_1 \neq \lambda_2 t_2$. This completes the proof of the Lemma.

To dispose of the second alternative allowed by Lemma (6.4), we consider the second derivatives of the function $\mathcal{L}_a$ at a critical point.

**Lemma 6.8.** *Suppose, in the notation above, that $\mathcal{L}_a$ and its first derivative both vanish at $g_0$, that $\lambda_1 r_1 = \lambda_2 r_2$ but $\lambda_1 t_1 \neq \lambda_2 t_2$. Then $g_0$ is not a local minimum of $\mathcal{L}_a$.*

To see this, we introduce local co-ordinates $u_1, u_2$ on a neighbourhood of $g_0$ in the space of simple PL functions. For $u_1, u_2 > 0$ we have a simple PL function

$$G_{u_1,u_2}(x_1, x_2) = \rho \max\left(0, 1 - \lambda_1 u_1 x_1 - \lambda_2 u_2 x_2\right),$$

where $\rho = 1/\sqrt{\lambda_1^2 u_1^2 + \lambda_2^2 u_2^2}$. Thus $g_0$ is the function $G_{(\lambda_1 t_1)^{-1},(\lambda_2 t_2)^{-1}}$. Straightforward calculations show that

$$\mathcal{F}_a(G_{u_1,u_2}) = \rho\left(\frac{1}{2}F(u_1, u_2) - \lambda_1 r_1 u_1 - \lambda_2 r_2 u_2 - r_0\right) \tag{6.9}$$

where

$$F(u_1, u_2) = \frac{1}{u_1} + \frac{1}{u_2} - \frac{C}{u_1 u_2}$$

with $C = a/3\lambda_1\lambda_2$. Notice that the factor $\rho$ in (6.9) is not relevant to the analysis of the critical point and that the terms involving $r_i$ are linear, so their second derivatives vanish. The hypotheses of the Lemma imply that the partial derivatives $\frac{\partial F}{\partial u_1}, \frac{\partial F}{\partial u_2}$, evaluated at the point $u_1 = (\lambda_1 t_1)^{-1}, u_2 = (\lambda_2 t_2)^{-1}$, are equal but that $u_1 \neq u_2$. Now

$$\frac{\partial F}{\partial u_1} = -\frac{1}{u_1^2}\left(1 - \frac{C}{u_2}\right), \quad \frac{\partial F}{\partial u_2} = -\frac{1}{u_2^2}\left(1 - \frac{C}{u_1}\right),$$

so we have

$$u_1^2 - Cu_1 = u_2^2 - Cu_2.$$

Since the $u_i$ are positive it is clear from the graph of the function $u^2 - Cu$ that we must have $u_1, u_2 < C$. Now consider the second derivative

$$\frac{\partial^2 F}{\partial u_1^2} = \frac{2}{u_1^3}\left(1 - \frac{C}{u_2}\right).$$

If $u_2 < C$ then $\frac{\partial^2 F}{\partial u_1^2}$ is negative so $F$ does not have a local minimum at $(u_1, u_2)$ and this implies that $\mathcal{L}_a$ does not have a local minimum at $g_0$.

Lemmas (6.4) and (6.8) together complete the proof of Proposition (6.1) in case (3). We next move on to case (1). If both $p$ and $q$ are vertices there is nothing to prove, since $g_0$ is then itself rational. Suppose $q$ is a vertex and $p$ is not. We restrict $\mathcal{L}_a$ to the set of simple PL functions whose crease passes through the fixed point $q$. We may suppose that we are in the same situation as that considered for case (3) above, except that now $t_1$ is a boundary point of the face $J_1$; so $t_1$ is rational. The conditions that $\mathcal{F}_a$ and its derivative with respect to $u_2$ vanish tell us that

$$\frac{1}{u_2} - \frac{C}{u_1 u_2}$$

is rational. Thus $u_2$ is rational unless $u_1 = C$. In this latter case the derivative $\frac{\partial \mathcal{F}_a}{\partial u_2}$ vanishes identically for all $u_2$. Thus any simple PL function with a crease through $q$ and any point of the face $J_2$ is also an extremal, and in particular we can find a rational extremal.

Finally we consider case (2) when the crease of $g_0$ passes through two parallel faces of $P$. Making a rational affine transformation, we may suppose that these are segments in the lines $x_2 = 1, x_2 = -1$, each lying in the half-plane $x_1 > 0$. The meaure $d\sigma$ is given by $\lambda dx_1$ on these segments for some $\lambda > 0$. Let the crease of $g_0$ be defined by the equation

$$x_1 - s = \theta x_2.$$

Let $S$ be the quadrilateral defined by the inequalities

$$x_1 > 0, \quad -1 < x_2 < 1, \quad x_1 - s < \theta x_2.$$

We consider the measure $d\sigma$ on the boundary of $S$, given by $\lambda dx_1$ on the "horizontal" faces and zero elsewhere. The same argument as before shows that the mass and moments of $P^+, (\partial P)^+$ are equal to those of $S, \partial S$ modulo rationals. Let $r_0$ be the difference between the mass of $S$ and the mass of $\partial S$ and let $r_1, r_2$ be the differences between the moments. A short calculation shows that the centre of mass of $S$ is the point

$$\left( \frac{1}{6s}(3s^2 + \theta^2), \frac{\theta}{3s} \right).$$

From this one readily shows that

$$r_0 = 2(\lambda - a)s, \quad r_2 = 2\left(\lambda - \frac{a}{3}\right)\theta, \quad r_1 = 2(\lambda - a)s^2 + 2\left(\lambda - \frac{a}{3}\right)\theta^2.$$

As before, we know that $r_0, r_1, r_2$ are rational. If $\lambda$ is not equal to $a$ or $a/3$ the first two of these immediately give that $s, \theta$ are rational and we are done. The two degenerate possibilities are similar to the final possibility considered above in case (1). If $\lambda = a$ then $\theta$ is rational but we get no constraint on $s$. However in this case the $r_i$ do not depend on $s$ and this means that the functional $\mathcal{F}_a(g)$ does not change as we vary $g$ through simple PL functions with crease parallel to that of $g_0$, provided we do not cross any vertices of $P$. Thus while the original $g_0$ may not be rational we can find a rational zero of $\mathcal{F}_a$. Similarly if $\lambda = a/3$ the functional does not change as we vary $g$ through simple PL functions with crease through the fixed point $(s, 0)$ and we can find a rational zero.

## 7.

### 7.1 Conclusion of proof

We will now put together the various components in the proof of Theorem 1.1. As a preliminary, we examine the notion of "K-convergence" of Kahler potentials over a toric surface, which enters into the statement of the Theorem. Thus we suppose we have a sequence $u_\alpha$ of symplectic potentials which correspond to Kahler potentials $\psi_\alpha$ via the Legendre transform.

**Lemma 7.1.1.** *The sequence $\psi_\alpha$ has a K-convergent subsequence if and only if there is a subsequence $\alpha'$ and affine-linear functions $\lambda_{\alpha'}$ on $\mathbf{R}^n$ such that the symplectic potentials $\widetilde{u}_{\alpha'} = u_{\alpha'} + \lambda_{\alpha'}$ satisfy the following two conditions.*

(1) *There is a constant $C$ such that when $Q$ is either $P$ or a face of $P$,*

$$C \geq \min_Q \widetilde{u}_{\alpha'} \geq -C,$$

*for all $\alpha'$.*

(2) *For each $x \in P$ there is a $C_x$ such that*

$$\widetilde{u}_{\alpha'}(x) \leq C_x.$$

To prove this Lemma, first recall that the actions of addition of constants and of the complex torus $T^c$ considered in the definition of K-convergence go over to the action of addition of affine-linear functions on the symplectic potentials. Thus there is no loss in ignoring these actions on either side. Now consider the Kahler potentials $\phi_\alpha$ on the open orbit in $V$. The derivatives of these take values in the bounded set $P$, so are bounded. Thus a necessary and sufficient condition for there to be a subsequence converging pointwise on the open orbit is that there is a subsequence with $\phi_{\alpha'}(0)$ bounded. But, by definition,

$$\phi_\alpha(0) = \min_P u_\alpha,$$

so this is equivalent to $\min_P u_{\alpha'}$ bounded. Likewise, a necessary and sufficient condition that there be a subsequence of the $\phi_\alpha$ converging pointwise on the codimension-1 orbit corresponding to a face $Q$ of $P$ is that there be a subsequence $\alpha'$ with $\min_Q u_{\alpha'}$ bounded. Thus the pointwise convergence criterion in the definition of K-convergence corresponds exactly to the condition in item (1) of the Lemma. To complete the proof we need to see that the second condition in the defintion of K-convergence corresponds to the second item in the statement of the Lemma. Thus consider a fixed point of the action on $V$, corresponding to a vertex of $P$. By applying an integral, affine-linear, transformation, there is no loss in generality in supposing that this vertex is the origin in $\mathbf{R}^2$, and that, near to the origin, $P$ is the positive quadrant. This means that the local co-ordinates $(w_1, w_2)$ around the fixed point are the same as our standard co-ordinates on the orbit and

$$\xi_i = \log |w_i|.$$

Now consider a point $(\eta_1, \eta_2)$ near to the origin in $P$. By definition

$$u_\alpha(x) = \max_\xi \left( \sum x_i \xi_i - \phi(\xi) \right).$$

Thus $u_\alpha(x) \leq C$ if and only if

$$\phi(\xi) \geq \sum x_i \xi_i - C,$$

which is in turn equivalent to

$$\phi(w_1, w_2) \geq x_1 \log |w_1| + x_2 \log |w_2| - C.$$

This effectively finishes the proof: we leave for the reader the task of working out how the calculation above transforms for the other fixed points.

We are ready for the proof of Theorem 1.1. Given the toric variety $(V, L)$ we consider the linear functional $\mathcal{L}_a$ on the normalised functions in $\mathcal{C}_\infty$ and two cases:

(1) $\mathcal{L}_a(f) > 0$ for all nonzero $f$. Then according to (5.2.2) there is an $\epsilon > 0$ such that
$$\mathcal{L}_a f \geq \epsilon \int_{\partial P} f d\sigma,$$
and by (5.1.2) the functional $\mathcal{F}_a$ is bounded below, i.e the Mabuchi functional is bounded below on the space of $T$-invariant potentials $\mathcal{H}^T$. Let $\phi^{(\alpha)}$ be a minimising sequence, as considered in Theorem 1.1. Modulo the actions of $T^c$ and of addition of constants, we can suppose that the $\phi^{(\alpha)}$ are the Legendre duals of normalised symplectic potential functions $u^{(\alpha)}$. These form a minimising sequence for $\mathcal{F}_a$ on $\mathcal{S}$ and hence also a minimising sequence for $\mathcal{F}_a$ on $\mathcal{C}_\infty$ by (3.3.11). Thus they satisfy a bound
$$\int_{\partial P} u^{(\alpha)} d\sigma \leq K$$
by (5.1.8). By Corollary (5.2.5) we can suppose, taking a subsequence, that the $u^\alpha$ converge uniformly on compact subsets of $P$ and on the interior of each face of $\partial P$. This means that the two criteria of Lemma (7.1.1) are satisfied so, by that Lemma, the $\phi_\alpha$ have a K-convergent subsequence.

(2) $\mathcal{L}_a f \leq 0$ for some nonzero $f$. If strict equality holds then, by the density result (5.2.8), we can suppose that $f$ is a PL function and then, clearly, that it is also a rational PL function. If there is no $f$ for which strict inequality holds, i.e., if $\mathcal{L}_a g \geq 0$ for all $g$ but $\mathcal{L}_a f = 0$, then by (5.3.1) and (6.1) we can suppose that $f$ is a rational simple PL function. In either case there is a nontrivial rational PL function $f$ with $\mathcal{L}_a(f) \leq 0$. This function defines a toric test configuration by with positive Futaki invariant by (4.2.1), and hence $(V, L)$ is not K-stable. This completes the proof of our main result (1.1).

Although we are not yet able to prove a full converse to Theorem 1.1, there is a simple partial converse. First we have:

**Proposition 7.1.2.** *Suppose there is a function $f \in \mathcal{C}_1$ with $\mathcal{L}_a(f) < 0$. Then the Mabuchi functional is not bounded below on the invariant metrics and the manifold $V$ does not admit any Kahler metric of constant scalar curvature in the given cohomology class.*

To see this, we can suppose (by an approximation argument) that $f$ lies in $\mathcal{C}_\infty$. If $u$ is some fixed element of $\mathcal{S}$ we consider the sequence

$u_k = u + kf$ . Then

$$\mathcal{F}_a(u_k) \leq \mathcal{F}_a(u) + k\mathcal{L}_a(f) \to -\infty$$

hence $\mathcal{F}_a$ is not bounded below on $\mathcal{C}_\infty$. By (3.3.11) the functional is not bounded below on $\mathcal{S}$ either, hence the Mabuchi functional is not bounded below on the invariant metrics. The last assertion follows from (3.3.4) together with Lichnerowicz's Theorem (which implies that any constant scalar curvature metric must be $T$-invariant).

To formulate a sharper result, we say that a polarised toric variety is *K-stable with respect to toric degenerations* if there is no destabilising configuration of the kind constructed in Section 4. Then we have:

**Proposition 7.1.3.** *Suppose* $(V, L)$ *is a toric variety such that the Mabuchi functional is bounded below on the invariant metrics* $\mathcal{H}^T$ *and any minimising sequence has a* $K$*-convergent subsequence. Then* $(V, L)$ *is K-stable with respect to toric degenerations.*

Thus the gap which needs to be filled to give a complete analytic criterion for K-stability is to show that this condition (K-stability) is equivalent to "K-stability with respect to toric degenerations".

To prove (7.1.3), suppose the variety is not K-stable with respect to toric degenerations. This means that there is a non-affine rational, piecewise linear function with $\mathcal{L}_a(f) \leq 0$. Strict inequality cannot hold, since then the Mabuchi functional would not be bounded below by the Proposition above. Thus $\mathcal{L}_a(f) = 0$ and we can suppose $f$ is a simple PL function by (5.3.1). We approximate $f$ by a sequence of convex functions $f_\alpha$ which are smooth up to the boundary, such that

$$\|f_\alpha - f\|_{L^\infty(\overline{P})} \leq 1/\alpha^2.$$

Let $u_\alpha$ be functions in $\mathcal{S}$ forming a minimising sequence for $\mathcal{F}_a$ and define

$$\widetilde{u}_\alpha = u_\alpha + \alpha f_\alpha.$$

Then

$$\mathcal{F}_a(\widetilde{u}_\alpha) \leq \mathcal{F}_a(u_\alpha) + C/\alpha,$$

for some $C$, so $\widetilde{u}_\alpha$ is another minimising sequence. By the hypothesis, and Lemma (7.1.1), we can without loss of generality suppose that the $\min_P u_\alpha$ are bounded and for each $x \in P$ the $u_\alpha(x)$ are bounded above. This implies that, for each $x$, the $u_\alpha(x)$ are bounded above and below. Likewise, we can find affine-linear functions $\lambda_\alpha$ such that the $\widetilde{u}_\alpha + \lambda_\alpha$

are bounded above and below for each $x \in P$. Thus, for each $x \in P$, $(\lambda_\alpha + \alpha f_\alpha)(x)$ is a bounded sequence. It follows that the $\lambda_\alpha(x)$ are bounded, for each $x$ in the set where $f = 0$. Since this set has nonempty interior, we deduce that the $\lambda_\alpha$ are bounded on $\overline{P}$, since they are affine-linear. But this contradicts the fact that $\alpha f_\alpha$ tends to infinity at some points of $P$.

Note that our notion of "K-convergence" does not capture the full strength of the convergence we have proved for the symplectic potentials, but in this respect our main Theorem is anyway of a provisional nature, since we expect that much stronger results are true.

## 7.2 Discussion

(1) Our main result deals with the two dimensional case but the general scheme of the argument works in all dimensions. To have a Theorem like (1.1) in higher dimensions one needs to extend the results of Sections (5.3) and 6. These are in principle elementary questions about convex functions, which one can address without knowing anything about toric and Kahler geometry.

(2) We give an explicit example to illustrate the theory we have developed. We start with the complex projective plane, represented as a toric variety by the triangle

$$P_0 = \{(x_1, x_2) : x_1, x_2 > 0, x_1 + x_2 < 1\}.$$

We obtain new toric varieties by repeatedly blowing up the points corresponding to the vertices of $P_0$. We recall that, in general, if $P$ is a polygon corresponding to a toric surface, and $v$ is a vertex of $P$, there are vectors $e_1, e_2$ forming a $\mathbf{Z}$-basis for $\mathbf{Z}^2$ such that the two edges of $P$ meeting at $v$ lie in the rays $\{v + te_i : t \geq 0\}$. Blowing up the surface at the point corresponding to $v$ gives a new toric variety with an associated polygon $\widetilde{P}$ obtained by removing the triangle with vertices $v, v+\epsilon e_1, v+\epsilon e_2$ from $P$. Here $\epsilon$ is any sufficiently small rational number (so that some multiple of $\widetilde{P}$ is an integral polygon). For all this, see [12]. If we apply this procedure $n$ times to the origin, and to the vertices that are generated from it, we get a polygon with vertices $(1,0), (0,1)$ and a chain of $2^n$ vertices

$$((\alpha_1, 0), (\alpha_2, \beta_2), \ldots, (0, \beta_{2^n}),$$

say, where the rational numbers $\alpha_i$ are decreasing and the $\beta_i$ are increasing. The values of the $\alpha_i$ and $\beta_i$ depend on the blow-up parameters chosen at each stage but the sequence of slopes

$$\tau(i) = \frac{\beta_i - \beta_{i-1}}{\alpha_{i-1} - \alpha_i},$$

does not depend on these choices. Conversely, if we have a polygon with a sequence of vertices of this kind, and with this sequence of slopes, the polygon arises from a blow-up, for suitable choices of the parameters. It is easy to check that there is always an index $i_n$ such that $\tau(i_n) = \frac{n-2}{2n-3}$, and then $\tau(2^n + 2 - i_n) = \frac{2n-3}{n-2}$.

We define a 9-sided polygon $Q_n$ as follows. We replace the vertex $(0, 0)$ of the original triangle $P_0$ by three vertices,

$$(1/4, 0), (r_n, r_n), (0, 1/4)$$

where

$$r_n = \frac{n-2}{4(3n-5)}.$$

This value of $r_n$ is chosen so that the two new edges have slopes $n - 2/2n - 3$ and $2n - 3/n - 2$. There is obviously a cyclic symmetry of the original triangle $P_0$, permuting the vertices—this just corresponds to the symmetry between the three homogeneous co-ordinates on the projective plane. We replace each of the other vertices of $P_0$ by a triple of new vertices as above, maintaining this symmetry. Now $Q_n$ has a canonical measure $d\sigma$ on its boundary, and we define $a_n$ to be the ratio of the mass of the boundary and the area of the interior, as usual. The moment conditions are satisfied because of the symmetry condition. Thus we have a functional $\mathcal{L} = \mathcal{L}_{a_n}$, which vanishes on affine-linear functions.

**Lemma 7.2.1.** *For sufficiently large $n$ the linear functional $\mathcal{L}$ is nonpositive. In fact if $f$ is a simple convex function with crease along the line through $(1/4, 0), (0, 1/4)$ then $\mathcal{L}(f) < 0$ for large $n$.*

To see this we choose $f$ to be zero on the region $\{x_1 + x_2 \geq 1/4\}$. Thus $\mathcal{L}(f)$ has two contributions:

(1) the integral of $f$ over the edges of slope $(n-2/2n-3), (2n-3/n-2)$ with respect to the measure $d\sigma$;

(2) minus $a_n$ times the integral of $f$ over the triangle $T$ with two edges as above and the third edge the line through $(1/4, 0)$ and $(0, 1/4)$.

Now, for the first contribution, the integers $n-2, 2n-3$ are coprime so the measure $d\sigma$ on these edges is $O(n^{-1})$ times the Euclidean measure, thus the first contribution is $O(n^{-1})$. For the second contribution, first note that the $a_n$ are bounded away from zero, since the total mass of the boundary of $Q_n$ is at least 6 times that of the segment $(1/4, 1/2)$ along the $x_1$-axis, which is independent of $n$, and the area of $Q_n$ is less than the area of $P_0$. The triangle $T$ contains the triangle with vertices $(1/4, 0), (1/12, 1/12), (0, 1/4)$ so the integral of $f$ over $T$ is also bounded away from zero. So when $n$ is large enough the magnitude of the second contribution must exceed that of the first and the Lemma is proved.

(In fact a little calculation shows that the conclusion of Lemma (7.2.1) holds for $n \geq 5$.)

Now fix $n$ large enough for the conclusion of Lemma (7.2.1) to hold. The polygon $Q_n$ does not itself arise from a blow-up of the plane but since the slopes of the new sides were chosen to arise in the sequence $(\tau(i))$ we can approximate $Q_n$ arbitrarily closely by such a polygon, in an obvious way. That is, we succesively blow up the three fixed points of the projective plane and choose the blow-up parameters defining all the other sides to be very small. We make the whole construction symmetrical under the cyclic symmetry of $P_0$ to ensure that the moment condition is satisfied. By continuity, the linear functional defined by this polygon must be negative on $f$, if the approximation to $Q_n$ is close enough. This means, by (7.1.2), that the corresponding toric variety cannot have a $T$-invariant metric of constant scalar curvature, even though its Futaki invariant vanishes (or, in other words, it has no extremal Kahler metric).

(3) Most of the ideas in the body of this paper apply in a more general setting. We consider a bounded open convex set $\Omega \subset \mathbf{R}^n$ with Lipschitz boundary, a positive measure $d\sigma$ on $\partial\Omega$ in the measure-class of $(n-1)$-dimensional Lebesgure measure and a bounded function $A$ on $\Omega$. Then we can define a space $\mathcal{C}_\infty$ and functionals $\mathcal{L}_A$ and $\mathcal{F}_A$ in the obvious way. We expect that the following is true:

**Conjecture 7.2.2.** *If $\mathcal{L}_A(f) > 0$ for all non-affine functions in $\mathcal{C}_\infty$ then $\mathcal{F}_A$ is bounded below and the minimum is attained in $\mathcal{C}_\infty$ by a function which satisfies the equation $S(u) = A$.*

(4) A generalisation which is pertinent to Kahler geometry arises in the study of extremal metrics. Here the function $A$ is an affine-linear function on $\mathbf{R}^n$. That is, given a toric variety $V$ corresponding to a polytope $P \subset \mathbf{R}^n$ with measure $d\sigma$ on $\partial P$, there is obviously a unique

affine-linear function $A = A_P$ satisfying the moment conditions (5.1.1). The existence of an extremal Kahler metric on $V$ is equivalent to the existence of a function in $\mathcal{S}$ minimising $\mathcal{F}_A$. Most of the discussion of the paper goes over to this case, with an appropriate modification of the definition of K-stability. However there is one point of difficulty: the results of (5.3) apply to *positive* functions $A$, but for some polytopes $A_P$ is not positive throughout $P$. Thus we leave the extension of our results to extremal Kahler metrics for the future.

Suppose now that $Q$ is a polyhedron in $\mathbf{R}^2$ and $d\sigma$ is a measure supported on *some* of the faces of $\partial Q$, equal to the canonical measure considered before on the faces on which it does not vanish. Let $V_Q$ be the toric variety defined by $Q$. The faces of $\partial Q$ correspond to curves in $V_Q$: we let $D$ be the divisor defined by the union of the curves for which $d\sigma$ vanishes on the corresponding face of the boundary. There is a unique affine-linear function $A = A_Q$ satisfying the moment conditions (5.1.1). Let $\mathcal{L}$ be the linear functional defined by $A_Q$ and $d\sigma$.

**Conjecture 7.2.3.** *If $\mathcal{L}$ is strictly positive on convex, non-affine functions then $\mathcal{F}$ is bounded below and there is a smooth minimiser, which corresponds to a complete extremal metric on $V_Q \setminus D$.*

The general picture one might expect is that if a toric variety is not $K$-stable then there should be a canonical decomposition of the associated polygon into subpolygons. On the boundary of each of these subpolygons we have a measure—the usual one on the faces which come from the boundary of $P$ and zero on the faces created by the decomposition. These subpolygons should be of two kinds; either they satisfy the hypothesis of the conjecture or, exceptionally, they should be parallelograms in which two opposite faces are faces of $P$. We expect that the "Calabi flow" (or other minimising procedure) generates a sequence of metrics with diameter tending to infinity, breaking the manifold into pieces. On the pieces corresponding to subpolygons of the first kind the metrics should converge to the complete extremal metrics of Conjecture (7.2.3). On the pieces corresponding to the parallelograms the metrics should collapse, with behaviour modelled on a product $S^1 \times B$, where the length of the circle tends to zero.

(4a) There is an alternative variational formulation of the PDE $S(u) = A$, which is related to almost-complex structures. Suppose we are in the setting of (2) above, with a function $A$ on a convex set $\Omega$ and measure $d\sigma$ on $\partial\Omega$. Suppose we can find a symmetric tensor field $V^{ij}$ on $\Omega$, with $V^{ij} > 0$ everywhere, which represents the linear functional

$\mathcal{L}_A$ in the sense that

$$\mathcal{L}_A(u) = \int_\Omega V^{ij} f_{,ij} d\mu,$$

for all $f \in \mathcal{C}_\infty$. (Here we use the summation convention over repeated indices, and $u_{,ij}$ denotes the second partial derivative.) Essentially this means that $V^{ij}$ satisfies the distributional equation

$$(V^{ij})_{,ij} = A - [d\sigma]$$

where $[d\sigma]$ is regarded as a distribution supported on $\partial\Omega$. Clearly this implies that $\mathcal{L}_A > 0$ on $\mathcal{C}_\infty$. It is reasonable to expect that a converse is true, so that if $\mathcal{L}_A$ is positive on $\mathcal{C}_1$ we can find such a tensor $V^{ij}$. This is clearly a weaker conjecture than (7.2.2), which requires a solution of the special form $V_{ij} = u_{,ij}$. In any case, let us suppose that we do have a solution $V^{ij}$, and let $\mathcal{V}$ be the space of all such solutions: a convex subset in an affine space. We consider the functional

$$\mathcal{N}(V) = \int_\Omega \log\det V^{ij}\ d\mu$$

on $\mathcal{V}$. This is a convex function and so has at most one critical point, a global minimum, on $\mathcal{V}$. To find the appropriate Euler-Lagrange equations we use:

**Lemma 7.2.4.** *The general solution of the equation $V^{ij}_{,ij} = 0$, for symmetric tensor fields $V^{ij}$, on $\Omega$ has the form*

$$V^{ij} = T^{ijk}_{,k} + T^{jik}_{,k}$$

*for an arbitrary tensor field $T^{ijk}$ which is skew in the indices $j, k$.*

This is a simple exercise, using two applications of the Poincaré Lemma.

Now the derivative of the functional $\mathcal{N}$ at a point $V^{ij}$ in the direction $\epsilon^{ij}$ is given by

$$\int_\Omega V_{ij}\epsilon_{ij}\ d\mu.$$

Thus if $V^{ij}$ is a critical point for $\mathcal{N}$ on $\mathcal{V}$ we must have

$$\int_\Omega V_{ij}(T^{ijk}_{,k} + T^{jik}_{,k})\ d\mu = 0,$$

for all compactly supported tensors $T^{ijk}$, skew in $j, k$. Integrating by parts, this tells us that $V_{ij,k}$ is symmetric in $j, k$. Using the Poincaré Lemma again, we can find a tensor field $\theta_i$ such that $V_{ij} = \theta_{i,j}$. Now the condition that $V$ is symmetric means, by the Poincaré Lemma, that $\theta_i = u_{,i}$ for some function $u$ on $\Omega$. In sum, solving the equation of Conjecture (7.2.2) is equivalent to finding a minimiser of the functional $\mathcal{N}$ on $\mathcal{V}$.

In the case when the domain is a polytope corresponding to a toric variety $V$, the discussion above corresponds to considering $T$-invariant almost-complex structures on $V$, as explained in (3.1). We call the function $V^{ij}_{,ij}$ the Hermitian scalar curvature of the almost-complex structure defined by $V_{ij}$. Thus finding a Kahler metric of constant scalar curvature is equivalent to minimising the function $\mathcal{N}$ on the set of almost-complex structures of constant Hermitian scalar curvature. Notice also that, in the converse direction, our examples of manifolds which are not K-stable give examples where there is not even an almost-complex structure of constant Hermitian scalar curvature.

(5) Our main result, Theorem (1.1), yields a new real-valued invariant of $K$-stable toric surfaces: the infimum of the functional $\mathcal{L}_a$. This seems to be special to the toric case, since in general the Mabuchi functional is only defined up to an overall constant, so even if it is bounded below, its infimum is not well-defined as a real number.

## Appendix: Proof of Proposition (4.1.3)

If $X \subset \mathbf{R}^n$ is a finite union of compact polytopes we define $\nu(X)$ to be the number of lattice points in the interior of $X$ plus one half the number of lattice points in the boundary. By applying the basic result (4.1.2) to a polytope $P$ and to its boundary faces one sees that Proposition (4.1.3) is equivalent to:

**Proposition A1.** *For an integer polytope $P \subset \mathbf{R}^n$,*

$$\nu(kP) = k^n \, Vol(P) + O(k^{n-2}).$$

This is the statement that we prove here. If $X_1$ and $X_2$ have disjoint interiors then one easily sees that

$$\nu(k(X_1 \cup X_2)) = \nu(kX_1) + \nu(kX_2) + O(k^{n-2}),$$

since the points that are counted differently in the sums lie in the $(n-2)$ skeleton of $k(X_1 \cup X_2)$. Since any integer polytope can be decomposed into a union of integer simplices it suffices to prove (A1) in the case when $P$ is a simplex. We now use induction on the dimension $n$. The statement is readily verified for $n = 1$ so we suppose $n \geq 2$ and that the result has been proved for lower-dimensional simplices.

**Lemma A2.** *Suppose a function $f$ on the positive integres satisfies*

$$f(k+1) - f(k-1) = Wk^{n-1} + O(k^{n-3}).$$

*Then $f(k) = \frac{W}{2n} k^n + O(k^{n-2})$.*

We leave the proof of this to the reader. Using this Lemma, and (4.1.2), it suffices to show that

$$\nu((k+1)P) - \nu((k-1)P) = Wk^{n-1} + O(k^{n-3}),$$

for some constant $W$. We can suppose that one vertex of our $n$-simplex $P$ is the origin in $\mathbf{R}^n$ and that the other vertices lie on the hyperplane $\{x_n = h\}$, where $(x_1, \ldots x_n)$ are standard co-ordinates on $\mathbf{R}^n$ and $h$ is a positive integer. Thus $P$ is a cone on a copy of an integer $(n-1)$-simplex $Q \subset \mathbf{R}^{n-1}$. Comparing the counts involved in the definition of the two terms we have

$$\begin{aligned} &\nu((k+1)P)) - \nu((k-1)P) \\ &= \nu(kQ) + \frac{1}{2}\left(\nu((k+1)Q) + \nu((k-1)Q)\right) \\ &\quad + \sum_{i=1}^{h-1} \left(\nu((k+i/h)Q) + \nu(k-i/h)Q)\right) \\ &\quad + \frac{1}{4}(N((k+1)\partial Q - N((k-1)\partial Q)), \end{aligned}$$

where $N((k+1)\partial Q)$ denotes the number of lattice points on the boundary of $(k+1)Q$. The first term $\nu(kQ)$ has the desired form, by the induction hypothesis, and the difference $N((k+1)\partial Q) - N((k-1)\partial Q)$ is $O(k^{n-3})$, so it suffices to prove that for $1 \leq i \leq h$:

(A3) $$\nu((k+i/h)Q) + \nu((k-i/h)Q) = W_i k^{n-1} + O(k^{n-3}).$$

We claim that, in fact, for any real number $\rho$

(A4) $$\nu((k+\rho)Q) + \nu((k-\rho)Q) = 2\nu(kQ) + O(k^{n-3}),$$

this implies (A3), by the induction hypothesis. To see (A4), consider a face $F_\alpha$ of $Q$, defined by an equation $\lambda_\alpha = c_\alpha$, where $\lambda_\alpha$ is a primitive integral linear function on $\mathbf{R}^{n-1}$. We can find an integer vector $e_\alpha$ such that $\lambda_\alpha(e_\alpha) = 1$. This choice defines a reflection map $R_{k,\alpha}$ on $\mathbf{R}^{n-1}$, preserving the integer lattice, with fixed set the hyperplane containing the corresponding face of $kQ$. For simplicity of exposition, we suppose that the origin in $\mathbf{R}^{n-1}$ lies in the interior of $Q$—the reader may check that the argument works without this assumption. Write $A_k$ for the complement of the interior of $(k-1)Q$ in $(k+1)Q$. For $\delta > 0$ let $A_k^\delta$ be the complement of the $\delta$-neighbourhood of the $(n-3)$-skeleton of $(k+1)Q$ in $A_k$. We can fix a $\delta$, independent of $k$, such that $A_k^\delta$ decomposes into a disjoint union of connected components $A_k^\delta(\alpha)$ associated to the faces $F_\alpha$; thus $A_k^\delta(\alpha)$ is contained in a region $\{kc_\alpha - \rho_\alpha \leq \lambda_\alpha(x) \leq kc_\alpha + \rho_\alpha\}$. Finally, let

$$B_k^\delta(\alpha) = A_k^\delta(\alpha) \cap R_\alpha(A_k^\delta(\alpha)),$$

and let $B_k = \bigcup_\alpha B_k^\delta(\alpha)$. Define an involution $R$ on $B_k$ to be given by $R_\alpha$ on $B_k^\delta(\alpha)$. The complement $A_k \setminus B_k$ is contained in an $\epsilon$-neighbourhood of the $(n-3)$-skeleton of $kQ$, for some fixed $\epsilon$ independent of $k$. Thus $\nu((k+\rho)Q - \nu(kQ)$ and $\nu(kQ) - \nu((k-\rho)Q)$ can be computed, up to $O(k^{n-3})$, by counting only those points in $B_k$. The map $R$ matches up the points in $B_k$ counted in computing $\nu((k+\rho)Q) - \nu(kQ)$ with those counted in computing $\nu(kQ) - \nu((k-\rho)Q)$, so the two terms agree up to $O(k^{n-3})$, as asserted in (A4).

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