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MEAN CURVATURE FLOWS OF LAGRANGIAN SUBMANIFOLDS WITH CONVEX POTENTIALS

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Abstract

This article studies the mean curvature flow of Lagrangian submanifolds. In particular, we prove the following global existence and convergence theorem: if the potential function of a Lagrangian graph in T^{2n} is convex, then the flow exists for all time and converges smoothly to a flat Lagrangian submanifold. We also discuss various conditions on the potential function that guarantee global existence and convergence.

1. Introduction

The mean curvature flow is an evolution process under which a submanifold evolves in the direction of its mean curvature vector. It can be considered as the gradient flow of the area functional in the space of submanifolds. The critical points of the area functional are minimal submanifolds.

In mirror symmetry, a distinguished class of minimal submanifolds called "special Lagrangians" are desirable in any complex n dimensional Calabi-Yau manifold with a parallel holomorphic $(n, 0)$ form Ω. A special Lagrangian is calibrated by $\mathrm{Re}\,\Omega$, which means $*\mathrm{Re}\,\Omega = 1$, where $*$ is the Hodge $*$ operator on the submanifold. A simple derivation using Stokes' Theorem shows a special Lagrangian minimizes area in its homology class. To produce special Lagrangians, it is thus natural to consider the mean curvature flow. We remark that the existence of Lagrangian minimizers in Kähler-Einstein surfaces was proved by Schoen-Wolfson[9] using variational methods.

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It is conjectured by Thomas and Yau in [17] that a stable Lagrangian isotopy class in a Calabi-Yau manifold contains a smooth special Lagrangian and the deformation process can be realized by the mean curvature flow. One of the stability condition is in terms of the range of $*\mathrm{Re}\,\Omega$. In [19] (see also [23]), the second author proves the following regularity theorem:

Theorem 1.1. *Let (X, Ω) be a Calabi-Yau manifold and Σ be a compact Lagrangian submanifold. If $*Re\,\Omega > 0$ on Σ, the mean curvature flow of Σ does not develop any Type* I *singularity.*

In particular, this theorem implies no neckpinching will occur in the flow. We remark that without this condition, neckpinching is possible by an example of Schoen-Wolfson [10]. It is thus of great interest to identify initial conditions that guarantee the long-time existence and convergence of the flow.

The mean curvature flow of Lagrangian surfaces in four-manifolds is studied in [16] and [20] independently. The first author [16] proves the long-time existence and smooth convergence theorem for graphs of area preserving diffeomorphisms in the nonpositive curvature case assuming an angle condition. In [20], the second author proves the long-time existence for graphs of area preserving diffeomorphisms between Riemann surfaces and uniform convergence when the diffeomorphism is homotopic to identity (smooth convergence for spheres). This gives a natural deformation retract of the group of symplectomorphism of Riemann surfaces. The maximum principle for parabolic equations is important in both papers [20] and [16]. The new ingredient in [20] is the blow-up analysis of the mean curvature flow developed in [19]. This has been applied to prove long-time existence and convergence theorems for general graphic mean curvature flows in arbitrary dimension and codimension in [21].

In this article, we prove the following global existence and convergence theorem in arbitrary dimension.

Theorem A. *Let Σ be a Lagrangian submanifold in T^{2n}. Suppose Σ is the graph of $f : T^n \to T^n$ and the potential function u of f is convex. Then the mean curvature flow of Σ exists for all time and converges smoothly to a flat Lagrangian submanifold.*

The potential u is only a locally defined function and the convexity of u will be explained more explicitly in §2. The mean curvature flow can be written locally as a fully non-linear parabolic equation for the potential u:

$$\frac{du}{dt} = \frac{1}{\sqrt{-1}} \ln \frac{\det(I + \sqrt{-1} D^2 u)}{\sqrt{\det(I + (D^2 u)^2)}}. \tag{1.1}$$

The Dirichlet problem for the elliptic version of (1.1) was solved by Caffarelli, Nirenberg and Spruck in [1].

The general existence theorem in [21] specialized to the Lagrangian case holds under the assumption that $\prod(1 + \lambda_i^2) < 4$ where $\lambda_i' s$ are eigenvalues of $D^2 u$. The first author proves the convexity of u is preserved in [15] and he also shows the existence and convergence theorem assuming u is convex and the eigenvalues of $D^2 u$ are less than one. The method in [15] indeed implies stronger results.

The core of the proof is to get control of $D^2 u$. It is interesting that there are two ways to interpret $D^2 u$. First we can identify it with a symmetric two-tensor on the submanifold Σ. One can then calculate the evolution equation with respect to the rough Laplacian on symmetric two tensors. Applying Hamilton's maximum principle [3] shows that the positive definiteness of the symmetric two-tensor is preserved along the flow. A stronger positivity gives the uniform C^2 bound of u.

On the other hand, since $f = \nabla u$, the graph of $D^2 u = df$ is the tangent space of the graph of f. Recall the Gauss map for a submanifold assigns each point to its tangent space. It was proved in [24] that the Gauss map of any mean curvature flow is a harmonic map heat flow and thus any convex region of the Grassmannian is preserved along the flow. We relate $D^2 u > 0$ to a convex region in the Grassmannian and this also gives uniform C^2 bound of u. The convergence part uses Krylov's $C^{2,\alpha}$ estimate [7] for nonlinear parabolic equations.

Since the geometry of a Lagrangian submanifold is invariant under the unitary group $\mathrm{U}(n)$, this gives other equivalent conditions to the convexity of u that also imply global existence and convergence. This is explained in §4.

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2. Preliminaries

We first derive the evolution equation of $f = \nabla u$ from the equation of the potential function u. For more material on the special Lagrangian equation, we refer to Harvey-Lawson [5].

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be a domain. $u : \Omega \times [0, T) \to \mathbb{R}$ is said to satisfy the special Lagrangian evolution equation if

$$\frac{du}{dt} = \frac{1}{\sqrt{-1}} \ln \frac{\det(I + \sqrt{-1}D^2u)}{\sqrt{\det(I + (D^2u)^2)}}. \tag{2.1}$$

$\frac{\det(I+\sqrt{-1}D^2u)}{\sqrt{\det(I+(D^2u)^2)}}$ is a unit complex number, so the right-hand side is always real.

It is not hard to see $u_i = \frac{\partial u}{\partial x^i}$ satisfies the following evolution equation.

$$\frac{du_i}{dt} = g^{jk}u_{ijk} \tag{2.2}$$

where $g^{jk} = g^{-1}_{jk}$ and $g_{jk} = \delta_{jk} + u_{jl}u_{kl}$. Indeed, the right-hand side of Equation (2.2) is the mean curvature form $H_i = g^{jk}h_{ijk}$, i.e., the trace of the second fundamental tensor h_{ijk} because in our local coordinates we have $h_{ijk} = u_{ijk}$. It is well-known that the mean curvature form H is closed. Locally (e.g., see Section 2.6 in [14]) H can be expressed by the differential $d\alpha$ of the Lagrangian angle $\alpha = \frac{1}{\sqrt{-1}} \ln \frac{\det(I+\sqrt{-1}D^2u)}{\sqrt{\det(I+(D^2u)^2)}}$, i.e., the right-hand side in (2.1). Then (2.2) follows from

$$\frac{d}{dt}du = d\frac{du}{dt} = d\alpha = H.$$

Equation (2.2) is indeed the nonparametric form of a graphic mean curvature flow, see [25] or [21] for the derivation of the general case. The graph of ∇u is then a Lagrangian submanifold in $\mathbb{C}^n \cong \mathbb{R}^n \oplus \mathbb{R}^n$ evolving by the mean curvature flow. It is well-known that being Lagrangian is preserved along the mean curvature flow, see for example [12] or [14]. The complex structure J on $\mathbb{C}^n$ is chosen so that the second summand $\mathbb{R}^n$ is the image under J of the first summand. Equation (2.2) is equivalent up to tangential diffeomorphisms to the original flow. On the other hand if Σ_t is a family of Lagrangian submanifolds moved by the mean curvature flow in $\mathbb{C}^n$ so that each Σ_t can be written as a graph over the base $\mathbb{R}^n$, it is not hard to check by integration that the potential u satisfies the special Lagrangian evolution equation locally.

Next we explain the convex potential condition. Suppose $f : T^n \to T^n$ is given so that the graph of f is a Lagrangian submanifold of $T^{2n} \cong T^n \times T^n$. The tangent space of T^{2n} is identified with $\mathbb{C}^n \cong \mathbb{R}^n \oplus \mathbb{R}^n$. The differential df is a linear map from the first $\mathbb{R}^n$ to the second $\mathbb{R}^n$, so is the complex structure J. The Lagrangian condition implies the bilinear form $\langle df(\cdot), J(\cdot)\rangle$ is symmetric. Therefore there is a locally defined potential function u of f. We shall identify D^2u with the bilinear form $\langle df(\cdot), J(\cdot)\rangle$.

Definition 2.2. The eigenvalues of D^2u are the eigenvalues of the symmetric bilinear form $\langle df(\cdot), J(\cdot)\rangle$. u is convex if $\langle df(v), J(v)\rangle > 0$ for any $v \in \mathbb{R}^n$.

Therefore an eigenvalue λ of D^2u satisfies $df(v) = \lambda J(v)$ for some nonzero $v \in \mathbb{R}^n$.

3. Proof of Theorem A

Let Σ be a Lagrangian submanifold of T^{2n} that can be written as the graph of the base T^n. We assume the potential u of Σ is convex. Let Σ_t be the mean curvature flow of Σ and u_t be the potential of Σ_t. u_t then satisfies the special Lagrangian evolution equation locally. We suppress the subindex t when it is clear the quantity is time-dependent.

The proof of Theorem A is divided into three parts. First we prove the convexity of the potential u is preserved, then we derive a C^2 estimate of u, at last we prove the long time existence and convergence. It is worth noting that we do not deal with D^2u directly. Instead we interpret the properties of D^2u in terms of the restriction of a symmetric two-tensor to Σ.

3.1 Preserving convexity of u

First we relate the convexity of u to the positive definiteness of a symmetric two-tensor on Σ. Recall the tangent space of T^{2n} is identified with $\mathbb{C}^n \cong \mathbb{R}^n \oplus \mathbb{R}^n$ and the complex structure J maps the first real space to the second one. Let π_1 and π_2 denote the projection onto the first and second summand in the splitting. Define the two-tensor

$$S(X, Y) = \langle J\pi_1(X), \pi_2(Y)\rangle$$

for any $X, Y \in \mathbb{C}^n \cong T(T^{2n})$. $S(X, Y)$ is symmetric for any X, Y in the same Lagrangian subspace of $\mathbb{C}^n$. This is because $\omega(X, Y) =$

$\langle J(\pi_1(X) + \pi_2(X)), \pi_1(Y) + \pi_2(Y)\rangle = 0.$

Lemma 3.1. *Given any compact Lagrangian submanifold $F : \Sigma \to T^{2n}$ that can be written as a graph over the base T^n. The potential u is convex if and only if F^*S is positive definite.*

Proof. Suppose $F(\Sigma)$ is the graph of $f = T^n \to T^n$. We identity Σ with the base T^n and choose the embedding $F(x) = (x, f(x))$. df is a linear map, $df : \mathbb{R}^n \to \mathbb{R}^n$. We have $dF(v) = v + df(v)$, $\pi_1(dF(v)) = v$ and $\pi_2(dF(v)) = df(v)$. Therefore F^*S becomes a symmetric two-tensor and is the same as $\langle Jv, df(v)\rangle = \langle df(v), Jv\rangle$. f has a locally defined potential u. By Definition 2.2, the positive definiteness of F^*S is the same as the convexity of u. q.e.d.

Next, we prove the positivity of F^*S is preserved using two methods. The first method calculates the evolution equation of F^*S and applies Hamilton's maximum principle; this is also proved in [15]. The second method uses the fact that the Gauss map of a mean curvature flow forms a harmonic map heat flow and the geometry of the Lagrangian Grassmannian developed in [24].

3.1.1 First method: the maximum principle for tensors

We recall the general evolution equation for the pull back of a parallel two-tensor of the ambient space from [19] (§2, Equation (2.3)).

Lemma 3.2. *Let $F : \Sigma \times [0, T) \to M$ be a mean curvature flow in M and S be a parallel two-tensor on M. ∇ denotes the Levi-Civita connection on M. For any tangent vector of M, $(\cdot)^T$ denotes the tangential part in $T\Sigma$ and $(\cdot)^\perp$ is the normal part in $N\Sigma$. Then*

$$\begin{aligned}(3.1)\qquad &\left(\frac{d}{dt} - \Delta\right) F^*S(X,Y)\\ &= S((\nabla_X H)^T, Y) + S(X, (\nabla_Y H)^T)\\ &\quad - S((\nabla_{e_k}(\nabla_{e_k} X)^\perp)^T, Y) - S(X, (\nabla_{e_k}(\nabla_{e_k} Y)^\perp)^T)\\ &\quad - 2S((\nabla_{e_k} X)^\perp, (\nabla_{e_k} Y)^\perp)\end{aligned}$$

for any $X, Y \in T\Sigma$ and any orthonormal basis $\{e_k\}$ for $T\Sigma$, where Δ is the rough Laplacian on two-tensors over Σ.

Now back to our setting when Σ is Lagrangian in T^{2n}. $\{Je_k\}$ forms an orthonormal basis for $N_p\Sigma$. We define the second fundamental form by

$$h_{kij} = \langle \nabla_{e_k} e_i, J(e_j)\rangle.$$

Thus $(\nabla_{e_k} e_i)^\perp = h_{kij} J(e_j)$ and $(\nabla_{e_k} J(e_i))^T = -h_{ikl} e_l$. Denote

$$H_j = \langle H, J(e_j)\rangle.$$

Thus $(\nabla_{e_i} H)^T = -H_l h_{ijl} e_j$.

Plugging these into Equation (3.1), we derive

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) S(e_i, e_j) = &-H_p h_{ipl} S(e_l, e_j) - H_p h_{jpl} S(e_i, e_l)\\ &+ h_{pki} h_{pkl} S(e_l, e_j) + h_{pkj} h_{pkl} S(e_i, e_l)\\ &- 2h_{kil} h_{kjm} S(J(e_l), J(e_m)).\end{aligned}$$

Recall $S(X,Y) = \langle J\pi_1(X), \pi_2(Y)\rangle$, so

$$S(J(e_l), J(e_m)) = \langle J\pi_1(J(e_l)), \pi_2(J(e_m))\rangle.$$

Since $J\pi_1 = \pi_2 J$ and $J\pi_2 = \pi_1 J$, we derive

$$\begin{aligned}S(J(e_l), J(e_m)) &= \langle JJ\pi_2(e_l), J\pi_1(e_m)\rangle\\ &= -\langle \pi_2(e_l), J\pi_1(e_m)\rangle = -S(e_l, e_m).\end{aligned} \tag{3.2}$$

The last step is because $S(\cdot,\cdot)$ is symmetric on any Lagrangian subspace.

Therefore, we obtain

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) S_{ij} = &(h_{pki} h_{pkl} - H_p h_{ipl}) S_{lj}\\ &+ (h_{pkj} h_{pkl} - H_p h_{jpl}) S_{il} + 2h_{kil} h_{kjm} S_{lm}.\end{aligned} \tag{3.3}$$

Now $H_p h_{ipl} - h_{pki} h_{pkl} = R_{il}$ is indeed the Ricci curvature on Σ. This equation is also derived in [15]. Since $h_{kil} h_{kjm} S_{lm}$ is positive definite if S_{ij} is, the positivity of S_{ij} being preserved is a direct consequence of Hamilton's maximum principle for tensors [3].

3.1.2 Second method: geometry of Lagrangian Grassmannian

In the following, we give another proof using the geometry of the Lagrangian Grassmannian. Let $\mathrm{LG}(n)$ denote the Lagrangian Grassmannian of all oriented Lagrangian subspaces of $\mathbb{C}^n$. Let $\gamma_t : \Sigma_t \to \mathrm{LG}(n)$ be the Gauss map of the mean curvature flow. Recall from [24] that γ_t is a harmonic map heat flow, thus any (Grassmannian) convex subset is preserved. That $S_{ij} > 0$ being preserved will follow from the following theorem.

Theorem 3.1. $\Xi = \{P \in \mathrm{LG}(n) \mid \min_{v\in P} S(v,v) \geq 0\}$ *is a convex subset of* $\mathrm{LG}(n)$ *with respect to the Grassmannian metric.*

Proof. Let P be an arbitrary point in the Lagrangian Grassmannian $\mathrm{LG}(n)$. P represents a Lagrangian subspace in $\mathbb{C}^n$ and a local coordinate chart near P is parametrized by $\mathfrak{S}$, the space of $n \times n$ symmetric matrices of the form $Z = [z_{ij}]_{i,j=1\dots n}$. They represent the collection of all Lagrangian subspaces that can be written as a graph over P. By [27], the invariant metric on $\mathrm{LG}(n)$ is given by

$$ds^2 = Tr[(I + Z^2)^{-1} dZ]^2.$$

We choose an orthonormal basis $\{e_i\}$ for P, so that $P = e_1 \wedge \cdots \wedge e_n \in \mathrm{LG}(n)$. By [27], a geodesic parametrized by arc length is given as $P(s)$ spanned by $\{e_i + z_{ik}(s)Je_k\}_{i=1\dots n}$such that $Z = [z_{ij}(s)]$ is a $n \times n$ matrix which satisfies the following ordinary differential equation:

$$Z'' - 2Z'Z(I + Z^2)^{-1}Z' = 0. \tag{3.4}$$

Note that $Z = Z^T$ for Lagrangian Grassmannians.

Now we check the convexity of Ξ. Let P be a boundary point of Ξ, so S is nonnegative definite on P. Let $v \in P$ be a zero eigenvector of S so that $S(v,v) = 0$. Consider a (Grassmannian) geodesic $P(s)$ through P and an extension of v, v_s on $P(s)$ and let $f(s) = S(v_s, v_s)$. To check the convexity, it suffices to show for any geodesic $P(s)$ we can find an (arbitrary) extension v_s with $|v_s| = 1$ and $f'(0) = 0$ so that $f''(0) < 0$. We remark that an arbitrary extension of v is good enough as the minimum function $\min_{v\in P} S(v,v)$ is always less than or equal to $f(s)$ along $P(s)$.

Denote

$$S_{ij}(s) = S(e_i + z_{ik}(s)Je_k, e_j + z_{jl}(s)Je_l) \tag{3.5}$$

and

$$g_{ij}(s) = \langle e_i + z_{ik}(s)Je_k, e_j + z_{jl}(s)Je_l\rangle = \delta_{ij} + z_{ik}(s)z_{jk}(s).$$

For any

$$v_s = v^i(s)(e_i + z_{ik}(s)Je_k),$$

we have

$$|v_s|^2 = v^i(s)v^j(s)g_{ij}(s) \tag{3.6}$$

and

$$f(s) = S(v_s, v_s) = v^i(s)v^j(s)S_{ij}(s).$$

We recall that $z_{ij}(0) = 0$ and by Equation (3.4) $z''_{ij}(0) = 0$. Denote $z'_{ij}(0) := \mu_{ij}$; μ_{ij} is an arbitrary $n \times n$ symmetric matrix. In the following, we calculate the derivative at $s = 0$.

$$S'_{ij}(0) = \mu_{ik}S(Je_k, e_j) + \mu_{jl}S(e_i, Je_l),$$
$$S''_{ij}(0) = 2\mu_{ik}\mu_{jl}S(Je_k, Je_l) = -2\mu_{ik}\mu_{jl}S_{kl},$$

$$g'_{ij}(0) = 0, \quad \text{and} \quad g''_{ij}(0) = 2\mu_{ik}\mu_{jk}.$$

We choose v_s with $(v^i)'(0) = 0$ so that the length $|v_s|^2$ is constant up to first order at $s = 0$.

The second derivative of f can be calculated in the following.

$$f''(0) = (S_{ij})''v^iv^j + S_{ij}(v^i)''v^j + S_{ij}v^i(v^j)''.$$

Since v is a zero eigenvector of S on P, $S_{ij}v^i = 0$ for each j, it follows that $f''(0) = -2\mu_{ik}\mu_{jl}S_{kl}v^iv^j$ is nonpositive and the theorem is proved. q.e.d.

3.2 C^2 estimate

First we reinterpret the C^2 bound of u in terms of the tensor S_{ij}.

Lemma 3.3. *Given any $\epsilon > 0$, $F^*S - \epsilon g > 0$ on Σ is equivalent to a uniform bound of D^2u.*

Proof. To see what $S_{ij} - \epsilon g_{ij} > 0$ means in terms of the eigenvalues of D^2u, we choose a particular orthonormal basis for $T_p\Sigma$ at a point p that we are interested. The tangent space of Σ is the graph of $df : \mathbb{R}^n \to \mathbb{R}^n$. Recall the complex structure J is chosen so that the target $\mathbb{R}^n$ is the image under J of the domain $\mathbb{R}^n$. Because $\langle df(\cdot), J(\cdot)\rangle$ is symmetric, we can find an orthonormal basis $\{a_i\}_{i=1\ldots n}$ for the domain $\mathbb{R}^n$ so that

$$df(a_i) = \lambda_i J(a_i).$$

Then

$$\left\{ e_i = \frac{1}{\sqrt{1+\lambda_i^2}}(a_i + \lambda_i J(a_i)) \right\}_{i=1,\ldots,n} \tag{3.7}$$

becomes an orthonormal basis for $T_p\Sigma$ and $\{J(e_i)\}_{i=1,\ldots,n}$ becomes an orthonormal basis for the normal bundle $N_p\Sigma$.

We compute for each i,

$$S(e_i, e_i) = \langle J\pi_1(e_i), \pi_2(e_i)\rangle = \frac{\lambda_i}{1+\lambda_i^2}.$$

Now $\frac{\lambda_i}{1+\lambda_i^2} > \epsilon$ implies a uniform upper bound on $\lambda_i' s$, the eigenvalues of $D^2 u$. q.e.d.

We also prove the condition $S_{ij} - \epsilon g_{ij} > 0$ is preserved along the mean curvature flow using the two methods. In the first method, we rewrite Equation (3.3) in an evolving orthonormal frame as in [4] and obtain

$$\left(\frac{d}{dt} - \Delta\right) S_{ij} = h_{pki}h_{pkl}S_{lj} + h_{pkj}h_{pkl}S_{il} + 2h_{kil}h_{kjm}S_{lm} \tag{3.8}$$

In the evolving orthonormal frame $g_{ij} = \delta_{ij}$, thus

(3.9)
$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right)(S_{ij} - \epsilon\delta_{ij}) &= h_{pki}h_{pkl}(S_{lj} - \epsilon\delta_{lj}) + h_{pkj}h_{pkl}(S_{il} - \epsilon\delta_{il}) \\ &\quad + 2h_{kil}h_{kjm}(S_{lm} - \epsilon\delta_{lm}) + 4\epsilon h_{pki}h_{pkj}\end{aligned}$$

and the result follows from the maximum principle again.

In the second method, we consider the tensor $E = S - \epsilon g$ and set $f(s) = E(v_s, v_s)$, after a similar calculation we have

$$\begin{aligned}(f)''(0) &= (E_{ij})''v^i v^j + E_{ij}(v^i)''v^j + E_{ij}v^i(v^j)'' \\ &= -2\mu_{ik}\mu_{jl}S_{kl}v^i v^j + 2\epsilon\mu_{ik}\mu_{jk}v^i v^j = -2\mu_{ik}\mu_{jl}E_{kl}v^i v^j\end{aligned}$$

A similar argument shows $\{P \in \mathrm{LG}(n) \mid \min_{v\in P} E(v,v) \geq 0\}$ is a convex subset for any $\epsilon \geq 0$.

3.3 Long time existence

We utilize the $C^{2,\alpha}$ estimate in space and the $C^{1,\alpha}$ estimate in time for nonlinear parabolic equations by Krylov [7] or [6] (see Section 5.5) to prove the long time existence. To apply it, we still need to check the uniform C^1 estimate for u in time and the concavity of

$$\frac{1}{\sqrt{-1}} \ln \frac{\det(I + \sqrt{-1}D^2u)}{\sqrt{\det(I + (D^2u)^2)}}$$

in the space of symmetric, positive definite matrices with the flat metric. The latter can be checked using a lemma of Caffarelli, Nirenberg and Spruck in [1] (Section 3 page 276), see also [15]. The former follows from Equation (2.2), because $\frac{du}{dt} = \alpha$ is given by the Lagrangian angle and it is well-known [13] that for the parametric mean curvature flow the Lagrangian angle satisfies the evolution equation

$$\frac{d}{dt}\alpha = \Delta\alpha \tag{3.10}$$

so that the maximum principle implies a uniform bound of $\frac{du}{dt}$.

With the $C^{2,\alpha}$ bound in space and the $C^{1,\alpha}$ bound in time, the convergence now follows from standard Schauder estimates and Simon's theorem [11]. Equation (3.11) then implies the limit is a flat Lagrangian submanifold.

We remark the long time existence also follows from the blow-up analysis in [21]. By Equation (3.9) in [21] (see also Equation (2.4) in [18]) and the positivity of λ_i, we obtain

$$\left(\frac{d}{dt} - \Delta\right) \ln \sqrt{\det(I + (D^2u)^2)} \leq |A|^2. \tag{3.11}$$

We integrate this inequality against the backward heat kernel and study the blow-up behavior at any possible singular points. The right hand side $|A|^2$ helps us to conclude any parabolic blow-up limit is totally geodesic and long time existence follows from White's regularity theorem [26].

4. Other equivalent conditions

As was remarked in [22] (Section 2) and [18] (see the remark at the end of the paper), the condition u being convex corresponds to a region

V on the Lagrangian Grassmannian. Since the geometry of a Lagrangian submanifold is invariant under the unitary group $\mathrm{U}(n)$, Theorem A applies whenever the Gauss map of a Lagrangian submanifold lies in a $\mathrm{U}(n)$ orbit of V. To be more precise, consider S as a bilinear form defined on $\mathbb{C}^n \cong T(T^{2n})$, given any $U \in \mathrm{U}(n)$ we may consider S_U defined by

$$S_U(\cdot,\cdot) = S(U(\cdot), U(\cdot)).$$

Notice that $JU = UJ$ as linear transformations on $\mathbb{C}^n$. It is not hard to see S_U again defines a symmetric bilinear form on any Lagrangian subspace. Also

$$S(J(X), J(Y)) = -S(X, Y)$$

for any X, Y in a Lagrangian subspace.

Now $F^*S_U > 0$ for $F : \Sigma \to T^{2n}$ implies the submanifold Σ can be locally written as a graph over a different Lagrangian plane with a convex potential function. The new Lagrangian plane is indeed the image of the domain $\mathbb{R}^n$ under U. This corresponds to choosing a different base point in the Lagrangian Grassmannian in §3.

Corollary A. *Let $F : \Sigma \to T^{2n}$ be a Lagrangian submanifold. Suppose there exists an $U \in \mathrm{U}(n)$ such that F^*S_U is positive definite on Σ. Then the mean curvature flow of Σ exists for all time and converges smoothly to a flat Lagrangian submanifold.*

Suppose Σ is the graph of $f : T^n \to T^n$ then the condition $F^*S_U > 0$ can be expressed in terms of the eigenvalues of the potential function u.

Recall from [22], given any splitting of $\mathbb{C}^n$, an element $U \in \mathrm{U}(n)$ can be represented by a $2n \times 2n$ block matrix

$$\begin{bmatrix} P & -Q \\ Q & P \end{bmatrix}$$

with

$$PP^T + QQ^T = I, -PQ^T + QP^T = 0.$$

Corresponding to the slitting $T(T^{2n}) = \mathbb{C}^n = T\Sigma \oplus N\Sigma$ and the bases (Equation (3.7)) $\left\{ e_i = \frac{1}{\sqrt{1+\lambda_i^2}}(a_i + \lambda_i J(a_i)), Je_i \right\}_{i=1,\dots,n}$, we have $Ue_i = \sum_k P_{ki}e_k + \sum_l Q_{li}Je_l$. Then

$$S_U(e_i, e_i) = \sum_k (P_{ki}^2 - Q_{ki}^2)\frac{\lambda_k}{1+\lambda_k^2} + \sum_k P_{ki}Q_{ki}\left(\frac{1-\lambda_k^2}{1+\lambda_k^2}\right).$$

Therefore the positive definiteness of S_U is the same as requiring the above expression to be positive for each i. Take $P = Q = \frac{1}{\sqrt{2}}I$ which amounts to rotating each complex plane by $\frac{\pi}{4}$, we obtain $S_U(e_i, e_i) = \frac{1}{2}\left(\frac{1-\lambda_i^2}{1+\lambda_i^2}\right)$. Therefore we have

Corollary B. *Let Σ be a Lagrangian submanifold in T^{2n}. Suppose Σ is the graph of $f : T^n \to T^n$ and the absolute values of the eigenvalues of the potential function u are less than one. Then the mean curvature flow of Σ exists for all time and converges smoothly to a flat Lagrangian submanifold. In particular, during the evolution the absolute values of all eigenvalues stay less than one.*

That the flow preserves the property of u having eigenvalues of absolute value less than one was also shown in [14], Theorem 2.6.3.

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