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ZOLL MANIFOLDS AND COMPLEX SURFACES

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Abstract

We classify compact surfaces with torsion-free affine connections for which every geodesic is a simple closed curve. In the process, we obtain completely new proofs of all the major results [4] concerning the Riemannian case. In contrast to previous work, our approach is twistor-theoretic, and depends fundamentally on the fact that, up to biholomorphism, there is only one complex structure on $\mathbb{CP}_2$.

1. Introduction

A *Zoll metric* on a smooth manifold M is a Riemannian metric g whose geodesics are all simple closed curves of equal length. This terminology [15] celebrates Otto Zoll's (now century-old) discovery [34] that S^2 admits many such metrics besides the obvious metrics of constant curvature [4]. Indeed, in terms of cylindrical coordinates $(z, \theta) \in [-1, 1] \times [0, 2\pi]$,

$$g = \frac{[1 + f(z)]^2}{1 - z^2} dz^2 + (1 - z^2) d\theta^2 \tag{1}$$

defines a Zoll metric on S^2 for any smooth odd function

$$f : [-1, 1] \to (-1, 1), \quad f(-z) = -f(z)$$

which vanishes at the end-points of the interval. A formal perturbation argument of Funk [12] later indicated that, modulo isometries and rescalings, the general Zoll metric on S^2 depends on one odd function

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$f : S^2 \to \mathbb{R}$. This formal calculation was later turned into a theorem by Guillemin [15], whose proof depends, e.g., on an implicit function theorem of Nash-Moser type. Because the function f is required to satisfy $f(-\vec{x}) = -f(\vec{x})$, however, these constructions never give rise to nonstandard Zoll metrics on $\mathbb{RP}^2$. Indeed, the so-called Blaschke conjecture, proved by Leon Green [13], asserts that, up to isometries and rescaling, the only Zoll metric on $\mathbb{RP}^2$ is the standard one. For an outstanding survey of these results, as well as an exploration of their higher-dimensional Riemannian generalizations, see [4].

The aims of the present article are twofold. First of all, instead of limiting ourselves to the study of Riemannian metrics, we will more generally consider torsion-free affine connections ∇, and ask how many such connections on a given manifold M have the property that all of their geodesics are simple closed curves. In order to make this a sensible problem, however, one must first observe that for any 1-form β on M, the torsion-free affine connection $\hat{\nabla}$ defined by

$$\hat{\nabla}_{\mathbf{u}}\mathbf{v} = \nabla_{\mathbf{u}}\mathbf{v} + \beta(\mathbf{u})\mathbf{v} + \beta(\mathbf{v})\mathbf{u}$$

has exactly the same *unparameterized* geodesics as the connection ∇; two connections related in this manner are said to be *projectively equivalent*, and obviously one should therefore only try to classify such connections modulo projective equivalence. In this rather general setting, our methods will allow us to obtain results very much like the classical Riemannian results alluded to above. Indeed, in §2, we begin by showing that the only compact surfaces which admit Zoll projective connections are S^2 and $\mathbb{RP}^2$. In §3, we then go on to show that, modulo diffeomorphisms, there is only one such projective class of connections on $\mathbb{RP}^2$. Finally, in §4, we prove that there is a nontrivial moduli space of such projective classes on S^2, locally parameterized by the space of vector fields on $\mathbb{RP}^2$.

But even in the Riemannian case, we seem to have something fundamentally new to contribute to the subject, as our proofs rest on foundations completely different from those used by our predecessors. Blaschke's unsuccessful approach to the problem of classifying Zoll metrics on $\mathbb{RP}^2$ amounted to a direct attempt to identify the space of all geodesics with the standard dual projective plane $\mathbb{RP}^{2*}$, the points of which which are by definition the real projective lines $\mathbb{RP}^1$ in $\mathbb{RP}^2$. The essence of our method is instead to use *complex*, rather than real, projective geometry to solve the problem. Indeed, we will construct a complex 2-manifold from any given Zoll structure, modeled on the dual complex projective plane $\mathbb{CP}_2^*$. The punch line of the proof is then that, up to biholomorphism, there is [3, 33] only one complex structure on $\mathbb{CP}_2$. Our proof of the generalized Blaschke conjecture then proceeds by recognizing the points of $\mathbb{RP}^2$ as the set of those complex projective lines $\mathbb{CP}_1$ in this $\mathbb{CP}_2$ which are invariant under the action of a certain anti-holomorphic involution. By contrast, the flexibility of Zoll structures on S^2 arises because the points in this case are instead represented by holomorphic disks with boundary on a totally real embedding of $\mathbb{RP}^2$ in $\mathbb{CP}_2$; deformations of this embedding then correspond to deformations of the Zoll structure. In this way, we are not only able to construct the general small deformation of the standard Zoll structure without recourse to Nash-Moser, but, more importantly, we are also able to glean a significant amount of information regarding arbitrary Zoll structures, even when they are quite far from the model case.

Finally, by way of an appendix, this article ends where it began, with a discussion of the axisymmetric case. After all, since we have chosen to generalize Zoll's problem by focusing on projective structures, it is only fitting that we should also generalize Zoll's construction by writing down all the axisymmetric Zoll projective structures on S^2 in closed form. In the process, we are able to show how the conceptual framework used in §4 can be implemented in concrete, calculational terms. We hope that our discussion of this special case will not only help clarify our general approach, but also make it seem all the more compelling.

2. Zoll projective structures

We begin by recalling the notion [27] of projective equivalence of affine connections.

Definition 2.1. Two torsion-free affine connections ∇ and $\hat{\nabla}$ on a manifold M are said to be *projectively equivalent* if they have the same geodesics, considered as *unparameterized* curves.

This condition may be re-expressed as the requirement that

$$\hat{\nabla}_{\mathbf{v}}\mathbf{v} \propto \mathbf{v} \iff \nabla_{\mathbf{v}}\mathbf{v} \propto \mathbf{v}.$$

We therefore have [27]:

Proposition 2.2. *Two C^k torsion-free affine connections ∇ and $\hat{\nabla}$ are projectively equivalent iff*

$$\hat{\nabla}_{\mathbf{u}}\mathbf{v} = \nabla_{\mathbf{u}}\mathbf{v} + \beta(\mathbf{u})\mathbf{v} + \beta(\mathbf{v})\mathbf{u}$$

for some C^k 1-form β.

Here a connection is said to be of differentiability class C^k with respect to a fixed C^{k+2} structure if the covariant derivative of any C^{k+1} vector field is a C^k tensor field; this is equivalent to requiring that the Christoffel symbols

$$\Gamma^j_{k\ell} = \left\langle dx^j, \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^\ell} \right\rangle$$

are all C^k functions in any admissible local coordinate system. We also note, in passing, that the torsion-free condition employed here can been imposed without any loss of generality; given an *arbitrary* affine connection, one can construct a unique torsion-free connection with precisely the same *parameterized* geodesics by replacing the Christoffel symbols with their symmetrizations:

$$\Gamma^j_{k\ell} \rightsquigarrow \hat{\Gamma}^j_{k\ell} = \frac{1}{2}\left(\Gamma^j_{k\ell} + \Gamma^j_{\ell k}\right) .$$

Definition 2.3. A C^k *projective structure* on a smooth manifold is the projective equivalence class $[\nabla]$ of some torsion-free C^k affine connection ∇.

By Definition 2.1, a projective structure $[\nabla]$ on M defines a certain family of geodesics, which are to be thought of as abstract immersed curves in M, without preferred parameterizations; conversely, this system of geodesics uniquely characterizes the projective structure in question.

In this paper, we will be interested in projective structures for which every geodesic is a simple closed curve.

Definition 2.4. Let ∇ be a C^1 torsion-free affine connection on a smooth manifold M. We will say that the projective equivalence class $[\nabla]$ of ∇ is a *Zoll projective structure* if the image $\mathfrak{C}$ of any maximal geodesic of ∇ is an embedded circle $S^1 \subset M$.

If $c : (a, b) \looparrowright M$ is any immersed curve, its derivative dc/dt is nonzero at every point, so that $[dc/dt]$ is a well-defined element of the *projectivized tangent bundle*

$$\mathbb{P}TM = (TM - 0_M)/\mathbb{R}^\times;$$

thus $t \mapsto [dc/dt]$ defines a curve $\widetilde{c} : \mathbb{R} \to \mathbb{P}TM$, called the *canonical lift* of c. Given a C^k Zoll projective structure $[\nabla]$ on M, the canonical lifts of its geodesics give us a C^k foliation $\mathcal{F}$ of $\mathbb{P}TM$ by circles. Let N denote the leaf space of this foliation.

Definition 2.5. Let $(M, [\nabla])$ be an n-manifold with C^k Zoll projective structure. We will say that $[\nabla]$ is *tame* if the corresponding foliation $\mathcal{F}$ of $\mathbb{P}TM$ by lifted geodesics is *locally trivial*, in the sense that each leaf has a neighborhood which is C^k diffeomorphic to $\mathbb{R}^{2n-2} \times S^1$ in such a manner that every leaf corresponds to a circle of the form $\{\mathrm{pt}\} \times S^1$.

These local trivializations give N the structure of a C^k $(2n-2)$-manifold in a canonical manner, making the quotient map $\nu : \mathbb{P}TM \to N$ into a C^k submersion. We will call the surface N the space of (undirected) geodesics of the tame Zoll projective structure $[\nabla]$. The situation is encapsulated by a diagram

$$\begin{tikzcd} & \mathbb{P}TM \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ M & & N \end{tikzcd}$$

which we shall refer to as the (real) *double fibration* of $[\nabla]$. Here $\mu : \mathbb{P}TM \to M$ of course denotes the bundle projection. Notice that, by construction, the tangent spaces of the fibers of μ and ν are everywhere linearly independent:

$$(\ker \mu_*) \cap (\ker \nu_*) = 0.$$

Moreover, the restriction of ν to any fiber of μ gives us an embedding $\mathbb{RP}^{n-1} \hookrightarrow N$.

Fortunately, as we will show in Theorem 2.16 below, this desirable picture applies to every compact Zoll surface. A key step in this direction is the following:

Proposition 2.6. *Any Zoll projective structure* $[\nabla]$ *on a compact* orientable *surface* M^2 *is tame.*

Proof. Because M is assumed to be a compact surface, $\mathbb{P}TM$ is a compact 3-manifold, and the Zoll projective structure $[\nabla]$ gives us a foliation $\mathcal{F}$ of $\mathbb{P}TM$ by circles. However, a theorem of Epstein [10] asserts that any foliation of a compact 3-manifold by circles is a Seifert fibration. Thus any leaf of $\mathcal{F}$ has a basis of neighborhoods modeled on

$$(\mathbb{C} \times S^1)/\mathbb{Z}_m,$$

where the $\mathbb{Z}_m$ action on $\mathbb{C} \times S^1 \subset \mathbb{C}^2$ is generated by $(z_1, z_2) \mapsto (e^{2\pi i\ell/m} z_1, e^{2\pi i/m} z_2)$, for some integer ℓ. All we therefore need to show is that no leaf is nontrivially covered by nearby leaves.

Now, because we have assumed that M is orientable, any geodesic circle $\mathfrak{C}$ has a tubular neighborhood diffeomorphic to the cylinder $S^1 \times \mathbb{R}$. Moreover, by Epstein's result, the lift of $\mathfrak{C}$ to $\mathbb{P}TM$ has a standard neighborhood whose projection to M is contained in the given cylindrical neighborhood. Thus, any geodesic circle $\mathfrak{C}'$ with initial point and tangent sufficiently close to those of $\mathfrak{C}$ will remain within our cylindrical neighborhood, and indeed will do so in such a manner that the projection $\mathfrak{C}' \to \mathfrak{C}$ induced by $S^1 \times \mathbb{R} \to S^1$ has nonzero derivative everywhere, and so will be a covering map. However, our tubular neighborhood $S^1 \times \mathbb{R}$ can be identified with $\mathbb{R}^2 - 0$ in such a manner that $\mathfrak{C}$ becomes the unit circle, and the degree of the covering becomes the winding number of $\mathfrak{C}'$ around the origin. But since $\mathfrak{C}'$ has been transformed into an embedded curve in the plane, the Jordan curve theorem tells us that its winding number around the origin has absolute value ≤ 1. Thus the covering map in question must have degree 1. The associated foliation $\mathcal{F}$ of $\mathbb{P}TM$ is therefore trivial in a neighborhood of the lift of $\mathfrak{C}$. q.e.d.

Next, we wish to determine precisely which compact surfaces admit Zoll projective structures. Our solution to this problem begins with the following simple observation:

Lemma 2.7. *Let $[\nabla]$ be a tame Zoll projective structure on an n-manifold M. Let $\varpi : \widetilde{M} \to M$ be the universal cover of M. Then $[\varpi^*\nabla]$ is a tame Zoll projective structure on $\widetilde{M}$.*

Proof. If $(M, [\nabla])$ is a tame Zoll manifold, all the lifted geodesics are freely homotopic embedded circles in $\mathbb{P}TM$; this is true because $\mathbb{P}TM$ is connected, and is the union of 'trivializing' open sets for the foliation $\mathcal{F}$, in which all the circular leaves are freely homotopic. Hence all the geodesic circles in M are freely homotopic. Moreover, by considering the geodesic circles through a given point $p \in M$, one obtains a base-point homotopy between any geodesic circle $\mathfrak{C} \subset M$ and its reverse-parameterized version $\overline{\mathfrak{C}}$. Hence $\mathfrak{C}$ either represents an element of order 1 or 2 in $\pi_1(M, p)$. Thus either $\mathfrak{C}$ or a 2-fold cover $\hat{\mathfrak{C}} \to \mathfrak{C}$ lifts to the universal cover $\widetilde{M}$ as an embedded circle, and this circle is geodesic with respect to the pull-back connection $\varpi^*\nabla$. Acting on each such lift by the action of $\pi_1(M)$, we thus see that every geodesic of $(\widetilde{M}, [\varpi^*\nabla])$ is an embedded circle, and $[\varpi^*\nabla]$ is therefore a Zoll projective structure on $\widetilde{M}$.

It remains to show that $[\varpi^*\nabla]$ is tame. To see this, first observe that the foliation $\mathcal{F}$ of $\mathbb{P}TM$ pulls back to the foliation $\hat{\mathcal{F}}$ of $\mathbb{P}T\widetilde{M}$ given by lifted geodesics of $[\varpi^*\nabla]$. Moreover, the induced map $\hat{\varpi} : \mathbb{P}T\widetilde{M} \to \mathbb{P}TM$ is a covering map. If $U \subset \mathbb{P}TM$ is any connected open set, and if $\hat{U} \subset \mathbb{P}T\widetilde{M}$ is any connected component of $\hat{\varpi}^{-1}(U)$, then $\hat{\varpi}|_{\hat{U}} : \hat{U} \to U$ is also a covering map. But if U is a trivializing neighborhood for $\mathcal{F}$, then the finite cover $\hat{U}$ of $U \approx S^1 \times \mathbb{R}^{2n-2}$ will therefore provide a local trivialization of $\hat{\mathcal{F}}$. Since $\mathbb{P}T\widetilde{M}$ is covered by such neighborhoods, this shows that $(\widetilde{M}, [\varpi^*\nabla])$ is tame, as claimed. q.e.d.

This leads to constraints on the topology of M.

Lemma 2.8. *Suppose that the n-manifold M admits a tame Zoll projective structure $[\nabla]$. Then M is compact, and has finite fundamental group. Moreover, every two points x and x' of M are joined by a geodesic of ∇.*

Proof. Choose an arbitrary point $x \in M$. In $\mathbb{P}TM$, consider the union

$$\hat{X} = \nu^{-1}\left(\nu\left[\mu^{-1}(x)\right]\right)$$

of the lifts of geodesics through x. Then $\hat{X}$ is a compact differentiable n-manifold. But since $\mu^{-1}(x) \subset \hat{X}$ is an $\mathbb{RP}^{n-1}$ whose normal bundle is the universal line bundle, $\hat{X}$ may be blown down along $\mu^{-1}(x)$ to produce a new compact differentiable n-manifold[1] X. Moreover, μ induces a differentiable map $\wp : X \to M$. Indeed, if $\check{x} \in X$ denotes the point obtained by blowing down $\mu^{-1}(x)$, then, in a neighborhood of $\check{x}$, $\wp$ is modeled on the exponential map of ∇ near $0 \in T_xM$. In particular, $\check{x}$ is a regular point of $\wp$. But, because $[\nabla]$ is Zoll, a geodesic circle can pass through x *only once*, so it follows that $\wp^{-1}(x) = \{\check{x}\}$. Thus x is a regular value of the proper map $\wp$ with $\#\wp^{-1}(x) = 1$. This shows that the mod-2 degree of the proper map $\wp$ is $1 \in \mathbb{Z}_2$. In particular, $\wp$ is onto, and $M = \wp(X)$ is therefore compact. The very definition of the surjective map $\wp$ now tells us that any point x' of M is joined to x by some geodesic of ∇.

Since the universal cover $\widetilde{M}$ also admits a tame Zoll projective structure by Lemma 2.7, the above argument now also shows that $\widetilde{M}$ is compact. Hence the universal covering map $\varpi : \widetilde{M} \to M$ is finite-to-one, and $\pi_1(M)$ is therefore finite, as claimed. q.e.d.

[1]We remark in passing that it is not difficult to show that X is always diffeomorphic to $\mathbb{RP}^n$.

Applying this to the two-dimensional case, we obtain the following:

Proposition 2.9. *A compact surface M^2 admits a Zoll projective structure iff M is diffeomorphic to either S^2 or $\mathbb{RP}^2$.*

Proof. By pulling the projective structure back to a double cover $\widetilde{M}$ of M if necessary, we obtain a Zoll projective structure on a compact orientable surface $\widetilde{M}$, and this pulled-back structure is then tame by Proposition 2.6. This forces $\widetilde{M}$, and hence M, to have finite fundamental group by Lemma 2.8. The classification of compact surfaces then tells us that M must be diffeomorphic to either S^2 or $\mathbb{RP}^2$. Conversely, the Levi-Civita connection $\triangledown$ of the standard, homogeneous metric determines a Zoll projective structure $[\triangledown]$ on either of these spaces. q.e.d.

The following information thus becomes pertinent to our discussion:

Lemma 2.10. *If $M = S^2$, $|\pi_1(\mathbb{P}TM)| = 4$. If $M = \mathbb{RP}^2$, $|\pi_1(\mathbb{P}TM)| = 8$.*

Proof. The unit tangent bundle of S^2 may be identified with SO(3) by thinking of the first column of an orthogonal matrix as a point of $S^2 \subset \mathbb{R}^3$, and the second column as a unit tangent vector at that point. Thus $\mathbb{P}TS^2$ may be identified with $\mathrm{SO}(3)/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ action is generated by left multiplication by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} .$$

Lifting to the universal cover $\mathrm{Sp}(1) = S^3 \subset \mathbb{H}^\times$ of SO(3), we thus have $\mathbb{P}TS^2 = \mathrm{Sp}(1)/\mathbb{Z}_4$, where the $\mathbb{Z}_4$ is generated by i. Hence $\pi_1(\mathbb{P}TS^2) \cong \mathbb{Z}_4$ has order 4, as claimed.

The antipodal map on S^2 acts on the unit tangent bundle via

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{SO}(3) ,$$

and this lifts to Sp(1) as $\pm k$. Thus $\mathbb{P}T\mathbb{RP}^2 = \mathrm{Sp}(1)/\{\pm 1, \pm i, \pm j, \pm k\}$, and hence $\pi_1(\mathbb{P}T\mathbb{RP}^2) \cong \{\pm 1, \pm i, \pm j, \pm k\}$ has order 8, as claimed. q.e.d.

In particular, $\pi_1(\mathbb{P}TM^2)$ must be finite. Hence:

Proposition 2.11. *Let $(M, [\nabla])$ be a compact surface with tame Zoll projective structure. Then its space N of unoriented geodesics is diffeomorphic to $\mathbb{RP}^2$.*

Proof. The group homomorphism

$$\nu_\natural : \pi_1(\mathbb{P}TM) \to \pi_1(N)$$

induced by the fibration ν is surjective, since each fiber of ν is path connected. But Proposition 2.9 and Lemma 2.10 together tell us that $\mathbb{P}TM$ has finite fundamental group. Hence $\pi_1(N)$ is finite, and the classification of 2-manifolds therefore tells us that N must be diffeomorphic to either S^2 or $\mathbb{RP}^2$. But we also know that N is not simply connected, since it has a nontrivial cover $\widetilde{N}$, given by the space of *directed* geodesics of $[\nabla]$. This shows that $N \approx \mathbb{RP}^2$, as claimed. q.e.d.

Next, we would like to understand the topological structure of the S^1-bundle

$$\nu : \mathbb{P}TM \to N.$$

Our method will simultaneously allow us to analyze the conjugate points of the projective structure $[\nabla]$. Let us thus begin by recalling the notion of a Jacobi field.

If ∇ is a torsion-free connection on a manifold M, and if $c : (a,b) \to M$ is an affinely parameterized geodesic of ∇, then a *Jacobi field* along c is by definition a vector field $\mathbf{y} \in \Gamma(c^*TM)$ along c which satisfies the linear differential equation

$$\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}\mathbf{y} = R_{\mathbf{v}\mathbf{y}}\mathbf{v},$$

where R denotes the curvature tensor of ∇, and where the standard tangent vector

$$\mathbf{v} = \frac{dc}{dt}$$

of our parameterized geodesic satisfies the *auto-parallel condition*

$$\nabla_{\mathbf{v}}\mathbf{v} = 0. \tag{2}$$

It is not difficult to see that $\mathbf{y}$ is a Jacobi field iff it is locally the joining vector field for a 1-parameter family of geodesics of ∇. More precisely, for any $[a', b'] \subset (a,b)$, there is an $\varepsilon > 0$ and a differentiable map

$$\begin{array}{rcl} \hat{c} : [a', b'] \times (-\varepsilon, \varepsilon) & \to & M \\ (t, u) & \mapsto & \hat{c}(t,u) \end{array}$$

with $\hat{c}(t, 0) = c(t)$, such that, setting

$$\widetilde{\mathbf{v}} = \frac{\partial \hat{c}}{\partial t}, \quad \widetilde{\mathbf{y}} = \frac{\partial \hat{c}}{\partial u},$$

one has

$$\nabla_{\widetilde{\mathbf{v}}}\widetilde{\mathbf{v}} = 0$$

and

$$\widetilde{\mathbf{y}}|_{u=0} = \mathbf{y}.$$

The notion of a Jacobi field is not actually projectively invariant, but there is a closely related concept which *is*.

Definition 2.12. Let $[\nabla]$ be a C^1 projective connection on M, and let $\mathfrak{C} \looparrowright M$ be any geodesic of $[\nabla]$. Then a section $\mathfrak{Y}$ of the normal bundle $TM/T\mathfrak{C}$ of $\mathfrak{C}$ will be called a *Jacobi class* on $\mathfrak{C}$ iff, near any given point $p \in \mathfrak{C}$,

$$\mathfrak{Y} \equiv \mathbf{y} \bmod T\mathfrak{C}$$

for some locally defined Jacobi field $\mathbf{y}$.

In other words, $\mathfrak{Y}$ is a Jacobi class iff it locally joins infinitesimally separated *unparameterized* geodesics. Thought of this way, it thus becomes immediately apparent that the notion of Jacobi class is projectively invariant.

Definition 2.13. Let $[\nabla]$ be a C^1 projective connection on M, and let $\mathfrak{C} \looparrowright M$ be any geodesic of $[\nabla]$. We will say that two points $p, q \in \mathfrak{C}$ are *conjugate* along $\mathfrak{C}$ iff there is a Jacobi class $\mathfrak{Y}$ on $\mathfrak{C}$ with $\mathfrak{Y}(p) = \mathfrak{Y}(q) = 0$.

Very roughly, conjugate points are thus the places where two infinitesimally separated geodesics of $[\nabla]$ meet.

Let us now make all of this more explicit in the special case of $\dim M = 2$. If $\mathfrak{C} \looparrowright M$ is a geodesic of an affine connection ∇ on a surface M, the normal bundle $TM/T\mathfrak{C}$ is a real line bundle $E \to \mathfrak{C}$. Since $T\mathfrak{C} \subset TM$ is parallel, ∇ defines a connection $\mathfrak{D}$ on E. Let us take an affine parameterization $c : (a, b) \to \mathfrak{C}$, so that $\mathbf{v} = dc/dt$ satisfies (2). Let us then trivialize $c^*E \to (a, b)$ by means of $[\mathbf{e}]$, where $\mathbf{e} \not\propto \mathbf{v}$ is a generic parallel section of c^*TM, and where the brackets $[\ \cdot\ ]$ indicate the equivalence class mod $T\mathfrak{C}$. Defining $\kappa : (a, b) \to \mathbb{R}$ by

$$\kappa = r(\mathbf{v}, \mathbf{v}),$$

where $r_{ab} = R^c{}_{acb}$ is the Ricci tensor of ∇, we then have

$$R_{\mathbf{ve}}\mathbf{v} \equiv -\kappa\mathbf{e} \bmod \mathbf{v},$$

so that $y(t)\mathbf{e} \equiv \mathbf{y} \bmod T\mathfrak{C}$ for some Jacobi field $\mathbf{y}$ iff $y : (a,b) \to \mathbb{R}$ satisfies the second order linear differential equation

$$\frac{d^2y}{dt^2} + \kappa y = 0. \tag{3}$$

More abstractly, (3) becomes

$$\mathfrak{D}_{\mathbf{v}}\mathfrak{D}_{\mathbf{v}}\mathfrak{Y} + r(\mathbf{v},\mathbf{v})\mathfrak{Y} = 0 \tag{4}$$

in terms of the connection $\mathfrak{D}$ induced on the normal bundle E, and this in turn generalizes to becomes

$$\mathfrak{D}_{\mathbf{v}}\mathfrak{D}_{\mathbf{v}}\mathfrak{Y} - \mathfrak{D}_{\nabla_{\mathbf{v}}\mathbf{v}}\mathfrak{Y} + r(\mathbf{v},\mathbf{v})\mathfrak{Y} = 0 \tag{5}$$

if we drop the auto-parallel condition (2) on our tangent field $\mathbf{v}$. Let us remark that if ∇ is replaced by the projectively equivalent connection $\hat{\nabla}$ defined by

$$\hat{\nabla}_{\mathbf{u}}\mathbf{v} = \nabla_{\mathbf{u}}\mathbf{v} + \beta(\mathbf{u})\mathbf{v} + \beta(\mathbf{v})\mathbf{u},$$

one then has

$$\begin{aligned}
\hat{\mathfrak{D}}_{\mathbf{v}}\hat{\mathfrak{D}}_{\mathbf{v}}\mathfrak{Y} &= \mathfrak{D}_{\mathbf{v}}\mathfrak{D}_{\mathbf{v}}\mathfrak{Y} + 2\beta(\mathbf{v})\mathfrak{D}_{\mathbf{v}}\mathfrak{Y} + \left[\mathbf{v}\beta(\mathbf{v}) + \beta(\mathbf{v})^2\right]\mathfrak{Y},\\
\hat{\mathfrak{D}}_{\hat{\nabla}_{\mathbf{v}}\mathbf{v}}\mathfrak{Y} &= \mathfrak{D}_{\nabla_{\mathbf{v}}\mathbf{v}}\mathfrak{Y} + 2\beta(\mathbf{v})\mathfrak{D}_{\mathbf{v}}\mathfrak{Y} + \left[\beta(\nabla_{\mathbf{v}}\mathbf{v}) + 2\beta(\mathbf{v})^2\right]\mathfrak{Y},\\
\hat{r}(\mathbf{v},\mathbf{v}) &= r(\mathbf{v},\mathbf{v}) + (n-1)\left[\beta(\nabla_{\mathbf{v}}\mathbf{v}) - \mathbf{v}\beta(\mathbf{v}) + \beta(\mathbf{v})^2\right],
\end{aligned}$$

so that blind, brute-force calculation does indeed show that (5) is projectively invariant in dimension $n = 2$, as previously deduced by pure thought.

Now the vector space of solutions of (3) is two dimensional, corresponding to choices of y and y' at an arbitrary base-point of the interval (a,b). Let $\{y_1, y_2\}$ be an arbitrary basis for this solution space, and consider the Wronskian

$$W(t) = \begin{vmatrix} y_1(t) & y_1'(t) \\ y_2(t) & y_2'(t) \end{vmatrix} = y_1y_2' - y_2y_1'.$$

The differential equation (3) then tells us that

$$\begin{aligned}
\frac{dW}{dt} &= y_1'y_2' + y_1y_2'' - y_2'y_1' - y_2y_1''\\
&= y_1(-\kappa y_2) - y_2(-\kappa y_1) = 0,
\end{aligned}$$

so that $W(t)$ is constant. Moreover, this constant must be nonzero, since y_1 and y_2 have linearly independent initial values at the base-point. The map

$$\begin{array}{rcl} \phi : (a,b) & \longrightarrow & \mathbb{RP}^1 \\ t & \mapsto & [y_1(t) : y_2(t)] \end{array}$$

is therefore well-defined for all t, since y_1 and y_2 cannot simultaneous vanish. Moreover, ϕ is an *immersion*, since

$$\frac{d}{dt}\left(\frac{y_1}{y_2}\right) = \frac{W}{y_2^2} \quad \text{and} \quad \frac{d}{dt}\left(\frac{y_2}{y_1}\right) = -\frac{W}{y_1^2}$$

are never zero. Geometrically, ϕ may be interpreted as sending $x \in (a,b)$ to the set of Jacobi classes $\mathfrak{Y}$ with $\mathfrak{Y}(x) = 0$, since a Jacobi class

$$y(t) = \lambda_2 y_1(t) - \lambda_1 y_2(t) \not\equiv 0$$

vanishes at x iff $[\lambda_1 : \lambda_2] = \phi(x) := [y_1(x) : y_2(x)]$. In particular, two points are conjugate along $c\,[(a,b)]$ iff they have the same image under ϕ.

For a tame C^k Zoll projective structure $[\nabla]$ on a surface M^2, there are two linearly independent Jacobi classes defined along the entirety of any closed geodesic $\mathfrak{C}$; indeed, if $y \in N$ represents $\mathfrak{C}$ in the space of geodesics, T_yN is naturally in one-to-one correspondence with the space of Jacobi classes along $\mathfrak{C}$ via $\mu_* \circ (\nu_*)^{-1}$. The above construction thus gives us a C^{k+1} covering map $\phi : \mathfrak{C} \to \mathbb{RP}^1$ for every geometrically closed geodesic $\mathfrak{C}$, and this map is uniquely defined modulo the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{RP}^1$. The order of the covering ϕ will be called the *conjugacy number* of the geodesic, since it exactly counts how many points of $\mathfrak{C}$ are conjugate to $x \in \mathfrak{C}$, of course including x itself. We will now see that this number actually has a rather deeper meaning.

Proposition 2.14. *Let $[\nabla]$ be a tame C^k Zoll projective connection, $1 \leq k \leq \infty$, on a compact 2-manifold M, and consider the C^{k-1} map*

$$\begin{array}{rcl} \varphi : \mathbb{P}TM & \longrightarrow & \mathbb{P}TN \\ z & \mapsto & \nu_* \left(\ker \mu_{*z}\right), \end{array}$$

where μ_* and ν_* denote the derivatives of μ and ν, respectively. Then φ is a covering map. Moreover, the order of the covering φ exactly equals the conjugacy number of any closed geodesic $\mathfrak{C} \subset M$. In particular, all the geodesics of $[\nabla]$ have the same conjugacy number.

Proof. Let us first notice that we have a commutative diagram

$$\begin{tikzcd} \mathbb{P}TM \arrow[rr, "\varphi"] \arrow[dr, "\nu"'] & & \mathbb{P}TN \arrow[dl, "\pi"] \\ & N & \end{tikzcd}$$

where π denotes the relevant canonical projection. Moreover, since N is by definition the leaf space of the foliation $\mathcal{F}$, we also know that φ maps each leaf of $\mathcal{F}$ to a different fiber of π.

Now the tangent space of N at any point can be canonically identified with the space of Jacobi classes on the corresponding geodesic in M. With this identification, φ then sends a point of a geodesic $\mathfrak{C}$ (identified, by lifting, with a leaf of $\mathcal{F}$) to the set of Jacobi classes which vanish at that point. In other words, on each leaf of $\mathcal{F}$, thought of as a geodesic $\mathfrak{C} \subset M$ of $[\nabla]$, φ precisely coincides with the map ϕ described above. This shows that φ immerses each leaf in $\mathbb{P}TN$ as a fiber of π. Since ν is a submersion, it follows, for $k \geq 2$, that φ_* is injective, and hence that φ is a local diffeomorphism; for $k = 1$, one instead may observe that φ must be injective on some neighborhood of any point, and so must be a local homeomorphism by the open mapping theorem. But since $\mathbb{P}TM$ is compact, this implies that φ is a covering map. Moreover, the order of this covering is precisely the number of points on a leaf of $\mathcal{F}$ which are sent to the same point of a fiber π. This shows that the order of covering φ is precisely the conjugacy number of any geodesic of $[\nabla]$.

q.e.d.

If X is any manifold, let us use $\mathbb{S}TX$ to denote the sphere bundle $(TX - 0_X)/\mathbb{R}^+$. In other words, $\mathbb{S}TX$ may be thought of as the set of unit tangent vectors for an arbitrary Riemannian metric on X.

Theorem 2.15. *If $[\nabla]$ is any C^k Zoll projective structure, $k \geq 1$, on $M \approx S^2$, its conjugacy number is two, and there is a C^{k-1} diffeomorphism $\mathbb{P}TM \approx \mathbb{S}TN$ such that ν becomes the canonical projection $\mathbb{S}TN \to N$. Moreover, the real line bundle* $\ker \mu_*$ *over $\mathbb{P}TM$ is trivial.*

Proof. Let us first recall that Proposition 2.6 tells us that $[\nabla]$ is tame. But now, with a nod to Lemma 2.10, we see that the covering map $\varphi : \mathbb{P}TM \to \mathbb{P}TN$ has order

$$\frac{|\pi_1(\mathbb{P}TN)|}{|\pi_1(\mathbb{P}TM)|} = \frac{|\pi_1(\mathbb{P}T\mathbb{RP}^2)|}{|\pi_1(\mathbb{P}TS^2)|} = \frac{8}{4} = 2,$$

and the conjugacy number is therefore 2, by Proposition 2.14.

Now notice that the real line bundle $\ker \mu_*$ over $\mathbb{P}TS^2$ is trivial. Indeed, after the choice of a metric and orientation, $\mathbb{P}TS^2$ can be identified with the SO(2) bundle of oriented orthonormal frames divided by $\langle -\mathbf{1}\rangle \subset \mathrm{SO}(2)$, and carries an induced $\mathfrak{so}(2)$ action which trivializes $\ker \mu_*$. Let $\mathbf{v}$ denote the vector field which generates this action. Imitating our construction of φ, we now obtain a diagram

$$\begin{tikzcd} \mathbb{P}TM \arrow[rr, "\hat{\varphi}"] \arrow[dr, "\nu"'] & & \mathbb{S}TN \arrow[dl, "\hat{\pi}"] \\ & N & \end{tikzcd}$$

by defining $\hat{\varphi}(z) = \mathbb{R}^+\nu_*(\mathbf{v}_z)$; here $\hat{\pi} : \mathbb{S}TN \to N$ of course denotes the canonical projection. Now $\hat{\varphi}$ is a covering map, since it lifts φ. But

$$\frac{|\pi_1(STN)|}{|\pi_1(\mathbb{P}TM)|} = \frac{|\pi_1(ST\mathbb{RP}^2)|}{|\pi_1(\mathbb{P}TS^2)|} = \frac{4}{4} = 1,$$

so it now follows that $\hat{\varphi}$ is a homeomorphism if $k = 1$, and a diffeomorphism if $k \geq 2$. q.e.d.

This finally allows us to definitively dispense with the tame condition.

Theorem 2.16. *Any C^1 Zoll projective structure on a compact surface M^2 is tame.*

Proof. Proposition 2.6 covers the orientable case, so we may henceforth assume that M is nonorientable. Proposition 2.9 then tells us that M is diffeomorphic to $\mathbb{RP}^2$, so we have $M = \widetilde{M}/\langle a\rangle$, where $\widetilde{M} \approx S^2$, and where $a : \widetilde{M} \to \widetilde{M}$ corresponds to the antipodal map on S^2. Any Zoll projective structure on M then pulls back to a tame Zoll projective structure on $\widetilde{M}$, each of whose geodesics is sent to some geodesic by a. If $\widetilde{N} \approx \mathbb{RP}^2$ is the space of unoriented geodesics of $\widetilde{M}$, then a thus induces a diffeomorphism $\hat{a} : \widetilde{N} \to \widetilde{N}$. We claim that $\hat{a}$ is in fact the identity.

Suppose not. Then $\hat{a}$ generates a nontrivial $\mathbb{Z}_2$ action. But any action by a finite group of diffeomorphisms is isometric with respect to some Riemannian metric, and so has fixed-point set consisting of a disjoint union of closed submanifolds. In our case, the fixed-point set would be a finite union of disjoint circles and points. Moreover, the quotient $\widetilde{N}/\mathbb{Z}_2$ would have Euler characteristic

$$\chi(\widetilde{N}/\mathbb{Z}_2) = \frac{\chi(\widetilde{N}) + m}{2} = \frac{1+m}{2},$$

where m is the number of isolated fixed points. Since the Euler characteristic is an integer, this shows that $\hat{a}$ has at least one isolated fixed point. At such an isolated fixed point, the derivative of $\hat{a}$ must be -1, as this is the unique order-2 element of $O(2)$ with trivial $+1$-eigenspace.

Back in $\widetilde{M}$, this fixed point would correspond to a geodesic circle $\mathfrak{C}$ with $a(\mathfrak{C}) = \mathfrak{C}$, along which a_* induced the action $\mathfrak{Y} \mapsto -\mathfrak{Y}$ on the vector space of Jacobi classes. In particular, the zero locus of a Jacobi class $\mathfrak{Y} \not\equiv 0$ would necessarily be sent to itself by a_*. But since $\widetilde{M}$ has conjugacy number 2 by Theorem 2.15, and since a has no fixed points, this means that a acts on $\mathfrak{C}$ by sending each point to the unique other point to which it is conjugate. Now trivialize the normal bundle $E = T\widetilde{M}/T\mathfrak{C}$, so that we can talk about whether a nonzero element of L is 'positive' or 'negative'. Then, since any Jacobi class $\mathfrak{Y} \not\equiv 0$ meets the zero section of E transversely in exactly 2 points, the subsets of $\mathfrak{C}$ given by $\mathfrak{Y} > 0$ and $\mathfrak{Y} < 0$ are necessarily intervals, and are necessarily interchanged by the fixed-point-free map a. But since $\mathfrak{Y} \mapsto -\mathfrak{Y}$ under a_*, this shows that a_* acts on the normal bundle E in an *orientation-preserving* manner. Moreover, a_* is also orientation-preserving on $T\mathfrak{C}$, since $a : \mathfrak{C} \to \mathfrak{C}$ has no fixed point. Hence a acts on $\widetilde{M}$ in an orientation-preserving manner — contradicting the fact that, by construction, a is an orientation-reversing map!

This contradiction shows that $\hat{a}$ must be the identity on $\widetilde{N}$. Hence a_* induces an action on $\mathbb{P}T\widetilde{M}$ which sends each leaf to itself, and holonomy around any leaf in $\mathbb{P}TM$ is therefore trivial. Hence the given Zoll projective structure on $M \approx \mathbb{RP}^2$ is tame, as claimed. q.e.d.

In particular, it now makes sense to talk about the conjugacy number of any Zoll projective structure on $\mathbb{RP}^2$.

Theorem 2.17. *If $[\nabla]$ is any C^k Zoll projective structure, $k \geq 1$, on $M \approx \mathbb{RP}^2$, its conjugacy number is 1. Moreover, there is a C^{k-1} diffeomorphism $\mathbb{P}TM \approx \mathbb{P}TN$ such that ν becomes the canonical projection $\mathbb{P}TN \to N$, and such that $\ker \mu_* \to \mathbb{P}TM$ becomes the 'tautological' real line bundle $L \to \mathbb{P}TN$, whose frame bundle bundle is the principal $\mathbb{R}^\times$-bundle $(TN - 0_N) \to \mathbb{P}TN$.*

Proof. Since $[\nabla]$ is tame by Theorem 2.16, we are free to consider the covering map $\varphi : \mathbb{P}TM \to \mathbb{P}TN$ of Proposition 2.14. By construction, the tautological line bundle $L \to \mathbb{P}TN$ then satisfies $\varphi^* L = \ker \mu_*$. Since the order of this covering is

$$\frac{|\pi_1(\mathbb{P}TN)|}{|\pi_1(\mathbb{P}TM)|} = \frac{|\pi_1(\mathbb{P}T\mathbb{RP}^2)|}{|\pi_1(\mathbb{P}T\mathbb{RP}^2)|} = 1,$$

we conclude that φ is a homeomorphism, and the conjugacy number is therefore 1 by Proposition 2.14. Moreover, the same argument also shows that φ is actually a diffeomorphism if $k \geq 2$. q.e.d.

Corollary 2.18. *For any Zoll projective structure $[\nabla]$ on $M \approx \mathbb{RP}^2$, any two distinct points are joined by a unique geodesic circle $\mathfrak{C}$.*

Proof. As in the proof of Lemma 2.8, let

$$\hat{X} = \nu^{-1}\left(\nu\left[\mu^{-1}(x)\right]\right)$$

be the union of the lifts of geodesics through x. Then $\hat{X}$ is a compact differentiable surface and may be blown down along $\mu^{-1}(x)$ to produce a new smooth compact surface X. Since $\hat{X}$ is a circle bundle over the circle $\ell_x = \nu\left[\mu^{-1}(x)\right]$, and since a neighborhood of $\mu^{-1}(x)$ is a Möbius band B, it follows that X contains a Möbius band $B' = \hat{X} - B$, and hence is not orientable.

On the other hand, Theorem 2.17 tells us that each geodesic in M has conjugacy number 1, and hence no point $x' \neq x$ is conjugate to x along any geodesic. Hence the canonical projection $\hat{X} \to M$ is an immersion away from $\mu^{-1}(x)$, and the induced map $\wp : X \to M$ is therefore an immersion everywhere. Since X is compact, $\wp$ is therefore a covering map. Since X is not simply connected and $\pi_1(M) = \mathbb{Z}_2$, it follows that $\wp$ is a one-to-one and onto. But, by the very definition of $\wp$, this means that there is one and only one geodesic between x and any other point $x' \neq x$ in M. q.e.d.

Corollary 2.19. *Let $(M^2, [\nabla])$ be a compact surface with Zoll projective structure. Let $\mathfrak{C} \subset M$ be any geodesic circle. Then the following conditions are equivalent:*

- $\langle w_1(M), [\mathfrak{C}]\rangle = 1 \in \mathbb{Z}_2$*;*
- *the conjugacy number of $\mathfrak{C}$ is odd;*
- *M is not orientable;*
- *M is diffeomorphic to $\mathbb{RP}^2$.*

Proof. At points where a Jacobi class $\mathfrak{Y} \not\equiv 0$ vanishes along $\mathfrak{C}$, the covariant derivative $\mathfrak{D}_{\mathsf{v}}\mathfrak{Y}$ must be nonzero, since $\mathfrak{Y}$ satisfies (3). Thus the mod-2 reduction of the conjugacy number of $\mathfrak{C}$ calculates $\langle w_1(E), [\mathfrak{C}]\rangle$, where $E = TM/T\mathfrak{C}$ is the normal bundle, and this of course coincides with $\langle w_1(M), [\mathfrak{C}]\rangle := \langle w_1(TM), [\mathfrak{C}]\rangle$, since $T\mathfrak{C}$ is trivial. But Theorems 2.17 and 2.15 tell us that the only possible values of the conjugacy number are 1 and 2, and that the value of the conjugacy number determines whether M is diffeomorphic to $\mathbb{RP}^2$ or S^2. q.e.d.

The same argument also yields the following:

Corollary 2.20. *Let $(M^2, [\nabla])$ be a compact surface with Zoll projective structure. Let $\mathfrak{C} \subset M$ be a geodesic circle. Then the following conditions are equivalent:*

- $\langle w_1(M), [\mathfrak{C}]\rangle = 0 \in \mathbb{Z}_2$*;*
- *the conjugacy number of $\mathfrak{C}$ is even;*
- *M is orientable;*
- *M is diffeomorphic to S^2.*

Let us now take a moment to compare our definitions with those previously used by others in the Riemannian context [4, 15].

Proposition 2.21. *Let (M^2, g) be a compact surface with C^k Riemannian metric, $2 \leq k \leq \infty$. Let ∇ be the Levi-Civita connection of g. Then $[\nabla]$ is a C^{k-1} Zoll projective structure on M iff the geodesics of g are all simple closed curves of equal length.*

Proof. If $[\nabla]$ is a Zoll projective structure, Theorem 2.16 then tells us it is tame, and its geodesic circles are therefore freely homotopic to one another *through geodesic circles*. But the affinely parameterized closed geodesics of g are precisely those differentiable maps $c : S^1 \to M$ which are critical points of the energy functional

$$E(c) = \int_{S^1} g(c'(t), c'(t))dt \;;$$

thus the energy is necessarily constant for any 1-parameter family of closed geodesics. This shows that the geodesic circles of g must all have equal energy, and hence equal length. q.e.d.

We conclude this section with an aside which plays no rôle whatsoever in what follows, but which, in light of Proposition 2.21, has a certain intrinsic interest. Given a Zoll projective structure $[\nabla]$ on a compact surface M, it is natural to ask whether there is a connection ∇ representing $[\nabla]$ such that every affinely parameterized geodesic is *periodic*. The answer is affirmative.

Proposition 2.22. *If $[\nabla]$ is any Zoll projective structure on a compact surface M^2, then there is a torsion-free affine connection $\nabla \in [\nabla]$ for which each affinely parameterized geodesic extends as a periodic function $c : \mathbb{R} \to M$.*

Proof. If $M = S^2$, let ω be an arbitrary area form on M, and let ∇ be [27] the unique connection in the equivalence class such that $\nabla\omega = 0$. If $c : [a, b] \to M$ is an affine parameterization of a geodesic of ∇, with $c(b) = c(a)$ and $c'(b) = \lambda c'(a)$, then any parallel vector field $\mathbf{e}$ along c must satisfy $\mathbf{e}(b) = \lambda^{-1}\mathbf{e}(a) \bmod c'$. Now the Zoll condition guarantees the existence of a two-parameter family of solutions of (4) which satisfy the "periodicity" condition

$$\mathfrak{Y}|_{c(b)} = \mathfrak{Y}|_{c(a)}, \quad \mathfrak{D}\mathfrak{Y}|_{c(b)} = \mathfrak{D}\mathfrak{Y}|_{c(a)}.$$

Every solution of (3) must therefore satisfy

$$y(b) = \lambda y(a), \quad y'(b) = \lambda^2 y'(a).$$

Hence the Wronskian $W = y_1 y_2' - y_2 y_1'$ of two linearly independent solutions of (3) must satisfy $W(b) = \lambda^3 W(a)$. But W is constant! Thus $\lambda = 1$, and the given geodesic is therefore periodic. But this argument applies to *any* geodesic on M. Hence every geodesic of the chosen connection ∇ is periodic, and the claim follows if $M = S^2$.

The case of $\mathbb{RP}^2$ now follows easily; one simply takes the area form ω on S^2 to be anti-invariant under the antipodal map $a : S^2 \to S^2$, and then notices that the corresponding connection ∇ then descends to $\mathbb{RP}^2$. q.e.d.

3. The Blaschke conjecture revisited

If $[\nabla]$ is a Zoll projective structure on a compact surface M, we saw in §2 that its space of unoriented geodesics N is diffeomorphic to $\mathbb{RP}^2$. Now notice that N also comes equipped with a family

$$\ell_x = \nu[\mu^{-1}(x)]$$

of embedded circles $\ell_x \subset N$, $x \in M$. (For any given $x \in M$, this is to say that ℓ_x consists precisely of the geodesics passing through x.) If we were simply given N and this family of curves, we could then completely reconstruct the given projective structure on M. Indeed, M could be redefined as the parameter space or 'moduli space' of these curves ℓ_x, and the geodesics $\mathfrak{C}_y \subset M$ would then become the set of curves ℓ_x passing through some given point $y \in N$. The utility of this point of view might seem to be rather questionable, however, as there is no obvious geometric structure one might impose on N in order to keep track of which embedded circles $\ell \subset N$ are to be the elements of the family $\{\ell_x\}_{x\in M}$. However, our main observation, extrapolated from a twistor correspondence due to Hitchin [19] and the first author [20], is that one *can* naturally keep track of these curves by 'complexifying' the picture, and embedding N in a complex 2-manifold $\mathcal{N}$.

Let us suppose we are given a C^2 Zoll projective structure $[\nabla]$ on $M = \mathbb{RP}^2$. Consider the $\mathbb{CP}_1$-bundle

$$\mathbb{P}T_{\mathbb{C}}M = (\mathbb{C} \otimes TM - 0_M)/\mathbb{C}^\times,$$

and observe that the circle bundle

$$\mathbb{P}TM = (TM - 0_M)/\mathbb{R}^\times$$

is a hypersurface in the 4-manifold $\mathbb{P}T_{\mathbb{C}}M$. For brevity, we introduce the notation

$$\mathcal{Z} = \mathbb{P}T_{\mathbb{C}}M, \quad Z = \mathbb{P}TM.$$

Because each fiber of $\mathbb{P}T_{\mathbb{C}}M$ has a canonical complex structure $J^{\|}$, the normal bundle of $\mathbb{P}TM \subset \mathbb{P}T_{\mathbb{C}}M$ is just $J^{\|}(\ker \mu_*)$, where $\mu : \mathbb{P}TM \to M$ is the bundle projection. Now recall that our Zoll projective structure gives us a foliation $\mathcal{F}$ of $\mathbb{P}TM$ by circles, and the leaves of $\mathcal{F}$ are precisely the fibers of a C^2 submersion $\nu : \mathbb{P}TM \to N \approx \mathbb{RP}^2$. Moreover, Theorem 2.17 tells us that there is a C^1 diffeomorphism $\varphi : \mathbb{P}TM \to \mathbb{P}TN$ such that the real line bundle $\ker \mu_*$ becomes the pull-back $\varphi^* L$ of the tautological line bundle $L \to \mathbb{P}TN$. The latter line bundle is by definition a subbundle of π^*TN, where $\pi : \mathbb{P}TN \to N$ is the canonical projection; namely, for any nonzero vector $\mathbf{v} \in T_yN$, the fiber over $[\mathbf{v}] \in \mathbb{P}TN$ is $L_{[\mathbf{v}]} = \mathrm{span}(\mathbf{v}) \subset T_yN$. In particular, there is a tautological C^1 'blowing down' map $\psi : L \to TN$ which is a diffeomorphism away from the zero section $\mathbb{P}TN$ of L, but collapses this zero section to the zero section N of TN via $\pi : \mathbb{P}TN \to N$. On the other hand, the tubular neighborhood theorem tells us that $Z = \mathbb{P}TM$ has a neighborhood $\hat{\mathcal{V}}$ in $\mathcal{Z} = \mathbb{P}T_{\mathbb{C}}M$ which is C^∞ diffeomorphic to the total space of $J^{\|} \ker \mu_*$, in such a manner that the derivative along Z is the identity. Letting $\mathcal{V}$ denote the total space of $TN = T\mathbb{RP}^2$, we then have a C^1 map $\widetilde{\psi} : \hat{\mathcal{V}} \to \mathcal{V}$ which corresponds to ψ via our C^1 diffeomorphism $J^{\|} \ker \mu_* \to L$. We may now define a new C^1 compact 4-manifold

$$\mathcal{N} = \mathcal{U} \cup_{\widetilde{\psi}} \mathcal{V}$$

by gluing together $\mathcal{U} := \mathcal{Z} - Z$ and $\mathcal{V} = TN$ via $\widetilde{\psi}$. By construction, we also have a C^1 'blowing down' map

$$\Psi : \mathcal{Z} \to \mathcal{N},$$

given by the identity on $\mathcal{U}$ and by $\widetilde{\psi}$ on $\hat{\mathcal{V}}$.

If we suppose that $[\nabla]$ is C^k for $k > 2$, the above construction allows us to impose a C^{k-1} structure on $\mathcal{N}$ in such a manner that Ψ becomes a C^{k-1} map. While this will actually turn out to be technically useful, the reader should be warned, however, that such a C^{k-1} structure is in no sense *natural* or *canonical*, because it depends on the $(k-1)$-jet of our identification of the tubular neighborhood $\hat{\mathcal{V}}$ with $L \to \mathbb{P}TN$, and such a choice is uniquely specified by the geometry only when $k = 2$; for this reason, we will refer to such a choice as a *provisional* C^{k-1} structure. Fortunately, however, this apparent shortcoming will soon be remedied. Indeed, the thrust of our argument is that that $[\nabla]$ induces a certain complex structure J on $\mathcal{N}$, and so endows $\mathcal{N}$ with a canonical C^∞ structure. In order to see this, we will proceed by first constructing a certain involutive complex distribution $\mathbf{D}$ on $\mathbb{P}T_{\mathbb{C}}M$, and then analyzing its image under Ψ.

Since $\mathcal{Z} = \mathbb{P}T_{\mathbb{C}}M$, we have a bundle projection, which we will denote by $\hat{\mu} : \mathcal{Z} \to M$. The subbundle $\mathbf{V} = \ker \hat{\mu}_* \subset T\mathcal{Z}$ will be called the *vertical subbundle*. Now choose a connection ∇ representing the given projective structure $[\nabla]$, and let $\mathbf{H} \subset T\mathcal{Z}$ be the *horizontal subbundle*, corresponding to parallel transport with respect to ∇, so that we have a direct-sum decomposition

$$T\mathcal{Z} = \mathbf{V} \oplus \mathbf{H}.$$

Complexifying these bundles, we thus have

$$T_{\mathbb{C}}\mathcal{Z} = \mathbf{V}_{\mathbb{C}} \oplus \mathbf{H}_{\mathbb{C}},$$

where $T_{\mathbb{C}}\mathcal{Z} = \mathbb{C} \otimes T\mathcal{Z}$, etc. Notice that the derivative of the projection also gives us a canonical isomorphism

$$\hat{\mu}_* : \mathbf{H}_{\mathbb{C}} \xrightarrow{\cong} \hat{\mu}^* T_{\mathbb{C}} M.$$

Using this picture, we will now define two line subbundles

$$\mathbf{L}_j \subset T_{\mathbb{C}}\mathcal{Z} = \mathbb{C} \otimes T\mathcal{Z}, \quad j = 1, 2.$$

To this end, let us first recall that each fiber of $\mathcal{Z} \to M$ is a $\mathbb{CP}_1$, so that we have a fiber-wise complex structure tensor

$$J^{\|} : \mathbf{V} \to \mathbf{V}, \quad (J^{\|})^2 = -\mathbf{1},$$

and we define $\mathbf{L}_1 \subset \mathbf{V}_{\mathbb{C}}$ to be the $(-i)$-eigenspace of $J^{\|}$:

$$\mathbf{L}_1 = \mathbf{V}^{0,1}_{J^{\|}}.$$

On the other hand, each element of $\mathcal{Z} = \mathbb{P}T_{\mathbb{C}}M$ may be identified with a 1-dimensional complex-linear subspace of $T_{\mathbb{C}}M$, and this picture gives us a tautological line subbundle $\mathbf{L}_2$ of $\mathbf{H}_{\mathbb{C}} \cong \hat{\mu}^* T_{\mathbb{C}} M$:

$$\mathbf{L}_2|_{[\mathbf{w}]} = (\hat{\mu}_{*[\mathbf{w}]})^{-1}(\text{span } \mathbf{w}). \tag{6}$$

Set

$$\mathbf{D} = \mathbf{L}_1 \oplus \mathbf{L}_2 \subset T_{\mathbb{C}}\mathcal{Z}. \tag{7}$$

Then $\mathbf{D}$ is a C^2 distribution of complex 2-planes on $\mathcal{Z}$. We will now see that $\mathbf{D}$ is involutive, in the sense that

$$[C^1(\mathbf{D}), C^1(\mathbf{D})] \subset C^0(\mathbf{D}).$$

Moreover, $\mathbf{D}$ will turn out to be unchanged if we replace ∇ with a projectively equivalent connection $\hat{\nabla}$.

Indeed, let $(x^1, x^2) : \Omega \to \mathbb{R}^2$ be a local coordinate system on $\Omega \subset M$, and let

$$\Gamma^j_{k\ell} = \left\langle \; dx^j \;,\; \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^\ell} \; \right\rangle$$

be the corresponding Christoffel symbols of the connection ∇. We can then introduce local coordinates $(x^1, x^2, \zeta) : \hat{\mu}^{-1}(\Omega) \to \mathbb{R}^2 \times \mathbb{C}$ on $\hat{\mu}^{-1}(\Omega) \subset \mathcal{Z}$ by

$$\left[\left(\frac{\partial}{\partial x^1} + \zeta \frac{\partial}{\partial x^2} \right) \Big|_{(x^1,x^2)} \right] \longleftrightarrow (x^1, x^2, \zeta).$$

Then, in these coordinates, $\mathbf{L}_1$ is spanned by $\partial/\partial\bar{\zeta}$, whereas $\mathbf{L}_2$ is spanned by

$$\Xi_0 = \frac{\partial}{\partial x^1} + \zeta \frac{\partial}{\partial x^2} + Q(x, \zeta, \zeta) \frac{\partial}{\partial \zeta} + Q(x, \zeta, \overline{\zeta}) \frac{\partial}{\partial \overline{\zeta}},$$

where

$$Q(x, u, v) = -\Gamma^2_{11} - \Gamma^2_{12}(u + v) - \Gamma^2_{22} uv + \Gamma^1_{11} v + \Gamma^1_{12} v(u + v) + \Gamma^1_{22} uv^2$$

encodes the Christoffel symbols $\Gamma^j_{k\ell}$ of our chart, which are of course functions of $x = (x^1, x^2)$. In particular, $\mathbf{D}$ is spanned by $\partial/\partial\overline{\zeta}$ and

$$\Xi = \frac{\partial}{\partial x^1} + \zeta \frac{\partial}{\partial x^2} + P(x, \zeta) \frac{\partial}{\partial \xi}, \tag{8}$$

where $\zeta = \xi + i\eta$ and where

$$\begin{aligned} P(x, \zeta) &= Q(x, \zeta, \zeta) \\ &= -\Gamma^2_{11} + \left[\Gamma^1_{11} - 2\Gamma^2_{12}\right] \zeta + \left[2\Gamma^1_{12} - \Gamma^2_{22}\right] \zeta^2 + \Gamma^1_{22} \zeta^3 \end{aligned} \tag{9}$$

is evidently of the same differentiability class as ∇. But

$$\left[\frac{\partial}{\partial \overline{\zeta}} \;,\; \Xi \right] = \left[\frac{\partial}{\partial \overline{\zeta}} \;,\; \frac{\partial}{\partial x^1} + \zeta \frac{\partial}{\partial x^2} + P(x, \zeta) \frac{\partial}{\partial \xi} \right] = 0 \;,$$

because

$$\frac{\partial}{\partial \overline{\zeta}} \zeta = 0, \qquad \frac{\partial}{\partial \overline{\zeta}} P(x, \zeta) = 0.$$

It therefore follows that $\mathbf{D} = \text{span}\, \{\Xi, \partial/\partial\bar{\zeta}\}$ is involutive, as claimed.

Notice that the replacement

$$\Gamma^i_{jk} \rightsquigarrow \Gamma^i_{jk} + \delta^i_j \beta_k + \beta_j \delta^i_k$$

leaves $P(x, \zeta)$ unaltered. Thus replacing ∇ with a projectively equivalent connection $\hat{\nabla}$ leaves Ξ unchanged, and $\mathbf{D} = \mathrm{span}\{\Xi, \partial/\partial\bar{\zeta}\}$ is therefore projectively invariant.

The distribution $\mathbf{D}$ does not quite define a complex structure on $\mathcal{Z}$, because certain real tangent vectors are elements of $\mathbf{D}$. Indeed, notice that, because $\mathbf{D}$ is the direct sum of $\mathbf{L}_1 \subset \mathbf{V}_{\mathbb{C}}$ and $\mathbf{L}_2 \subset \mathbf{H}_{\mathbb{C}}$, and because the projections $T_{\mathbb{C}}M \to \mathbf{V}_{\mathbb{C}}$ and $T_{\mathbb{C}}M \to \mathbf{H}_{\mathbb{C}}$ commute with complex conjugation, any real element of $\mathbf{D}$ must have real components in $\mathbf{L}_1$ and $\mathbf{L}_2$. But since $\mathbf{L}_1$ contains no nonzero real element, we therefore have

$$\mathbf{D} \cap \overline{\mathbf{D}} = (\mathbf{L}_1 \cap \overline{\mathbf{L}}_1) + (\mathbf{L}_2 \cap \overline{\mathbf{L}}_2) = \mathbf{L}_2 \cap \overline{\mathbf{L}}_2.$$

On the other hand, Equation (6) tells us that $\mathbf{L}_2$ contains a nonzero real element precisely at the hypersurface $Z = \mathbb{P}TM$ in $\mathcal{Z} = \mathbb{P}T_{\mathbb{C}}M$:

$$\dim(\mathbf{D}_z \cap \overline{\mathbf{D}}_z) = \begin{cases} 0, & z \notin Z \\ 1, & z \in Z. \end{cases} \tag{10}$$

Indeed, $\mathbf{L}_2|_Z$ is simply the complexification $\mathbb{C} \otimes \ker \nu_*$ of the tangent space of the foliation $\mathcal{F}$ of $\mathbb{P}TM$ by lifted geodesics. This observation gives a somewhat more geometric explanation for the previously noted projective invariance of $\mathbf{D}$. Indeed, in Equation (9) we carefully chose our complex vector field Ξ so that at the locus Z, given by $\eta = 0$, Ξ is real and tangent to $\mathcal{F}$, with coefficients that are holomorphic in $\zeta = \xi + i\eta$, and so determined by the behavior of Ξ along $\eta = 0$.

Proposition 3.1. *Let $[\nabla]$ be a Zoll projective structure which is represented by a C^3 connection ∇ on $M \approx \mathbb{RP}^2$. Then there is a unique integrable almost-complex structure J on $\mathcal{N}$ such that*

$$\Psi_*[\mathbf{D}] \subset T^{0,1}(\mathcal{N}, J).$$

The unique C^∞ structure on $\mathcal{N}$ associated with its maximal atlas of J-compatible complex charts is compatible with the previously-constructed C^1 structure on $\mathcal{N}$, so that $\Psi : \mathcal{Z} \to \mathcal{N}$ remains a C^1 map relative to this smooth structure; moreover, Ψ actually becomes C^3 on the open dense set $\mathcal{Z} - Z$. Moreover, if $[\nabla]$ is represented by a $C^{k,\alpha}$ connection ∇ on M, $3 \leq k \leq \infty$, $0 < \alpha < 1$, and if $\mathcal{N}$ is again given the natural C^∞ structure associated with J, then $\Psi : \mathcal{Z} \to \mathcal{N}$ is actually a $C^{k+1,\alpha}$ map on $\mathcal{Z} - Z$.

Remark. With the same hypotheses, we will later also show (remark, page 488) that Ψ is actually $C^{k+1,\alpha}$ on *all* of $\mathcal{Z}$.

Proof. We begin by defining J point-wise. On the open set $\mathcal{N}-N = \Psi(\mathcal{Z}-Z)$, we may do this by first observing that

$$T_{\mathbb{C}}(\mathcal{N}-N) = \Psi_*\mathbf{D} \oplus \overline{\Psi_*\mathbf{D}}$$

by (10) and the fact that $\Psi|_{\mathcal{Z}-Z}$ is a diffeomorphism; on $\mathcal{N}-N$, we now set

$$J = \begin{bmatrix} -i & 0 \\ 0 & +i \end{bmatrix}$$

with respect to this direct sum decomposition. On the other hand, since $\mathcal{V} \subset \mathcal{N}$ is, by definition, a copy of the total space of $TN \to N$, we have a canonical identification

$$T\mathcal{N}|_N = TN \oplus TN,$$

where the first factor is tangent to N, and where the second factor is transverse to it; and along $N \subset \mathcal{N}$ we can therefore set

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

with respect to this second direct sum decomposition. This defines the almost-complex structure J at all points of $\mathcal{N}$.

While it is not yet even yet clear that this J is continuous, it is at least easy to see that $\Psi_*\mathbf{D} \subset T^{0,1}(\mathcal{N}, J)$. Indeed, by construction, $\Psi_*\mathbf{D} = T^{0,1}(\mathcal{N}, J)$ away from N. On the other hand, $\Psi_*\mathbf{D} = \Psi_*\mathbf{V}^{0,1}$ along Z, and since we used $J^{\|}$ to pick out the normal factor of $T\mathcal{Z}|_Z = TZ \oplus L$ before blowing down, $\Psi_* \circ J^{\|} = J \circ \Psi_*$ on $\mathbf{V}|_Z$, and it follows that $\Psi_*\mathbf{D} \subset T^{0,1}(\mathcal{N}, J)$ along Z, too. Moreover, J is certainly the *only* almost-complex structure with this property, since, for any $y \in N$,

$$T_y\mathcal{N} = \Psi_*\mathbf{V}_x \oplus \Psi_*\mathbf{V}_{x'}$$

whenever $x \neq x'$ are distinct points of the geodesic $\mathfrak{C}_y \subset M$ represented by y.

Now since $[\nabla]$ has been assumed to be C^3, we can can give $\mathcal{N}$ a 'provisional' C^2 structure, compatible with its fixed C^1 structure, relative to which Ψ becomes a C^2 map. We now claim that J is actually Lipschitz continuous in the associated charts on $\mathcal{N}$. Of course, this is is only a nontrivial statement near a point $y \in N$, since the restriction of

J to $\mathcal{N} - N$ corresponds, via Ψ, to a C^3 almost-complex structure on $\mathcal{Z} - Z$.

Now let us recall that we have written down an explicit local framing $(\Xi, \partial/\partial\overline{\zeta})$ of $\mathbf{D}$ such that $[\Xi, \partial/\partial\overline{\zeta}] = 0$, and such that Ξ is real along $Z = \mathbb{P}TM$ and spans the tangent space of the foliation $\mathcal{F}$ there. Giving an arbitrary leaf $\hat{\mathfrak{C}}_y$ a parameter t such that $\Xi = d/dt$ along the leaf, then, for any C^2 function f on $\mathcal{N}$ we have

$$\begin{aligned}
\frac{d}{dt}\left[\Psi_*\left(\frac{\partial}{\partial\overline{\zeta}}\right)f\right] &= \frac{d}{dt}\frac{\partial}{\partial\overline{\zeta}}\Psi^* f \\
&= \Xi\frac{\partial}{\partial\overline{\zeta}}\Psi^* f \\
&= \frac{\partial}{\partial\overline{\zeta}}\Xi\Psi^* f \\
&= \frac{\partial}{\partial\overline{\zeta}}\left[\Psi_*(\Xi)f\right].
\end{aligned}$$

Thus, setting $\zeta = \xi + i\eta$,

$$\frac{d}{dt}\left[\Psi_*\left(\frac{\partial}{\partial\overline{\zeta}}\right)\right] = \frac{\partial}{\partial\overline{\zeta}}\left[\Psi_*(\Xi)\right] = \frac{i}{2}\frac{\partial}{\partial\eta}\left[\Psi_*(\Xi)\right]$$

at $y \in N$, since $\Psi_*(\Xi) \equiv 0$ along Z, where $\eta = 0$. Here the right-hand side should be interpreted as the invariant derivative *at a zero* of a section of a vector bundle on $\Sigma_x := \Psi[\hat{\mu}^{-1}(x)] \cong \mathbb{CP}_1$. On the other hand,

$$\Psi_*\left(\frac{\partial}{\partial\overline{\zeta}}\right) \in T^{0,1}_y(\mathcal{N}, J)$$

for all t, by our previous discussion, so it follows that

$$\left.\frac{\partial}{\partial\eta}\left[\Psi_*(\Xi)\right]\right|_{\eta=0} \in T^{0,1}_y(\mathcal{N}, J),$$

too. Along Σ_x, we therefore have, near an arbitrary point $y \in N$, two continuous sections of $T^{0,1}$ given by $\mathbf{e}_1 = \Psi_*(\partial/\partial\overline{\zeta})$ and

$$\mathbf{e}_2 = \begin{cases} \left[\Psi_*(\Xi)\right]/\eta & \eta \neq 0 \\ \frac{\partial}{\partial\eta}\left[\Psi_*(\Xi)\right] & \eta = 0. \end{cases}$$

These sections are linearly independent at every point, and so span $T^{0,1}_y$, because $\det(\Psi_*)$ only vanishes to first order at Z. Moreover, since Ψ

appears to be C^2 in our coordinates, these sections are both continuously differentiable in our chart, with derivatives that may be expressed in any coordinate system in terms of partial derivatives of Ψ of order ≤ 2. Hence J is also differentiable, and in particular is Lipschitz, along Σ_x, with Lipschitz constant controlled by the partial derivatives of Ψ of order ≤ 2. Since the family $\{\Sigma_x\}$ sweeps out all the radial lines in our tubular neighborhood TN of $N \subset \mathcal{N}$, if follows that the tensor field J on $\mathcal{N}$ is Lipschitz.

In particular, the partial derivatives of the components of J are smooth bounded functions on the complement of a submanifold of $\mathcal{N}$, and so, by extension across a set of measure zero, can be considered as locally bounded measurable functions on $\mathcal{N}$. By an elementary integration by parts argument, these bounded measurable functions are then precisely the distributional partial derivatives of the relevant components. The Nijenhuis tensor

$$\tau(\mathbf{v}, \mathbf{w}) = [\mathbf{v}, \mathbf{w}] - [J\mathbf{v}, J\mathbf{w}] + J[\mathbf{v}, J\mathbf{w}] + J[J\mathbf{v}, \mathbf{w}]$$

of our almost-complex structure J is therefore well-defined in the distributional sense, and has L^∞_{loc} components. But this means that τ vanishes in the distributional sense, since by construction J is integrable away from a subset $N \subset \mathcal{N}$ of measure zero. However, Hill and Taylor [18] have recently shown that the Newlander-Nirenberg theorem holds for Lipschitz almost-complex structures for which $\tau = 0$ in just this distributional sense. Thus every point of $\mathcal{N}$ has a neighborhood on which we can find a pair (z^1, z^2) of differentiable complex-valued functions with $dz^k \in \Lambda^{1,0}(\mathcal{N}, J)$ and $dz^1 \wedge dz^2 \neq 0$. Taking these to be the complex coordinate systems gives $\mathcal{N}$ the structure of a compact complex surface. In particular, this gives $\mathcal{N}$ a specific real-analytic structure, and hence a specific C^∞ structure.

Finally, we address the smoothness of $\Psi : \mathcal{Z} \to \mathcal{N}$. Suppose that ∇ is of differentiability class $C^{k,\alpha}$, and suppose that f is a holomorphic function on some open subset of $\mathcal{N}$; we then consider the function $\Psi^* f$ on $\mathcal{Z} = \mathbb{P}T_{\mathbb{C}}M$. Now, by [18], f is a C^1 function with respect to our (original, unchanged) C^1 structure on $\mathcal{N}$, and $\Psi^* f$ is therefore a C^1 function, since Ψ was C^1 by construction. Moreover, since $\Psi_* \mathbf{D} \subset T^{0,1}\mathcal{N}$, $\Psi^* f = 0$ solves the Cauchy-Riemann equations $\bar{\partial}_{\mathbf{D}}(\Psi^* f) = 0$ with respect to the $C^{k,\alpha}$ almost-complex structure which $\mathbf{D}$ determines on $\mathcal{Z} - Z$. But since $\bar{\partial}_{\mathbf{D}} + \bar{\partial}^*_{\mathbf{D}}$, defined with respect to an arbitrary $C^{k,\alpha}$ Hermitian metric on $\mathcal{Z} - Z$, is a first-order elliptic system with $C^{k,\alpha}$ coefficients, elliptic regularity [24] tells us that $\Psi^* f$ is $C^{k+1,\alpha}$ on $\mathcal{Z} - Z$.

Applying these observations when f is any local complex coordinate z^j on $\mathcal{N}$ then shows that Ψ belongs to the claimed differentiability class.

q.e.d.

Remark. The above proof uses a powerful recent analytic theorem in order to obtain the result without too much hard work. Most readers will find it reassuring, however, that older technology may instead be used to prove a workable version of the proposition at the price of a half-dozen derivatives and a certain amount of careful calculation. Moreover, this approach has the added benefit of providing some immediate added information concerning the regularity of Ψ along $Z \subset \mathcal{Z}$. In particular, those primarily interested in the C^∞ case might well prefer the following elementary argument.

Suppose that ∇ is a C^k connection, where $k = 2\ell + 2$. Choose $C^{2\ell+2}$ local real coordinates $(\check{y}^1, \check{y}^2)$ on $U \subset N$, and pull them back to $Z = \mathbb{P}TM$ so as to obtain $C^{2\ell+2}$ functions $y^\jmath = \nu^* \check{y}^\jmath$ on $\nu^{-1}U \subset Z$. By construction, these solve the equation $\Xi y^\jmath = 0$. We now extend the $y^\jmath$ as $C^{\ell+2}$ complex-valued functions $\mathfrak{z}^\jmath$ defined on an open set in $\mathcal{Z}$ by requiring that $\partial y^\jmath / \partial \bar{\zeta}$ vanish to order $\ell - 1$ along Z. This completely specifies the ℓ-jet of the function, and we must have

$$\mathfrak{z}^\jmath(x^1, x^2, \xi, \eta) = \sum_{r=0}^{\ell} \frac{i^r}{r!} \eta^r \left. \frac{\partial^r y^\jmath}{\partial \xi^r} \right|_{(x^1, x^2, \xi)} + O(\eta^{\ell+1}).$$

Indeed, this recipe does indeed give us

$$\begin{aligned}
\frac{\partial \mathfrak{z}^\jmath}{\partial \bar{\zeta}} &= \frac{1}{2} \left(\frac{\partial}{\partial \xi} + i \frac{\partial}{\partial \eta} \right) \left(\sum_{r=0}^{\ell} \frac{i^r}{r!} \eta^r \frac{\partial^r y^\jmath}{\partial \xi^r} + O(\eta^{\ell+1}) \right) \\
&= \frac{1}{2} \sum_{r=0}^{\ell} \frac{i^r}{r!} \eta^r \frac{\partial^{r+1} y^\jmath}{\partial \xi^{r+1}} - \frac{1}{2} \sum_{r=1}^{\ell} \frac{i^{r-1}}{(r-1)!} \eta^{r-1} \frac{\partial^r y^\jmath}{\partial \xi^r} + O(\eta^\ell) \\
&= \frac{1}{2} \sum_{r=0}^{\ell} \frac{i^r}{r!} \eta^r \frac{\partial^{r+1} y^\jmath}{\partial \xi^{r+1}} - \frac{1}{2} \sum_{r=0}^{\ell-1} \frac{i^r}{r!} \eta^r \frac{\partial^{r+1} y^\jmath}{\partial \xi^{r+1}} + O(\eta^\ell) \\
&= \frac{1}{2} \frac{i^\ell}{\ell!} \eta^\ell \frac{\partial^{\ell+1} y^\jmath}{\partial \xi^{\ell+1}} + O(\eta^\ell) \\
&= O(\eta^\ell),
\end{aligned}$$

and since the cancellation is a term-by-term matter, uniqueness of the ℓ-jet follows. But since our condition on the ℓ-jet is obviously independent of the choice of coordinates (x^1, x^2) on M, global existence now follows by patching together any such local choices via a partition of unity.

The uniqueness argument also has another useful consequence. Notice that there certainly are C^2 coordinates $(\widetilde{\mathfrak{z}}^1, \widetilde{\mathfrak{z}}^2)$ for our provisional C^2 structure on $\mathcal{N}$ whose restrictions to N are the $\check{y}^\jmath$, and which are satisfy $\overline{\partial}_J \widetilde{\mathfrak{z}}^\jmath = 0$ to 0^{th} order along N, since the restriction of J to $T\mathcal{N}|_N$ is C^{k-1}. But pulling these back to $\mathcal{Z}$ would gives us C^2 functions killed by $\partial/\partial\overline{\zeta}$ to 0^{th} order along Z, and the $\ell = 1$ version of the above calculation therefore gives

$$\Psi^* \widetilde{\mathfrak{z}}^\jmath = \mathfrak{z}^\jmath + O(\eta^2).$$

It follows that $(\mathfrak{z}^1, \mathfrak{z}^2)$ is actually a C^1 complex-valued coordinate system on $\mathcal{N}$. Our strategy will now be to analyze the the almost-complex structure J by thinking of $(x^1, x^2, \xi, \eta) \mapsto (\mathfrak{z}^1, \mathfrak{z}^2)$ as a representation of Ψ in special coordinates

To this end, we next observe that, since $[\Xi, \frac{\partial}{\partial\overline{\zeta}}] = 0$, the $C^{\ell+1}$ function $\Xi\mathfrak{z}^\jmath$ satisfies

$$\frac{\partial^m}{\partial\overline{\zeta}^m} \Xi \mathfrak{z}^\jmath = \Xi \frac{\partial^m}{\partial\overline{\zeta}^m} \mathfrak{z}^\jmath = \Xi O(\eta^{\ell-m+1}) = O(\eta^{\ell-m+1}),$$

so that

$$\left(\frac{\partial}{\partial\xi} + i\frac{\partial}{\partial\eta}\right)^m (\Xi\mathfrak{z}^\jmath)\bigg|_{\eta=0} \equiv 0,$$

for $m = 0, \ldots, \ell$. But since $\Xi\mathfrak{z}^\jmath \equiv 0$ along $\eta = 0$, this tells us that

$$\frac{\partial^m}{\partial\eta^m}(\Xi\mathfrak{z}^\jmath)\bigg|_{\eta=0} \equiv 0$$

for $m = 0, \ldots, \ell$, and hence that

$$\Xi\mathfrak{z}^\jmath = O(\eta^{\ell+1}).$$

Now we have already shown, by an elementary argument, that the almost-complex structure J is characterized, in a point-wise manner, by the fact that $\Psi_*\partial/\partial\overline{\zeta}$ and $\Psi_*\Xi$ are always elements of $T^{0,1}(\mathcal{N}, J)$. Since $\mathrm{span}\{\partial/\partial\mathfrak{z}^1, \partial/\partial\mathfrak{z}^2\}$ contains the image of $\partial/\partial\overline{\zeta}$ and (trivially) Ξ along the locus N given by $\Im m\mathfrak{z}^\jmath = 0$, we must therefore have

$$T^{0,1}(\mathcal{N}, J)|_{\Im m\mathfrak{z}^\jmath=0} = \mathrm{span}\left\{\frac{\partial}{\partial\overline{\mathfrak{z}}^\jmath}\right\}_{\jmath=1,2},$$

and
$$T^{*1,0}(\mathcal{N},J)|_{\Im m \mathfrak{z}^\jmath=0} = \mathrm{span}\,\{d\mathfrak{z}^\jmath\}_{\jmath=1,2}\,.$$
Elsewhere,
$$T^{*1,0}(\mathcal{N},J) = \mathrm{span}\left\{d\mathfrak{z}^\jmath - \sum_\imath a^\jmath_\imath d\bar{\mathfrak{z}}^\imath\right\}_{\jmath=1,2}$$
and
$$T^{0,1}(\mathcal{N},J) = \mathrm{span}\left\{\frac{\partial}{\partial\bar{\mathfrak{z}}^\jmath} + \sum_\imath a^\imath_\jmath \frac{\partial}{\partial \mathfrak{z}^\imath}\right\}_{\jmath=1,2},$$
where the $a^\jmath_\imath$ are to be found by solving the equation
$$\begin{bmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{bmatrix}\begin{bmatrix} \Xi\bar{\mathfrak{z}}^1 & \frac{\partial\bar{\mathfrak{z}}^1}{\partial\bar\zeta} \\ \Xi\bar{\mathfrak{z}}^2 & \frac{\partial\bar{\mathfrak{z}}^2}{\partial\bar\zeta}\end{bmatrix} = \begin{bmatrix} \Xi\mathfrak{z}^1 & \frac{\partial\mathfrak{z}^1}{\partial\bar\zeta} \\ \Xi\mathfrak{z}^2 & \frac{\partial\mathfrak{z}^2}{\partial\bar\zeta}\end{bmatrix}.$$
But
$$\frac{\partial}{\partial\bar\zeta}\bar{\mathfrak{z}}^\jmath = \frac{\partial y^\jmath}{\partial\xi} + O(\eta),$$
and
$$\begin{aligned}\Xi\bar{\mathfrak{z}}^\jmath &= \Xi\left(-i\eta\frac{\partial y^\jmath}{\partial\bar\xi} + O(\eta^2)\right)\\ &= -i\eta[\Xi, \frac{\partial}{\partial\xi}]y^\jmath = i\eta\frac{\partial y^\jmath}{\partial x^2} + i\eta P'(\xi)\frac{\partial y^\jmath}{\partial\xi} + O(\eta^2),\end{aligned}$$
so that
$$\begin{aligned}\begin{vmatrix} \Xi\bar{\mathfrak{z}}^1 & \frac{\partial\bar{\mathfrak{z}}^1}{\partial\bar\zeta} \\ \Xi\bar{\mathfrak{z}}^2 & \frac{\partial\bar{\mathfrak{z}}^2}{\partial\bar\zeta}\end{vmatrix} &= i\eta\begin{vmatrix} \frac{\partial y^1}{\partial x^2} + P'(\xi)\frac{\partial y^1}{\partial\xi} & \frac{\partial y^1}{\partial\xi} \\ \frac{\partial y^2}{\partial x^2} + P'(\xi)\frac{\partial y^2}{\partial\xi} & \frac{\partial y^2}{\partial\xi}\end{vmatrix} + O(\eta^2)\\ &= i\eta\frac{\partial(y^1,y^2)}{\partial(x^2,\xi)} + O(\eta^2).\end{aligned}$$
But $\partial(y^1,y^2)/\partial(x^2,\xi) \neq 0$ everywhere, since Ξ is always linearly independent from $\partial/\partial x^2$ and $\partial/\partial\xi$. Thus
$$\begin{aligned}&\begin{bmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{bmatrix}\\ &= \begin{bmatrix} \Xi\mathfrak{z}^1 & \frac{\partial\mathfrak{z}^1}{\partial\bar\zeta} \\ \Xi\mathfrak{z}^2 & \frac{\partial\mathfrak{z}^2}{\partial\bar\zeta}\end{bmatrix}\begin{bmatrix} \Xi\bar{\mathfrak{z}}^1 & \frac{\partial\bar{\mathfrak{z}}^1}{\partial\bar\zeta} \\ \Xi\bar{\mathfrak{z}}^2 & \frac{\partial\bar{\mathfrak{z}}^2}{\partial\bar\zeta}\end{bmatrix}^{-1}\end{aligned}$$

$$= \begin{bmatrix} O(\eta^{\ell+1}) & O(\eta^{\ell}) \\ O(\eta^{\ell+1}) & O(\eta^{\ell}) \end{bmatrix} \frac{1}{i\eta}\left(\frac{\partial(x^2,\xi)}{\partial(y^1,y^2)} + O(\eta)\right) \begin{bmatrix} \frac{\partial\bar{\mathfrak{z}}^2}{\partial\bar{\zeta}} & -\frac{\partial\bar{\mathfrak{z}}^1}{\partial\bar{\zeta}} \\ -\Xi\bar{\mathfrak{z}}^2 & \Xi\bar{\mathfrak{z}}^1 \end{bmatrix}$$
$$= \left(\frac{\partial(\xi,x^2)}{\partial(y^1,y^2)} + O(\eta)\right) \begin{bmatrix} O(\eta^{\ell}) & O(\eta^{\ell-1}) \\ O(\eta^{\ell}) & O(\eta^{\ell-1}) \end{bmatrix} \begin{bmatrix} O(\eta^0) & O(\eta^0) \\ O(\eta) & O(\eta) \end{bmatrix}$$
$$= O(\eta^{\ell-1}).$$

More precisely, for (x^1, x^2, ξ, η) in any fixed compact set, there is a constant C such that
$$|a_\imath^\jmath| < C \mid \eta \mid^{\ell-1}.$$
For the corresponding set in $\mathcal{N}$, this becomes the statement that
$$|a_\imath^\jmath| < C_1 \mid \Im m\ \vec{\mathfrak{z}} \mid^{\ell-1}.$$
But since Ψ is a proper map, it only takes a finite number of closed coordinate balls to cover the inverse image of any compact set in $\mathcal{N}$, and hence we have
$$|a_\imath^\jmath| < C_2 \mid \Im m\ \vec{\mathfrak{z}} \mid^{\ell-1}$$
as long as $\vec{\mathfrak{z}} = (\mathfrak{z}^1, \mathfrak{z}^2)$ is constrained to lie in any fixed compact set.

Since $(x^1, x^2, \xi, \eta) \mapsto (\mathfrak{z}^1, \mathfrak{z}^2)$ is a $C^{\ell+1}$ diffeomorphism away from $\eta = 0$, the $a_\imath^\jmath$ are $C^{\ell+1}$ functions of the $(\mathfrak{z}^1, \mathfrak{z}^2)$ away from $\Im m\ \mathfrak{z}^1 = \Im m\ \mathfrak{z}^2 = 0$, and on the other hand we have seen that they vanish to order $\ell-2$ along this bad locus. Thus the $a_\imath^\jmath$ are $C^{\ell-2}$ functions of the $\mathfrak{z}^\jmath$, and the complex structure J on $\mathcal{N}$ is $C^{\ell-2}$ in these coordinates. If $\ell-2 \geq 1$, the Nijenhuis tensor therefore vanishes identically by continuity, since it is already known to vanish on an open dense set. If $\ell - 2 \geq 4$, or in other words if $[\nabla]$ is at least C^{14}, we may therefore apply the original Newlander-Nirenberg theorem [25] to get $C^{\ell-2}$ functions (z^1, z^2) of $(\mathfrak{z}^1, \mathfrak{z}^2)$ which are holomorphic with respect to J. The Malgrange refinement [21] of Newlander-Nirenberg may similarly be applied if $\ell - 2 \geq 2$, or in other words if $[\nabla]$ is at least C^{10}. The rest of the proof then proceeds as before. Notice, however, that this second argument also directly verifies that $\Psi : \mathcal{Z} \to \mathcal{N}$ is at least $C^{[k/2]-3}$ along $Z \subset \mathcal{Z}$.

Having constructed our compact complex surface $\mathcal{N}$, we will now try to unmask its identity. To this end, recall that we originally assembled $\mathcal{N}$ from two open sets, $\mathcal{U} = \mathcal{Z} - Z$ and $\mathcal{V} \approx T\mathbb{RP}^2$. However, $\mathcal{U}$ may be identified with the space of all almost-complex structures[2] on M,

[2]Indeed, the fact that $\mathbf{D}$ is a complex structure on $\mathcal{U}$ thus naturally arises in the context of the O'Brian-Rawnsley generalization [26] of the Atiyah-Hitchin-Singer approach [1] to twistor theory.

since an almost-complex structure is completely characterized by its $(0,1)$-tangent space, and in dimension 2 this may be taken to be any 1-dimensional subspace of $T_{\mathbb{C}}M$ which is not spanned by a real vector. Thus $\mathcal{U} \to M$ may be identified with the space of pairs $([g], \circlearrowleft)$, where g is a Riemannian metric on some tangent space T_xM, $[g]$ is its conformal class, and $\circlearrowleft$ denotes a choice of orientation of T_xM. Since the space of Riemannian metrics is a convex cone, $\mathcal{U}$ therefore canonically deform retracts to the set of point-wise orientations $\circlearrowleft$ on M, once we choose a single 'background' Riemannian metric h on M. But the 2-fold cover $\{\circlearrowleft\} \to M$ is evidently just S^2, since $M = \mathbb{RP}^2$ by assumption. This shows that $\mathcal{U}$ is homotopy equivalent to S^2.

With this observation in hand, we are now in a position to list some identifying traits of our complex surface $(\mathcal{N}, J)$.

Proposition 3.2. *Let $[\nabla]$ be a Zoll projective structure on $M = \mathbb{RP}^2$, and let $N \approx \mathbb{RP}^2$ denote the corresponding space of unoriented geodesics. Then there is a compact complex surface $\mathcal{N}$ and an embedding $N \hookrightarrow \mathcal{N}$ such that:*

- $\pi_1(\mathcal{N}) = 0$*;*
- *there is an anti-holomorphic involution $\sigma : \mathcal{N} \to \mathcal{N}$ with fixed-point set N;*
- *for all $x \in M$, there is a σ-invariant complex curve $\Sigma_x \subset \mathcal{N}$, $\Sigma_x \cong \mathbb{CP}_1$, such that*
$$\ell_x = \Sigma_x \cap N;$$
- *the Σ_x all represent the same element of $\pi_2(\mathcal{N})$; and*
- *if x and x' are distinct points of M, then Σ_x and $\Sigma_{x'}$ are transverse, and meet in exactly one point.*

Proof. By construction, $\mathcal{N} = \mathcal{U} \cup \mathcal{V}$, where $\mathcal{U} = \mathcal{Z} - Z$ and $\mathcal{V} = TN \approx T\mathbb{RP}^2$. But we have just seen that $\mathcal{U}$ deform retracts to S^2. Moreover, $\mathcal{V}$ deform retracts to $N \approx \mathbb{RP}^2$, and the inclusion map $\jmath : \mathcal{U} \cap \mathcal{V} \hookrightarrow \mathcal{V}$ is homotopic to the bundle projection $\wp : (TN - 0_N) \to N$. Because $\mathcal{U}$ is simply connected and $\mathcal{U} \cap \mathcal{V}$ is connected, the Seifert-van Kampen theorem tells us that

$$\pi_1(\mathcal{N}) = \frac{\pi_1(\mathcal{V})}{\jmath_\natural[\pi_1(\mathcal{U} \cap \mathcal{V})]} = \frac{\pi_1(N)}{\wp_\natural[\pi_1(TN - 0_N)]}.$$

But $\wp_\natural : \pi_1(TN - 0_N) \to \pi_1(N)$ is surjective, since the fibers of $\wp$ are path connected. Hence $\mathcal{N}$ is simply connected.

Complex conjugation $\mathbb{P}T_{\mathbb{C}}M \to \mathbb{P}T_{\mathbb{C}}M$ sends the distribution $\mathbf{D}$ to its conjugate $\overline{\mathbf{D}}$. The induced involution $\sigma : \mathcal{N} \to \mathcal{N}$ is therefore anti-holomorphic, and obviously has fixed point set precisely consisting of N.

For each $x \in M$, set $\Sigma_x = \Psi(\mathbb{P}T_{x\mathbb{C}}M)$. Then Σ_x is an embedded genus 0 complex curve in $\mathcal{N}$. Since the fibers of $\mathbb{P}T_{\mathbb{C}}M$ are all homotopic, so are their images in $\mathcal{N}$. Moreover, since the fibers of $\mathbb{P}T_{\mathbb{C}}M$ are all disjoint, we must have $\Sigma_x \cap \Sigma_{x'} \subset N$. But, by construction, $\Sigma_x \cap N = \ell_x$, and so

$$\Sigma_x \cap \Sigma_{x'} = (\Sigma_x \cap N) \cap (\Sigma_{x'} \cap N) = \ell_x \cap \ell_{x'},$$

and if $x \neq x'$ this consists of precisely one point y, representing the unique geodesic joining x to x'; cf. Corollary 2.18. Now Σ_x and $\Sigma_{x'}$ are both σ-invariant, so $T_y\Sigma_x \cap T_y\Sigma_{x'}$ is invariant under the complex anti-linear involution σ_* of $T_y\mathcal{N}$, which we may identify with complex conjugation on $\mathbb{C} \otimes T_yN$. But since $T_y\ell_x \cap T_y\ell_{x'} = 0$, its complexification $T_y\Sigma_x \cap T_y\Sigma_{x'}$ is also zero, and Σ_x and $\Sigma_{x'}$ therefore intersect transversely, at the unique point y, exactly as claimed. q.e.d.

We now come to the key step in our proof, which is to observe that $\mathcal{N}$ must be biholomorphic to $\mathbb{CP}_2$. It is a deep and remarkable fact [33] that, up to biholomorphism, $\mathbb{CP}_2$ is the only simply connected complex surface of Euler characteristic 3, and it might therefore be tempting to now invoke this powerful result, much as we will later do in §4 below. However, we will actually need to know a great deal about the biholomorphism $F : \mathcal{N} \to \mathbb{CP}_2$, and for this reason it is in every sense more satisfactory to instead make use of the following low-tech lemma, based on the classical ideas of Castelnuovo, Enriques and Kodaira; cf. [3, Proposition V.4.3]. As a courtesy to the reader, as well as to emphasize the elementary nature of the result, we include a short, complete proof.

Lemma 3.3. *Let $\mathcal{S}$ be a simply connected compact complex surface, equipped with a fixed homology class $\mathbf{a} \in H_2(\mathcal{S}, \mathbb{Z})$ such that $\mathbf{a} \cdot \mathbf{a} = 1$. For every $p \in \mathcal{S}$, suppose that there exists a nonsingular, embedded complex curve $\Sigma \subset \mathcal{S}$ of genus 0 passing through p, with homology class $[\Sigma] = \mathbf{a}$. Then $\mathcal{S}$ is biholomorphic to $\mathbb{CP}_2$, in such a manner that all of the given curves become projective lines.*

Proof. Since the Frölicher spectral sequence of any complex surface degenerates at the E_1 level [3, Theorem IV.2.7], we have

$$H^1(\mathcal{S},\mathbb{C}) \cong H^1(\mathcal{S},\mathcal{O}) \oplus H^0(\mathcal{S},\Omega^1),$$

so the assumption that $\pi_1(\mathcal{S}) = 0$ immediately implies that $H^1(\mathcal{S},\mathcal{O}) = 0$. But the divisor line bundle $\mathcal{O}(\Sigma)$ of any of the curves $\Sigma \subset \mathcal{S}$ fits into an exact sequence

$$0 \to \mathcal{O} \xrightarrow{f\cdot} \mathcal{O}(\Sigma) \to n_\Sigma \to 0 \tag{11}$$

of sheaves on $\mathcal{S}$, where n_Σ is the normal sheaf of Σ, extended to $\mathcal{S}$ by 0, and where $f\cdot$ denotes multiplication by a holomorphic section f of $\mathcal{O}(\Sigma)$ which vanishes only at Σ, with $df \neq 0$ along Σ. Now the normal bundle of Σ has degree $\mathbf{a}\cdot\mathbf{a} = 1$, and thus n_Σ can be identified with the unique degree-1 holomorphic line bundle $\mathcal{O}(1)$ on $\mathbb{CP}_1$. Since $H^1(\mathcal{S},\mathcal{O}) = 0$, the long exact sequence in cohomology induced by (11) therefore gives us the short exact sequence

$$0 \to \mathbb{C} \xrightarrow{f\cdot} \Gamma(\mathcal{S},\mathcal{O}(\Sigma)) \to \Gamma(\mathbb{CP}_1,\mathcal{O}(1)) \to 0. \tag{12}$$

In particular, $H^0(\mathcal{S},\mathcal{O}(\Sigma)) \cong \mathbb{C}^3$; moreover, there is a holomorphic section of $\mathcal{O}(\Sigma)$ which is nonzero at any given point of $\mathcal{S}$. The associated map

$$F : \mathcal{S} \to \mathbb{P}[H^0(\mathcal{S},\mathcal{O}(\Sigma))^*] \cong \mathbb{CP}_2,$$

is thus everywhere defined. Also notice that $F(\Sigma)$ is a projective line $\mathcal{P} \subset \mathbb{CP}_2$, and that the derivative of F is of maximal rank at any point p of Σ, since (12) allows us to produce two sections of $\mathcal{O}(\Sigma)$, f and another one, which vanish at p, but have linearly independent derivatives there.

Since $H^1(\mathcal{S},\mathcal{O}) = 0$, the exact sequence

$$\cdots \to H^1(\mathcal{S},\mathcal{O}) \to H^1(\mathcal{S},\mathcal{O}^\times) \xrightarrow{c_1} H^2(\mathcal{S},\mathbb{Z}) \to \dots$$

tells us that holomorphic line bundles on $\mathcal{S}$ are classified by their first Chern classes. But if Σ and Σ' are two complex curves in the homology class $\mathbf{a}$, their divisor line bundles $\mathcal{O}(\Sigma)$ and $\mathcal{O}(\Sigma')$ both have Chern class equal to the Poincaré dual of $\mathbf{a}$. Thus $\mathcal{O}(\Sigma) \cong \mathcal{O}(\Sigma')$, and $\Gamma(\mathcal{S},\mathcal{O}(\Sigma)) = \Gamma(\mathcal{S},\mathcal{O}(\Sigma'))$. The holomorphic map $F : \mathcal{S} \to \mathbb{CP}_2$ determined by Σ therefore also maps Σ' biholomorphically to a projective line $\mathcal{P}'$, and the derivative of F has maximal rank at every point of Σ'. Since, by hypothesis, we may find such a curve through any point, F is a local biholomorphism. But since $\mathcal{S}$ is compact, F is therefore a covering map; and since $\mathbb{CP}_2$ is simply connected, we conclude that F is a biholomorphism. q.e.d.

Theorem 3.4. *Let $(M, [\nabla])$ be a compact 2-manifold with Zoll projective structure of odd conjugacy number. Assume that ∇ is of differentiability class $C^{k,\alpha}$, for some $k \geq 3$, and some $\alpha \in (0,1)$. Then there is a $C^{k+2,\alpha}$ diffeomorphism $\Phi : M \xrightarrow{\approx} \mathbb{RP}^2$ such that $[\nabla] = [\Phi^* \triangledown]$, where $\triangledown$ is the Levi-Civita connection $\triangledown$ of the standard, constant curvature Riemannian metric h on $\mathbb{RP}^2$.*

Proof. By Proposition 3.2, the entire complex surface $\mathcal{N}$ is swept out by the genus zero curves Σ_x, $x \in M$, and the homology class $[\Sigma_x] \in H_2(\mathcal{N}, \mathbb{Z})$ is independent of x. Moreover, this homology class has self-intersection

$$[\Sigma_x] \cdot [\Sigma_x] = [\Sigma_x] \cdot [\Sigma_{x'}] = 1,$$

since Σ_x and $\Sigma_{x'}$ intersect transversely in one point whenever $x \neq x'$. Lemma 3.3 therefore tells us that there is a biholomorphism $F : \mathcal{N} \to \mathbb{CP}_2$ which sends each of the complex curves Σ_x to a corresponding projective line $\mathbb{CP}_1 \subset \mathbb{CP}_2$.

Now the anti-holomorphic involution $\sigma : \mathcal{N} \to \mathcal{N}$ induces an anti-holomorphic involution $\widetilde{\sigma} = F \circ \sigma \circ F^{-1} : \mathbb{CP}_2 \to \mathbb{CP}_2$. By taking the Jacobian determinant of this map, we then obtain an anti-holomorphic involution $\widetilde{\sigma}^* : K \to K$ of the canonical line bundle $K = \Lambda^{2,0}$ of $\mathbb{CP}_2$. But K has a unique holomorphic cube-root $K^{1/3}$, the frame bundle of which is the universal cover of the frame bundle of K; and covering space theory now tells us that $\widetilde{\sigma}^*$ has three possible anti-holomorphic lifts $\varrho : K^{1/3} \to K^{1/3}$, differing by multiplicative factors of a cube-root of unity. Choose any such lift, and observe that ϱ^2 is the identity on any fiber over the fixed-point locus $F(N)$ of $\widetilde{\sigma}$; since $F(N)$ is totally real and of maximal dimension, the principle of analytic continuation therefore implies that the holomorphic map ϱ^2 must therefore be the identity. The anti-linear map

$$\varrho^* : \Gamma(\mathbb{CP}_2, \mathcal{O}(K^{-1/3})) \to \Gamma(\mathbb{CP}_2, \mathcal{O}(K^{-1/3}))$$

therefore satisfies $(\varrho^*)^2 = \mathbf{1}$. It is therefore diagonalizable over $\mathbb{R}$, with eigenvalues ± 1, and, because it is anti-linear, it can be put in the form

$$(z_1, z_2, z_3) \mapsto (\bar{z}_1, \bar{z}_2, \bar{z}_3)$$

by choosing a suitable basis for $\Gamma(\mathbb{CP}_2, \mathcal{O}(K^{-1/3})) \cong \Gamma(\mathbb{CP}_2, \mathcal{O}(1)) \cong \mathbb{C}^3$. But $[z_1 : z_2 : z_3]$ gives us a set of homogeneous coordinates on $\mathbb{CP}_2$, so we have succeeded in identifying $\sigma : \mathcal{N} \to \mathcal{N}$ with the standard complex conjugation on $\mathbb{CP}_2$. In the process, we have thereby identified N with $\mathbb{RP}^2 \subset \mathbb{CP}_2$, and each complex curves Σ_x with a complex projective line $\mathbb{CP}_1$ which is invariant under complex conjugation.

Now let $\mathbb{CP}_2^* = \mathbb{P}(\mathbb{C}^{3*})$ denote the dual projective plane of $\mathbb{CP}_2 = \mathbb{P}(\mathbb{C}^3)$, and consider the map

$$\begin{array}{rcl} \Phi_0 : M & \to & \mathbb{CP}_2^* \\ x & \mapsto & F(\Sigma_x)^\perp, \end{array}$$

where $\perp$ denotes the usual correspondence between lines in $\mathbb{CP}_2$ and points in $\mathbb{CP}_2^*$. We claim that Φ_0 is of differentiability class $C^{k+2,\alpha}$. Indeed, let $\mathcal{C} \subset \mathcal{U}$ be a (noncompact) holomorphic curve which is transverse to the fibers of $\hat{\mu}$, obtained by setting some local complex coordinate $\mathfrak{z}^1$ equal to zero. Since the almost complex structure on $\mathcal{U}$ is of class $C^{k,\alpha}$, elliptic regularity tells us that the local complex coordinates $(\mathfrak{z}^1, \mathfrak{z}^2)$ are of class $C^{k+1,\alpha}$, and $\mathcal{C}$ is therefore representable as the image of a $C^{k+1,\alpha}$ map from an open set in $\mathbb{C}$ to $\mathcal{U}$. But the projection from $\mathcal{C}$ to M is a local diffeomorphism, and so $\mathcal{C}$ may locally be thought of as the graph of a $C^{k+1,\alpha}$ local section ς of $\mathcal{U} \to M$. But such a section is precisely a local almost-complex structure on M of differentiability class $C^{k+1,\alpha}$. Since the map $F \circ \Psi \circ \varsigma$ is holomorphic with respect to this $C^{k+1,\alpha}$ almost-complex structure, it is therefore of class $C^{k+2,\alpha}$ by elliptic regularity. But on the domain of this function, $F(\Sigma_x)^\perp$ is the unique line joining $F(\Psi(\varsigma(x))$ to its complex conjugate, and so can be expressed in homogeneous coordinates as

$$\Phi_0(x) = F(\Psi(\varsigma(x)) \times \overline{F(\Psi(\varsigma(x))},$$

where $\times : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^{3*}$ is the vector cross-product. Since M is covered by the domains of such local almost-complex structures ς, this shows that Φ_0 is $C^{k+2,\alpha}$ on all of M.

Now notice that Φ_0 is also an immersion, because Ψ is a diffeomorphism on $\mathcal{U}$, and the section of the normal bundle of $\Sigma_x \subset \mathcal{N}$ corresponding to a nonzero element of T_xM is therefore never identically zero. Moreover, because each $F(\Sigma_x)$ is invariant under complex conjugation, $\Phi_0(M)$ actually lies in the real dual projective plane $\mathbb{RP}^{2*} \subset \mathbb{CP}_2^*$. Thus, Φ_0 actually gives us a $C^{k+2,\alpha}$ immersion

$$\Phi : M \to \mathbb{RP}^{2*}$$

which can be described as

$$x \mapsto F(\ell_x)^\perp.$$

But since M is a compact 2-manifold, this immersion must be a covering map, and since $\pi_1(M) \cong \pi_1(\mathbb{RP}^{2*}) = \mathbb{Z}_2$, it follows that Φ is a diffeomorphism. Moreover,Φ sends the geodesic $\mathfrak{C}_y$ to the set of projective lines through the point $F(y) \in \mathbb{RP}^2$, or in other words to the projective line $F(y)^\perp$ in $\mathbb{RP}^{2*}$. This shows that $\Phi_*\nabla$ has the same geodesics as the Levi-Civita connection $\triangledown$ of the standard metric h on $\mathbb{RP}^{2*}$, so that $\Phi^*\triangledown$ is projectively equivalent to ∇. Identifying $\mathbb{RP}^2$ with $\mathbb{RP}^{2*}$ via any isometry now proves the claim. q.e.d.

Remark. Much the same trick used to check the regularity of Φ_0 also allows one to show that $\Psi : \mathcal{Z} \to \mathcal{N}$ is actually $C^{k+1,\alpha}$ along Z. Indeed, let ς_0, ς_1 and ς_∞ be three smooth sections of $\mathcal{U} \to M$ over a coordinate domain $U \subset M$ whose values are all distinct at each point. In terms of our local coordinates (x^1, x^2, ζ), these correspond to three complex-valued functions $\zeta_\ell(x) = \zeta(\varsigma_\ell)$, $\ell = 0, 1, \infty$, whose values are all distinct, and never real. Set

$$\widetilde{\zeta}(x,\zeta) = \frac{[\zeta - \zeta_0(x)][\zeta_\infty(x) - \zeta_1(x)]}{[\zeta_1(x) - \zeta_0(x)][\zeta_\infty(x) - \zeta]},$$

so that $\widetilde{\zeta}(x, \zeta_\ell(x)) = \ell$ for each $x = (x^1, x^2)$ and $\ell = 0, 1, \infty$. Choose an inhomogeneous coordinate system on $\mathbb{CP}_2$ such that $z^1(F(\Psi(\varsigma_\ell(0, 0))))$, $\ell = 0, 1, \infty$, are all finite and distinct, and, for x in a neighborhood of 0, set

$$(z^1_\ell(x), z^2_\ell(x)) = F \circ \Psi(\varsigma_\ell(x^1, x^2)), \quad \ell = 0, 1, \infty.$$

Then, in these coordinates, $F \circ \Psi$ must explicitly be given by

$$(x,\zeta) \mapsto \left(\frac{\lambda z^1_0(x) + \widetilde{\zeta}(x,\zeta) z^1_\infty(x)}{\lambda(x) + \widetilde{\zeta}(x,\zeta)}, \frac{\lambda z^2_0(x) + \widetilde{\zeta}(x,\zeta) z^2_\infty(x)}{\lambda(x) + \widetilde{\zeta}(x,\zeta)} \right),$$

where

$$\lambda(x^1, x^2) = \frac{z^1_\infty(x) - z^1_1(x)}{z^1_1(x) - z^1_0(x)},$$

since each $\mathbb{CP}_1$ fiber of $\mathcal{Z} \to M$ is sent to holomorphically to a projective line in $\mathbb{CP}_2$ by $F \circ \Psi$. If ∇ is $C^{k,\alpha}$, this shows, albeit quite indirectly, that Ψ is $C^{k+1,\alpha}$ on all of $\mathcal{Z}$, and not just on $\mathcal{U} = \mathcal{Z} - Z$. Needless to say, however, a direct analytic proof of this fact, perhaps along the lines of [7], would be highly desirable.

If we start with a Zoll metric g on $M = \mathbb{RP}^2$, rather than just a Zoll projective structure, the complex surface $\mathcal{N}$ comes equipped with a certain additional complex curve $\mathcal{Q} \subset \mathcal{N}$. Indeed, let us consider the locus

$$\mathcal{C} = \{[v] \in \mathbb{P}T_{\mathbb{C}}M \mid g(v,v) = 0\},$$

where g has been extended from TM to $T_{\mathbb{C}}M$ as a *complex bilinear* form, and set

$$\mathcal{Q} = \Psi[\mathcal{C}].$$

In any inhomogeneous coordinate ζ on the fiber $\mathbb{P}T_{x\mathbb{C}}M$, $g(v,v)$ becomes a quadratic polynomial of degree 2, and the corresponding locus in $\mathbb{P}T_{x\mathbb{C}}M$ thus consists of two points, perhaps counted with multiplicity. However, since g is real, $\mathcal{C}$ is invariant under complex conjugation, so a root of multiplicity two would have to lie in the real slice $\mathbb{P}T_xM$; but the latter is impossible, since g is a positive-definite inner product on T_xM. Thus $\mathcal{C}$ intersects each fiber of $\mathbb{P}T_{\mathbb{C}}M$ in precisely two points, neither of which is in $\mathbb{P}TM$. Indeed, if we choose to think of $\mathcal{U} = \mathcal{Z} - Z$ as the bundle of all point-wise almost-complex structures on M, $\mathcal{C}$ is consists precisely of those almost-complex structures which are orthogonal transformations of T_xM with respect to g; and there are exactly two of these for each x, corresponding to the two possible orientations of T_xM.

Now $\mathcal{C}$ is horizontal with respect to the Levi-Civita connection ∇, since parallel transport preserves g. This not only implies that $\mathcal{C}$ meets each fiber of $\mathbb{P}T_{\mathbb{C}}M$ transversely, but also, more importantly, that there is a nonzero element Ξ_0 of $\mathbf{D}$ which is tangent to $\mathcal{C}$ at each point. Thus $\mathcal{C}$ is a complex curve in $\mathbb{P}T_{\mathbb{C}}M - \mathbb{P}TM$, and its diffeomorphic image $\mathcal{Q} = \Psi[\mathcal{C}]$ is a complex submanifold of $\mathcal{N}$. Since $\mathcal{C}$ is invariant under complex conjugation, the corresponding curve $\mathcal{Q} \subset \mathcal{N}$ is therefore invariant under the action of $\sigma : \mathcal{N} \to \mathcal{N}$. Moreover, since $\mathcal{C}$ meets each fiber of $\mathbb{P}T_{\mathbb{C}}M$ transversely, in two points $\notin \mathbb{P}TM$, it follows that $\mathcal{Q}$ meets Σ_x transversely in two points, for any $x \in M$.

Also notice that the bundle projection $\hat{\mu} : \mathbb{P}T_{\mathbb{C}}M \to M$ induces a 2-to-1 covering map $\varpi : \mathcal{C} \to M \approx \mathbb{RP}^2$, so $\mathcal{C}$ is therefore compact — and indeed, must be diffeomorphic to S^2. Moreover, this covering map ϖ is a *conformal map* from the Riemann surface $\mathcal{C}$ to the Riemannian manifold (M,g), since

$$\varpi_*[T^{0,1}_{[v]}\mathcal{C}] = \mathrm{span}(v) \subset T_{\mathbb{C}}M,$$

and $g(v,v) = 0$. With this observation in hand, we may now prove the following:

Theorem 3.5. *Let (M, g) be a $C^{k,\alpha}$ Riemannian 2-manifold whose geodesics are all embedded circles of length π, where $k \geq 4$ and $0 < \alpha < 1$. If M is not simply connected, there is a $C^{k+1,\alpha}$ diffeomorphism $\Phi : M \xrightarrow{\approx} \mathbb{RP}^2$ such that $g = \Phi^* h$, where h is the standard curvature 1 Riemannian metric on $\mathbb{RP}^2$.*

Proof. With these hypotheses, the Hopf-Rinow theorem tells us that M is necessarily compact, since, for any $x \in M$, the closed disk of radius $\pi/2$ in T_xM will surject onto M under the exponential map. Proposition 2.21, therefore tells us that $[\nabla]$ is a Zoll projective structure on the compact surface M. Now assume henceforth that M is not simply connected, We then know that $M \approx \mathbb{RP}^2$ by Proposition 2.9, and that $[\nabla]$ has conjugacy number 1 by Theorem 2.17.

Since the Christoffel symbols of the Levi-Civita connection ∇ of g are expressed in terms of the first derivatives of g, Theorem 3.4 applies provided we assume that g is of class $C^{k,\alpha}$, $4 \leq k$, $0 < \alpha < 1$. Moreover, the proof of Theorem 3.4 tells us that that there is a biholomorphism $F : \mathcal{N} \to \mathbb{CP}_2$ such that the $F(\Sigma_x)$ is a projective lines $\mathbb{CP}_1 \subset \mathbb{CP}_2$ for each $x \in M$, and such that $F \circ \sigma \circ F^{-1}$ is the complex conjugation map

$$[z^1 : z^2 : z^3] \mapsto [\bar{z}^1 : \bar{z}^2 : \bar{z}^3].$$

Thus, $F(\mathcal{Q})$ is a nonsingular compact complex curve in $\mathbb{CP}_2$ which is invariant under complex conjugation, and which meets certain projective lines transversely, in two points. Hence $F(\mathcal{Q})$ is a nonsingular conic, and so is the zero locus of a quadratic polynomial

$$0 = q(z) = \sum_{j,k=1}^{3} q_{jk} z^j z^k.$$

But since $F(\mathcal{Q})$ is invariant under complex conjugation, it is also the zero locus of $\overline{q(\bar{z})}$, so that both

$$\sum_{j,k=1}^{3} (\Re e\ q_{jk}) z^j z^k \quad \text{and} \quad \sum_{j,k=1}^{3} (\Im m\ q_{jk}) z^j z^k$$

vanish along $F(\mathcal{Q})$; and at least one of these quadratic forms is nontrivial, since $q \not\equiv 0$. Thus $F(\mathcal{Q})$ is the zero locus of a real quadratic form, represented by a real symmetric 3×3 matrix $A = [a_{jk}]$. But any such A is similar, over $GL(3, \mathbb{R})$, to a diagonal matrix with entries $\in \{1, 0, -1\}$. On the other hand, since $F(\mathcal{Q}) \cap \mathbb{RP}^2 = \emptyset$, the quadratic form represented by A must be definite. Thus, by a suitable real change of coordinates, we may arrange for our map $F : \mathcal{N} \to \mathbb{CP}_2$ to send $\mathcal{Q}$ to the standard conic $\mathcal{Q}_0$ given by

$$(z^1)^2 + (z^2)^2 + (z^3)^2 = 0$$

without sacrificing any of the previously-used properties of F.

On the other hand, we can repeat the entire construction for the standard metric h on $\mathbb{RP}^2$. The map $\Phi : M \to \mathbb{RP}^2$ constructed in Theorem 3.4 is then characterized by

$$\Phi(x) = \widetilde{x} \Longleftrightarrow F(\Sigma_x) = \widetilde{F}(\widetilde{\Sigma}_{\widetilde{x}})$$

where untilded letters pertain to (M, g) and tilded ones pertain to $(\mathbb{RP}^2, h)$. But since we have arranged for both $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ to map biholomorphically to $\mathcal{Q}_0 \subset \mathbb{CP}_2$, it follows that

$$\begin{aligned} F\left[\Psi[\varpi^{-1}(x)]\right] &= F(\Sigma_x) \cap \mathcal{Q}_0 \\ \widetilde{F}\left[\widetilde{\Psi}[\widetilde{\varpi}^{-1}(\widetilde{x})]\right] &= \widetilde{F}(\widetilde{\Sigma}_{\widetilde{x}}) \cap \mathcal{Q}_0. \end{aligned}$$

The holomorphic map

$$\hat{\Phi} = \left((\widetilde{F} \circ \widetilde{\Psi})\Big|_{\widetilde{\mathcal{C}}} \right)^{-1} \circ (F \circ \Psi) : \mathcal{C} \to \widetilde{\mathcal{C}}$$

therefore makes the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\hat{\Phi}"] \arrow[d, "\varpi"'] & \widetilde{\mathcal{C}} \arrow[d, "\widetilde{\varpi}"] \\ M \arrow[r, "\Phi"] & \mathbb{RP}^2 \end{tikzcd}$$

commute, and, since ϖ and $\widetilde{\varpi}$ are both conformal maps, it follows that Φ is also conformal. In other words, $\Phi^* g = e^{2u} h$ for some smooth function $u : M \to \mathbb{R}$. The Levi-Civita connection $\widetilde{\nabla}$ of $\Phi^* h$ is thus related to the Levi-Civita connection ∇ of g by

$$\widetilde{\nabla}_{\mathbf{v}} \mathbf{w} - \nabla_{\mathbf{v}} \mathbf{w} = du(\mathbf{v})\mathbf{w} + du(\mathbf{w})\mathbf{v} + g(\mathbf{v}, \mathbf{w}) \ \mathrm{grad}_g u.$$

However, the proof of Theorem 3.4 tells us that $\widetilde{\nabla}$and ∇ are also projectively equivalent; that is,

$$\widetilde{\nabla}_{\mathbf{v}} \mathbf{w} - \nabla_{\mathbf{v}} \mathbf{w} = \beta(\mathbf{v})\mathbf{w} + \beta(\mathbf{w})\mathbf{v}$$

for some 1-form β. Thus

$$\beta(\mathbf{v})\mathbf{w} + \beta(\mathbf{w})\mathbf{v} = du(\mathbf{v})\mathbf{w} + du(\mathbf{w})\mathbf{v} + g(\mathbf{v}, \mathbf{w}) \ \mathrm{grad}_g u$$

for all vectors $\mathbf{v}$ and $\mathbf{w}$. But if, for example, we take $\mathbf{v}$ and $\mathbf{w}$ to be orthonormal, with $\beta(\mathbf{w}) = 0$, we then have $\beta(\mathbf{v})\ \mathbf{w} = du(\mathbf{v})\ \mathbf{w} + du(\mathbf{w})\ \mathbf{v}$, so that $du(\mathbf{v}) = \beta(\mathbf{v})$ and $du(\mathbf{w}) = 0 = \beta(\mathbf{w})$; thus du and β must have the same components in the basis $(\mathbf{v}, \mathbf{w})$, and hence $\beta = du$. But if instead we take $\mathbf{w} = \mathbf{v} \neq 0$, we instead obtain

$$2\ du(\mathbf{v})\ \mathbf{v} + |\mathbf{v}|^2\ \mathrm{grad}_g u = 2\ \beta(\mathbf{v})\ \mathbf{v},$$

and the substitution $\beta = du$ then tells us that $\mathrm{grad}_g u = 0$. Hence u is constant. But, by hypothesis, g is normalized so that its geodesic circles all have the same length as those of h. The constant e^{2u} must therefore equal 1, and Φ is therefore an isometry between (M, g) and $(\mathbb{RP}^2, h)$. q.e.d.

This is essentially equivalent [4] to the classical Blaschke conjecture first proved by Leon Green [13] in the early 1960s.

Corollary 3.6 (Blaschke Conjecture)**.** *Let (M, g) be a compact $C^{k,\alpha}$ Riemannian 2-manifold for which the cut locus of each point $x \in M$ is a one-point set $\{x'\} \subset M$. If $k \geq 4$ and $0 < \alpha < 1$, there is a $C^{k+1,\alpha}$ diffeomorphism $\Phi : M \xrightarrow{\approx} S^2$ such that $g = c\Phi^* h$, where h is the standard curvature 1 Riemannian metric on S^2, and c is some positive constant.*

Proof. On a compact Riemannian manifold, any minimizing geodesic segment necessarily has finite length, so every arc-length-parameterized geodesic emanating from x must arrive at the cut locus $\{x'\}$, and must first do so precisely at time $\mathrm{dist}(x, x')$. But since x' represents the first conjugate point on each geodesic leaving x, we see, by following these geodesics backwards, that x is an element of the cut locus of x', and our hypothesis therefore implies that the cut locus of x' is *exactly* $\{x\}$. Thus $x \mapsto x'$ is an involution $\imath : M \to M$. Moreover, every geodesic of M is a simple closed curve, and $\imath$ maps every such geodesic circle to itself, by a rotation of 180°. In particular, $\imath$ is an isometry, and is therefore smooth. Moreover, $\mathrm{dist}(x, \imath(x))$ is independent of x along any particular geodesic, and thus is constant on M. Thus the geodesics of the the quotient Riemannian metric on $M/\langle\imath\rangle$ are all simple closed curves of equal length. After a suitable rescaling, Theorem 3.5 therefore tells us that the nonsimply-connected Zoll manifold $M/\langle\imath\rangle$ becomes isometric to the standard $\mathbb{RP}^2$, and hence that M becomes isometric to the standard S^2. q.e.d.

Remark. It is interesting to examine the minimal level of differentiability needed for our proof of Theorem 3.5. If we just assume that g is of class C^4, then the proof goes through, although the constructed map Φ is *a priori* also only C^4. Nonetheless, Φ^*h is still C^3, and its Gauss curvature is therefore the pull-back of the Gauss curvature of h. This shows any C^4 Zoll metric g on $\mathbb{RP}^2$ must have constant curvature. However, Green's proof [13] actually draws the same conclusion even if h is merely assumed to be C^3. It would thus be extremely gratifying if there were some way of improving the present arguments so as to make them work, e.g., when $[\nabla]$ is merely assumed to be of class C^2!

4. Zoll structures on the 2-sphere

In light of our success in understanding Zoll structures of odd conjugacy number, it now seems reasonable to ask what our techniques can tell us about the even case. Let us therefore suppose that we are given a C^3 Zoll projective structure $[\nabla]$ of even conjugacy number on a compact 2-manifold M. By Corollary 2.20, M is then diffeomorphic to S^2. Let us fix some orientation of M, and observe that

$$\mathcal{U} = \mathcal{Z} - Z = \mathbb{P}T_{\mathbb{C}}M - \mathbb{P}TM$$

can once again be identified with the space of all point-wise almost-complex structures on M. Thus

$$\mathcal{U} = \mathcal{U}_+ \cup \mathcal{U}_-,$$

where $\mathcal{U}_+$ (respectively, $\mathcal{U}_-$) consists of those almost-complex structures which are compatible (respectively, incompatible) with the given orientation of M. These are both connected sets; indeed, either can be identified with the space of all point-wise conformal structures on M. Let us now consider the compact 4-manifold-with-boundary

$$\mathcal{Z}_+ := \mathcal{U}_+ \cup Z,$$

with $\partial \mathcal{Z}_+ = Z$. We can identify $\mathcal{Z}_+$ with the nonzero, semi-positive elements of $\odot^2 T^*M$, modulo rescaling. Relative to some chosen 'background' metric h on $M \approx S^2$, we can then identify $\mathcal{Z}_+ \to M$ as the unit disk bundle in the traceless, symmetric bilinear forms $\odot_0^2 T^*M$. From a topological view-point, this allows us to think of $\mathcal{Z}_+$ as the unique oriented 2-disk bundle of Euler class 4 over S^2.

Let us now give the normal bundle $J^{\|}\ker\mu_*$ of $Z = \partial\mathcal{Z}_+$ the 'inward pointing' orientation, and then give $\ker\mu_*$ the corresponding orientation. Having made such a choice, Theorem 2.15 then tells us that $\nu : Z \to N$ can be canonically identified with the circle bundle $\mathbb{S}TN \to N$, in such a way that $J^{\|}\ker\mu_*$ is canonically identified with the pullback of the (trivial) tautological line bundle over $\mathbb{S}TN$, meaning the subbundle $L \subset \hat{\pi}^*TN$, where $\hat{\pi} : \mathbb{S}TN \to N$ is the canonical projection, whose fiber at $[\mathbf{v}] \in \mathbb{P}TN$ is $\mathrm{span}(\mathbf{v})$. Now, with respect to the canonical 'outward pointing' orientation of $L \to \mathbb{S}TN$, let L^+ be the $[0,\infty)$-bundle consisting of vectors which are not inward pointing. By the tubular neighborhood theorem, $Z = \partial\mathcal{Z}_+$ has a neighborhood $\hat{\mathcal{V}}$ in $\mathcal{Z}_+$ which can be identified with L^+ via a C^1 diffeomorphism whose derivative along the zero section of L is given by our previous identification of $J^{\|}\ker\mu_*$ and L. But we have an obvious C^1 'blowing down' map $\psi : L^+ \to TN$, and, letting $\mathcal{V}$ denote the total space of TN, this now corresponds to a C^1 map $\widetilde{\psi} : \hat{\mathcal{V}} \to \mathcal{V}$ which is a diffeomorphism on the complement of Z. We may now define a differentiable 4-manifold

$$\mathcal{N} = \mathcal{U}_+ \cup_{\widetilde{\psi}} \mathcal{V}$$

by gluing together $\mathcal{U}_+$ and $\mathcal{V} = TN$ via $\widetilde{\psi}$. By construction, we have a surjective C^1 'blowing down' map

$$\Psi : \mathcal{Z}_+ \to \mathcal{N},$$

given by the identity on $\mathcal{U}_+$ and by $\widetilde{\psi}$ on $\hat{\mathcal{V}}$. In particular we know that $\mathcal{N}$ is compact. Moreover, if $[\nabla]$ is C^k, we can once again impose a 'provisional' C^{k-1} structure on $\mathcal{N}$ so that Ψ will become a C^{k-1} map.

Now $\mathcal{Z}$ still carries an involutive complex distribution $\mathbf{D}$, and the proof of Proposition 3.1, supplemented by the remark on pp. 479-482, then proves the following:

Proposition 4.1. *Let $[\nabla]$ be a Zoll projective structure which is represented by a C^3 connection ∇ on $M \approx S^2$. Then there is a unique integrable almost-complex structure J on $\mathcal{N}$ such that*

$$\Psi_*[\mathbf{D}] \subset T^{0,1}(\mathcal{N}, J).$$

The unique C^∞ structure on $\mathcal{N}$ associated with its maximal atlas of J-compatible complex charts is compatible with the previously-constructed

C^1 structure on $\mathcal{N}$, so that $\Psi : \mathcal{Z} \to \mathcal{N}$ remains a C^1 map relative to this smooth structure. Moreover, if ∇ is of class C^{2k+6}, then Ψ is C^k.

In order to unmask the identity of the complex surface $(\mathcal{N}, J)$, we will now call in the heavy artillery, in the form of the following fundamental result, due to Yau [33]. We include the synopsis of a complete proof, both as a courtesy to the reader, and for our own enjoyment.

Lemma 4.2 (Yau). *Let $\mathcal{S}$ be a simply connected compact complex surface with $b_2(\mathcal{S}) = 1$. Then $\mathcal{S}$ is biholomorphic to $\mathbb{CP}_2$.*

Proof. Any compact, oriented, simply connected 4-manifold $\mathcal{S}$ has Euler characteristic $\chi(\mathcal{S}) = 2 + b_2(\mathcal{S})$, so that $\chi(\mathcal{S}) = 3$ if $b_2(\mathcal{S}) = 1$. On the other hand, if $b_2(\mathcal{S}) = 1$, the signature $\tau(\mathcal{S})$ is evidently ± 1, where the $\pm$ sign indicates whether the intersection form of $\mathcal{S}$ is positive or negative definite. But our $\mathcal{S}$ is assumed to admit a complex structure, so its first Chern class has self-intersection

$$c_1^2(\mathcal{S}) = 2\chi(\mathcal{S}) + 3\tau(\mathcal{S}) = 6 \pm 3 > 0,$$

and the intersection form $H^2(\mathcal{S}, \mathbb{Z}) \times H^2(\mathcal{S}, \mathbb{Z}) \to \mathbb{Z}$ therefore cannot be negative definite. Thus $\tau(\mathcal{S}) = 1$, and $c_1^2(\mathcal{S}) = 6 + 3 = 9$. Since this same calculation also shows that there is a holomorphic line bundle of positive self-intersection, Grauert's criterion implies [3] that $\mathcal{S}$ is projective algebraic. But since $H^2(\mathcal{S}, \mathbb{Z}) \subset H^2(\mathcal{S}, \mathbb{R}) \cong \mathbb{R}$, and $c_1(\mathcal{S}) \neq 0$, this can only happen if $c_1(\mathcal{S}) = \pm[\omega]$ for some Kähler form ω.

Now if we had $c_1(\mathcal{S}) = -[\omega]$, the Aubin/Yau theorem [2, 33] would tell us that $\mathcal{S}$ admitted a Kähler-Einstein metric of negative Ricci curvature. However, one has the Gauss-Bonnet-like formula

$$\chi - 3\tau = \frac{1}{8\pi^2} \int_{\mathcal{S}} \left[3|W_-|^2 - \frac{|\mathring{r}|^2}{2} \right] d\mu$$

for any Kähler metric on any compact complex surface, where $\mathring{r}$ is the trace-free Ricci-curvature, and where the anti-self-dual Weyl curvature W_- is the only piece of the curvature tensor not determined by the Ricci tensor. For our manifold, $\chi = 3\tau$, whereas $\mathring{r}$ vanishes for any Einstein metric, so we would conclude that $W_- \equiv 0$. Our Kähler-Einstein manifold would therefore necessarily have negative sectional curvature, and so would have contractible universal cover. But $\mathcal{S}$ has been assumed to be compact and simply connected, so this is a contradiction.

We must therefore have $c_1(\mathcal{S}) = [\omega]$ for some Kähler metric. Set $L = K^{-1/3}$, where $K = \Lambda^{2,0}$ is once again the canonical bundle, so that L is the unique positive line bundle on $\mathcal{S}$ with $c_1^2(L) = 1$. By the Kodaira vanishing theorem, $H^p(\mathcal{S}, \mathcal{O}(L)) = 0$ for $p > 0$, and the Hirzebruch-Riemann-Roch theorem therefore tells us that

$$h^0(\mathcal{S}, \mathcal{O}(L)) = \left\langle \left(1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12}\right) \exp\left(\frac{c_1}{3}\right), [\mathcal{S}] \right\rangle = \frac{11}{36}c_1^2 + \frac{1}{12}c_2 = 3.$$

Moreover, if $\Sigma \subset \mathcal{S}$ is the curve cut out by the vanishing of any nontrivial holomorphic section of L, then, because L is positive on every curve and satisfies $L \cdot L = 1$, Σ can have only one irreducible component, and the zero of the section can only have multiplicity 1 at a generic point of Σ. If $\hat{\Sigma}$ is the normalization of Σ, the pull-backs of these sections therefore give us a 2-dimensional space of sections of the degree-1 line bundle $L|_{\hat{\Sigma}}$. But since $\hat{\Sigma}$ is connected, Abel's theorem tells us that this gives us a biholomorphism $\hat{\Sigma} \to \mathbb{CP}_1$; and since this map is induced by pull-backs of sections from $\mathcal{S}$, $\hat{\Sigma} \to \mathcal{S}$ is an embedding, so that $\Sigma = \hat{\Sigma}$ is a nonsingular embedded curve. Moreover, there is no point of Σ at which every section of L vanishes. This shows that the linear system $|L|$ has empty base locus, and the sections of L therefore give us a well-defined holomorphic map

$$F : \mathcal{S} \to \mathbb{P}[H^0(\mathcal{S}, \mathcal{O}(L))^*] \cong \mathbb{CP}_2.$$

But since the inverse image of any $\mathbb{CP}_1 \subset \mathbb{CP}_2$ is a smooth complex curve Σ which is carried biholomorphically onto its image, this map is a degree-1 holomorphic submersion, and is therefore a biholomorphism. q.e.d.

Let us next recall that a differentiable n-dimensional submanifold X of a complex n-manifold (Y^{2n}, J) is said to be *totally real* if $T_pX \cap J(T_pX) = 0$ at each $p \in X$. When $n = 2$, which is the case of interest to us here, this is equivalent to the statement that T_pX is never a 1-dimensional complex subspace of $(T_pY, J) \cong \mathbb{C}^2$. This is of course an open condition on T_pX; indeed, for $n = 2$, this claim essentially follows from the observation that $Gr_1(\mathbb{C}^2) = \mathbb{CP}_1$ is a closed submanifold of $Gr_2(\mathbb{R}^4) \cong (S^2 \times S^2)/\mathbb{Z}_2$. Consequently, every submanifold $X' \subset Y$ which C^1-close to a given totally real submanifold $X \subset Y$ will also be totally real.

It will also be convenient to introduce some terminology specifically tailored to the discussion of differentiable embeddings of $\mathbb{RP}^2$ into $\mathbb{CP}_2$.

Definition 4.3. A differentiable embedding $\jmath : \mathbb{RP}^2 \hookrightarrow \mathbb{CP}_2$ will be said to be *weakly unknotted* if there exists a diffeomorphism $\phi : \mathbb{CP}_2 \to$

$\mathbb{CP}_2$ such that $\jmath = \phi \circ j$, where $j : \mathbb{RP}^2 \hookrightarrow \mathbb{CP}_2$ is the standard embedding $[x : y : z] \mapsto [x : y : z]$.

Remark. By composing with complex conjugation $\mathbb{CP}_2 \to \mathbb{CP}_2$ if necessary, we may always arrange for ϕ to induce the identity on homology. But since two self-homeomorphisms of a simply connected compact 4-manifold are C^0-isotopic iff they induce the same maps on homology [11], our diffeomorphism ϕ would then be in the identity component of the homeomorphism group of $\mathbb{CP}_2$. Thus any weakly unknotted embedding of $\mathbb{RP}^2$ in $\mathbb{CP}_2$, as defined above, may be moved through locally flat *topological* embeddings so as to "unknot" it into the standard $\mathbb{RP}^2$. *A priori*, however, it might still be impossible to carry out this unknotting process by a path of *smooth* embeddings.

Theorem 4.4. *Let $[\nabla]$ be a C^3 Zoll projective structure on an oriented surface $M \approx S^2$. Then, up to a projective linear transformation, the projective structure $[\nabla]$ uniquely determines a differentiable, totally real, weakly unknotted embedding of the space of geodesics $N \approx \mathbb{RP}^2$ into $\mathbb{CP}_2$. If $[\nabla]$ is C^∞, so is the embedding. Moreover, the image of each of the circles $\ell_x \subset N$, $x \in M$, bounds a holomorphic embedding of the disk $D^2 \hookrightarrow \mathbb{CP}_2$, and the interiors of these disks foliate the complement $\mathbb{CP}_2 - N$.*

Proof. By construction, the smooth 4-manifold $\mathcal{N}$ can be obtained by gluing the unit disk bundle in $T\mathbb{RP}^2$ to the Euler-class-4 D^2 bundle over S^2 via an orientation-reversing diffeomorphism of their common boundary, which is the Lens space $X = S^3/\mathbb{Z}_4$. However, the diffeomorphism type of the pair $(\mathcal{N}, N)$ only depends on the isotopy class of the diffeomorphism $X \to X$. But the group of orientation-preserving diffeomorphisms of X is connected [6], so it follows that the diffeotype of the pair $(\mathcal{N}, N)$ is independent of which Zoll projective structure $[\triangledown]$ on S^2 we use. However, the standard structure $[\nabla]$ gives us the pair $(\mathbb{CP}_2, \mathbb{RP}^2)$. Thus there is a diffeomorphism $\phi : \mathbb{CP}_2 \to \mathcal{N}$ with $\phi(\mathbb{RP}^2) = N$.

In particular, this argument says that $\mathcal{N}$ is diffeomorphic to $\mathbb{CP}_2$. Lemma 4.2 therefore tells us that there is a *biholomorphism* $F : \mathcal{N} \to \mathbb{CP}_2$, and this F is unique modulo composition with elements of $\mathrm{PSL}(3, \mathbb{C})$. The promised embedding $N \hookrightarrow \mathbb{CP}_2$ is then given by $F|_N$, whereas the promised disks are the images of the the fibers of $\mathcal{Z}_+ \to M$ under $F \circ \Psi$. Moreover, since the diffeomorphism $\phi = F \circ \phi : \mathbb{CP}_2 \to \mathbb{CP}_2$ sends $\mathbb{RP}_2$ to $F(N)$, our embedding $F|_N$ is weakly unknotted, and we are done. q.e.d.

Now, in order to invert the above construction, let us instead suppose that we are given a totally real submanifold $N \approx \mathbb{RP}^2$ of $\mathbb{CP}_2$, and attempt to construct a suitable family of holomorphic disks $D \hookrightarrow \mathbb{CP}_2$ with boundary $\partial D = S^1 \hookrightarrow N$; these circles in N will then eventually become the curves ℓ_x corresponding to a Zoll projective structure on S^2. Our method of accomplishing this will be to invoke the inverse function theorem, and so will apply only when the given embedding $N \hookrightarrow \mathbb{CP}_2$ is C^1 close to the standard embedding $\mathbb{RP}^2 \hookrightarrow \mathbb{CP}_2$. Thus, relative to a choice of tubular neighborhood, we will henceforth assume that N is represented by by a section of the normal bundle of $\mathbb{RP}^2$. This allows us a further technical simplification, since N will automatically be totally real provided the corresponding section has sufficiently small C^1-norm. We remark that the normal bundle of $\mathbb{RP}^2 \subset \mathbb{CP}_2$ can be canonically identified, via the complex structure, with $T\mathbb{RP}^2$. Thus the freedom in choosing $N \subset \mathbb{CP}_2$ can be conveniently parameterized by the space of vector fields on $\mathbb{RP}^2$ of sufficiently small C^1-norm.

For the standard projective structure on S^2, the disks in question are obtained by considering those complex projective lines $\mathbb{CP}_1 \subset \mathbb{CP}_2$ which are complexifications of some real projective line $\mathbb{RP}^1 \subset \mathbb{RP}^2$, and then choosing one of the hemispheres into which such a $\mathbb{CP}_1$ is divided by the corresponding $\mathbb{RP}^1$. In order to understand these disks more explicitly, let us begin with the standard homogeneous coordinates $[z_1 : z_2 : z_3]$ on $\mathbb{CP}_2$, with the usual convention that $\mathbb{RP}^2$ is represented by z_1, z_2, z_3 real, and consider the affine chart $(\mathfrak{z}_1, \mathfrak{z}_2)$ on $\mathbb{CP}_2$ defined by

$$\mathfrak{z}_1 = \frac{z_1 - iz_2}{z_1 + iz_2}, \qquad \mathfrak{z}_2 = \frac{z_3}{z_1 + iz_2}.$$

This chart realizes $\mathbb{RP}^2 - [0 : 0 : 1]$ as the Möbius band $B \subset \mathbb{C}^2$ given by

$$\mathfrak{z}_1\bar{\mathfrak{z}}_1 = 1, \qquad \mathfrak{z}_1\bar{\mathfrak{z}}_2 = \mathfrak{z}_2.$$

Note that we may also parameterize B by

$$\begin{aligned} \mathfrak{z}_1 &= e^{i\theta} \\ \mathfrak{z}_2 &= te^{i\theta/2}, \end{aligned}$$

where the real coordinates (θ, t) are best thought of as really taking values in the *abstract* Möbius band $\mathbb{R}^2/\mathbb{Z}$ corresponding to the $\mathbb{Z}$-action generated by

$$(\theta, t) \mapsto (\theta + 2\pi, -t).$$

Now the projective line $z_3 = 0$ in $\mathbb{CP}_2$ corresponds, in this picture, to the complex affine line $\mathfrak{z}_2 = 0$; and one hemisphere of this $\mathbb{CP}_1$ is the disk $|\mathfrak{z}_1| \leq 0$ in this affine complex line, the boundary of which is the circle $\theta \mapsto (e^{i\theta}, 0)$ in B. How many other ways can one holomorphically embed the disk D into $\mathbb{C}^2$ so that its boundary $\partial D = S^1$ lands in B, and is homotopic in B to $e^{i\theta} \mapsto (e^{i\theta}, 0)$? Projecting any such disk to the $\mathfrak{z}_1$ axis would give us a holomorphic map $D \to \mathbb{C}$ whose boundary values define a degree-1 map $S^1 \to S^1$. Any such map is a Möbius transformation

$$\zeta \mapsto \frac{a\zeta + b}{\overline{a} + \bar{b}\zeta}, \quad |a|^2 - |b|^2 = 1. \tag{13}$$

Thus, after composition with a Möbius transformation, any such disk is the graph $\mathfrak{z}_2 = f(\mathfrak{z}_1)$ of a holomorphic function f on the unit disk $|\mathfrak{z}_1| \leq 1$. However, the requirement that $f(\partial D)$ lie in B says that

$$f(e^{i\theta}) = e^{i\theta}\overline{f(e^{i\theta})}.$$

If f has power series expansion

$$f(\mathfrak{z}_1) = \sum_{\ell=0}^{\infty} a_\ell \mathfrak{z}_1^\ell,$$

our boundary condition becomes

$$\sum_{\ell=0}^{\infty} a_\ell e^{i\ell\theta} = \sum_{\ell=-\infty}^{1} \overline{a}_{-\ell+1} e^{i\ell\theta}.$$

Hence, setting $a = a_0$, every such disk is the graph

$$\mathfrak{z}_2 = a + \bar{a}\mathfrak{z}_1$$

of an affine linear function restricted to the unit disk $|\mathfrak{z}_1| \leq 1$. Each of these disks exactly represents one hemisphere of the projective line $\mathbb{CP}_1 \subset \mathbb{CP}_2$ given by

$$z_3 = (2\, \Re e\ a)\ z_1 + (-2\, \Im m\ a)\ z_2,$$

and the boundaries of these disks are thus precisely the real projective lines $\mathbb{RP}^1 \subset \mathbb{RP}^2$ which do not pass through the point $[0 : 0 : 1]$ which was excluded by our choice of coordinates. By considering all possible permutations of the homogeneous coordinates z_1, z_2, z_3, one obtains the entire family of disks corresponding to the points of S^2 equipped with its standard projective structure.

We now consider the problem of constructing an analogous family of disks with boundaries on a submanifold $N \subset \mathbb{CP}_2$ which is C^1-near to $\mathbb{RP}^2 \subset \mathbb{CP}_2$. To do this, it is enough to completely analyze the corresponding problem arising when intersection of the Möbius band B and a large ball is replaced with a section of its normal bundle, since N is covered by a finite number of pieces of this form.

To this end, we will begin by considering maps of the circle S^1 to the abstract Möbius band $\mathbb{R}^2/\mathbb{Z}$ with winding number 1. For reasons of technical transparency, we will consider maps of Sobolev class L^2_k, where $k \geq 1$. Let us recall that the Cauchy-Schwarz inequality immediately implies the Sobolev embedding theorem in this case, since any smooth, real valued function f on the line satisfies

$$|f(a) - f(b)| \leq \left(\int_a^b \left| \frac{df}{dx} \right|^2 dx \right)^{1/2} |a - b|^{1/2}, \tag{14}$$

whence $L^2_k(S^1) \subset C^{k-1,\frac{1}{2}}(S^1)$. In particular, maps from the circle of class L^2_k are continuous, and it thus makes sense to talk about winding numbers of such maps. Moreover, this shows that point-wise multiplication of functions gives us a continuous bilinear map $L^2_k(S^1) \times L^2_k(S^1) \to L^2_k(S^1)$. Also note that the composition of of any C^k function with an L^2_k function is again an L^2_k function.

We will freely identify $L^2_k(S^1)$ with the real Hilbert space of real-valued L^2_k functions of $\theta \in [0, 2\pi]$ with $u(2\pi) = u(0)$, and we will also need to consider the real Hilbert space $\widetilde{L}^2_k(S^1)$ of L^2_k sections of the Möbius band, which we may think of as functions of $\theta \in [0, 2\pi]$ with $u(2\pi) = -u(0)$. Since any continuous section of the Möbius band must have a zero, (14) tells us that any $u \in \widetilde{L}^2_k$, $k \geq 1$, satisfies

$$\sup |u| \leq \sqrt{\pi} \left(\int_0^{2\pi} \left| \frac{du}{d\theta} \right|^2 d\theta \right)^{1/2} \leq \sqrt{\pi} \|u\|_{L^2_k},$$

so the elements u of the ball of radius $R/\sqrt{\pi}$ in $\widetilde{L}^2_k$ maybe thought of as defining a section $\theta \mapsto (\theta, u(\theta))$ of the *finite* Möbius strip

$$B^R = (\mathbb{R} \times [-R, R]) / \mathbb{Z},$$

where the $\mathbb{Z}$ action is again generated by $(\theta, t) \mapsto (\theta + 2\pi, -t)$. We will use $C^k(B^R)$ to denote the real Banach space of C^k real-valued functions on this strip, and

$$\widetilde{C}^k(B^R) = \left\{ h : \mathbb{R} \times [-R, R] \xrightarrow{C^k} \mathbb{R} \mid h(\theta + 2\pi, -t) = -h(\theta, t) \right\}$$

to denote the real Banach space of C^k sections of the nontrivial real line bundle on B^R, the Banach-space norms being of course the suprema of the absolute values of all partial derivatives of order $\leq k$.

Any pair $(h_1, h_2) \in C^{k+1}(B^R) \times \widetilde{C}^{k+1}(B^R)$ defines an embedding $B^R \hookrightarrow \mathbb{C}^2$ by

$$(\theta, t) \mapsto \left(e^{h_1(\theta,t)+i\theta}, [t + ih_2(\theta, t)]e^{i\theta/2} \right),$$

and any C^{k+1} submanifold $N \subset \mathbb{CP}_2$ which is sufficiently close to the standard $\mathbb{RP}^2 \subset \mathbb{CP}_2$ can be written as a finite union of images of such embeddings of finite strips via suitable systems of inhomogeneous coordinates. The general L^2_k embedding of S^1 inside this strip with winding number 1 can then be written as

$$\theta \mapsto \Big(e^{h_1(\theta+u_1(\theta),u_2(\theta))+i[\theta+u_1(\theta)]}, \\ [u_2(\theta) + ih_2(\theta + u_1(\theta), u_2(\theta))]e^{i(\theta+u_1(\theta))/2} \Big)$$

for $u_1 \in L^2_k(S^1)$ and $u_2 \in \widetilde{L}^2_k(S^1)^R$, where $\widetilde{L}^2_k(S^1)^R$ denotes the open ball of radius $R/\sqrt{\pi}$ centered at the origin in $\widetilde{L}^2_k(S^1)$. This motivates us to consider the maps of Banach manifolds

$$\mathfrak{F}_1, \mathfrak{F}_2 : L^2_k(S^1) \times \widetilde{L}^2_k(S^1)^R \times C^{k+\ell}(B^R) \times \widetilde{C}^{k+\ell}(B^R) \\ \longrightarrow L^2_k(S^1, \mathbb{C}) \times L^2_k(S^1, \mathbb{C}),$$

given by

$$[\mathfrak{F}_1(u_1, u_2, h_1, h_2)](\theta) = \exp\left[h_1\Big(\theta + u_1(\theta), u_2(\theta)\Big) + i\Big(\theta + u_1(\theta)\Big) \right]$$

and

$$[\mathfrak{F}_2(u_1, u_2, h_1, h_2)](\theta) \\ = \left[u_2(\theta) + ih_2\Big(\theta + u_1(\theta), u_2(\theta)\Big) \right] \exp\left(i\frac{\theta + u_1(\theta)}{2} \right).$$

These maps are both C^ℓ; in particular, for $\ell \geq 1$ they have bounded continuous derivatives given by

$$(\dot{u}_1, \dot{u}_2, \dot{h}_1, \dot{h}_2)] \overset{\mathfrak{F}_{1*}}{\longmapsto} \left[\dot{h}_1(\theta + u_1, u_2) + \left(i + \frac{\partial h_1}{\partial \theta}\right)\dot{u}_1 + \frac{\partial h_1}{\partial t}\dot{u}_2\right] e^{h_1 + i(\theta + u_1)}$$

and

$$(\dot{u}_1, \dot{u}_2, \dot{h}_1, \dot{h}_2)] \overset{\mathfrak{F}_{2*}}{\longmapsto} \Bigg[\left(\frac{iu_2 - h_2}{2} + i\frac{\partial h_2}{\partial \theta}\right)\dot{u}_1 + \left(1 + i\frac{\partial h_2}{\partial t}\right)\dot{u}_2 + i\dot{h}_2(\theta + u_1, u_2)\Bigg] e^{i(\theta + u_1)/2},$$

where h_1, h_2, and their first partial derivatives with respect to θ and t are understood to be evaluated at $(\theta + u_1(\theta), u_2(\theta))$, and thus are functions of class L^2_k which depend continuously on (u_1, u_2, h_1, h_2). In particular, notice that the derivatives of these maps at the origin are respectively given by

$$[\mathfrak{F}_{1*\mathbf{0}}(\dot{u}_1, \dot{u}_2, \dot{h}_1, \dot{h}_2)](\theta) = \left[\dot{h}_1(\theta, 0) + i\dot{u}_1(\theta)\right] e^{i\theta}$$

and

$$[\mathfrak{F}_{2*\mathbf{0}}(\dot{u}_1, \dot{u}_2, \dot{h}_1, \dot{h}_2)](\theta) = \left[\dot{u}_2(\theta) + i\dot{h}_2(\theta, 0)\right] e^{i\theta/2}.$$

Next, we introduce the orthogonal projection

$$\Pi : L^2(S^1, \mathbb{C}) \to L^2{\downarrow}$$

to the closed linear subspace

$$L^2{\downarrow} = \left\{ \sum_{\ell < 0} a_\ell e^{i\ell\theta} \;\middle|\; a_\ell \in \mathbb{C}, \; \sum_{\ell < 0} |a_\ell|^2 < \infty \right\} \subset L^2(S^1, \mathbb{C})$$

of *negative frequency* functions given by

$$\Pi\left(\sum_{\ell=-\infty}^{\infty} a_\ell e^{i\ell\theta}\right) = \sum_{\ell=-\infty}^{-1} a_\ell e^{i\ell\theta}.$$

This a bounded linear operator, and indeed has operator norm 1 . Notice that the kernel of Π precisely consists of those L^2 function on the circle which arise as the boundary values of holomorphic functions on the disk. Set

$$L^2_k{\downarrow} = \left\{ \sum_{\ell<0} a_\ell e^{i\ell\theta} \;\middle|\; a_\ell \in \mathbb{C},\ \sum_{\ell<0} \ell^{2k}|a_\ell|^2 < \infty \right\} = L^2_k(S^1,\mathbb{C}) \cap L^2{\downarrow}.$$

and notice that

$$\Pi : L^2_k(S^1,\mathbb{C}) \to L^2_k{\downarrow}$$

is also bounded, and indeed again has operator norm 1.

Similarly, let us define

$$п : L^2_k(S^1,\mathbb{C}) \to \mathbb{C}$$

by

$$п\left(\sum_{\ell=-\infty}^{\infty} a_\ell e^{i\ell\theta}\right) = a_0.$$

Remark. The linear map Π is closely related to the Hilbert transform on the circle, and can be explicitly be realized [29] as the singular integral operator

$$[\Pi(u)](\theta) = u(\theta) - \frac{e^{-i\theta}}{2\pi}\ p.v.\int_0^{2\pi} \frac{u(\phi)d\phi}{e^{i(\phi-\theta)}-1}.$$

This can be used [17] to show that Π is also bounded with respect to in $C^{k,\alpha}$ norms. However, we have chosen, in the spirit of [5], to emphasize Sobolev norms here, as this has the advantage of keeping the technical details to a minimum.

Now, for $k, \ell \geq 1$, consider the C^ℓ map

$$L^2_k(S^1) \times \widetilde{L}^2_k(S^1)^R \times C^{k+\ell}(B^R) \times \widetilde{C}^{k+\ell}(B^R) \xrightarrow{\mathfrak{F}} L^2_k{\downarrow} \times L^2_k{\downarrow} \times C^{k+\ell}(B^R) \times \widetilde{C}^{k+\ell}(B^R) \times \mathbb{C} \times \mathbb{C} \times \mathbb{R}$$

of real Banach manifolds defined by

$$\mathfrak{F} = (\Pi \circ \mathfrak{F}_1) \times (\Pi \circ \mathfrak{F}_2) \times Л \times \widetilde{Л} \times (п \circ \mathfrak{F}_1) \times (п \circ \mathfrak{F}_2) \times ш,$$

where

$$Л : L^2_k(S^1) \times \widetilde{L}^2_k(S^1)^R \times C^{k+\ell}(B^R) \times \widetilde{C}^{k+\ell}(B^R) \longrightarrow C^{k+\ell}(B^R)$$

and

$$\widetilde{Л} : L^2_k(S^1) \times \widetilde{L}^2_k(S^1)^R \times C^{k+\ell}(B^R) \times \widetilde{C}^{k+\ell}(B^R) \longrightarrow \widetilde{C}^{k+\ell}(B^R)$$

are the factor projections, while

$$ш : L^2_k(S^1) \times \widetilde{L}^2_k(S^1)^R \times C^{k+\ell}(B^R) \times \widetilde{C}^{k+\ell}(B^R) \longrightarrow \mathbb{R}$$

is given by

$$ш(u_1, u_2, h_1, h_2) = \frac{1}{2\pi}\int_0^{2\pi} u_1(\theta)\, d\theta.$$

Since Π, Л, $\widetilde{Л}$, п and ш are all bounded linear operators, this map is C^ℓ, with derivative given by

$$\mathfrak{F}_* = (\Pi \circ \mathfrak{F}_{1*}) \times (\Pi \circ \mathfrak{F}_{2*}) \times Л \times \widetilde{Л} \times (п \circ \mathfrak{F}_{1*}) \times (п \circ \mathfrak{F}_{2*}) \times ш.$$

In particular, for any

$$\dot{u}_1 = b_0 + \sum_{\ell=1}^{\infty} b_\ell \cos(\ell\theta) + c_\ell \sin(\ell\theta)$$

in $L^2_k(S^1)$, and any

$$\dot{u}_2 = \sum_{\ell=0}^{\infty} \widetilde{b}_\ell \cos\left[(\ell + \tfrac{1}{2})\theta\right] + \widetilde{c}_\ell \sin\left[(\ell + \tfrac{1}{2})\theta\right]$$

in $\widetilde{L}^2_k(S^1)$, we see that the derivative of $\mathfrak{F}$ at the origin is given explicitly by

$$\mathfrak{F}_{*\mathbf{0}} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{h}_1 \\ \dot{h}_2 \end{bmatrix} = \begin{bmatrix} \Pi\left(\dot{h}_1(\theta,0)e^{i\theta}\right) + \sum_{\ell=2}^{\infty} \frac{-c_\ell + ib_\ell}{2} e^{-i(\ell-1)\theta} \\ \Pi\left(i\dot{h}_2(\theta,0)e^{i\theta/2}\right) + \sum_{\ell=1}^{\infty} \frac{\widetilde{b}_\ell + i\widetilde{c}_\ell}{2} e^{-i\ell\theta} \\ \dot{h}_1 \\ \dot{h}_2 \\ п\left(\dot{h}_1(\theta,0)e^{i\theta}\right) + \frac{-c_1 + ib_1}{2} \\ п\left(i\dot{h}_2(\theta,0)e^{i\theta/2}\right) + \frac{\widetilde{b}_0 + i\widetilde{c}_0}{2} \\ b_0 \end{bmatrix}.$$

Since $\mathfrak{F}_{*\mathbf{0}}$ manifestly has bounded inverse, the Banach-space inverse function theorem [28] tells us that there is an open neighborhood $\mathfrak{U}$ of

$\mathbf{0} \in L^2_k(S^1) \times \widetilde{L}^2_k(S^1)^R \times C^{k+\ell}(B^R) \times \widetilde{C}^{k+\ell}(B^R)$ and an open neighborhood $\mathfrak{V}$ of $\mathbf{0} \in L^2_k\!\downarrow \times L^2_k\!\downarrow \times C^{k+\ell}(B^R) \times \widetilde{C}^{k+\ell}(B^R) \times \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ such that

$$\mathfrak{F}|_{\mathfrak{U}} : \mathfrak{U} \longrightarrow \mathfrak{V}$$

is a C^ℓ diffeomorphism. For any h_1, h_2 of sufficiently small $C^{k+\ell}$ norm, we therefore obtain a 5-parameter family of holomorphic disks $D \to \mathbb{CP}_2$ with boundaries on the graph of (h_1, h_2) by considering the unique disks with boundary values specified by $(\mathfrak{F}|_{\mathfrak{U}})^{-1}[\mathfrak{V} \cap (\{(0,0,h_1,h_2)\} \times \mathbb{C} \times \mathbb{C} \times \mathbb{R})]$. On the other hand, not all of these disks correspond to geometrically distinct unparameterized disks, since any parameterized disk gives rise to a 3-parameter family of other parameterized disks by composition with Möbius transformations of the form (13). However, we can easily kill this "gauge freedom" by instead considering the 2-parameter family of disks whose boundary values are the circles

$$(\mathfrak{F}|_{\mathfrak{U}})^{-1}(0,0,h_1,h_2,-w^2,w,0), \quad w \in \mathbb{C}, |w| < \varepsilon.$$

The other disks in our original 5-parameter family can then all be obtained by composing the disks in this 2-parameter family with Möbius transformations. Notice, however, that we have now carefully constructed our disks so that their centers are on the complex curve

$$\mathfrak{z}_1 + \mathfrak{z}_2^2 = 0$$

in $\mathbb{C}^2$, and that our parameter w exactly sweeps out a neighborhood of the origin in this curve. However, this curve is just an affine chart on the conic $\mathcal{Q} \subset \mathbb{CP}_2$ given by

$$z_1^2 + z_2^2 + z_3^2 = 0.$$

Now the subgroup $\mathrm{SO}(3) \subset \mathrm{PSL}(3,\mathbb{C})$ preserves both $\mathcal{Q}$ and $\mathbb{RP}^2 \subset \mathbb{CP}_2$, and acts transitively on both $\mathcal{Q}$ and the set of real projective lines $\mathbb{RP}^1 \subset \mathbb{RP}^2$. Thus, by considering only affine charts $(\mathfrak{z}_1, \mathfrak{z}_2)$ related to our original choice by the action of $\mathrm{SO}(3)$, we can construct a collection of families of disks so that their centers run through a finite open cover of $\mathcal{Q} \approx S^2$, in a uniform manner depending on the submanifold $N \subset \mathbb{CP}_2$, thought of as the graph of a section of the normal bundle of $\mathbb{RP}^2 \subset \mathbb{CP}_2$ of sufficiently small $C^{k+\ell}$ norm, corresponding to (h_1, h_2) in local coordinates. Since $\mathfrak{F}|_{\mathfrak{U}}$ is a diffeomorphism, we can also arrange that these disks coincide up to Möbius transformations on overlaps by at worst restricting to a smaller open set of N's in the $C^{k+\ell}$ topology. This yields the following result:

Proposition 4.5. *If $N \subset \mathbb{CP}_2$ is the image of any embedding $\mathbb{RP}^2 \hookrightarrow \mathbb{CP}_2$ which is sufficiently close to the standard one in the C^{2k-1} topology, then N contains a unique family of embedded oriented circles $\ell_x \subset N$, $x \in S^2$, each of which bounds an embedded holomorphic disk $D^2 \subset \mathbb{CP}_2$, and each of which is L^2_k close (and hence C^{k-1} close) to the image of an oriented real projective line $\mathbb{RP}^1 \hookrightarrow \mathbb{RP}^2$. Moreover, if $k \geq 2$, the corresponding family of holomorphic disks can be realized by a fiber-wise holomorphic, C^{k-1} map from the unit disk bundle in the $\mathcal{O}(4)$ complex line bundle over $S^2 = \mathbb{CP}_1$. These disks are all embedded, and their interiors foliate $\mathbb{CP}_2 - N$.*

Proof. Locally, our family of disks has been found by using $\mathfrak{F}^{-1}$ to construct a C^{k-1} map from an open set $W \subset \mathbb{C}$ to the space of L^2_k maps from the circle to N which bound maps of the 2-disk. But, provided that $k \geq 2$, the inclusion $L^2_k \hookrightarrow C^{k-1}$ is a bounded linear map, and the maximum principle tells us that we therefore have a C^{k-1} map from W into the C^{k-1} maps of the disk to $\mathbb{CP}_2$. But any such map is given by a C^{k-1} map $W \times D^2 \to \mathbb{CP}_2$. Since we have also arranged for the centers of our disks to land on the conic $\mathcal{Q}$, our various local families of disks are related by Möbius transformations which fix the origin, and so are elements of $U(1)$; moreover, these transformations are C^{k-1} functions of our parameters, and so determine a C^{k-1} disk bundle over $\mathcal{Q} \approx S^2$.

Now our family of disks is a C^{k-1} map $\mathfrak{f}$ from this disk bundle to $\mathbb{CP}_2$, and sends the zero section to $\mathcal{Q}$. In our $(\mathfrak{z}^1, \mathfrak{z}^2)$ coordinates, each our disks is C^{k-1} close to a disk in a complex line $\mathfrak{z}^2 = a + \bar{a}\mathfrak{z}^1$. By possibly shrinking our neighborhood of N's, we can thus arrange that each is embedded, and transverse to $\mathcal{Q}$. Similarly, we can arrange for the derivative of $\mathfrak{f}$ to be nonzero everywhere, since locally the map is C^1 close to our model example. Moreover, each of our N's can be obtained from $\mathbb{RP}^2$ by applying a self-diffeomorphism of $\mathbb{CP}_2$ which is C^{k-1} close to the identity, and the push-forward of the local functions $|\mathfrak{z}^1|^2$ by these diffeomorphisms will result in functions which are subharmonic on each disk of the family, and the maximum principle therefore shows each of the disks will meet N only along its boundary. Thus $\mathfrak{f}$ gives us a proper local diffeomorphism, and hence a covering map, from the interior of the disk bundle to $\mathbb{CP}_2 - N$; but $\mathbb{CP}_2 - N$ is simply connected, so $\mathfrak{f}$ is a diffeomorphism on the interior of our disk bundle. In particular, the zero section of our disk bundle, which is sent to $\mathcal{Q}$, has self-intersection $\mathcal{Q} \cdot \mathcal{Q} = 2^2 = 4$, so our disk bundle has first Chern class 4, and so must be C^{k-1} isomorphic to the unit disk bundle in $\mathcal{O}(4)$. q.e.d.

Thus, we have constructed a family of curves $\ell_x \subset N$, $x \in S^2$, which bound holomorphic disks. We now wish to consider the curves $\mathfrak{C}_y \subset S^2$, $y \in N$, obtained by considering the set of all ℓ_x's passing through y, and we would like to assert that these must be the geodesics of a unique Zoll projective connection $[\nabla]$ on $M = S^2$. Our proof of this assertion will hinge on:

Lemma 4.6. *Let M be a smooth connected 2-manifold, and let $\varpi : \mathcal{X} \to M$ be a smooth $\mathbb{CP}_1$-bundle. Let $\rho : \mathcal{X} \to \mathcal{X}$ be an involution which commutes with the projection ϖ, and has as fixed-point set $\mathcal{X}_\rho$ an S^1-bundle over M which disconnects $\mathcal{X}$ into two closed 2-disk bundles $\mathcal{X}_\pm$ with common boundary $\mathcal{X}_\rho$. Suppose that $\text{Д} \subset T_{\mathbb{C}}\mathcal{X}$ is a distribution of complex 2-planes on $\mathcal{X}$ such that:*

- $\rho^* \text{Д} = \overline{\text{Д}}$;
- *the restriction of Д to $\mathcal{X}_+$ is C^k, $k \geq 1$, and involutive*;
- $\text{Д} \cap \ker \varpi_*$ *is the $(0,1)$ tangent space of the $\mathbb{CP}_1$ fibers of ϖ; and*
- *the restriction of Д to a fiber of $\mathcal{X}$ has $c_1 = -3$ with respect to the complex orientation.*

Then there is a unique C^{k-1} projective structure $[\nabla]$ on M such that Д is obtained from the associated involutive distribution $\mathbf{D}$ on $\mathbb{P}T_{\mathbb{C}}M$ given by the recipe (7), pulled back by a uniquely determined C^k diffeomorphism $\phi : \mathcal{X} \to \mathbb{P}T_{\mathbb{C}}M$ which makes the diagrams

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\phi"] \arrow[dr] & & \mathbb{P}T_{\mathbb{C}}M \arrow[dl] \\ & M & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathcal{X} \arrow[r, "\phi"] \arrow[d, "\rho"'] & \mathbb{P}T_{\mathbb{C}}M \arrow[d, "c"] \\ \mathcal{X} \arrow[r, "\phi"] & \mathbb{P}T_{\mathbb{C}}M \end{tikzcd}$$

commute, where $c : \mathbb{P}T_{\mathbb{C}}M \to \mathbb{P}T_{\mathbb{C}}M$ denotes the usual complex conjugation map.

Proof. Let us begin by noticing that, since $\text{Д} = \rho^*\overline{\text{Д}}$ is continuous on the closed sets $\mathcal{X}_+$ and $\mathcal{X}_-$, it is continuous on all of $\mathcal{X}$.

Now let L_1 be the $(0,1)$ tangent space of the fibers. By hypothesis, $L_1 \subset \text{Д}$, so that $L_2 = \text{Д}/L_1$ is a well defined complex line bundle.

Also notice that, since $Д \cap \ker \varpi_* = L_1$, the fibers of L_2 are carried injectively into $T_{\mathbb{C}}M$ by ϖ_*. We may therefore define a continuous map $\phi : \mathcal{X} \to \mathbb{P}T_{\mathbb{C}}M$ by $z \mapsto \varpi_*(L_2|_z) = \varpi_*(D_z)$; moreover, this ϕ makes the above diagrams commute.

Now let ζ be a smooth, fiber-wise holomorphic coordinate on $\mathcal{X}$, and notice that the corresponding vertical vector field $\partial/\partial\overline{\zeta}$ is a smooth section of D. Next, near any point of the interior of $\mathcal{X}_+$, let $\mathfrak{w}$ be any other local section of $Д$ which is linearly independent from $\partial/\partial\overline{\zeta}$, and then notice that the involutivity hypothesis $[C^1(Д), C^1(Д)] \subset C^0(Д)$ tells us that

$$\frac{\partial}{\partial\overline{\zeta}}(\varpi_*(\mathfrak{w})) = \varpi_*(\mathcal{L}_{\frac{\partial}{\partial\overline{\zeta}}}\mathfrak{w}) = \varpi_*\left(\left[\frac{\partial}{\partial\overline{\zeta}}, \mathfrak{w}\right]\right) \equiv 0 \bmod \varpi_*(\mathfrak{w}),$$

so that ϕ is fiber-wise holomorphic on the interior of $\mathcal{X}_+$. Since $\phi = c \circ \phi \circ \rho$, it thus follows that ϕ is also fiber-wise holomorphic on the interior $\mathcal{X}_-$. But since ϕ is also continuous across $\mathcal{X}_\rho = \mathcal{X}_+ \cap \mathcal{X}_-$, this implies that ϕ is actually fiber-wise holomorphic on all of $\mathcal{X}$.

Now the restriction of L_2 to $\varpi^{-1}(x)$ is the pull-back, via ϕ, of the tautological $\mathcal{O}(-1)$ line bundle over $\mathbb{P}(\mathbb{C} \otimes T_xM) \cong \mathbb{CP}_1$. Since L_1 is the $(0,1)$ tangent space of $\varpi^{-1}(x)$, and $\varpi^{-1}(x) \cong \mathbb{CP}_1$, $c_1(L_1) = -2$ on any fiber of ϖ. On the other hand, $c_1(D) = -3$ on $\varpi^{-1}(x)$, by hypothesis. Adjunction therefore tells us that $c_1(L_2) = -1$ on any fiber. However, $c_1(\mathcal{O}(-1)) = -1$ on $\mathbb{CP}_1$, and we have just observed that the $\phi^*c_1(\mathcal{O}(-1)) = c_1(L_2)$. This shows that the fiber-wise degree of ϕ is $(-1)/(-1) = +1$. But since ϕ is also fiber-wise holomorphic, it follows that ϕ maps each fiber of $\mathcal{X}$ biholomorphically to the corresponding fiber of $\mathbb{P}T_{\mathbb{C}}M$. This in turn implies that ϕ is C^k on all of $\mathcal{X}$, since it sends any three pointwise-distinct local C^k sections of $\mathcal{X}_+$ to three pointwise-distinct local C^k sections of $\mathbb{P}T_{\mathbb{C}}M$, and ϕ is then algebraically determined by its value along these sections.

Let us now try to analyze the distribution of complex 2-planes $\phi_*Д$ on $\mathcal{Z} = \mathbb{P}T_{\mathbb{C}}M$. To this end, let us begin by choosing an arbitrary C^{k-1} torsion-free affine connection ∇_0 on M, and then considering the corresponding C^{k-1} integrable distribution of complex 2-planes $\mathbf{D}_0$ on $\mathcal{Z}$ given by (7). By construction, $\phi_*Д$ and $\mathbf{D}_0$ both intersect the vertical in the $(0,1)$ tangent spaces of the fibers. Moreover, letting $\mathbf{V}^{0,1}$ denote the $(0,1)$ vertical tangent bundle of $\mathbb{P}T_{\mathbb{C}}M$, $\mathbf{D}_0/\mathbf{V}^{0,1} = (\phi_*Д)/\mathbf{V}^{0,1} = \mathcal{O}(-1)$, where $\mathcal{O}(-1)$ of course denotes the tautological line bundle. Thus there is a unique continuous section γ of $\mathbf{V}^{1,0} \otimes \mathcal{O}(1)$ such that $\mathfrak{w} \in \mathbf{D}_0$ iff $\mathfrak{w} + \gamma(\pi_*\mathfrak{w}) \in \phi_*Д$; here we have used the notation $\mathbf{V}^{1,0} =$

$\overline{\mathbf{V}^{0,1}}$ and $\mathbf{V}^{1,0} \otimes \mathcal{O}(1) = \mathcal{H}om(\mathcal{O}(-1), \mathbf{V}^{1,0})$. Moreover, the regularity of $\varPi$ guarantees that γ is C^{k-1} away from the real slice $\mathbb{P}TM \subset \mathbb{P}T_{\mathbb{C}}M$. Now, let $\mathfrak{w}$ be a C^{k-1} local section of $\mathbf{D}_0$ for which $\pi_*\mathfrak{w}$ is a fiber-wise holomorphic section of $\mathcal{O}(-1)$; such a section may always be constructed by multiplying a generic section by a suitable complex-valued function. Set $f\partial/\partial\zeta = \gamma(\pi_*\mathfrak{w})$. Then, away from the real slice, the involutivity of $\phi_*\varPi$ and $\mathbf{D}_0$ then tells us that

$$\left[\frac{\partial}{\partial\bar{\zeta}}, \mathfrak{w}\right] \equiv 0 \bmod \frac{\partial}{\partial\bar{\zeta}}$$

and

$$\left[\frac{\partial}{\partial\bar{\zeta}}, \mathfrak{w} + f\frac{\partial}{\partial\zeta}\right] \equiv 0 \bmod \frac{\partial}{\partial\bar{\zeta}}\ ,$$

so that

$$\frac{\partial f}{\partial\bar{\zeta}}\frac{\partial}{\partial\zeta} \equiv 0 \bmod \frac{\partial}{\partial\bar{\zeta}},$$

and hence $\partial f/\partial\bar{\zeta} = 0$. This shows that γ is fiber-wise holomorphic away from the real slice. But γ is also continuous across the real slice. It follows that γ is fiber-wise holomorphic on all of $\mathbb{P}T_{\mathbb{C}}M$.

Now any holomorphic section of $(T^{1,0}\mathbb{CP}_1) \otimes \mathcal{O}(1) \cong \mathcal{O}(3)$ arises from a unique trace-free element of $\mathbb{C}^2 \otimes \odot^2(\mathbb{C}^2)^*$. Thus γ is uniquely expressible as a trace-free symmetric tensor field

$$\mathrm{r} \in T_{\mathbb{C}}M \otimes \odot^2 T^*_{\mathbb{C}}M.$$

Since γ is C^{k-1} away from the real slice, it follows that r must be C^{k-1}. Moreover, because $\phi_*\varPi$ and $\mathbf{D}_0$ are both sent to their complex conjugates by c, so is γ, and r is therefore real-valued. Setting $\nabla = \nabla_0 + \mathrm{r}$ now gives us a C^{k-1} torsion-free affine connection on M such that $\phi_*\varPi$ coincides with the distribution $\mathbf{D}$ defined by (7). Since this last requirement certainly also determines ∇ up to projective equivalence, we are therefore done. q.e.d.

This allows us to finally show that our constructed families of holomorphic disks actually give us Zoll projective structures.

Theorem 4.7. *Let N be any embedding of $\mathbb{RP}^2$ into $\mathbb{CP}_2$ which is C^{2k+5} close to the standard one. Let $\{\ell_x \mid x \in S^2\}$ be the constructed family of circles which bound holomorphic disks. For each $y \in N$, set*

$$\mathfrak{C}_y = \{x \in S^2 \mid y \in \ell_x\}.$$

Then there is a unique C^k Zoll projective structure $[\nabla]$ on S^2 for which every $\mathfrak{C}_y$ is a geodesic.

Proof. Let $\mathcal{X}_+$ be the unit disk bundle in $\mathcal{O}(4)$, and let $\mathcal{X}$ be its double, obtained by identifying two copies of $\mathcal{X}_+$ along their boundaries. Let $\mathcal{X}_-$ be the second copy of $\mathcal{X}_+$, and let $\rho : \mathcal{X} \to \mathcal{X}$ be the smooth map which interchanges $\mathcal{X}_+$ and $\mathcal{X}_-$. (Notice that one may think of $\mathcal{X} \to S^2$ as the fourth Hirzebruch surface; thus, while $\mathcal{X}$ is itself diffeomorphic to $S^2 \times S^2$, the 'real slice' $\mathcal{X}_\rho \to S^2$ is the circle bundle of Euler class 4.)

Next, we consider the constructed family of holomorphic disks $\mathfrak{f} : \mathcal{X}_+ \to \mathbb{CP}_2$ with boundary on N. Let $\mathfrak{f}_*^{1,0} : T_{\mathbb{C}}\mathcal{X}_+ \to \mathfrak{f}^*T^{1,0}\mathbb{CP}_2$ be the $(1,0)$ component of its derivative. Since $\mathfrak{f}_*^{1,0}$ is C^{k+1} close to the corresponding surjective morphism arising in the model case of the linear embedding $\mathbb{RP}^2 \hookrightarrow \mathbb{CP}_2$, it is also surjective for every embedding in an appropriate neighborhood with respect to the topology in question. Thus we may arrange for $\mathit{\Pi} = \ker \mathfrak{f}_*^{1,0}$ to be a C^{k+1} distribution of complex 2-planes on $\mathcal{X}_+$ for each of the embeddings in question. Moreover, $\mathit{\Pi}$ is involutive on the interior of $\mathcal{X}_+$, since $\mathfrak{f}$ is a diffeomorphism there, and sends $\mathit{\Pi}$ to the involutive distribution $T^{0,1}\mathbb{CP}_2$.

Along $\mathcal{X}_\rho = \partial\mathcal{X}_+$, note that $\mathit{\Pi}$ is spanned by $\partial/\partial\overline{\zeta}$ and the distribution of real lines tangent to the fibers of

$$\mathfrak{f}|_{\partial\mathcal{X}_+} : \mathcal{X}_\rho \to N.$$

We may therefore extend $\mathit{\Pi}$ to all of $\mathcal{X}$ by declaring it equal to $\rho^*\overline{\mathit{\Pi}}$ on $\mathcal{X}_-$. The resulting distribution is C^0 close to the one corresponding to the model case, and so has $c_1(\mathit{\Pi}) = -3$ on every fiber of $\mathcal{X}$. Thus the hypotheses of Lemma 4.6 are all fulfilled, and we therefore obtain a unique C^k projective structure $[\nabla]$ on $M = S^2$ for which $\mathit{\Pi}$ corresponds to $\mathbf{D}$ via ϕ. But ϕ sends $\mathcal{X}_\rho$ diffeomorphically to $\mathbb{P}TM$, and the fibers of $\mathfrak{f}|_{\partial\mathcal{X}_+}$ are thereby sent to a foliation $\mathcal{F}$ of $\mathbb{P}TM$ by circles which is horizontal with respect to $[\nabla]$, and must coincide with the foliation by lifted $[\nabla]$-geodesics. Because each fiber of $\mathcal{X}_\rho \to M$ is sent injectively to an embedded circle $\ell_x \subset N$, no leaf of $\mathcal{F}$ meets a fiber of μ twice. Since each such leaf is also compact, the projective structure $[\nabla]$ is therefore Zoll. The space of geodesics $\widetilde{N}$ of $[\nabla]$ is then a compact manifold diffeomorphic to $\mathbb{RP}^2$, and comes equipped with a tautological submersion to N; this map is necessarily a covering map, and hence is a diffeomorphism by comparison of fundamental groups. In particular, the $\mathfrak{C}_y$ are precisely the geodesics of the constructed projective structure. q.e.d.

We now address the issue of determining when a given projective structure can be represented by the Levi-Civita connection of a Riemannian metric.

Suppose that g is a Zoll metric on $M \approx S^2$. Then, in analogy with the construction on page 489, we obtain a preferred holomorphic curve $\mathcal{C} \subset \mathcal{Z}_+$ of genus zero and self-intersection $+4$ by considering $T^{0,1}M$ for the unique complex structure compatible with h and the fixed orientation of M. The image $\mathcal{Q} = \Psi[\mathcal{C}]$ of this Riemann surface is then an embedded, nonsingular rational curve of self-intersection 4 in in $\mathcal{N} \cong \mathbb{CP}_2$, and so must be a nonsingular conic.[3] After a projective linear transformation, we may thus identify $\mathcal{Q}$ with the smooth conic given by

$$z_1^2 + z_2^2 + z_3^2 = 0.$$

Henceforth, we will impose this choice as a matter of convention.

Now observe that the Riemann surface $\mathcal{C}$ is one of the two connected components of the locus $g(\mathbf{v}, \mathbf{v}) = 0$ in $\mathbb{P}T_{\mathbb{C}}M$. The complement of this locus is doubly covered by

$$UT_{\mathbb{C}}M = \{\mathbf{v} \in T_{\mathbb{C}}M \mid g(\mathbf{v}, \mathbf{v}) = 1\},$$

which we will think of as a fiber-wise complexification of the unit tangent bundle of (M, g). However, $UT_{\mathbb{C}}M$ may be canonically identified, using g, with

$$UT^*_{\mathbb{C}}M = \{\eta \in T^*_{\mathbb{C}}M \mid g^{-1}(\eta, \eta) = 1\},$$

and we may thus equip $UT_{\mathbb{C}}M$ with a complex-valued 2-form Υ obtained by restricting $d\Theta$ to $UT^*_{\mathbb{C}}M$, where $\Theta = y_1 dx^1 + y_2 dx^2$ is the tautological complex-valued 1-form on $T^*_{\mathbb{C}}M$. Moreover, it is not hard to see that $\mathbf{D} = \ker \Upsilon$ on $UT_{\mathbb{C}}M$, since, taking geodesic normal coordinates around an arbitrary point, we have

$$g = (dx^1)^2 + (dx^2)^2 + O(|x|^2),$$

so that

$$\Upsilon = d\left(\frac{dx^1 + \zeta dx^2}{\sqrt{1 + \zeta^2 + O(|x|^2)}}\right) = \frac{d\zeta \wedge (\zeta dx^1 - dx^2)}{(1 + \zeta^2)^{3/2}} + O(x^1, x^2),$$

[3] Note that in this Zoll metric case, Lemma 4.2 may therefore be replaced with the classical observation [3, Proposition V.4.3] that a compact complex surface which contains a rational curve of positive self-intersection is birationally equivalent to $\mathbb{CP}_2$.

and $\ker \Upsilon$ is therefore spanned by $\partial/\partial\overline{\zeta}$ and $\Xi = \partial/\partial x^1 + \zeta\ \partial/\partial x^2 + O(x^1, x^2)$. Away from the real slice, Υ is therefore a closed form of type $(2,0)$ with respect to $\mathbf{D}$, and hence is holomorphic; and, by the last calculation, $\Upsilon \otimes \Upsilon$ descends to $\mathbb{P}T_{\mathbb{C}}M - \mathcal{C}$ so as to have a pole of order 3 along $\mathcal{C}$. On the other hand, the restriction of Υ to the unit circle bundle of M is real-valued, and descends to the space of oriented geodesics by symplectic reduction [4, 32]. Thus Υ gives rise to a continuous 2-form on the double cover of $\mathbb{CP}_2 - \mathcal{Q}$ which is holomorphic on the the complement of N, and so holomorphic everywhere. Hence $\Upsilon \otimes \Upsilon$ is a well-defined meromorphic section of K^2 on $\mathbb{CP}_2$, with polar locus $3\mathcal{Q}$. It follows that

$$\Upsilon = \lambda \frac{z_1\ dz_2 \wedge dz_3 + z_2\ dz_3 \wedge dz_1 + z_3\ dz_1 \wedge d_2}{(z_1^2 + z_2^2 + z_3^2)^{3/2}} \tag{15}$$

for some constant $\lambda \in \mathbb{C}$.

Now, by our construction of $\mathcal{N}$, $\mathbb{CP}_2 - \mathcal{Q}$ deform-retracts to N, and also of course deform-retracts to the standard $\mathbb{RP}^2$. Suitably oriented double covers $\widetilde{N} \to N$ and $S^2 \to \mathbb{RP}^2$ are therefore homotopic in the universal (double) cover of $\mathbb{CP}_2 - \mathcal{Q}$, since both generate $\pi_2(\mathbb{CP}_2 - \mathcal{Q})$. Since Υ is closed, this tells us that

$$\int_{\widetilde{N}} \Upsilon = \int_{S^2} \lambda \frac{z_1\ dz_2 \wedge dz_3 + z_2\ dz_3 \wedge dz_1 + z_3\ dz_1 \wedge d_2}{(z_1^2 + z_2^2 + z_3^2)^{3/2}} = 4\pi\lambda\ .$$

However, the restriction of Υ to $\widetilde{N}$ is real, so this shows that λ must be real, too. We thus conclude that the Riemannian condition implies that N is *Lagrangian* with respect to the sign-ambiguous symplectic structure

$$\omega = \pm \Im m \left(\frac{z_1\ dz_2 \wedge dz_3 + z_2\ dz_3 \wedge dz_1 + z_3\ dz_1 \wedge d_2}{(z_1^2 + z_2^2 + z_3^2)^{3/2}} \right) \tag{16}$$

on $\mathbb{CP}_2 - \mathcal{Q}$.

As we shall now see, the converse is also true:

Theorem 4.8. *Let $N \hookrightarrow \mathbb{CP}_2$ be a totally real embedding of $\mathbb{RP}^2$ which arises from a $C^{k,\alpha}$ projective structure $[\nabla]$ on $M \approx S^2$, $k \geq 3$, $\alpha \in (0,1)$. Then there is a $C^{k+1,\alpha}$ Riemannian metric g on M whose Levi-Civita connection ∇ belongs to the projective class $[\nabla]$ iff, after a* $\mathrm{PSL}(3,\mathbb{C})$ *transformation of $\mathbb{CP}_2$, the surface N avoids the conic $\mathcal{Q}$, and is Lagrangian with respect to the signed symplectic structure ω on $\mathbb{CP}_2 - \mathcal{Q}$. Moreover, such a Lagrangian embedding completely determines the metric g up to an overall multiplicative constant.*

Proof. In light of our previous discussion, it suffices to check the assertion in the 'if' direction. Thus, suppose the surface N corresponding to a given projective structure $[\nabla]$ on $M = S^2$ *does* avoid the conic $\mathcal{Q}$ and *is* Lagrangian with respect to the sign-ambiguous symplectic structure ω defined by (16). Because N is weakly unknotted by Theorem 4.4, there is a diffeomorphism $\phi : \mathbb{CP}_2 \to \mathbb{CP}_2$ which is C^0-isotopic to the identity and which carries carries $\mathbb{RP}^2$ to N, per the remark on page 497. Since ϕ_*^{-1} therefore identifies $H_2(\mathbb{CP}_2 - N, \mathbb{Z}) \to H_2(\mathbb{CP}_2, \mathbb{Z})$ with the injective homomorphism $H_2(\mathbb{CP}_2 - \mathbb{RP}^2, \mathbb{Z}) \to H_2(\mathbb{CP}_2, \mathbb{Z})$, the homology class of $\mathcal{Q}$ therefore generates $H_2(\mathbb{CP}_2 - N, \mathbb{Z})$. On the other hand, the divisor defined by the interior of any one of our holomorphic disks generates $H^2(\mathbb{CP}_2 - N, \mathbb{Z}) = H^2(\mathcal{U}_+, \mathbb{Z})$. Hence each of the holomorphic disks associated with $[\nabla]$ has intersection number $+1$ with the holomorphic curve $\mathcal{Q}$, and hence geometrically intersects $\mathcal{Q}$ transversely in a unique point. This gives us a diffeomorphism $\mathcal{Q} \approx M$, and hence fixes a conformal structure $[g]$ on M. Since $\mathcal{C} = \Psi^{-1}(\mathcal{Q})$ is a holomorphic curve in $\mathcal{U}_+$, and because the complex structure of $\mathcal{U}_+$ is of class $C^{k,\alpha}$, elliptic regularity tells us that $\mathcal{C}$ is a $C^{k+1,\alpha}$ section of $\mathcal{U}_+$. But $\mathcal{U}_+$ is precisely the bundle of oriented almost-complex structures on M, so this construction defines a $C^{k+1,\alpha}$ almost-complex structure on M. Our diffeomorphism $M \to \mathcal{Q}$ is holomorphic with respect to this almost-complex structure, and so is a map of class $C^{k+2,\alpha}$ by elliptic regularity. Hence the constructed conformal structure $[g]$ is of class $C^{k+1,\alpha}$.

Let (x^1, x^2) be $C^{k+2,\alpha}$ isothermal local coordinates on $(M, [g])$, obtained for free by taking $x^1 + ix^2$ to be a complex coordinate system on $\mathcal{Q}$. Relative to these coordinates, the conformal structure $[g]$ is then represented by the Euclidean metric $(dx^1)^2 + (dx^2)^2$, and the loci $\zeta = \pm i$ represent $\mathcal{C} = \Psi^{-1}(\mathcal{Q}) \subset \mathcal{U}_+$ and its complex conjugate $\overline{\mathcal{C}} \subset \mathcal{U}_-$. The $C^{k,\alpha}$ function $P(x, \zeta)$ of Equation (9) must therefore vanish when $\zeta = \pm i$, and we may therefore uniquely define a 1-form

$$\gamma = \gamma_1 \, dx^1 + \gamma_2 \, dx^2$$

of class $C^{k,\alpha}$ on our coordinate domain by requiring that

$$P(x, \zeta) = (1 + \zeta^2)(\gamma_2 - \gamma_1 \zeta).$$

Setting

$$\Gamma^j_{k\ell} = \gamma_k \delta^j_\ell + \gamma_\ell \delta^j_k - \gamma^j \delta_{k\ell},$$

we then may observe that $P(x, \zeta)$ is related to the Γ's by Equation (9).

Hence the torsion-free connection $\triangledown$ defined[4] by these Christoffel symbols in particular gives rise to the same foliation $\mathcal{F}$ of the real locus $\mathbb{P}TM$ defined by $P(x,\zeta)$. It follows that $\triangledown$ belongs to the given projective equivalence class $[\nabla]$, since it defines the correct family of geodesics.

For some choice of real constant $\lambda > 0$, let Υ now be defined by Equation (15). Pull this singular, multi-valued holomorphic $(2,0)$-form back to $\mathcal{U}_+ \cup \mathbb{P}TM$ via Ψ. The resulting 2-form annihilates $\mathbf{D}$, and we therefore have

$$\Psi^*\Upsilon = v(x,\zeta)\Big(d\zeta - P(x,\zeta)dx^1\Big)\wedge\Big(dx^2 - \zeta dx^1\Big) \tag{17}$$

for some unknown differentiable nonzero function v which is defined, up to sign, in the region $\Im m\ \zeta \geq 0$, $\zeta \neq i$. Notice that v is real along the locus $\Im m\ \zeta = 0$, because by assumption $\Im m\ \Upsilon$ annihilates TN. Now recall that Υ is closed, and observe that the condition $d(\Psi^*\Upsilon) = 0$ can be written as the pair of equations

$$\frac{\partial v}{\partial \bar{\zeta}} = 0 \tag{18}$$

and

$$\frac{\partial}{\partial \zeta}\Big(vP\Big) + \frac{\partial f}{\partial x^1} + \zeta\frac{\partial v}{\partial x^2} = 0. \tag{19}$$

Since $\Upsilon \otimes \Upsilon$ is meromorphic on $\mathbb{CP}_2$, with polar locus $3\mathcal{Q}$, Equation (18) tells us that, for each fixed $x = (x^1, x^2)$, $v^2(x,\zeta)$ is meromorphic in $\zeta = \xi + i\eta$ for $\eta > 0$, with only one pole, located at $\zeta = +i$ and of order 3. Because v^2 is real along $\eta = 0$, we may extend v^2 to all ζ so as to obtain a meromorphic function whose only poles are located $\zeta = \pm i$, both with order 3. However, we can also apply the same arguments to the function

$$\hat{v}(x,\hat{\zeta}) = -\frac{v(1/\hat{\zeta})}{\hat{\zeta}^3}$$

obtained in (17) by interchanging x^1 and x^2, and replacing ζ with $\hat{\zeta} = 1/\zeta$. Thus v^2 must have a zero of order 6 at $\zeta = \infty$, and we conclude that

$$v(x^1,x^2,\zeta) = \pm\frac{u(x^1,x^2)}{(1+\zeta^2)^{3/2}}$$

[4]We remark in passing that $\triangledown$ is actually the unique element of $[\nabla]$ which is also a *Weyl connection* for $[g]$, in the sense that that $\nabla_{\mathbf{v}} g \propto g \quad \forall \mathbf{v}$. Notice that the construction so far only depends on the choice of a conic $\mathcal{Q}$ avoiding N, and so can be carried out, e.g., for any Zoll projective structure on S^2 which is sufficiently close to the standard one.

for a unique positive differentiable function $u(x^1, x^2)$. Equation (19) thus tells us that

$$(1+\zeta^2)^{3/2}u(x^1,x^2)\frac{\partial}{\partial\zeta}\left(\frac{\gamma_2-\gamma_1\zeta}{(1+\zeta^2)^{1/2}}\right)+\frac{\partial u}{\partial x^1}+\zeta\frac{\partial u}{\partial x^2}=0,$$

or in other words that

$$(\gamma_1+\gamma_2\zeta)u(x)=\frac{\partial u}{\partial x^1}+\zeta\frac{\partial u}{\partial x^2}.$$

Thus $\gamma = d\log u$, and ∇ is therefore exactly the Levi-Civita connection of the metric

$$g=u^2\Big[(dx^1)^2+(dx^2)^2\Big].$$

Moreover, since γ is $C^{k,\alpha}$, it follows that u and g are of class $C^{k+1,\alpha}$.

Direct calculation now reveals that the restriction

$$\hat{\omega}=d\left(\frac{u\ dx^1+u\xi\ dx^2}{\sqrt{1+\xi^2}}\right)$$

of the symplectic form of T^*M to the unit cotangent bundle exactly coincides with the restriction of our expression (17) for $\Psi^*\Upsilon$ to the real slice $\zeta=\xi$. In particular, our construction of g is therefore coordinate-independent, because the set of unit covectors of g is precisely the image of $\Psi^*\Upsilon(\partial/\partial\theta,\cdot)$, where

$$\frac{\partial}{\partial\theta}=(1+\xi^2)\frac{\partial}{\partial\xi}$$

denotes the vertical vector field on $\mathbb{P}TM$ which generates the standard SO(2) action associated with the conformal class $[g]$. q.e.d.

Now the double cover of $\mathbb{CP}_2-\mathcal{Q}$ may explicitly be identified with the affine quadric $\mathcal{A}\subset\mathbb{C}^3$ given by

$$z_1^2+z_2^2+z_3^2=1,$$

and on $\mathcal{A}$ the symplectic form (16) simplifies to become

$$\omega=\Im m\Big(z_1\ dz_2\wedge dz_3+z_2\ dz_3\wedge dz_1+z_3\ dz_1\wedge dz_2\Big).$$

Now set $\vec{z}=\vec{x}+i\vec{y}$, and observe that $\mathcal{A}$ is defined by the pair of equations

$$\vec{x}\cdot\vec{y}=0,\quad |\vec{x}|=\sqrt{1+|\vec{y}|^2}$$

for $\vec{x}, \vec{y} \in \mathbb{R}^3$. This allows us to identify $\mathcal{A}$ with

$$T^*S^2 = \{(\vec{q}, \vec{p}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \vec{q} \cdot \vec{q} = 1\ ,\ \vec{q} \cdot \vec{p} = 0\}$$

via the diffeomorphism $\mathfrak{G} : T^*S^2 \to \mathcal{A}$ given by

$$\begin{aligned} \vec{x} &= \sqrt{1 + |\vec{p}|^2}\ \vec{q}, \\ \vec{y} &= \vec{p} \times \vec{q}. \end{aligned}$$

The pull-back $\mathfrak{G}^*\omega$ is then just the classical symplectic structure $\sum dp_j \wedge dq_j$ on T^*S^2. Indeed, since $\mathfrak{G}$ is SO(3)-equivariant with respect to the obvious, symplectic SO(3)-actions on T^*S^2 and $\mathcal{A} \subset \mathbb{C}^3$, it suffices to check this assertion along the curve $\vec{p} = (0, t, 0)$, $\vec{q} = (0, 0, 1)$, where the derivative $\mathfrak{G}_*$ acts by

$$\begin{aligned}
\frac{\partial}{\partial p_1} &=: \mathbf{e}_1 \mapsto \widetilde{\mathbf{e}}_1 := -\frac{\partial}{\partial y_2} \\
\frac{\partial}{\partial q_1} &=: \mathbf{e}_2 \mapsto \widetilde{\mathbf{e}}_2 := \sqrt{1+t^2}\frac{\partial}{\partial x_1} - t\frac{\partial}{\partial y_3} \\
\frac{\partial}{\partial p_2} &=: \mathbf{e}_3 \mapsto \widetilde{\mathbf{e}}_3 := \frac{\partial}{\partial y_1} + \frac{t}{\sqrt{1+t^2}}\frac{\partial}{\partial x_3} \\
\frac{\partial}{\partial q_2} - t\frac{\partial}{\partial p_3} &=: \mathbf{e}_4 \mapsto \widetilde{\mathbf{e}}_4 := \sqrt{1+t^2}\frac{\partial}{\partial x_2}
\end{aligned}$$

and where the components of $\sum dp_j \wedge dq_j$ and ω relative to the bases $\mathbf{e}$ and $\widetilde{\mathbf{e}}$, respectively, are in both cases just the entries of the matrix

$$\begin{bmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix}.$$

We remark, in passing, that the existence of the symplectomorphism $\mathfrak{G}$ illustrates a general result due to Weinstein [31]: any Lagrangian submanifold X of a symplectic manifold Y has a tubular neighborhood which is symplectomorphic to a neighborhood of the zero section of T^*X, equipped with its classical symplectic form. In the present case, however, one should also observe that, because our formula for $\mathfrak{G}$ involves the cross product in $\mathbb{R}^3$, the nontrivial deck transformation $\vec{z} \to -\vec{z}$ of $\mathcal{A}$ becomes the involution of T^*S^2 given by $\phi \mapsto -a^*\phi$, where $a : S^2 \to S^2$ is the antipodal map.

Now any embedding $S^2 \hookrightarrow T^*S^2$ which is C^1 close to the zero section is of course the graph of a 1-form ϕ on S^2. The condition for such a graph to be Lagrangian is that $d\phi = 0$, and, because $H^1(S^2) = 0$, this is equivalent to saying that $\phi = df$ for some function $f : S^2 \to \mathbb{R}$. On the other hand, in order for this $S^2 \hookrightarrow \mathcal{A}$ to double-cover an embedding $\mathbb{RP}^2 \hookrightarrow \mathbb{CP}_2 - \mathcal{Q}$ we must require that $\phi = -a^*\phi$, and in the Lagrangian case there will then be a unique f with $f = -a^*f$ such that $\phi = df$. Thus smooth Zoll metrics g of total area 4π on S^2 near the standard metric h correspond to smooth odd functions $f : S^2 \to \mathbb{R}$, $f(-\vec{x}) = -f(\vec{x})$. This of course simply reconfirms the functional freedom first predicted by Funk [12] and subsequently rigorously demonstrated by Guillemin [15].

5. Concluding remarks

A number of important technical issues remain to be resolved in connection with our treatment of Zoll structures on S^2. We have shown that one can associate a totally real embedding of $\mathbb{RP}^2$ in $\mathbb{CP}_2$ with each Zoll projective connection on S^2, and that, conversely, those embeddings which are sufficiently close to the standard one can be used to determine a projective connection on S^2. However, one loses a ridiculous number of derivatives in following the story full circle, back to one's starting point. Ideally, one might hope that $C^{k,\alpha}$ Zoll projective structures on S^2 should exactly correspond to $C^{k+1,\alpha}$ surfaces $N \subset \mathbb{CP}_2$. Alas, we are at present quite far from being able to make such an assertion in either direction.

What is worse, we do not at present know that our family of disks either exists or is unique when N is very far from the the standard $\mathbb{RP}^2$. Nonetheless, optimism might well be appropriate in the present instance. Let us thus throw caution to the wind, and hazard the following:

Conjecture 5.1. *The moduli space of Zoll metrics on S^2 is connected. Moreover, once we introduce a "marking" consisting of an orthonormal frame at some base-point, the moduli space of marked Zoll Riemannian structures of fixed total area is in natural* 1-1 *correspondence with the set of totally real Lagrangian embeddings $\mathbb{RP}^2 \hookrightarrow (\mathbb{CP}_2 - \mathcal{Q}, \omega)$ which are homotopic to the standard embedding.*

In fact, while it seems clear enough that the set of $N \subset \mathbb{CP}_2$ carrying suitable families of embedded holomorphic disks is open, there would be numerous technical difficulties involved in trying to show that it is also closed — e.g., sequences of embedded disks may have singular limits, and one tends, in the limit, to lose regularity of the dependence of families on parameters . Moreover, one would need to know that the relevant set of Lagrangian $\mathbb{RP}^2$'s in $(\mathbb{CP}_2 - \mathcal{Q}, \omega)$ is actually connected for this program to ultimately succeed. Fortunately, however, the latter is similar to problems already solved by Eliashberg [8, 9] and his co-workers, so there is ample reason to hope that such a program might be viable.

One might also want to hazard an analogous conjecture about Zoll projective structures. However, this would seem to be a considerably more difficult problem, as there is as yet no good mechanism for trying to show that two weakly unknotted embeddings of $\mathbb{RP}^2$ in $\mathbb{CP}_2$ are actually isotopic. On the other hand, Gromov's h-principle [14, 8] at least provides a rather complete reduction of questions concerning isotopy through totally real submanifolds to questions of isotopy in the more elementary sense.

It seems improbable that the methods we have developed here will shed much light on higher-dimensional Zoll manifolds, at least in the near term. However, our techniques certainly have obvious extensions which could be brought to bear on Zoll-like Lorentzian 3-manifolds [16], special classes of split-signature Einstein manifolds [22] and certain problems in Yang-Mills fields [23]. We look forward to watching the further development of our circle of ideas in connection with these problems.

Appendix A: Axisymmetric Zoll structures

The main result of this appendix is a formula for the general axisymmetric Zoll projective structure on S^2 obtained by perturbing the standard round structure. We first give the formulæ for the connection, and prove that it gives rise to a Zoll projective structure. We then go on to show how these examples arise from the twistor correspondence. The inclusion of the latter discussion is fundamentally more a matter of honesty than of logical exposition, as it primarily reveals how these examples were in fact discovered. However, we also hope that the reader will find this discussion useful insofar as it provides a carefully worked-out family of concrete examples which illustrate the twistor correspondence which plays such an essential rôle in the body of the paper.

A.1 Axisymmetric examples

In order to introduce the formulae, we first recall Zoll's original family of axisymmetric metrics expressed here in spherical polar coordinates

$$g = (F-1)^2 d\phi^2 + \sin^2\phi \; d\theta^2.$$

This metric is the same as that given by (1), after the coordinate transformation $z = \cos\phi$ and the substitution $F(\phi) = -f(\cos\phi)$.

We will express the general Zoll projective structure in terms of the difference between a compatible affine connection and the metric connection of the above metric. Consider the orthonormal frame

$$(\mathbf{e}_1, \mathbf{e}_2) = \left(\frac{1}{F-1}\frac{\partial}{\partial\phi}, \frac{1}{\sin\phi}\frac{\partial}{\partial\theta}\right)$$

and dual co-frame $(\theta^1, \theta^2) = ((F-1)d\phi, \sin\phi d\theta)$. In this frame, it is straightforward to calculate that the connection 1-form is

$$\omega = \frac{\cot\phi}{F-1}\theta^2.$$

The associated Levi-Civita connection, ∇^g, turns out to give the most general axisymmetric Zoll projective structure that is compatible with a metric (at least close to the round metric).

In general, a compatible torsion-free affine connection for a projective structure can be given by a connection ∇ such that, with

$$\gamma^k_{ij} = \left\langle \theta^k, (\nabla_i - \nabla^g_i)\mathbf{e}_j \right\rangle,$$

γ^k_{ij} is symmetric on the ij indices (so that ∇ is torsion-free) and trace free; this last condition corresponds to fixing the connection in the projective equivalence class by requiring that it preserve the metric volume form.

The general axisymmetric Zoll projective structure close to the round sphere will turn out to be given by the choice

$$\gamma^i_{11} = 0\,, \quad \gamma^1_{22} = -\frac{h^2\cot\phi}{F-1}\,, \quad \gamma^1_{21} = \frac{1}{3(F-1)}\left(\frac{\partial h}{\partial\phi} - \frac{2h}{\sin\phi\cos\phi}\right)$$

where all the other components of γ^k_{ij} are determined by the trace and symmetry conditions and $h = h(\phi)$ is a smooth function of ϕ vanishing in some small neighborhood of 0 and π and odd under $\phi \to \pi - \phi$.

This information can be encapsulated in the geodesic spray. This is the vector field on the projective tangent bundle $\mathbb{P}TS^2$ that at $(v, x) \in \mathbb{P}T_xS^2$ is the horizontal lift of the vector v at x. We parametrize the fiber of $\mathbb{P}TS^2$ by ζ corresponding to the vector $\mathbf{e}_1 + \zeta\mathbf{e}_2$. Then the geodesic spray from the projective structure above is given by

$$\begin{aligned}(20)\quad \Xi = \frac{\partial}{\partial\phi} + \left(\frac{F-1}{\sin\phi}\right)\zeta\frac{\partial}{\partial\theta} \\ - \zeta\left((1+\zeta^2(1+h^2))\cot\phi - \zeta\left(\frac{\partial h}{\partial\phi} - \frac{2h}{\sin\phi\cos\phi}\right)\right)\frac{\partial}{\partial\zeta}\end{aligned}$$

on $\mathbb{P}TS^2$ defines a Zoll projective structure if the smooth functions $F(\phi)$ and $h = h(\phi)$ are respectively odd and even under $\phi \to \pi - \phi$. For regularity at $\phi = \pi/2$, we further require that h should vanish (and hence to second order) at $\phi = \pi/2$. For regularity at $\phi = 0, \pi$, we assume that F and h vanish in some small neighborhood of these values. This is actually stronger than necessary, but makes the proof of the Zoll property more straightforward; the minimal requirement would be to just stipulate that they be smooth functions of $\cos\phi$ that vanish at $\phi = 0$ and π.

The metric case occurs when $h = 0$, and in this case there is the preferred overall scaling factor that gives the arc-length parameterization; this arises on dividing by $(F-1)\sqrt{(1+\zeta^2)}$. To see that the above gives a multiple of the geodesic spray in this case, coordinatize the tangent bundle by $(\mu_1, \mu_2) \to \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2$. Then the horizontal lift of $\mathbf{e}_1$ is just $\mathbf{e}_1$ since $\omega(\mathbf{e}_1) = 0$ and the horizontal lift of $\mathbf{e}_2$ is $\mathbf{e}_2 - \omega(\mathbf{e}_2)(\mu_1\frac{\partial}{\partial\mu_2} - \mu_2\frac{\partial}{\partial\mu_1})$. Thus, using the affine coordinate $\zeta = \mu_2/\mu_1$ on the projective tangent bundle, the geodesic spray will be

$$\mathbf{e}_1 + \zeta\left(\mathbf{e}_2 - (1+\zeta^2)\frac{\cot\theta}{F-1}\frac{\partial}{\partial\zeta}\right)$$

and this can be seen to be proportional to the formula given above when $h = 0$ as required. If we wish to normalize the horizontal part to have unit length, then we must divide by $\sqrt{(1+\zeta^2)}$ and this will give the overall factor required to give proper length parameterization.

We first give a direct proof of the Zoll property, and then in the subsequent sections we show how the formula arises from the twistor construction. (The direct proof of the Zoll property below in fact will use equations arising in the twistor derivation below, but it is easily checked that these follow directly from the form of the geodesic spray above. It is difficult, however, to see how they might have been anticipated without the twistor construction.)

Theorem A.1. *Equation* (20) *defines a Zoll projective structure for all smooth odd functions F and even functions h with $h(\pi/2) = 0$ and both h and F vanishing in some neighborhood of $\phi = 0$.*

The proof is divided into two parts. We first analyze the flow of the projection of the geodesic spray under $q : \mathbb{P}TS^2 \to \mathbb{RP}^1 \times [0, \pi]$, $q(\zeta, \phi, \theta) = (\zeta, \phi)$, to the space of orbits of $\partial/\partial\theta$ in $\mathbb{P}TS^2$. We show first that the orbits of the projected flow are circles, and secondly that the lifts of these to orbits of the full geodesic spray are also circles in $\mathbb{P}TS^2$.

1) We first study the integral curves of $q_*\Xi$ for $(\zeta, \phi) \in \mathbb{RP}^1 \times [0, \pi/2]$. Introduce the angular coordinate $\psi \in [0, \pi)$ on $\mathbb{RP}^1$ by $\zeta = \tan\psi$ so that ψ is a smooth coordinate near $\zeta = \infty$. Then, the flow becomes

(21)

$$\begin{aligned} \dot\phi &= \sin\phi\cos\psi\,, \\ \dot\psi &= -\sin\psi\left((1 + h^2\sin^2\psi)\cos\phi - \cos\psi\sin\psi\left(\frac{\partial h}{\partial\phi}\sin\phi - \frac{2h}{\cos\phi}\right)\right) \end{aligned}$$

where $\dot\phi = d\phi/dt$ for the time parameter t along the flow defined by

$$\frac{d}{dt} = \sin\phi\cos\psi q_*\Xi\,.$$

These additional factors yield a smooth flow by inspection noting in particular that our requirement that $h(\pi/2) = 0$ implies that $h/\cos\phi$ is smooth. Note also that this flow is invariant under the reflection in $\phi = \pi/2$: $(\psi, \phi, t) \to (\psi, \pi - \phi, -t)$.

[The perceptive reader might have noticed that the direction of the flow changes sign across the identification of $\psi = \pi$ with $\psi = 0$. Although the flow defines a smooth distribution in the projective tangent bundle away from the fixed points, to obtain a flow with continuous direction, we would need to work on the double cover obtained by factoring the tangent bundle by the positive scalings. This will not be a problem in the following as $\psi = 0$ or π is a flow line.]

Lemma A.2. *The flow of $q_*\Xi$ has fixed points at $(\psi, \phi) = (0, 0)$, $(0, \pi)$ and $(\pi/2, \pi/2)$. The integral curves of $q_*\Xi$ are smoothly embedded curves in $(\zeta, \phi) \in \mathbb{RP}^1 \times [0, \pi]$ on which, for $\phi \in [0, \pi/2]$ (resp. $\phi \in [\pi/2, \pi]$) the coordinate ϕ decreases (resp. increases) from $\pi/2$ to a unique minimum (resp. maximum) value and then increases (resp. decreases) again to $\pi/2$. The extrema occur when $\psi = \pi/2$.*

The fixed points are where both the right-hand sides vanish, so that, from $\dot{\phi} = 0$ we obtain either $\phi = 0, \pi$ or $\psi = \pi/2$. At $\phi = 0, \pi$, $h = 0$ and so we find $\dot{\psi} = \mp \sin\psi$, i.e., a fixed point at $\psi = 0 (= \zeta)$. At $\psi = \pi/2$, we find that $\dot{\psi} = 0$ iff $\cos\phi = 0$, i.e., $\phi = \pi/2$.

It is clear from the first of Equations (21) that for $\phi \in (0, \pi)$, $\dot{\phi}$ only vanishes when $\psi = \pi/2$. The second derivative at $\psi = \pi/2$ can be calculated to give

$$\frac{\partial^2 \phi}{\partial \psi^2} = (1 + h^2) \cot\phi$$

and it can be seen that this second derivative $\partial^2\phi/\partial\psi^2$ is positive for $\phi \in (0, \pi/2)$ and so this must be a minimum. Similarly on $\phi \in (\pi/2, \pi)$, ϕ can only be a maximum at a stationary point. Thus, on an integral curve in $\phi \in (0, \pi/2)$, ϕ will descend to a unique minimum value, at which $\zeta = \infty$ and then increase again. q.e.d.

The key issue now is as to whether we can make these integral curves join up into a circle. Firstly note that $\psi = 0$ and $\phi = 0, \pi$ are all flow lines, and these are the only flow lines limiting onto the fixed points $(0, 0)$ and $(0, \pi)$ as we have assumed that $h = 0$ in a neighborhood of $\phi = 0$ and of π, and this means that the flow lines in those neighborhoods are precisely those of the flat case, and these are precisely the level curves of $\sin\phi \sin\psi$.

Let us suppose that a curve starts at some value of $\psi \in (0, \pi/2)$. Then ϕ will descend to a minimum and either (a) increase up to $\pi/2$ again, or (b) the minimum will be $\phi = 0$. In case (a), the reflection of the orbit under the involution $(\psi, \phi, t) \to (\psi, \pi - \phi, -t)$ will be an orbit in $\phi \in (\pi/2, \pi)$ and this will join up to make a circular orbit. Case (b) will be the case $\psi = 0$ since the orbit must intersect $\phi = 0$ at $\psi = 0$, since the complement of that point in $\phi = 0$ is a regular orbit on its own, but the only orbit in a neighborhood of $\phi = 0$ that intersects this fixed point is $\psi = 0$ (or $\phi = 0$).

Thus, all the orbits of the flow are circles, except the above mentioned fixed points and special orbits that limit onto the fixed points; this gives the flow diagram 1.

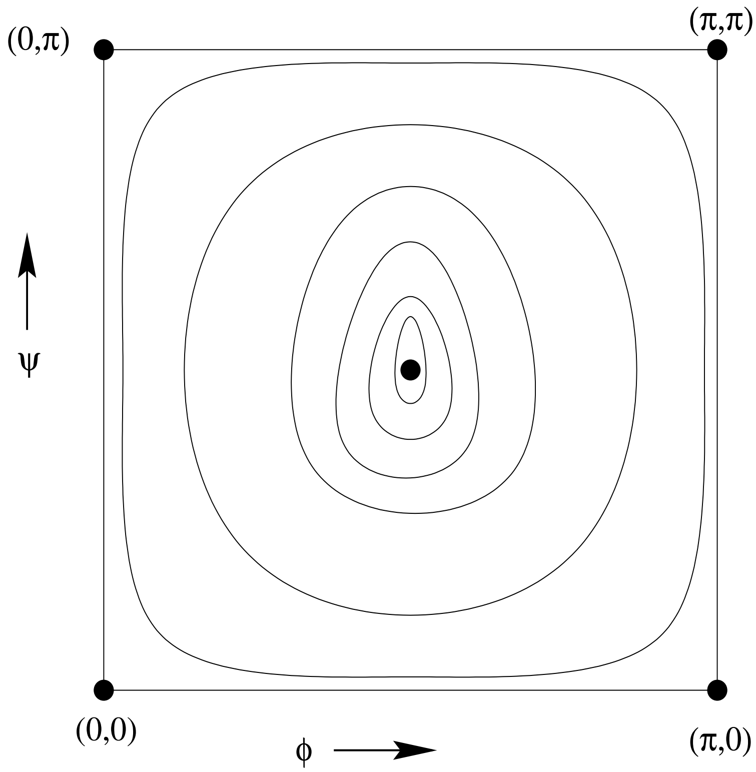

Figure 1: The flow diagram for the projected flow.

2) We now wish to show that these orbits in the (ψ, ϕ) plane only lift to give closed S^1 orbits in the full projective tangent bundle of the sphere. In the above coordinates, the equation for θ will become

$$\dot{\theta} = (F - 1) \sin \psi \,. \tag{22}$$

In order for the geodesics to be circles, we need to prove that the integral of the right-hand side around an integral curve of $q_* \Xi$ is 0 modulo 2π for each integral curve. The first and second terms in the right-hand side of Equation (22) are respectively odd and even under $\theta \to \pi - \theta$. Since the integral curves of $q_* \Xi$ are even, the first part will automatically integrate to zero. We need to show, then, that the second part will in fact integrate to 0 modulo 2π on all integral curves.

To integrate $\dot{\theta} = - \sin \psi$, from Equations (26) and (27) in the lifting part of the twistor construction, we note that, with $1/a = (h - i)|\cos\phi|$

$$\omega = \frac{1}{2} \arg \frac{(1 - \zeta/\bar{a})}{1 - \zeta/a}$$

satisfies

$$\dot{\omega} = -\sin\psi\,.$$

(We leave it to the assiduous reader to show that Equations (26) and (27) below follow independently of the twistor construction.) Thus, $\theta = \omega$ is the solution to the even part of the θ flow. However, ω is the argument of a single valued complex function on the (ψ, ϕ)–plane, and so, when we do a complete circuit around an integral curve of $q_*\Xi$ returning to our original point, the argument must return to zero modulo 2π. q.e.d.

A.2 The twistor construction in the axisymmetric case

In §A.2.1 we study the structure of the action of axisymmetry on the twistor space and the correspondence for the round metric. In §A.2.2 we give the axisymmetric deformations of the real slice. The subsequent subsection §A.2.3 is devoted to constructing the holomorphic disks, and then finally in §A.2.4 the associated projective structure is constructed.

A.2.1 The round sphere

We consider the action of the standard rotation on $\mathbb{R}^3$, its complexified action on $\mathbb{C}^3$ and induced action on $\mathbb{CP}_2$. With coordinates $(z, \widetilde{z}, z_0)$, the S^1 action is generated by the real part of the holomorphic vector field

$$\frac{\partial}{\partial\theta} = i\left(z\frac{\partial}{\partial z} - \widetilde{z}\frac{\partial}{\partial\widetilde{z}}\right),$$

where $\mathbb{R}^3$ is taken to be $\widetilde{z} = \bar{z}$ and $z_0 = \bar{z}_0$. If we remove the the fixed points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, the generic orbits form the pencil of conics $(1 - w)z_0^2 = wz\widetilde{z}$ that are tangent to the line $z = 0$ at $(0, 1, 0)$ and also to the line $\widetilde{z} = 0$ at $(1, 0, 0)$. The degenerate orbits consist of the double line $z_0 = 0$ at $w = 0$ and the pair of lines $z = 0$ and $\widetilde{z} = 0$ at $w = 1$. They determine a fibration of $\mathbb{CP}_2 - \{(1, 0, 0), (0, 1, 0)\}$ over $\mathbb{CP}_1$ with affine coordinate w, and, away from the exceptional fibers at $w = 0, 1$, we can coordinatize $\mathbb{CP}_2$ with $(w, \xi) = (z_0^2/(z_0^2 + z\widetilde{z}), z/z_0)$. In these coordinates $\frac{\partial}{\partial\theta} = i\xi\frac{\partial}{\partial\xi}$.

The real slice, $\mathbb{RP}^2$, is given by $w \in [0,1]$ and $|\xi|^2 = -1 + 1/w$. Note that the orbit $z_0 = 0$ intersects $\mathbb{RP}^2$ in a real line, whereas the orbit $\{z = 0\} \cup \{\tilde{z} = 0\}$ intersects $\mathbb{RP}^2$ in a single point. All the other real orbits are contractible circles in $\mathbb{RP}^2$.

Introduce spherical polar coordinates (ϕ, θ) on S^2 so that the symmetry is $\frac{\partial}{\partial\theta}$. We can coordinatize the fibers of the projective tangent bundle by ζ so that ζ corresponds to the vector $\frac{\partial}{\partial\phi} + \frac{\zeta}{\sin\phi}\frac{\partial}{\partial\theta}$. (These coordinates will then only break down at the fixed points.) The lines in $\mathbb{CP}_2$ corresponding to points of S^2 are $2z_0 = \tan\phi(e^{i\theta}z + e^{-i\theta}\tilde{z})$. In terms of ζ, and the coordinates (w, ξ) on $\mathbb{CP}_2$, the holomorphic disks are the images of the upper-half plane in ζ under

$$w = \frac{\zeta^2 \sin^2\phi}{1+\zeta^2}, \qquad \xi = e^{i\theta}\frac{\zeta\cos\phi + i}{\zeta\sin\phi}, \tag{23}$$

and when ζ is real the image lies in $\mathbb{RP}^2$.[5]

It is worth noting for later use that, on these disks,

$$\zeta = \sqrt{w/(\sin^2\phi - w)}$$

defines the square root in the upper-half plane.

In these coordinates, the geodesic spray takes the form:

$$\Xi = \frac{\partial}{\partial\phi} + \frac{\zeta}{\sin\phi}\frac{\partial}{\partial\theta} - \cot\phi(1+\zeta^2)\zeta\frac{\partial}{\partial\zeta}.$$

It should also be noted that the conserved quantity associated to the axial symmetry $\frac{\partial}{\partial\theta}$ and metric $g = d\phi^2 + \sin^2\phi\, d\theta^2$ is

$$\frac{g(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi} + \frac{\zeta}{\sin\phi}\frac{\partial}{\partial\theta})}{\sqrt{g(\frac{\partial}{\partial\phi} + \frac{\zeta}{\sin\phi}\frac{\partial}{\partial\theta}), \frac{\partial}{\partial\phi} + \frac{\zeta}{\sin\phi}\frac{\partial}{\partial\theta})}} = \sqrt{w}.$$

This formula can also be derived intrinsically on $\mathbb{CP}_2$; namely, $\sqrt{w}$ is the Hamiltonian for $\frac{\partial}{\partial\phi}$ using the symplectic form associated to the conic $\mathcal{Q}$ defined in Equation (16).

[5] A global and invariant formulation can be obtained in index notation by letting z_i, $i = 1, \dots 3$ be homogeneous coordinates on $\mathbb{CP}_2$, and x^i coordinates on $\mathbb{R}^3$, then the open disk in $\mathbb{CP}_2$ corresponding to x^i on S^2 is given by the condition that $iz_i\bar{z}_j\varepsilon^{ijk}$ be a positive multiple of x^k.

A.2.2 Deformation of the real slice

We will represent a circle invariant deformed embedding of $\mathbb{RP}^2$ into $\mathbb{CP}_2$ as the set given by

$$w = \gamma(\phi)\,, \text{ and } |\xi|^2 = e^{g(\phi)}\left|\frac{1-\gamma(\phi)}{\gamma(\phi)}\right|$$

for $\phi \in [0, \pi/2]$. Here g is a smooth real function with compact support in $(0, \pi/2)$ and $\gamma : [0, \pi/2] \to \mathbb{C}$ is a smooth embedded curve from $w = 0$ to $w = 1$ such that $\gamma(\phi) = \sin^2\phi$ on the complement of some compact subset of $(0, \pi/2]$.

In the homogeneous case, $\gamma(\phi) = \sin^2\phi$, and $g = 0$. The compact support of the deviation from the homogeneous case will guarantee smoothness of this deformation near the degenerate fiber $z_0 = 0$. In particular, the embedding of $\mathbb{RP}^2$ into $\mathbb{CP}_2$ near the fixed line $z_0 = 0$, is the same as the canonical embedding, and so the holomorphic disks near those at $z_0 = 0$ will be those above in Equation (23) and so we will not need to concern ourselves with singular behavior there.

These assumptions amount to the assumption that our S^1–invariant Zoll projective structure on S^2 will have two fixed points corresponding to $\phi = 0, \pi$ in a neighborhood of each of which the projective structure will be that of the round sphere, and exactly one of the S^1 orbits will be a geodesic with $\phi = \pi/2$.

In the metric case we will have that $\gamma(\phi) = \sin^2\phi$ since the square of the conserved quantity is determined by the geometry of the action on $\mathbb{CP}_2$ relative to its fixed symplectic structure. It will necessarily be equal to w, and will be real on the real slice. The nontrivial information in this case is contained only in the function $g(\phi)$.

For later convenience, we extend γ and g to $\phi \in [0, \pi]$ by $\gamma(\phi) = \gamma(\pi-\phi)$ and $g(\phi) = g(\pi-\phi)$. The data of the location of the deformation of $\mathbb{RP}^2$ could be represented more economically by expressing the curve γ as a graph of the imaginary part over the real interval $[0, \pi/2]$. However the formulation above will allow us to make a convenient choice of the coordinate ϕ later.

A.2.3 Construction of the holomorphic disks

The problem of finding the deformed disks with boundary on the deformed real slice decomposes into two parts: firstly that of finding the projection of the disk to the w–Riemann sphere with boundary on the projection of the real slice, the curve γ, and secondly, the problem of lifting the disk to $\mathbb{CP}_2$.

1) The projected disks must have their boundary on some subinterval of the curve γ. This subinterval must include the end at $\phi = 0$: this end corresponds to the line $z_0 = 0$ and each boundary of a disk must be homologous to this line, but because these are all generators of the homology of $\mathbb{RP}^2$, they must intersect each other at least once.

Thus, the first task is to find, for each $\phi \in [0, \pi/2]$, a map $\zeta \to w(\zeta, \phi)$ from the upper half plane into the w-Riemann sphere such that the boundary of the disk is mapped to the image of the interval $[0, \phi]$ under γ.

To analyze this, first consider the conformal map

$$w \to v(w, \phi) = \sqrt{\frac{w}{\gamma(\phi) - w}}\,,$$

where we fix the branch of the square root by requiring that, near $w = 0$, $v\sqrt{\gamma(\phi)}$ lies in the upper-half-plane (there is no obstruction to choosing $\sqrt{\gamma(\phi)}$ as ϕ varies so that it is positive for small ϕ). In the v-Riemann sphere, the image of $\gamma([0, \phi])$ is a continuously differentiable embedded circle tangent to $\sqrt{\gamma(\phi)}\times$ the real axis at the origin and passing through the point $v = \infty$. It will be smooth except possibly at 0 and ∞. Thus the branch defined above is well-defined and determines a region V_ϕ in the v-plane as the image of the complement of $\gamma([0, \phi])$.

By the Riemann mapping theorem there will exist a conformal map from the upper-half-plane in ζ to V_ϕ and hence to the complement of $\gamma([0, \phi])$ in the w-Riemann sphere. It will be smooth with nonvanishing derivative up to and including the boundary on the v-Riemann sphere except possibly at 0 and ∞ where it is nevertheless guaranteed to be continuous [30, p. 340]. It is worth emphasizing that while Proposition 4.5 guarantees that the disks will be smoothly embedded in $\mathbb{CP}_2$, but they will be tangent to the fibers of the projection along the orbits of the complexified axisymmetry at $w = 0$ and $\gamma(\phi)$. Hence, the projection of the disks to the w-Riemann sphere will be smooth up to $\gamma([0, \phi])$ except at the points 0 and $\gamma(\phi)$ which will be ramification points of order 2. Using a Möbius transformation of the upper-half plane to itself, this map $w(\zeta, \phi)$ can be chosen so that

$$w(\zeta, \phi) = \zeta^2 \sin^2 \phi + O(\zeta^3)\,,$$

at $\zeta = 0$ and $w(\zeta, \phi) = \gamma(\phi) - k(\phi)\gamma'(\phi)\zeta^{-2} + O(\zeta^{-3})$ at $\zeta = \infty$ for some real $k(\phi) > 0$.

For later use we define the function $s(\zeta,\phi)$ for $\zeta\in\mathbb{R}$, $\phi\in[0,\pi]$ by the condition that

$$\gamma(s(\zeta,\phi))=w(\zeta,\phi).$$

In the following we extend both $w(\zeta,\phi)$ and $s(\zeta,\phi)$ to $\phi\in[0,\pi]$ so that they are even functions under $\phi\to\pi-\phi$.

2) We now wish to find the lift of these conformal mappings to disks in $\mathbb{CP}_2$ with boundary on the deformed real slice. To do this we need to obtain $\xi(\zeta,\phi,\theta)$ holomorphic on the upper-half-plane in ζ such that, for $\zeta\in\mathbb{R}$,

$$|\xi(\zeta,\phi,\theta)|^2=e^{g(s(\zeta,\phi))}\left|\frac{1-\gamma(s(\zeta,\phi))}{\gamma(s(\zeta,\phi))}\right|.$$

By symmetry we must have $\xi(\zeta,\phi,\theta)=e^{i\theta}\xi(\zeta,\phi,0)$.

The orbits of the complexified axisymmetry corresponding to $w\neq 0,1$ are regular orbits. Thus for $w(\zeta,\phi)\neq 0,1$, the lift $\xi(\zeta,\phi,\theta)$ cannot meet $\xi=0$ or ∞ since $\xi=0$ is part of the orbit $w=1$ and $\xi=\infty$ is the orbit $w=0$. However, as $w\to 0$ we must have, by the above condition on the real slice, $|\xi|^2\to|(1-w)/w|\to\infty$. Furthermore, if $\phi=\pi/2$, $w=1$ is a real point on the boundary of the conformal mapping and must therefore lift to the real point $\xi=\widetilde{\xi}=0$. Conversely, at $w=1$, but $\phi\neq\pi/2$, the point $w=1$ is not a real point on the disk and so we cannot have both $\xi=0$ and $\widetilde{\xi}=0$. Hence either we will have $\xi=0$ and $\widetilde{\xi}\neq 0$, or $\xi\neq 0$ and $\widetilde{\xi}=0$. We can therefore assume that, by continuity from the round sphere case, $\xi\neq 0$ for $\phi\in[0,\pi/2)$, and $\widetilde{\xi}\neq 0$ for $\phi\in(\pi/2,\pi]$.

By taking logs, the problem of lifting the conformal maps to disks in $\mathbb{CP}_2$, can be reduced to an abelian problem. However, we cannot proceed completely naively as we will still have $\xi\to\infty$ as $w\to 0$, although we can guarantee that either ξ or $\widetilde{\xi}$ will be nonvanishing. We work first on $\phi\in(0,\pi/2)$ so that $\xi\neq 0$, and divide that problem into a part that is regular on taking logs, and one that can be handled explicitly. Set

$$\xi(\zeta,\phi,\theta)=e^{i\theta+G(\zeta,\phi)}\Gamma(\zeta,\phi)$$

then we wish to find $G(\zeta,\phi)$ that is holomorphic for $\Im m\zeta>0$ such that for ζ real

$$\Re e G(\zeta,\phi)=g(s(\zeta,\phi))$$

and similarly we wish to find $\Gamma(\zeta,\phi)$ holomorphic on the upper half plane in ζ, such that for ζ real

$$|\Gamma(\zeta,\phi)|^2=\left|\frac{1-\gamma(s(\zeta,\phi))}{\gamma(s(\zeta,\phi))}\right|.$$

The first problem is solved in a standard way by a contour integral along the real axis

$$G(\zeta,\phi)=\frac{1}{2\pi i}\oint\frac{\Re eG(\mu,\phi)}{\mu-\zeta}d\mu-\frac{1}{2\pi i}P.V.\int\frac{\Re eG(\mu,\phi)}{\mu}d\mu$$

where the purpose of the last term is to remove the ambiguity associated with the addition of a constant (in ζ but perhaps with ϕ–dependence) to the imaginary part of G. This choice ensures $\Im mG(0,\phi)=0$.

The problem for Γ cannot be solved so simply in the above way. First we define the complex function $a(\phi)$ in the upper half plane by the condition $w(a,\phi)=1$, i.e., the image in the ζ plane of $w=1$. Then the function

$$\Gamma(\zeta,\phi)=i\sqrt{\frac{(1-\zeta/\bar a)}{(1-\zeta/a)}\frac{(1-w)}{w}}$$

makes sense for ζ in the upper half plane since the function whose root is taken does not vanish on the upper half plane . We choose the branch for the square root that tends towards $i/\zeta\sin\phi$ as ϕ and ζ tend to zero. Then Γ as defined is nonvanishing, holomorphic in the upper half plane and has the required modulus when $\zeta\in\mathbb{R}$ as then $|(1-\zeta/\bar a)/(1-\zeta/a)|=1$.

For $\phi\in[\pi/2,\pi]$ we work with $\widetilde{\xi}$ as that will be nonzero on this interval. However,

$$|\widetilde{\xi}(\zeta,\phi,\theta)|^2=\frac{|1-w|^2}{|w^2\xi^2|}=e^{-g(s(\zeta,\phi)}\frac{|1-\gamma(s,\zeta,\phi)|}{|\gamma(s,\zeta,\phi)|}\,.$$

and so the solution will be

$$\widetilde{\xi}=e^{-i\theta-G(\zeta,\pi-\phi)}\Gamma(\zeta,\pi-\phi)\,,$$

where the Γ and G are the functions obtained above.

A.2.4 Construction of the projective structure

To reconstruct the corresponding projective connection on S^2, we wish to construct the vector field determining the geodesic spray on the correspondence space, $\mathbb{P}TS^2$. We use coordinates (ϕ,θ) on S^2, and $\zeta\in\mathbb{R}$ on the fibers of $\mathbb{P}TS^2$. We construct the geodesic spray Ξ in two steps:

1) Under the projection $q:(\zeta,\phi,\theta)\to(\zeta,\phi)$, Ξ projects to $q_*\Xi=\frac{\partial}{\partial\phi}-p(\zeta,\phi)\frac{\partial}{\partial\zeta}$ for some $p(\zeta,\phi)$. The function w is constant along the geodesic spray so that $q_*\Xi w=0$ which gives $p=\partial_\phi w/\partial_\zeta w$.

When $\zeta \in \mathbb{R}$, $w = \gamma(s(\zeta, \phi)$, so

$$p(\phi, \zeta) = \frac{\gamma' \partial s/\partial \phi}{\gamma' \partial s/\partial \zeta} = \frac{\partial s/\partial \phi}{\partial s/\partial \zeta}$$

is real. Thus p can be extended meromorphically over the ζ Riemann sphere by defining it in the lower-half plane to be the complex conjugate of the pullback under $\zeta \to \bar{\zeta}$. The fact that it is real for $\zeta \in \mathbb{R}$ ensures continuity and hence holomorphy there. It does, however, have simple poles at $\zeta = 0, \infty$ as $\frac{\partial}{\partial \zeta} w$ has simple zeroes there. However, the chosen form at $\zeta = 0$ implies that in fact, p vanishes at $\zeta = 0$. Thus, since $p\frac{\partial}{\partial \zeta}$ is globally holomorphic except a simple pole at $\zeta = \infty$ (as a vector field on the Riemann sphere), zero at $\zeta = 0$ and real for ζ real, we can write

$$p = \zeta(\Gamma_2 \zeta^2 + \Gamma_1 \zeta + \cot \phi)$$

where Γ_1 and Γ_2 are real functions of ϕ and the $\cot \phi$ follows from the expansion at $\zeta = 0$.

Note here that since w and s are even functions under $\phi \to \pi - \phi$, Γ_2, Γ_1 and $\cot \phi$ are odd as they involve the ϕ derivatives of s.

We will need the fact later that Γ_1 and Γ_2 can be expressed in terms of $a(\phi)$ and its first derivative by using the condition

$$q_* \Xi(\zeta - a)|_{\zeta = a} = 0 \tag{24}$$

which follows from the fact that $\zeta = a$ corresponds to $w = 1$ which is a holomorphic curve in $\mathbb{CP}_2$. This yields the equation

$$\frac{\partial}{\partial \phi} a + a(\Gamma_2 a^2 + \Gamma_1 a + \cot \phi) = 0$$

and this together with its complex conjugate yields

$$\Gamma_2 = \frac{\sin \phi}{\bar{a} - a} \frac{\partial}{\partial \phi} \left(\frac{a - \bar{a}}{|a|^2 \sin \phi} \right), \quad \Gamma_1 = \frac{\sin \phi}{\bar{a} - a} \left(\bar{a} \frac{\partial}{\partial \phi} \left(\frac{1}{a \sin \phi} \right) - c.c. \right). \tag{25}$$

The number of free functions here is two: either the pair Γ_1 and Γ_2 or, equivalently, the real and imaginary parts of a. This is to be compared to the one free function we have in the data of the curve $\gamma(\phi)$ in the reduced twistor space and the second free function we have in choosing the coordinate ϕ, which, up to now, has been arbitrary (at least away from $\phi = 0, \pi/2$). We will fix this coordinate freedom subsequently.

2) The next step is to lift $q_*\Xi$ to the vector field Ξ on the full correspondence space $\mathbb{P}TS^2$ that annihilates also ξ or equivalently $\widetilde{\xi}$. We will have

$$\Xi = q_*\Xi - \frac{(q_*\Xi\xi)}{\partial_\theta \xi}\frac{\partial}{\partial\theta} = q_*\Xi - \frac{(q_*\Xi\xi)}{i\xi}\frac{\partial}{\partial\theta} = q_*\Xi + i(q_*\Xi \log\xi)\frac{\partial}{\partial\theta} \,.$$

In order to proceed further, note that the coefficient of $\frac{\partial}{\partial\theta}$ is $iq_*\Xi\log\xi$, and this is (a) holomorphic over upper-half-plane in ζ, and (b) is real for $\zeta \in \mathbb{R}$ since the imaginary part of

$$iq_*\Xi \log\xi|_{\zeta=\bar\zeta} = i\left(\frac{\partial}{\partial\phi} - \frac{\partial s/\partial\phi}{\partial s/\partial\zeta}\frac{\partial}{\partial\zeta}\right)\log\xi$$

is just $q_*\Xi\log|\xi|$ but $\log|\xi| = \Re e \log\Gamma + \Re G$ is a function of ζ and ϕ only through s, and such functions of s alone are annihilated by $q_*\Xi$ by construction. Thus, the imaginary part of the right hand side of the above equation vanishes for $\zeta \in \mathbb{R}$. Hence, we can extend it meromorphically over the ζ–Riemann sphere by setting it to be the complex conjugate of the pullback under $\zeta \to \bar\zeta$ for $\Im\zeta < 0$ and noting that reality at $\zeta \in \mathbb{R}$ implies continuity and hence holomorphy across the real axis.

The function $iq_*\Xi\log\xi$ divides into two parts:

$$iq_*\Xi\log\xi = iq_*\Xi G(\zeta,\phi) + iq_*\Xi\log\Gamma \,,$$

and since w is constant along $q_*\Xi$, the second part reduces to

$$iq_*\Xi\log\Gamma = \frac{i}{2}q_*\Xi\log\frac{1-\zeta/\bar a}{1-\zeta/a} \,.$$

They are both holomorphic on the full ζ sphere, except with poles at $\zeta = \infty$ since $q_*\Xi$ has one there. However, they will also have a simple zero at $\zeta = 0$ since the imaginary parts of G and the above expression for $iq_*\Xi\log\Gamma$ vanish there by construction. (The possible apparent poles in $iq_*\Xi\log\Gamma$ are removable as a consequence of Equation (24).) Therefore

$$iq_*\Xi G = \frac{F(\phi)}{\sin\phi}\zeta \,, \qquad \text{and} \qquad iq_*\Xi\log\Gamma = \beta(\phi)\zeta \tag{26}$$

for some real functions F and β and the geodesic spray is

$$\Xi = \frac{\partial}{\partial\phi} + \left(\frac{F}{\sin\phi} + \beta\right)\zeta\frac{\partial}{\partial\theta} - \zeta(\Gamma_2\zeta^2 + \Gamma_1\zeta + \cot\phi)\frac{\partial}{\partial\zeta} \,.$$

Using the above and Equations (24) and (25) we calculate directly that

$$\beta = -\Gamma_2 \Im m a \,.$$

When $\phi \in [\pi/2, \pi]$ we should note first that G and Γ are even functions under $\phi \to \pi - \phi$. Hence, F and β are, as defined, odd functions. However, there is a further sign change on using $\widetilde{\xi}$ instead of ξ for β which yields an *even* contribution for β and odd for F and p, i.e., for $\phi \in [\pi/2, \pi]$

$$\Xi = \frac{\partial}{\partial\phi} + \left(-\frac{F(\pi-\phi)}{\sin\phi} + \beta(\pi-\phi)\right)\zeta\frac{\partial}{\partial\theta} + p(\pi-\phi,\zeta)\frac{\partial}{\partial\zeta}\,.$$

We now fix the choice of the coordinate ϕ which up to now has been arbitrary except near $\phi = 0$ and $\pi/2$. We do this by imposing

$$\Im m\frac{1}{a} = -|\cos\phi|$$

(note that a must always be in the upper half plane , and must be even under $\phi \to \pi - \phi$). This gives

$$\beta = -1/\sin\phi \tag{27}$$

Introduce the function $h(\phi)$ by

$$\Re e\frac{1}{a} = h|\cos\phi|$$

and this leads to the formulae

$$\Gamma_1 = -\frac{\partial}{\partial\phi}h + \frac{2h}{\sin\phi\cos\phi}\,, \quad \Gamma_2 = \cot\phi\left(1+h^2\right)\,.$$

This leads to our final formula for the geodesic spray

$$\begin{aligned}\Xi = {} & \frac{\partial}{\partial\phi} + \frac{F-1}{\sin\phi}\zeta\frac{\partial}{\partial\theta} \\ & - \left((1+\zeta^2+\zeta^2h^2)\cot\phi - \zeta\left(\frac{\partial h}{\partial\phi} - \frac{2h}{\sin\phi\cos\phi}\right)\right)\zeta\frac{\partial}{\partial\zeta}\end{aligned} \tag{28}$$

where F must be odd under $\phi \to \pi - \phi$ and h must be even. For regularity, h should vanish to second order at $\phi = \pi/2$. From the assumption that the twistor data was zero in some small neighborhood of the fixed line $z_0 = 0$, we also deduce that the functions h and F should vanish in some small neighborhood of $\phi = 0, \pi$. This is the formula that leads to the expressions given at the beginning of this appendix.

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